# Interpolation and stability estimates for edge and face virtual elements of general order

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**Abstract**

We develop interpolation error estimates for general order standard and serendipity edge and face virtual elements in two and three dimensions. Contextually, we investigate the stability properties of the associated $L^2$ discrete bilinear forms. These results are fundamental tools in the analysis of general order virtual elements, e.g., for electromagnetic problems.

**Keywords:** edge and face virtual element spaces; serendipity spaces; polytopal meshes; interpolation properties; stability analysis.
**AMS classification:** 65N12; 65N15.

## 1 Introduction

The virtual element method (VEM) [6] can be interpreted as an extension of the finite element method (FEM) to polytopal meshes. Trial and test spaces typically contain a polynomial subspace *plus* other nonpolynomial functions that are never computed explicitly. Rather, these functions are evaluated via cleverly chosen degrees of freedom (DoFs) and allow for the design of (nodal, edge, face ...) conforming global spaces. Such DoFs can be used to compute certain polynomial projections and stabilizations: the former are needed for the polynomial consistency of the scheme; the latter for its well-posedness.

A preliminary version of $\mathbf{H}(\mathrm{div})$ virtual elements was first introduced for 2D problems in Ref. [18] as the extension of Raviart-Thomas or Brezzi-Douglas-Marini elements to polygonal meshes. In order to cope with a sufficiently wide range of problems in mixed form and electromagnetic problems, see for instance Refs. [14, 27], in Ref. [7] the authors developed several variants of $\mathbf{H}(\mathrm{div})$ and $\mathbf{H}(\mathbf{curl})$ VE spaces in two and three dimensions. Furthermore, serendipity edge and face virtual element spaces were first considered in Ref. [9]; serendipity spaces allow for a reduction of the number of internal DoFs without affecting the convergence and stability properties of the VEM. This fact has a paramount impact on the performance of the method in the three dimensional case, notably in the reduction of the face DoFs, as bulk DoFs in 3D can be removed by static condensation. Although the spaces introduced in Ref. [9] are more efficient than those in Ref. [7], they have the important drawback of missing the full discrete De-Rham diagram, only recovering part of it. This shortcoming was finally handled in a series of paper, which represent the current "state of the art" of VEM De Rham complexes, dealing with the general order 2D case [3], the lowest order 3D case [5], and the 3D general order case [4]. All these papers also treat the magnetostatic equations as a simple model problem; more involved problems can be found, e.g., in Refs. [13, 21]. The lowest order case [5] was published independently of the general order case [4] not only with the aim of reaching different communities, but also because the former case allows for a simpler definition of the VE spaces.

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Compared to its nodal counterpart [10,15,17,19,20,22,28], the interpolation and stability theory for edge and face virtual elements is still rather limited. In Ref. [12], interpolation estimates for $\mathbf{H}(\mathrm{div})$ virtual element spaces in 2D were proved, while $\mathbf{H}(\mathbf{curl}^2)$ virtual element spaces in 2D were tackled in Ref. [34]. The extension to two-dimensional face virtual elements with curved edges, including interpolation properties, was considered in Ref. [24]. Most importantly, both interpolation error estimates and stability properties for the lowest order edge and face virtual element spaces of Ref. [5] were derived in Ref. [11] in two and three dimensions.

The aim of this paper is to prove interpolation estimates and stability properties for general order standard and serendipity edge and face virtual element spaces in 2D and 3D [3, 4, 7, 9]. Amongst the several variants of $\mathbf{H}(\mathrm{div})$ and $\mathbf{H}(\mathbf{curl})$ spaces, we focus on those in Ref. [4]. The ideas outlined in the paper can be extended to other settings as well.

Compared with the proofs for the lowest order spaces [11], the general order case hides many additional difficulties of technical nature. For instance, many more DoFs types (moments of various kinds on edges, faces, volumes) appear and serendipity spaces are employed. Indeed, while in the lowest order spaces the serendipity construction can be avoided by a simpler, yet equivalent, definition, it is in the general order case that the peculiar definition of serendipity VE spaces appears in its full complexity. To the authors knowledge, this is the first contribution where the interpolation and stability analysis of serendipity VE spaces (of any kind) is tackled. Although many relevant ideas are contained in the proofs of the "lesser" lemmas, we give here a short guideline of our main results:

- Theorems 3.3 and 3.8 contain interpolation estimates for 2D standard and serendipity edge elements, respectively;
- Theorems 3.9 and 3.10 quickly extend the above results to 2D standard and serendipity face elements, respectively;
- Theorems 4.5 and 4.6 contain interpolation estimates for 3D standard and serendipity edge elements, respectively;
- Theorem 4.2 contains interpolation estimates for 3D standard face elements;
- Theorems 5.1 and 5.2 contain the stability estimates for 2D standard and serendipity edge spaces, respectively;
- Remark 6 extends the stability estimates to 2D standard and serendipity face spaces;
- Theorem 5.5 and Remark 7 contain the stability estimates for 3D standard and serendipity edge spaces, respectively;
- Theorem 5.3 contains the stability estimates for 3D standard face spaces.

The remainder of the paper is organized as follows: in Section 2, we introduce the necessary functional spaces and mesh assumptions, and recall some technical results needed for the error estimates; in Sections 3 and 4, we prove the interpolation error estimates for edge and face virtual element spaces in 2D and 3D, respectively; in Section 5, we define several stabilizations for edge and face virtual element spaces, and prove their stability properties.

# 2 Preliminaries

The outline of this section is as follows: in Section 2.1, we introduce the functional space setting; in Section 2.2, we detail the assumptions on the regularity of the mesh decompositions; in Sections 2.3, 2.4, and 2.5, we state some technical results, namely polynomial inverse inequalities and decompositions, Sobolev trace inequalities, and Poincaré and Friedrichs inequalities, respectively.

## 2.1 Sobolev spaces

Throughout the paper, given $m, p \in \mathbb{N}_0$ and a bounded Lipschitz domain $D \subseteq \mathbb{R}^d$ ($d = 1, 2, 3$) with boundary $\partial D$, we shall use standard notations [16] for the scalar Sobolev space $W^{m,p}(D)$ equipped with the norm $\|\cdot\|_{W^{m,p}(D)}$ and the seminorm $|\cdot|_{W^{m,p}(D)}$. If $p = 2$, we denote $W^{m,2}(D)$ by $H^m(D)$ equipped with the norm $\|\cdot\|_{m,D}$, the seminorm $|\cdot|_{m,D}$, and the inner product $(\cdot,\cdot)_D$. We set $H^0(D) = L^2(D)$; in the corresponding norm, we omit the subscript 0. Let $H^{-m}(D)$ be the dual space of $H^m(D)$ equipped with the negative norm $\|\cdot\|_{-m,D}$. For $k \in \mathbb{N}_0$, $\mathbb{P}_k(D)$ denotes the space of polynomials of degree at most $k$ on $D$ and $\pi_{k,d}$ its dimension. We set $\mathbb{P}_{-\ell}(D) = \{0\}$ for all $\ell \in \mathbb{N}$. Moreover, $\mathbb{P}^0_k(D)$ denotes the subspace of $\mathbb{P}_k(D)$ of functions with zero average on either $\partial D$ or $D$. We shall use the boldface to denote vector variables and spaces; for example, $\mathbf{v}$, $\mathbf{H}^m(D)$, and $\mathbf{L}^2(D)$ denote the vector version of a function $v$, a Sobolev space, and a Lebesgue space.

With an abuse of notation, we denote local sets of coordinates in two and three dimensions by $[x_1, x_2]$ and $[x_1, x_2, x_3]$, respectively. Given a function $\phi : F \subseteq \mathbb{R}^2 \to \mathbb{R}$ and a field $\mathbf{v} = [v_1, v_2]^T : F \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, we define the operators

$$
\begin{aligned}
&\boldsymbol{\nabla}_F \phi = \left[\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}\right]^T, \quad \mathbf{curl}_F\, \phi = \left[\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1}\right]^T, \quad \Delta_F \phi = \frac{\partial^2 \phi}{\partial^2 x_1} + \frac{\partial^2 \phi}{\partial^2 x_2},\\
&\mathrm{rot}_F\, \mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \mathrm{div}_F\, \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}.
\end{aligned}
$$

In three dimensions, given a function $\phi : \mathbb{R}^3 \to \mathbb{R}$ and a field $\mathbf{v} = [v_1, v_2, v_3]^T : \mathbb{R}^3 \to \mathbb{R}^3$, we define

$$
\begin{aligned}
&\boldsymbol{\nabla} \phi = \left[\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3}\right]^T, \quad \Delta \phi = \frac{\partial^2 \phi}{\partial^2 x_1} + \frac{\partial^2 \phi}{\partial^2 x_2} + \frac{\partial^2 \phi}{\partial^2 x_3},\\
&\mathbf{curl}\, \mathbf{v} = \left[\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right]^T, \quad \mathrm{div}\, \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}.
\end{aligned}
$$

Next, given a polygon $F$ and a polyhedron $E$, we denote the usual div, rot, and **curl** spaces by $\mathbf{H}(\mathrm{div}_F, F)$, $\mathbf{H}(\mathrm{rot}_F, F)$, $\mathbf{H}(\mathrm{div}, E)$, and $\mathbf{H}(\mathbf{curl}, E)$.

## 2.2 Mesh regularity assumptions

Let $\mathcal{T}_h$ be a sequence of decompositions of a given polyhedral domain $\Omega \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$ into nonoverlapping polygonal/polyhedral elements $E$. For each $E$, we denote its two-dimensional boundary by $\partial E$ and the one-dimensional boundary of each face $F$ in $\partial E$ by $\partial F$. For any geometric object $D$ of dimension $d$ ($d = 1, 2, 3$), i.e., an edge $e$, a face $F$, or an element $E$, we denote its barycenter, its measure (length, area, or volume, respectively), and its diameter by $\mathbf{b}_D$, $|D|$, and $h_D$, respectively. We denote the unit outer normal to the boundary $\partial E$ by $\mathbf{n}_{\partial E}$ and the restriction to the face $F$ of $\mathbf{n}_{\partial E}$ by $\mathbf{n}_F$. For each face $F$, we also denote the unit outer normal to $\partial F$ in the plane containing $F$ by $\mathbf{n}_{\partial F}$ and the restriction to the edge $e$ of $\mathbf{n}_{\partial F}$ in the plane containing $F$ by $\mathbf{n}_e$. Further, the unit tangential vector $\mathbf{t}_e$ along the edge $e$ is defined as the vector pointed in counter-clockwise sense of $\mathbf{n}_e$ (for example, $\mathbf{t}_e = (-n_2, n_1)$ if $\mathbf{n}_e = (n_1, n_2)$ in two dimensions), and $\mathbf{t}_{\partial F}$ is locally defined by $\mathbf{t}_{\partial F}|_e := \mathbf{t}_e$.

Henceforth, we demand the following mesh regularity assumption:

**(M)** For $d = 2$, there exists a uniform constant $\rho > 0$ such that, for every polygon $F$,

- (i) $F$ is star-shaped with respect to a disk of radius $\geq \rho h_F$;
- (ii) every edge $e$ of $\partial F$ satisfies $h_e \geq \rho h_F$.

For $d = 3$, there exists a uniform constant $\rho > 0$ such that, for every element $E$,

- (i) $E$ is star-shaped with respect to a ball of radius $\geq \rho h_E$;
- (ii) every face $F$ of $\partial E$ is star-shaped with respect to a disk with radius $\geq \rho h_F$;
- (iii) for every face $F$ of $\partial E$, every edge $e$ of $\partial F$ satisfies $h_e \geq \rho h_F \geq \rho^2 h_E$.

In certain cases that will be indicated explicitly, we shall also require the following uniform convexity condition:

(**MC**) in two dimensions, every polygonal element $F$ is convex and there exists a constant $\varepsilon > 0$ such that each internal angle $\theta$ of element $F$ satisfies $\varepsilon \leq \theta \leq \pi - \varepsilon$; in the three dimensional case, each face $F$ of the mesh satisfies such condition.

*Remark* 1. An immediate consequence of the above mesh regularity assumptions is that each three-dimensional element $E$ or each two-dimensional face $F$ are uniformly Lipschitz domains that admit a shape-regular tessellation $\widetilde{\mathcal{T}}_h$ into simplices, i.e., a partition of $E$ into tetrahedra or $F$ into triangles. Such a decomposition is obtained by connecting each edge/face (in two and three dimensions, respectively) with the center of the ball in assumption (**M**).

In what follows, given two positive quantities $a$ and $b$, we use the short-hand notation "$a \lesssim b$" if there exists a positive constant $c$ independent of the discretization parameters such that "$a \leq c\, b$". Moreover, we write "$a \approx b$" if and only if "$a \lesssim b$" and "$b \lesssim a$". When keeping track of the constant is necessary, we shall use explicit generic constants $C$, $C'$, $C_1, \cdots$ that are independent of the mesh and may vary at different occurrences. Furthermore, $D$ will denote a generic polytopal domain (polygon in $\mathbb{R}^2$ or polyhedron in $\mathbb{R}^3$) representing either an element or a face of the mesh, thus satisfying the above assumptions (**M**).

Throughout, the explanation of the identities and upper and lower bounds will appear either in the preceding text or as an equation reference above the equality symbol "$=$" or the inequality symbols "$\leq$", "$\geq$" etc, whichever we believe it is easier for the reader.

### 2.3 Polynomial properties

The following polynomial inverse estimates in a polytopal domain $D \subset \mathbb{R}^d$ ($d = 2, 3$) are valid: for all $p_k \in \mathbb{P}_k(D)$,

$$\|p_k\|_{\partial D} \lesssim h_D^{-\frac{1}{2}} \|p_k\|_D, \qquad |p_k|_{1,D} \lesssim h_D^{-1} \|p_k\|_D, \qquad \|p_k\|_D \lesssim h_D^{-1} \|p_k\|_{-1,D}. \tag{1}$$

Furthermore, for each piecewise polynomial $p_k$ of degree at most $k$ over $\partial D$, we have

$$\|p_k\|_{\partial D} \lesssim h_D^{-\frac{1}{2}} \|p_k\|_{-\frac{1}{2},\partial D}, \tag{2}$$

where $\|\cdot\|_{-\frac{1}{2},\partial D}$ denotes the scaled $H^{-\frac{1}{2}}(\partial D)$ dual norm

$$|\cdot|_{-\frac{1}{2},\partial D} := \sup_{\varphi \in H^{\frac{1}{2}}(\partial D)} \frac{(\cdot, \varphi)_{\partial D}}{|\varphi|_{\frac{1}{2},\partial D} + h_D^{\frac{1}{2}} \|\varphi\|_{\partial D}}.$$

The proof of the above inverse estimates hinges upon the existence of a shape-regular simplicial tessellation, see Remark 1, and standard polynomial inverse estimates on simplices as in Section 3.6 of Ref. [33].

Let $b_D$ be the cubic ($d = 2$) or quartic ($d = 3$) piecewise bubble function associated with the shape-regular tessellation of the element $D$, see Remark 1, with unitary $L^\infty$ norm. The following result which establishes standard estimate for bubble functions will be useful:

$$\|p_k\|_D^2 \lesssim \int_D b_D p_k^2 \lesssim \|p_k\|_D^2 \quad \forall p_k \in \mathbb{P}_k(D). \tag{3}$$

A proof of this result is obtained by using Theorem 3.3 in Ref. [1] and standard manipulations.

Moreover, the following decompositions of polynomial vector spaces are valid; see, e.g., Refs. [3, 7]. Given a polygon $F$, we have

$$(\mathbb{P}_k(F))^2 = \mathbf{curl}_F\, \mathbb{P}_{k+1}(F) \oplus \mathbf{x}\mathbb{P}_{k-1}(F), \tag{4}$$

which implies that $\mathrm{div}_F$ is an isomorphism between $\{\mathbf{x}\mathbb{P}_k(F)\}$ and $\mathbb{P}_k(F)$. Moreover,

$$(\mathbb{P}_k(F))^2 = \boldsymbol{\nabla}_F \mathbb{P}_{k+1}(F) \oplus \mathbf{x}^\perp \mathbb{P}_{k-1}(F), \tag{5}$$

which implies that $\mathrm{rot}_F$ is an isomorphism between $\{\mathbf{x}^\perp \mathbb{P}_k(F)\}$ and $\mathbb{P}_k(F)$.

Given a polyhedron $E$, we have

$$(\mathbb{P}_k(E))^3 = \mathbf{curl}\,(\mathbb{P}_{k+1}(E))^3 \oplus \mathbf{x}\mathbb{P}_{k-1}(E), \tag{6}$$

which implies that div is an isomorphism between $\{\mathbf{x}\mathbb{P}_k(E)\}$ and $\mathbb{P}_k(E)$. Furthermore,

$$(\mathbb{P}_k(E))^3 = \boldsymbol{\nabla}\mathbb{P}_{k+1}(E) \oplus \mathbf{x} \wedge (\mathbb{P}_{k-1}(E))^3, \tag{7}$$

which implies that for each $\mathbf{p}_k \in (\mathbb{P}_k(E))^3$ with $\mathrm{div}\,\mathbf{p}_k = 0$, there exists $\mathbf{q}_k \in (\mathbb{P}_k(E))^3$ such that $\mathbf{curl}(\mathbf{x} \wedge \mathbf{q}_k) = \mathbf{p}_k$.

## 2.4 Trace inequalities

The following trace inequalities are valid; see, e.g., in [29, Theorem A.20]: given a polytopal domain $D$, representing either an element or a face of the mesh, there hold

$$\|v\|_{\partial D} \lesssim h_D^{-\frac{1}{2}}\|v\|_D + h_D^{\delta-\frac{1}{2}}|v|_{\delta,D} \qquad \forall v \in H^\delta(D), \frac{1}{2} < \delta < \frac{3}{2} \tag{8}$$

$$|v|_{\varepsilon,\partial D} \lesssim h_D^{-(\varepsilon+\frac{1}{2})}\|v\|_D + |v|_{\varepsilon+\frac{1}{2},D} \qquad \forall v \in H^{\varepsilon+\frac{1}{2}}(D), 0 < \varepsilon < 1. \tag{9}$$

If additionally $1/2 < \delta \leq 1$ and $v$ has zero average on either $\partial D$ or $D$, then we have

$$\|v\|_{\partial D} \lesssim h_D^{\delta-\frac{1}{2}}|v|_{\delta,D}. \tag{10}$$

For functions $v$ with zero average on either $\partial D$ or $D$, we also recall the multiplicative trace inequality

$$\|v\|_{\partial D} \lesssim \|v\|_D^{\frac{1}{2}}|v|_{1,D}^{\frac{1}{2}}. \tag{11}$$

Let $F$ be a polygon and $E$ be a polyhedron, respectively, representing either a face $F$ or an element $E$ of the mesh, thus satisfying the above assumptions (**M**). For $\mathbf{w} \in \mathbf{H}(\mathrm{div}_F, F)$, $\mathbf{v} \in \mathbf{H}(\mathrm{rot}_F, F)$, $\boldsymbol{\phi} \in \mathbf{H}(\mathrm{div}, E)$, $\boldsymbol{\psi} \in \mathbf{H}(\mathbf{curl}, E)$, and $\boldsymbol{\chi} \in \mathbf{H}(\mathrm{div}, E) \cap \mathbf{H}(\mathbf{curl}, E)$, the following trace inequalities are valid; the following trace inequalities are valid; see, e.g., Theorems 3.29 and 3.24 in [23, 27], and page 367 in [23]:

$$\|\mathbf{v} \cdot \mathbf{t}_{\partial F}\|_{-\frac{1}{2},\partial F} \lesssim \|\mathbf{v}\|_F + h_F\|\mathrm{rot}_F\,\mathbf{w}\|_F, \tag{12}$$

$$\|\boldsymbol{\phi} \cdot \mathbf{n}_{\partial E}\|_{-\frac{1}{2},\partial E} \lesssim \|\boldsymbol{\phi}\|_E + h_E\|\mathrm{div}\,\boldsymbol{\phi}\|_E, \tag{13}$$

$$\|\boldsymbol{\psi} \wedge \mathbf{n}_{\partial E}\|_{-\frac{1}{2},\partial E} \lesssim \|\boldsymbol{\psi}\|_E + h_E\|\mathbf{curl}\,\boldsymbol{\psi}\|_E, \tag{14}$$

$$\|\boldsymbol{\chi}\wedge\mathbf{n}_{\partial E}\|_{\partial E} \lesssim h_E^{-\frac{1}{2}}\|\boldsymbol{\chi}\|_E + h_E^{\frac{1}{2}}\|\mathrm{div}\,\boldsymbol{\chi}\|_E + h_E^{\frac{1}{2}}\|\mathbf{curl}\,\boldsymbol{\chi}\|_E + \|\boldsymbol{\chi}\cdot\mathbf{n}_{\partial E}\|_{\partial E}. \tag{15}$$

All constants involved in the bounds above are uniform, i.e. independent of the particular element $E$ or face $F$ in $\{\mathcal{T}_h\}_h$, since the mesh assumptions **(M)** guarantee that the parameters associated to the star-shaped and Lipschitz properties are uniform in the mesh family.

## 2.5 Poincaré and Friedrichs inequalities

For each $v \in H^1(D)$, $D \subseteq \mathbb{R}^d$ $(d = 2, 3)$, if $v$ has zero average on either $\partial D$ or $D$, then we have the following Poincaré inequality; see, e.g., Section 5.3 in Ref. [16]:

$$h_D^{-1}\|v\|_D \lesssim |v|_{1,D}. \tag{16}$$

Let $E \in \mathcal{T}_h$ be a polyhedral element and $\mathbf{v} \in \mathbf{H}(\mathbf{curl}, E) \cap \mathbf{H}(\mathrm{div}, E)$ be a divergence free function satisfying $\mathbf{v} \wedge \mathbf{n}_{\partial E} \in \mathbf{L}^2(\partial E)$. Then, the following Friedrichs inequality is valid; see, e.g., Corollary 3.51 in Ref. [27] or Lemma 2.2 in Ref. [11]:

$$h_E^{-1}\|\mathbf{v}\|_E \lesssim h_E^{-\frac{1}{2}}\|\mathbf{v} \wedge \mathbf{n}_{\partial E}\|_{\partial E} + \|\mathbf{curl}\,\mathbf{v}\|_E. \tag{17}$$

Similarly, let $\mathbf{v} \in \mathbf{H}(\mathbf{curl}, E) \cap \mathbf{H}(\mathrm{div}, E)$ be a divergence free function satisfying $\mathbf{v} \cdot \mathbf{n}_{\partial E} \in L^2(\partial E)$. Then, the following Friedrichs inequality is also valid; see Corollary 3.51 in Ref. [27]:

$$h_E^{-1}\|\mathbf{v}\|_E \lesssim h_E^{-\frac{1}{2}}\|\mathbf{v} \cdot \mathbf{n}_{\partial E}\|_{\partial E} + \|\mathbf{curl}\,\mathbf{v}\|_E. \tag{18}$$

# 3 Interpolation properties of edge and face virtual element spaces in 2D

Here, we prove interpolation properties of general order for standard and serendipity edge and face virtual element spaces on polygons. These polygons can be interpreted as elements of a two-dimensional mesh or as faces of a three-dimensional mesh; we shall often refer to them as "faces". In what follows, we shall concentrate on interpolation and stability results on local elements, since the corresponding global results follow by a summation on all the elements. In Section 3.1, we begin with edge virtual element spaces on polygons; in Section 3.2, we consider the serendipity edge virtual element space in 2D, which allows us to reduce the number of internal DoFs of the standard edge virtual element space introduced in Section 3.1; in Section 3.3, we extend the results of edge virtual element spaces to face virtual element spaces in 2D.

## 3.1 Standard edge virtual element space on polygons

Given a face $F$ and an integer $k \geq 1$, the edge virtual element space is defined as [4]

$$\begin{aligned}\mathbf{V}^e_k(F) = \big\{\mathbf{v}_h \in \mathbf{L}^2(F) : \operatorname{div}_F \mathbf{v}_h \in \mathbb{P}_k(F),\, \operatorname{rot}_F \mathbf{v}_h \in \mathbb{P}_{k-1}(F),& \\ \mathbf{v}_h \cdot \mathbf{t}_e \in \mathbb{P}_k(e) \quad \forall e \subseteq \partial F\big\}.&\end{aligned} \tag{19}$$

The following linear operators are a set of unisolvent DoFs:

- the moments $\displaystyle\int_e \mathbf{v}_h \cdot \mathbf{t}_e p_k \qquad \forall p_k \in \mathbb{P}_k(e),\ \forall e \subseteq \partial F;$ (20)
- the moments $\displaystyle\int_F \mathbf{v}_h \cdot \mathbf{x}_F p_k \qquad \forall p_k \in \mathbb{P}_k(F);$ (21)
- the rot-moments $\displaystyle\int_F \operatorname{rot}_F \mathbf{v}_h p^0_{k-1} \qquad \forall p^0_{k-1} \in \mathbb{P}^0_{k-1}(F)$ only for $k > 1$, (22)

where $\mathbf{x}_F := \mathbf{x} - \mathbf{b}_F$.

The inclusion $(\mathbb{P}_k(F))^2 \subseteq \mathbf{V}^e_k(F)$ is valid and the $\mathbf{L}^2$ projection $\mathbf{\Pi}^{0,F}_{k+1} : \mathbf{V}^e_k(F) \to (\mathbb{P}_{k+1}(F))^2$ is computable by the DoFs (20)–(22); see Refs. [3,4].

*Remark* 2 (Generality of the approach). To keep the theoretical analysis as clear as possible, we chose the $\mathbf{V}^e_k(F)$ that corresponds to that of Ref. [4]. We might have employed other definitions; see, e.g., Refs. [3,9]. This would simply result in a change of the polynomial orders appearing in (19), and (20)–(22): the notation would be heavier but the theoretical extension would trivially follow the same steps here shown for (19). This same consideration applies to all the virtual element spaces introduced in the following.

We begin with the proof of the following auxiliary bound for functions belonging to $\mathbf{V}^e_k(F)$.

**Lemma 3.1.** *For each $\boldsymbol{v}_h \in \boldsymbol{V}^e_k(F)$, we have*

$$\|\boldsymbol{v}_h\|_F \lesssim h_F \|\operatorname{rot}_F \boldsymbol{v}_h\|_F + h_F^{\frac{1}{2}} \|\boldsymbol{v}_h \cdot \boldsymbol{t}_{\partial F}\|_{\partial F} + \sup_{p_k \in \mathbb{P}_k(F)} \frac{\int_F \boldsymbol{v}_h \cdot \boldsymbol{x}_F p_k}{\|\boldsymbol{x}_F p_k\|_F}. \tag{23}$$

*Proof.* Since $\operatorname{rot}_F \mathbf{curl}_F = -\Delta_F$, the following Helmholtz decomposition of $\mathbf{v}_h$ is valid:

$$\mathbf{v}_h = \mathbf{curl}_F\, \rho + \boldsymbol{\nabla}_F \sigma, \tag{24}$$

where $\rho \in H^1(F) \setminus \mathbb{R}$ and $\sigma \in H^1(F)$ satisfy weakly

$$-\Delta_F \rho = \operatorname{rot}_F \mathbf{v}_h \text{ in } F, \quad \mathbf{curl}_F\, \rho \cdot \mathbf{t}_{\partial F} = \mathbf{v}_h \cdot \mathbf{t}_{\partial F} \text{ on } \partial F, \tag{25}$$

and

$$\Delta_F \sigma = \operatorname{div}_F \mathbf{v}_h \text{ in } F, \quad \sigma = 0 \text{ on } \partial F. \tag{26}$$

By the orthogonality $(\mathbf{curl}_F\,\rho, \boldsymbol{\nabla}_F\sigma)_F = 0$, we also have

$$\|\mathbf{v}_h\|_F^2 = \|\mathbf{curl}_F\,\rho\|_F^2 + \|\boldsymbol{\nabla}_F\sigma\|_F^2. \tag{27}$$

We show an upper bound on the two terms on the right-hand side of (27): using $\mathrm{rot}_F\,\mathbf{curl}_F = -\Delta_F$ and $\|\boldsymbol{\nabla}_F\rho\|_F = \|\mathbf{curl}_F\,\rho\|_F$,[1]

$$\begin{aligned}
\|\mathbf{curl}_F\,\rho\|_F^2 &\overset{\text{IBP}}{=} -\int_F \rho(\Delta_F\rho) + \int_{\partial F}\rho(\mathbf{curl}_F\,\rho\cdot\mathbf{t}_{\partial F}) \overset{(25)}{\lesssim} \|\rho\|_F\|\mathrm{rot}_F\,\mathbf{v}_h\|_F + \|\rho\|_{\partial F}\|\mathbf{v}_h\cdot\mathbf{t}_{\partial F}\|_{\partial F}\\
&\overset{(10),(16)}{\lesssim} h_F\|\boldsymbol{\nabla}_F\rho\|_F\|\mathrm{rot}_F\,\mathbf{v}_h\|_F + h_F^{\frac12}\|\boldsymbol{\nabla}_F\rho\|_F\|\mathbf{v}_h\cdot\mathbf{t}_{\partial F}\|_{\partial F}\\
&\lesssim \left(h_F\|\mathrm{rot}_F\,\mathbf{v}_h\|_F + h_F^{\frac12}\|\mathbf{v}_h\cdot\mathbf{t}_{\partial F}\|_{\partial F}\right)\|\mathbf{curl}_F\,\rho\|_F.
\end{aligned} \tag{28}$$

By using (4), the fact that $\mathrm{div}_F\,\mathbf{v}_h \in \mathbb{P}_k(F)$, and a scaling argument, there exists a polynomial $q_k \in \mathbb{P}_k(F)$ such that

$$\mathrm{div}_F\,(\mathbf{x}_F q_k) = \mathrm{div}_F\,\mathbf{v}_h \text{ and } \|\mathbf{x}_F q_k\|_F \lesssim h_F\|\mathrm{div}_F\,\mathbf{v}_h\|_F. \tag{29}$$

We have the following inverse estimate involving edge virtual element functions:

$$\|\mathrm{div}_F\,\mathbf{v}_h\|_F \lesssim h_F^{-1}\|\mathbf{v}_h\|_F \qquad \forall \mathbf{v}_h \in \mathbf{V}^e_{k-1}(F). \tag{30}$$

To prove (30), we split the face $F$ into a shape-regular sub-triangulation $\widetilde{\mathcal{T}}_h$; see Remark 1. Let $b_F$ be the usual positive cubic bubble function over each triangle $\widetilde{F} \in \widetilde{\mathcal{T}}_h$ scaled such that $\|b_F\|_{\infty,\widetilde{F}} = 1$. By using that $\mathrm{div}_F\,\mathbf{v}_h \in \mathbb{P}_k(F)$, and the polynomial inverse inequalities (3) and (1), we have

$$\|\mathrm{div}_F\,\mathbf{v}_h\|_F^2 \lesssim (b_F \mathrm{div}_F\,\mathbf{v}_h, \mathrm{div}_F\,\mathbf{v}_h)_F = -(\boldsymbol{\nabla}_F(b_F\mathrm{div}_F\mathbf{v}_h), \mathbf{v}_h)_F \lesssim h_F^{-1}\|\mathrm{div}_F\,\mathbf{v}_h\|_F\|\mathbf{v}_h\|_F,$$

which proves (30).

Next, we cope with the second term on the right-hand side of (27):

$$\begin{aligned}
\|\boldsymbol{\nabla}_F\sigma\|_F^2 &\overset{\text{IBP},(26),(29)}{=} -\int_F \mathrm{div}_F\,(\mathbf{x}_F q_k)\sigma \overset{\text{IBP},(26)}{=} \int_F (\mathbf{x}_F q_k)\cdot\boldsymbol{\nabla}_F\sigma\\
&\overset{(24)}{=} \int_F (\mathbf{x}_F q_k)\cdot(\mathbf{v}_h - \mathbf{curl}_F\,\rho)\\
&\le \|\mathbf{x}_F q_k\|_F \sup_{p_k\in\mathbb{P}_k(F)} \frac{\int_F \mathbf{v}_h\cdot\mathbf{x}_F p_k}{\|\mathbf{x}_F p_k\|_F} + \|\mathbf{x}_F q_k\|_F\|\mathbf{curl}_F\,\rho\|_F\\
&\overset{(29),(30)}{\lesssim} \left(\sup_{p_k\in\mathbb{P}_k(F)} \frac{\int_F \mathbf{v}_h\cdot\mathbf{x}_F p_k}{\|\mathbf{x}_F p_k\|_F} + \|\mathbf{curl}_F\,\rho\|_F\right)\|\mathbf{v}_h\|_F.
\end{aligned} \tag{31}$$

Substituting (28) and (31) into (27), and using (27) and (28) again, we can obtain (23). □

The following bound, which generalizes Lemma 4.4 in Ref. [11] will be useful in the sequel.

**Lemma 3.2.** *For each face $F \subseteq \partial E$ and given $\varepsilon > 0$, let $\boldsymbol{v} \in \boldsymbol{H}^\varepsilon(F)\cap\boldsymbol{H}(\mathrm{rot}_F, F)$ such that $\boldsymbol{v}\cdot\boldsymbol{t}_e$ is integrable on each edge of $F$. Then, the following bound is valid: for all $e$ in $\partial F$ and $p_k$ in $\mathbb{P}_k(e)$,*

$$\left|\int_e (\boldsymbol{v}\cdot\boldsymbol{t}_e)p_k\right| \lesssim \|p_k\|_{L^\infty(e)}\left(\|\boldsymbol{v}\|_F + h_F^\varepsilon|\boldsymbol{v}|_{\varepsilon,F} + h_F\|\mathrm{rot}_F\,\boldsymbol{v}\|_F\right). \tag{32}$$

*The last term on the right-hand side can be neglected if $\varepsilon > 1/2$.*

*Proof.* The inequality is trivial for $\varepsilon > \frac12$ by using the trace inequality (8). Therefore, we assume $0 < \varepsilon \le \frac12$. Recalling Remark 1, we split the face $F$ into a shape-regular triangulation $\widetilde{\mathcal{T}}_h$.

[1] Henceforth, IBP stands for integration by parts

Let $T \in \widetilde{\mathcal{T}}_h$ be the triangle such that $e \subseteq \partial T$. We first prove the following inequality: for all fixed $p > 2$ and $p_k \in \mathbb{P}_k(e)$ $\forall e \subseteq \partial F$,

$$\left| \int_e (\mathbf{v} \cdot \mathbf{t}_e) p_k \right| \lesssim \|p_k\|_{L^\infty(e)} \left( h_F^{1-2/p} \|\mathbf{v}\|_{L^p(T)} + h_F \|\mathrm{rot}_F\, \mathbf{v}\|_T \right). \tag{33}$$

Let $\hat{T}$ be the affine equivalent reference element to the triangle $T$ and $\hat{e}$ be the edge of $\hat{T}$ corresponding to the edge $e \subseteq \partial T$ through the Piola transform; see Definition 3.4.1 in Ref. [16]. Let $\hat{q}_k : \hat{T} \to \mathbb{R}$ be the prolongation of $\hat{p}_k$ ( $\hat{\cdot}$ denoting the usual pull-back of $\cdot$ ; see Remark 3.4.2 in Ref. [16]) by the constant extension along the normal direction to $\hat{e}$. From the trace theorem on Lipschitz domains [16], the trace operator is surjective from $W^{1,p'}(\hat{T})$ to $W^{1/p,p'}(\partial\hat{T})$, where $p'$ denotes the dual index to $p$, i.e. $1/p + 1/p' = 1$, $p > 2$. Further, the space $W^{1/p,p'}(\partial\hat{T})$ contains piecewise discontinuous functions over $\partial\hat{T}$ since $p > 2$. In particular, there exists a function $\hat{w}$ such that $\hat{w} = 1$ on $\hat{e}$, $\hat{w} = 0$ on $\partial\hat{T}/\hat{e}$, and $\|\hat{w}\|_{W^{1,p'}(\hat{T})} < \infty$. The function $\hat{w}\hat{q}_k$ belongs to $W^{1,p'}(\hat{T})$.

Using a scaling argument, an integration by parts, the Hölder inequality, and the norm equivalence of polynomial functions with fixed degree on the reference triangle $\hat{T}$, we have

$$\begin{aligned}
&\left| \int_e (\mathbf{v} \cdot \mathbf{t}_e) p_k \right| \lesssim h_F \left| \int_{\hat{e}} (\hat{\mathbf{v}} \cdot \hat{\mathbf{t}}_{\hat{e}}) \hat{p}_k \right| = h_F \left| \int_{\partial\hat{T}} (\hat{\mathbf{v}} \cdot \hat{\mathbf{t}}_{\hat{e}})(\hat{w}\hat{q}_k) \right| \\
&= h_F \left| \int_{\hat{T}} \hat{\mathrm{rot}}_{\hat{F}}\, \hat{\mathbf{v}} (\hat{w}\hat{q}_k) - \int_{\hat{T}} \hat{\mathbf{v}} \cdot \mathbf{curl}_{\hat{F}} (\hat{w}\hat{q}_k) \right| \\
&\lesssim h_F \left( \|\hat{\mathrm{rot}}_{\hat{F}} \hat{\mathbf{v}}\|_{\hat{T}} \|\hat{w}\hat{q}_k\|_{\hat{T}} + \|\hat{\mathbf{v}}\|_{L^p(\hat{T})} |\hat{w}\hat{q}_k|_{W^{1,p'}(\hat{T})} \right) \lesssim h_F \Bigg( \|\hat{\mathrm{rot}}_{\hat{F}} \hat{\mathbf{v}}\|_{\hat{T}} \|\hat{w}\|_{\hat{T}} \|\hat{q}_k\|_{L^\infty(\hat{T})} \\
&\quad + \|\hat{\mathbf{v}}\|_{L^p(\hat{T})} \left( \|\hat{w}\|_{W^{1,p'}(\hat{T})} \|\hat{q}_k\|_{L^\infty(\hat{T})} + \|\hat{w}\|_{L^{p'}(\hat{T})} \|\hat{q}_k\|_{W^{1,\infty}(\hat{T})} \right) \Bigg) \\
&\lesssim h_F \left( \|\hat{\mathrm{rot}}_{\hat{F}} \hat{\mathbf{v}}\|_{\hat{T}} + \|\hat{\mathbf{v}}\|_{L^p(\hat{T})} \right) \|\hat{w}\|_{W^{1,p'}(\hat{T})} \|\hat{q}_k\|_{L^\infty(\hat{T})} \\
&\lesssim \|p_k\|_{L^\infty(e)} \left( h_F^{1-2/p} \|\mathbf{v}\|_{L^p(T)} + h_F \|\mathrm{rot}_F \mathbf{v}\|_T \right),
\end{aligned}$$

which completes the proof of (33). By taking $p = 2/(1-\varepsilon) > 2$ in (33), noting that $T \subseteq F$, and using the (scaled) embedding $H^\varepsilon(F) \hookrightarrow L^p(F)$, we get (32). □

The DoFs interpolation operator $\widetilde{\mathbf{I}}_h^e$ on the space $\mathbf{V}_k^e(F)$ is well defined for each function $\mathbf{v}$ in $\mathbf{H}^s(F) \cap \mathbf{H}(\mathrm{rot}_F, F)$ with $\mathbf{v} \cdot \mathbf{t}_e$ integrable on each edge. We impose

$$\int_e (\mathbf{v} - \widetilde{\mathbf{I}}_h^e \mathbf{v}) \cdot \mathbf{t}_e p_k = 0 \qquad \forall p_k \in \mathbb{P}_k(e),\ \forall e \subseteq \partial F; \tag{34a}$$

$$\int_F (\mathbf{v} - \widetilde{\mathbf{I}}_h^e \mathbf{v}) \cdot \mathbf{x}_F p_k = 0 \qquad \forall p_k \in \mathbb{P}_k(F); \tag{34b}$$

$$\int_F \mathrm{rot}_F\, (\mathbf{v} - \widetilde{\mathbf{I}}_h^e \mathbf{v}) p_{k-1}^0 = 0 \qquad \forall p_{k-1}^0 \in \mathbb{P}_{k-1}^0(F) \ \text{ only for } k > 1. \tag{34c}$$

Next, we prove interpolation properties of the operator $\widetilde{\mathbf{I}}_h^e$.

**Theorem 3.3.** *For each $\boldsymbol{v} \in \boldsymbol{H}^s(F)$, $0 < s \le k+1$, with $\mathrm{rot}_F\, \boldsymbol{v} \in H^r(F)$, $0 \le r \le k$, and $\boldsymbol{v} \cdot \boldsymbol{t}_e$ integrable on each edge, we have*

$$\|\boldsymbol{v} - \widetilde{\boldsymbol{I}}_h^e \boldsymbol{v}\|_F \lesssim h_F^s |\boldsymbol{v}|_{s,F} + h_F \|\mathrm{rot}_F\, \boldsymbol{v}\|_F, \tag{35}$$

$$\|\mathrm{rot}_F\, (\boldsymbol{v} - \widetilde{\boldsymbol{I}}_h^e \boldsymbol{v})\|_F \lesssim h_F^r |\mathrm{rot}_F\, \boldsymbol{v}|_{r,F}. \tag{36}$$

*The second term on the right-hand side of* (35) *can be neglected if* $s \ge 1$.

*Proof.* For each $p_{k-1} \in \mathbb{P}_{k-1}(F)$, we write

$$\int_F \mathrm{rot}_F\, (\mathbf{v} - \widetilde{\mathbf{I}}_h^e \mathbf{v}) p_{k-1} \overset{\mathrm{IBP},(34a),(34c)}{=} 0.$$

This and the fact that $\mathrm{rot}_F\,(\widetilde{\mathbf{I}}_h^e\mathbf{v}) \in \mathbb{P}_{k-1}(F)$ imply that

$$\mathrm{rot}_F\,(\widetilde{\mathbf{I}}_h^e\mathbf{v}) = \Pi_{k-1}^{0,F}(\mathrm{rot}_F\,\mathbf{v}). \tag{37}$$

Then, (36) follows from standard polynomial approximation properties.

Next, we focus on (35). By (34a) and the fact that $\widetilde{\mathbf{I}}_h^e\mathbf{v}\cdot\mathbf{t}_e \in \mathbb{P}_k(e)$, we have

$$\Pi_k^{0,e}(\mathbf{v}\cdot\mathbf{t}_e) = \widetilde{\mathbf{I}}_h^e\mathbf{v}\cdot\mathbf{t}_e \quad \forall e\subseteq\partial F. \tag{38}$$

Since $\mathbf{\Pi}_k^{0,F}\mathbf{v} \in (\mathbb{P}_k(F))^2 \subseteq \mathbf{V}_k^e(F)$, we have

$$\begin{aligned}\|\mathbf{\Pi}_k^{0,F}\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v}\|_F &\overset{(23)}{\lesssim} h_F\|\mathrm{rot}_F\,(\mathbf{\Pi}_k^{0,F}\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\|_F\\ &+ h_F^{\frac12}\|(\mathbf{\Pi}_k^{0,F}\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\cdot\mathbf{t}_{\partial F}\|_{\partial F} + \sup_{p_k\in\mathbb{P}_k(F)}\frac{\int_F(\mathbf{\Pi}_k^{0,F}\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\cdot\mathbf{x}_Fp_k}{\|\mathbf{x}_Fp_k\|_F}.\end{aligned}$$

As for the boundary term, also using (38), we have

$$\begin{aligned}h_F^{\frac12}\|(\mathbf{\Pi}_k^{0,F}\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\cdot\mathbf{t}_{\partial F}\|_{\partial F} &\lesssim h_F^{\frac12}\sum_{e\subset\partial F}\sup_{p_k\in\mathbb{P}_k(e)}\frac{\big((\mathbf{\Pi}_k^{0,F}\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\cdot\mathbf{t}_{\partial F},p_k\big)_e}{\|p_k\|_e}\\ &= h_F^{\frac12}\sum_{e\subset\partial F}\sup_{p_k\in\mathbb{P}_k(e)}\frac{\big((\mathbf{v}-\mathbf{\Pi}_k^{0,F}\mathbf{v})\cdot\mathbf{t}_{\partial F},p_k\big)_e}{\|p_k\|_e}.\end{aligned}$$

Using (32) with $\varepsilon = s$ and a polynomial inverse inequality, we deduce

$$\begin{aligned}h_F^{\frac12}\|(\mathbf{\Pi}_k^{0,F}\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\cdot\mathbf{t}_{\partial F}\|_{\partial F} \lesssim&\|\mathbf{v}-\mathbf{\Pi}_k^{0,F}\mathbf{v}\|_F + h_F^s|\mathbf{v}-\mathbf{\Pi}_k^{0,F}\mathbf{v}|_{s,F}\\ &+ h_F\|\mathrm{rot}_F\,(\mathbf{v}-\mathbf{\Pi}_k^{0,F}\mathbf{v})\|_F.\end{aligned}$$

Further, the definition of $\widetilde{\mathbf{I}}_h^e$ in (34) entails

$$\int_F(\mathbf{\Pi}_k^{0,F}\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\cdot\mathbf{x}_Fp_k = \int_F(\mathbf{\Pi}_k^{0,F}\mathbf{v}-\mathbf{v})\cdot\mathbf{x}_Fp_k.$$

Thus, we write

$$\begin{aligned}\|\mathbf{\Pi}_k^{0,F}\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v}\|_F \lesssim&\|\mathbf{v}-\mathbf{\Pi}_k^{0,F}\mathbf{v}\|_F + h_F^s|\mathbf{v}-\mathbf{\Pi}_k^{0,F}\mathbf{v}|_{s,F}\\ &+ h_F\|\mathrm{rot}_F\,(\mathbf{v}-\mathbf{\Pi}_k^{0,F}\mathbf{v})\|_F + h_F\|\mathrm{rot}_F\,(\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\|_F.\end{aligned} \tag{39}$$

If $s \geq 1$, then we apply (39), (36) with $r = s-1$, and standard polynomial approximation properties, leading to

$$\begin{aligned}\|\mathbf{\Pi}_k^{0,F}\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v}\|_F \lesssim& \|\mathbf{v}-\mathbf{\Pi}_k^{0,F}\mathbf{v}\|_F + h_F|\mathbf{v}-\mathbf{\Pi}_k^{0,F}\mathbf{v}|_{1,F} + h_F\|\mathrm{rot}_F\,(\mathbf{v}-\mathbf{\Pi}_k^{0,F}\mathbf{v})\|_F\\ &+h_F\|\mathrm{rot}_F\,(\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\|_F \lesssim h_F^s\,(|\mathbf{v}|_{s,F}+|\mathrm{rot}_F\,\mathbf{v}|_{s-1,F}) \lesssim h_F^s|\mathbf{v}|_{s,F}.\end{aligned} \tag{40}$$

Instead, if $0 < s < 1$, we replace the term $\mathbf{\Pi}_k^{0,F}\mathbf{v}$ by $\mathbf{\Pi}_0^{0,F}\mathbf{v}$ in (39). Then, we apply (36) with $r = 0$ and standard polynomial approximation properties, yielding

$$\begin{aligned}\|\mathbf{\Pi}_0^{0,F}\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v}\|_F \lesssim& \|\mathbf{v}-\mathbf{\Pi}_0^{0,F}\mathbf{v}\|_F + h_F^s|\mathbf{v}-\mathbf{\Pi}_0^{0,F}\mathbf{v}|_{s,F} + h_F\|\mathrm{rot}_F\,(\mathbf{v}-\mathbf{\Pi}_0^{0,F}\mathbf{v})\|_F\\ &+h_F\|\mathrm{rot}_F\,(\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\|_F \lesssim h_F^s|\mathbf{v}|_{s,F} + h_F\|\mathrm{rot}_F\,\mathbf{v}\|_F.\end{aligned} \tag{41}$$

Bounds (40) and (41) combined with a triangle inequality and standard polynomial approximation results prove the assertion (35). □

## 3.2 Serendipity edge virtual element space on polygons

As in Refs. [4, 8, 9], we set $\eta_F$ as the minimum number of straight lines necessary to cover the boundary of $F$ and define $\beta_F := k + 1 - \eta_F$. Next, we introduce a well defined projection $\mathbf{\Pi}^e_S : \mathbf{V}^e_k(F) \to (\mathbb{P}_k(F))^2$ as [4]

$$\int_{\partial F} [(\mathbf{v}_h - \mathbf{\Pi}^e_S \mathbf{v}_h) \cdot \mathbf{t}_{\partial F}][\boldsymbol{\nabla}_F p_{k+1} \cdot \mathbf{t}_{\partial F}] = 0 \quad \forall p_{k+1} \in \mathbb{P}_{k+1}(F); \tag{42a}$$

$$\int_{\partial F} (\mathbf{v}_h - \mathbf{\Pi}^e_S \mathbf{v}_h) \cdot \mathbf{t}_{\partial F} = 0; \tag{42b}$$

$$\int_F \mathrm{rot}_F\, (\mathbf{v}_h - \mathbf{\Pi}^e_S \mathbf{v}_h) p^0_{k-1} = 0 \quad \forall p^0_{k-1} \in \mathbb{P}^0_{k-1}(F) \text{ only for } k > 1; \tag{42c}$$

$$\int_F (\mathbf{v}_h - \mathbf{\Pi}^e_S \mathbf{v}_h) \cdot \mathbf{x}_F p_{\beta_F} = 0 \qquad \forall p_{\beta_F} \in \mathbb{P}_{\beta_F}(F) \text{ only for } \beta_F \geq 0. \tag{42d}$$

*Remark* 3. To handle the serendipity VEM in the present section we assume the additional (uniform) convexity condition (**MC**) in Section 2.2. For the particular case $\beta_F < 0$, such a condition could be relaxed at the price of additional technicalities that we prefer to avoid.

Based on the space $\mathbf{V}^e_k(F)$ in (19) and the projection operator $\mathbf{\Pi}^e_S$ in (42), we define the serendipity edge virtual element space on the face $F$ as

$$\mathbf{SV}^e_k(F) = \left\{ \mathbf{v}_h \in \mathbf{V}^e_k(F) : \int_F (\mathbf{v}_h - \mathbf{\Pi}^e_S \mathbf{v}_h) \cdot \mathbf{x}_F p = 0 \ \ \forall p \in \mathbb{P}_{\beta_F|k}(F) \right\}, \tag{43}$$

where $\mathbb{P}_{\beta_F|k}(F)$ is chosen to satisfy $\mathbb{P}_k(F) = \mathbb{P}_{\beta_F} \oplus \mathbb{P}_{\beta_F|k}(F)$. It can be checked that $(\mathbb{P}_k(F))^2 \subseteq \mathbf{SV}^e_k(F) \subseteq \mathbf{V}^e_k(F)$. A set of unisolvent DoFs $\{\mathrm{dof}^F_i\}_{i=1}^{N_d}$ for the space $\mathbf{SV}^e_k(F)$ with $N_d = N_e \pi_{k,1} + \pi_{k-1,2} + \pi_{\beta_F,2} - 1$ is given by (20), (22), and the internal moments

$$\int_F \mathbf{v}_h \cdot \mathbf{x}_F p_{\beta_F} \quad \forall p_{\beta_F} \in \mathbb{P}_{\beta_F}(F) \text{ only for } \beta_F \geq 0. \tag{44}$$

This choice reduces the internal DoFs of the standard edge virtual element space $\mathbf{V}^e_k(F)$ by $(\pi_{k,2} - \pi_{\beta_F,2})$. Notably, we can compute the moments of order up to $\beta_F$ given in (44), whereas the remaining moments of order up to $k$ can be computed by those of the projection $\mathbf{\Pi}^e_S$; see (43).

By Proposition 5.2 in Ref. [3], we have that a set of unisolvent DoFs $\{\mathrm{DoF}^F_i\}_{i=1}^{N_D}$ with $N_D = 2\pi_{k,2}$ for the space $(\mathbb{P}_k(F))^2$ is given by the functionals used to define $\mathbf{\Pi}^e_S$ in (42).

For sufficiently large constants $\gamma, \hat{\gamma} \in \mathbb{R}^+$, which we shall fix in the proofs of Corollary 3.5 and Lemma 3.6 below, we introduce a norm $|||\cdot|||_F$ on $(\mathbb{P}_k(F))^2$ induced by (42):

$$\begin{aligned} |||\,\mathbf{s}_k|||_F &:= \widetilde{\gamma} \left| \int_{\partial F} \mathbf{s}_k \cdot \mathbf{t}_{\partial F} \right| + \gamma \sup_{p^0_{k-1} \in \mathbb{P}^0_{k-1}(F)} \frac{h_F \int_F \mathrm{rot}_F\, \mathbf{s}_k p^0_{k-1}}{\|p^0_{k-1}\|_F} \\ &+ \hat{\gamma} \sup_{p_{k+1} \in \mathbb{P}_{k+1}(F)} \frac{h_F^{\frac{1}{2}} \int_{\partial F} (\mathbf{s}_k \cdot \mathbf{t}_{\partial F})(\boldsymbol{\nabla}_F p_{k+1} \cdot \mathbf{t}_{\partial F})}{\|\boldsymbol{\nabla}_F p_{k+1} \cdot \mathbf{t}_{\partial F}\|_{\partial F}} + \sup_{p_{\beta_F} \in \mathbb{P}_{\beta_F}(F)} \frac{h_F^{-1} \int_F \mathbf{s}_k \cdot \mathbf{x}_F p_{\beta_F}}{\|p_{\beta_F}\|_F}, \end{aligned} \tag{45}$$

where $\widetilde{\gamma} := \gamma h_F / |F|^{\frac{1}{2}}$.

By the mesh regularity assumptions in Section 2.2, $h_F / |F|^{\frac{1}{2}}$ is a uniformly bounded constant. Further, the operator $|||\cdot|||_F$ can be applied to all sufficiently smooth functions.

We first prove a critical polynomial estimate that we shall employ in the following analysis.

**Lemma 3.4.** *If the assumption (**MC**) in Section 2.2 is valid, then the following bound holds true:*

$$\|p_k\|_F \lesssim h_F^{\frac{1}{2}} \|p_k\|_{\partial F} + \sup_{p_{\beta_F} \in \mathbb{P}_{\beta_F}(F)} \frac{\int_F p_k p_{\beta_F}}{\|p_{\beta_F}\|_F} \qquad \forall p_k \in \mathbb{P}_k(F), \tag{46}$$

*where $C$ only depends on $\varepsilon$, $k$, and the shape-regularity parameter $\rho$.*

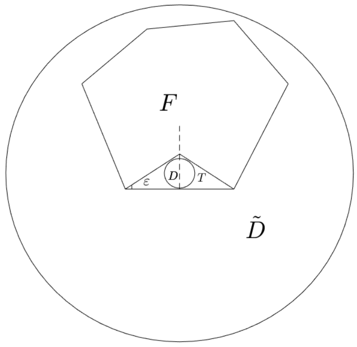


**Figure 1:** The sample figure on the element $F \in \mathrm{S}$.

*Proof.* It suffices to prove the result when $h_F = 1$ and then use a scaling argument. It is not restrictive to assume that $F$ has a vertex in the origin of the $[x, y]$ coordinate axes and an edge lies on the "$y = 0$" axis. Given any vertex $v_i$ of $F$, we denote its coordinates by $[v_{i,x}, v_{i,y}]$. We define the set of admissible polygons

$$\begin{aligned}\mathrm{S} := \big\{F:\ &F \text{ is a convex polygon with } \eta_F \text{ edges and vertices counter-clockwise} \\ &\text{ordered } \{v_1, v_2, \cdots, v_{\eta_F}\} \text{ with } v_1 = (0,0), v_{2,y} = 0;\ \text{furthermore } h_F = 1, \\ &h_e \geq \rho\ \forall e \subseteq \partial F,\ \varepsilon < \theta < \pi - \varepsilon \text{ for each internal angle } \theta \text{ of } F\big\}.\end{aligned}$$

We also define the (injective) application $\mathcal{I} : \mathrm{S} \to \mathbb{R}^{2\eta_F}$ by

$$F \longmapsto [v_{1,x}, v_{1,y}, v_{2,x}, v_{2,y}, \cdots, v_{\eta_F,x}, v_{\eta_F,y}].$$

Under the geometric assumptions of Section 2.2, $\mathcal{I}(\mathrm{S})$ is a bounded and closed subset in $\mathbb{R}^{2\eta_F}$. For each polygon $F \in \mathrm{S}$, we denote the edge connecting $v_i$ to $v_{i+1}$ by $e_i$, with the usual notation $v_{\eta_F+1} = v_1$. By the assumptions that $h_e \geq \rho\ \forall e \subseteq \partial F$ and each internal angle $\theta$ of the convex polygon $F$ satisfies $\varepsilon < \theta < \pi - \varepsilon$, there exists an isosceles triangle $T$ with basis $e_1$ and height $h \geq \beta$ (for a uniformly positive constant $\beta$) that is contained in all $F$ of $S$. Therefore, it exists a disk $D \subseteq T \subseteq F$ such that its radius is uniformly bounded by $h_F$ from below. Meanwhile, we denote the disk of radius $R = 1$ that is concentric with $D$ and containing $F$ by $\widetilde{D}$; see Figure 1 for a graphical example. We have

$$D \subseteq F \subseteq \widetilde{D} \qquad \forall F \in \mathrm{S}. \tag{47}$$

We are now in the position of proving (46) by contradiction. If (46) were false, then we could find a sequence of elements $\{F_m\}_{m\in\mathbb{N}}$ in S and a sequence of polynomials $\{p_m\}_{m\in\mathbb{N}} \in \mathbb{P}_k(F_m)$ such that

$$\|p_m\|_{F_m} = 1, \qquad \|p_m\|_{\partial F_m} \leq \frac{1}{m}, \qquad \sup_{p_\beta \in \mathbb{P}_{\beta_F}(F)} \frac{\int_F p_m p_{\beta_F}}{\|p_{\beta_F}\|_F} \leq \frac{1}{m}\ \ \forall m \in \mathbb{N}. \tag{48}$$

Since $\mathcal{I}(\mathrm{S})$ is bounded and closed, there exists a subsequence $\mathcal{I}(F_{m_j}) \subseteq \mathbb{R}^{2\eta_F}$ that converges to $\mathcal{I}(F)$ for some $F \in \mathrm{S}$ as $j \to +\infty$. In particular, all vertexes of $F_{m_j}$ converge to those of $F \in \mathrm{S}$ as $j \to +\infty$. By (47) and (48), we have

$$\|p_{m_j}\|_D \leq \|p_{m_j}\|_{F_{m_j}} = 1,$$

which implies that $\{p_{m_j}\}_{j\in\mathbb{N}} \in \mathbb{P}_k(D)$ is a bounded sequence.

Then, there exists a subsequence $\{p_{m_{j_l}}\}_{l\in\mathbb{N}}$ such that $p_{m_{j_l}} \to p_k \in \mathbb{P}_k(D)$ as $l \to +\infty$. By (47) and standard polynomial properties, it follows that

$$1 = \|p_{m_{j_l}}\|_{F_{m_{j_l}}} \leq \|p_{m_{j_l}}\|_{\widetilde{D}} \lesssim \|p_{m_{j_l}}\|_D.$$

By taking $l \to +\infty$, this yields

$$p_k \neq 0 \text{ in } D \subseteq F. \tag{49}$$

Since the ordered vertices of $F_{m_j}$ converge to those of $F$ due to $\mathcal{I}(F_{m_j}) \to \mathcal{I}(F)$, we have the boundary convergence $\partial F_{m_j} \to \partial F$ as $j \to +\infty$. By (48), we know that the subsequences $\{p_{m_{j_l}}\}_{l\in\mathbb{N}}$ and $\{\partial F_{m_{j_l}}\}_{l\in\mathbb{N}}$ satisfy

$$\|p_{m_{j_l}}\|_{\partial F_{m_{j_l}}} \leq \frac{1}{m_{j_l}},$$

which entails that $p_k|_{\partial F} = 0$ by taking $l \to +\infty$. Then, there exists $\hat{p}_{\beta_F} \in \mathbb{P}_{\beta_F}(F)$ such that

$$p_k = b_{\eta_F}\hat{p}_{\beta_F}, \tag{50}$$

where $b_{\eta_F}$ is the polynomial of degree $\eta_F$ that vanishes identically on $\partial F$ and is equal to 1 at the barycenter of the element $F$. Since $F$ is convex, we have $b_{\eta_F} > 0$ in $F$; see, e.g., Ref. [5]. Letting $\ell \to +\infty$, recalling the last inequality of (48), and combining the resulting inequality and (50) together, we arrive at

$$\int_F b_{\eta_F}(\hat{p}_{\beta_F})^2 = 0,$$

which implies that $\hat{p}_{\beta_F} \equiv 0$. By (50), it follows that

$$p_k \equiv 0 \text{ in } F.$$

Yet, this and (49) contradict each other, whence the assertion follows. □

**Corollary 3.5.** *Under the same assumptions of Lemma 3.4, for $\hat{\gamma}$ sufficiently large and independent of $F$, and each $p_k^0 \in \mathbb{P}_k^0(F)$, we have*

$$\|\boldsymbol{\nabla}_F p_k^0\|_F \lesssim \hat{\gamma} h_F^{\frac{1}{2}} \|\boldsymbol{\nabla}_F p_k^0 \cdot \boldsymbol{t}_{\partial F}\|_{\partial F} + \sup_{p_{\beta_F}\in\mathbb{P}_{\beta_F}(F)} \frac{h_F^{-1}\int_F \boldsymbol{\nabla}_F p_k^0 \cdot \boldsymbol{x}_F p_{\beta_F}}{\|p_{\beta_F}\|_F}.$$

*Proof.* We write

$$\begin{aligned}
&\hat{\gamma} h_F^{\frac{1}{2}} \|\boldsymbol{\nabla}_F p_k^0 \cdot \mathbf{t}_{\partial F}\|_{\partial F} + \sup_{p_{\beta_F}\in\mathbb{P}_{\beta_F}(F)} \frac{h_F^{-1}\int_F \boldsymbol{\nabla}_F p_k^0 \cdot \mathbf{x}_F p_{\beta_F}}{\|p_{\beta_F}\|_F} \\
&\overset{(16),\text{IBP}}{\geq} C'\hat{\gamma} h_F^{-\frac{1}{2}} \|p_k^0\|_{\partial F} + \sup_{p_{\beta_F}\in\mathbb{P}_{\beta_F}(F)} \frac{h_F^{-1}\int_F p_k^0 \operatorname{div}(\mathbf{x}_F p_{\beta_F}) - h_F^{-1}\int_{\partial F} p_k^0 \mathbf{x}_F \cdot \mathbf{n}_{\partial F} p_{\beta_F}}{\|p_{\beta_F}\|_F} \\
&\overset{(4)}{\geq} (\hat{\gamma}C' - C'') h_F^{-\frac{1}{2}} \|p_k^0\|_{\partial F} + \sup_{p'_{\beta_F}\in\mathbb{P}_{\beta_F}(F)} \frac{h_F^{-1}\int_F p_k^0 p'_{\beta_F}}{\|p'_{\beta_F}\|_F} \overset{(46)}{\geq} C h_F^{-1}\|p_k^0\|_F \overset{(1)}{\geq} C\|\boldsymbol{\nabla}_F p_k^0\|_F,
\end{aligned}$$

where we have chosen $\hat{\gamma}$ sufficiently large. □

Next, we prove lower and upper bounds on the operator $|||\cdot|||_F$ introduced in (45) with respect to the $L^2$ norm $\|\cdot\|_F$.

**Lemma 3.6.** *For given $\varepsilon > 0$, the following bounds are valid:*

$$\|\boldsymbol{s}_k\|_F \lesssim |||\boldsymbol{s}_k|||_F \qquad \forall \boldsymbol{s}_k \in (\mathbb{P}_k(F))^2, \tag{51}$$

$$|||\boldsymbol{v}_h|||_F \lesssim \|\boldsymbol{v}_h\|_F \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_k^e(F), \tag{52}$$

$$|||\boldsymbol{v}|||_F \lesssim \|\boldsymbol{v}\|_F + h_F^\varepsilon |\boldsymbol{v}|_{\varepsilon,F} + h_F\|\operatorname{rot}_F \boldsymbol{v}\|_F \quad \forall \boldsymbol{v} \in \boldsymbol{H}^\varepsilon(F) \cap \boldsymbol{H}(\operatorname{rot}_F, F). \tag{53}$$

*Proof.* First, we prove (51). From (5) and $\mathbf{s}_k \in (\mathbb{P}_k(F))^2$, there exist $q_{k+1}^0 \in \mathbb{P}_{k+1}^0(F)$ and $q_{k-1} \in \mathbb{P}_{k-1}(F)$ such that

$$\mathbf{s}_k = \boldsymbol{\nabla}_F q_{k+1}^0 + \mathbf{x}_F^\perp q_{k-1}. \tag{54}$$

Define $\widetilde{\operatorname{rot}_F \mathbf{s}_k} := \operatorname{rot}_F \mathbf{s}_k - \frac{1}{|F|}\int_F \operatorname{rot}_F \mathbf{s}_k$ and observe that

$$\int_F \operatorname{rot}_F \mathbf{s}_k \widetilde{\operatorname{rot}_F \mathbf{s}_k} = \int_F \widetilde{\operatorname{rot}_F \mathbf{s}_k}\widetilde{\operatorname{rot}_F \mathbf{s}_k}. \tag{55}$$

By taking $p^0_{k-1} = \widetilde{\mathrm{rot}_F \mathbf{s}_k}$ and $p_{k+1} = q^0_{k+1}$ that realize (54) in the second and third terms involving supremum of (45) and using the property (55), we write

$$
\begin{aligned}
|||\mathbf{s}_k|||_F \geq \widetilde{\gamma} \left| \int_{\partial F} \mathbf{s}_k \cdot \mathbf{t}_{\partial F} \right| &+ \frac{\gamma h_F \int_F \widetilde{\mathrm{rot}_F \mathbf{s}_k} \widetilde{\mathrm{rot}_F \mathbf{s}_k}}{\|\widetilde{\mathrm{rot}_F \mathbf{s}_k}\|_F} \\
&+ \frac{\hat{\gamma} h_F^{\frac{1}{2}} \int_{\partial F} ((\boldsymbol{\nabla}_F q^0_{k+1} + \mathbf{x}_F^{\perp} q_{k-1}) \cdot \mathbf{t}_{\partial F})(\boldsymbol{\nabla}_F q^0_{k+1} \cdot \mathbf{t}_{\partial F})}{\|\boldsymbol{\nabla}_F q^0_{k+1} \cdot \mathbf{t}_{\partial F}\|_{\partial F}} \\
&+ \sup_{p_{\beta_F} \in \mathbb{P}_{\beta_F}(F)} \frac{h_F^{-1} \int_F (\boldsymbol{\nabla}_F q^0_{k+1} + \mathbf{x}_F^{\perp} q_{k-1}) \cdot \mathbf{x}_F p_{\beta_F}}{\|p_{\beta_F}\|_F} \\
\geq \widetilde{\gamma} \left| \int_{\partial F} \mathbf{s}_k \cdot \mathbf{t}_{\partial F} \right| &+ \gamma h_F \|\widetilde{\mathrm{rot}_F \mathbf{s}_k}\|_F + \hat{\gamma} h_F^{\frac{1}{2}} \|\boldsymbol{\nabla}_F q^0_{k+1} \cdot \mathbf{t}_{\partial F}\|_{\partial F} \\
&- \hat{\gamma} h_F^{\frac{1}{2}} \|\mathbf{x}_F^{\perp} q_{k-1} \cdot \mathbf{t}_{\partial F}\|_{\partial F} + \sup_{p_{\beta_F} \in \mathbb{P}_{\beta_F}(F)} \frac{h_F^{-1} \int_F \boldsymbol{\nabla}_F q^0_{k+1} \cdot \mathbf{x}_F p_{\beta_F}}{\|p_{\beta_F}\|_F}.
\end{aligned}
\tag{56}
$$

We estimate every term on the right-hand side of (56) from below. We begin with the term involving $\|\widetilde{\mathrm{rot}_F \mathbf{s}_k}\|_F$:

$$
\begin{aligned}
\|\mathrm{rot}_F \mathbf{s}_k\|_F &\leq \|\widetilde{\mathrm{rot}_F \mathbf{s}_k}\|_F + \left\| \frac{1}{|F|} \int_F \mathrm{rot}_F \mathbf{s}_k \right\|_F \\
&= \|\widetilde{\mathrm{rot}_F \mathbf{s}_k}\|_F + \frac{1}{|F|} \left\| \int_{\partial F} \mathbf{s}_k \cdot \mathbf{t}_{\partial F} \right\|_F = \|\widetilde{\mathrm{rot}_F \mathbf{s}_k}\|_F + \frac{1}{|F|^{\frac{1}{2}}} \left| \int_{\partial F} \mathbf{s}_k \cdot \mathbf{t}_{\partial F} \right|.
\end{aligned}
\tag{57}
$$

Further, using (1) and the fact that $\mathrm{rot}_F \mathbf{s}_k = \mathrm{rot}_F (\mathbf{x}_F^{\perp} q_{k-1})$, we obtain

$$
h_F^{\frac{1}{2}} \|\mathbf{x}_F^{\perp} q_{k-1} \cdot \mathbf{t}_{\partial F}\|_{\partial F} \lesssim \|\mathbf{x}_F^{\perp} q_{k-1}\|_F \lesssim h_F \|\mathrm{rot}_F (\mathbf{x}_F^{\perp} q_{k-1})\|_F = h_F \|\mathrm{rot}_F \mathbf{s}_k\|_F.
$$

Inserting this and (57) in (56), recalling that $\widetilde{\gamma} = (\gamma h_F)/|F|^{\frac{1}{2}}$, and using $\mathrm{rot}_F \mathbf{s}_k = \mathrm{rot}_F (\mathbf{x}^{\perp} q_{k-1})$, Corollary 3.5, and (54), we arrive at

$$
\begin{aligned}
|||\, \mathbf{s}_k|||_F &\geq \widetilde{\gamma} \left| \int_{\partial F} \mathbf{s}_k \cdot \mathbf{t}_{\partial F} \right| + \gamma h_F \|\mathrm{rot}_F \mathbf{s}_k\|_F - \frac{\gamma h_F}{|F|^{\frac{1}{2}}} \left| \int_{\partial F} \mathbf{s}_k \cdot \mathbf{t}_{\partial F} \right| \\
&\quad + \hat{\gamma} h_F^{\frac{1}{2}} \|\boldsymbol{\nabla}_F q^0_{k+1} \cdot \mathbf{t}_{\partial F}\|_{\partial F} - C \hat{\gamma} h_F \|\mathrm{rot}_F \mathbf{s}_k\|_F + \sup_{p_{\beta_F} \in \mathbb{P}_{\beta_F}(F)} \frac{h_F^{-1} \int_F \boldsymbol{\nabla}_F q^0_{k+1} \cdot \mathbf{x}_F p_{\beta_F}}{\|p_{\beta_F}\|_F} \\
&\geq (\gamma - C\hat{\gamma}) h_F \|\mathrm{rot}_F \mathbf{s}_k\|_F + \hat{\gamma} h_F^{\frac{1}{2}} \|\boldsymbol{\nabla}_F q^0_{k+1} \cdot \mathbf{t}_{\partial F}\|_{\partial F} + \sup_{p_{\beta_F} \in \mathbb{P}_{\beta_F}(F)} \frac{h_F^{-1} \int_F \boldsymbol{\nabla}_F q^0_{k+1} \cdot \mathbf{x}_F p_{\beta_F}}{\|p_{\beta_F}\|_F} \\
&\geq (\gamma - C\hat{\gamma}) h_F \|\mathrm{rot}_F \mathbf{s}_k\|_F + C \|\boldsymbol{\nabla}_F q^0_{k+1}\|_F \gtrsim \|\mathbf{x}_F^{\perp} q_{k-1}\|_F + \|\boldsymbol{\nabla}_F q^0_{k+1}\|_F \gtrsim \|\mathbf{s}_k\|_F,
\end{aligned}
$$

where we have fixed the parameter $\gamma = 2C\hat{\gamma}$. The parameter $\hat{\gamma}$ was fixed in the proof of Corollary 3.5, sufficiently large but independent of $F$. Thus, (51) follows.

Before proceeding with the proof of the other two bounds, we observe the validity of the following inverse estimate on the space $\mathbf{SV}^e_k(F)$, which can be proven as inequality (30):

$$
\|\mathrm{rot}_F \mathbf{v}_h\|_F \lesssim h_F^{-1} \|\mathbf{v}_h\|_F \qquad \forall \mathbf{v}_h \in \mathbf{SV}^e_k(F). \tag{58}
$$

Estimate (52) is proven using (45), (2) recalling that $\mathbf{v}_h \cdot \mathbf{t}_{\partial F}$ is a piecewise polynomial, (12), and (58):

$$
|||\mathbf{v}_h|||_F \lesssim h_F^{\frac{1}{2}} \|\mathbf{v}_h \cdot \mathbf{t}_{\partial F}\|_{\partial F} + \|\mathbf{v}_h\|_F + h_F \|\mathrm{rot}_F \mathbf{v}_h\|_F \lesssim \|\mathbf{v}_h\|_F + h_F \|\mathrm{rot}_F \mathbf{v}_h\|_F \lesssim \|\mathbf{v}_h\|_F.
$$

As for estimate (53), from (45), (32), the inequality $\|\boldsymbol{\nabla}_F p_{k+1} \cdot \mathbf{t}_e\|_{L^\infty(e)} \lesssim h_e^{-\frac{1}{2}} \|\boldsymbol{\nabla}_F p_{k+1} \cdot \mathbf{t}_e\|_e$ for all $e$ in $\partial F$, and the fact that the number of edges on each face $F$ is uniformly bounded, it follows that

$$
\begin{aligned}
|||\mathbf{v}|||_F &\lesssim \sum_{e\subseteq\partial F}\left|\int_e \mathbf{v}\cdot\mathbf{t}_e\right| + h_F\|\mathrm{rot}_F\,\mathbf{v}\|_F + \|\mathbf{v}\|_F + \sup_{p_{k+1}\in\mathbb{P}_{k+1}(F)} \frac{h_F^{\frac{1}{2}}\sum_{e\subseteq\partial F}\int_e(\mathbf{v}\cdot\mathbf{t}_e)(\boldsymbol{\nabla}_F p_{k+1}\cdot\mathbf{t}_{\partial F})}{\|\boldsymbol{\nabla}_F p_{k+1}\cdot\mathbf{t}_{\partial F}\|_{\partial F}} \\
&\lesssim \|\mathbf{v}\|_F + h_F^{\varepsilon}|\mathbf{v}|_{\varepsilon,F} + h_F\|\mathrm{rot}_F\,\mathbf{v}\|_F.
\end{aligned}
$$

□

The following result contains a useful estimate for the projection $\mathbf{\Pi}_S^e$.

**Theorem 3.7.** *For each $\boldsymbol{v}\in\boldsymbol{H}^s(F)$, $0<s\le k+1$ with $\mathrm{rot}_F\,\boldsymbol{v}\in L^2(F)$, we have*

$$
\|\boldsymbol{v}-\mathbf{\Pi}_S^e\boldsymbol{v}\|_F \lesssim h_F^s|\boldsymbol{v}|_{s,F} + h_F\|\mathrm{rot}_F\,\boldsymbol{v}\|_F. \tag{59}
$$

*The second term on the right-hand side can be neglected if $s\ge 1$.*

*Proof.* For any $\mathbf{p}_k\in(\mathbb{P}_k(F))^2$, from (51), the fact that $|||\mathbf{\Pi}_S^e\cdot|||_F$ is equal to $|||\cdot|||_F$, and finally (53), we obtain

$$
\begin{aligned}
\|\mathbf{v}-\mathbf{\Pi}_S^e\mathbf{v}\|_F &\le \|\mathbf{v}-\mathbf{p}_k\|_F + \|\mathbf{\Pi}_S^e(\mathbf{v}-\mathbf{p}_k)\|_F \lesssim \|\mathbf{v}-\mathbf{p}_k\|_F + |||\mathbf{\Pi}_S^e(\mathbf{v}-\mathbf{p}_k)|||_F \\
&= \|\mathbf{v}-\mathbf{p}_k\|_F + |||\mathbf{v}-\mathbf{p}_k|||_F \\
&\lesssim \|\mathbf{v}-\mathbf{p}_k\|_F + h_F^s|\mathbf{v}-\mathbf{p}_k|_{s,F} + h_F\|\mathrm{rot}_F\,(\mathbf{v}-\mathbf{p}_k)\|_F.
\end{aligned} \tag{60}
$$

If $s\ge 1$, then (60) and standard polynomial approximation estimates yield

$$
\|\mathbf{v}-\mathbf{\Pi}_S^e\mathbf{v}\|_F \lesssim h_F^s|\mathbf{v}|_{s,F}.
$$

Instead, if $0<s<1$, then we replace $\mathbf{p}_k$ by the average vector constant $\mathbf{p}_0$ of $\mathbf{v}$ over $F$ in (60). The Poincaré inequality gives

$$
\begin{aligned}
\|\mathbf{v}-\mathbf{\Pi}_S^e\mathbf{v}\|_F &\lesssim \|\mathbf{v}-\mathbf{p}_0\|_F + h_F^s|\mathbf{v}-\mathbf{p}_0|_{s,F} + h_F\|\mathrm{rot}_F\,(\mathbf{v}-\mathbf{p}_0)\|_F \\
&\lesssim h_F^s|\mathbf{v}|_{s,F} + h_F\|\mathrm{rot}_F\,\mathbf{v}\|_F.
\end{aligned}
$$

□

We define an interpolation operator $\mathbf{I}_h^e$ for functions in $\mathbf{SV}_k^e(F)$ by requiring that the values of the DoFs (20), (22), and (44) of $\mathbf{I}_h^e\mathbf{v}$ are equal to those of $\mathbf{v}$. Combining (20) with (22), we obtain the following property:

$$
\mathrm{rot}_F\,(\mathbf{I}_h^e\mathbf{v}) = \Pi_{k-1}^{0,F}(\mathrm{rot}_F\,\mathbf{v}). \tag{61}
$$

We prove the following interpolation estimates for $\mathbf{I}_h^e$ on the serendipity edge virtual element space $\mathbf{SV}_k^e(F)$.

**Theorem 3.8.** *For each $\boldsymbol{v}\in\boldsymbol{H}^s(F)$, $0<s\le k+1$, with $\mathrm{rot}_F\,\boldsymbol{v}\in H^r(F)$, $0\le r\le k$, we have*

$$
\|\boldsymbol{v}-\boldsymbol{I}_h^e\boldsymbol{v}\|_F \lesssim h_F^s|\boldsymbol{v}|_{s,F} + h_F\|\mathrm{rot}_F\,\boldsymbol{v}\|_F, \tag{62}
$$

$$
\|\mathrm{rot}_F\,(\boldsymbol{v}-\boldsymbol{I}_h^e\boldsymbol{v})\|_F \lesssim h_F^r|\mathrm{rot}_F\,\boldsymbol{v}|_{r,F}. \tag{63}
$$

*The second term on the right-hand side of* (62) *can be neglected if $s\ge 1$.*

*Proof.* As for (63), by (61) and standard polynomial approximation properties, we have

$$
\|\mathrm{rot}_F\,(\mathbf{v}-\mathbf{I}_h^e\mathbf{v})\|_F = \|\mathrm{rot}_F\,\mathbf{v}-\Pi_{k-1}^{0,F}(\mathrm{rot}_F\,\mathbf{v})\|_F \lesssim h_F^r|\mathrm{rot}_F\,\mathbf{v}|_{r,F}.
$$

The remainder of the proof is devoted to proving (62). Observe that (37) and (61) imply $\mathrm{rot}_F(\widetilde{\mathbf{I}}_h^e\mathbf{v}-\mathbf{I}_h^e\mathbf{v})=0$, which yields the existence of a function $\phi\in H^1(F)$ such that $\widetilde{\mathbf{I}}_h^e\mathbf{v}-\mathbf{I}_h^e\mathbf{v}=\nabla_F\phi$, satisfying weakly

$$
\Delta_F\phi = \mathrm{div}_F\,(\widetilde{\mathbf{I}}_h^e\mathbf{v}-\mathbf{I}_h^e\mathbf{v}) \text{ in } F, \quad \phi=0 \text{ on } \partial F. \tag{64}
$$

The boundary conditions in (64) follow from the fact that

$$\partial_t \phi_{|\partial F} = (\widetilde{\mathbf{I}}_h^e \mathbf{v} - \mathbf{I}_h^e \mathbf{v}) \cdot \mathbf{t}_{|\partial F} = 0,$$

since the definitions of $\mathbf{I}_h^e$ and $\widetilde{\mathbf{I}}_h^e$ entail

$$(\widetilde{\mathbf{I}}_h^e \mathbf{v} - \mathbf{I}_h^e \mathbf{v}) \cdot \mathbf{t}_e = 0 \quad \forall e \subseteq \partial F.$$

Since

$$\|\widetilde{\mathbf{I}}_h^e \mathbf{v} - \mathbf{I}_h^e \mathbf{v}\|_F = \|\boldsymbol{\nabla}_F \phi\|_F, \tag{65}$$

it suffices to estimate the right-hand side of (65). By the fact that $\mathrm{div}_F\,(\widetilde{\mathbf{I}}_h^e \mathbf{v} - \mathbf{I}_h^e \mathbf{v}) \in \mathbb{P}_k(F)$ and (4), there exists a polynomial $q_k \in \mathbb{P}_k(F)$ such that

$$\mathrm{div}_F\,(\mathbf{x}_F q_k) = \mathrm{div}_F\,(\widetilde{\mathbf{I}}_h^e \mathbf{v} - \mathbf{I}_h^e \mathbf{v}), \tag{66}$$

with

$$\|\mathbf{x}_F q_k\|_F \overset{(30)}{\lesssim} h_F \|\mathrm{div}_F\,(\widetilde{\mathbf{I}}_h^e \mathbf{v} - \mathbf{I}_h^e \mathbf{v})\|_F \lesssim \|\widetilde{\mathbf{I}}_h^e \mathbf{v} - \mathbf{I}_h^e \mathbf{v}\|_F. \tag{67}$$

Moreover, $\boldsymbol{\Pi}_S^e \mathbf{I}_h^e \mathbf{v} = \boldsymbol{\Pi}_S^e \mathbf{v}$ since $\mathbf{I}_h^e \mathbf{v}$ and $\mathbf{v}$ share the same DoFs (20), (22), (44), and the value of the projection $\boldsymbol{\Pi}_S^e$ only depends on such DoFs. Thus, we write

$$\begin{aligned}
\|\boldsymbol{\nabla}_F \phi\|_F^2 &= (\widetilde{\mathbf{I}}_h^e \mathbf{v} - \mathbf{I}_h^e \mathbf{v}, \boldsymbol{\nabla}_F \phi)_F \overset{\mathrm{IBP}}{=} -(\mathrm{div}_F\,(\widetilde{\mathbf{I}}_h^e \mathbf{v} - \mathbf{I}_h^e \mathbf{v}), \phi)_F \\
&\overset{(66)}{=} -(\mathrm{div}_F\,(\mathbf{x}_F q_k), \phi)_F \overset{(64)}{=} (\mathbf{x}_F q_k, \boldsymbol{\nabla}_F \phi)_F = (\mathbf{x}_F q_k, \widetilde{\mathbf{I}}_h^e \mathbf{v} - \mathbf{I}_h^e \mathbf{v})_F \\
&\overset{(34b),(43)}{=} (\mathbf{x}_F q_k, \mathbf{v} - \boldsymbol{\Pi}_S^e \mathbf{I}_h^e \mathbf{v})_F \overset{(42d)}{=} (\mathbf{x}_F q_k, \mathbf{v} - \boldsymbol{\Pi}_S^e \mathbf{v})_F \lesssim \|\mathbf{x}_F q_k\|_F \|\mathbf{v} - \boldsymbol{\Pi}_S^e \mathbf{v}\|_F \\
&\overset{(67)}{\lesssim} \|\widetilde{\mathbf{I}}_h^e \mathbf{v} - \mathbf{I}_h^e \mathbf{v}\|_F \|\mathbf{v} - \boldsymbol{\Pi}_S^e \mathbf{v}\|_F \overset{(59)}{\lesssim} (h_F^s |\mathbf{v}|_{s,F} + h_F \|\mathrm{rot}_F\, \mathbf{v}\|_F)\, \|\widetilde{\mathbf{I}}_h^e \mathbf{v} - \mathbf{I}_h^e \mathbf{v}\|_F,
\end{aligned} \tag{68}$$

where the term $\|\mathrm{rot}_F\, \mathbf{v}\|_F$ can be ignored if $s \geq 1$.

Substituting (68) into (65), and by using the triangle inequality and (35), estimate (62) follows. □

### 3.3 Face virtual element spaces on polygons

Since 2D face virtual element spaces can be viewed as a $\pi/2$ rotation of the 2D edge ones, we can extend all above definitions and results to standard and serendipity face virtual element spaces in 2D; see Refs. [3, 4, 7, 9]. The face virtual element space on the face $F$ is defined as

$$\mathbf{V}_k^f(F) = \left\{\mathbf{v}_h \in \mathbf{L}^2(F) : \mathrm{div}_F\, \mathbf{v}_h \in \mathbb{P}_{k-1}(F),\ \mathrm{rot}_F\, \mathbf{v}_h \in \mathbb{P}_k(F),\ \mathbf{v}_h \cdot \mathbf{n}_e \in \mathbb{P}_k(e) \quad \forall e \subseteq \partial F\right\},$$

and is endowed with the unisolvent DoFs [3, 4]

- $\displaystyle\int_e \mathbf{v}_h \cdot \mathbf{n}_e p_k \qquad \forall p_k \in \mathbb{P}_k(e),\ \forall e \subseteq \partial F;$ (69)
- $\displaystyle\int_F \mathbf{v}_h \cdot \mathbf{x}_F^{\perp} p_k \qquad \forall p_k \in \mathbb{P}_k(F);$ (70)
- $\displaystyle\int_F \mathrm{div}_F\, \mathbf{v}_h p_{k-1}^0 \qquad \forall p_{k-1}^0 \in \mathbb{P}_{k-1}^0(F)$ only for $k > 1$. (71)

We define the DoFs interpolation operator $\widetilde{\mathbf{I}}_h^f$ on the space $\mathbf{V}_k^f(F)$ by requiring that the values of the DoFs (69), (70), and (71) of $\widetilde{\mathbf{I}}_h^f \mathbf{v}$ are equal to those of $\mathbf{v} \in \mathbf{H}^s(F) \cap \mathbf{H}(\mathrm{div}_F, F)$, $s > 0$. We can easily extend the interpolation estimates of edge virtual element spaces, to the face case; see Theorem 3.3.

**Theorem 3.9.** *For each* $\boldsymbol{v} \in \boldsymbol{H}^s(F)$, $0 < s \leq k+1$ *with* $\mathrm{div}_F\,\boldsymbol{v} \in H^r(F)$, $0 \leq r \leq k$, *we have*

$$\|\boldsymbol{v} - \widetilde{\boldsymbol{I}}_h^f \boldsymbol{v}\|_F \lesssim h_F^s |\boldsymbol{v}|_{s,F} + h_F \|\mathrm{div}_F\,\boldsymbol{v}\|_F, \tag{72}$$

$$\|\mathrm{div}_F\,(\boldsymbol{v} - \widetilde{\boldsymbol{I}}_h^f \boldsymbol{v})\|_F \lesssim h_F^r |\mathrm{div}_F\,\boldsymbol{v}|_{r,F}. \tag{73}$$

*The second term on the right-hand side of* (72) *can be neglected if* $s \geq 1$.

By rotating everything by $\pi/2$ corresponding to edge elements, we can also introduce a well defined projection $\boldsymbol{\Pi}_S^f : \mathbf{V}_k^f(F) \to (\mathbb{P}_k(F))^2$ by

$$\begin{aligned}
&\int_{\partial F} [(\mathbf{v}_h - \boldsymbol{\Pi}_S^f \mathbf{v}_h)\cdot \mathbf{n}_{\partial F}][\mathbf{curl}_F\, p_{k+1} \cdot \mathbf{n}_{\partial F}] = 0 \qquad \forall p_{k+1} \in \mathbb{P}_{k+1}(F);\\
&\int_{\partial F} (\mathbf{v}_h - \boldsymbol{\Pi}_S^f \mathbf{v}_h)\cdot \mathbf{n}_{\partial F} = 0;\\
&\int_F \mathrm{div}_F\,(\mathbf{v}_h - \boldsymbol{\Pi}_S^f \mathbf{v}_h) p_{k-1}^0 = 0 \qquad \forall p_{k-1}^0 \in \mathbb{P}_{k-1}^0(F) \text{ only for } k > 1;\\
&\int_F (\mathbf{v}_h - \boldsymbol{\Pi}_S^f \mathbf{v}_h)\cdot \mathbf{x}_F^\perp p_{\beta_F} = 0 \qquad \forall p_{\beta_F} \in \mathbb{P}_{\beta_F}(F) \text{ only for } \beta_F \geq 0.
\end{aligned}$$

Eventually, we introduce the serendipity face virtual element space on the face $F$

$$\mathbf{SV}_k^f(F) = \left\{ \mathbf{v}_h \in \mathbf{V}_k^f(F) : \int_F (\mathbf{v}_h - \boldsymbol{\Pi}_S^f \mathbf{v}_h)\cdot \mathbf{x}_F^\perp p = 0 \ \ \forall p \in \mathbb{P}_{\beta_F|k}(F) \right\},$$

which is endowed with the set of unisolvent DoFs (69) and (71), plus the moments

$$\int_F \mathbf{v}_h \cdot \mathbf{x}_F^\perp p_{\beta_F} \quad \forall p_{\beta_F} \in \mathbb{P}_{\beta_F}(F) \ \text{ only for } \beta_F \geq 0. \tag{74}$$

We define the DoFs interpolation operator $\mathbf{I}_h^f$ on the serendipity face virtual element space $\mathbf{SV}_k^f(F)$ by requiring that the values of the DoFs (69), (71), and (74) of $\mathbf{I}_h^f \mathbf{v}$ are equal to those of $\mathbf{v}$. We inherit interpolation estimates from serendipity edge spaces. In fact, the following result is proven as the rotated version of Theorem 3.8 (and thus also needs the additional mesh assumption (**MC**)).

**Theorem 3.10.** *For each* $\boldsymbol{v} \in \boldsymbol{H}^s(F)$, $0 < s \leq k+1$, *with* $\mathrm{div}_F\,\boldsymbol{v} \in H^r(F)$, $0 \leq r \leq k$, *we have*

$$\|\boldsymbol{v} - \boldsymbol{I}_h^f \boldsymbol{v}\|_F \lesssim h_F^s |\boldsymbol{v}|_{s,F} + h_F \|\mathrm{div}_F\,\boldsymbol{v}\|_F, \tag{75}$$

$$\|\mathrm{div}_F\,(\boldsymbol{v} - \boldsymbol{I}_h^f \boldsymbol{v})\|_F \lesssim h_F^r |\mathrm{div}_F\,\boldsymbol{v}|_{r,F}. \tag{76}$$

*The second term on the right-hand side of* (75) *can be neglected if* $s \geq 1$.

# 4 Interpolation properties of edge and face virtual element spaces in 3D

In this section, we prove interpolation properties for general order face and edge virtual element spaces on polyhedra. More precisely we consider standard face virtual element spaces in Section 4.1; standard edge virtual element spaces in Section 4.2; serendipity edge virtual element space in Section 4.3.

## 4.1 Standard face virtual element space on polyhedrons

We consider the face virtual element space [4]

$$\begin{aligned}
\mathbf{V}_{k-1}^f(E) = \big\{ \mathbf{v}_h \in \mathbf{L}^2(E) : \mathrm{div}\,\mathbf{v}_h \in \mathbb{P}_{k-1}(E), \mathbf{curl}\,\mathbf{v}_h \in (\mathbb{P}_k(E))^3,&\\
\mathbf{v}_h \cdot \mathbf{n}_F \in \mathbb{P}_{k-1}(F)\ \forall F \subseteq \partial E \big\},&
\end{aligned}$$

and endow it with the unisolvent set of DoFs [4, 7]

- $\displaystyle\int_F \mathbf{v}_h \cdot \mathbf{n}_F p_{k-1}$ $\qquad\forall p_{k-1} \in \mathbb{P}_{k-1}(F),\ \forall F \subseteq \partial E;$
- $\displaystyle\int_E \mathbf{v}_h \cdot \mathbf{x}_E \wedge \mathbf{p}_k$ $\qquad\forall \mathbf{p}_k \in (\mathbb{P}_k(E))^3;$
- $\displaystyle\int_E \operatorname{div} \mathbf{v}_h p^0_{k-1}$ $\qquad\forall p^0_{k-1} \in \mathbb{P}^0_{k-1}(E)$ only for $k > 1$.

A simple computation reveals that the $\mathbf{L}^2$ projection $\mathbf{\Pi}^{0,E}_{k+1}$:$\mathbf{V}^f_{k-1}(E) \to (\mathbb{P}_{k+1}(E))^3$ is computable by means of such DoFs.

We first prove the following auxiliary bound for functions in $\mathbf{V}^f_{k-1}(E)$.

**Lemma 4.1.** *For each $\boldsymbol{v}_h \in \boldsymbol{V}^f_{k-1}(E)$, we have*

$$\|\boldsymbol{v}_h\|_E \lesssim h_E \|\operatorname{div} \boldsymbol{v}_h\|_E + h_E^{\frac{1}{2}} \|\boldsymbol{v}_h \cdot \boldsymbol{n}_{\partial E}\|_{\partial E} + \sup_{\boldsymbol{p}_k \in (\mathbb{P}_k(E))^3} \frac{\int_E \boldsymbol{v}_h \cdot \boldsymbol{x}_E \wedge \boldsymbol{p}_k}{\|\boldsymbol{x}_E \wedge \boldsymbol{p}_k\|_E}. \tag{77}$$

*Proof.* The following Helmholtz decomposition of $\mathbf{v}_h$ is valid; see Proposition 3.1 in Ref. [11]:

$$\mathbf{v}_h = \mathbf{curl}\, \boldsymbol{\rho} + \boldsymbol{\nabla}\psi, \tag{78}$$

where the function $\psi \in H^1(E) \setminus \mathbb{R}$ satisfies weakly

$$\Delta\psi = \operatorname{div} \mathbf{v}_h \text{ in } E, \quad \boldsymbol{\nabla}\psi \cdot \mathbf{n}_{\partial E} = \mathbf{v}_h \cdot \mathbf{n}_{\partial E} \text{ on } \partial E,$$

and the function $\boldsymbol{\rho} \in \mathbf{H}(\mathbf{curl}, E) \cap \mathbf{H}(\text{div,E})$ satisfies weakly

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{\rho} = \mathbf{curl}\, \mathbf{v}_h \text{ in } E, \quad \operatorname{div} \boldsymbol{\rho} = 0 \text{ in } E, \quad \boldsymbol{\rho} \wedge \mathbf{n}_{\partial E} = 0 \text{ on } \partial E. \tag{79}$$

We have

$$(\mathbf{curl}\boldsymbol{\rho}, \boldsymbol{\nabla}\psi)_E = 0, \qquad \|\mathbf{v}_h\|^2_E = \|\mathbf{curl}\, \boldsymbol{\rho}\|^2_E + \|\boldsymbol{\nabla}\psi\|^2_E. \tag{80}$$

By using (78), an integration by parts, (10) and (16), it is immediate that

$$\begin{aligned}\|\boldsymbol{\nabla}\psi\|^2_E &\overset{(78),(80)}{=} (\boldsymbol{\nabla}\psi, \mathbf{v}_h)_E \overset{\text{IBP}}{=} \int_{\partial E} \mathbf{v}_h \cdot \mathbf{n}_{\partial E}\psi - \int_E \operatorname{div} \mathbf{v}_h \psi \lesssim \|\mathbf{v}_h \cdot \mathbf{n}_{\partial E}\|_{\partial E} \|\psi\|_{\partial E} \\ &+ \|\operatorname{div} \mathbf{v}_h\|_E \|\psi\|_E \overset{(10),(16)}{\lesssim} \Big(h_E \|\operatorname{div} \mathbf{v}_h\|_E + h_E^{\frac{1}{2}} \|\mathbf{v}_h \cdot \mathbf{n}_{\partial E}\|_{\partial E}\Big) \|\boldsymbol{\nabla}\psi\|_E.\end{aligned} \tag{81}$$

Since $\mathbf{curl}\, \mathbf{v}_h \in (\mathbb{P}_k(E))^3$ with $\operatorname{div}(\mathbf{curl}\, \mathbf{v}_h) = 0$, (7) implies the existence of $\mathbf{q}_k \in (\mathbb{P}_k(E))^3$ such that

$$\mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k) = \mathbf{curl}\, \mathbf{v}_h \text{ and } \|\mathbf{x}_E \wedge \mathbf{q}_k\|_E \lesssim h_E \|\mathbf{curl}\, \mathbf{v}_h\|_E. \tag{82}$$

The following inverse estimate inequality involving face virtual element functions is the three dimensional version of (58) and is based on the existence of a shape-regular decomposition of $E$ into tetrahedra (see Remark 1):

$$\|\mathbf{curl}\, \mathbf{v}_h\|_E \lesssim h_E^{-1} \|\mathbf{v}_h\|_E \quad \forall \mathbf{v}_h \in \mathbf{V}^f_{k-1}(E). \tag{83}$$

Next, we estimate the first term on the right-hand side of (80):

$$\begin{aligned}\|\mathbf{curl}\, \boldsymbol{\rho}\|^2_E &\overset{\text{IBP}}{=} \int_E \boldsymbol{\rho} \cdot \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{\rho} \overset{(79)}{=} \int_E \boldsymbol{\rho} \cdot \mathbf{curl}\, \mathbf{v}_h \overset{(82)}{=} \int_E \boldsymbol{\rho} \cdot \mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k) \\ &\overset{\text{IBP},(78)}{=} \int_E (\mathbf{v}_h - \boldsymbol{\nabla}\psi) \cdot (\mathbf{x}_E \wedge \mathbf{q}_k) + \int_{\partial E} (\boldsymbol{\rho} \wedge \mathbf{n}_{\partial E}) \cdot (\mathbf{x}_E \wedge \mathbf{q}_k) \\ &\overset{(79)}{\leq} \Big( \sup_{\mathbf{p}_k \in (\mathbb{P}_k(E))^3} \frac{\int_E \mathbf{v}_h \cdot \mathbf{x}_E \wedge \mathbf{p}_k}{\|\mathbf{x}_E \wedge \mathbf{p}_k\|_E} + \|\boldsymbol{\nabla}\psi\|_E \Big) \|\mathbf{x}_E \wedge \mathbf{q}_k\|_E \\ &\overset{(82),(83)}{\lesssim} \Big( \sup_{\mathbf{p}_k \in (\mathbb{P}_k(E))^3} \frac{\int_E \mathbf{v}_h \cdot \mathbf{x}_E \wedge \mathbf{p}_k}{\|\mathbf{x}_E \wedge \mathbf{p}_k\|_E} + \|\boldsymbol{\nabla}\psi\|_E \Big) \|\mathbf{v}_h\|_E.\end{aligned} \tag{84}$$

Bound (77) easily follows by combining (80), (81), and (84). $\square$

The DoFs interpolation operator $\widetilde{\mathbf{I}}_h^f$ on the space $\mathbf{V}_{k-1}^f(E)$ is well defined for functions in $\mathbf{H}^s(E) \cap \mathbf{H}(\mathrm{div}, E)$, $s > 1/2$:

$$\int_F (\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v}) \cdot \mathbf{n}_F p_{k-1} = 0 \qquad \forall p_{k-1} \in \mathbb{P}_{k-1}(F),\ \forall F \subseteq \partial E; \tag{85a}$$

$$\int_E (\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v}) \cdot \mathbf{x}_E \wedge \mathbf{p}_k = 0 \qquad \forall \mathbf{p}_k \in (\mathbb{P}_k(E))^3; \tag{85b}$$

$$\int_E \mathrm{div}\,(\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v}) p_{k-1}^0 = 0 \qquad \forall p_{k-1}^0 \in \mathbb{P}_{k-1}^0(E) \text{ only for } k > 1. \tag{85c}$$

From (85a) and (85c), we have

$$\mathrm{div}\,(\widetilde{\mathbf{I}}_h^f \mathbf{v}) = \Pi_{k-1}^{0,E}(\mathrm{div}\,\mathbf{v}). \tag{86}$$

Next, we prove interpolation estimates for the three-dimensional face virtual element space $\mathbf{V}_{k-1}^f(E)$.

**Theorem 4.2.** *For each $\boldsymbol{v} \in \boldsymbol{H}^s(E)$, $1/2 < s \leq k$, with $\mathrm{div}\,\boldsymbol{v} \in H^r(E)$, $0 \leq r \leq k$,we have*

$$\|\boldsymbol{v} - \widetilde{\boldsymbol{I}}_h^f \boldsymbol{v}\|_E \lesssim h_E^s |\boldsymbol{v}|_{s,E} + h_E \|\mathrm{div}\,\boldsymbol{v}\|_E, \tag{87}$$

$$\|\mathrm{div}\,(\boldsymbol{v} - \widetilde{\boldsymbol{I}}_h^f \boldsymbol{v})\|_E \lesssim h_E^r |\mathrm{div}\,\boldsymbol{v}|_{r,E}. \tag{88}$$

*The second term on the right-hand side of* (87) *can be neglected if $s \geq 1$.*

*Proof.* By (86) and standard polynomial approximation properties, we immediately get (88). Hence, we focus on bound (87).

First, we observe that (85a) implies

$$h_E^{\frac{1}{2}} \|(\mathbf{\Pi}_{k-1}^{0,E}\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v}) \cdot \mathbf{n}_{\partial E}\|_{\partial E} \leq h_E^{\frac{1}{2}} \|(\mathbf{\Pi}_{k-1}^{0,E}\mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{\partial E}\|_{\partial E}. \tag{89}$$

Using the facts that $\mathbf{\Pi}_{k-1}^{0,E}\mathbf{v} \in (\mathbb{P}_{k-1}(E))^3 \subseteq \mathbf{V}_{k-1}^f(E)$ and $\widetilde{\mathbf{I}}_h^f \mathbf{v} \cdot \mathbf{n}_F \in \mathbb{P}_{k-1}(F)$, (77), and (85), it follows that

$$\begin{aligned}
\|\mathbf{\Pi}_{k-1}^{0,E}\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v}\|_E &\overset{(77)}{\lesssim} h_E \|\mathrm{div}\,(\mathbf{\Pi}_{k-1}^{0,E}\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v})\|_E \\
&+ h_E^{\frac{1}{2}} \|(\mathbf{\Pi}_{k-1}^{0,E}\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v}) \cdot \mathbf{n}_{\partial E}\|_{\partial E} + \sup_{\mathbf{p}_k \in (\mathbb{P}_k(E))^3} \frac{\int_E (\mathbf{\Pi}_{k-1}^{0,E}\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v}) \cdot \mathbf{x}_E \wedge \mathbf{p}_k}{\|\mathbf{x}_E \wedge \mathbf{p}_k\|_E} \\
&\overset{(85b),(89)}{\lesssim} h_E \|\mathrm{div}\,(\mathbf{v} - \mathbf{\Pi}_{k-1}^{0,E}\mathbf{v})\|_E + h_E \|\mathrm{div}\,(\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v})\|_E \\
&+ h_E^{\frac{1}{2}} \|(\mathbf{v} - \mathbf{\Pi}_{k-1}^{0,E}\mathbf{v}) \cdot \mathbf{n}_{\partial E}\|_{\partial E} + \|\mathbf{v} - \mathbf{\Pi}_{k-1}^{0,E}\mathbf{v}\|_E.
\end{aligned} \tag{90}$$

We apply the triangle inequality and (90) to obtain

$$\begin{aligned}
\|\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v}\|_E &\leq \|\mathbf{v} - \mathbf{\Pi}_{k-1}^{0,E}\mathbf{v}\|_E + \|\mathbf{\Pi}_{k-1}^{0,E}\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v}\|_E \lesssim \|\mathbf{v} - \mathbf{\Pi}_{k-1}^{0,E}\mathbf{v}\|_E \\
&+ h_E \|\mathrm{div}\,(\mathbf{v} - \mathbf{\Pi}_{k-1}^{0,E}\mathbf{v})\|_E + h_E \|\mathrm{div}\,(\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v})\|_E + h_E^{\frac{1}{2}} \|(\mathbf{v} - \mathbf{\Pi}_{k-1}^{0,E}\mathbf{v}) \cdot \mathbf{n}_{\partial E}\|_{\partial E}.
\end{aligned} \tag{91}$$

If $s \geq 1$, standard polynomial approximation properties lead to

$$\begin{aligned}
\|\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v}\|_E &\overset{(91),(8)}{\lesssim} \|\mathbf{v} - \mathbf{\Pi}_{k-1}^{0,E}\mathbf{v}\|_E + h_E \|\mathrm{div}\,(\mathbf{v} - \mathbf{\Pi}_{k-1}^{0,E}\mathbf{v})\|_E + h_E \|\mathrm{div}\,(\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v})\|_E \\
&+ h_E |\mathbf{v} - \mathbf{\Pi}_{k-1}^{0,E}\mathbf{v}|_{1,E} \overset{(88)}{\lesssim} h_E^s \left(|\mathbf{v}|_{s,E} + |\mathrm{div}\,\mathbf{v}|_{s-1,E}\right) \lesssim h_E^s |\mathbf{v}|_{s,E}.
\end{aligned}$$

Instead, if $1/2 < s < 1$, we replace the term $\mathbf{\Pi}_{k-1}^{0,E}\mathbf{v}$ by $\mathbf{\Pi}_0^{0,E}\mathbf{v}$ in (90) and (91), use standard polynomial approximation properties, and write

$$\begin{aligned}
\|\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v}\|_E &\overset{(8)}{\lesssim} \|\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v}\|_E + h_E \|\mathrm{div}\,(\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v})\|_E \\
&+ h_E \|\mathrm{div}\,(\mathbf{v} - \widetilde{\mathbf{I}}_h^f \mathbf{v})\|_E + h_E^s |\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v}|_{s,E} \overset{(88)}{\lesssim} h_E^s |\mathbf{v}|_{s,E} + h_E \|\mathrm{div}\,\mathbf{v}\|_E.
\end{aligned}$$

□

## 4.2 Standard edge virtual element space on polyhedrons

As in Ref. [4, 9], we first introduce the boundary space

$$\mathcal{B}_k(\partial E)=\left\{\mathbf{v}_h\in\mathbf{L}^2(\partial E):\mathbf{v}_h^F\in\mathbf{V}_k^e(F)\ \forall F\subseteq\partial E, \mathbf{v}_h\cdot\mathbf{t}_e \text{ is continuous } \forall e\subseteq\partial F\right\}, \tag{92}$$

where $\mathbf{v}_h^F$ denotes the tangential component of the vector $\mathbf{v}_h$ over $F$ given by

$$\mathbf{v}_h^F=(\mathbf{v}_h-(\mathbf{v}_h\cdot\mathbf{n}_F)\mathbf{n}_F)|_F. \tag{93}$$

The standard edge virtual element space in 3D is defined as [4]

$$\begin{aligned}\mathbf{V}_k^e(E)=\{\mathbf{v}_h\in\mathbf{L}^2(E):\ &\operatorname{div}\mathbf{v}_h\in\mathbb{P}_{k-1}(E), \mathbf{curl}\,\mathbf{curl}\,\mathbf{v}_h\in(\mathbb{P}_k(E))^3,\\ &\mathbf{v}_h^F\in\mathbf{V}_k^e(F)\ \forall F\subseteq\partial E, \mathbf{v}_h\cdot\mathbf{t}_e \text{ is continuous } \forall e\subseteq\partial F\}.\end{aligned}$$

We endow the space $\mathbf{V}_k^e(E)$ with the following set of DoFs:

- $\displaystyle\int_e \mathbf{v}_h\cdot\mathbf{t}_e p_k \qquad \forall p_k\in\mathbb{P}_k(e),\ \forall e\subseteq\partial F;$ (94)
- $\displaystyle\int_F \mathbf{v}_h^F\cdot\mathbf{x}_F^F p_k \qquad \forall p_k\in\mathbb{P}_k(F);$ (95)
- $\displaystyle\int_F \operatorname{rot}_F\mathbf{v}_h^F p_{k-1}^0 \qquad \forall p_{k-1}^0\in\mathbb{P}_{k-1}^0(F)$ only for $k>1$; (96)
- $\displaystyle\int_E \mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{x}_E\wedge\mathbf{p}_k \qquad \forall\mathbf{p}_k\in(\mathbb{P}_k(E))^3;$ (97)
- $\displaystyle\int_E \mathbf{v}_h\cdot\mathbf{x}_E p_{k-1} \qquad \forall p_{k-1}\in\mathbb{P}_{k-1}(E).$ (98)

The unisolvence of the above DoFs is proven in Section 8.6 of Ref. [9]. From Proposition 3.7 in Ref. [4], the $\mathbf{L}^2$ projection $\mathbf{\Pi}_k^{0,E}$ from $\mathbf{V}_k^e(E)$ to $(\mathbb{P}_k(E))^3$ can be computed by such DoFs.

Next, we recall a well-posedness result for **curl-curl** systems; for the sake of completeness, we discuss its proof.

**Lemma 4.3.** *For any given $\boldsymbol{v}_h\in\boldsymbol{V}_k^e(E)$, the problem*

$$\begin{cases}\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\rho}=\mathbf{curl}\,\boldsymbol{v}_h, & \operatorname{div}\boldsymbol{\rho}=0 \quad \text{in } E,\\ \mathbf{curl}\,\boldsymbol{\rho}\wedge\boldsymbol{n}_{\partial E}=\boldsymbol{v}_h\wedge\boldsymbol{n}_{\partial E}, & \boldsymbol{\rho}\cdot\boldsymbol{n}_{\partial E}=0 \quad \text{on } \partial E,\end{cases} \tag{99}$$

*has a unique solution $\boldsymbol{\rho}$ in $\boldsymbol{H}(\mathbf{curl},E)\cap\boldsymbol{H}(\operatorname{div},E)$. Moreover, the following a priori bound is valid:*

$$\|\boldsymbol{\rho}\|_E+h_E\|\mathbf{curl}\,\boldsymbol{\rho}\|_E\lesssim h_E^2\|\mathbf{curl}\,\boldsymbol{v}_h\|_E+h_E^{\frac{3}{2}}\|\boldsymbol{v}_h\wedge\boldsymbol{n}_{\partial E}\|_{\partial E}. \tag{100}$$

*Proof.* To see that (99) is well-posed, we introduce the auxiliary variable $\boldsymbol{\sigma}:=\mathbf{curl}\,\boldsymbol{\rho}$. Then, (99) can be equivalently decomposed into the two following problems:
• for given $\mathbf{v}_h\in\mathbf{V}_k^e(E)$, find $\boldsymbol{\sigma}\in\mathbf{H}(\mathbf{curl},E)\cap\mathbf{H}(\operatorname{div},E)$ such that

$$\begin{cases}\mathbf{curl}\,\boldsymbol{\sigma}=\mathbf{curl}\,\mathbf{v}_h, & \operatorname{div}\boldsymbol{\sigma}=0 \quad \text{in } E,\\ \boldsymbol{\sigma}\wedge\mathbf{n}_{\partial E}=\mathbf{v}_h\wedge\mathbf{n}_{\partial E} & \qquad\qquad \text{on } \partial E;\end{cases} \tag{101}$$

• find $\boldsymbol{\rho}\in\mathbf{H}(\mathbf{curl},E)\cap\mathbf{H}(\operatorname{div},E)$ such that

$$\begin{cases}\mathbf{curl}\,\boldsymbol{\rho}=\boldsymbol{\sigma}, & \operatorname{div}\boldsymbol{\rho}=0 \quad \text{in } E,\\ \boldsymbol{\rho}\cdot\mathbf{n}_{\partial E}=0 & \qquad\qquad \text{on } \partial E.\end{cases} \tag{102}$$

Since the above div-**curl** systems are uniquely solvable [2, 7], (99) has a unique solution. Next, we prove (100). We first observe that

$$\|\mathbf{curl}\,\boldsymbol{\rho}\|_E\overset{(102)}{=}\|\boldsymbol{\sigma}\|_E\overset{(17),(101)}{\lesssim}h_E\|\mathbf{curl}\,\mathbf{v}_h\|_E+h_E^{\frac{1}{2}}\|\mathbf{v}_h\wedge\mathbf{n}_{\partial E}\|_{\partial E}. \tag{103}$$

Furthermore, we have

$$\|\boldsymbol{\rho}\|_E \overset{(18),(102)}{\lesssim} h_E\|\mathbf{curl}\,\boldsymbol{\rho}\|_E \overset{(103)}{\lesssim} h_E^2\|\mathbf{curl}\,\mathbf{v}_h\|_E + h_E^{\frac{3}{2}}\|\mathbf{v}_h \wedge \mathbf{n}_{\partial E}\|_{\partial E}. \tag{104}$$

The assertion follows combining (103) and (104). □

We could have proved Lemma 4.3 by writing (99) in mixed form [26,30]. In the following result, we prove an auxiliary bound for functions in $\mathbf{V}_k^e(E)$.

**Lemma 4.4.** *For each $\boldsymbol{v}_h \in \boldsymbol{V}_k^e(E)$, we have*

$$\begin{aligned}\|\boldsymbol{v}_h\|_E \lesssim \sum_{F\subseteq\partial E}&\left(h_F^{\frac{3}{2}}\|\mathbf{curl}\,\boldsymbol{v}_h\cdot\boldsymbol{n}_F\|_F + h_F\|\boldsymbol{v}_h^F\cdot\boldsymbol{t}_{\partial F}\|_{\partial F} + \sup_{p_k\in\mathbb{P}_k(F)}\frac{h_F^{\frac{1}{2}}\int_F \boldsymbol{v}_h^F\cdot\boldsymbol{x}_F^F p_k}{\|\boldsymbol{x}_F^F p_k\|_F}\right)\\ &+\sup_{\boldsymbol{p}_k\in(\mathbb{P}_k(E))^3}\frac{h_E\int_E \mathbf{curl}\,\boldsymbol{v}_h\cdot\boldsymbol{x}_E\wedge\boldsymbol{p}_k}{\|\boldsymbol{x}_E\wedge\boldsymbol{p}_k\|_E} + \sup_{p_{k-1}\in\mathbb{P}_{k-1}(E)}\frac{\int_E \boldsymbol{v}_h\cdot\boldsymbol{x}_E p_{k-1}}{\|\boldsymbol{x}_E p_{k-1}\|_E}.\end{aligned} \tag{105}$$

*Proof.* We first prove that there exist $\psi \in H^1(E)\setminus\mathbb{R}$ and $\boldsymbol{\rho}\in\mathbf{H}(\mathbf{curl},E)\cap\mathbf{H}(\mathrm{div},E)$ such that the following Helmholtz decomposition of $\mathbf{v}_h$ is valid:

$$\mathbf{v}_h = \mathbf{curl}\,\boldsymbol{\rho} + \boldsymbol{\nabla}\psi. \tag{106}$$

To prove (106), we define a function $\psi\in H^1(E)$ satisfying weakly

$$\Delta\psi = \mathrm{div}\,\mathbf{v}_h \text{ in } E,\quad \psi = 0 \text{ on } \partial E, \tag{107}$$

and a function $\boldsymbol{\rho}\in\mathbf{H}(\mathbf{curl},E)\cap\mathbf{H}(\mathrm{div},E)$ satisfying weakly

$$\begin{cases}\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\rho} = \mathbf{curl}\,\mathbf{v}_h, & \mathrm{div}\,\boldsymbol{\rho}=0 \quad \text{in } E,\\ \mathbf{curl}\,\boldsymbol{\rho}\wedge\mathbf{n}_{\partial E} = \mathbf{v}_h\wedge\mathbf{n}_{\partial E}, & \boldsymbol{\rho}\cdot\mathbf{n}_{\partial E}=0 \quad \text{on } \partial E.\end{cases} \tag{108}$$

Lemma 4.3 implies the well posedness of (108). Identity (106) easily follows from (107), (108), and the fact that $E$ is simply connected. We also have

$$(\mathbf{curl}\,\boldsymbol{\rho},\boldsymbol{\nabla}\psi)_E = 0,\qquad \|\mathbf{v}_h\|_E^2 = \|\mathbf{curl}\,\boldsymbol{\rho}\|_E^2 + \|\boldsymbol{\nabla}\psi\|_E^2. \tag{109}$$

Since $\|\boldsymbol{\rho}^F\|_F = \|\boldsymbol{\rho}\wedge\mathbf{n}_F\|_F$ for all $F$ in $\partial E$, cf. (93), we obtain

$$\begin{aligned}\|\mathbf{curl}\,\boldsymbol{\rho}\|_E^2 &\overset{\text{IBP}}{=} \int_E \boldsymbol{\rho}\cdot\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\rho} - \int_{\partial E}(\mathbf{curl}\,\boldsymbol{\rho}\wedge\mathbf{n}_{\partial E})\cdot\boldsymbol{\rho}\\ &\overset{(108),(93)}{=} \int_E \boldsymbol{\rho}\cdot\mathbf{curl}\,\mathbf{v}_h - \sum_{F\subset\partial E}\int_F(\mathbf{v}_h\wedge\mathbf{n}_F)\cdot\boldsymbol{\rho}^F\\ &\le \|\boldsymbol{\rho}\|_E\|\mathbf{curl}\,\mathbf{v}_h\|_E + \|\mathbf{v}_h\wedge\mathbf{n}_{\partial E}\|_{\partial E}\|\boldsymbol{\rho}\wedge\mathbf{n}_{\partial E}\|_{\partial E}\\ &\overset{(15),(108)}{\lesssim} \|\boldsymbol{\rho}\|_E\|\mathbf{curl}\,\mathbf{v}_h\|_E + \left(h_E^{-\frac{1}{2}}\|\boldsymbol{\rho}\|_E + h_E^{\frac{1}{2}}\|\mathbf{curl}\,\boldsymbol{\rho}\|_E\right)\|\mathbf{v}_h\wedge\mathbf{n}_{\partial E}\|_{\partial E}\\ &\overset{(18),(108)}{\lesssim} \left(h_E\|\mathbf{curl}\,\mathbf{v}_h\|_E + h_E^{\frac{1}{2}}\|\mathbf{v}_h^F\|_{\partial E}\right)\|\mathbf{curl}\,\boldsymbol{\rho}\|_E.\end{aligned} \tag{110}$$

In view of (6) and $\mathrm{div}\,\mathbf{v}_h\in\mathbb{P}_{k-1}(E)$, there exists $q_{k-1}\in\mathbb{P}_{k-1}(E)$ such that

$$\mathrm{div}\,(\mathbf{x}_E q_{k-1}) = \mathrm{div}\,\mathbf{v}_h \qquad \text{and} \qquad \|\mathbf{x}_E q_{k-1}\|_E \lesssim h_E\|\mathrm{div}\,\mathbf{v}_h\|_E. \tag{111}$$

We obtain

$$\begin{aligned}\|\boldsymbol{\nabla}\psi\|_E^2 &\overset{\text{IBP},(107)}{=} -\int_E \mathrm{div}\,\mathbf{v}_h\psi \overset{(111)}{=} -\int_E \mathrm{div}\,(\mathbf{x}_E q_{k-1})\psi\\ &\overset{\text{IBP},(107),(106)}{=} \int_E(\mathbf{x}_E q_{k-1})\cdot(\mathbf{v}_h - \mathbf{curl}\,\boldsymbol{\rho}) \le \|\mathbf{x}_E q_{k-1}\|_E\|\mathbf{curl}\,\boldsymbol{\rho}\|_E + \int_E \mathbf{v}_h\cdot\mathbf{x}_E q_{k-1}\\ &\overset{(110),(111)}{\lesssim} h_E\left(h_E\|\mathbf{curl}\,\mathbf{v}_h\|_E + h_E^{\frac{1}{2}}\|\mathbf{v}_h^F\|_{\partial E} + \sup_{p_{k-1}\in\mathbb{P}_{k-1}(E)}\frac{\int_E \mathbf{v}_h\cdot\mathbf{x}_E p_{k-1}}{\|\mathbf{x}_E p_{k-1}\|_E}\right)\|\mathrm{div}\,\mathbf{v}_h\|_E.\end{aligned} \tag{112}$$

Recall that the 2D and 3D spaces here analyzed constitute an exact complex [4], whence $\mathbf{curl}\,\mathbf{v}_h \in \mathbf{V}^f_{k-1}(E)$. Since $\mathbf{v}_h \in \mathbf{V}^e_k(F)$ for each $F$ in $\partial E$, we have

$$\begin{aligned}\|\mathbf{v}_h^F\|_F &\overset{(23)}{\lesssim} h_F\|\mathrm{rot}_F\,\mathbf{v}_h^F\|_F + h_F^{\frac{1}{2}}\|\mathbf{v}_h^F\cdot\mathbf{t}_{\partial F}\|_{\partial F} + \sup_{p_k\in\mathbb{P}_k(F)}\frac{\int_F \mathbf{v}_h^F\cdot\mathbf{x}_F^F p_k}{\|\mathbf{x}_F^F p_k\|_F},\\ \|\mathbf{curl}\,\mathbf{v}_h\|_E &\overset{(77)}{\lesssim} h_E^{\frac{1}{2}}\|\mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{n}_{\partial E}\|_{\partial E} + \sup_{\mathbf{p}_k\in(\mathbb{P}_k(E))^3}\frac{\int_E \mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{x}_E\wedge\mathbf{p}_k}{\|\mathbf{x}_E\wedge\mathbf{p}_k\|_E}.\end{aligned}\tag{113}$$

By the fact that $\mathrm{div}\,\mathbf{v}_h \in \mathbb{P}_{k-1}(E)$ and employing arguments similar to those used in proving (83), we have the following inverse estimate involving edge virtual element functions in 3D:

$$\|\mathrm{div}\,\mathbf{v}_h\|_E \lesssim h_E^{-1}\|\mathbf{v}_h\|_E \quad \forall \mathbf{v}_h \in \mathbf{V}^e_k(E).$$

We plug this and (113) in (112), and deduce

$$\begin{aligned}\|\boldsymbol{\nabla}\psi\|_E^2 \lesssim \Bigg[ &\sup_{p_{k-1}\in\mathbb{P}_{k-1}(E)}\frac{\int_E \mathbf{v}_h\cdot\mathbf{x}_E p_{k-1}}{\|\mathbf{x}_E p_{k-1}\|_E}\\ &+ h_E^{\frac{1}{2}}\sum_{F\subseteq\partial E}\left(h_F\|\mathrm{rot}_F\,\mathbf{v}_h^F\|_F + h_F^{\frac{1}{2}}\|\mathbf{v}_h^F\cdot\mathbf{t}_{\partial F}\|_{\partial F} + \sup_{p_k\in\mathbb{P}_k(F)}\frac{\int_F \mathbf{v}_h^F\cdot\mathbf{x}_F^F p_k}{\|\mathbf{x}_F^F p_k\|_F}\right)\\ &+ h_E\left(h_E^{\frac{1}{2}}\|\mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{n}_{\partial E}\|_{\partial E} + \sup_{\mathbf{p}_k\in(\mathbb{P}_k(E))^3}\frac{\int_E \mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{x}_E\wedge\mathbf{p}_k}{\|\mathbf{x}_E\wedge\mathbf{p}_k\|_E}\right)\Bigg]\|\mathbf{v}_h\|_E.\end{aligned}\tag{114}$$

Inserting (110) and (114) into (109), using $h_F \approx h_E$, and noting that $\mathrm{rot}_F\,\mathbf{v}_h^F = (\mathbf{curl}\,\mathbf{v}_h)|_F\cdot\mathbf{n}_F$ for all $F$ in $\partial E$, yield

$$\begin{aligned}\|\mathbf{v}_h\|_E^2 \lesssim \Bigg[ &h_E\|\mathbf{curl}\,\mathbf{v}_h\|_E + h_E^{\frac{1}{2}}\|\mathbf{v}_h^F\|_{\partial E} + \sup_{p_{k-1}\in\mathbb{P}_{k-1}(E)}\frac{\int_E \mathbf{v}_h\cdot\mathbf{x}_E p_{k-1}}{\|\mathbf{x}_E p_{k-1}\|_E}\\ &+ h_E^{\frac{1}{2}}\sum_{F\subseteq\partial E}\left(h_F\|\mathrm{rot}_F\,\mathbf{v}_h^F\|_F + h_F^{\frac{1}{2}}\|\mathbf{v}_h^F\cdot\mathbf{t}_{\partial F}\|_{\partial F} + \sup_{p_k\in\mathbb{P}_k(F)}\frac{\int_F \mathbf{v}_h^F\cdot\mathbf{x}_F^F p_k}{\|\mathbf{x}_F^F p_k\|_F}\right)\\ &+ h_E\left(h_E^{\frac{1}{2}}\|\mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{n}_{\partial E}\|_{\partial E} + \sup_{\mathbf{p}_k\in(\mathbb{P}_k(E))^3}\frac{\int_E \mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{x}_E\wedge\mathbf{p}_k}{\|\mathbf{x}_E\wedge\mathbf{p}_k\|_E}\right)\Bigg]\|\mathbf{v}_h\|_E\\ \lesssim \Bigg[ &\sum_{F\subseteq\partial E}\left(h_F^{\frac{3}{2}}\|\mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{n}_F\|_F + h_F\|\mathbf{v}_h^F\cdot\mathbf{t}_{\partial F}\|_{\partial F} + \sup_{p_k\in\mathbb{P}_k(F)}\frac{h_F^{\frac{1}{2}}\int_F \mathbf{v}_h^F\cdot\mathbf{x}_F^F p_k}{\|\mathbf{x}_F^F p_k\|_F}\right)\\ &+ \sup_{\mathbf{p}_k\in(\mathbb{P}_k(E))^3}\frac{h_E\int_E \mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{x}_E\wedge\mathbf{p}_k}{\|\mathbf{x}_E\wedge\mathbf{p}_k\|_E} + \sup_{p_{k-1}\in\mathbb{P}_{k-1}(E)}\frac{\int_E \mathbf{v}_h\cdot\mathbf{x}_E p_{k-1}}{\|\mathbf{x}_E p_{k-1}\|_E}\Bigg]\|\mathbf{v}_h\|_E.\end{aligned}$$

□

For each sufficiently regular $\mathbf{v}$, we define the DoFs interpolation operator $\widetilde{\mathbf{I}}_h^e$ on $\mathbf{V}^e_k(E)$ by

$$\int_e(\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\cdot\mathbf{t}_e p_k = 0 \qquad \forall p_k\in\mathbb{P}_k(e),\ \forall e\subseteq\partial F;\tag{115a}$$

$$\int_F(\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})^F\cdot\mathbf{x}_F^F p_k = 0 \qquad \forall p_k\in\mathbb{P}_k(F);\tag{115b}$$

$$\int_F \mathrm{rot}_F\,(\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})^F p_{k-1}^0 = 0 \qquad \forall p_{k-1}^0\in\mathbb{P}_{k-1}^0(F)\ \text{ only for } k>1;\tag{115c}$$

$$\int_E \mathbf{curl}\,(\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\cdot\mathbf{x}_E\wedge\mathbf{p}_k = 0 \qquad \forall\mathbf{p}_k\in(\mathbb{P}_k(E))^3;\tag{115d}$$

$$\int_E(\mathbf{v}-\widetilde{\mathbf{I}}_h^e\mathbf{v})\cdot\mathbf{x}_E p_{k-1} = 0 \qquad \forall p_{k-1}\in\mathbb{P}_{k-1}(E).\tag{115e}$$

Next, we prove interpolation estimates for the operator $\widetilde{\mathbf{I}}_h^e$. The following result includes different requirements on the regularity of the objective function; see also Remark 4. Below, given any non-negative real number $s$, the symbol $[s]$ will denote the highest integer strictly smaller than $s$ ($[\cdot]$ differs from the floor$(\cdot)$ function; for instance, $[1] = 0$ while floor$(1) = 1$).

**Theorem 4.5.** *For each $\boldsymbol{v} \in \boldsymbol{H}^s(E)$, $1/2 < s \le k+1$, with $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^r(E)$, $1/2 < r \le k$, for $\widehat{r} = \min\{r, [s]\}$, we have*

$$\|\boldsymbol{v} - \widetilde{\boldsymbol{I}}_h^e \boldsymbol{v}\|_E \lesssim h_E^s |\boldsymbol{v}|_{s,E} + h_E^{\widehat{r}+1} |\mathbf{curl}\,\boldsymbol{v}|_{\widehat{r},E} + h_E \|\mathbf{curl}\,\boldsymbol{v}\|_E, \tag{116}$$

$$\|\mathbf{curl}\,(\boldsymbol{v} - \widetilde{\boldsymbol{I}}_h^e \boldsymbol{v})\|_E \lesssim h_E^r |\mathbf{curl}\,\boldsymbol{v}|_{r,E}. \tag{117}$$

*The third term on the right-hand side of* (116) *can be neglected if $s \ge 1$.*

*Proof.* Following Proposition 4.2 in Ref. [4], we have

$$\mathbf{curl}\,(\widetilde{\mathbf{I}}_h^e \mathbf{v}) = \widetilde{\mathbf{I}}_h^f (\mathbf{curl}\,\mathbf{v}). \tag{118}$$

Recalling (87), bound (117) immediately follows.

Next, we prove bound (116). We define the natural number $\bar{k} = [s] \le k$ and consider $\mathbf{\Pi}_{\bar{k}}^E$ the (vector valued version of the) projection operator from $H^s(E)$ in $\mathbb{P}_{\bar{k}}(E)$ defined in Ref. [32]. Such an operator guarantees the following approximation properties

$$\|\mathbf{v} - \mathbf{\Pi}_{\bar{k}}^E \mathbf{v}\|_E \lesssim h_E^s |\mathbf{v}|_{s,E}\,, \quad \|\mathbf{curl}\,\mathbf{v} - \mathbf{curl}\,\mathbf{\Pi}_{\bar{k}}^E \mathbf{v}\|_E \lesssim h_E^{\widehat{r}} |\mathbf{curl}\,\mathbf{v}|_{\widehat{r},E}. \tag{119}$$

To show the second bound in (119), it suffices to recall the properties of $\mathbf{\Pi}_{\bar{k}}^E \mathbf{v}$. In particular, see [32, eqs. (2.1) and (2.2)], all its partial derivatives (up to order $\bar{k}$) have the same average as those of $\mathbf{v}$. This implies that also the derivatives (up to one order less) of the **curl** of the two functions have the same average. The estimate follows from iterative applications of the Poincaré inequality.

Since $\mathbf{\Pi}_{\bar{k}}^E \mathbf{v} \in (\mathbb{P}_k(E))^3 \subseteq \mathbf{V}_k^e(E)$, we obtain

$$\begin{aligned}
\|\mathbf{\Pi}_{\bar{k}}^E \mathbf{v} - \widetilde{\mathbf{I}}_h^e \mathbf{v}\|_E &\overset{(105)}{\lesssim} \sum_{F \subseteq \partial E} h_F^{\frac{3}{2}} \|\mathbf{curl}\,(\mathbf{\Pi}_{\bar{k}}^E \mathbf{v} - \widetilde{\mathbf{I}}_h^e \mathbf{v}) \cdot \mathbf{n}_F\|_F \\
&+ \sum_{F \subseteq \partial E} h_F \Big( \|(\mathbf{\Pi}_{\bar{k}}^E \mathbf{v} - \widetilde{\mathbf{I}}_h^e \mathbf{v})^F \cdot \mathbf{t}_{\partial F}\|_{\partial F} + \sup_{p_k \in \mathbb{P}_k(F)} \frac{h_F^{\frac{1}{2}} \int_F (\mathbf{\Pi}_{\bar{k}}^E \mathbf{v} - \widetilde{\mathbf{I}}_h^e \mathbf{v})^F \cdot \mathbf{x}_F^F p_k}{\|\mathbf{x}_F^F p_k\|_F} \Big) \\
&+ \sup_{\mathbf{p}_k \in (\mathbb{P}_k(E))^3} \frac{h_E \int_E \mathbf{curl}\,(\mathbf{\Pi}_{\bar{k}}^E \mathbf{v} - \widetilde{\mathbf{I}}_h^e \mathbf{v}) \cdot \mathbf{x}_E \wedge \mathbf{p}_k}{\|\mathbf{x}_E \wedge \mathbf{p}_k\|_E} + \sup_{p_{k-1} \in \mathbb{P}_{k-1}(E)} \frac{\int_E (\mathbf{\Pi}_{\bar{k}}^E \mathbf{v} - \widetilde{\mathbf{I}}_h^e \mathbf{v}) \cdot \mathbf{x}_E p_{k-1}}{\|\mathbf{x}_E p_{k-1}\|_E} := \sum_{i=1}^5 T_i.
\end{aligned} \tag{120}$$

We estimate the five terms on the right-hand side of (120) separately. First, we observe

$$\mathbf{curl}\,(\widetilde{\mathbf{I}}_h^e \mathbf{v})|_F \cdot \mathbf{n}_F \overset{(118),(85a)}{=} \Pi_{k-1}^{0,F}((\mathbf{curl}\,\mathbf{v})|_F \cdot \mathbf{n}_F) \quad \forall F \subseteq \partial E. \tag{121}$$

As for the term $T_1$, the triangle inequality and the trivial continuity of the projector $\Pi_{k-1}^{0,F}$ in the $L^2$ norm implies

$$\begin{aligned}
T_1 &\le \sum_{F \subseteq \partial E} h_F^{\frac{3}{2}} \Big( \|\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_{\bar{k}}^E \mathbf{v}) \cdot \mathbf{n}_F\|_F + \|\mathbf{curl}\,\mathbf{v} \cdot \mathbf{n}_F - \Pi_{k-1}^{0,F}(\mathbf{curl}\,\mathbf{v} \cdot \mathbf{n}_F)\|_F \Big) \\
&\overset{(121)}{\lesssim} \sum_{F \subseteq \partial E} h_F^{\frac{3}{2}} \|\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_{\bar{k}}^E \mathbf{v}) \cdot \mathbf{n}_F\|_F.
\end{aligned} \tag{122}$$

We estimate the terms $T_3$, $T_4$, and $T_5$ as follows:

$$\begin{aligned}
\sum_{i=3}^5 T_i &\overset{(115b),(115d),(115e)}{=} \sum_{F \subseteq \partial E} \sup_{p_k \in \mathbb{P}_k(F)} \frac{h_F^{\frac{1}{2}} \int_F (\mathbf{v} - \mathbf{\Pi}_{\bar{k}}^E \mathbf{v})^F \cdot \mathbf{x}_F^F p_k}{\|\mathbf{x}_F^F p_k\|_F} \\
&+ \sup_{p_{k-1} \in \mathbb{P}_{k-1}(E)} \frac{\int_E (\mathbf{v} - \mathbf{\Pi}_{\bar{k}}^E \mathbf{v}) \cdot \mathbf{x}_E p_{k-1}}{\|\mathbf{x}_E p_{k-1}\|_E} + \sup_{\mathbf{p}_k \in (\mathbb{P}_k(E))^3} \frac{h_E \int_E \mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_{\bar{k}}^E \mathbf{v}) \cdot \mathbf{x}_E \wedge \mathbf{p}_k}{\|\mathbf{x}_E \wedge \mathbf{p}_k\|_E} \\
&\lesssim \sum_{F \subseteq \partial E} h_F^{\frac{1}{2}} \|(\mathbf{v} - \mathbf{\Pi}_{\bar{k}}^E \mathbf{v})^F\|_F + \|\mathbf{v} - \mathbf{\Pi}_{\bar{k}}^E \mathbf{v}\|_E + h_E \|\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_{\bar{k}}^E \mathbf{v})\|_E.
\end{aligned} \tag{123}$$

Inserting (122) and (123) into (120) yields

$$\|\mathbf{\Pi}_k^E\mathbf{v} - \widetilde{\mathbf{I}}_h^e\mathbf{v}\|_E \lesssim \sum_{F\subseteq\partial E} \Big( h_F^{\frac{3}{2}} \|\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}) \cdot \mathbf{n}_F\|_F + h_F^{\frac{1}{2}} \|(\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v})^F\|_F \Big) \\ + \|\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}\|_E + h_E\|\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v})\|_E + T_2. \tag{124}$$

We are left to estimate the term $T_2$. If $s > 1$, then by (115a), (8) with $1/2 < \delta < \min\{1, s - 1/2\}$, and (9) with $\varepsilon = \delta$, we have

$$\begin{aligned} T_2 &= h_F\|(\mathbf{\Pi}_k^E\mathbf{v} - \widetilde{\mathbf{I}}_h^e\mathbf{v})^F \cdot \mathbf{t}_{\partial F}\|_{\partial F} \lesssim \sum_{F\subseteq\partial E} h_F^{\frac{1}{2}} \|(\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v})^F\|_F + h_F^{\delta+\frac{1}{2}} |(\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v})^F|_{\delta,F}) \\ &\lesssim \sum_{F\subseteq\partial E} h_F^{\frac{1}{2}} (h_E^{-\frac{1}{2}} \|\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}\|_E + h_E^{\delta-\frac{1}{2}} |\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}|_{\delta,E}) \\ &\quad + h_F^{\delta+\frac{1}{2}} (h_E^{-(\delta+\frac{1}{2})} \|\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}\|_E + |\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}|_{\delta+\frac{1}{2},E}) \\ &= \|\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}\|_E + h_E^{\delta} |\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}|_{\delta,E} + h_E^{\delta+\frac{1}{2}} |\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}|_{\delta+\frac{1}{2},E}. \end{aligned} \tag{125}$$

Substituting (125) into (124), and using (8), (119), and standard polynomial approximation properties, we obtain

$$\begin{aligned} \|\mathbf{\Pi}_k^E\mathbf{v} - \widetilde{\mathbf{I}}_h^e\mathbf{v}\|_E &\lesssim \sum_{F\subseteq\partial E} \big( h_F^{\frac{3}{2}} \|\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}) \cdot \mathbf{n}_F\|_F + h_F^{\frac{1}{2}} \|(\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v})^F\|_F \big) \\ &+ h_E\|\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v})\|_E + \|\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}\|_E + h_E^{\delta} |\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}|_{\delta,E} + h_E^{\delta+\frac{1}{2}} |\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}|_{\delta+\frac{1}{2},E} \\ &\lesssim h_E\|\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v})\|_E + h_E^{\widehat{r}+1} |\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v})|_{\widehat{r},E} + \|\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}\|_E \\ &+ h_E^s |\mathbf{v} - \mathbf{\Pi}_k^E\mathbf{v}|_{s,E} \lesssim h_E^{\widehat{r}+1} |\mathbf{curl}\,\mathbf{v}|_{\widehat{r},E} + h_E^s |\mathbf{v}|_{s,E}. \end{aligned} \tag{126}$$

Instead, if $1/2 < s \leq 1$, we replace the term $\mathbf{\Pi}_k^{0,E}$ by $\mathbf{\Pi}_0^{0,E}$ in (120) and (124). For the term $T_2$, by the fact that $(\mathbf{\Pi}_0^{0,E}\mathbf{v} - \widetilde{\mathbf{I}}_h^e\mathbf{v})^F \cdot \mathbf{t}_e \in \mathbb{P}_0(e)\ \forall e \subseteq \partial F$, (115a), (32) and the property that $\|p_k\|_{L^\infty(e)} \leq C h_e^{-\frac{1}{2}} \|p_k\|_e$, we arrive at

$$\begin{aligned} T_2 &\lesssim \sum_{F\subseteq\partial E} h_F \sum_{e\subseteq\partial F} \sup_{p_0\in\mathbb{P}_0(e)} \frac{((\mathbf{\Pi}_0^{0,E}\mathbf{v} - \widetilde{\mathbf{I}}_h^e\mathbf{v})^F \cdot \mathbf{t}_e, p_0)_e}{\|p_0\|_e} \\ &= \sum_{F\subseteq\partial E} h_F \sum_{e\subseteq\partial F} \sup_{p_0\in\mathbb{P}_0(e)} \frac{((\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v})^F \cdot \mathbf{t}_e, p_0)_e}{\|p_0\|_e} \\ &\lesssim \sum_{F\subseteq\partial E} h_F^{\frac{1}{2}} \Big( \|(\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v})^F\|_F + h_F^{\varepsilon} |(\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v})^F|_{\varepsilon,F} + h_F \|\mathrm{rot}_F\,(\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v})^F\|_F \Big). \end{aligned}$$

Combining this and (124), we choose $\varepsilon = s - \frac{1}{2}$ and apply (8) with $\delta = r$, (10) with $\delta = s$, (9), and standard polynomial approximation properties, yielding

$$\begin{aligned} &\|\mathbf{\Pi}_0^{0,E}\mathbf{v} - \widetilde{\mathbf{I}}_h^e\mathbf{v}\|_E \\ &\lesssim \sum_{F\subseteq\partial E} \Big( h_F^{\frac{3}{2}} \|\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v}) \cdot \mathbf{n}_F\|_F + h_F^{\frac{1}{2}} \|(\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v})^F\|_F \\ &\quad + h_F^{\varepsilon+\frac{1}{2}} |(\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v})^F|_{\varepsilon,F} \Big) + \|\mathbf{v} - \mathbf{\Pi}_0^E\mathbf{v}\|_E + h_E\|\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v})\|_E \\ &\lesssim h_E\|\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v})\|_E + h_E^{r+1} |\mathbf{curl}\,(\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v})|_{r,E} + \|\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v}\|_E \\ &\quad + h_E^s |\mathbf{v} - \mathbf{\Pi}_0^{0,E}\mathbf{v}|_{s,E} \lesssim h^s |\mathbf{v}|_{s,E} + h_E^{r+1} |\mathbf{curl}\,\mathbf{v}|_{r,E} + h_E \|\mathbf{curl}\,\mathbf{v}\|_E. \end{aligned} \tag{127}$$

The assertion follows from a triangle inequality and standard polynomial approximation properties. $\square$

*Remark* 4. Theorem 4.5 represents an optimal approximation result in the $H_{\mathrm{curl}}$ norm. For low values of $s$ it requires some additional regularity on $\mathbf{curl}\,\mathbf{v}$ due to the definition of the interpolation operator. This requirement could be slightly relaxed employing arguments similar to those used in the proof of Lemma 3.2, requiring that $\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}_F$ is integrable on the element faces. This comes at the price of more cumbersome technicalities, which we prefer to avoid. Besides, for $s > 3/2$, we can choose $r = s-1$ in (116) and eliminate also the second term on the right-hand side. This same remark applies also to Theorem 4.6 below.

### 4.3 Serendipity edge virtual element space on polyhedrons

We first change the boundary space $\mathcal{B}_k(\partial E)$ in (92) into its serendipity version:

$$\mathcal{B}_k^S(\partial E) = \left\{\mathbf{v}_h \in \mathbf{L}^2(\partial E) : \mathbf{v}_h^F \in \mathbf{SV}_k^e(F)\ \forall F \subseteq \partial E, \mathbf{v}_h\cdot\mathbf{t}_e \text{ is continuous } \forall e \subseteq \partial F\right\}.$$

The serendipity edge virtual element space in 3D is defined as [4, 9]

$$\begin{aligned}\mathbf{SV}_k^e(E) = \big\{\mathbf{v}_h \in \mathbf{L}^2(E) : \operatorname{div}\mathbf{v}_h \in \mathbb{P}_{k-1}(E), \mathbf{curl}\,\mathbf{curl}\,\mathbf{v}_h \in (\mathbb{P}_k(E))^3,\\ \mathbf{v}_h^F \in \mathbf{SV}_k^e(F)\ \forall F \subseteq \partial E, \mathbf{v}_h\cdot\mathbf{t}_e \text{ is continuous } \forall e \subseteq \partial F\big\}.\end{aligned}$$

We endow $\mathbf{SV}_k^e(E)$ with the unisolvent DoFs (94), (96), (97), (98), and

- $\displaystyle\int_F \mathbf{v}_h^F\cdot\mathbf{x}_F^F p_{\beta_F}\forall p_{\beta_F} \in \mathbb{P}_{\beta_F}(F)$ only for $\beta_F \geq 0$.

For each sufficiently regular $\mathbf{v}$, the DoFs interpolation operator $\mathbf{I}_h^e$ on the space $\mathbf{SV}_k^e(E)$ can be defined through the above DoFs enforcing the same conditions (115a), (115c), (115d), and (115e), and substituting (115b) by

$$\int_F (\mathbf{v} - \mathbf{I}_h^e\mathbf{v})^F\cdot\mathbf{x}_F^F p_{\beta_F} = 0 \qquad\qquad \forall p_{\beta_F} \in \mathbb{P}_{\beta_F}(F) \text{ only for } \beta_F \geq 0.$$

From Proposition 4.2 in Ref. [4], we have

$$\mathbf{curl}\,(\mathbf{I}_h^e\mathbf{v}) = \widetilde{\mathbf{I}}_h^f(\mathbf{curl}\,\mathbf{v}). \tag{128}$$

Next, we prove interpolation estimates for the operator $\mathbf{I}_h^e$ on the serendipity edge virtual element space $\mathbf{SV}_k^e(E)$.

**Theorem 4.6.** *For each $\boldsymbol{v} \in \boldsymbol{H}^s(E)$, $1/2 < s \leq k+1$, with $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^r(E)$, $1/2 < r \leq k$, we have*

$$\|\boldsymbol{v} - \boldsymbol{I}_h^e\boldsymbol{v}\|_E \lesssim h_E^s|\boldsymbol{v}|_{s,E} + h_E^{\widehat{r}+1}|\mathbf{curl}\,\boldsymbol{v}|_{\widehat{r},E} + h_E\|\mathbf{curl}\,\boldsymbol{v}\|_E, \tag{129}$$

$$\|\mathbf{curl}\,(\boldsymbol{v} - \boldsymbol{I}_h^e\boldsymbol{v})\|_E \lesssim h_E^r|\mathbf{curl}\,\boldsymbol{v}|_{r,E}. \tag{130}$$

*where $\widehat{r} = \min\{r, [s]\}$. The third term on the right-hand side of* (116) *can be neglected if $s \geq 1$.*

*Proof.* The proof of bound (130) is essentially identical to that of (117); see (128).

Next, we prove bound (129). By the inclusion that $\mathbf{SV}_k^e(E) \subseteq \mathbf{V}_k^e(E)$, bound (105) holds true for functions in $\mathbf{SV}_k^e(E)$. Thus, for all $\mathbf{v}_h$ in $\mathbf{SV}_k^e(E)$, also making use of (42d) and (43), we can write

$$\begin{aligned}\|\mathbf{v}_h\|_E \overset{(105)}{\lesssim} \sum_{F\subseteq\partial E}\bigg(h_F^{\frac{3}{2}}\|\mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{n}_F\|_F + h_F\|\mathbf{v}_h^F\cdot\mathbf{t}_{\partial F}\|_{\partial F} + \sup_{p_k\in\mathbb{P}_k(F)}\frac{h_F^{\frac{1}{2}}\int_F \boldsymbol{\Pi}_S^e\mathbf{v}_h^F\cdot\mathbf{x}_F^F p_k}{\|\mathbf{x}_F^F p_k\|_F}\bigg)\\ + \sup_{\mathbf{p}_k\in(\mathbb{P}_k(E))^3}\frac{h_E\int_E \mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{x}_E\wedge\mathbf{p}_k}{\|\mathbf{x}_E\wedge\mathbf{p}_k\|_E} + \sup_{p_{k-1}\in\mathbb{P}_{k-1}(E)}\frac{\int_E \mathbf{v}_h\cdot\mathbf{x}_E p_{k-1}}{\|\mathbf{x}_E p_{k-1}\|_E}.\end{aligned}$$

Let $\mathbf{\Pi}_k^E$ be the operator introduced in the proof of Theorem 4.5. By replacing $\mathbf{v}_h$ with $\mathbf{\Pi}_k^E\mathbf{v}-\mathbf{I}_h^e\mathbf{v}$, we can write

$$\|\mathbf{\Pi}_k^E\mathbf{v}-\mathbf{I}_h^e\mathbf{v}\|_E \lesssim \sum_{F\subseteq\partial E} h_F^{\frac{3}{2}}\|\mathbf{curl}\,(\mathbf{\Pi}_k^E\mathbf{v}-\mathbf{I}_h^e\mathbf{v})\cdot\mathbf{n}_F\|_F \tag{131}$$
$$+\sum_{F\subseteq\partial E} h_F\Big(\|(\mathbf{\Pi}_k^E\mathbf{v}-\mathbf{I}_h^e\mathbf{v})^F\cdot\mathbf{t}_{\partial F}\|_{\partial F}+\sup_{p_k\in\mathbb{P}_k(F)}\frac{h_F^{\frac{1}{2}}\int_F \mathbf{\Pi}_S^e(\mathbf{\Pi}_k^E\mathbf{v}-\mathbf{I}_h^e\mathbf{v})^F\cdot\mathbf{x}_F^F p_k}{\|\mathbf{x}_F^F p_k\|_F}\Big)$$
$$+\sup_{\mathbf{p}_k\in(\mathbb{P}_k(E))^3}\frac{h_E\int_E\mathbf{curl}\,(\mathbf{\Pi}_k^E\mathbf{v}-\mathbf{I}_h^e\mathbf{v})\cdot\mathbf{x}_E\wedge\mathbf{p}_k}{\|\mathbf{x}_E\wedge\mathbf{p}_k\|_E}+\sup_{p_{k-1}\in\mathbb{P}_{k-1}(E)}\frac{\int_E(\mathbf{\Pi}_k^E\mathbf{v}-\mathbf{I}_h^e\mathbf{v})\cdot\mathbf{x}_E p_{k-1}}{\|\mathbf{x}_E p_{k-1}\|_E}.$$

The difference between (120) and (131) resides only in the third term on the right-hand side, whence we only discuss its upper bound. The other four terms are dealt with exactly as in the proof of Theorem 4.5.

Due to the definition of the interpolation operator $\mathbf{I}_h^e$, the functions $(\mathbf{I}_h^e\mathbf{v})^F$ and $\mathbf{v}^F$ share the same DoFs on each face $F$ of $E$. Since the value of the projection $\mathbf{\Pi}_S^e$ only depends on such DoFs, we have $\mathbf{\Pi}_S^e(\mathbf{I}_h^e\mathbf{v})^F=\mathbf{\Pi}_S^e\mathbf{v}^F$. This allows us to write

$$\|\mathbf{\Pi}_S^e(\mathbf{\Pi}_k^E\mathbf{v}-\mathbf{I}_h^e\mathbf{v})^F\|_F=\|\mathbf{\Pi}_S^e(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})^F\|_F \overset{(51)}{\lesssim} |||\mathbf{\Pi}_S^e(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})^F|||_F$$
$$\overset{(42)}{=}|||(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})^F|||_F\overset{(53)}{\lesssim}\|(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})^F\|_F+h_F^\varepsilon|(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})^F|_{\varepsilon,F}+h_F\|\mathrm{rot}_F(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})^F\|_F.$$

This yields

$$\sum_{F\subseteq\partial E}\sup_{p_k\in\mathbb{P}_k(F)}\frac{h_F^{\frac{1}{2}}\int_F\mathbf{\Pi}_S^e(\mathbf{\Pi}_k^E\mathbf{v}-\mathbf{I}_h^e\mathbf{v})^F\cdot\mathbf{x}_F^F p_k}{\|\mathbf{x}_F^F p_k\|_F}\lesssim\sum_{F\subseteq\partial E}h_F^{\frac{1}{2}}\|\mathbf{\Pi}_S^e(\mathbf{\Pi}_k^E\mathbf{v}-\mathbf{I}_h^e\mathbf{v})^F\|_F \tag{132}$$
$$\lesssim\sum_{F\subseteq\partial E}h_F^{\frac{1}{2}}\big(\|(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})^F\|_F+h_F^\varepsilon|(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})^F|_{\varepsilon,F}+h_F\|\mathrm{rot}_F(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})^F\|_F\big).$$

Inserting (122), (123), and (132) into (131), we derive

$$\|\mathbf{\Pi}_k^E\mathbf{v}-\mathbf{I}_h^e\mathbf{v}\|_E\lesssim\sum_{F\subseteq\partial E}\Big(h_F^{\frac{3}{2}}\|\mathbf{curl}\,(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})\cdot\mathbf{n}_F\|_F+h_F^{\frac{1}{2}}\|(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})^F\|_F$$
$$+h_F^{\varepsilon+\frac{1}{2}}|(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})^F|_{\varepsilon,F}\Big)+\|\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v}\|_E+h_E\|\mathbf{curl}\,(\mathbf{v}-\mathbf{\Pi}_k^E\mathbf{v})\|_E+T_2.$$

Bound (129) now follows from the same arguments as in (126)-(127). □

*Remark* 5. Differently from the 2D case, we proved interpolation estimates in 3D for face and edge elements for functions in $H^s$ with $s>1/2$. One might possibly try to design quasi-interpolation estimates for functions with minimal regularity, i.e., in $H^s$, $s>0$, and some extra regularity condition on the divergence/curl, for instance by taking the steps from the recent work [25] on finite elements. Such additional developments are beyond the scope of this work.

# 5 Stability theory of the discrete bilinear forms

In this section, we focus on the stability properties of $L^2$ discrete VEM bilinear forms proposed for the discretization of electromagnetic problems in 2D and 3D [3, 4]. In Section 5.1, we define computable stabilizations for the VEM discretization of $L^2$ bilinear forms associated with face and edge virtual element spaces in 2D, and prove their stability properties; in Section 5.2, we consider the corresponding results in 3D. Note that here we focus the attention on stability forms that have a "functional" expression with explicit integrals and projections (i.e. do not depend on the particular basis chosen for the VE space). With some additional work, the present results could be also easily extended to dofi-dofi type stabilizations, which are instead related to the basis adopted for the test polynomial spaces in the DoFs definition.

## 5.1 The stability in 2D edge and face virtual element spaces

For each face $F$, we introduce the discrete $L^2$ bilinear form $(\cdot,\cdot)_F : \mathbf{V}_k^e(F) \times \mathbf{V}_k^e(F) \to \mathbb{R}$ as [3,4]

$$[\mathbf{v}_h, \mathbf{w}_h]_{e,F} := (\mathbf{\Pi}_k^{0,F}\mathbf{v}_h, \mathbf{\Pi}_k^{0,F}\mathbf{w}_h)_F + S_e^F((\mathbf{I}-\mathbf{\Pi}_k^{0,F})\mathbf{v}_h, (\mathbf{I}-\mathbf{\Pi}_k^{0,F})\mathbf{w}_h). \tag{133}$$

In (133), $S_e^F(\cdot,\cdot)$ denotes any symmetric positive definite bilinear form computable via the DoFs of $\mathbf{V}_k^e(F)$ such that there exist two positive constant $C_1$ and $C_2$ independent of the mesh size for which

$$C_1\|\mathbf{v}_h\|_F^2 \le S_e^F(\mathbf{v}_h,\mathbf{v}_h) \le C_2\|\mathbf{v}_h\|_F^2 \quad \forall \mathbf{v}_h \in \mathbf{V}_k^e(F). \tag{134}$$

There are many stabilization choices in the literature. We here analyze the following (computable) stabilization $S_e^F : \mathbf{V}_k^e(F) \times \mathbf{V}_k^e(F) \to \mathbb{R}$ given by

$$S_e^F(\mathbf{v}_h,\mathbf{w}_h) = h_F \sum_{e\subseteq\partial F} (\mathbf{v}_h\cdot\mathbf{t}_e, \mathbf{w}_h\cdot\mathbf{t}_e)_e + h_F^2(\mathrm{rot}_F\,\mathbf{v}_h, \mathrm{rot}_F\,\mathbf{w}_h)_F + (\mathbf{\Pi}_{k+1}^{0,F}\mathbf{v}_h, \mathbf{\Pi}_{k+1}^{0,F}\mathbf{w}_h)_F.$$

**Theorem 5.1.** *The stabilization $S_e^F(\cdot,\cdot)$ satisfies the stability bounds in* (134).

*Proof.* The lower bound in (134) is proven as follows:

$$\begin{aligned}
\|\mathbf{v}_h\|_F &\overset{(23)}{\lesssim} h_F\|\mathrm{rot}_F\,\mathbf{v}_h\|_F + h_F^{\frac{1}{2}}\|\mathbf{v}_h\cdot\mathbf{t}_{\partial F}\|_{\partial F} + \sup_{p_k\in\mathbb{P}_k(F)} \frac{\int_F \mathbf{v}_h\cdot\mathbf{x}_F p_k}{\|\mathbf{x}_F p_k\|_F} \\
&\lesssim h_F\|\mathrm{rot}_F\,\mathbf{v}_h\|_F + h_F^{\frac{1}{2}}\|\mathbf{v}_h\cdot\mathbf{t}_{\partial F}\|_{\partial F} + \|\mathbf{\Pi}_{k+1}^{0,F}\mathbf{v}_h\|_F,
\end{aligned}$$

Next, we observe that the inverse inequality (58) is valid for functions in $\mathbf{V}_k^e(F)$ as well. We deduce the upper bound in (134):

$$\begin{aligned}
&h_F^{\frac{1}{2}}\|\mathbf{v}_h\cdot\mathbf{t}_{\partial F}\|_{\partial F} + h_F\|\mathrm{rot}_F\,\mathbf{v}_h\|_F + \|\mathbf{\Pi}_{k+1}^{0,F}\mathbf{v}_h\|_F \overset{(2)}{\lesssim} \|\mathbf{v}_h\cdot\mathbf{t}_{\partial F}\|_{-\frac{1}{2},\partial F} \\
&+ h_F\|\mathrm{rot}_F\,\mathbf{v}_h\|_F + \|\mathbf{\Pi}_{k+1}^{0,F}\mathbf{v}_h\|_F \overset{(12)}{\lesssim} \|\mathbf{v}_h\|_F + h_F\|\mathrm{rot}_F\,\mathbf{v}_h\|_F \overset{(58)}{\lesssim} \|\mathbf{v}_h\|_F.
\end{aligned}$$

□

In the serendipity case, we can still define a discrete bilinear form on $\mathbf{SV}_k^e(F) \times \mathbf{SV}_k^e(F)$ as in (133), substituting the stabilization $S_e^F(\cdot,\cdot)$ by the (computable) serendipity stabilization

$$S_e^{s,F}(\mathbf{v}_h,\mathbf{w}_h) = h_F \sum_{e\subseteq\partial F} (\mathbf{v}_h\cdot\mathbf{t}_e, \mathbf{w}_h\cdot\mathbf{t}_e)_e + h_F^2(\mathrm{rot}_F\,\mathbf{v}_h, \mathrm{rot}_F\,\mathbf{w}_h)_F + (\mathbf{\Pi}_S^e\mathbf{v}_h, \mathbf{\Pi}_S^e\mathbf{w}_h)_F.$$

**Theorem 5.2.** *The stabilization $S_e^{s,F}(\cdot,\cdot)$ satisfies the bounds*

$$C_1\|\boldsymbol{v}_h\|_F^2 \le S_e^{s,F}(\boldsymbol{v}_h,\boldsymbol{v}_h) \le C_2\|\boldsymbol{v}_h\|_F^2 \quad \forall \boldsymbol{v}_h \in \boldsymbol{SV}_k^e(F).$$

*Proof.* The proof follows along the same lines of that of Theorem 5.1. The only difference resides in the lower bound, while treating the term involving the supremum. It suffices to observe that, due to (42d) and (43), we have

$$\sup_{p_k\in\mathbb{P}_k(F)} \frac{\int_F \mathbf{v}_h\cdot\mathbf{x}_F p_k}{\|\mathbf{x}_F p_k\|_F} = \sup_{p_k\in\mathbb{P}_k(F)} \frac{\int_F \mathbf{\Pi}_S^e\mathbf{v}_h\cdot\mathbf{x}_F p_k}{\|\mathbf{x}_F p_k\|_F},$$

and then apply the Cauchy-Schwarz inequality. □

*Remark* 6. The stability theory of standard and serendipity face virtual element spaces in 2D follows from the above stability bounds for edge virtual element spaces, changing "$\mathbf{t}_e$" into "$\mathbf{n}_e$" and "$\mathrm{rot}_F$" into "$\mathrm{div}_F$".

## 5.2 The stability in 3D edge and face virtual element spaces

We first prove stability properties for 3D face virtual element space. We introduce the symmetric, positive definite, and computable bilinear form $S_f^E(\cdot,\cdot)$ on $\mathbf{V}_{k-1}^f(E)\times\mathbf{V}_{k-1}^f(E)$ defined by

$$\begin{aligned} S_f^E(\mathbf{v}_h,\mathbf{w}_h) = h_E \sum_{F\subseteq\partial E} (\mathbf{v}_h\cdot\mathbf{n}_F,\mathbf{w}_h\cdot\mathbf{n}_F)_F + h_E^2(\operatorname{div}\mathbf{v}_h,\operatorname{div}\mathbf{w}_h)_E \\ + (\mathbf{\Pi}_{k+1}^{0,E}\mathbf{v}_h,\mathbf{\Pi}_{k+1}^{0,E}\mathbf{w}_h)_E. \end{aligned} \tag{135}$$

We define the local discrete bilinear form on $\mathbf{V}_{k-1}^f(E)\times\mathbf{V}_{k-1}^f(E)$:

$$[\mathbf{v}_h,\mathbf{w}_h]_{f,E} := (\mathbf{\Pi}_{k-1}^{0,E}\mathbf{v}_h,\mathbf{\Pi}_{k-1}^{0,E}\mathbf{w}_h)_E + S_f^E((\mathbf{I}-\mathbf{\Pi}_{k-1}^{0,E})\mathbf{v}_h,(\mathbf{I}-\mathbf{\Pi}_{k-1}^{0,E})\mathbf{w}_h), \tag{136}$$

which is computable and approximates the $L^2$ bilinear form $(\cdot,\cdot)_E$. Recalling Lemma 4.1 and employing the same arguments as those used in the proof of Theorem 5.1, we have the following stability property.

**Theorem 5.3.** *The following stability bounds are valid:*

$$C_1\|\boldsymbol{v}_h\|_E^2 \le S_f^E(\boldsymbol{v}_h,\boldsymbol{v}_h) \le C_2\|\boldsymbol{v}_h\|_E^2 \quad \forall\boldsymbol{v}_h\in\boldsymbol{V}_{k-1}^f(E). \tag{137}$$

Next, we consider the stability analysis for the VEM discrete form associated with the 3D edge virtual element space [4]. The VEM discrete form of the $L^2$ bilinear form $(\cdot,\cdot)_E$ on $\mathbf{V}_k^e(E)\times\mathbf{V}_k^e(E)$ is defined by

$$[\mathbf{v}_h,\mathbf{w}_h]_{e,E} := (\mathbf{\Pi}_k^{0,E}\mathbf{v}_h,\mathbf{\Pi}_k^{0,E}\mathbf{w}_h)_E + S_e^E((\mathbf{I}-\mathbf{\Pi}_k^{0,E})\mathbf{v}_h,(\mathbf{I}-\mathbf{\Pi}_k^{0,E})\mathbf{w}_h), \tag{138}$$

where $S_e^E(\cdot,\cdot)$ is a symmetric, positive definite, and computable bilinear form defined by

$$\begin{aligned} S_e^E(\mathbf{v}_h,\mathbf{w}_h) = \sum_{F\subseteq\partial E}\Big(h_F^2(\mathbf{v}_h\cdot\mathbf{t}_{\partial F},\mathbf{w}_h\cdot\mathbf{t}_{\partial F})_{\partial F} + h_F(\mathbf{\Pi}_{k+1}^{0,F}\mathbf{v}_h^F,\mathbf{\Pi}_{k+1}^{0,F}\mathbf{w}_h^F)_F\Big) \\ + h_E^2 S_f^E(\mathbf{curl}\,\mathbf{v}_h,\mathbf{curl}\,\mathbf{w}_h). \end{aligned}$$

Before proving stability properties for the discrete bilinear form $[\cdot,\cdot]_{e,E}$, we extend the inverse inequalities involving edge and face virtual element functions in Lemma 5.3 of Ref. [11] to the general order case. Such estimates are critical in the following.

**Lemma 5.4.** *The following inverse inequalities hold true:*

$$\|\boldsymbol{v}_h\|_E \lesssim h_E^{-1}\|\boldsymbol{v}_h\|_{-1,E} \qquad \forall\boldsymbol{v}_h\in\boldsymbol{V}_{k-1}^f(E), \tag{139}$$

$$\|\boldsymbol{v}_h^F\|_F \lesssim h_F^{-\frac{1}{2}}\|\boldsymbol{v}_h^F\|_{-\frac{1}{2},F} \qquad \forall\boldsymbol{v}_h\in\boldsymbol{V}_k^e(E),\ \forall F\subseteq\partial E. \tag{140}$$

*Proof.* We first prove (139). Recalling (82), for each $\mathbf{v}_h\in\mathbf{V}_{k-1}^f(E)$, there exists $\mathbf{q}_k\in(\mathbb{P}_k(E))^3$ with $\operatorname{div}\mathbf{q}_k=0$ such that

$$\mathbf{curl}\,(\mathbf{v}_h-\mathbf{x}_E\wedge\mathbf{q}_k)=\mathbf{0},\ \|\mathbf{x}_E\wedge\mathbf{q}_k\|_E\lesssim h_E\|\mathbf{curl}(\mathbf{x}_E\wedge\mathbf{q}_k)\|_E\lesssim h_E\|\mathbf{curl}\,\mathbf{v}_h\|_E. \tag{141}$$

Moreover, the following polynomial inequality holds true:

$$\|\mathbf{x}_E\wedge\mathbf{q}_k\|_{-1,E} = \sup_{0\ne\mathbf{v}\in\mathbf{H}_0^1(E)}\frac{(\mathbf{x}_E\wedge\mathbf{q}_k,\mathbf{v})_E}{|\mathbf{v}|_{1,E}} \overset{(16)}{\lesssim} h_E\|\mathbf{x}_E\wedge\mathbf{q}_k\|_E. \tag{142}$$

**Part 1: proving the auxiliary bound** (147) **below.** From (141), there exists a function $\psi\in H^1(E)\setminus\mathbb{R}$ such that

$$\mathbf{v}_h-\mathbf{x}_E\wedge\mathbf{q}_k = \boldsymbol{\nabla}\psi. \tag{143}$$

Such a function is defined by

$$\Delta\psi = \mathrm{div}\,(\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k) \text{ in } E, \quad \boldsymbol{\nabla}\psi \cdot \mathbf{n}_{\partial E} = (\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k) \cdot \mathbf{n}_{\partial E} \text{ on } \partial E.$$

Observing that $\nabla\psi|_{\partial E} \cdot \mathbf{n}_{\partial E}$ is a piecewise polynomial, we have

$$\begin{aligned}
\|\boldsymbol{\nabla}\psi\|_E^2 &\overset{\text{IBP}}{=} \int_{\partial E} \boldsymbol{\nabla}\psi \cdot \mathbf{n}_{\partial E}\psi - \int_E \psi\Delta\psi \lesssim \|\boldsymbol{\nabla}\psi \cdot \mathbf{n}_{\partial E}\|_{\partial E}\|\psi\|_{\partial E} + \|\Delta\psi\|_E\|\psi\|_E \\
&\overset{(2),(13)}{\lesssim} h_E^{-\frac{1}{2}} \left(\|\boldsymbol{\nabla}\psi\|_E + h_E\|\Delta\psi\|_E\right) \|\psi\|_{\partial E} + \|\Delta\psi\|_E\|\psi\|_E \\
&\overset{(1)}{\lesssim} h_E^{-\frac{1}{2}} \left(\|\boldsymbol{\nabla}\psi\|_E + \|\Delta\psi\|_{-1,E}\right) \|\psi\|_{\partial E} + h_E^{-1}\|\Delta\psi\|_{-1,E}\|\psi\|_E \\
&\lesssim \left(h_E^{-\frac{1}{2}}\|\psi\|_{\partial E} + h_E^{-1}\|\psi\|_E\right)\|\boldsymbol{\nabla}\psi\|_E \overset{(11)}{\lesssim} \left(h_E^{-\frac{1}{2}}\|\psi\|_E^{\frac{1}{2}}\|\boldsymbol{\nabla}\psi\|_E^{\frac{1}{2}} + h_E^{-1}\|\psi\|_E\right)\|\boldsymbol{\nabla}\psi\|_E.
\end{aligned}$$

Also using (16), this implies

$$\|\boldsymbol{\nabla}\psi\|_E \lesssim h_E^{-\frac{1}{2}}\|\psi\|_E^{\frac{1}{2}}\|\boldsymbol{\nabla}\psi\|_E^{\frac{1}{2}} + h_E^{-1}\|\psi\|_E^{\frac{1}{2}}h_E^{\frac{1}{2}}\|\boldsymbol{\nabla}\psi\|_E^{\frac{1}{2}} = h_E^{-\frac{1}{2}}\|\psi\|_E^{\frac{1}{2}}\|\boldsymbol{\nabla}\psi\|_E^{\frac{1}{2}}. \tag{144}$$

Further, by using the continuous inf-sup condition of the Stokes problem, see, e.g., Section 8.2.1 in Ref. [14], we have the following upper bound on $\|\psi\|_E$:

$$\|\psi\|_E \lesssim \sup_{\boldsymbol{\xi}\in\mathbf{H}_0^1(E)} \frac{(\psi, \mathrm{div}\,\boldsymbol{\xi})_E}{|\boldsymbol{\xi}|_{1,E}} = \sup_{\boldsymbol{\xi}\in\mathbf{H}_0^1(E)} \frac{(\boldsymbol{\nabla}\psi, \boldsymbol{\xi})_E}{|\boldsymbol{\xi}|_{1,E}} = \|\boldsymbol{\nabla}\psi\|_{-1,E}. \tag{145}$$

Combining (144) and (145), we arrive at

$$\|\boldsymbol{\nabla}\psi\|_E \lesssim h_E^{-1}\|\boldsymbol{\nabla}\psi\|_{-1,E}. \tag{146}$$

Using (143), (146) yields

$$\|\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k\|_E \lesssim h_E^{-1}\|\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k\|_{-1,E}. \tag{147}$$

**Part 2: proving** (139)**.** We introduce the auxiliary function $\mathbf{z} \in \mathbf{H}_0^1(E)$ that realizes the supremum in the definition of $\|\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k\|_{-1,E}$, i.e., let $\mathbf{z}$ be the function in $\mathbf{H}_0^1(E)$ such that

$$\|\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k\|_{-1,E} \lesssim (\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k, \mathbf{z})_E \quad \text{with} \quad |\mathbf{z}|_{1,E} = 1. \tag{148}$$

As in Remark 1, we split $E$ into shape-regular tetrahedra $\widetilde{\mathcal{T}}_h$. Define $\psi_E$ as the square of the piecewise quartic bubble function over $\widetilde{\mathcal{T}}_h$, scaled such that $\|\psi_E\|_{L^\infty(E)} = 1$. We take $\widetilde{\mathbf{w}}_E = \psi_E\mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k)$ and defined its scaled version $\mathbf{w}_E = \widetilde{\mathbf{w}}_E/|\mathbf{curl}\,\widetilde{\mathbf{w}}_E|_{1,E}$. We have $\mathbf{w}_E \in \mathbf{H}_0^2(E)$ and $|\mathbf{curl}\,\mathbf{w}_E|_{1,E} = 1$. Furthermore, by (143), we get

$$(\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k, \mathbf{curl}\,\mathbf{w}_E)_E = (\boldsymbol{\nabla}\psi, \mathbf{curl}\,\mathbf{w}_E)_E = 0. \tag{149}$$

We write

$$\begin{aligned}
&(\mathbf{x}_E \wedge \mathbf{q}_k, \mathbf{curl}\,\mathbf{w}_E)_E \overset{\text{IBP}}{=} (\mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k), \mathbf{w}_E)_E \\
&= \frac{(\mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k), \psi_E\mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k))_E}{|\mathbf{curl}\,(\psi_E\mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k))|_{1,E}} \overset{(1)}{\geq} \frac{(\mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k), \psi_E\mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k))_E}{h_E^{-1}\|\mathbf{curl}\,(\psi_E\mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k))\|_E} \\
&\overset{(1),(3)}{\geq} \frac{C\|\mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k)\|_E^2}{h_E^{-2}\|\mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k)\|_E} = Ch_E^2\|\mathbf{curl}\,(\mathbf{x}_E \wedge \mathbf{q}_k)\|_E \\
&\overset{(141)}{\geq} Ch_E\|\mathbf{x}_E \wedge \mathbf{q}_k\|_E \overset{(142)}{\geq} C_1\|\mathbf{x}_E \wedge \mathbf{q}_k\|_{-1,E}.
\end{aligned} \tag{150}$$

From the definition of negative norm $\|\cdot\|_{-1,E}$, the fact that $\mathbf{curl}\,\mathbf{w}_E \in \mathbf{H}_0^1(E)$, and (149), we can write

$$\begin{aligned}
\|\mathbf{v}_h\|_{-1,E} &= \sup_{\boldsymbol{\xi}\in\mathbf{H}_0^1(E)} \frac{(\mathbf{v}_h, \boldsymbol{\xi})_E}{|\boldsymbol{\xi}|_{1,E}} = \sup_{\boldsymbol{\xi}\in\mathbf{H}_0^1(E)} \frac{(\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k, \boldsymbol{\xi})_E + (\mathbf{x}_E \wedge \mathbf{q}_k, \boldsymbol{\xi})_E}{|\boldsymbol{\xi}|_{1,E}} \\
&\geq \frac{(\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k, \mathbf{z} + \alpha\mathbf{curl}\,\mathbf{w}_E)_E + (\mathbf{x}_E \wedge \mathbf{q}_k, \mathbf{z} + \alpha\mathbf{curl}\,\mathbf{w}_E)_E}{|\mathbf{z} + \alpha\mathbf{curl}\,\mathbf{w}_E|_{1,E}} \\
&\geq \frac{(\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k, \mathbf{z})_E + (\mathbf{x}_E \wedge \mathbf{q}_k, \mathbf{z})_E + (\mathbf{x}_E \wedge \mathbf{q}_k, \alpha\mathbf{curl}\,\mathbf{w}_E)_E}{1 + \alpha},
\end{aligned} \tag{151}$$

where $\alpha$ is a positive constant, which we shall fix in what follows. Next, we obtain

$$
\begin{aligned}
\|\mathbf{v}_h\|_{-1,E} &\overset{(151),(148),(150)}{\geq} \frac{C\|\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k\|_{-1,E} - \|\mathbf{x}_E \wedge \mathbf{q}_k\|_{-1,E} + C_1\alpha\|\mathbf{x}_E \wedge \mathbf{q}_k\|_{-1,E}}{1+\alpha} \\
&= \frac{C}{1+\alpha}\|\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k\|_{-1,E} + \frac{C_1\alpha - 1}{1+\alpha}\|\mathbf{x}_E \wedge \mathbf{q}_k\|_{-1,E} \\
&\overset{(147),(1)}{\geq} Ch_E(\|\mathbf{v}_h - \mathbf{x}_E \wedge \mathbf{q}_k\|_E + \|\mathbf{x}_E \wedge \mathbf{q}_k\|_E) \geq Ch_E\|\mathbf{v}_h\|_E.
\end{aligned}
$$

where we have fixed $\alpha = 2/C_1$. This completes the proof of (139).

**Part 3: proving** (140)**.** We first recall that $\mathbf{v}_h^F$ belongs to $\mathbf{V}_k^e(F)$ for each $\mathbf{v}_h \in \mathbf{V}_k^e(E)$. Next, we observe that the inverse estimate (139) for functions in $\mathbf{V}_{k-1}^f(E)$, implies an 2D analogous counterpart on the space $\mathbf{V}_k^f(F)$:

$$\|\mathbf{v}_h\|_F \lesssim h_F^{-1}\|\mathbf{v}_h\|_{-1,F} \quad \forall \mathbf{v}_h \in \mathbf{V}_k^f(F).$$

The counterpart for the 2D edge virtual element space $\mathbf{V}_k^e(F)$ is obtained via a "rotation" argument as in Section 3.3:

$$\|\mathbf{v}_h\|_F \lesssim h_F^{-1}\|\mathbf{v}_h\|_{-1,F} \quad \forall \mathbf{v}_h \in \mathbf{V}_k^e(F).$$

Hence, we arrive at

$$\|\mathbf{v}_h^F\|_F \lesssim h_F^{-1}\|\mathbf{v}_h^F\|_{-1,F} \quad \forall \mathbf{v}_h \in \mathbf{V}_k^e(E),\ \forall F \subseteq \partial E.$$

The assertion follows from classical results in space interpolation theory [31]. □

With these tools at hand, we can prove the following stability property.

**Theorem 5.5.** *The following stability bounds are valid:*

$$C_1\|\boldsymbol{v}_h\|_E^2 \leq S_e^E(\boldsymbol{v}_h, \boldsymbol{v}_h) \leq C_2\|\boldsymbol{v}_h\|_E^2 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_k^e(E). \tag{152}$$

*Proof.* The lower bound in (152) is proved as follows:

$$
\begin{aligned}
\|\mathbf{v}_h\|_E &\overset{(105)}{\lesssim} \sum_{F\subseteq\partial E}\left(h_F^{\frac{3}{2}}\|\mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{n}_F\|_F + h_F\|\mathbf{v}_h^F\cdot\mathbf{t}_{\partial F}\|_{\partial F} + \sup_{p_k\in\mathbb{P}_k(F)}\frac{h_F^{\frac{1}{2}}\int_F \mathbf{v}_h^F\cdot\mathbf{x}_F^F p_k}{\|\mathbf{x}_F^F p_k\|_F}\right) \\
&+ \sup_{\mathbf{p}_k\in(\mathbb{P}_k(E))^3}\frac{h_E\int_E \mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{x}_E\wedge\mathbf{p}_k}{\|\mathbf{x}_E\wedge\mathbf{p}_k\|_E} + \sup_{p_{k-1}\in\mathbb{P}_{k-1}(E)}\frac{\int_E \mathbf{v}_h\cdot\mathbf{x}_E p_{k-1}}{\|\mathbf{x}_E p_{k-1}\|_E} \\
&\overset{(135)}{\lesssim} h_E S_f^E(\mathbf{curl}\,\mathbf{v}_h, \mathbf{curl}\,\mathbf{v}_h)^{\frac{1}{2}} + \sum_{F\subseteq\partial E}\left(h_F\|\mathbf{v}_h^F\cdot\mathbf{t}_{\partial F}\|_{\partial F} + h_F^{\frac{1}{2}}\|\boldsymbol{\Pi}_{k+1}^{0,F}\mathbf{v}_h^F\|_F\right),
\end{aligned}
$$

As for the upper bound in (152), we write

$$
\begin{aligned}
&\sum_{F\subseteq\partial E}\left(h_F\|\mathbf{v}_h^F\cdot\mathbf{t}_{\partial F}\|_{\partial F} + h_F^{\frac{1}{2}}\|\boldsymbol{\Pi}_{k+1}^{0,F}\mathbf{v}_h^F\|_F\right) + h_E S_f^E(\mathbf{curl}\,\mathbf{v}_h, \mathbf{curl}\,\mathbf{v}_h)^{\frac{1}{2}} \\
&\overset{(2),(137)}{\lesssim} \sum_{F\subseteq\partial E}\left(h_F^{\frac{1}{2}}\|\mathbf{v}_h^F\cdot\mathbf{t}_{\partial F}\|_{-\frac{1}{2},\partial F} + h_F^{\frac{1}{2}}\|\boldsymbol{\Pi}_{k+1}^{0,F}\mathbf{v}_h^F\|_F\right) + h_E\|\mathbf{curl}\,\mathbf{v}_h\|_E + \|\mathbf{v}_h\|_E \\
&\overset{(12)}{\lesssim} \sum_{F\subseteq\partial E}\left(h_F^{\frac{1}{2}}\|\mathbf{v}_h^F\|_F + h_F^{\frac{3}{2}}\|\mathrm{rot}\,\mathbf{v}_h^F\|_F + h_F^{\frac{1}{2}}\|\boldsymbol{\Pi}_{k+1}^{0,F}\mathbf{v}_h^F\|_F\right) + h_E\|\mathbf{curl}\,\mathbf{v}_h\|_E + \|\mathbf{v}_h\|_E \\
&\overset{(58),(140)}{\lesssim} \|\mathbf{v}_h\wedge\mathbf{n}_{\partial E}\|_{-\frac{1}{2},\partial E} + h_E\|\mathbf{curl}\,\mathbf{v}_h\|_E + \|\mathbf{v}_h\|_E \overset{(14),(139)}{\lesssim} \|\mathbf{curl}\,\mathbf{v}_h\|_{-1,E} + \|\mathbf{v}_h\|_E \\
&= \sup_{\boldsymbol{\psi}\in\mathbf{H}_0^1(E)}\frac{(\mathbf{curl}\,\mathbf{v}_h, \boldsymbol{\psi})_E}{|\boldsymbol{\psi}|_{1,E}} + \|\mathbf{v}_h\|_E \overset{\mathrm{IBP}}{\lesssim} \|\mathbf{v}_h\|_E.
\end{aligned}
$$

□

*Remark* 7. Following the definition of $S_e^{s,E}(\cdot,\cdot)$, we can also define the following alternative stabilization for the case of the serendipity edge virtual element space in 3D:

$$\begin{aligned} S_e^{s,E}(\mathbf{v}_h, \mathbf{w}_h) = \sum_{F \subseteq \partial E} \big( & h_F^2 (\mathbf{v}_h \cdot \mathbf{t}_{\partial F}, \mathbf{w}_h \cdot \mathbf{t}_{\partial F})_{\partial F} + h_F (\mathbf{\Pi}_S^e \mathbf{v}_h^F, \mathbf{\Pi}_S^e \mathbf{w}_h^F)_F \big) \\ & + h_E^2 S_f^E(\mathbf{curl}\, \mathbf{v}_h, \mathbf{curl}\, \mathbf{w}_h). \end{aligned} \tag{153}$$

The advantage of the variant above is that, if we substitute $(\mathbf{I} - \mathbf{\Pi}_{k-1}^{0,E})$ by $(\mathbf{I} - \mathbf{\Pi}_S^e)$ in the stabilization term of the scalar product (138), then the second addendum in definition (153) will vanish, thus leading to a lighter form. Employing analogous arguments, we can prove the same stability bounds as in Theorem 5.5 also for choice (153).

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