# Cherlin's conjecture on finite primitive binary permutation groups 

Nick Gill<br>Department of Mathematics, University of South Wales,<br>Treforest CF37 1DL, U.K.<br>nick.gill@southwales.ac.uk

Martin W. Liebeck<br>Department of Mathematics, Imperial College London,<br>London SW7 2AZ, UK<br>m.liebeck@imperial.ac.uk

Pablo Spiga<br>Dipartimento di Matematica e Applicazioni,<br>University of Milano-Bicocca,<br>Via Cozzi 55,<br>20125 Milano, Italy<br>pablo.spiga@unimib.it

July 13, 2021

In memory of Jan Saxl.

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## Chapter 1

## Introduction

In this monograph, we are concerned with the problem of classifying the finite primitive binary permutation groups. Let $G$ be a permutation group on the set $\Omega$. Given a positive integer $n$, given $I:=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)$ in the Cartesian product $\Omega^{n}$ and given $g \in G$, we write

$$
I^{g}:=\left(\omega_{1}^{g}, \omega_{2}^{g}, \ldots, \omega_{n}^{g}\right)
$$

Moreover, for every $1 \leq i<j \leq n$, we let $I_{i j}:=\left(\omega_{i}, \omega_{j}\right)$ be the 2 -subtuple of $I$ corresponding to the $i^{\text {th }}$ and to the $j^{\text {th }}$ coordinate. Now, the permutation group $G$ on $\Omega$ is called binary if, for all positive integers $n$, and for all $I$ and $J$ in $\Omega^{n}$, there exists $g \in G$ such that $I^{g}=J$ if and only if for all 2-subtuples, $I_{i j}$, of $I$, there exists an element $g_{i j}$ such that $I_{i j}^{g_{i j}}=J_{i j}$.

Cherlin has proposed a conjecture listing the finite primitive binary permutation groups [20]. The conjecture is as follows, and our task is to complete the proof of this conjecture.
Conjecture 1.1. A finite primitive binary permutation group must be one of the following:

1. a symmetric group $\operatorname{Sym}(n)$ acting naturally on $n$ elements;
2. a cyclic group of prime order acting regularly on itself;
3. an affine orthogonal group $V \rtimes \mathrm{O}(V)$ with $V$ a vector space over a finite field equipped with a nondegenerate anisotropic quadratic form, acting on itself by translation, with complement the full orthogonal group $\mathrm{O}(V)$.

The terminology of Conjecture 1.1 is fully explained in subsequent sections. In particular, we give two equivalent definitions of the adjective "binary" in $\$ 1.1$ and all three families listed in Conjecture 1.1 are fully discussed in $\$ 1.2$.

The O'Nan-Scott-Aschbacher theorem describes the structure of finite primitive permutation groups: there are five families of these. Thus, to prove Conjecture 1.1, it is sufficient to prove it for each of these families.

Cherlin himself gave a proof of the conjecture for the family of affine permutation groups, i.e. when $G$ has an abelian socle [21]. Wiscons then studied the remaining cases and showed that Conjecture 1.1 reduces to the following statement concerning almost simple groups [106].
Conjecture 1.2. If $G$ is a finite binary almost simple primitive group on $\Omega$, then $G=\operatorname{Sym}(\Omega)$.
We recall that an almost simple group $G$ is a finite group that has a unique minimal normal subgroup $S$ and, moreover, the group $S$ is non-abelian and simple. Note that $S$ is the socle of $G$.

We now invoke the Classification of Finite Simple Groups which says that a non-abelian simple group is either an alternating group, $\operatorname{Alt}(n)$ with $n \geq 5$; a simple group of Lie type; or one of 26 sporadic groups.

In [46], Conjecture 1.2 was proved for groups with socle a simple alternating group; in [34, Conjecture 1.2 was proved for groups with socle a sporadic simple group. In this monograph we deal with the remaining family.

Theorem 1.3. Let $G$ be an almost simple group with socle a finite group of Lie type and assume that $G$ has a primitive and binary action on a set $\Omega$. Then $|\Omega| \in\{5,6,8\}$ and $G \cong \operatorname{Sym}(\Omega)$.

The examples in Theorem 1.3 arise via the isomorphisms

1. $G \cong \mathrm{SL}_{2}(4) .2 \cong \mathrm{PGL}_{2}(5) \cong \operatorname{Sym}(5)$ and $|\Omega|=5$;
2. $G \cong \operatorname{Sp}_{4}(2) \cong \mathrm{PSL}_{2}(9) .2 \cong \operatorname{Sym}(6)$ and $|\Omega|=6$;
3. $G \cong \mathrm{SL}_{4}(2) .2 \cong \operatorname{Sym}(8)$ and $|\Omega|=8$.

Note that, here, we have not tried to list all isomorphisms between classical groups and the symmetric groups listed in Theorem 1.3. The listed isomorphisms are the ones that crop up in the proof that follows; there are many further isomorphisms with classical groups not listed in the theorem (for example $\left.\mathrm{SO}_{4}^{-}(2) \cong \Gamma \mathrm{O}_{3}(4) \cong \operatorname{Sym}(5)\right)$.

A special case of Theorem 1.3 has already appeared in the literature; in [34], the theorem is proved for the case where $G$ is almost simple with socle a finite group of Lie type of rank 1 .

Theorem 1.3 is the final piece in the jigsaw. We can now assert that Cherlin's conjecture is true 1
Corollary 1.4. Conjecture 1.1 is true.
As will become clear, once the various equivalent definitions of the word "binary" have been introduced, a proof of Conjecture 1.1 is equivalent to a classification of the finite primitive binary relational structures. In particular we have the following (the definition of homogeneous relational structure can be found in Definitions 1.1.1 and 1.1.5):

Corollary 1.5. Let $\mathcal{R}$ be a homogeneous binary relational structure with vertex set $\Omega$, such that $G=$ $\operatorname{Aut}(\mathcal{R})$ acts primitively on $\Omega$. Then the action of $G$ on $\Omega$ is one of the actions listed in Conjecture 1.1.

We have not completely described the relational structure $\mathcal{R}$ in our statement of Corollary 1.5- to do this, we would need to specify the relations in $\mathcal{R}$. We will not do this here, but we can at least start the task, making use of the fact that all relations of $\mathcal{R}$ must be unions of orbits of $G$ on $\Omega^{2}$.

Consider the first family listed in Conjecture 1.1, where $G=\operatorname{Sym}(\Omega)$. In this case $G$ has two orbits on $\Omega^{2}$ : the set $D$, of distinct pairs, and the set $R$, of repeated pairs. Thus the binary relational structures with all relations some union of $D$ and $R$ are:

$$
(\Omega),(\Omega, D),(\Omega, R),(\Omega, D, R),(\Omega, R, D) \text { and }(\Omega, D \cup R)
$$

One can check directly that every one of these is homogeneous and has automorphism group isomorphic to $\operatorname{Sym}(\Omega)$. One needs to repeat this analysis for the other two families; in these cases enumerating orbits and ascertaining which of the resulting structures are homogeneous is much more difficult.

For the remainder of this chapter we have three basic aims: first we seek to give the basic theory of relational complexity for permutation groups including, in particular, the definition of a binary action, and of a binary permutation group. We will also describe some of the key examples.

[^0]Second, we will give some motivation for interest in our result - thus we will survey some related results in the study of relational structures, and in group theory. We will also briefly discuss Cherlin's original motivation for studying binary permutation groups, which arises from model theoretic considerations.

In neither of these first two aspects do we make any claim for originality - instead we seek to draw the key definitions and examples together into one place. Much of the material of this kind that we present below was worked out by Cherlin in his papers [20, 21, 26].

Our third aim in this chapter is to present some of the results and methods concerning binary permutation groups that we consider to be most essential. These will be used in subsequent chapters when we commence our proof of Theorem 1.3,

The remainder of this monograph is occupied with a proof of Theorem 1.3. In Chapter 2 we give a number of general background results concerning groups of Lie type; in Chapter 3 we prove the theorem for the exceptional groups of Lie type; in Chapter 4 we prove the theorem for the classical groups of Lie type.

## Acknowledgements

All three authors were supported in this work by the Engineering and Physical Sciences Research Council grant number EP/R028702/1.

All three of us wish to express our thanks to Gregory Cherlin for his help and encouragement of our work, and for his creation of the beautiful mathematics that inspired our research in the first place. Thanks are also due to Joshua Wiscons for a number of helpful discussions.

NG and PS would like to thank their PhD students, Scott Hudson and Bianca Lodá; their research into the relational complexity of permutation groups has shed a great deal of light.

### 1.1 Basics: The definition of relational complexity

The notion of relational complexity can be defined in two different ways. Our job in this section is to present these definitions, and to show that they are equivalent. Throughout this section $G$ is a permutation group on a set $\Omega$ of size $t<\infty$. Note that when we write "permutation group" we are assuming that the associated action of $G$ on $\Omega$ is faithful - in other words we can think of $G$ as a subgroup of $\operatorname{Sym}(\Omega)$.

### 1.1.1 Relational structures

The first approach towards relational complexity is via the concept of a relational structure [21]. Recall that, for a positive integer $\ell, \Omega^{\ell}$ denotes the set of $\ell$-tuples with entries in $\Omega$.

Definition 1.1.1. A relational structure $\mathcal{R}$ is a tuple $\left(\Omega, R_{1}, \ldots, R_{k}\right)$, where $\Omega$ is a set, $k$ is a non-negative integer and, for each $i \in\{1, \ldots, k\}$, there exists an integer $\ell_{i} \geq 2$ such that $R_{i} \subseteq \Omega^{\ell_{i}}$.

The set $\Omega$ is called the vertex set of the structure, while the sets $R_{1}, \ldots, R_{k}$ are referred to as relations; in addition, for each $i$, the integer $\ell_{i}$ is the arity of relation $R_{i}$. We say that the relational structure $\mathcal{R}$ is of arity $\ell$, where $\ell=\max \left\{\ell_{1}, \ldots, \ell_{k}\right\}$.

Example 1.1.2. If a relation, or a relational structure is of arity 2 (resp. 3), then it is commonly called binary (resp. ternary). Binary relational structures which contain a single relation are nothing more nor less than directed graphs: if $\mathcal{R}=\left(\Omega, R_{1}\right)$ is one such, then the elements of the vertex set $\Omega$ are of course the vertices, and each pair in $R_{1}$ can be thought of as a directed edge between two elements of $\Omega$. (Note that by "graph" here we implicitly mean a graph with no multiple edges.)

When considering a binary relational structure with more than one relation, it is sometimes helpful to think of it as a directed graph in which there are several different "edge colours" - each relation corresponding to a different "colour".

The notions of isomorphism and automorphism are generalizations of the corresponding definitions for graphs.

Definition 1.1.3. Let $\mathcal{R}=\left(\Omega, R_{1}, \ldots, R_{k}\right)$ and $\mathcal{S}=\left(\Lambda, S_{1}, \ldots, S_{k}\right)$ be relational structures. An isomorphism $h: \mathcal{R} \rightarrow \mathcal{S}$ is a bijection $h: \Omega \rightarrow \Lambda$ such that

$$
\left(\omega_{1}, \ldots, \omega_{\ell_{i}}\right) \in R_{i} \Longleftrightarrow\left(\omega_{1}^{h}, \ldots, \omega_{\ell_{i}}^{h}\right) \in S_{i}
$$

An automorphism $g$ of $\mathcal{R}$ is an element of $\operatorname{Sym}(\Omega)$ that is also an isomorphism $g: \mathcal{R} \rightarrow \mathcal{R}$. It is clear that the set of all automorphisms of $\mathcal{R}$ forms a group under composition of bijections; we denote this group by $\operatorname{Aut}(\mathcal{R})$, and note that it is a subgroup of $\operatorname{Sym}(\Omega)$.

Note that we have only defined isomorphisms between relational structures that have the same number of relations; the definition also implies that the (ordered) list of relation-arities must be the same for isomorphic relational structures $\frac{2}{2}$

Our focus will be on those relational structures that exhibit the maximum possible level of symmetry - this requires the notion of homogeneity. To state this definition we must first explain what is meant by "an induced substructure" - once again this notion is a direct analogue of the same idea for graphs.

Definition 1.1.4. Let $\mathcal{R}=\left(\Omega, R_{1}, \ldots, R_{k}\right)$ be a relational structure, with $R_{i}$ a relation of arity $\ell_{i}$ for each $i=1, \ldots, k$. Let $\Gamma$ be a subset of $\Omega$. The induced substructure on $\Gamma$ is the relational structure $\mathcal{R}_{\Gamma}=\left(\Gamma, R_{1}^{\prime}, \ldots, R_{k}^{\prime}\right)$ where $R_{i}^{\prime}=\Gamma^{\ell_{i}} \cap R_{i}$.

So, to clarify what we said above: if $\mathcal{R}=\left(\Omega, R_{1}\right)$ is a binary structure with a single relation (i.e. a directed graph), and $\Gamma$ is a subset of the vertex set $\Omega$, then $\mathcal{R}_{\Gamma}$ is precisely the induced subgraph on $\Gamma$.

Definition 1.1.5. A relational structure $\mathcal{R}=\left(\Omega, R_{1}, \ldots, R_{k}\right)$ is called homogeneous if, for all $\Gamma, \Gamma^{\prime} \subset \Omega$ and for all isomorphisms $h: \mathcal{R}_{\Gamma} \rightarrow \mathcal{R}_{\Gamma^{\prime}}$, there exists $g \in \operatorname{Aut}(\mathcal{R})$ such that $\left.g\right|_{\Gamma}=h$.

The following example will be important shortly.
Example 1.1.6. Given a permutation group $G$ on a set $\Omega$ of size $t$, we define a relational structure $\mathcal{R}_{G}=\left(\Omega, R_{1}, \ldots, R_{k}\right)$, where the relations $R_{1}, \ldots, R_{k}$ are precisely the orbits of the group $G$ on the sets $\Omega^{2}, \ldots, \Omega^{t-1}$.

Observe, first, that by definition any element of $G$ maps an element of relation $R_{i}$ to an element of relation $R_{i}$, for all $i \in\{1, \ldots, k\}$; we conclude that $G \leq \operatorname{Aut}\left(\mathcal{R}_{G}\right)$.

On the other hand, suppose that $h \in \operatorname{Aut}\left(\mathcal{R}_{G}\right)$, and let $r=\left(\omega_{1}, \ldots, \omega_{t-1}\right)$ be a tuple of distinct elements in $\Omega$ lying in relation $R_{j}$, for some $j$. The image of this tuple under $h$ also lies in $R_{j}$; since $R_{j}$ is an orbit of $G$, this implies that there exists $g \in G$ such that for all $i \in\{1, \ldots, t-1\}, \omega_{i}^{h}=\omega_{i}^{g}$. It follows that $\omega_{t}^{h}=\omega_{t}^{g}$, where $\omega_{t}$ is the only element of $\Omega$ not represented in the tuple $r$. We conclude that $h=g$ and so, in particular, $G=\operatorname{Aut}\left(\mathcal{R}_{G}\right)$.

Finally, suppose that $\Gamma$ and $\Delta$ are proper subsets of $\Omega$ of size $s$ such that the associated induced relational structures are isomorphic, i.e. there exists an isomorphism $h:\left(\mathcal{R}_{G}\right)_{\Gamma} \rightarrow\left(\mathcal{R}_{G}\right)_{\Delta}$. Let $r_{\gamma}=\left(\gamma_{1}, \ldots, \gamma_{s}\right)$ be a tuple containing all of the distinct elements of $\Gamma$, and observe that $r_{\gamma}$ lies in a relation $R_{j}$ of $\mathcal{R}_{G}$, for some $j$. Indeed, by construction, $r_{\gamma}$ lies in the corresponding relation $R_{j}$ of $\left(\mathcal{R}_{G}\right)_{\Gamma}$, and so $\left(r_{\gamma}\right)^{h}$ lies in the corresponding relation $R_{j}$ of $\left(\mathcal{R}_{G}\right)_{\Delta}$, and hence also lies in the relation $R_{j}$ of $\mathcal{R}_{G}$. In particular, since $R_{j}$ is an orbit of $G$, we conclude that there exists $g \in G$ such that for all $i \in\{1, \ldots, s\}, \gamma_{i}^{h}=\gamma_{i}^{g}$. Since $G=\operatorname{Aut}\left(\mathcal{R}_{G}\right)$, we conclude that $\mathcal{R}_{G}$ is homogeneous.

[^1]We are ready to give our first definition of relational complexity. Before stating it, we remind the reader that we are assuming that $G$ is a permutation group on a set $\Omega$, and we recall that if $\mathcal{R}$ is any relational structure with vertex set $\Omega$, then $\operatorname{Aut}(\mathcal{R})$ is also a permutation group on $\Omega$.

Definition 1.1.7. The structural relational complexity of a permutation group $G$ is equal to the smallest integer $s \geq 2$ for which there exists a homogeneous relational structure $\mathcal{R}=\left(\Omega, R_{1}, \ldots, R_{k}\right)$ of arity $s$ such that $\operatorname{Aut}(\mathcal{R})$ is permutation isomorphic to $G$.

Note that Example 1.1 .6 implies, in particular, that if $|\Omega| \geq 3$, then the structural relational complexity of $G$ is well-defined, and is bounded above by $|\Omega|-1$ (and is at least 2). In what follows, we will write $\operatorname{SRC}(G, \Omega)$ for the structural relational complexity of the permutation group $G$.

One might wonder why we have required that $\operatorname{SRC}(G, \Omega) \geq 2$. The reason is that, in the next section we will define a different statistic $\operatorname{TRC}(G, \Omega)$ using a completely different approach, and we will also require that $\operatorname{TRC}(G, \Omega) \geq 2$. We will then show that $\operatorname{SRC}(G, \Omega)=\operatorname{TRC}(G, \Omega)$ for all permutation groups $G$ on a set $\Omega$. Were we to omit the requirement that $\operatorname{SRC}(G, \Omega) \geq 2$ and $\operatorname{TRC}(G, \Omega) \geq 2$, there would be a number of actions for which $\operatorname{SRC}(G, \Omega) \neq \operatorname{TRC}(G, \Omega)$, for instance the natural action of $\operatorname{Sym}(\Omega)$.

### 1.1.2 Tuples

In this section we give an alternative approach to the notion of relational complexity based on [26]. We then show that it coincides with the approach of the previous section. As before $G$ is a permutation group on a finite set $\Omega$.

Definition 1.1.8. Let $2 \leq r \leq n$ be positive integers, and let $I=\left(I_{1}, \ldots, I_{n}\right)$ and $J=\left(J_{1}, \ldots, J_{n}\right)$ be elements of $\Omega^{n}$. We say that $I$ and $J$ are $r$-subtuple complete with respect to $G$ if, for all $k_{1}, k_{2}, \ldots, k_{r}$ integers with $1 \leq k_{1}, k_{2}, \ldots, k_{r} \leq n$, there exists $g \in G$ with $I_{k_{i}}^{g}=J_{k_{i}}$ for $i \in\{1, \ldots, r\}$. In this case we write $I \widetilde{r} J$.

Note that if $I \widetilde{r} J$ and $u \leq r$, then $I \widetilde{u} J$.
Definition 1.1.9. The permutation group $G$ has tuple relational complexity equal to $s$ if the following two conditions hold:

1. if $n \geq s$ is any integer and $I, J$ are elements of $\Omega^{n}$ such that $I \widetilde{s} J$, then there exists $g \in G$ such that $I^{g}=J$.
2. $s \geq 2$ is the smallest integer for which (1) holds.

We write $\operatorname{TRC}(G, \Omega)$ for the tuple relational complexity of the permutation group $G$.
Put another way, the tuple relational complexity of $G$ is the smallest integer $s \geq 2$ such that

$$
I \underset{s}{\sim} J \Longrightarrow I \widetilde{n} J
$$

for any integer $n \geq s$, and any pair of $n$-tuples $I$ and $J$.
It is not immediately clear, a priori, that $\operatorname{TRC}(G, \Omega)$ exists for every permutation group $G$ on the set $\Omega$. The next lemma deals with this concern.

Lemma 1.1.10. If $\operatorname{SRC}(G, \Omega)=s$, then $\operatorname{TRC}(G, \Omega)$ exists and is bounded above by $s$.
Proof. Let $n \geq 2$ be some integer, and let $I$ and $J$ be subsets of $\Omega^{n}$ such that $I \widetilde{s} J$. We must prove that there exists $g \in G$ such that $I^{g}=J$.

Let $\mathcal{R}$ be a homogeneous relational structure of arity $s$ for which $G=\operatorname{Aut}(\mathcal{R})$. Write $\{I\}$ (resp. $\{J\}$ ) for the underlying set associated with the $n$-tuple $I$ (resp. $J$ ); as $s \geq 2$, these sets must be of equal cardinality bounded above by $n$. Now consider the induced substructures $\mathcal{R}_{\{I\}}$ and $\mathcal{R}_{\{J\}}$ and consider the map $h: \mathcal{R}_{\{I\}} \rightarrow \mathcal{R}_{\{J\}}$ for which $h\left(I_{i}\right)=J_{i}$ for all $i \in\{1, \ldots, n\}$.

We claim that $h$ is an isomorphism of relational structures. Let $\left(I_{i_{1}}, \ldots, I_{i_{u}}\right)$ be an element of some relation $R_{j}$ in $\mathcal{R}_{\{I\}}$. Note that $u \leq s$ and recall that $I \widetilde{u} J$ with respect to the action of $G$. Thus there exists $g \in G$ such that

$$
\left(J_{i_{1}}, \ldots, J_{i_{u}}\right)=\left(I_{i_{1}}, \ldots, I_{i_{u}}\right)^{g} .
$$

Then, since $g \in \operatorname{Aut}(\mathcal{R})$, we conclude that $\left(J_{i_{1}}, \ldots, J_{i_{u}}\right)$ is an element of relation $R_{j}$ in $\mathcal{R}_{\{J\}}$. We conclude that $h$ is an isomorphism as required.

Now, since $\mathcal{R}$ is homogeneous, there exists $g \in G=\operatorname{Aut}(\mathcal{R})$ such that $g_{\mid\{I\}}=h$; in particular $I^{g}=J$, as required.

Lemma 1.1.11. $\operatorname{SRC}(G, \Omega) \leq \operatorname{TRC}(G, \Omega)$.
Proof. Let $r=\operatorname{TRC}(G, \Omega)$. Define $\mathcal{R}=\left(\Omega, R_{1}, \ldots, R_{k}\right)$, where $R_{1}, \ldots, R_{k}$ are the orbits of $G$ on $\Omega^{i}$ for all $i \in\{2, \ldots, r\}$.

Clearly $G \leq \operatorname{Aut}(\mathcal{R})$. Suppose that $\sigma \in \operatorname{Aut}(\mathcal{R})$, and let $I=\left(\omega_{1}, \ldots, \omega_{t}\right)$ be a $t$-tuple of distinct elements of $\Omega$, where $t=|\Omega|$ (so every entry of $\Omega$ occurs as an entry in $I$ ). Then $I \widetilde{r} I^{\sigma}$, and so there exists $g \in G$ such that $I^{g}=I^{\sigma}$. This implies that $\sigma=g$, and so $\operatorname{Aut}(\mathcal{R}) \leq G$. We conclude that $G=\operatorname{Aut}(\mathcal{R})$.

We must show that $\mathcal{R}$ is homogeneous. Let $\Gamma$ and $\Delta$ be subsets of $\Omega$ of size $s$ such that there exists an isomorphism $\varphi: \mathcal{R}_{\Gamma} \rightarrow \mathcal{R}_{\Delta}$. Furthermore, let $I=\left(\gamma_{1}, \ldots, \gamma_{s}\right)$ be an $s$-tuple of distinct elements of $\Gamma$. Suppose first $s \leq r$. Since $\mathcal{R}$ contains all the orbits of $G$ on $\Omega^{s}$ and since $\mathcal{R}_{\Gamma} \cong \mathcal{R}_{\Delta}$, we deduce that $I$ and $\varphi(I)$ are in the same $G$-orbit, that is, there exists $g \in G$ such that $I^{g}=\varphi(I)$. Thus $\varphi=\left.g\right|_{\Gamma}$, as required. Suppose next $s>r$. Since all $r$-subtuples of $I$ occur as relations in $\mathcal{R}$ and since $\mathcal{R}_{\Gamma} \cong \mathcal{R}_{\Delta}$, we conclude that $I \widetilde{r} \varphi(I)$. Since $r=\operatorname{TRC}(G, \Omega)$, we deduce $I \underset{s}{\sim} \varphi(I)$. As before, this implies that there exists $g \in G=\operatorname{Aut}(\mathcal{R})$ such that $I^{g}=\varphi(I)$; in other words $\varphi=\left.g\right|_{\Gamma}$, as required.

Corollary 1.1.12. $\operatorname{SRC}(G, \Omega)=\operatorname{TRC}(G, \Omega)$.
In light of this corollary, we now drop the distinction between the two types of relational complexity:
Definition 1.1.13. The relational complexity of $G$ is equal to the tuple relational complexity of $G$ (and hence also equal to the structural relational complexity of $G$ ), and is denoted $\operatorname{RC}(G, \Omega)$.

In particular, a permutation group $G \leq \operatorname{Sym}(\Omega)$ is called binary if $\operatorname{RC}(G, \Omega)=2$.
Our definition of relational complexity has, to this point, pertained only to permutation groups, i.e. to faithful group actions. It is convenient to extend this definition now to any group action:

Definition 1.1.14. Suppose that a group $G$ acts on a set $\Omega$. The relational complexity of the action, denoted $\operatorname{RC}(G, \Omega)$, is the relational complexity of the permutation group induced by the action of $G$ on $\Omega$.

Note, finally, that in [26] the word arity is used as a synonym for relational complexity.

### 1.2 Basics: Some key examples

Our focus in this monograph is on actions with small relational complexity, thus the examples we present below are skewed in this direction. In particular, all of the actions listed in Conjecture 1.1 are discussed.

As we shall see, there are times when the structural definition of relational complexity is easiest to work with, and times when we prefer the tuple definition.

Before we outline the primary examples, we need to say a few words about the third family in Conjecture 1.1. This family consists of all groups isomorphic to an affine orthogonal group $V \rtimes \mathrm{O}(V)$ with $V$ a vector space over a finite field equipped with a non-degenerate anisotropic quadratic form, acting on itself by translation, with complement the full orthogonal group $\mathrm{O}(V)$. It is a straightforward consequence of the classification of non-degenerate quadratic forms that if $V$ admits an anisotropic quadratic form $Q$ (i.e.
one for which $Q(\mathbf{v}) \neq 0$ for all $\mathbf{v} \in V \backslash\{\mathbf{0}\})^{3}$, then $\operatorname{dim}(V) \leq 2$. We will split this family into two smaller families according to whether $\operatorname{dim}(V)$ is 1 or 2 ;

3a. $\operatorname{dim}(V)=1$ : the associated group $G$ is isomorphic to $\mathbb{F}_{q} \rtimes C_{2}$, where $C_{2}$ acts as -1 on the finite field $\mathbb{F}_{q}$ with $q$ elements, and the action is on $\Omega=\mathbb{F}_{q}$. For $G$ to be primitive we require that $q$ is prime, and we obtain that $G$ is isomorphic to the dihedral group of order $2 q$, with the action being on the $q$-gon, as usual.

3b. $\operatorname{dim}(V)=2$ and the associated quadratic form is of minus type: the associated group $G$ is isomorphic to $\mathbb{F}_{q}^{2} \rtimes \mathrm{O}_{2}^{-}(q) \cong \mathbb{F}_{q}^{2} \rtimes D_{2(q+1)}$, where $D_{2(q+1)}$ is a dihedral group of order $2(q+1)$.
First, let us observe that the relational complexity of the natural action of the symmetric group is as small as it can possibly be.

Example 1.2.1. Consider the natural action of $G=\operatorname{Sym}(t)$ on the set $\Omega=\{1, \ldots, t\}$. Define

$$
R=\{(i, j) \mid 1 \leq i, j \leq t \text { and } i \neq j\}
$$

Then $\mathcal{R}=(\Omega, R)$ is the complete directed graph, $\mathcal{R}$ is homogeneous and $G=\operatorname{Aut}(\mathcal{R})$. We conclude immediately that $\operatorname{RC}(G, \Omega)=2$.

Note that the first family of permutation groups listed in Conjecture 1.1 is precisely the family of finite symmetric groups in their natural action.

In many group-theoretic respects, the alternating group is very like the symmetric group. The next example shows that relational complexity does not conform to this rule-of-thumb: while, as we have just seen, the natural action of the symmetric group has relational complexity as small as it can possibly be, the natural action of the alternating group has relational complexity as large as it can possibly be.

Example 1.2.2. Consider the natural action of $G=\operatorname{Alt}(t)$ on the set $\Omega=\{1, \ldots, t\}$. Consider the tuples

$$
I=(1,2,3, \ldots, t) \text { and } J=(2,1,3, \ldots, t)
$$

It is straightforward to check that $I \widetilde{t-2} J$; it is equally clear that the only permutation $h$ for which $I^{h}=J$ is $h=(1,2) \notin G$. We conclude that $\operatorname{RC}(G, \Omega) \geq t-1$. Now Example 1.1.6 implies that $\operatorname{RC}(G, \Omega)=t-1$.

The previous two examples are a salutary warning that, in general, relational complexity behaves badly with respect to subgroups. All is not lost however: Lemma 1.6 .2 shows that the relational complexity of a group is related to that of some of its subgroups.

Our first aim is to understand the actions listed in Conjecture 1.1. Note that the Families 2 and 3a (using the notation at the start of this section) consist of primitive actions with very small point-stabilizers (size 1 and 2, respectively). In the next couple of examples we consider this situation.

Example 1.2.3. If $G$ acts regularly on $\Omega$, then $\mathrm{RC}(G, \Omega)$ is binary.
Proof: Suppose that $I=\left(I_{1}, \ldots, I_{n}\right)$ and $J=\left(J_{1}, \ldots, J_{n}\right)$ satisfy $I \widetilde{2} J$. For $i \in\{1, \ldots, n-1\}$, let $g_{i}$ be an element of $G$ that satisfies $I_{i}^{g_{i}}=J_{i}$ and $I_{i+1}^{g_{i}}=J_{i+1}$. The regularity of $G$ implies that, for $j \in\{1, \ldots, n\}$, there is a unique element of $G$ satisfying $I_{j}^{g}=J_{j}$. This fact, applied with $j=2$, implies that $g_{1}=g_{2}$; then applied with $j=3$, implies that $g_{2}=g_{3}$, and so on. Thus $g_{1}=\cdots=g_{n-1}$; calling this element $g$, we see that $I^{g}=J$ and we conclude that $I \widetilde{n} J$, as required.

Recall that the only regular primitive actions are associated with cyclic groups of prime order; we see, then, that the second family of groups in Conjecture 1.1 are precisely the regular primitive groups.

[^2]Example 1.2.4. Suppose that $G$ is transitive and a point-stabilizer $H$ has size 2, and suppose that $x$ is the non-trivial element in $H$. Let $C=x^{G}$ be the conjugacy class of $x$ in $G$. Then

$$
\mathrm{RC}(G)= \begin{cases}2, & \text { if } C \nsubseteq C^{2} \\ 3, & \text { otherwise }\end{cases}
$$

Proof: It is an easy exercise to verify that, under these assumptions, $\mathrm{RC}(G) \leq 3$. One can use, for instance, Lemma 1.5.1 below.

Since $\operatorname{RC}(G) \leq 3$, it is clear that a pair of $n$-tuples will be $n$-subtuple complete if and only if they are 3 -subtuple complete. Thus, if there exists an $n$-tuple that is 2 -subtuple complete but not $n$-subtuple complete, then there must exist a 3 -tuple that is 2 -subtuple complete but not 3 -subtuple complete.

Suppose that $G$ is not binary, and let $(P, Q)=\left(\left(P_{1}, P_{2}, P_{3}\right),\left(Q_{1}, Q_{2}, Q_{3}\right)\right)$ be a pair of 3 -tuples that is 2 -subtuple complete but not 3 -subtuple complete. Then there is, by assumption, an element $g$ of $G$ that maps ( $P_{1}, P_{2}$ ) to ( $Q_{1}, Q_{2}$ ). Replacing $Q$ by $Q^{g^{-1}}$ and relabelling, we conclude that there exists a pair

$$
\left(\left(P_{1}, P_{2}, P_{3}\right),\left(P_{1}, P_{2}, P_{4}\right)\right)
$$

that is 2-subtuple complete but not 3 -subtuple complete, in particular $P_{3} \neq P_{4}$. Write $H_{i}$ for the stabilizer of $P_{i}$, and let $x_{i}$ be the non-trivial element of $H_{i}$. Then we must have

$$
P_{3}^{x_{1}}=P_{3}^{x_{2}}=P_{4} .
$$

Since $(P, Q)$ is not 3 -subtuple complete, $x_{1} \neq x_{2}$, otherwise $P^{x_{1}}=Q$. Moreover, since $P_{3}^{x_{1} x_{2}}=P_{3}$, we conclude that $x_{1} x_{2}$ is the non-trivial element in $H_{3}$. Thus $C \subseteq C^{2}$, as required.

Suppose now that $C \subseteq C^{2}$. Let $x_{1}, x_{2}, x_{3} \in C$ with $x_{3}=x_{1} x_{2}$. In particular, there exist three points $P_{1}, P_{2}$ and $P_{3}$ with $G_{P_{1}}=\left\langle x_{1}\right\rangle, G_{P_{2}}=\left\langle x_{2}\right\rangle$ and $G_{P_{3}}=\left\langle x_{3}\right\rangle$. Set $P_{4}:=P_{3}^{x_{1}}$. We claim that $\left(\left(P_{1}, P_{2}, P_{3}\right),\left(P_{1}, P_{2}, P_{4}\right)\right)$ is a pair of 3 -tuples that is 2 -subtuple complete. In fact,

$$
\begin{aligned}
\left(P_{1}, P_{2}\right)^{1_{G}} & =\left(P_{1}, P_{2}\right), \\
\left(P_{1}, P_{3}\right)^{x_{1}} & =\left(P_{1}^{x_{1}}, P_{3}^{x_{1}}\right)=\left(P_{1}, P_{4}\right), \\
\left(P_{2}, P_{3}\right)^{x_{2}} & =\left(P_{2}^{x_{2}}, P_{3}^{x_{2}}\right)=\left(P_{2}, P_{3}^{x_{3} x_{2}}\right)=\left(P_{2}, P_{3}^{x_{1}}\right)=\left(P_{2}, P_{4}\right) .
\end{aligned}
$$

If this pair is 3 -subtuple complete, then there exists $g \in G$ with $P_{1}^{g}=P_{1}, P_{2}^{g}=P_{2}$ and $P_{3}^{g}=P_{4}$. In particular, $g \in\left\langle x_{1}\right\rangle \cap\left\langle x_{2}\right\rangle$. If $g=1$, then $P_{3}=P_{4}=P_{3}^{x_{1}}$ and hence $x_{1} \in\left\langle x_{3}\right\rangle$. This gives $x_{1}=x_{3}$ and hence $x_{2}=1$ because $x_{1} x_{2}=x_{3}$. However, this is a contradiction. Thus $g=x_{1}=x_{2}$ and hence $x_{3}=x_{1} x_{2}=1$, again a contradiction. Therefore, $\left(\left(P_{1}, P_{2}, P_{3}\right),\left(P_{1}, P_{2}, P_{4}\right)\right)$ is a pair of 3 -tuples that are 2 -subtuple complete but that are not 3 -subtuple complete; hence $G$ is not binary.

There is an important special case which occurs when point-stabilizers are of size 2 , and $G$ has a regular normal subgroup $N$. In this case it follows immediately that $C \nsubseteq C^{2}$ (where $C$ is as in Example 1.2.4), and thus $\operatorname{RC}(G, \Omega)=2$. Such an action is primitive if and only if $N$ is of prime order, and we now see that Family 3a pertaining to Conjecture 1.1 is precisely this 4

Our next example addresses Family 3b in Conjecture 1.1 ,

[^3]Example 1.2.5. This example is Lemma 1.1 of [21]. We identify $\Omega$ with a vector space $V$ over a field $F$, such that $V$ is endowed with a quadratic form $Q$ such that $Q$ is anisotropic, i.e. $Q(v) \neq 0$ for all $v \in V \backslash\{0\}$. We set $G=V \rtimes \mathrm{O}(V)$, where $\mathrm{O}(V)$ is the isometry group of the form $Q$, and the semidirect product is the natural one, as is the action of $G$ on $\Omega=V$.

Let us see that this action is binary. Let $n$ be a positive integer, and assume that $\mathbf{u}=\left(u_{0}, \ldots, u_{n}\right)$ and $\mathbf{u}^{\prime}=\left(u_{0}^{\prime}, \ldots, u_{n}^{\prime}\right)$ satisfy $\mathbf{u} \widetilde{2} \mathbf{u}^{\prime}$. Let us show that $\mathbf{u} \widetilde{n+1} \mathbf{u}^{\prime}$. We may suppose, without loss of generality that $u_{0}=u_{0}^{\prime}=0$.

Note that $\mathbf{u} \widetilde{2} \mathbf{u}^{\prime}$ implies that $Q\left(u_{i}\right)=Q\left(u_{i}^{\prime}\right)$ for all $i \in\{1, \ldots, n\}$. What is more, since the isometry group also preserves the polar form $\beta$ of $Q, \mathbf{u} \widetilde{\tau_{2}} \mathbf{u}^{\prime}$ also implies that

$$
\beta\left(u_{i}, u_{j}\right)=\beta\left(u_{i}^{\prime}, u_{j}^{\prime}\right),
$$

for any $1 \leq i, j \leq n$. This, in turn, implies that

$$
\begin{equation*}
Q\left(\sum_{j=1}^{n} c_{j} u_{j}\right)=Q\left(\sum_{j=1}^{n} c_{j} u_{j}^{\prime}\right) \tag{1.2.1}
\end{equation*}
$$

for any choice of scalars $c_{1}, \ldots, c_{n} \in F$.
Let $W=\operatorname{span}(\mathbf{u})$, and let $W^{\prime}=\operatorname{span}\left(\mathbf{u}^{\prime}\right)$ and suppose, without loss of generality, that $u_{1}, \ldots, u_{m}$ is a basis for $W$ (for $m=\operatorname{dim}(W))$. We claim that then $u_{1}^{\prime}, \ldots, u_{m}^{\prime}$ is a basis for $W^{\prime}$. To see this, it is enough to show that if $u_{1}, \ldots, u_{k}$ are linearly independent, then so too are $u_{1}^{\prime}, \ldots, u_{k}^{\prime}$. Suppose that $c_{1}, \ldots, c_{k} \in F$ such that $c_{1} u_{1}^{\prime}+\cdots+c_{k} u_{k}^{\prime}=0$. Then, clearly,

$$
Q\left(c_{1} u_{1}^{\prime}+\cdots+c_{k} u_{k}^{\prime}\right)=Q(0)=0
$$

But, by the observation above, this implies that $Q\left(c_{1} u_{1}+\cdots+c_{k} u_{k}\right)=0$, which implies that $c_{1} u_{1}+\cdots+$ $c_{k} u_{k}=0$, which in turn implies that $c_{1}=\cdots=c_{k}=0$. The claim follows.

Now we can define an isometry $f: W \rightarrow W^{\prime}$ by setting $f\left(u_{i}\right)=u_{i}^{\prime}$ for $i \in\{1, \ldots, m\}$, and extending linearly. Then Witt's Lemma implies that there exists $g \in \mathrm{O}(V)$ such that $u_{i}^{g}=u_{i}^{\prime}$ for all $i \in\{1, \ldots, m\}$. Let us now consider $m<i \leq n$. Write $u_{i}=\sum_{j=1}^{m} c_{j} u_{j}$ and now, observe that (1.2.1) yields that

$$
Q\left(u_{i}^{\prime}-\sum_{j=1}^{m} c_{j} u_{j}^{\prime}\right)=Q\left(u_{i}-\sum_{j=1}^{n} c_{j} u_{j}\right)=Q(0)=0
$$

Now the fact that $Q$ is anisotropic implies that $u_{i}^{\prime}-\sum_{j=1}^{m} c_{j} u_{j}^{\prime}=0$, and we conclude that $u_{i}^{g}=u_{i}^{\prime}$, as required.
All of the examples considered so far have been transitive. Let us briefly consider what can happen with intransitive actions.

Example 1.2.6. Suppose that the action of $G$ on $\Omega$ is intransitive with orbits $\Delta_{1}, \ldots, \Delta_{v}$. It is immediate from the definition that

$$
\mathrm{RC}(G, \Omega) \geq \max \left\{\mathrm{RC}\left(G, \Delta_{1}\right), \mathrm{RC}\left(G, \Delta_{2}\right), \ldots, \mathrm{RC}\left(G, \Delta_{v}\right)\right\}
$$

$G$. Now a well-known formula asserts that

$$
a_{i j v}=\frac{\left|C_{i}\right|\left|C_{j}\right|}{|G|} \sum_{\chi \in \operatorname{Irrc}(G)} \frac{\chi\left(g_{i}\right) \chi\left(g_{j}\right) \chi\left(g_{v}^{-1}\right)}{\chi(1)} .
$$

We conclude, therefore, that if a point-stabilizer $H=\langle x\rangle$ has size 2 , then $\operatorname{RC}(G)=2$ if and only if

$$
\sum_{\chi \in \operatorname{Irrc}(G)} \frac{\chi(x)^{3}}{\chi(1)}=0 .
$$

On the other hand, let $n \geq 3$ and consider the intransitive action of $G=\operatorname{Sym}(n)$ with two orbits, where the action on the first orbit is the natural one of degree $n$, and the second orbit is of size 2 . Clearly the action of $G$ on each orbit is binary; on the other hand, one can check directly that $\mathrm{RC}(G, \Omega)=n=t-2$.

This example suggests that the problem of calculating the relational complexity of intransitive actions may be rather difficult.

### 1.2.1 Existing results on relational complexity

Results on relational complexity above and beyond the basic examples discussed above are hard to obtain. Nearly all of the important results are due to Cherlin, and his co-authors, and we briefly mention some of these here. The first result is stated in [20], with a small correction in [21].

Theorem 1.2.7. Let $\Omega$ be the set of all $k$-subsets of a the set $\{1, \ldots, n\}$ with $2 k \leq n$. If $G=\operatorname{Sym}(n)$, then $\operatorname{RC}(G, \Omega)=2+\left\lfloor\log _{2} k\right\rfloor$. If $G=\operatorname{Alt}(n)$, then

$$
\operatorname{RC}(G, \Omega)= \begin{cases}n-1, & \text { if } k=1 \\ \max (n-2,3), & \text { if } k=2 \\ n-2, & \text { if } k \geq 3 \text { and } n=2 k+2 \\ n-3, & \text { otherwise }\end{cases}
$$

The actions of the symmetric and alternating groups on partitions, rather than $k$-sets, are currently being studied by Cherlin and Wiscons [24]. The only general result to date is for $\operatorname{Sym}(2 n)$ and $\operatorname{Alt}(2 n)$ acting on $\Omega$, the set of partitions of $2 n$ into $n$ blocks of size 2 (so, for $G=\operatorname{Sym}(2 n)$, this is the action on cosets of a maximal imprimitive subgroup of form $\operatorname{Sym}(2)$ wr $\operatorname{Sym}(n))$. The result they have obtained for $n \geq 2$ is as follows:

$$
\begin{aligned}
\operatorname{RC}(\operatorname{Sym}(2 n), \Omega) & =n ; \\
\operatorname{RC}(\operatorname{Alt}(2 n), \Omega) & = \begin{cases}2, & n=2 ; \\
4, & n \in\{3,4\} ; \\
n, & n>3 \text { and } n \equiv 0,1,3,5 \quad(\bmod 6) ; \\
n-1, & n>4 \text { and } n \equiv 2,4 \quad(\bmod 6)\end{cases}
\end{aligned}
$$

As we shall see below (Theorem 1.5.2), when considering large relational complexity, an important family of actions involves groups $G$ which are subgroups of $\operatorname{Sym}(m) \operatorname{wr} \operatorname{Sym}(r)$ containing $(\operatorname{Alt}(m))^{r}$, where the action of $\operatorname{Sym}(m)$ is on $k$-subsets of $\{1, \ldots, m\}$ and the wreath product has the product action of degree $t=\binom{m}{k}^{r}$. The particular situation where $G=\operatorname{Sym}(m) \operatorname{wr} \operatorname{Sym}(r)$ is studied in [26]. We summarise some of the results there, using the notation just established.

Theorem 1.2.8. Let $G=\operatorname{Sym}(m)$ wr $\operatorname{Sym}(r)$ acting on a set $\Omega$ of size $t=\binom{m}{k}^{r}$, as described.

1. If $m=2$, then $k=1$ and $\operatorname{RC}(G, \Omega)=2+\left\lfloor\log _{2} r\right\rfloor$.
2. If $k=1$, then $\mathrm{RC}(G, \Omega) \leq m+\left\lfloor\log _{2} r\right\rfloor$.
3. $\mathrm{RC}(G, \Omega) \leq\left\lfloor 2+\log _{2} k\right\rfloor\left\lfloor 1+\log _{2} r\right\rfloor$ with equality if $m \geq 2 k\left\lfloor 1+\log _{2} r\right\rfloor$.

The particular situation where $k=1$ and $G=\operatorname{Sym}(m) \operatorname{wr} \operatorname{Sym}(r)$ (so we are considering the natural product action of degree $m^{r}$ ) has been taken much further in a series of papers by Saracino [87, 88, 89]. Saracino's results effectively yield an exact value for the relational complexity of this family of actions. We do not write this value here as the precise formulation of the results is slightly involved; instead we refer to [26, §6] and to the papers of Saracino, particularly the first.

### 1.3 Motivation: On homogeneity

In his paper [20], Cherlin chooses a quote from Aschbacher as an epigraph. This quote, plus some more, goes as follows:

Define an object $X$ in a category $\mathfrak{C}$ to possess the Witt property if, whenever $Y$ and $Z$ are subobjects of $X$ and $\alpha: Y \rightarrow Z$ is an isomorphism, then $\alpha$ extends to an automorphism of $X$. Witt's Lemma says that orthogonal spaces, symplectic spaces, and unitary spaces have the Witt property in the category of spaces with forms and isometries. All objects in the category of sets and functions have the Witt property. But in most categories few objects have the Witt property; those that do are very well behaved indeed. If $X$ is an object with the Witt property and $G$ is its group of automorphisms, then the representation of $G$ on $X$ is usually an excellent tool for studying $G$. [3, pp. 81, 82]

One should think of "the Witt property" as a generalization of the notion of homogeneity which we have introduced in the specific setting of relational structures. The study of homogeneous objects in different categories has a long and interesting history ${ }^{5}$

Before discussing this history, let us delve a little deeper into why such objects have received attention: Aschbacher's answer is given above. This approach has its roots in the Erlangen Programme of Klein, in which the key features of a particular "geometry" define, and are defined by, the group of automorphisms of said geometry. The idea here is that one studies the geometry in question, one deduces information about the geometry, which one then reinterprets as information about the associated group; one can use this information about the group to deduce further information about the geometry and so on. Thus the process of mathematical inquiry moves back-and-forth between geometrical study and algebraic (group theoretic).

The efficacy of this approach varies considerably - if an object has a very small automorphism group for instance, then group theory may provide very little insight. On the other hand, as Aschbacher suggests, this approach is most spectacularly successful when the object in question is homogeneous. Indeed the two examples which Aschbacher mentions clearly illustrate the success of this approach.

First, we note that the category of sets and functions have the Witt property. If we restrict ourselves to finite objects in this category, then the associated automorphism groups are the finite symmetric groups, $\operatorname{Sym}(n)$. Of course, all of the basic group-theoretical information about these groups is most naturally expressed in the language of their natural (homogeneous) action on a set of size $n$. This includes their conjugacy class structure (via cycle type), and their subgroup structure (via the O'Nan-Scott-Aschbacher Theorem [2, 91]; see also (71]).

Second, in the category of spaces with forms, basic linear algebra asserts that objects associated with a zero form (i.e. naked vector spaces) have the Witt property; Witt's Lemma extends this to cover objects associated with either a non-degenerate quadratic or non-degenerate sesquilinear form. Again, restricting ourselves to finite such objects, we obtain the finite classical groups as the associated automorphism groups. As before, the basic group-theoretical properties of these groups are most naturally expressed in the language of their natural homogeneous action on the associated vector space. This includes their conjugacy class structure (via rational canonical form for $\mathrm{GL}_{n}(q)$, and the variants due to Wall for the other classical groups [101), and their subgroup structure (via Aschbacher's Theorem [1).

In light of all this, a natural question when studying some (permutation) group $G$ is whether we can find an object in some category on which $G$ acts homogeneously. Example 1.1.6 gives an easy answer to this: it turns out that there is always such an object in the category of relational structures. The bad news is that the object provided by Example 1.1.6 is little more than an encoding of the complete structure of the permutation group in terms of a relational structure - studying the structure $\mathcal{R}_{G}$ will hardly be easier than studying the original group and its associated action.

[^4]The investigation of relational complexity seeks to remedy this disappointing state of affairs: given a group $G$ and an associated action, $\mathrm{RC}(G, \Omega)$ gives us an indication of the efficiency with which we can build a relational structure on which $G$ can act homogeneously. From this point of view, an "efficient" representation of $G$ acting homogeneously on a relational structure is one for which the arity of the structure is as small as possible.

There is an alternative way of viewing efficiency in this context where one is, instead, interested in using relational structures with as few relations as possible (but not necessarily worrying about the arity of the relations used). We will not pursue this point of view here, but we refer to [52] (for the primitive case) and to [22] (for the general case), for results that pertain to this approach.

### 1.3.1 Existing results on homogeneity

We briefly review some important results on homogeneity for particular finite relational structures.
The classification of homogeneous graphs was partially completed by Sheehan [94], and then completely by Gardiner 42]. Indeed, Gardiner's result applies to a wider class of graphs than those we would call homogeneous. This classification was then extended by Lachlan to homogeneous digraphs [60].

In order to state these results we need some terminology: a digraph, $\Gamma$, is an ordered pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is a non-empty set, and $E(\Gamma)$ is an irreflexive binary relation on that set. The digraph is symmetric (resp. anti-symmetric) if, whenever $(x, y) \in E(\Gamma)$, we have $(y, x)$ in (resp. not in) $E(\Gamma)$. So a symmetric digraph is the object commonly called a graph in the literature.

If $\Gamma$ and $\Delta$ are two digraphs, then we can construct two new digraphs with vertex set $V(\Gamma) \times V(\Delta)$ :

1. in the composition of $\Gamma$ and $\Delta, \Gamma[\Delta]$, vertices $\left(u_{1}, v_{1}\right)$ and $\left(u_{2}, v_{2}\right)$ are connected if and only if $\left(u_{1}, u_{2}\right) \in E(\Gamma)$, or $u_{1}=u_{2}$ and $\left(v_{1}, v_{2}\right) \in E(\Delta)$;
2. in the direct product of $\Gamma$ and $\Delta, \Gamma \times \Delta$ vertices $\left(u_{1}, v_{1}\right)$ and $\left(u_{2}, v_{2}\right)$ are connected if and only if $\left(u_{1}, u_{2}\right) \in E(\Gamma)$ and $\left(v_{1}, v_{2}\right) \in E(\Delta)$.

We write $K_{n}$ for the complete (symmetric di)graph on $n$ vertices. We also define two infinite families of graphs, both indexed by a parameter $n \in \mathbb{Z}$ with $n \geq 3$ :

1. $\Lambda_{n}$ is the digraph with vertex set $\{0,1, \ldots, n-1\}$ and $(x, y) \in E\left(\Gamma_{n}\right)$ if and only if $x-y \equiv 1(\bmod n)$;
2. $\Delta_{n}$ is the symmetric digraph with vertex set $\{0,1, \ldots, n-1\}$ and $(x, y) \in E\left(\Delta_{n}\right)$ if and only if $x-y \equiv \pm 1(\bmod n)$.

Thus $\Lambda_{n}$ is the directed cycle on $n$ vertices, and $\Delta_{n}$ is the undirected cycle on $n$ vertices. Let $\mathcal{S}$ (resp. $\mathcal{A}$ ) denote the set of homogeneous symmetric (resp. antisymmetric) digraphs. We write $\bar{\Gamma}$ for the complement of $\Gamma$. Then Gardiner's result is the following:

Theorem 1.3.1. A digraph $\Gamma$ is in $\mathcal{S}$ if and only if $\Gamma$ or $\bar{\Gamma}$ is isomorphic to one of

$$
\Delta_{5}, K_{3} \times K_{3}, K_{m}\left[\overline{K_{n}}\right]
$$

where $m, n \in \mathbb{Z}^{+}$.
Now we will state Lachlan's result in three stages. First we need to define three "sporadic homogeneous digraphs"; this is done in Figure 1.1.

Second we classify the homogeneous antisymmetric digraphs.
Theorem 1.3.2. A digraph $\Gamma$ is in $\mathcal{A}$ if and only if $\Gamma$ is isomorphic to one of

$$
\Lambda_{4}, \overline{K_{n}}, \overline{K_{n}}\left[\Lambda_{3}\right], \Lambda_{3}\left[\overline{K_{n}}\right], H_{0}
$$

where $n \in \mathbb{Z}^{+}$.


Figure 1.1: Three homogeneous digraphs. The presence of an undirected edge $\{v, w\}$ in the diagrams for $H_{0}$ and $H_{1}$ indicates that both directed edges between $v$ and $w$ are present. In the diagram for $H_{2}$ we have omitted most of the directed edges. To obtain the remaining edges, note first that each vertex in $H_{2}$ has a unique mate, to which it is connected by an undirected edge (indicated in the diagram). Next, let $v$ and $w$ be vertices, and let $w^{\prime}$ be the mate of $w$. Finally, if $(v, w)$ is a directed edge, then $\left(w^{\prime}, v\right)$ is a directed edge, and if $(w, v)$ is a directed edge, then $\left(v, w^{\prime}\right)$ is a directed edge. This leads to the insertion of another 36 directed edges.

Finally we can state Lachlan's classification of homogeneous digraphs.
Theorem 1.3.3. A digraph $\Gamma$ is homogeneous if and only if $\Gamma$ or $\bar{\Gamma}$ is isomorphic to a digraph with one of the following forms:

$$
K_{n}[A], A\left[K_{n}\right], S, \Lambda_{3}[S], S\left[\Lambda_{3}\right], H_{1}, H_{2},
$$

where $n \in \mathbb{Z}^{+}, A \in \mathcal{A}$ and $S \in \mathcal{S}$.
Lachlan's result, expressed in our terms, is almost a classification of those homogeneous relational structures $\mathcal{R}=\left(\Omega, R_{1}\right)$ such that $R_{1}$ is binary. We write "almost" because Lachlan imposes the condition that $R_{1}$ is irreflexive whereas we make no such restriction. Nonetheless, given that in this monograph we are focusing on transitive actions, Lachlan's result is sufficient: any relational structure $\mathcal{R}=\left(\Omega, R_{1}\right)$ for which $R_{1}$ is binary and $\operatorname{Aut}(\mathcal{R})$ is transitive on $\Omega$, will either be precisely of the form listed in Theorem 1.3.3, or else will be of the form listed in Theorem 1.3 .3 with the addition of a loop at every vertex. We have made no attempt to extend this classification to the situation where $\operatorname{Aut}(\mathcal{R})$ is not transitive on $\Omega$ although we note that in this situation, $\operatorname{Aut}(\mathcal{R})$ would have exactly two orbits on $\Omega$ - one corresponding to vertices with loops, one corresponding to vertices without.

The groups $\operatorname{Aut}(\Gamma)$ for $\Gamma$ appearing in Theorem 1.3 .3 have not been explicitly listed to our knowledge. We will not calculate this list, but we can at least start the task: It is easy to check that $\operatorname{Aut}\left(\Lambda_{n}\right)$ is the cyclic group of order $n, \operatorname{Aut}\left(\Delta_{n}\right)$ is the dihedral group of order $2 n$ and $\operatorname{Aut}\left(K_{n}\right)$ is the symmetric group of degree $n$. It is slightly more involved to check the larger sporadic examples; the automorphism group and the action on points (which is necessarily binary) are as follows:

1. $\operatorname{Aut}\left(K_{3} \times K_{3}\right)=\operatorname{Sym}(3) \operatorname{wr} \operatorname{Sym}(2)$ in the product action on 9 points;
2. $\operatorname{Aut}\left(H_{0}\right) \cong \mathrm{SL}_{2}(3)$ acting on the 8 cosets of a Sylow 3-subgroup;
3. $\operatorname{Aut}\left(H_{1}\right)$ is the semidihedral group of order 16 - it has presentation $\left\langle x, y \mid x^{8}=y^{2}, x^{y}=x^{3}\right\rangle$ - in an action of degree 8 ;
4. $\operatorname{Aut}\left(H_{2}\right) \cong \operatorname{Alt}(4) \rtimes C_{4}$ where $C_{4}=\langle x\rangle$ acts by conjugation on Alt(4) via $g^{x}=g^{(1,2,3,4)}$ for all $g \in \operatorname{Alt}(4)$; as an abstract $\operatorname{group} \operatorname{Aut}\left(H_{2}\right) \cong(\operatorname{Alt}(4) \times 2) .2$, and the action is of degree 12 .

To complete the enumeration of the automorphism groups of homogeneous digraphs, we would need to study the automorphisms of the various graphs arising from the composition of two others: for instance, we would need to calculate $\operatorname{Aut}\left(A\left[K_{n}\right]\right)$ and $\operatorname{Aut}\left(K_{n}[A]\right)$ for each $A \in \mathcal{A}$. We will not do this.

There are a multitude of results that extend Gardiner, Sheehan and/or Lachlan's results to finite (di)graphs with automorphism groups that satisfy weaker properties than homogeneity. We particularly mention [48] which considers so-called set-homogeneous digraphs. In a different direction Cherlin has classified the homogeneous countable digraphs [25] extending work of Lachlan and Woodrow classifying the homogeneous countable graphs [63, and of Lachlan classifying the homogeneous countable tournaments 61.

Analogues for some of the given results exist for relational structures containing a single relation which may not be binary. Lachlan and Tripp have classified the homogeneous 3-graphs 64] and Cameron has done the same for homogeneous $k$-graphs with $k \geq 6$ [17]; these results are analogues of Gardiner's result for ternary relational structures with a single relation. Devillers has studied a rather similar problem in her work on homogeneous Steiner systems, however the notion of homogeneity considered there is different to ours 37.

### 1.4 Motivation: On model theory

Cherlin's conjecture arises from model theory considerations rooted in Lachlan's theory of finite homogeneous relational structures (see, for instance, [59, 62]). We give a brief summary of some of the main ideas; the origin of nearly everything we consider here is [20].

Let us consider a family of theorems indexed by parameters $k$ and $\ell$, with $k, \ell \in \mathbb{Z}^{+}$and $\ell \geq 2$. Theorem $(k, \ell)$ is a full classification of the homogeneous relational structures with at most $\ell$ relations, and with arity at most $k$. So, for instance, the first theorem we are likely to consider is Theorem $(2,1)$ which (modulo the transitivity assumption we discussed above) is just Theorem 1.3.3, a result of Lachlan himself that classifies finite binary relational structures with one relation; in other words finite simple homogeneous directed graphs.

Lachlan's theory of finite homogeneous relational structures asserted a number of facts about the form of these theorems, and about the relationships between them. With regard to the form of the theorem, Lachlan's theory asserts that each theorem can be written as follows:
"A finite homogeneous relational structure of arity at most $k$ with at most $\ell$ relations lies in one of a number of infinite families, or else is one of a finite number of sporadic individuals."

The power of this assertion is in the restrictions which Lachlan placed upon the definition of the word "family": a family of finite homogeneous relational structures in Lachlan's sense is an infinite collection of structures that can be constructed from a single infinite relational structure via a set of explicitly described operations.

With regard to the relationships between these theorems, Lachlan's theory gives us information about what the word "sporadic" means in these theorems. Specifically he asserts that any sporadic individual cropping up in Theorem ( $k, \ell$ ), say, will appear later as part of an infinite family in Theorem $\left(k^{\prime}, \ell^{\prime}\right)$ for some $k^{\prime} \geq k$ and $\ell^{\prime} \geq \ell$. Thus the "sporadic-ness" of a particular homogeneous relational structure is, in some sense, not genuine - rather, it is an artefact of restricting our investigations to particular values of $k$ and $\ell$.

The significance of all of this from a group-theoretic point of view lies in Cherlin's observation that every finite permutation group can be viewed as the automorphism group of a homogeneous relational structure - we demonstrated one way of seeing this in Example 1.1.6. This observation allows us to shift our point of view on the family of theorems studied by Lachlan: we can think of them as being about finite permutation groups.

In this setting the parameters $k$ and $\ell$ can be seen as providing some kind of stratification on the universe of finite permutations groups, and Lachlan's results concerning "families" and "sporadic-ness" can be seen
as statements about groups as well as structures. Finally, we can rewrite the theorems themselves from a group-theoretic point of view; they take the following form:
"Let $G$ be the automorphism group of a homogeneous relational structure $\mathcal{R}$ on a set $\Omega$ of arity at most $k$ with at most $\ell$ relations. Then, viewed as a permutation group on $\Omega, G$ lies in one of a number of infinite families, or else is one of a finite number of sporadic individuals."

With this set-up, any given permutation group $G$ will occur in an infinite number of Theorems $(k, \ell)$. Typically, though, we are interested in the first such occurrence: we are interested in the pair $(k, \ell)$ for which $k$ is minimal, and having fixed $k$ as this minimal value, we then seek the minimum possible value of $\ell$. The resulting pair $(k, \ell)$ is a measure of the complexity of $G$ from the model-theoretic point of view or, using the point of view espoused in \$1.3, gives a measure of the efficiency with which $G$ can be represented as the automorphism group of a homogeneous relational structure.

Of course, plenty remains: we know that these theorems about finite permutation groups exist; we know their form, and we know something about the relationships that exist between them. We would like to know the statements of these theorems, and we would like to prove them!

As described in the previous section, this last task has only been completed for Theorem (2,1) (and, even then, with a small caveat). The main theorem of this monograph completes the task of ascertaining which groups appear as primitive permutation groups in any Theorem $(2, \ell)$.

### 1.5 Motivation: Other important statistics

It turns out that relational complexity is closely connected to a number of other permutation group statistics, some of which have received a great deal of attention in the literature. Our reference for the following definitions is [5].

For $\Lambda=\left\{\omega_{1}, \ldots, \omega_{k}\right\} \subseteq \Omega$ and for $G \leq \operatorname{Sym}(\Omega)$, we write $G_{(\Lambda)}$ or $G_{\omega_{1}, \omega_{2}, \ldots, \omega_{k}}$ for the point-wise stabilizer. If $G_{(\Lambda)}=\{1\}$, then we say that $\Lambda$ is a base. The size of the smallest possible base is known as the base size of $G$ and is denoted $\mathrm{b}(G)$.

We say that a base is a minimal base if no proper subset of it is a base. We denote the maximum size of a minimal base by $\mathrm{B}(G)$.

Given an ordered sequence of elements of $\Omega,\left[\omega_{1}, \omega_{2}, \ldots, \omega_{k}\right]$, we can study the associated stabilizer chain:

$$
G \geq G_{\omega_{1}} \geq G_{\omega_{1}, \omega_{2}} \geq G_{\omega_{1}, \omega_{2}, \omega_{3}} \geq \cdots \geq G_{\omega_{1}, \omega_{2}, \ldots, \omega_{k}}
$$

If all the inclusions given above are strict, then the stabilizer chain is called irredundant. If, furthermore, the group $G_{\omega_{1}, \omega_{2}, \ldots, \omega_{k}}$ is trivial, then the sequence $\left[\omega_{1}, \omega_{2}, \ldots, \omega_{k}\right]$ is called an irredundant base. The size of the longest possible irredundant base is denoted $\mathrm{I}(G)$.

Finally, let $\Lambda$ be any subset of $\Omega$. We say that $\Lambda$ is an independent set if its point-wise stabilizer is not equal to the point-wise stabilizer of any proper subset of $\Lambda$. We define the height of $G$ to be the maximum size of an independent set, and we denote this quantity by $\mathrm{H}(G)$.

Note that if $G$ is a transitive permutation group on a set $\Omega$, then $\mathrm{H}(G)=1$ if and only if $G$ is regular; similarly, $\mathrm{H}(G)=2$ if and only if the stabilizer of a point is a non-trivial TI-subgroup of $G$. (Recall that $X$ is said to be a non-trivial TI-subgroup of a group $G$ if $X$ is a proper subgroup of $G$ and $X \cap X^{g}=1$, for every $g \in G \backslash N_{G}(X)$.)

There is a basic connection between the four statistics we have defined so far:

$$
\begin{equation*}
\mathrm{b}(G) \leq \mathrm{B}(G) \leq \mathrm{H}(G) \leq \mathrm{I}(G) \leq \mathrm{b}(G) \log t \tag{1.5.1}
\end{equation*}
$$

Recall that in this document, if the base is not specified, then "log" always means "log to the base 2 "; recall, also, that $t=|\Omega|$. Let us see why (1.5.1) is true:

The first inequality is obvious. For the second, suppose that $\Lambda$ is a minimal base; then $\Lambda$ is an independent set. For the third, suppose that $\Lambda:=\left\{\omega_{1}, \omega_{2}, \ldots, \omega_{k}\right\}$ is an independent set and observe that

$$
G>G_{\omega_{1}}>G_{\omega_{1}, \omega_{2}}>G_{\omega_{1}, \omega_{2}, \omega_{3}}>\cdots>G_{\omega_{1}, \omega_{2}, \ldots, \omega_{k}}
$$

is a strictly decreasing sequence of stabilizers. In particular, $\left[\omega_{1}, \omega_{2}, \ldots, \omega_{k}\right]$ is irredundant and we may extend this irredundant sequence to an irredundant base. Hence $\mathrm{H}(G) \leq \mathrm{I}(G)$.

The fourth inequality has been attributed to Blaha [7] who, in turn, describes it as an "observation of Babai" [4]. Suppose that $G$ has a base of size $b=\mathrm{b}(G)$. Then, in particular $|G| \leq t^{b}$. On the other hand, any irredundant base and any independent set have size at most $\log |G|$. We conclude that $\mathrm{I}(G) \leq \log \left(t^{b}\right)$, and the result follows.

We are ready to connect relational complexity to the four statistics we have just defined. The key result is the following.

Lemma 1.5.1. $\mathrm{RC}(G) \leq \mathrm{H}(G)+1$.
Proof. Let $h=H(G)$ and consider a pair $(I, J) \in \Omega^{n}$ such that $I \widetilde{h+1} J$. We must show that $I \widetilde{n} J$.
Observe that we can reorder the tuples without affecting their subtuple completeness. Hence, without loss of generality, we can assume that

$$
G_{I_{1}}>G_{I_{1}, I_{2}}>\cdots>G_{I_{1}, I_{2}, \ldots, I_{\ell}}
$$

for some $\ell \leq h$ and then this chain stabilizers, i.e.

$$
G_{I_{1}, \ldots, I_{\ell}}=G_{I_{1}, \ldots, I_{\ell+j}},
$$

for all $1 \leq j \leq n-\ell$. From the assumption of $h$-subtuple completeness it follows that there exists an element $g \in G$ such that $I_{i}^{g}=J_{i}$ for all $1 \leq i \leq \ell$ and observe that the set of all such elements $g$ forms a coset of $G_{I_{1}, \ldots, I_{\ell}}$.

The assumption of $(h+1)$-subtuple completeness implies, moreover, that for all $1 \leq j \leq n-\ell$ there exists $g_{j} \in G$ such that

$$
\left\{\begin{array}{l}
I_{i}^{g_{j}}=J_{i}, \quad \text { for } \quad 1 \leq i \leq \ell, \\
I_{\ell+j}^{g_{j}}=J_{\ell+j} .
\end{array}\right.
$$

The set of all such elements $g_{j}$ forms a coset of $G_{I_{1}, \ldots, I_{\ell}, I_{\ell+j}}$, which is, again, a coset of $G_{I_{1}, \ldots, I_{\ell}}$. Since any coset of $G_{I_{1}, \ldots, I_{\ell}}$ is defined by the image of the points $I_{1}, \ldots, I_{\ell}$ under an element of the coset, we conclude that elements of the same coset of $G_{I_{1}, \ldots, I_{\ell}}$ map $I_{\ell+j}$ to $J_{\ell+j}$ for all $1 \leq j \leq n-\ell$. In particular, $I \widetilde{n} J$, as required.

Lemma 1.5.1 has been exploited in [44], where an upper bound on the height of a primitive permutation group is proved, from which the obvious upper bound on relational complexity is deduced. The main result on height is the following:

Theorem 1.5.2. Let $G$ be a finite primitive group of degree $t$. Then one of the following holds:

1. $G$ is a subgroup of $\operatorname{Sym}(m) \operatorname{wr} \operatorname{Sym}(r)$ containing $(\operatorname{Alt}(m))^{r}$, where the action of $\operatorname{Sym}(m)$ is on $k$ subsets of $\{1, \ldots, m\}$ and the wreath product has the product action of degree $t=\binom{m}{k}^{r}$;
2. $\mathrm{H}(G)<9 \log t$.

Note that various members of the family listed at item (1) of Theorem 1.5.2 genuinely violate the bound at item (2): for example, when $r=k=1$, we obtain the groups $\operatorname{Sym}(t)$ and $\operatorname{Alt}(t)$ in their natural action, for which the height is $t-1$ and $t-2$, respectively. In fact, though, we do not know the exact height of the groups listed at item (1) for all possible values of $k, m$ and $r$.

The proof of Theorem 1.5 .2 exploits the rich array of results in the literature giving bounds on $\mathrm{b}(G)$ for various families of permutation groups. In particular, use is made of the proof of the Cameron-Kantor conjecture [19] by Liebeck and Shalev [80, and of Cameron's follow-up conjecture giving a value for the associated constant [18] by many authors [11, 13, 14, 15]. These results mean that, in the almost simple case, work is only required for the so-called "standard actions".

Theorem 1.5.2 is an analogue of an existing result for $\mathrm{b}(G)$ [70; now (1.5.1) and Lemma 1.5.1 yield analogues for $\mathrm{B}(G)$ and $\mathrm{RC}(G)$. With this result for $\mathrm{RC}(G)$, and with the proof of Conjecture 1.1, we now have a good handle on those permutation groups $G$ for which $\operatorname{RC}(G)$ is either very large, or as small as possible. In the case where $\operatorname{RC}(G)$ is large, work remains to be done to ascertain the relational complexity of the groups listed at item (1) of the theorem; the most important results in this direction can be found in [26], and we summarised some of these above in Theorem 1.2.8.

The relationship between the various statistics occurring in (1.5.1), and between these statistics and $\mathrm{RC}(G)$ is an intriguing area of investigation, although not one that has hitherto received much attention. Cherlin and Wiscons have started to study some of these questions, and we mention two of their remarks [23):

1. From computational evidence, it appears that $\mathrm{RC}(G)$ and $H(G)$ are "close" (say, $\mathrm{RC}(G) \geq \mathrm{H}(G)-3)$. The obvious exceptions to this rule of thumb are the symmetric groups in their natural action; more generally, among primitive groups of degree at most 100 , the only groups for which $\mathrm{RC}(G)<\mathrm{H}(G)-3$ are various members of the family listed at item (1) of Theorem 1.5.2,
2. Again, from computational evidence, more often than not, it appears that $\mathrm{B}(G)$ and $\mathrm{H}(G)$ coincide for primitive groups. Moreover, for all primitive groups of degree at most $100, \mathrm{H}(G)-\mathrm{B}(G) \leq 3$.

We shy away from making conjectures about the general pattern for larger $n$ but, still, these lines of inquiry seem promising.

### 1.6 Methods: basic lemmas

Most of the results in this section were first written down in [34, 45, 46]. All of these papers were focused on showing that certain group actions are not binary, hence the lemmas we present here tend to yield lower bounds for relational complexity.

As always $G$ is a group acting on a set $\Omega$. In what follows, we will write $I, J \in \Omega^{n}$ to mean that $n \geq 2$ is a positive integer and $I, J$ are elements of $\Omega^{n}$; we will always assume that $I=\left(I_{1}, \ldots, I_{n}\right)$ and $J=\left(J_{1}, \ldots, J_{n}\right)$. We will write $I \widetilde{k} J$ to mean that the pair $(I, J)$ is $k$-subtuple complete; we will write $I \widetilde{k, n} J$ to mean that the pair $(I, J)$ is $k$-subtuple complete but not $n$-subtuple complete with respect to the action of $G$.

### 1.6.1 Relational complexity and subgroups

Examples 1.2 .1 and 1.2 .2 serve as a warning that relational complexity can behave badly with respect to arbitrary subgroups of the group $G$. Nonetheless, something can still be said.

Lemma 1.6.1. Let $G$ be a transitive permutation group on $\Omega$ and let $M$ be a point-stabilizer in this action. Let $\Lambda$ be a non-trivial orbit of $M$. Then

$$
\mathrm{RC}(G, \Omega) \geq \mathrm{RC}(M, \Lambda)
$$

Note, in particular, that if $G$ is binary, then the action of $M$ on all non-trivial suborbits must be binary. This will be useful later, particularly when we consider actions in which $G$ is very large and $M$ relatively small (for instance, $G=E_{8}(2)$, and $M=\operatorname{Aut}\left(\mathrm{PSU}_{3}(8)\right)$ ), in which case it is sometimes possible to use magma to list all of the transitive binary actions of $M$.

Proof. Write $\alpha$ for an element of $\Omega$ stabilized by $M$. Let $r=\operatorname{RC}(M, \Lambda)$; then there exist $I, J \in \Lambda^{n}$ such that $I \underset{r-1, n}{\sim} J$ with respect to the action of $M$ on $\Lambda$. But now observe that if we define

$$
I^{*}=\left(\alpha, I_{1}, \ldots, I_{n}\right) \text { and } J^{*}=\left(\alpha, J_{1}, \ldots, J_{n}\right),
$$

then $I^{*} \underset{r-1, n+1}{\sim} J^{*}$, and the result follows.
We write $(G: M)$ here, and below, to mean the set of right cosets of $M$ in $G$.
Lemma 1.6.2. Let $M<H<G$. Then $\operatorname{RC}(G:(G: M)) \geq \operatorname{RC}(H,(H: M))$.
Proof. Write $r=\mathrm{RC}(H:(H: M))$, and observe that $\Lambda=(H: M)$ is a subset of $\Omega=(G: M)$. Then there exist $I, J \in \Lambda^{n}$ such that $I \underset{r-1, n}{\sim} J$ with respect to the action of $H$.

We must show that $I \underset{r-1, n}{\sim} J$ with respect to the action of $G$. That $I_{r-1}^{\sim} J$ with respect to the action of $G$ is immediate. Suppose that $I \widetilde{n} J$ with respect to the action of $G$. Then there exists $g \in G$ such that $I_{i}^{g}=J_{i}$ for all $i \in\{1, \ldots, n\}$. Since $I_{i}, J_{i} \in(H: M)$ for all $i \in\{1, \ldots, n\}$, we must have $g \in H$. But then $I \widetilde{n} J$ with respect to the action of $H$, which is a contradiction.

### 1.6.2 Relational complexity and subsets

For $\Lambda$ a subset of $\Omega$ we write $G_{\Lambda}$ for the set-wise stabilizer of $\Lambda$, and $G_{(\Lambda)}$ for the point-wise stabilizer of $\Lambda$. We write $G^{\Lambda}$ for the permutation group induced on $\Lambda$ by $G_{\Lambda}$; note that $G^{\Lambda} \cong G_{\Lambda} / G_{(\Lambda)}$.

In this section we present some results connecting $\operatorname{RC}(G, \Omega)$ with $\operatorname{RC}\left(G^{\Lambda}, \Lambda\right)$.
Definition 1.6.3. Let $t:=|\Omega|$. For $k \in \mathbb{Z}^{+}$with $k \geq 2$, we say that the action of $G$ on $\Omega$ is strongly non- $k$-ary if there exist $I, J \in \Omega^{t}$ such that $I \widetilde{k, t} J$, and all elements of $I$ (resp. $J$ ) are distinct.

Note that this definition requires the existence of $I, J \in \Omega^{t}$ with $I \widetilde{k, t} J$ and with every element of $\Omega$ occurring as an entry of $I$ (and, therefore, also of $J$ ). If $k=2$, then we tend to write strongly non-binary as a synonym for strongly non- $k$-ary.

The notion of a strongly non- $k$-ary set is connected to a classical notion in permutation group theory which was introduced by Wielandt [103].
Definition 1.6.4. Let $G \leq \operatorname{Sym}(\Omega)$ and let $k \in \mathbb{Z}^{+}$. The $k$-closure of $G$ is the set

$$
G^{(k)}=\left\{\sigma \in \operatorname{Sym}(\Omega) \mid \forall I \in \Omega^{k}, \text { there exists } g \in G, I^{g}=I^{\sigma}\right\} .
$$

We say that $G$ is $k$-closed if $G=G^{(k)}$.
Observe that $G^{(k)}$ is the largest subgroup of $\operatorname{Sym}(\Omega)$ that has the same orbits on the set of $k$-tuples of $\Omega$ as $G$. Now the connection with strongly non- $k$-ary sets is as follows.
Lemma 1.6.5. The group $G$ is strongly non-k-ary if and only if $G$ is not $k$-closed.
Proof. Write $\Omega:=\left\{\omega_{1}, \ldots, \omega_{t}\right\}$. If $G$ is not $k$-closed, then there exists $\sigma \in G^{(k)} \backslash G$. Now, it is easy to verify that $I:=\left(\omega_{1}, \ldots, \omega_{t}\right)$ and $J:=I^{\sigma}=\left(\omega_{1}^{\sigma}, \ldots, \omega_{t}^{\sigma}\right)$ are $k$-subtuple complete (because $\left.\sigma \in G^{(k)}\right)$ and are not $t$-subtuple complete (because $\sigma \notin G$ ). Thus $I_{k, t} J$, and we conclude that the action of $G$ on $\Omega$ is strongly non- $k$-ary. The converse is similar.

The most important example, for us, of a permutation group that is not $k$-closed is as follows.
Example 1.6.6. Let $G$ be a $k$-transitive permutation group on $\Omega$, for some integer $k \geq 2$. The definition implies that $G^{(k)}=\operatorname{Sym}(\Omega)$.

We immediately conclude that $\operatorname{Alt}(\Omega)$ is not $(t-2)$-closed, and we obtain (again) that $\operatorname{RC}(\operatorname{Alt}(\Omega), \Omega) \geq$ $t-1$.

Recall that the Classification of Finite Simple Groups implies that examples of $k$-transitive permutation groups that do not contain $\operatorname{Alt}(\Omega)$ only exist for $k \leq 5$. What is more, all such groups are classified for $k \geq 2$ (see, for instance [38, §7.7]).

The next lemma shows how we will use the notion of a strongly non- $k$-ary permutation group in what follows.

Lemma 1.6.7. Let $\Lambda \subseteq \Omega$. If $G^{\Lambda}$ is strongly non-k-ary, then $\operatorname{RC}(G, \Omega)>k$.
Proof. Suppose that $|\Lambda|=\ell$, and let $I, J$ be $\ell$-tuples of distinct elements of $\Lambda$ such that $I \widetilde{k, \ell} J$ with respect to the action of $G^{\Lambda}$. It is enough to show that $I \widetilde{k, \ell} J$ with respect to the action of $G$. It is immediate that $I \widetilde{k} J$ with respect to the action of $G$. On the other hand, if $I \widetilde{\ell} J$, then there exists $g \in G$ such that $I^{g}=J$. Since $I$ contains all elements of $\Lambda$, we conclude that $g \in G_{\Lambda}$ which contradicts the fact that $I \chi_{\ell} J$ with respect to the action of $G^{\Lambda}$.

### 1.6.3 Strongly non-binary subsets

Our final few results apply specifically to the study of binary actions. As usual $G$ acts on a set $\Omega$, and we refer to a subset $\Lambda \subseteq \Omega$ as strongly non-binary if $G^{\Lambda}$ is strongly non-binary.

The next lemma details our first example of such a subset. This example was first described in [46]; its key properties are a consequence of Example 1.6 .6 and Lemma 1.6.7.
Lemma 1.6.8. Suppose that there exists a subset $\Lambda \subseteq \Omega$ such that $|\Lambda| \geq 2$ and $G^{\Lambda}$ is a 2-transitive proper subgroup of $\operatorname{Sym}(\Lambda)$. Then $G^{\Lambda}$ is strongly non-binary and the action of $G$ on $\Omega$ is not binary.

In subsequent chapters, our focus is on proving that certain actions are not binary. Lemma 1.6 .8 means that we will be interested in finding subsets which have 2-transitive set-wise stabilizers. The next lemma requires no proof, but we include it as it clarifies when such subsets exist.

Lemma 1.6.9. Let $K$ be some 2-transitive group, and let $K_{0}$ be a point-stabilizer in $K$. Let $H$ be a subgroup of $G$ and suppose that $\varphi: H \rightarrow K$ is a surjective homomorphism. Let $M$ be the stabilizer in $G$ of a point $\omega \in \Omega$ and let $C$ be the core of $H \cap M$ in $H$. If $\operatorname{Ker}(\varphi)=C$ and $\varphi(H \cap M)=K_{0}$, then $H$ acts 2 -transitively on the orbit $\omega^{H}$.

The next lemma is a useful tool in finding subsets on which a set-stabilizer acts 2 -transitively (recall that, when $r \geq 2$, the affine special linear group $\operatorname{ASL}_{r}(q)$ is 2-transitive in its natural action on $q^{r}$ points).
Lemma 1.6.10. Let $G$ be a finite group acting transitively on a set $\Omega$ with point-stabilizer $M$, and suppose that the following two conditions hold:
(i) $M$ has a subgroup $A \cong \operatorname{SL}_{r}(q)$, where $r \geq 2$, and
(ii) $G$ has a subgroup $S$ that is a central quotient of $\mathrm{SL}_{r+1}(q)$, such that $A \leq S$ (the natural completely reducible embedding) and $S \not \leq M$.
Then there is a subset $\Delta$ of $\Omega$ such that $|\Delta|=q^{r}$ and $G^{\Delta} \geq \operatorname{ASL}_{r}(q)$.
Proof. We have $A \leq S \cap M<S$. Since $A$ is embedded in $S$ via the natural completely reducible embedding, we have $S \cap M \leq P_{i}(S)$ with $i \in\{1, r\}$, where $P_{i}(S)$ is a maximal parabolic subgroup of $S$ stabilizing a 1-dimensional or an $r$-dimensional subspace. Say $i=1$ (the case $i=r$ is entirely similar). Then writing matrices with respect to a suitable basis,

$$
S \cap M \leq P_{1}(S)=\left\{\left(\begin{array}{cc}
Y & v \\
0 & \lambda
\end{array}\right): Y \in \mathrm{GL}_{r}(q), v \in \mathbb{F}_{q}^{r}, \operatorname{det}(Y) \lambda=1\right\}
$$

where $A$ is the subgroup obtained by setting $\lambda=1, \operatorname{det}(Y)=1$ and $v=0$. Define

$$
U=\left\{\left(\begin{array}{ll}
I & 0 \\
a & 1
\end{array}\right): a \in \mathbb{F}_{q}^{r}\right\}
$$

and set $\Delta=\{M u: u \in U\} \subseteq \Omega$ (where we identify $\Omega$ with the set ( $G: M$ ) of right cosets of $M$ in $G$ ).
Since $M \cap U=1$, the cosets $M u(u \in U)$ are all distinct, and so $|\Delta|=q^{r}$. Since $A$ normalizes $U$ and $A \leq M$, the subgroup $U A \cong q^{r} . \operatorname{SL}_{r}(q)$ stabilizes $\Delta$, and since $U A \cap M=A$, we have $(U A)^{\Delta}=\operatorname{ASL}_{r}(q) \leq$ $G^{\Delta}$.

It turns out that in the context of almost simple groups, it is convenient to use a variant of Lemma 1.6.8 where we don't just seek proper 2-transitive subgroups of $\operatorname{Sym}(\Omega)$, but also exclude $\operatorname{Alt}(\Omega)$ from our consideration. To that end we include the following definition which first appeared in [46.

Definition 1.6.11. A subset $\Lambda \subseteq \Omega$ is a $G$-beautiful subset if $G^{\Lambda}$ is a 2 -transitive subgroup of $\operatorname{Sym}(\Lambda)$ that is isomorphic to neither $\operatorname{Alt}(\Lambda)$ nor $\operatorname{Sym}(\Lambda)$.

In what follows, if the group $G$ is clear from the context, we will speak of a beautiful subset rather than a $G$-beautiful subset of $\Omega$. Observe that a beautiful subset of $\Omega$ is a strongly non-binary subset. The reason for the stronger definition is explained by the following result.

Lemma 1.6.12. Suppose that $G$ is almost simple with socle $S$. If $\Omega$ contains an $S$-beautiful subset, then $G$ is not binary.

Proof. Let $\Lambda$ be an $S$-beautiful subset and observe that $\Lambda$ has cardinality at least 5 . Then, since $S$ is normal in $G$, the group $\left(S_{\Lambda} G_{(\Lambda)}\right) / G_{(\Lambda)}$ is a normal subgroup of $G_{\Lambda} / G_{(\Lambda)}$. This implies that $G_{\Lambda} / G_{(\Lambda)}$ is (isomorphic to) a 2 -transitive proper subgroup of $\operatorname{Sym}(\Lambda)$. Then Lemma 1.6 .8 implies that $G$ is not binary.

Although in this paper we do not need to deal with $\mathcal{C}_{1}$-actions for classical groups since they were dealt with in [46, we include the next lemma because it clearly illustrates the beautiful subsets method. The lemma has the added advantage of giving the reader an idea of how to deal with $\mathcal{C}_{1}$-actions in general. (These actions all yield to the method of beautiful subsets provided $n$ and $q$ are not too small.)

Lemma 1.6.13. Let $S=\operatorname{PSL}_{n}(q)$ and for $n=2$ assume $q>5$. Let $M$ be a maximal parabolic subgroup of $S$, and let $\Omega$ be the set of right cosets of $M$. Then $\Omega$ contains an $S$-beautiful subset.

Proof. Here $M$ is the stabilizer of a subspace $W$ of $V$, where $V$ is the natural $n$-dimensional module for $\mathrm{SL}_{n}(q)$. Since the action of $S$ on the $k$-dimensional subspaces of $V$ is permutation isomorphic to the action on the $(n-k)$-subspaces of $V$, we may assume that $\operatorname{dim}(W) \leq n / 2$.

If $\operatorname{dim}(W)=1$, then the action of $S$ on $\Omega$ is 2 -transitive. Now $\Omega$ itself is an $S$-beautiful subset, because we are assuming $q>5$ when $n=2$.

Suppose next that $\operatorname{dim}(W)>1$. Observe that this implies that $n \geq 4$. Let $W^{\prime}$ be a subspace of $W$ with $\operatorname{dim}\left(W^{\prime}\right)=\operatorname{dim}(W)-1$ and consider $\Lambda=\left\{W^{\prime \prime} \leq V \mid W^{\prime} \subset W^{\prime \prime}, \operatorname{dim}\left(W^{\prime \prime}\right)=\operatorname{dim}(W)\right\}$. Clearly, $S_{\Lambda}=\operatorname{Stab}_{S}\left(W^{\prime}\right)$ and the action of $S^{\Lambda}$ on $\Lambda$ is permutation isomorphic to the natural 2-transitive action of $\mathrm{GL}\left(V / W^{\prime}\right)$ on the 1 -dimensional subspaces of $V / W^{\prime}$. Since $\operatorname{dim}\left(V / W^{\prime}\right) \geq 3$, the action of $\mathrm{GL}\left(V / W^{\prime}\right)$ induces neither the alternating nor the symmetric group on the set $P_{1}\left(V / W^{\prime}\right)$ of 1-dimensional subspaces of $V / W^{\prime}$; therefore $\Lambda$ is a beautiful subset.

Our second example of a strongly non-binary subset is taken from [45, Example 2.2]
Example 1.6.14. Let $G$ be a subgroup of $\operatorname{Sym}(\Omega)$, let $g_{1}, g_{2}, \ldots, g_{r}$ be elements of $G$, and let $\tau, \eta_{1}, \ldots, \eta_{r}$ be elements of $\operatorname{Sym}(\Omega)$ with

$$
g_{1}=\tau \eta_{1}, g_{2}=\tau \eta_{2}, \ldots, g_{r}=\tau \eta_{r} .
$$

Suppose that, for every $i \in\{1, \ldots, r\}$, the support of $\tau$ is disjoint from the support of $\eta_{i}$; moreover, suppose that, for each $\omega \in \Omega$, there exists $i \in\{1, \ldots, r\}$ (which may depend upon $\omega$ ) with $\omega^{\eta_{i}}=\omega$. Suppose, in addition, $\tau \notin G$. Now, writing $\Omega=\left\{\omega_{1}, \ldots, \omega_{t}\right\}$, observe that

$$
\left(\left(\omega_{1}, \omega_{2}, \ldots, \omega_{t}\right),\left(\omega_{1}^{\tau}, \omega_{2}^{\tau}, \ldots, \omega_{t}^{\tau}\right)\right)
$$

is a non-binary witness. Thus the action of $G$ on $\Omega$ is strongly non-binary.

The next two lemmas which are taken from [34 are based on Example 1.6.14. In both cases, the given assumptions on the permutation group $G$ are enough to conclude that a strongly non-binary subset of the type described in Example 1.6 .14 must exist. In both lemmas, given a permutation or a permutation group $X$ on $\Omega$, we let $\operatorname{Fix}_{\Omega}(X)$ define the subset of $\Omega$ fixed point-wise by $X$; if $\Omega$ is clear from the context, we drop the label $\Omega$.

Lemma 1.6.15 ([34, Lemma 2.5]). Let $G$ be a transitive permutation group on $\Omega$, let $\alpha \in \Omega$ and let $p$ be a prime with $p$ dividing both $|\Omega|$ and $\left|G_{\alpha}\right|$ and with $p^{2}$ not dividing $\left|G_{\alpha}\right|$. Suppose that $G$ contains an elementary abelian p-subgroup $V=\langle g, h\rangle$ with $g \in G_{\alpha}$, with $\langle h\rangle$ and $\langle g h\rangle$ conjugate to $\langle g\rangle$ via $G$. Then $G$ is not binary.

In [34, Lemma 2.5], the hypothesis actually requires that $h$ and $g h$ are conjugate to $g$ via $G$; however the same proof yields the conclusion that $G$ is not binary under the weaker assuption that $\langle h\rangle$ and $\langle g h\rangle$ are conjugate to $\langle g\rangle$ in $G$, as stated in the lemma. We will need this strengthening in what follows.

Lemma 1.6.16 ([34, Lemma 2.6]). Let $G$ be a permutation group on $\Omega$ and suppose that $g$ and $h$ are $G$-conjugate elements of prime order $p$, and $g h^{-1}$ is conjugate to $g$ (and so to $h$ ). Suppose that $V=\langle g, h\rangle$ is elementary abelian of order $p^{2}$. Suppose, finally, that $G$ does not contain any elements of order $p$ that fix more points of $\Omega$ than $g$. If $|\operatorname{Fix}(V)|<|\operatorname{Fix}(g)|$, then $G$ is not binary.

### 1.7 Methods: Frobenius groups

It turns out that the presence of Frobenius groups can be a powerful tool in proving that certain actions are not binary. We give three lemmas in this direction; the first was proved independently by Wiscons, although a proof has not appeared in the literature ${ }^{6}$

Lemma 1.7.1. Let $G$ be a Frobenius permutation group on $\Omega$ (that is, $G$ acts transitively on $\Omega, G_{\omega} \neq 1$ for every $\omega \in \Omega$ and $G_{\omega \omega^{\prime}}=1$ for every $\omega, \omega^{\prime} \in \Omega$ with $\omega \neq \omega^{\prime}$ ). If $G$ is binary, then a Frobenius complement has order equal to 2 .

Proof. Throughout this proof we write $G=N \rtimes H$ where $N$ is the Frobenius kernel, and $H$ is a Frobenius complement (and point-stabilizer). Suppose that $G$ is binary. Let $a$ and $b$ be distinct non-trivial elements of $N$. We claim that the binary condition on triples implies that

$$
H H^{a} \cap H H^{b}=H .
$$

To see this, assume $H$ stabilizes $\alpha \in \Omega$. Let $\beta \in \alpha H^{a} \cap \alpha H^{b}$. Then the tuples $\left(\alpha, \alpha^{a}, \alpha^{b}\right) \widetilde{\widetilde{2}}\left(\beta, \alpha^{a}, \alpha^{b}\right)$. As $G$ is binary, there exists $g \in G$ mapping the first tuple to the second. Then $g \in H^{a} \cap H^{b}$ and since the action is Frobenius, $g=1$ and hence $\alpha=\beta$. So $\alpha H^{a} \cap \alpha H^{b}=\alpha$, and considering the isomorphic action on cosets of $H$ yields $H H^{a} \cap H H^{b}=H$.

We will show that if $|H|>2$, then this equality cannot hold. Suppose, then, that $h_{1}, h_{2}, h_{3}, h_{4} \in H$ are such that

$$
\begin{equation*}
h_{1} a^{-1} h_{2} a=h_{3} b^{-1} h_{4} b . \tag{1.7.1}
\end{equation*}
$$

Observe first that if the element represented by the two sides of this equation is equal to an element of $H$, then $a^{-1} h_{2} a$ must also be an element of $H$ and so $h_{2}$ is equal to 1 , as is $h_{4}$, and in addition $h_{1}=h_{3}$.

Thus it suffices to find a solution to (1.7.1) for which $h_{1} \neq h_{3}$. To do this we start by rearranging to obtain that

$$
h_{3}^{-1} h_{1} a^{-1} h_{2} a h_{4}^{-1}=b^{-1} h_{4} b h_{4}^{-1}
$$

[^5]and observe that the right-hand side lies in $N$. Thus the left-hand side lies in $N$. Doing some rearranging we find that the left-hand side can be rewritten as
$$
\left(h_{3}^{-1} h_{1} a^{-1} h_{1}^{-1} h_{3}\right)\left(h_{3}^{-1} h_{1} h_{2} a h_{2}^{-1} h_{1}^{-1} h_{3}\right)\left(h_{3}^{-1} h_{1} h_{2} h_{4}^{-1}\right) .
$$

Since the first two bracketed terms lie in $N$, we conclude that $h_{3}^{-1} h_{1} h_{2} h_{4}^{-1}=1$. This allows us to replace $h_{4}$ in (1.7.1) to get

$$
h_{1} a^{-1} h_{2} a=h_{3} b^{-1} h_{3}^{-1} h_{1} h_{2} b,
$$

which we rearrange one last time to obtain

$$
\begin{equation*}
\left(a^{-1}\right)\left(h_{2} a h_{2}^{-1}\right)\left(h_{2} b^{-1} h_{2}^{-1}\right)=h_{1}^{-1} h_{3} b^{-1} h_{3}^{-1} h_{1} \tag{1.7.2}
\end{equation*}
$$

where, again, we have put terms that lie in $N$ in brackets.
Now, for $h \in H \backslash\{1\}$, we define

$$
\begin{aligned}
\phi_{h}: & N \rightarrow N \\
n & \mapsto n^{-1} h n h^{-1} .
\end{aligned}
$$

We claim that this map is a bijection. We need only show injectivity: suppose that $n_{1}, n_{2} \in N$ with

$$
n_{1}^{-1} h n_{1} h^{-1}=n_{2}^{-1} h n_{2} h^{-1} .
$$

Then $n_{2} n_{1}^{-1} h n_{1} n_{2}^{-1}=h$ and hence $n_{1} n_{2}^{-1}$ centralizes $h$. Since we have a Frobenius action, we obtain that $n_{1}=n_{2}$, as required.

Now fix $b \in N$ and $h_{2} \in H \backslash\{1\}$ and consider (1.7.2). The first two bracketed terms correspond to $\phi_{h_{2}}(a)$ and the surjectivity of the function $\phi_{h_{2}}$ implies that the left-hand side of (1.7.2) ranges over all values of $N$ as $a$ varies across $N$. Recall, though, that we require that $a \neq b$ : this restriction tells us that the left-hand side equals all but one of the elements of $N$, as $a$ varies.

On the other hand if $H$ has orbits of size at least 3, we obtain that (1.7.2) has a solution in which $h_{1} \neq h_{3}$. We are done.

Lemma 1.7.2. Let $F \triangleleft G \leq \operatorname{Sym}(\Omega)$ with $F$ having an orbit $\Lambda \subseteq \Omega$ on which it acts as a Frobenius group. (As usual, $F^{\Lambda}$ is the permutation group induced by the action of $F$ on $\Lambda$.) Write $F^{\Lambda}=T \rtimes C$, where $T$ is the Frobenius kernel, and $C$ is a Frobenius complement. If $T$ is cyclic, and $C$ contains an element $x$ of order strictly greater than 2 , then $G$ is not binary.

Proof. Let $\alpha \in \Lambda$. Since $\Lambda$ is a block of imprimitivity for $G$, the group $G_{\alpha}$ must preserve $\Lambda$ set-wise. Observe that $G_{\Lambda}$ normalizes $F$, because $F \unlhd G$. In particular, $F^{\Lambda} \unlhd G^{\Lambda}$. Since the non-identity elements of $T$ are precisely those elements of $F^{\Lambda}$ that are fixed-point-free, $G^{\Lambda}$ also normalizes $T$. Thus $T$ is a regular normal subgroup of $G^{\Lambda}$. As $T$ acts regularly on $\Lambda$, from the Frattini argument we obtain $G^{\Lambda}=T \rtimes G_{\alpha}^{\Lambda}$.

We can, therefore, identify $T$ with $\Lambda$ in such a way that the action of $G_{\alpha}^{\Lambda}$ on $\Lambda$ is permutation isomorphic to the conjugation-action of $G_{\alpha}^{\Lambda}$ on $T$. To see this, define

$$
\begin{aligned}
\theta: T & \rightarrow \Lambda \\
y & \mapsto \alpha^{y},
\end{aligned}
$$

and observe that, for $y \in T$ and $g \in G_{\alpha}$,

$$
\theta\left(y^{g}\right)=\alpha^{\left(y^{g}\right)}=\alpha^{g^{-1} y g}=\alpha^{y g}=\left(\alpha^{y}\right)^{g}=(\theta(y))^{g} .
$$

With this set-up, we write $n=|T|$ and we let $y$ be a generator of $T$. We will construct (for the action of $G$ ) a 2-subtuple complete pair of the form

$$
\begin{equation*}
\left(\left(1, y, y^{a}\right),\left(1, y, y^{b}\right)\right) \tag{1.7.3}
\end{equation*}
$$

We must choose $a$ and $b$ appropriately. Let $x \in C$ having order strictly greater than 2 . First, let $k \in \mathbb{Z}^{+}$ be such that $y^{x}=y^{k}$; note that $\operatorname{gcd}(k, n)=1$, and so $k$ is invertible modulo $n$. Then we set $a=\frac{1+k}{k} \in \mathbb{Z}_{n}$ and set $b=1+k$. Now observe that

$$
\begin{aligned}
(1, y)^{\mathrm{id}} & =(1, y) ; \\
\left(1, y^{a}\right)^{x} & =\left(1, y^{(k+1) / k}\right)^{x}=\left(1, y^{k+1}\right)=\left(1, y^{b}\right) ; \\
\left(y, y^{a}\right)^{y^{-1} x^{2} y} & =\left(y, y^{(k+1) / k}\right)^{y^{-1} x^{2} y}=\left(1, y^{1 / k}\right)^{x^{2} t}=\left(1, y^{k}\right)^{y}=\left(y, y^{k+1}\right)=\left(y, y^{b}\right) .
\end{aligned}
$$

We see immediately that the pair (1.7.3) is 2 -subtuple complete.
Note on the other hand that, provided $a \neq b$, this pair cannot be 3 -subtuple complete: suppose that an element $g \in G$ sends the first triple in (1.7.3) to the second. Then $g$ fixes 1 and, as we saw above, this means that the action of $g$ on $\Lambda$ is isomorphic to the action of $g$ by conjugation on $T$. Since $y^{g}=y$, we conclude that, if $\left(y^{a}\right)^{g}=y^{b}$, then we must have $a=b$ modulo $n$. But now observe that

$$
a=b \Longleftrightarrow \frac{1+k}{k}=1+k \Longleftrightarrow k^{2}=1
$$

Since we chose $x$ to have order strictly greater than 2 , we see that $k^{2} \neq 1$, and we conclude that (1.7.3) is a pair which is 2 -subtuple complete but not 3 -subtuple complete. The result follows.

Lemma 1.7.3. Let $F=T \rtimes C \leq G \leq \operatorname{Sym}(\Omega)$ with $C$ acting by conjugation fixed-point-freely on $T$. Suppose there exists $\alpha \in \Omega$ such that $F_{\alpha}=C$, and let $\Lambda$ be the orbit of $\alpha$ under $F$. Define

$$
m:=\min \left\{\left|G_{\gamma_{1}, \gamma_{2}}\right| \mid \gamma_{1}, \gamma_{2} \text { distinct elements of } \Lambda\right\} .
$$

If $\left\lceil\frac{(|C|-1)(|C|-2)}{|\Lambda|-2}\right\rceil \geq m$, then $G$ is not binary. In particular, if $|G: F| \leq\left\lceil\frac{(|C|-1)(|C|-2)}{|\Lambda|-2}\right\rceil$, then $G$ is not binary.

Proof. Observe that $F$ acts as a Frobenius group on $\Lambda$, where $T$ is the Frobenius kernel, and $C$ is a Frobenius complement. It is useful to observe that the regularity of $T$ on $\Lambda$ implies that, for every $c \in C$ and for every $\beta \in \Lambda$, there exists a unique $x \in T$ such that $\beta^{x c}=\beta$.

We study triples of the form

$$
\begin{equation*}
((\alpha, \beta, \gamma),(\alpha, \beta, \delta)) \tag{1.7.4}
\end{equation*}
$$

for $\alpha, \beta, \gamma, \delta \in \Lambda$. We make the following claim:
Claim: for any distinct pair of elements $(\alpha, \beta)$, there are at least $(|C|-1)(|C|-2)$ choices for $(\gamma, \delta)$ such that the set $\{\alpha, \beta, \gamma, \delta\}$ has size 4 , and the pair (1.7.4) is 2 -subtuple complete.

Proof of claim: First we consider the set of pairs of distinct non-trivial elements in $C$, i.e.

$$
C^{(2)}:=\left\{\left(c_{1}, c_{2}\right) \mid c_{1}, c_{2} \in C \backslash\{1\}, c_{1} \neq c_{2}\right\} .
$$

Now we construct a function $\phi: C^{(2)} \rightarrow \Omega^{2}$ as follows. For $\left(c_{1}, c_{2}\right) \in C^{(2)}$, we let $t_{1}$ be the unique nontrivial element of $T$ such that $t_{1} c_{1} \in G_{\beta}$. Now, since $c_{1} \neq c_{2}$, we can define $\gamma$ to be the unique point in $\Lambda$ fixed by $t_{1} c_{1} c_{2}^{-1}$. Observe that $\gamma$ is distinct from both $\alpha$ and $\beta$.

Next, we see that

$$
\gamma^{t_{1} c_{1} c_{2}^{-1}}=\gamma \Longleftrightarrow \gamma^{t_{1} c_{1}}=\gamma^{c_{2}} .
$$

We define $\delta:=\gamma^{c_{2}}$, and we set $\phi\left(c_{1}, c_{2}\right)=(\gamma, \delta)$. An easy argument shows that $\delta$ is distinct from all of $\alpha, \beta$ and $\gamma$. Furthermore we claim that, with these definitions the pair (1.7.4) is 2 -subtuple complete. Indeed, observe that

$$
(\alpha, \beta)^{1}=(\alpha, \beta),(\alpha, \gamma)^{c_{2}}=(\alpha, \delta) \text { and }(\beta, \gamma)^{t_{1} c_{1}}=(\beta, \delta)
$$

Thus every element $(\gamma, \delta)$ in the image of $\phi$ gives rise to a 2 -subtuple complete pair as in (1.7.4). Since the domain of $\phi, C^{(2)}$ has order $(|C|-1)(|C|-2)$, the claim will follow if we prove that $\phi$ is one-to-one.

Suppose, then, that $\phi\left(c_{1}, c_{2}\right)=(\gamma, \delta)=\phi\left(c_{1}^{\prime}, c_{2}^{\prime}\right)$. Let $t_{1}$ (resp. $t_{1}^{\prime}$ ) be the unique element of $T$ such that $t_{1} c_{1}$ (resp. $t_{1}^{\prime} c_{1}^{\prime}$ ) is in $G_{\beta}$. Then $t_{1} c_{1} c_{2}^{-1}$ and $t_{1}^{\prime} c_{1}^{\prime}\left(c_{2}^{\prime}\right)^{-1}$ fix $\gamma$. What is more $\gamma^{c_{2}}=\gamma^{c_{2}^{\prime}}=\delta$ and so $c_{2}^{\prime} c_{2}^{-1}$ fixes $\gamma$. However $c_{2}, c_{2}^{\prime} \in C=F_{\alpha}$ and so $c_{2}^{\prime} c_{2}^{-1}$ fixes two points of $\Lambda$. We conclude that $c_{2}=c_{2}^{\prime}$. But now, write $h_{1}:=t_{1} c_{1}$ and $h_{1}^{\prime}:=t_{1}^{\prime} c_{1}^{\prime}$; observe that $h_{1}, h_{1}^{\prime} \in G_{\beta}$ and $\gamma^{h_{1}}=\gamma^{h_{1}^{\prime}}$. As before we conclude that $h_{1}^{\prime} h_{1}^{-1}$ fixes $\beta$ and $\gamma$, and so $h_{1}=h_{1}^{\prime}$. Then $t_{1} c_{1}=t_{1}^{\prime} c_{1}^{\prime}$ and so $t_{1}^{-1} t_{1}^{\prime}=c_{1}^{\prime} c_{1}^{-1}$; since $T \cap C=\{1\}$, this gives $c_{1}=c_{1}^{\prime}$, as required.

The claim and the pigeon-hole principle imply that there exists some $\gamma \in \Lambda \backslash\{\alpha, \beta\}$ for which there are $k:=\left\lceil\frac{(|C|-1)(|C|-2)}{|\Lambda|-2}\right\rceil$ choices for $\delta$ such that all pairs of the form (1.7.4) are 2 -subtuple complete; call these elements $\delta_{1}, \ldots, \delta_{k}$. If $G$ is binary, then all of these pairs are 3 -subtuple complete and we conclude that the set $\left\{\gamma, \delta_{1}, \ldots, \delta_{k}\right\}$ is a subset of an orbit of $G_{\alpha, \beta}$. But this is only possible if $k+1 \leq m$, and the result follows.

### 1.8 Methods: On computation

We will use magma very frequently in what follows to verify that certain actions are not binary. The methods we use to do this are largely drawn from [34]. We give a brief summary of some of the key methods here. In what follows $G$ acts transitively on the set $\Omega$, and $M$ is the stabilizer of a point.

Test 1: Using the permutation character. Given $\ell \in \mathbb{N} \backslash\{0\}$, we denote by $\Omega^{(\ell)}$ the subset of the Cartesian product $\Omega^{\ell}$ consisting of the $\ell$-tuples $\left(\omega_{1}, \ldots, \omega_{\ell}\right)$ with $\omega_{i} \neq \omega_{j}$, for every two distinct elements $i, j \in\{1, \ldots, \ell\}$. We denote by $r_{\ell}(G)$ the number of orbits of $G$ on $\Omega^{(\ell)}$. The next result is Lemma 2.7 of [34.

Lemma 1.8.1. If $G$ is transitive and binary, then $r_{\ell}(G) \leq r_{2}(G)^{\ell(\ell-1) / 2}$ for each $\ell \in \mathbb{N}$.
Let $\pi: G \rightarrow \mathbb{N}$ be the permutation character of $G$, that is, $\pi(g)=\operatorname{fix}_{\Omega}(g)$ where fix ${ }_{\Omega}(g)$ is the cardinality of the fixed point set $\operatorname{Fix}_{\Omega}(g):=\left\{\omega \in \Omega \mid \omega^{g}=\omega\right\}$ of $g$. From the Orbit Counting Lemma, we have

$$
\begin{aligned}
r_{\ell}(G) & =\frac{1}{|G|} \sum_{g \in G} \operatorname{fix}_{\Omega}(g)\left(\operatorname{fix}_{\Omega}(g)-1\right) \cdots\left(\operatorname{fix}_{\Omega}-(\ell-1)\right) \\
& =\langle\pi(\pi-1) \cdots(\pi-(\ell-1)), 1\rangle_{G}
\end{aligned}
$$

where 1 is the principal character of $G$ and $\langle\cdot, \cdot\rangle_{G}$ is the natural Hermitian product on the space of $\mathbb{C}$-class functions of $G$.

Clearly whenever the permutation character of $G$ is available in magma, we can directly check the inequality in Lemma 1.8.1, and this is often enough to confirm that a particular action is not binary.

Test 2: using Lemma 1.6.5, By connecting the notion of strong-non-binariness to 2 -closure, Lemma 1.6 .5 yields an immediate computational dividend: there are built-in routines in magma to compute the 2 -closure of a permutation group.

Thus if $\Omega$ is small enough, say $|\Omega| \leq 10^{7}$, then we can easily check whether or not the group $G$ is 2 -closed. Thus we can ascertain whether or not $G$ is strongly non-binary.

Test 3: a direct analysis. The next test we discuss is feasible once again provided $|\Omega| \leq 10^{7}$. It simply tests whether or not 2 -subtuple-completeness implies 3 -subtuple completeness, and the procedure is as follows:

We fix $\alpha \in \Omega$, we compute the orbits of $G_{\alpha}$ on $\Omega \backslash\{\alpha\}$ and we select a set of representatives $\mathcal{O}$ for these orbits. Then, for each $\beta \in \mathcal{O}$, we compute the orbits of $G_{\alpha} \cap G_{\beta}$ on $\Omega \backslash\{\alpha, \beta\}$ and we select a set of representatives $\mathcal{O}_{\beta}$. Then, for each $\gamma \in \mathcal{O}_{\beta}$, we compute $\gamma^{G_{\alpha}} \cap \gamma^{G_{\beta}}$. Finally, for each $\gamma^{\prime} \in \gamma^{G_{\alpha}} \cap \gamma^{G_{\beta}}$, we test whether the two triples $(\alpha, \beta, \gamma)$ and $\left(\alpha, \beta, \gamma^{\prime}\right)$ are $G$-conjugate. If the answer is "no", then $G$ is not binary because by construction $(\alpha, \beta, \gamma)$ and $\left(\alpha, \beta, \gamma^{\prime}\right)$ are 2 -subtuple complete. In particular, in this circumstance, we can break all the "for loops" and deduce that $G$ is not binary.

If the answer is "yes", for every $\beta, \gamma, \gamma^{\prime}$, then we cannot deduce that $G$ is binary, but we can keep track of these cases for a deeper analysis. We observe that, if the answer is "yes", for every $\beta, \gamma, \gamma^{\prime}$, then

2-subtuple completeness implies 3-subtuple completeness. At this point, we may either use a different method for checking whether the permutation group is genuinely binary or, with a similar method, we can check whether 3 -subtuple completeness implies 4 -subtuple completeness. This test is very expensive in terms of time, therefore before starting this whole procedure, we do a preliminary check: for $10^{6}$ times, we select $\beta, \gamma, \gamma^{\prime}$ as above at random and we check this random triple.

Test 4: studying suborbits. Lemma 1.6 .1 implies that if $G$ is binary, then the action of $M$ on any suborbit is also binary. This fact is particularly useful for computation in situations where the group $G$ is very large compared to the group $M$.

In general, our approach is to demonstrate that there must be some suborbit on which the action of $M$ is not binary. For instance, this would follow in the case where $|\Omega|=|G: M|$ is divisible by some integer $d$, and all non-trivial transitive binary actions of $M$ are also of degree divisible by $d$.

This last approach sometimes fails for just a few possible actions of $M$; in this situation, provided the action of $G$ on $\Omega$ is primitive, the following lemma is often useful.

Lemma 1.8.2 ([103, Theorem 18.2]). Suppose that $G$ is a finite primitive subgroup of $\operatorname{Sym}(\Omega)$. Let $\Gamma$ be a non-trivial orbit of $M$. Then, every simple section of $M$ is isomorphic to a section of the group $M^{\Gamma}$ which $M$ induces on $\Gamma$. In particular, each composition factor of $M$ is isomorphic to a section of $M^{\Gamma}$.

This lemma means that when studying possible suborbits of our action we may disregard the actions of $M$ (on a set $\Gamma$ say) where $M$ has a simple section not isomorphic to a section of the group $M^{\Gamma}$. If the resulting set of actions are all not binary, then we can conclude that the action of $G$ on $\Omega$ is also not binary. The method is summarised in Lemma 3.1 of [34]:

Lemma 1.8.3. Let $G$ be a primitive group on a set $\Omega$, let $\alpha$ be a point of $\Omega$, let $M$ be the stabilizer of $\alpha$ in $G$ and let $d$ be an integer with $d \geq 2$. Suppose $M \neq 1$ and, for each transitive action of $M$ on a set $\Lambda$ satisfying:

1. $|\Lambda|>1$, and
2. every composition factor of $M$ is isomorphic to some section of $M^{\Lambda}$, and
3. either $M_{(\Lambda)}=1$ or, given $\lambda \in \Lambda$, the stabilizer $M_{\lambda}$ has a normal subgroup $N$ with $N \neq M_{(\Lambda)}$ and $N \cong M_{(\Lambda)}$, and
4. $M$ is binary in its action on $\Lambda$,
we have that d divides $|\Lambda|$. Then either $d$ divides $|\Omega|-1$ or $G$ is not binary.
Test 5: special primes. We have turned Lemmas 1.6 .15 and 1.6 .16 into a routine in magma. Both of these lemmas are rather convenient from a computational point of view because they do not require us to construct the permutation representation of $G$ on $(G: M)$. For example, the only critical step in the routine for Lemmas 1.6 .15 and Lemma 1.6 .16 is the construction of the centraliser in $G$ of an element $g$ in $M$ of prime order $p$. There is a stardard built-in command in magma for constructing centralizers. Most often than not, this command is sufficient for our computations. However, for very large groups, where it is computationally out of reach to use a general command for computing centralizers, we have constructed $C_{G}(g)$ with $a d$ hoc methods exploiting the subgroup structure of the group $G$ under consideration.

Test 6: $M$ very small. This method draws on the following lemma.
Lemma 1.8.4 ([46, Lemma 2.5]). Let $\omega_{0}, \omega_{1}, \omega_{2} \in \Omega$ with $G_{\omega_{0}} \cap G_{\omega_{1}}=1$. Suppose there exists $g \in$ $G_{\omega_{0}} \cap G_{\omega_{2}} G_{\omega_{1}}$ with $g \notin G_{\omega_{2}}$. Then the two triples $\left(\omega_{0}, \omega_{1}, \omega_{2}\right)$ and $\left(\omega_{0}, \omega_{1}, \omega_{2}^{g}\right)$ are 2-subtuple complete but are not 3-subtuple complete. In particular, $G$ is not binary.

This method is particularly useful when $M\left(G_{\omega_{0}}\right.$ in Lemma 1.8.4) is small compared to $G$ because in this case it is more likely that $G_{\omega_{0}} \cap G_{\omega_{1}}=1$, for some $\omega_{1}$. This method also has the benefit that it does not
require us to construct the permutation representation of $G$ on $(G: M)$, and that all the computations are performed locally. Since this method is designed to deal with the case that ( $G: M$ ) is large compared to $M$, we do not exhaustively check all triples $\omega_{0}, \omega_{1}, \omega_{2} \in(G: M)$. In practice, we let $\omega_{0}:=M$, we generate at random $g_{1}, g_{2} \in G$, we let $G_{\omega_{1}}:=M^{g_{1}}$ and $G_{\omega_{2}}:=M^{g_{2}}$ and we check whether Lemma 1.8.4 applies to $\omega_{0}, \omega_{1}, \omega_{2}$. We repeat this routine $10^{5}$ times and if at some point we find a triple satisfying Lemma 1.8.4, then $G$ acting on $(G: M)$ is not binary and we stop the routine. If, after the $10^{5}$ trials, we have not found any triple satisfying Lemma 1.8.4, then we turn to a different method.

## Chapter 2

## Preliminary results for groups of Lie type

In this chapter we collect a number of results that will be needed when we come to prove Theorem 1.3., All of these results involve the finite groups of Lie type, so let us first establish the notation that we will use in this chapter and those that follow.

Our notation for the classical groups is standard and is consistent with, for instance, [54, Table 2.1.B]. We write, for example, $\mathrm{SO}_{n}^{+}(q)$ to mean a group of special isometries associated with a +-type quadratic form on an $n$-dimensional vector space over the finite field $\mathbb{F}_{q}$ having $q$ elements, and we write $\operatorname{PSO}_{n}^{+}(q)$ for the projective version of the same. We write $\mathrm{SO}_{n}^{ \pm}(q)$ or $\mathrm{SO}_{n}^{\varepsilon}(q)$ if we wish to allow the quadratic form to have either + or - type.

We shall also use the general notation $\mathrm{Cl}_{n}(q)$ to denote a quasisimple classical group with natural module of dimension $n$ over the field $\mathbb{F}_{q}$ (over $\mathbb{F}_{q^{2}}$ for unitary groups).

Our Lie notation is also standard: we write $A_{n}(q), B_{n}(q), C_{n}(q)$, and so on, for quasisimple groups of Lie type associated with Dynkin diagrams of type $A_{n}, B_{n}, C_{n}, \ldots ;$ similarly we write ${ }^{2} A_{n}(q),{ }^{2} B_{2}(q)$, and so on, for twisted versions of the same. Note that the Lie notation does not specify the group up to isomorphism in all cases. For instance, $A_{n}(q)$ can stand for both $\mathrm{SL}_{n+1}(q)$ and $\mathrm{PSL}_{n+1}(q)$.) We write $A_{n}^{-}(q), D_{n}^{-}(q)$ and $E_{6}^{-}(q)$ as alternative notation for ${ }^{2} A_{n}(q),{ }^{2} D_{n}(q)$ and ${ }^{2} E_{6}(q)$ respectively, and we write $A_{n}^{ \pm}(q), D_{n}^{ \pm}(q), E_{6}^{ \pm}(q)$ or $A_{n}^{\varepsilon}(q), D_{n}^{\varepsilon}(q), E_{6}^{\varepsilon}(q)$ if we wish to consider both the twisted and untwisted version at the same time.

The results collected here are of six kinds:

1. Results concerning alternating sections: We consider a simple group of Lie type, $G$, and we specify for which values of $r$ the alternating group, $\operatorname{Alt}(r)$, is a section of $G$. These results will be used later, in conjunction with Definition 1.6.11, when we study the primitive actions of $G$ one frequently-used method for showing that these actions are not binary will be to show that they exhibit a beautiful subset.
2. Stabilizer results: We consider a group $G$, and we consider all faithful transitive actions of $G$ in which the stabilizer of a point, $H$, contains a particular element $g$. We will prove that, for an appropriately chosen $G$ and $g$, such an action is always not binary. We call these "stabilizer results" because these lemmas will typically be applied in later chapters in contexts where $G$ is a pointstabilizer and we are seeking to use Lemma 1.6.1. These applications motivate the choices of $G$ which we consider in this section.
3. Odd degree results: We consider a group $M$, normally a small group of Lie type, and we use magma to show that all of the transitive actions of odd degree of $M$ are not binary. Although it is not about groups of Lie type, we also include one result - Lemma 2.3 .2 - which does the same thing for the sporadic groups.
4. Centralizer results: We will present a number of results giving lower bounds for the size of a centralizer of a non-trivial element in a simple group of Lie type.
5. Automorphism results: We present a well-known result classifying the outer automorphisms of prime ofrder of finite groups of Lie type.
6. Fusion and factorization results: All these results will be used in conjunction with Lemma 1.6.10 to prove the existence of beautiful subsets (Definition 1.6.11).

We will use the stabilizer results in two ways when it comes to the proof of Theorem 1.3 , For the proof we study an almost simple group $G$ acting on the cosets of a maximal subgroup $M$. Now, the first use of our stabilizer results is direct: if $M$ contains the element $g$, then we immediately know that the action is not binary and we are done.

The second use is slightly less direct. In this case, we wish to apply our stabilizer results to the group $M$, rather than the group $G$ : so we pick a distinguished element $g \in M$ and appeal to our stabilizer results to assert that if $H$ is any core-free subgroup of $M$ that contains $g$, then the action of $M$ on $(M: H)$ is not binary. Next we use our centralizer results, to show that, in general $\left|C_{M}(g)\right|$ is smaller than the smallest centralizer in $G$. We conclude that there exists $x \in C_{G}(g) \backslash C_{M}(g)$. Now $M \cap M^{x}$ is a core-free subgroup of $M$ that contains $g$. We conclude that the action of $M$ on $\left(M: M \cap M^{x}\right)$ is not binary. Then Lemma 1.6.1 implies that the action of $G$ on $(G: M)$ is not binary.

This second method explains the selection of groups under consideration for our stabilizer results: for instance the group $G$ appearing in Lemma 2.2.1 is studied because such a group is maximal in $E_{8}(q)$.

The second method also applies to the odd degree results: if we are studying the action of a group $G$ on the cosets of a subgroup $M$ and we know (a) that $|G: M|$ is even, (b) that all odd-degree actions of $M$ are not binary, then Lemma 1.6 .1 implies that the action of $G$ on $(G: M)$ is not binary.

### 2.1 Results on alternating sections

Let $G$ be a simple group of Lie type. We wish to know for which values of $r$ the alternating group, $\operatorname{Alt}(r)$, is a section of $G$.

We first consider classical groups.
Lemma 2.1.1. Let $\mathrm{Cl}_{n}(q)$ be a simple classical group with natural module of dimension $n$ and $p$ is a prime number. If $\mathrm{Cl}_{n}(q)$ has a section isomorphic to the alternating group $\operatorname{Alt}(r)$, then

$$
\begin{equation*}
n \geq R_{p}(\operatorname{Alt}(r)), \tag{2.1.1}
\end{equation*}
$$

where $R_{p}(\operatorname{Alt}(r))$ denotes the smallest dimension of a non-trivial projective representation of $\operatorname{Alt}(r)$ over a field of characteristic $p$. In particular, for $r \geq 9$, we have

$$
R_{p}(\operatorname{Alt}(r))=r-1-\delta,
$$

where

$$
\delta= \begin{cases}1, & \text { if } p \mid r \\ 0, & \text { otherwise }\end{cases}
$$

For $5 \leq r \leq 8$, the values for $R_{p}(\operatorname{Alt}(r))$ are as in Table 2.1.1.

| $r$ | $R_{2}(\operatorname{Alt}(r))$ | $R_{3}(\operatorname{Alt}(r))$ | $R_{5}(\operatorname{Alt}(r))$ | $R_{p}(\operatorname{Alt}(r)), p \geq 7$ |
| :---: | :---: | :---: | :---: | :---: |
| 5 | 2 | 2 | 2 | 2 |
| 6 | 3 | 2 | 3 | 3 |
| 7 | 4 | 4 | 3 | 4 |
| 8 | 4 | 7 | 7 | 7 |

Table 2.1.1: Values for $R_{p}(\operatorname{Alt}(r))$, with $5 \leq r \leq 8$

Proof. The inequality $R_{p}(\operatorname{Alt}(r))=r-1-\delta$ follows from [40, Proposition 4.1]. The values of $R_{p}(\operatorname{Alt}(r))$ are well-known (see [54, Proposition 5.3.7]).

If $G$ is exceptional, then the following lemma gives the result that we need. (Here, $\delta_{x, y}$ is the usual Kronecker delta.)

Lemma 2.1.2. Let $G=G(q)$ be a finite simple group of exceptional Lie type as in the table below, where $q=p^{a}, a \geq 1$ and $p$ is a prime number. If $\operatorname{Alt}(r)$ is a section of $G$, then $r \leq N_{G}$, where $N_{G}$ is as in the table below.

| $G$ | $E_{8}(q)$ | $E_{7}(q)$ | $E_{6}^{\epsilon}(q)$ | $F_{4}(q)$ | $G_{2}(q),{ }^{3} D_{4}(q)$ | ${ }^{2} F_{4}(q)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N_{G}$ | $17+\delta_{p, 3}$ | $13+\delta_{p, 7}$ | $11+\delta_{p, 2}+\delta_{p, 5}$ | $10+\delta_{p, 11}$ | $6+\delta_{p, 5}$ | 6 |

Proof. Fix $r \geq 5$, and let $K \triangleleft H \leq G$ with $H / K \cong \operatorname{Alt}(r)$, and $|H|$ minimal. Choose a minimal subfield $\mathbb{F}_{q_{0}} \subseteq \mathbb{F}_{q}$ such that $H \leq G\left(q_{0}\right)$, and a maximal subgroup $M$ of $G\left(q_{0}\right)$ such that $H \leq M$.

Consider first $G\left(q_{0}\right)={ }^{2} F_{4}\left(q_{0}\right)$. The maximal subgroups are given by [83], from which it follows that $\operatorname{Alt}(r)$ is a section of one of the groups $\operatorname{Sp}_{4}\left(q_{0}\right)$ or $\mathrm{PSU}_{3}\left(q_{0}\right)$. Hence by Lemma 2.1.1 we have $4 \geq R_{2}(\operatorname{Alt}(r))$, forcing $r \leq 8$. As Alt(7) is not a section of $\operatorname{Sp}_{4}\left(q_{0}\right)$ or $\operatorname{PSU}_{3}\left(q_{0}\right)$ (see, for instance, [10]), we in fact have $r \leq 6$, as in the conclusion.

The cases where $G\left(q_{0}\right)=G_{2}\left(q_{0}\right)$ or ${ }^{3} D_{4}\left(q_{0}\right)$ are dealt with similarly, using [29, 56, 57] for the lists of maximal subgroups.

Now consider the remaining cases, where $G$ is of type $E_{8}, E_{7}, E_{6}^{\epsilon}$ or $F_{4}$. By the minimality of $H$ we have $K \leq \Phi(H)$, where $\Phi(H)$ is the Frattini subgroup of $H$. So, $K$ is nilpotent.

Suppose $Z(H) \neq 1$, and let $1 \neq x \in Z(H)$. Then $H \leq C_{G}(x)$, which is contained in a parabolic or a subsystem subgroup, and it follows that $\operatorname{Alt}(r)$ is a section of one of the following subgroups of $G$ :

| $G$ | Alt $(r)$ section of one of |
| :---: | :---: |
| $F_{4}(q)$ | $B_{4}(q), C_{3}(q)$ |
| $E_{6}^{\epsilon}(q)$ | $A_{5}^{\epsilon}(q), D_{5}^{\epsilon}(q)$ |
| $E_{7}(q)$ | $A_{7}^{ \pm}(q), D_{6}(q), E_{6}^{ \pm}(q)$ |
| $E_{8}(q)$ | $D_{8}(q), A_{8}^{ \pm}(q), E_{7}(q)$ |

Working down from $F_{4}(q)$, the bounds in the conclusion now follow using Lemma 2.1.1. (Note that the possibilities $r=18$ in $E_{8}(p=2)$ and $r=14$ in $E_{7}(p=2)$ are excluded by the fact that $D_{8}\left(2^{a}\right)$ (resp. $D_{6}\left(2^{a}\right)$ ) does not have a section isomorphic to Alt(18) (resp. Alt(14)) (see [54, (5.3.8)]).

Suppose finally that $Z(H)=1$. If $Z(K)=1$, then $K=1$ (as $K$ is nilpotent), so $H \cong \operatorname{Alt}(r)$, and the conclusion follows from [77, Table 10.1]. So assume $Z(K) \neq 1$. If $p$ divides $|Z(K)|$, then $H$ is contained in a parabolic subgroup of $G$ by [8], a case already considered above. Hence we may assume that $Z(K)$ has order divisible by a prime $s$ with $s \neq p$. As $Z(H)=1$, it must be the case that $H / K \cong \operatorname{Alt}(r)$ acts non-trivially on the elementary abelian group $E=\Omega_{1}\left(O_{s}(Z(K))\right)$. Say $E \cong\left(C_{s}\right)^{\kappa}$, of rank $\kappa$. Then $\kappa \geq R_{s}(\operatorname{Alt}(r))$. On the other hand, [27] shows that $\kappa \leq R+1$, where $R$ is the untwisted Lie rank of $G$. Hence

$$
R_{s}(\operatorname{Alt}(r)) \leq R+1,
$$

and the bounds for $r$ in the conclusion follow from this. This completes the proof.

### 2.2 Stabilizers containing certain elements

In this section we prove results that are (more or less) of the following kind: we suppose that $x$ is an element of a group $G$, and we prove that, if $H$ is any core-free subgroup of $G$ containing $x$, then the action of $G$ on $(G: H)$ is not binary. In the first subsection we consider groups $G$ of a variety of isomorphism types; in subsequent subsections, $G$ will always be almost simple.

### 2.2.1 Some groups that are not almost simple

Lemma 2.2.1. Let $S=\mathrm{PGL}_{2}(q) \times \operatorname{Sym}(5)$ with $q>5$, and suppose that $S \unlhd G$ with $G / S$ solvable. Let $L$ be the normal subgroup in $S$ that is isomorphic to $\mathrm{PGL}_{2}(q)$, and suppose that $g \in L$ has order $q-1$; let $M$ be a subgroup of $G$ that contains $g$. If the action of $G$ on $(G: M)$ is binary, then $M$ contains $L$.

Proof. Assume that the action of $G$ on $(G: M)$ is binary. Notice that the element $g$ normalizes, and acts fixed-point-freely by conjugation upon two unipotent subgroups of $L$ of order $q$; we call these $U_{1}$ and $U_{2}$.

Suppose that $M$ does not contain $U_{1}$. Since $g \in M$, we have $M \cap U_{1}=\{1\}$. Now, we define $\Lambda=\{M u \mid$ $\left.u \in U_{1}\right\}$. It is easy to see that $U_{1} \rtimes\langle g\rangle$ acts 2-transitively on $\Lambda$, which is a subset of $(G: M)$ of size $q$. Since $G$ is binary on $(G: M)$, the group $G^{\Lambda}$ is isomorphic to the symmetric group of degree $q$. As $q>5$ and $G / S$ is solvable, by Lemma 2.1.1, $G$ has no section isomorphic to $\operatorname{Alt}(q)$, which is a contradiction.

Thus $M$ contains $U_{1}$ and, by the same reasoning, $U_{2}$. But now $\left\langle U_{1}, U_{2}, g\right\rangle=L$ and we are done.
Lemma 2.2.2. Let $S=\operatorname{PSL}_{2}(q) \times \operatorname{PSL}_{2}(q)$ with $q \geq 4$ and $q \neq 5$, and suppose that $S=F^{*}(G)$, where $F^{*}(G)$ is the generalized Fitting subgroup of $G$. Let $L$ be a subgroup of $S$ isomorphic to $D_{t(q-1)} \times D_{t(q-1)}$ (where $t=(2, q)$ and where $D_{t(q-1)}$ denotes the dihedral group of order $t(q-1)$ ), and let $M$ be a subgroup of $G$ that contains $L$. If the action of $G$ on $(G: M)$ is binary, then $M \geq S$.

Proof. We write $S=S_{1} \times S_{2}$ and $L=L_{1} \times L_{2}$, where $D_{t(q-1)} \cong L_{i}<S_{i} \cong \operatorname{PSL}_{2}(q)$ for $i \in\{1,2\}$. Assume first that $q \notin\{4,7,9,11\}$.

Suppose, first, that $M \cap S=L$; we must show that the action of $G$ on $(G: M)$ is not binary. Let $H=\langle M, S\rangle=M S$. Lemma 1.6 .2 implies that it is sufficient to show that the action of $H$ on $(H: M)$ is not binary. Now observe that

$$
H / S=M S / S \cong M /(M \cap S)=M / L
$$

Thus $|H: M|=|S: L|$ and we can identify $(H: M)$ with the set of conjugates of $L$ in $S$, by using the map

$$
(H: M) \rightarrow\left\{L^{s} \mid s \in S\right\}, \quad M g \mapsto L^{g} .
$$

Now define

$$
\Gamma=\left\{L_{1} \times L_{2}^{g} \mid g \in S_{2}\right\} .
$$

The intersection of the elements of $\Gamma$ is $L_{1}$ and so $H_{\Gamma} \leq N_{H}\left(L_{1}\right)$. Since the reverse inclusion is also true, we deduce

$$
\begin{equation*}
H_{\Gamma}=N_{H}\left(L_{1}\right) . \tag{2.2.1}
\end{equation*}
$$

Observe that the action of $H_{\Gamma}$ on $\Gamma$ is isomorphic to the action of an almost simple group with socle $S_{2}=\mathrm{PSL}_{2}(q)$ on the cosets of a subgroup $M_{2}$ for which $M_{2} \cap S_{2} \cong D_{t(q-1)}$. When $q \notin\{4,7,9,11\}$, the action of $H_{\Gamma}$ on $\Gamma$ is primitive by [10, Table 8.1] and hence the main theorem of [34] implies that this action is not binary. Thus there is an integer $k \geq 3$ and two $k$-tuples $I, J \in \Gamma^{k}$ that are 2-subtuple complete but not $k$-subtuple complete with respect to the action of $H^{\Gamma}$. Using (2.2.1), one can see that any $h \in H$ for which $I^{h}=J$ must satisfy $h \in H_{\Gamma}$, and so $I, J$ are not $k$-subtuple complete with respect to the action of $H$. Thus the action of $H$ on $(H: M)$ is not binary, and so the action of $G$ on $(G: M)$ is not binary, as required.

We conclude that $L$ is a proper subgroup of $M \cap S$. We may assume, without loss of generality, that $M \cap S$ contains $S_{1}$ but not $S_{2}$. Then the action of $G$ on $(G: M)$, modulo the kernel, is isomorphic to the action of an almost simple group with socle $S_{2}=\mathrm{PSL}_{2}(q)$ on the cosets of a maximal subgroup $M_{2}$ for which $M_{2} \cap S_{2} \cong D_{t(q-1)}$. Once again the main theorem of [34] implies that this action is not binary.

The only remaining possibility is that $M \geq S$, as required.
Assume now that $q \in\{4,7,9,11\}$. With the help of magma, we have constructed all the groups $G$ with $F^{*}(G)=S \cong \operatorname{PSL}_{2}(q) \times \operatorname{PSL}_{2}(q)$ and all the subgroups $H$ of $G$ containing $D_{t(q-1)} \times D_{t(q-1)}$. Then we have verified, by witnessing non-binary triples, that the action of $G$ on $(G: H)$ is binary only when $S \leq H$.

### 2.2.2 Classical groups

Lemma 2.2.3. Let $G$ be almost simple with socle $S=\operatorname{PSL}_{2}(q)$, and let $x$ be the projective image of an element $\tilde{x}$ as given in Table 2.2.1 line 1. Let $M<G$ be core-free with $x \in M \cap S$. Then, provided $q \notin\{4,5\}$, the action of $G$ on $(G: M)$ is not binary.

Proof. For $q \in\{7,9,11,13,27\}$, we confirm the result using magma. In particular, for the rest of the proof we suppose $q=8$ or $q>13$ with $q \neq 27$.

Let $d=(2, q-1)$. The stated conditions imply that $M \cap S \in\left\{C_{(q-1) / d}, D_{2(q-1) / d}, B\right\}$, where $B$ is a Borel subgroup of $S$. We set $q=p^{f}$, for a prime $p$ and positive integer $f$, and we consider the three cases separately.
CASE 1: Suppose that $M \cap S=C_{(q-1) / d}$. In particular, $T:=M \cap S$ is a split torus in $\operatorname{PSL}_{2}(q)$. Since distinct split tori in $\mathrm{PGL}_{2}(q)$ intersect trivially, we conclude $\left|G_{\alpha, \beta}\right| \leq f$. On the other hand, let $B=U \rtimes T$, a Borel subgroup of $S$, and observe that $B$ acts as a Frobenius group on the set $\Lambda=\{M u \mid u \in U\} \subset(G: M)$. Clearly $|\Lambda|=q$; if $p=2$, then $d=1, \Lambda$ is a beautiful subset and we are done. Suppose, then, that $q$ is odd; Lemma 1.7 .3 implies that, if $G$ is binary, then

$$
\left\lceil\frac{\left(\frac{q-1}{2}-1\right)\left(\frac{q-1}{2}-2\right)}{q}\right\rceil<f .
$$

This implies $\lceil q / 4-2-15 / q\rceil<f$. It is easy to verify that, when $q>13$,

$$
\lceil q / 4-2-15 /(4 q)\rceil=\lceil q / 4-2\rceil \geq(q-7) / 4
$$

and hence, in particular, $q-7<4 f$. However, this inequality is never satisfied when $q>13$.
CASE 2: Suppose that $M \cap S=D_{2(q-1) / d}$. The analysis of the previous case still applies: for $q$ even, we obtain a beautiful subset again and are done; for $q$ odd, we proceed as before except that this time $\left|G_{\alpha, \beta}\right| \leq 2 f$, which implies that $q-7<8 f$. However, for $q>13$, this inequality is satisfied only when $q=27$, but we are excluding this case here.
Case 3: Suppose that $M \cap S=B$. Let $K=\langle M, S\rangle$. Then $(K: M)$ is a set of size $q+1$ that is stabilized by $K$ and on which $K$ acts 2 -transitively. By Lemma 2.1.1, any alternating section, $\operatorname{Alt}(r)$, of $\mathrm{P}^{2} \mathrm{~L}_{2}(q)$ has $r \leq 6$, hence $(K: M)$ is a beautiful subset of $(G: M)$, and we are done.

We shall also need the following variant of Lemma 2.2.3.
Lemma 2.2.4. Let $G$ be almost simple with socle $S=\operatorname{PSL}_{2}\left(q^{2}\right)$, and let $x$ be the projective image of the diagonal matrix $\operatorname{diag}\left(a, a^{-1}\right)$, where $a \in \mathbb{F}_{q^{2}}$ has order $(q-1,2)(q-1)$. Let $M<G$ be core-free with $x \in M \cap S$. Then, provided $q \geq 7$, the action of $G$ on $(G: M)$ is not binary.

Proof. For $\lambda \in \mathbb{F}_{q^{2}}$, define subgroups $U_{\lambda}^{ \pm}$of $S$ by

$$
U_{\lambda}^{+}=\left\{I+\lambda t E_{12}: t \in \mathbb{F}_{q}\right\}, \quad U_{\lambda}^{-}=\left\{I+\lambda t E_{21}: t \in \mathbb{F}_{q}\right\}
$$

where as usual $E_{i j}$ denotes the matrix with $i j$-entry 1 and 0 elsewhere. Then $T=\langle x\rangle$ normalizes $U_{\lambda}^{ \pm}$and acts transitively on $U_{\lambda}^{ \pm} \backslash\{1\}$. Since $\left\langle U_{\lambda}^{ \pm}: \lambda \in \mathbb{F}_{q^{2}}\right\rangle=S$, there exists $\lambda$ and $\epsilon= \pm$ such that $U=U_{\lambda}^{\epsilon} \not 又 M$. Then $U T$ acts 2-transitively on the set $\Lambda=\{M u \mid u \in U\} \subset(G: M)$ of size $q$, and since $\operatorname{Alt}(q)$ is not a section of $G$ for $q \geq 7$, it follows that the action of $G$ on $(G: M)$ ) is not binary for $q \geq 7$.

Lemma 2.2.5. Let $G$ contain a subgroup $S \cong \operatorname{SL}_{n}(q) / Z$, where $Z$ is a central subgroup of $\mathrm{SL}_{n}(q)$, and such that $n \geq 3$. Let $x \in S$ be the projection in $S$ of an element $\tilde{x} \in \mathrm{SL}_{n}(q)$ as given in Table 2.2.1 lines 2 and 3. Let $M<G$ with $x \in M$. Then one of the following holds:

1. $G$ contains a section isomorphic to $\operatorname{Sym}\left(q^{n-2}\right)($ if $q>2)$ or $\operatorname{Sym}\left(2^{n-1}\right)($ if $q=2)$;

| Line | $S / Z(S)$ | $\tilde{x}$ | Conditions |
| :---: | :---: | :---: | :---: |
| 1 | $\operatorname{PSL}_{2}(q)$ | $\left(\begin{array}{ll}a & \\ & a^{-1}\end{array}\right)$ | $a$ of order $q-1$ |
| 2 | $\mathrm{PSL}_{n}(q)$ <br> $n \geq 3$ <br> $q \geq 3$ | $\left(\begin{array}{ll}1 & \\ & A \\ & \\ & a^{-1}\end{array}\right)$ | $A \in \mathrm{GL}_{n-2}(q)$ <br> $A$ of order $q^{n-2}-1$ <br> $\operatorname{det}(A)=a \in \mathbb{F}_{q}$ |
| 3 | $\mathrm{SL}_{n}(2)$ <br> $n \geq 3$ | $\left(\begin{array}{ll}1 & \\ & A\end{array}\right)$ | $A \in \mathrm{GL}_{n-1}(2)$ <br> $A$ of order $2^{n-1}-1$ |

Table 2.2.1: Auxiliary table for Lemma 2.2.5
2. $M$ contains $S$;
3. the action of $G$ on $(G: M)$ is not binary.

Proof. We assume that none of the three possibilities hold, and we reach a contradiction. In particular, the action of $G$ on $(G: M)$ is binary. Since $S \cong \mathrm{SL}_{n}(q) / Z$, there exists a surjective group homomorphism $\pi: \mathrm{SL}_{n}(q) \rightarrow S$.
CASE 1: $q>2$. We observe first that $\langle\tilde{x}\rangle$ normalizes two distinct elementary abelian subgroups of $\mathrm{SL}_{n}(q)$ of order $q^{n-2}$, namely those having shape
$U_{1}=\left\{\left.\left(\begin{array}{ccccc}1 & u_{1} & \cdots & u_{n-2} & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1\end{array}\right) \right\rvert\, u_{1}, \ldots, u_{n-2} \in \mathbb{F}_{q}\right\}, U_{2}=\left\{\left.\left(\begin{array}{cccc}1 & & & \\ u_{1} & 1 & & \\ \vdots & & \ddots & \\ \\ u_{n-2} & & & 1 \\ 0 & & & 1\end{array}\right) \right\rvert\, u_{1}, \ldots, u_{n-2} \in \mathbb{F}_{q}\right\}$.
Observe that $\langle\tilde{x}\rangle$ acts by conjugation fixed-point-freely on each of these two groups. Let us suppose that $\pi\left(U_{1}\right) \not \leq M$. As $\pi(\tilde{x})=x \in M$, the fixed-point-freeness of the action yields $\pi\left(U_{1}\right) \cap M=\{1\}$.

Now let $\Lambda$ be the set of cosets of $M$ corresponding to $M \pi\left(U_{1}\right)$, that is, $\Lambda=\left\{M h \mid h \in \pi\left(U_{1}\right)\right\}$. Then $\Lambda$ is a set of size $q^{n-2}$ on which the group $M_{1}=\pi\left(U_{1} \rtimes\langle\tilde{x}\rangle\right)$ acts 2 -transitively. Since we are assuming that $G$ on $(G: M)$ is binary, $\Lambda$ is not a beautiful subset. Therefore, $G^{\Lambda} \geq \operatorname{Sym}\left(q^{n-2}\right)$; however this contradicts the fact that we are assuming that $G$ has no section isomorphic to $\operatorname{Sym}\left(q^{n-2}\right)$. Thus $\pi\left(U_{1}\right) \leq M$.

A similar argument applies to $U_{2}$. Thus $\pi\left(U_{2}\right) \leq M$ and hence $\left\langle\pi\left(U_{1}\right), \pi\left(U_{2}\right), x\right\rangle \leq M$. Observe that

$$
\left\langle\left(\begin{array}{ll}
1 & u \\
0 & 1
\end{array}\right), \left.\left(\begin{array}{ll}
1 & 0 \\
u & 1
\end{array}\right) \right\rvert\, u \in \mathbb{F}_{q}\right\rangle=\mathrm{SL}_{2}(q)
$$

Now, an easy inductive argument on $n$ shows that

$$
\left\langle U_{1}, U_{2}\right\rangle=\left\{\left.\left(\begin{array}{ll}
Z & 0 \\
0 & 1
\end{array}\right) \right\rvert\, Z \in \mathrm{SL}_{n-1}(q)\right\}
$$

and hence, from the definition of $\tilde{x}$, we obtain that $\left\langle U_{1}, U_{2}, \tilde{x}\right\rangle$ contains all matrices of the form

$$
\left(\begin{array}{ll}
Z & \\
& z^{-1}
\end{array}\right)
$$

where $Z \in \mathrm{GL}_{n-1}(q)$ has determinant $z \in \mathbb{F}_{q}$. But now we define two elementary abelian subgroups of $\mathrm{SL}_{n}(q)$ of order $q^{n-1}$, namely those having shape

$$
U_{3}=\left\{\left.\left(\begin{array}{cccc}
1 & & & u_{1} \\
& \ddots & & \vdots \\
& & \ddots & u_{n-1} \\
& & & 1
\end{array}\right) \right\rvert\, u_{1}, \ldots, u_{n-1} \in \mathbb{F}_{q}\right\}, U_{4}=\left\{\left.\left(\begin{array}{cccc}
1 & & & \\
& \ddots & & \\
& & \ddots & \\
u_{1} & \cdots & u_{n-1} & 1
\end{array}\right) \right\rvert\, u_{1}, \ldots, u_{n-1} \in \mathbb{F}_{q}\right\}
$$

Repeating the same argument as before, with $U_{1}$ and $U_{2}$ replaced by $U_{3}$ and $U_{4}$, we obtain that $M$ contains $\pi\left(U_{3}\right)$ and $\pi\left(U_{4}\right)$. But then $M \geq\left\langle\pi\left(U_{1}\right), \pi\left(U_{2}\right), \pi\left(U_{3}\right), \pi\left(U_{4}\right), x\right\rangle=S$, a contradiction, and we are done.
Case 2: $q=2$. Clearly, in this case, $Z=1$ and we may think of $\pi$ as the identity mapping. We define two elementary abelian subgroups of $\mathrm{SL}_{n}(2)$ of order $2^{n-1}$, namely those having shape

$$
U_{1}=\left\{\left.\left(\begin{array}{cccc}
1 & & & u_{1} \\
& \ddots & & \vdots \\
& & \ddots & u_{n-1} \\
& & & 1
\end{array}\right) \right\rvert\, u_{1}, \ldots, u_{n-1} \in \mathbb{F}_{q}\right\}, U_{2}=\left\{\left.\left(\begin{array}{cccc}
1 & & \\
& \ddots & \\
& & \ddots & \\
u_{1} & \cdots & u_{n-1} & 1
\end{array}\right) \right\rvert\, u_{1}, \ldots, u_{n-1} \in \mathbb{F}_{q}\right\}
$$

We suppose that $U_{1} \not \leq M$. As in the previous proof we use the fact that $\langle x\rangle$ normalizes, and acts fixed-point-freely on, $U_{1}$. As before, either $G$ contains a section isomorphic to $\operatorname{Sym}\left(2^{n-1}\right)$ (but this contradicts our hypothesis) or else we obtain a beautiful subset (but this contradicts again our hypothesis). Hence $M$ contains $U_{1}$ and, by the same argument, $U_{2}$. Since $\left\langle U_{1}, U_{2}\right\rangle=\mathrm{SL}_{n}(2)$, we obtain a contradiction and are done.

Applying Lemma 2.2.5 to the case where $G$ is almost simple, and applying [54, Proposition 5.3.7] to establish when $G$ may contain the relevant alternating section, we obtain the following result.

Lemma 2.2.6. Let $G$ be almost simple with socle $S=\operatorname{PSL}_{n}(q)$, and let $x$ be the projective image of an element $\tilde{x}$ as given in Table 2.2.1. Let $M<G$ be core-free with $x \in M \cap S$. Then, provided $(n, q) \notin$ $\{(2,4),(2,5)\}$ and $(G, M) \notin\{(\operatorname{Sym}(8), \operatorname{Alt}(7)),(\operatorname{Sym}(8), \operatorname{Sym}(7))\}$, the action of $G$ on $(G: M)$ is not binary.

Moreover, if $(n, q) \notin\{(2,3),(3,3)\}$, then $\left|C_{S}(x)\right|<q^{n}$.
Proof. Since $M$ is core-free in $G$, we have $S \not \leq M$. When $n=2$, the proof follows from Lemma 2.2.3, When $n \geq 3$, from Lemma 2.2.5, if the action of $G$ on $(G: M)$ is binary, then $G$ contains a section isomorphic to $\operatorname{Sym}\left(q^{n-2}\right)$ (if $q>2$ ) or $\operatorname{Sym}\left(2^{n-1}\right)$ (if $q=2$ ). From this it follows from Lemma 2.1.1 that $(n, q) \in\{(3,3),(3,4),(3,5),(3,2),(4,2)\}$. For these values of $(n, q)$, we have constructed all the permutation representations under consideration and we have checked that none is binary unless $(n, q)=(4,2)$ and $(G, M)$ is one for the cases listed in the statement.

When $\tilde{x}$ is as in Line 1 of Table 2.2.1, $\langle x\rangle$ is a torus in $\mathrm{PSL}_{2}(q)$ of cardinality $(q-1) / 2$ when $q$ is odd, and $q-1$ when $q$ is even. Thus $\left|C_{S}(x)\right| \leq q-1<q$, except when $q=3$. Similarly, using the fact that, if $A \in \operatorname{GL}_{k}(q)$ has order $q^{k}-1$ (that is, $\langle A\rangle$ is a Singer cycle), then $C_{\mathrm{GL}_{k}(q)}(A)=\langle A\rangle$, we deduce that $\left|C_{S}(x)\right|=\left(q^{n-1}-1\right)(q-1) /(n, q-1)<q^{n}-1$ when $\tilde{x}$ is as in Line 2 of Table 2.2.1 and $q \neq 3$, and $\left|C_{S}(x)\right|=2^{n-1}-1<2^{n}$ when $\tilde{x}$ is as in Line 3 of Table 2.2.1,

The fact that $\left|C_{S}(x)\right|<q^{n}$ will be important later on - in Lemma 2.2.5, and in the results that follow, we have tried to pick distinguished elements $x \in S$ for which $C_{S}(x)$ is relatively small.

For groups with socle $\operatorname{PSL}_{4}(q)$ we shall also need the following special result.
Lemma 2.2.7. Let $G$ be almost simple with socle $S=\mathrm{PSL}_{4}(q)$, and let $x$ be the projective image of $a$ diagonal matrix $\tilde{x}=\operatorname{diag}\left(1,1, a, a^{-1}\right)$, where $a \in \mathbb{F}_{q}$ has order $q-1$. Let $M<G$ be core-free with $x \in M \cap S$. Then, provided $q \geq 8$, the action of $G$ on $(G: M)$ is not binary.

Proof. The proof is very similar to that of Lemma 2.2.5, Let $T=\langle x\rangle$, and for $i \neq j$ define $U_{i j}=\left\{I+\alpha E_{i j}\right.$ : $\left.\alpha \in \mathbb{F}_{q}\right\}$, where $E_{i j}$ denotes the matrix with $i j$-entry 1 and 0 elsewhere. Then $T$ acts fixed-point-freely on the groups $U_{i j}$ for $i \in\{1,2\}, j \in\{3,4\}$ or vice versa. Since these subgroups $U_{i j}$ generate $S$, at least one of them is not contained in $M$. Hence we obtain a subset $\Delta$ of size $q$ on which $G_{\Delta}$ acts 2-transitively. If $\Delta$ is a beautiful subset then $(G,(G: M))$ is not binary. So suppose $\Delta$ is not beautiful. Then $\operatorname{Alt}(q)$ is a section of $S$, and moreover $\operatorname{Alt}(q-1)$ is a section of $M$. By Lemma 2.1.1. Alt $(q)$ is a section of $S$ only if $q \leq 8$; moreover, if $q=8$, then $\operatorname{Alt}(7)$ can only be a section of a maximal core-free subgroup $M$ of $G$ if $M$
is a subfield subgroup of type $\mathrm{PSL}_{4}(2)$ (see [10, Tables 8.8, 8.9]) - but such a subgroup does not contain the element $x$. Hence if $q \geq 8$ we have a contradiction, and the proof is complete.

We now need to prove an analogue of Lemma 2.2 .5 for the other classical groups, albeit subject to some conditions (including lower bounds on $n$ ). Some of the situations excluded by these conditions are studied in subsequent lemmas. In the statement and proof of the lemma, if $S$ is orthogonal or symplectic, we set $\mathbb{K}=\mathbb{F}_{q}$; if $S$ is unitary, then we set $\mathbb{K}=\mathbb{F}_{q^{2}}$. In either case, for a scalar $a \in \mathbb{K}$ we define $\bar{a}:=a^{q}$; for a matrix $A=\left(a_{i j}\right)_{i, j} \in \mathrm{GL}_{d}(\mathbb{K})$ we write $\bar{A}$ for the matrix $\left(\bar{a}_{i j}\right)_{i, j}$.

Lemma 2.2.8. Suppose that one of the following holds:

1. $G$ contains a subgroup $S \cong \mathrm{SU}_{n}(q) / Z$ where $Z$ is a central subgroup of $\mathrm{SU}_{n}(q)$ and $n \geq 5$;
2. $G$ contains a subgroup $S \cong \operatorname{Sp}_{n}(q) / Z$ where $Z$ is a central subgroup of $\operatorname{Sp}_{n}(q)$ and $n \geq 4$;
3. $G$ contains a subgroup $S \cong \Omega_{n}^{\varepsilon}(q), q$ is even and $n \geq 8$;
4. $G$ contains a subgroup $S \cong \mathrm{SO}_{n}^{\varepsilon}(q) / Z$ where $Z$ is a central subgroup of $\mathrm{SO}_{n}^{\varepsilon}(q)$, $q$ is odd and $n \geq 7$.

Let $k$ be the Witt index of the associated formed space. If $S \neq \mathrm{SU}_{n}(q) / Z$ with $n$ even, then we define $j=k$, otherwise $j=k-1$. We let $\mathcal{B}=\left\{e_{1}, \ldots, e_{j}, f_{1}, \ldots, f_{j}\right\} \cup Y$ be a hyperbolic basis; thus $Y$ is a set of linearly independent anisotropic vectors if $S \neq \mathrm{SU}_{2 j+2}(q) / Z$, otherwise $Y=\left\{e_{j+1}, f_{j+1}\right\}$. We set $y=|Y| \in\{0,1,2\}$.

Let $M<G$ with $x \in M$, where $x$ is the projective image in $S$ of

$$
\tilde{x}=\left(\begin{array}{ccccc}
1 & & & & \\
& A & & & \\
& & 1 & & \\
& & & \overline{A^{-T}} & \\
& & & & J_{y}
\end{array}\right),
$$

written with respect to $\mathcal{B}, A \in \mathrm{GL}_{j-1}(\mathbb{K})$ is of order $|\mathbb{K}|^{j-1}-1$, and $J_{y}$ is some $y$-by-y matrix. If $S$ is not unitary, then we can take $J_{y}$ to be the identity matrix; if $S$ is unitary, then $J_{y}$ is a matrix such that $\operatorname{det}(\tilde{x})=1$ (thereby ensuring that $x \in S$ ).

Then one of the following holds:

1. $G$ contains a section isomorphic to $\operatorname{Sym}\left(|\mathbb{K}|^{j-1}\right)$;
2. $M$ contains $S$;
3. the action of $G$ on $(G: M)$ is not binary.

In particular if $G$ is almost simple, $M$ is core-free and the action of $G$ on $(G: M)$ is binary, then $S$ is symplectic and one of the following holds:

1. $(k, q)=(2,2)$, or
2. $(k, q)=(2,3)$ and $M=\langle x\rangle$.

Note that if $S$ is orthogonal and $q$ is even, then [12, Lemmas 2.5.7 and 2.5.9] imply that $\tilde{x}$ lies in $\Omega_{n}^{\varepsilon}(q)$.
Proof. We suppose throughout that the action of $G$ on $(G: M)$ is binary. Our argument is the same for all families, more or less, but the details are different; we will, therefore, need to do some case work especially in the third stage of the proof.

Step 1. We observe first that $\langle x\rangle$ normalizes $U_{1}$ and $U_{2}$, two distinct elementary-abelian subgroups of $S$ of order $|\mathbb{K}|^{j-1}$, namely those having shape

$$
\left(\begin{array}{ccccccc}
1 & u_{1} & \cdots & u_{j-1} & & & \\
& 1 & & & & & \\
& & \ddots & & & & \\
& & & 1 & & & \\
& & & & 1 & & \\
& & & & -\overline{u_{1}} & 1 & \\
& & & & \vdots & & \ddots
\end{array}\right) \text {, and }\left(\begin{array}{cccccccc}
1 & & & & & & \\
u_{1} & 1 & & & & & & \\
\vdots & & \ddots & & & & & \\
u_{j-1} & & & 1 & & & & \\
& & & & & & & 1 \\
\hline u_{j-1} & -\overline{u_{1}} & \cdots & -\overline{u_{j-1}} \\
& & & & & & & 1
\end{array}\right) \text {, }
$$

respectively. In each case we write only the first $2 j$ rows and columns of each matrix - the remaining rows and columns are completed by setting off-diagonal entries to be 0 , and diagonal entries to be 1 . The resulting group is the set of all matrices obtained by allowing the parameters $u_{i}$ to range over $\mathbb{K}$.

Observe that $\langle x\rangle$ acts fixed-point-freely on each of these two groups. Let us suppose that $U_{1} \not \leq M$; then the fixed-point-freeness of the action means that $U_{1} \cap M=\{1\}$. Now let $\Lambda$ be the set of cosets $\left\{M u \mid u \in U_{1}\right\}$ of $M$ corresponding to $M U_{1}$. Then this is a set of size $|\mathbb{K}|^{j-1}$ on which the group $M_{1}=U_{1} \rtimes\langle x\rangle$ acts 2-transitively. Now Lemma 1.6 .8 implies that $G$ contains a section isomorphic to $\operatorname{Sym}\left(|\mathbb{K}|^{j-1}\right)$ and the result follows. The same argument works with $U_{2}$ so we may assume hereafter that $M$ contains $\left\langle U_{1}, U_{2}\right\rangle$.

Observe that

$$
\left\langle\left(\begin{array}{cccc}
1 & u & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -\bar{u} & 1
\end{array}\right), \left.\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
u & 1 & 0 & 0 \\
0 & 0 & 1 & -\bar{u} \\
0 & 0 & 0 & 1
\end{array}\right) \right\rvert\, u \in \mathbb{F}_{q}\right\rangle=\left\{\left.\left(\begin{array}{cc}
Z & 0 \\
0 & \bar{Z}^{-T}
\end{array}\right) \right\rvert\, Z \in \mathrm{SL}_{2}(q)\right\} .
$$

Now, an easy inductive argument on $j$ shows that

$$
\left\langle U_{1}, U_{2}\right\rangle=\left\{\left.\left(\begin{array}{cc}
Z & 0 \\
0 & \bar{Z}^{-T}
\end{array}\right) \right\rvert\, Z \in \mathrm{SL}_{j}(q)\right\}
$$

and hence, from the definition of $\tilde{x}, K=\left\langle U_{1}, U_{2}, x\right\rangle$ contains all matrices of the form

$$
\left(\begin{array}{ll}
Z & \\
& \bar{Z}^{-T}
\end{array}\right),
$$

where $Z \in \mathrm{GL}_{j}(\mathbb{K})$.
Step 2A. Next we define $U_{3}, \ldots, U_{j+2}, j$ elementary-abelian subgroups of $S$ of order $q^{j-1}$, namely those having shape


Note, first, that we have placed dotted lines to mark the point where the " $e$-vectors" change to " $f$-vectors"; note, second, that we have omitted columns and rows corresponding to basis elements from $Y$; note, third, that in the case where $S$ is symplectic the given matrices do not lie in $S$ - but this is fixed by removing all minus signs, and proceeding in the same way.

It is easy enough to see that $\langle x\rangle$ normalizes, and acts fixed-point freely on $U_{3}$. Similarly, $K$ contains a conjugate of $\langle x\rangle$ that acts fixed-point-freely on $U_{4}$, and so on. By the same argument as before, we have two possibilities:
(a) $M$ contains $U_{3}$;
(b) there is a set $\Lambda \subset \Omega$ such that $|\Lambda|=|\mathbb{K}|^{j-1}$ and on which $S^{\Lambda}$ acts 2-transitively; then Lemma 1.6.8 implies that $G$ contains a section isomorphic to $\operatorname{Sym}\left(|\mathbb{K}|^{j-1}\right)$ and the result follows.
Thus, again, we may assume that $M$ contains $U_{3}$.
Since the same argument works for $U_{4}, \ldots, U_{j+2}$, we conclude that the group $M$ must contain the group $W_{1}$, consisting of all matrices of the form

$$
\left(\begin{array}{ccc}
I & Z &  \tag{2.2.2}\\
& I & \\
& & I_{y}
\end{array}\right),
$$

where $Z$ is a $j$-by- $j$ matrix satisfying $Z=-\bar{Z}^{T}$ and having zero diagonal entries (or, in the case where $S$ is symplectic, $Z$ satisfies $Z=Z^{T}$ and has zero diagonal entries).

Step 2b. Now we repeat the argument of Step 2a but this time, all the matrices we use are the transposes of those in Step 2a. We conclude that $M$ must contain the group $W_{2}$, consisting of all matrices of the form

$$
\left(\begin{array}{lll}
I & &  \tag{2.2.3}\\
Z & I & \\
& & I_{y}
\end{array}\right)
$$

where $Z$ is a $j$-by- $j$ matrix satisfying $Z=-\bar{Z}^{T}$ and having zero diagonal entries (or, in the case where $S$ is symplectic, $Z$ satisfies $Z=Z^{T}$ and has zero diagonal entries).

Step 3. We use the fact that $M$ contains the group $\left\langle K, W_{1}, W_{2}\right\rangle$ and we split into cases, depending on the particular family of classical groups which we are dealing with.

Case 3A: $S$ is unitary. In this case an easy argument says that, since $M$ contains the group $K$, the group $M$ contains all matrices of the form (2.2.2), where $Z$ is a $j$-by- $j$ matrix satisfying $Z=-\bar{Z}^{T}$, i.e. we can drop the requirement that $Z$ has zero diagonal entries. The resulting set of matrices forms an elementary-abelian group $U$ of size $q^{j^{2}}$ which is the unipotent radical of a parabolic subgroup $P_{j}$ in $\mathrm{SU}_{2 j}(q)$.

The same argument works "with transposes" and we obtain that $M$ contains all matrices of the form (2.2.3), where $Z$ is a $j$-by- $j$ matrix satisfying $Z=-\bar{Z}^{T}$. We split into two cases, depending on the parity of $n$.

Assume, first, that $n=2 k+1$ with $k \geq 2$. Then $M$ contains the projective image of $M_{0} \cong \mathrm{SU}_{2 k}(q)$, where $M_{0}$ stabilizes the unique non-isotropic basis vector, $v$, in $Y$. Without loss of generality, we may suppose that $v$ has norm 1 .

Let $\alpha_{2}, \ldots, \alpha_{k} \in \mathbb{K}$ and for simplicity set $\alpha_{1}:=0$. For each $i \in\{1, \ldots, k\}$, let $\beta_{i, i} \in \mathbb{K}$ with $\beta_{i, i}+$ $\overline{\beta_{i, i}}+\alpha_{i} \overline{\alpha_{i}}=0$. (Observe that the existence of $\beta_{i, i}$ is guaranteed by Hilbert's Theorem 90.) For each $i, j \in\{1, \ldots, k\}$ with $i \neq j$, let $\beta_{i, j}=0$ when $i>j$ and $\beta_{i, j}=-\overline{\alpha_{i}} \alpha_{j}$ when $i<j$. Now, let $g \in S$ be the element fixing $e_{1}, \ldots, e_{k}$ pointwise and which satisfies

$$
\begin{aligned}
& v \mapsto v+\alpha_{2} e_{2}+\cdots+\alpha_{k} e_{k}, \\
& f_{i} \mapsto f_{i}-\overline{\alpha_{i}} v+\sum_{j=1}^{k} \beta_{i, j} e_{j} .
\end{aligned}
$$

In particular, the matrix form of $g$ with respect to the basis $\left(e_{1}, \ldots, e_{k}, f_{1}, \ldots, f_{k}, v\right)$ is

$$
\left(\begin{array}{ccc}
I & 0 & 0 \\
B & I & -\bar{d} \\
d^{T} & 0^{T} & 1
\end{array}\right), \text { where } d=\left(\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{k}
\end{array}\right), B=\left(\begin{array}{cccc}
\beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1, k} \\
\beta_{2,1} & \beta_{2,2} & \cdots & \beta_{2, k} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{k, 1} & \beta_{k, 2} & \cdots & \beta_{k, k}
\end{array}\right) .
$$

Let $U_{0}$ be the subgroup of $S$ consisting of all of these elements, as $\alpha_{2}, \ldots, \alpha_{k}$ run through $\mathbb{K}$. Let $T_{0}=$ $N_{M_{0}}\left(U_{0}\right)$ and observe that $T_{0}$ acts transitively on the non-identity elements of $U_{0}$. We conclude that either $M$ contains $U_{0}$ or else $U_{0} \rtimes T_{0}$ acts 2-transitively by right multiplication on the set of right cosets $\left\{M u \mid u \in U_{0}\right\}$, a set of size $\mathbb{K}^{k-1}$. In the latter case Lemma 1.6 .8 implies that $G$ contains a section isomorphic to $\operatorname{Sym}\left(|\mathbb{K}|^{k-1}\right)$ and the result follows. In the former case $M$ contains $U_{0}$ which in turn implies that $M \geq S$ and the result follows.

Assume, next, that $n=2 j+2$. Then $M$ contains the projective image of $M_{0} \cong \mathrm{SU}_{2 k-2}(q)$, where $M_{0}$ stabilizes the basis vectors $e_{k}$ and $f_{k}$. In this case we define two subgroups:

1. $U_{1}$ is the subgroup of $S$ whose elements $g$ fix $e_{1}, \ldots, e_{k-1}$ and which satisfy $e_{k} \mapsto e_{k}+\alpha_{1} e_{1}+\cdots+$ $\alpha_{k-1} e_{k-1}$ for some $\alpha_{1}, \ldots, \alpha_{k-1} \in \mathbb{K}$.
2. $U_{2}$ is the subgroup of $S$ whose elements $g$ fix $f_{1}, \ldots, f_{k-1}$ and which satisfy $f_{k} \mapsto f_{k}+\beta_{1} f_{1}+\cdots+$ $\beta_{k-1} f_{k-1}$ for some $\beta_{1}, \ldots, \beta_{k-1} \in \mathbb{K}$.

The same argument as for $n$ odd allows us to conclude that either $G$ contains a section isomorphic to $\operatorname{Sym}\left(|\mathbb{K}|^{k-1}\right)$ or else $M$ contains $U_{1}$ and $U_{2}$ and so contains $S$ and the result follows.

From here, we have, by definition, that $j=k$.
Case 3B: $S$ is symplectic and $q$ is odd. Again an easy argument asserts that, since $M$ also contains the group $K$, then $M$ must contain all matrices of the form (2.2.2), where $Z$ is any symmetric matrix. These matrices together form an elementary abelian group $U$ of size $q^{\frac{1}{2} k(k+1)}$, which is the unipotent radical of a parabolic subgroup $P_{k}$. Applying the same argument "with transposes" allows us to conclude that $M \geq S$, and the result follows.

Case 3C: $S$ is symplectic and $q$ is even. In this case, the set of matrices of the form (2.2.2), where $Z$ is symmetric with zero diagonal entries, forms an elementary abelian group $U$ of size $q^{\frac{1}{2} k(k-1)}$, which is the unipotent radical of a parabolic subgroup $P_{k}$ of an orthogonal group $L=\Omega_{2 k}^{+}(q)$ (this is the particular orthogonal group corresponding to the quadratic form for which our basis is hyperbolic).

Now, as in the odd case, we can apply the same argument to the transpose of these matrices to conclude that $M \cap S$ contains the group $L \cong \Omega_{2 k}^{+}(q)$. In particular $M \cap S$ is either $L, L .2$ or $S$. The result follows if $M \cap S=S$, so assume $M \cap S$ is either $L$ or L.2. We define an element $\tilde{g}$ whose action on $\left\langle e_{1}, f_{2}\right\rangle$ is given by the matrix $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ and which fixes all elements of $\mathcal{B} \backslash\left\{e_{1}, f_{1}\right\}$. Clearly $\tilde{g}$ is an element of $\operatorname{Sp}_{2 k}(q)$; we take $g$ to be the projective image of $\tilde{g}$ in $S$. Observe that $g$ centralizes $x$ but does not normalize $L$. We can, therefore, repeat all of the preceding argument using subgroups of $L^{g}$ instead of $L$. The same case is left: when $M \cap S$ contains both $L$ and $L^{g}$. Since $\left\langle L, L^{g}\right\rangle=S$ the result follows.

Case 3D: $S$ is orthogonal and $n=2 k$. In this case, $S=\Omega_{2 k}^{+}(q) / Z$ and the groups $W_{1}$ and $W_{2}$ are both unipotent radicals of parabolic subgroups $P_{k}$ in $S$. From this, we conclude that $M \geq S$ and the result follows.

Case 3E: $S$ is orthogonal, and $n \in\{2 k+1,2 k+2\}$. In this case, $S=\Omega_{2 k+1}(q)$ or $\Omega_{2 k+2}^{-}(q) / Z$ and, arguing à la Case 3D, we see that $M$ contains the projective image of $L \cong \Omega_{2 k}^{+}(q)$, where $L$ fixes all vectors in the non-degenerate subspace $\langle Y\rangle$. Recall that, by construction, the element $\tilde{x}$ fixes all vectors in $\langle Y\rangle$.

Suppose first that $q$ is odd, let $z \in Y$ and suppose that $\varphi(z, z)=\eta$ where $\varphi$ is the symmetric form associated with the covering group of $S$. We define an element $\tilde{g}$ whose action on $\left\langle e_{1}, z, f_{1}\right\rangle$ is given by the
matrix

$$
\left(\begin{array}{ccc}
1 & a & -\frac{1}{2} a \eta \\
& 1 & -a \eta \\
& & 1
\end{array}\right)
$$

where $a$ is some non-zero element of $\mathbb{F}_{q}$, and $\tilde{G}$ fixes all elements of $\mathcal{B} \backslash\left\{e_{1}, f_{1}, z\right\}$. Clearly $\tilde{g}$ is an element of $\mathrm{SO}_{n}^{\varepsilon}(q)$; we take $g$ to be the projective image of $\tilde{g}$ in $S$. Observe that $g$ centralizes $x$ but does not normalize $L$. We can, therefore, repeat all of the preceding argument using subgroups of $L^{g}$ instead of $L$. The same case is left: when $M \cap S$ contains both $L$ and $L^{g}$. Notice that we can repeat this argument for any choice of $z \in Y$ and any choice of $a \in \mathbb{F}_{q}$. It is straightforward to conclude that the resulting collection of conjugates of $L$ generates $S$ and, hence $M \geq S$ and the result follows.

Suppose next that $q$ is even, in which case $n=2 k+2, S=\Omega_{2 k+2}(q)$ and $Y=\langle x, y\rangle$. Let $Q$ be the quadratic form associated with the covering group of $S$ and consider the restriction of $Q$ to the subspace $W=\left\langle e_{1}, f_{1}, x, y\right\rangle$. Let $\tilde{g}$ be a linear transformation which fixes all elements of $\mathcal{B} \backslash\left\{e_{1}, f_{1}, x, y\right\}$ and, on $W$, restricts to an element of the group $J=\Omega_{4}^{-}(q)$ associated with $\left.Q\right|_{W}$. By [12, Lemma 2.5.9], $\tilde{g}$ is an element of the covering group of $S$ and we take $g$ to be its projective image in $G$. Again $g$ centralizes $x$ and, again, we must deal with the case where $M \cap S$ contains $L$ and $L^{g}$ for all such $g$. Thus we may assume that $M$ contains $L_{1}=\left\langle L^{g} \mid g \in J\right\rangle$. There are two possibilities: either $J$ normalizes $L_{1}$ or else we can repeat the same argument with $L_{1}$ in place of $L$ and we are able to assume that $M$ contains $L_{2}=\left\langle L^{g} \mid g \in J\right\rangle$. Repeating as many times as necessary we are left with the situation where $M$ contains a group $L_{\infty}$ that contains $L \cong \Omega_{2 k}^{+}(q)$ and is normalized by $J \cong \Omega_{4}^{-}(q)$.

Let $X=\left\langle e_{1}, f_{1}, e_{2}, f_{2}, x, y\right\rangle$ and consider the group $H \leq S$ that fixes every vector in $X^{\perp}$ and induces $\Omega_{6}^{-}(q)$ on $X$. Observe that $H$ contains $J$ and so, in particular, $J$ normalizes $H \cap L_{\infty}$. Then $H \cap L_{\infty}$ is a subgroup of $H=\Omega_{6}^{-}(q) \cong \mathrm{SU}_{4}(q)$ that contains a group isomorphic to $\Omega_{4}^{+}(q) \cong \mathrm{SL}_{2}(q) \times \mathrm{SL}_{2}(q)$ and is normalized in $H$ by a group isomorphic to $\Omega_{4}^{-}(q) \cong \mathrm{SL}_{2}\left(q^{2}\right)$. Checking [10, Tables 8.10 and 8.14] we conclude that $H \cap L_{\infty}$ is either $H$ or a subgroup of $H$ isomorphic to $\operatorname{Sp}_{4}(q)$. Suppose $H \cap L_{\infty} \cong \operatorname{Sp}_{4}(q)$. Checking [10. Table 8.10] we see that there is precisely one conjugacy class of subgroups of $H \cong \mathrm{SU}_{4}(q)$ isomorphic to $\mathrm{Sp}_{4}(q)$ hence, regarding $H$ as $\Omega_{6}^{-}(q)$ these are the stabilizers of non-singular vectors. Note that $H \cap L_{\infty}$ contains all $J$-conjugates of $H \cap L$. What is more the non-singular vectors fixed by $H \cap L$ are precisely those in $\langle x, y\rangle$. This, in turn, means that, for all $j \in J$, the non-singular vectors fixed by $(H \cap L)^{j}$ are precisely those in $\langle x, y\rangle^{j}$. Thus if $v$ is a non-singular vector fixed by $H \cap L_{\infty}$, then $v^{j} \in\langle x, y\rangle$ for all $j \in J$. Direct calculation (or using the fact that $J$ is irreducible on $W$ ) confirms that no such vector exists. We conclude that $H \cap L_{\infty}=H$.

Now observe that, working with respect to the basis $\mathcal{B}, M$ contains all of the fundamental root groups for $S$, and hence $M$ contains $S$ as required.

Finally, suppose that $G$ is almost simple and $M$ is core-free. Either the action of $G$ on $(G: M)$ is not binary (and we are done) or else $G$ contains a section isomorphic to $\operatorname{Sym}\left(|\mathbb{K}|^{j-1}\right)$. Lemma 2.1.1 (and [10]) yield the result barring only a few values of $k$ and $q$. In particular we use magma to verify the result when $S=\operatorname{PSp}_{n}(q)$ with $(k, q) \in\{(2,2),(2,3),(2,4),(2,5),(2,7),(3,2),(3,3),(4,2)\}$, when $S=\operatorname{PSU}_{n}(q)$ with $(k, q)=(2,2)$ and when $S$ is an orthogonal group with $(k, q) \in\{(3,2),(3,3),(4,2)\}$.

The following proposition deals with one of the lacunae in the previous: when $S$ is orthogonal, $q$ is odd, and $G$ does not contain $\operatorname{PSO}_{n}^{\varepsilon}(q)$. The statement of the proposition uses the notation established in the statement of the previous; to make matters more straightforward we assume that $G$ is almost simple.

Lemma 2.2.9. Suppose that $q$ is odd, and that $S=\mathrm{P} \Omega_{n}^{\varepsilon}(q) \unlhd G \leq \operatorname{Aut}\left(\mathrm{P} \Omega_{n}^{\varepsilon}(q)\right)$ with $n \geq 7$. Let $k$ be the Witt index of the associated formed space and let $M<G$ be core-free with $x \in M \cap S$, where $x$ is the
projective image of

$$
\tilde{x}=\left(\begin{array}{cccccc}
1 & & & & & \\
& A & & & & \\
& & \zeta & & & \\
& & & 1 & & \\
& & & & A^{-T} & \\
\\
& & & & & \zeta^{-1} \\
\\
& & & & & I_{y}
\end{array}\right)
$$

written with respect to $\mathcal{B}, A \in \mathrm{GL}_{k-2}(q)$ is of order $q^{k-2}-1, \zeta$ is a non-square in $\mathbb{F}_{q}$ and $I_{y}$ is the $y$-by-y identity matrix. If the action of $G$ on $(G: M)$ is binary, then

$$
(k, q) \in\{(3,3),(3,5),(3,7),(3,9),(4,3)\} .
$$

Proof. We refer, first, to [12, Lemma 2.5.7] to confirm that $\tilde{x}$ is indeed an element of $\Omega_{n}^{\varepsilon}(q)$. Now the action of $\tilde{x}$ on the subspace $W:=\left\langle e_{1}, \ldots, e_{k-1}, f_{1}, \ldots, f_{k-1}, Y\right\rangle$ is identical to that studied in the previous proposition; the arguments given there allow us to assume that $M$ contains (the projective image of) the group

$$
K:=\left\{g \in \Omega_{n}^{\varepsilon}(q)\left|e_{k}^{g}=e_{k}, f_{k}^{g}=f_{k}, g\right|_{W} \in \Omega(W)\right\} .
$$

We should be careful about exceptions however: studying the proof we see that our conclusion is valid only when $\operatorname{Alt}\left(q^{k-2}\right)$ is not a section in $S$. Now, Lemma 2.1.1 implies that exceptions occur only when $q^{k-2} \leq 2 k+4$; this yields the given list.

Now we study the normalizer in $K$ of four different elementary-abelian subgroups $U_{1}, \ldots, U_{4}$ of $S$ of order $q^{k-2}$. We choose these groups so that they stabilize the subspaces $E=\left\langle e_{1}, \ldots, e_{k}\right\rangle$ and $F=$ $\left\langle f_{1}, \ldots, f_{k}\right\rangle$. We require furthermore that $\langle Y\rangle$ is in the 1-eigenspace of each of the groups, thus to specify the elements of these groups it is enough to specify their action on the subspace $E$ :

$$
\begin{aligned}
& U_{1}:=\left\{g \mid e_{1}^{g}=e_{1} \text {; for all } i=2, \ldots, k-1 \text {, there exist } \alpha_{i} \text { such that } e_{i}^{g}=e_{i}+\alpha_{i} e_{k}\right\} ; \\
& U_{2}:=\left\{g \mid e_{2}^{g}=e_{2} ; \text { for all } i=1,3, \ldots, k-1 \text {, there exist } \alpha_{i} \text { such that } e_{i}^{g}=e_{i}+\alpha_{i} e_{k}\right\} ; \\
& U_{3}:=\left\{\begin{array}{l}
\left.g \left\lvert\, \begin{array}{l}
e_{1}^{g}=e_{1}, \ldots, e_{k-1}^{g}=e_{k-1} ; \\
\text { for all } i=2, \ldots, k-1, \text { there exist } \alpha_{i} \text { s.t. } e_{k}^{g}=e_{k}+\alpha_{2} e_{2}+\cdots+\alpha_{k-1} e_{k-1}
\end{array}\right.\right\} ; ~ ; ~ ; ~
\end{array}\right. \\
& U_{4}:=\left\{g \left\lvert\, \begin{array}{l}
e_{1}^{g}=e_{1}, \ldots, e_{k-1}^{g}=e_{k-1} ; \\
\text { for all } i=1,3, \ldots, k-1, \text { there exist } \alpha_{i} \text { s.t. } e_{k}^{g}=e_{k}+\alpha_{1} e_{1}+\alpha_{3} e_{3}+\cdots+\alpha_{k-1} e_{k-1}
\end{array}\right.\right\} .
\end{aligned}
$$

It is a simple matter to check that, for each $i=1, \ldots, 4, N_{K}\left(U_{i}\right)$ acts transitively on the non-trivial elements of $U_{i}$. Thus, by the same argument as before, we have three possibilities:
(a) $M$ contains $U_{i}$;
(b) $G$ admits a beautiful subset of size $q^{k-2}$;
(c) $S$ admits a section isomorphic to $\operatorname{Alt}\left(q^{k-2}\right)$.

The second possibility is ruled out because $G$ is binary on $(G: M)$ and the third is ruled out as before, except for the listed exceptions. Therefore, $M$ contains $U_{i}$ for each $i$ and hence $\left\langle K, U_{1}, \ldots, U_{4}\right\rangle=S$, and the result follows.

Lemmas 2.2.10 and 2.2.11 deal with some small rank cases that were not covered by Lemma 2.2.8.
Lemma 2.2.10. Let $G$ contain a subgroup $S \cong \mathrm{SU}_{3}(q) / Z$, where $Z$ is a central subgroup of $\mathrm{SU}_{3}(q)$ and $q>2$. We let $\mathcal{B}:=\left(e_{1}, f_{1}, x\right)$ be a hyperbolic basis for the underlying unitary space. Let

$$
\tilde{g}=\left(\begin{array}{lll}
t & & \\
& t^{-q} & \\
& & 1
\end{array}\right) \in \mathrm{SU}_{3}(q)
$$

where $t \in \mathbb{F}_{q}$ is of order $q-1$; let

$$
\tilde{g}^{\prime}=\left(\begin{array}{lll}
u & & \\
& u^{-q} & \\
& & u^{q-1}
\end{array}\right) \in \mathrm{SU}_{3}(q),
$$

where $u \in \mathbb{F}_{q}$ is of order $q^{2}-1$. Let $g$ and $g^{\prime}$ be the projective images of $\tilde{g}$ and $\tilde{g}^{\prime}$ in $S$. Let $M<G$ and, if $q$ is odd, then suppose that $g \in M$; if $q$ is even, then suppose that $g^{\prime} \in M$. Then one of the following holds:

1. $G$ contains a section isomorphic to $\operatorname{Sym}(q)$;

## 2. $M$ contains $S$;

3. the action of $G$ on $(G: M)$ is not binary.

In particular, if $G$ is almost simple with socle $S$ and $M$ is core-free, then the action of $G$ on $(G: M)$ is not binary.

Proof. Assume first that $q$ is odd and suppose that the action of $G$ on $(G: M)$ is binary. Write $T_{0}=\langle g\rangle$ and observe that $T_{0}$ normalizes the following groups of order $q$ :

$$
U_{1}:=\left\{\left.\left(\begin{array}{ccc}
1 & -\frac{1}{2} a^{2} & a \\
& 1 & \\
& -a & 1
\end{array}\right) \right\rvert\, a \in \mathbb{F}_{q}\right\} \text { and } U_{2}:=\left\{\left.\left(\begin{array}{ccc}
1 & \\
-\frac{1}{2} a^{2} & 1 & -a \\
a & 1
\end{array}\right) \right\rvert\, a \in \mathbb{F}_{q}\right\} .
$$

What is more, $U_{1} \rtimes T_{0}$ and $U_{2} \rtimes T_{0}$ are both Frobenius subgroups.
Suppose that $M$ does not contain the projective image of $U_{1}$. Then, since $M$ contains $T_{0}$, we conclude that $M \cap U_{1}=\{1\}$. Define $M_{1}$ to be the projective image in $S$ of $U_{1} \rtimes T_{0}$, and observe that $M_{1}$ acts 2-transitively on $\left(M_{1}: M \cap M_{1}\right)$. Let $\Lambda:=\left\{M u \mid u \in U_{1}\right\}$; we conclude that the set-wise stabilizer of $\Lambda$ acts 2 -transitive. Lemma 1.6 .8 implies that $G$ contains a section isomorphic to $\operatorname{Sym}(q)$ and the result follows.

Clearly the same argument applies if $M$ does not contain the projective image of $U_{2}$. Thus we may assume that $M$ contains the projective image of $\left\langle U_{1}, U_{2}\right\rangle$. This projective image is a subfield subgroup isomorphic to $\mathrm{SO}_{3}(q)$. Thus $\mathrm{SO}_{3}(q) \leq M$. Now observe that $g$ acts fixed-point-freely by conjugation on the conjugates $U_{i}^{g^{\prime}}$ for $i \in\{1,2\}$. Therefore, we may apply the argument above also to the groups $U_{1}^{g^{\prime}}$ and $U_{2}^{g^{\prime}}$. We conclude, again, that $G$ contains a section isomorphic to $\operatorname{Sym}(q)$, or else that $M$ contains $\left\langle\mathrm{SO}_{3}(q), U_{1}^{g^{\prime}}, U_{2}^{g^{\prime}}\right\rangle=S$ and the result follows.

Assume now that $q>2$ is even and $g^{\prime} \in M$. Let $X$ be the subgroup of $S$ that is isomorphic to $\mathrm{SU}_{2}(q)$ and acts trivially on $\langle x\rangle$, where $x$ is the third basis vector of the basis $\mathcal{B}$ for $V$. Then $\left(g^{\prime}\right)^{q+1}$ acts fixed-point-freely on the unipotent subgroups

$$
U_{1}:=\left\{\left.\left(\begin{array}{ccc}
1 & \alpha & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \right\rvert\, \alpha \in \mathbb{F}_{q}\right\}, U_{2}:=\left\{\left.\left(\begin{array}{ccc}
1 & 0 & 0 \\
\alpha & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \right\rvert\, \alpha \in \mathbb{F}_{q}\right\}
$$

of $X$. Therefore, arguing as usual, we can assume $\left\langle U_{1}, U_{2}\right\rangle=X \leq M$. Hence $Y=\left\langle X, g^{\prime}\right\rangle \leq M$.
Now $Y$ is a maximal subgroup of $S$ in the $\mathcal{C}_{1}$ class (see for instance [10]). Then [46, Prop. 4.2] implies that $(S: Y)$ contains a beautiful subset with respect to the action of $S$ and, checking the proof of 46, Prop. 4.4] we see that there is always a beautiful subset of size at least $q$. We conclude that either $M$ contains $S$ (and the result follows) or else $M \cap S=Y$ and ( $G: M$ ) contains a subset $\Lambda$ of size at least $q$ on which $S_{\Lambda}$ acts 2-transitively. Then Lemma 1.6 .8 implies that $G$ contains a section isomorphic to $\operatorname{Sym}(q)$ and the result follows.

Finally, suppose that $G$ is almost simple and $M$ is core-free. Either the action of $G$ on $(G: M)$ is not binary (and we are done) or else $G$ contains a section isomorphic to $\operatorname{Sym}(q)$. Lemma 2.1.1 implies that $q \leq 5$; we confirm the result for $q \in\{3,4,5\}$ with a magma computation.

Lemma 2.2.11. Let $G$ contain a subgroup $S \cong \mathrm{SU}_{4}(q) / Z$, where $Z$ is a central subgroup of $\mathrm{SU}_{4}(q)$ and $q>2$. We let $\mathcal{B}:=\left(e_{1}, e_{2}, f_{1}, f_{2}\right)$ be a hyperbolic basis for the underlying unitary space. Let $x \in S$ be the projection in $S$ of

$$
\tilde{x}=\left(\begin{array}{llll}
a & & & \\
& 1 & & \\
& & a^{-1} & \\
& & & 1
\end{array}\right)
$$

written with respect to $\mathcal{B}$, where $a$ is an element of $\mathbb{F}_{q}^{*}$ of order $q-1$. Let $M<G$ with $x \in M$. Then one of the following holds:

1. $G$ contains a section isomorphic to $\operatorname{Sym}(q)$;
2. $M$ contains $S$;
3. the action of $G$ on $(G: M)$ is not binary.

In particular, if $G$ is almost simple with socle $S$ and $M$ is core-free, then the action of $G$ on $(G: M)$ is not binary.

Proof. Suppose that the action of $G$ on $(G: M)$ is binary, and write $X=\langle x\rangle$. Let $y$ be any element of one of the following forms:

$$
\left(\begin{array}{cccc}
1 & \alpha & & \\
& 1 & & \\
& & 1 & \\
& & -\bar{\alpha} & 1
\end{array}\right) \text { or }\left(\begin{array}{cccc}
1 & & & \alpha \\
& 1 & -\bar{\alpha} & \\
& & 1 & \\
& & & 1
\end{array}\right)
$$

or the transpose of these forms (in each case $\alpha \in \mathbb{F}_{q}^{*}$ ). Now let $U=\left\langle y^{h} \mid h \in X\right\rangle$. In all four cases, $U$ is a group of order $q$ that is normalized by $X$. In the usual way, we conclude that either $M$ contains $U$, or else there is a subset, $\Lambda:=\{M u \mid u \in U\}$, of $(G: M)$, on which $G_{\Lambda}$ acts 2-transitively. In the latter case Lemma 1.6 .8 implies that $G$ contains a section isomorphic to $\operatorname{Sym}(q)$ and the result follows. On the other hand, if the former case holds for all four unipotent subgroups in question, then, since these four subgroups generate $S$, we conclude that $M$ contains $S$ and the result follows.

Finally, suppose that $G$ is almost simple and $M$ is core-free. Either the action of $G$ on $(G: M)$ is not binary (and we are done) or else $G$ contains a section isomorphic to $\operatorname{Sym}(q)$. Lemma [2.1.1 implies that $q \leq 8$. One can check directly that $\mathrm{SU}_{4}(8)$ does not contain a section isomorphic to $\operatorname{Alt}(8)$; we confirm the result for $q \in\{3,4,5,7\}$ with a magma computation.

The groups we deal with in Lemmas 2.2.12, 2.2.13 and 2.2.14 have already been considered in previous lemmas; however, here, we choose a different distinguished element and we prove that every faithful transitive action containing this element gives rise to a non-binary action.

Lemma 2.2.12. Let $S=\operatorname{Sp}_{4}(q)$ where $q=2^{a}$ with $a \geq 2$, and suppose that $S \leq G \leq \operatorname{Aut}(S)$. Let $g$ be the element

$$
\left(\begin{array}{llll}
a & & & \\
& b & & \\
& & b^{-1} & \\
& & & a^{-1}
\end{array}\right)
$$

written with respect to a hyperbolic basis $\left(e_{1}, e_{2}, f_{2}, f_{1}\right)$, where $a, b \in \mathbb{F}_{q}$ are of order $q-1$. Let $M$ be any core-free subgroup of $G$ that contains $g$. Then the action of $G$ on $(G: M)$ is not binary.

Proof. We assume that the action of $(G: M)$ is binary, and show a contradiction. Suppose, first, that $q \geq 8$. Let $T=\langle g\rangle$ and consider the groups $U_{1}, \ldots, U_{4}$, all of order $q$, which contain elements of shape

$$
\left(\begin{array}{llll}
1 & & & \\
& 1 & u & \\
& & 1 & \\
& & & 1
\end{array}\right), \quad\left(\begin{array}{llll}
1 & & & \\
& 1 & & \\
& u & 1 & \\
& & & 1
\end{array}\right), \quad\left(\begin{array}{llll}
1 & & & u \\
& 1 & & \\
& & 1 & \\
& & & 1
\end{array}\right), \quad\left(\begin{array}{llll}
1 & & & \\
& 1 & & \\
& & 1 & \\
u & & & 1
\end{array}\right)
$$

respectively. Observe that, for all $i=1, \ldots, 4$, the group $T$ normalizes $U_{i}$ and acts fixed-point-freely upon it. Thus, using our usual argument, either $G$ contains $U_{i}$ or $G$ has a section isomorphic to $\operatorname{Alt}(q)$. From Lemma 2.1.1, $G$ does not contain a section isomorphic to Alt(7). Thus $G$ contains $\left\langle U_{1}, U_{2}, U_{3}, U_{4}\right\rangle \cong$ $\mathrm{Sp}_{2}(q) \times \mathrm{Sp}_{2}(q)$, where $\left\langle U_{1}, U_{2}, U_{3}, U_{3}\right\rangle$ is the subgroup of $S$ that stabilizes the subspaces $\left\langle e_{1}, f_{1}\right\rangle$ and $\left\langle e_{2}, f_{2}\right\rangle$.

Now we repeat the argument with the groups $U_{5}, \ldots, U_{8}$, all of order $q$, which contain elements of shape

$$
\left(\begin{array}{cccc}
1 & u & & \\
& 1 & & \\
& & 1 & u \\
& & & 1
\end{array}\right), \quad\left(\begin{array}{cccc}
1 & & u & \\
& 1 & & \\
& & 1 & \\
& & & 1
\end{array}\right), \quad\left(\begin{array}{llll}
1 & & & \\
u & 1 & & \\
& & 1 & \\
& & u & 1
\end{array}\right), \quad\left(\begin{array}{llll}
1 & & & \\
& 1 & & \\
u & & 1 & \\
& u & & 1
\end{array}\right)
$$

respectively. As before we find that either there is a beautiful subset, or else $G$ contains each of the groups $U_{5}, \ldots, U_{8}$. The first possibility is ruled out as before because $S$ does not admit a section isomorphic to $\operatorname{Alt}(7)$. Therefore $M \geq\left\langle\mathrm{Sp}_{2}(q) \times \mathrm{Sp}_{2}(q), U_{5}, \ldots, U_{8}\right\rangle=S$ and hence $M$ contains $S$, a contradiction.

If $q=4$, then the result is confirmed using a magma computation.
Lemma 2.2.13. Let $S=\operatorname{PSU}_{3}(q)$ with $q>2$, and suppose that $S \leq G \leq \operatorname{Aut}(S)$. Let $T$ be the projective image of a maximal torus of $\mathrm{SU}_{3}(q)$ of order $(q+1)^{2}$, and let $T .2$ be a subgroup of $N_{S}(T)$. Let $M$ be any core-free subgroup of $G$ that contains T.2. Then the action of $G$ on $(G: M)$ is not binary.

Proof. When $q \in\{3,4,5\}$, the veracity of this lemma is verified with the auxiliary help of magma.
Suppose that $q>5$. Consulting [10, Table 8.5], we see that there are two possibilities for $M \cap S$ : either $M \cap S=T .2$ or $M \cap S=T . \operatorname{Sym}(3)$. In the latter case, $M$ is a maximal subgroup of $S M$. Therefore, the action of $S M$ on $(S M: M)$ is primitive, and we know that the action is not binary, thanks to the argument in [45, Section 6]. Thus, by Lemma 1.6.2, the action of $G$ on $(G: M)$ is also not binary.

It turns out that the argument in [45, Section 6] can be used for the case $M \cap S=T .2$ as well. First, recall that if $M \cap S=T . \operatorname{Sym}(3)$, then the action of $S M$ on $(S M: M)$ is permutation equivalent to the natural action of $S M$ on

$$
\begin{aligned}
& \left\{\left\{V_{1}, V_{2}, V_{3}\right\} \mid \operatorname{dim}_{\mathbb{F}_{q^{2}}}\left(V_{1}\right)=\operatorname{dim}_{\mathbb{F}_{q^{2}}}\left(V_{2}\right)=\operatorname{dim}_{\mathbb{F}_{q^{2}}}\left(V_{3}\right)=1, V=V_{1} \perp V_{2} \perp V_{3},\right. \\
& \left.V_{1}, V_{2}, V_{3} \text { non-degenerate }\right\} .
\end{aligned}
$$

If $M \cap S=T .2$, then the action of $S M$ on $(S M: M)$ is permutation equivalent to the natural action of $S M$ on

$$
\begin{gathered}
\Lambda:=\left\{\left(V_{1},\left\{V_{2}, V_{3}\right\}\right) \mid \operatorname{dim}_{\mathbb{F}_{q^{2}}}\left(V_{1}\right)=\operatorname{dim}_{\mathbb{F}_{q^{2}}}\left(V_{2}\right)=\operatorname{dim}_{\mathbb{F}_{q^{2}}}\left(V_{3}\right)=1, V=V_{1} \perp V_{2} \perp V_{3},\right. \\
\\
\left.V_{1}, V_{2}, V_{3} \text { non-degenerate }\right\} .
\end{gathered}
$$

Now fix $M \cap S=T .2$ and identify $(S M: M)$ with the given set $\Lambda$. Let $e_{1}, e_{2}, e_{3}$ be a basis of $V$ such that the matrix of the Hermitian form with respect to this basis is the identity. Thus $\lambda_{0}:=\left(\left\langle e_{1}\right\rangle,\left\{\left\langle e_{2}\right\rangle,\left\langle e_{3}\right\rangle\right\}\right) \in$ $\Lambda$.

Consider $\Lambda_{0}:=\left\{\left(V_{1},\left\{V_{2}, V_{3}\right\}\right) \in \Lambda \mid V_{1}=\left\langle e_{1}\right\rangle\right\}$. Clearly, $(S M)_{\Lambda_{0}}=(S M)_{\left\langle e_{1}\right\rangle},(S M)_{\Lambda_{0}} / Z\left((S M)_{\Lambda_{0}}\right)$ is almost simple with socle isomorphic to $\mathrm{PSL}_{2}(q)$ (here we are using $q>3$ ), and the action of $(S M)_{\Lambda_{0}}$ on $\Lambda_{0}$

| $G(q)$ | $E_{8}(q)$ | $E_{7}(q)$ | $E_{6}(q)$ | ${ }^{2} E_{6}(q)$ | $F_{4}(q)$ | $G_{2}(q)$ | ${ }^{3} D_{4}(q)$ |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m$ | 9 | 8 | 6 | 4 | 4 | 3 | 3 |
| $r$ | 7 | 5 | 4 | 2 | 2 | 2 | 2 |
| $N$ | $q^{8}$ | $q^{7}$ | $q^{8}$ | $q^{18}$ | $q^{10}$ | $q^{4}$ | $q^{10}$ |
| $q$ |  |  |  | $q>3$ | $q>3$ | $q>5$ | $q>5$ |

Table 2.2.2: Values of $m$ such that $\mathrm{SL}_{m}(q) / Z \leq G(q)$
is permutation equivalent to the action of $(S M)_{\left\langle e_{1}\right\rangle}$ on $\Lambda_{0}^{\prime}:=\left\{\left\{W_{1}, W_{2}\right\} \mid \operatorname{dim}\left(W_{1}\right)=\operatorname{dim}\left(W_{2}\right),\left\langle e_{1}\right\rangle^{\perp}=\right.$ $W_{1} \perp W_{2}, W_{1}, W_{2}$ non degenerate \}. Therefore $(S M)^{\Lambda_{0}}$ is an almost simple primitive group with socle isomorphic to $\mathrm{PSL}_{2}(q)$ and having degree $\left|\Lambda_{0}\right|=q(q-1) / 2$. Applying [45, Theorem 1.3] to $(S M)^{\Lambda_{0}}$, we obtain that $(S M)^{\Lambda_{0}}$ is not binary and hence there exist two $\ell$-tuples ( $\left\{W_{1,1}, W_{1,2}\right\}, \ldots,\left\{W_{\ell, 1}, W_{\ell, 2}\right\}$ ) and $\left(\left\{W_{1,1}^{\prime}, W_{1,2}^{\prime}\right\}, \ldots,\left\{W_{\ell, 1}^{\prime}, W_{\ell, 2}^{\prime}\right\}\right)$ in $\Lambda_{0}^{\ell}$ which are 2-subtuple complete for the action of $(S M)_{\Lambda_{0}}$ but not in the same $(S M)_{\Lambda_{0}}$-orbit. By construction the two $\ell$-tuples

$$
\begin{aligned}
& I:=\left(\left(\left\langle e_{1}\right\rangle,\left\{W_{1,1}, W_{1,2}\right\}\right),\left(\left\langle e_{1}\right\rangle,\left\{W_{2,1}, W_{2,2}\right\}\right), \ldots,\left(\left\langle e_{1}\right\rangle,\left\{W_{\ell, 1}, W_{\ell, 2}\right\}\right)\right), \\
& J:=\left(\left(\left\langle e_{1}\right\rangle,\left\{W_{1,1}^{\prime}, W_{1,2}^{\prime}\right\}\right),\left(\left\langle e_{1}\right\rangle,\left\{W_{2,1}^{\prime}, W_{2,2}^{\prime}\right\}\right), \ldots,\left(\left\langle e_{1}\right\rangle,\left\{W_{\ell, 1}^{\prime}, W_{\ell, 2}^{\prime}\right)\right\}\right)
\end{aligned}
$$

are in $\Lambda^{\ell}$ and are 2-subtuple complete. Moreover, $I$ and $J$ are not in the same $S M$-orbit. Thus $S M$ is not binary on $\Lambda=(S M: M)$. Now, $G$ is not binary on $(G: M)$ by Lemma 1.6.2,

Lemma 2.2.14. Let $S=\operatorname{PSp}_{4}(q)$ where $q \in\{3,5\}$, and suppose that $S \leq G \leq \operatorname{Aut}(S)$. Let $T$ be a torus of $S$ of size $\frac{1}{2}(q-1)^{2}$, and let $M$ be any core-free subgroup of $G$ that contains $T$. Then the action of $G$ on $(G: M)$ is not binary.

Proof. In each case we use magma: we consider all almost simple groups $G$ with socle one of these two groups; we then compute all the core-free subgroups $M$ having order divisible by $(q-1)^{2} / 2$; finally we prove, in all cases, that the action of $G$ on the right coset of $M$ is not binary.

To test this, we have divided our algorithm in two cases: when $|M|^{3} \leq|G|$, since we could not afford to determine the permutation representation explicitly having too many points available, we have generated, for $10^{6}$ times, two cosets $M g_{1}$ and $M g_{2}$ of $M$ in $G$, and we tested whether Lemma 1.8.4 applies with $\omega_{0}:=M, \omega_{1}:=M g_{1}$ and $\omega_{2}:=M g_{2}$ (observe that for this test we do not need to construct the permutation representation of $G$ on the right cosets of $M$ ); when $|M|^{3}>|G|$, we have constructed the permutation representation of $G$ on the cosets of $M$ and we looked (extensively) for pairs for the form $\left(\left(\omega_{1}, \omega_{2}, \ldots, \omega_{\ell}\right),\left(\omega_{1}^{\prime}, \omega_{2}^{\prime}, \ldots, \omega_{\ell}^{\prime}\right)\right)$, with $\ell \leq 4$, which are 2 -subtuple complete but not in the same orbit.

### 2.2.3 Exceptional groups

Lemma 2.2.15. Suppose that $G$ is almost simple with socle $G_{0}=G(q)$, an exceptional group of Lie type as in Table 2.2.2, and let $m$ be the value given in the table.
(i) Then $G_{0}$ has a subgroup $L \cong \operatorname{SL}_{m}(q) / Z$, where $Z$ is central in $\mathrm{SL}_{m}(q)$.
(ii) Adopt the assumptions on $q$ in the last line of Table 2.2.2, and let $x \in L$ be the element as in the statement of Lemma 2.2.5, of order $q^{m-2}-1$ (if $q>2$ ) or $2^{m-1}-1$ (if $q=2$ ). If $M$ is any core-free subgroup of $G$ that contains $x$, then the action of $G$ on $(G: M)$ is not binary.
(iii) If $x$ is the element in part (ii), then $\left|C_{G}(x)\right|<N$, where $N$ is as in Table 2.2.2.

Proof. (i) The existence of these subgroups $L$ follows easily from inspection of extended Dynkin diagrams (this fact will also be used in the proofs of Propositions 3.3.1 and 3.4.1, where we also provide additional comments).
(ii) Let $L \cong \mathrm{SL}_{m}(q) / Z$ be the subgroup of (i), and let $x \in L$ be the element as in the statement of Lemma 2.2.5. Suppose that $M$ is a core-free subgroup of $G$ containing $x$, and assume that the action of $G$ on $(G: M)$ is binary. We apply Lemma 2.2.5, by our assumptions on $q$, the only possibilities are
(a) $G$ has a section isomorphic to $\operatorname{Alt}\left(q^{m-2}\right)$ (if $q>2$ ) or $\operatorname{Alt}\left(2^{m-1}\right)($ if $q=2)$, or
(b) $M$ contains $L$.

The possibility (a) is excluded by Lemma 2.1 .2 together with our assumptions on $q$.
Hence $L \leq M$. Using Theorem 3.1.1, it is straightforward to see that any core-free maximal subgroup $H$ of $G(q)$ containing $M$ is either parabolic, or of maximal rank, or a subgroup $F_{4}(q)$ or $C_{4}(q)$ of ${ }^{2} E_{6}(q)$. (Actually this follows from [75] except for the case where $q=2$.) Hence we can argue exactly as in the proofs of Propositions 3.3.1, 3.4.1 and 3.6.1 (case (4) of the proof) that there are subgroups $A<S$ of $G$ with the following properties:
(1) $A \leq L$ and $A \not \leq H$ for any maximal subgroup $H$ of $G(q)$ containing $M$, and
(2) $A \cong \mathrm{SL}_{r}(q), S \cong \mathrm{SL}_{r+1}(q) / Z$, where $r$ is as in Table 2.2.2,

By Lemma 1.6.10, there is a subset $\Delta$ of $(G: M)$ such that $|\Delta|=q^{r}$ and $G^{\Delta} \geq \operatorname{ASL}_{r}(q)$. Since $\operatorname{Alt}\left(q^{r}\right)$ is not a section of $G$, by Lemma 2.1.2 and our assumptions on $q$, this contradicts the assumption that the action of $G$ on $(G: M)$ is binary, completing the proof of (ii).
(iii) There is a simple adjoint algebraic group $\bar{G}$ over the algebraic closure $\overline{\mathbb{F}}_{q}$, and a Frobenius endomorphism $F$ of $\bar{G}$, such that $G(q)$ is the socle of the fixed point group $\left(\bar{G}^{F}\right)^{\prime}$. The element $x$ lies in $L=\left(\bar{L}^{F}\right)^{\prime}$, where $\bar{L}$ is a subsystem subgroup $A_{m-1}$ of $\bar{G}$. The composition factors of the restriction of the Lie algebra $L(\bar{G})$ to $\bar{L}$ can be found in Tables 8.1 - 8.5 of [76], from which it is easy to work out the action of $x$ on $L(\bar{G})$, and hence obtain the upper bound $\operatorname{dim} C_{L(\bar{G})}(x) \leq R$, where $R$ is as follows:

$$
\begin{array}{c|ccccccc}
G(q) & E_{8}(q) & E_{7}(q) & E_{6}(q) & { }^{2} E_{6}(q) & F_{4}(q) & G_{2}(q) & { }^{3} D_{4}(q) \\
\hline R & 8 & 7 & 8 & 18 & 10 & 4 & 10
\end{array}
$$

Here is an example of such a computation for the case where $G(q)={ }^{2} E_{6}(q)$ : here $\bar{L}=A_{3}$ and

$$
L(\bar{G}) \downarrow A_{3}=L\left(A_{3}\right) / \lambda_{1}^{4} / \lambda_{3}^{4} / \lambda_{2}^{4} / 0^{7}
$$

where in this notation, $\lambda_{1}$ denotes the irreducible 4-dimensional module $V_{A_{3}}\left(\lambda_{1}\right)$, and so on. The action of $x$ on the module $\lambda_{1}$ is $\left(1, a, \lambda, \lambda^{q}\right)$, where $\lambda \in \overline{\mathbb{F}}_{q}$ has order $q^{2}-1$ and $a=\lambda^{-1-q}$. Hence $x$ has fixed point space of dimension 1 on $\lambda_{1}$ and $\lambda_{3}$, and of dimension 0 on $\lambda_{2}$ (which is $\wedge^{2}\left(\lambda_{1}\right)$ ). It follows that

$$
\operatorname{dim} C_{L(\bar{G})}(x) \leq 3+4+4+7=18
$$

In each case there is in fact a subgroup of $\bar{G}$ of dimension $R$ centralizing $x$ : for $E_{8}$ and $E_{7}$ this is just a maximal torus; for $E_{6}$ it is a subgroup $T_{5} A_{1}$ (where $T_{5}$ denotes a 1-dimensional torus); for ${ }^{2} E_{6}$ it is a subgroup $T_{2} A_{2} A_{2}$ (since $x \in A_{2}<A_{3}=\bar{L}$, and this $A_{2}$ centralizes $A_{2} A_{2}$ in $\bar{G}$ ); similarly in $F_{4}$ it is $T_{2} A_{2}$; in $G_{2}$ it is $T_{1} A_{1}$ and in ${ }^{3} D_{4}$ it is $T_{1} A_{1}^{3}$. Hence these subgroups are the full centralizers of $x$ in $\bar{G}$ (noting that $C_{\bar{G}}(x)$ is connected), and hence taking fixed points under the Frobenius endomorphism $F$, we see that $\left|C_{\bar{G}^{F}}(x)\right|$ is as follows:

| $G(q)$ | $\left\|C_{\bar{G}^{F}}(x)\right\|$ | $G(q)$ | $\left\|C_{\bar{G}^{F}}(x)\right\|$ |
| :---: | :---: | :---: | :---: |
| $E_{8}(q)$ | $\left(q^{7}-1\right)(q-1), q>2$ | ${ }^{2} E_{6}(q)$ | $\left(q^{2}-1\right)\left\|A_{2}\left(q^{2}\right)\right\|$ |
|  | $2^{8}-1, q=2$ | $F_{4}(q)$ | $\left(q^{2}-1\right)\left\|A_{2}(q)\right\|$ |
| $E_{7}(q)$ | $\left(q^{6}-1\right)(q-1), q>2$ | $G_{2}(q)$ | $(q-1)\left\|A_{1}(q)\right\|$ |
| $E_{6}(q)$ | $2^{7}-1, q=2$ | $\left.q^{4}-1\right)(q-1)\left\|A_{1}(q)\right\|, q>2$ |  |
|  | $\left(2^{5}-1\right)\left\|A_{1}(2)\right\|, q=2$ | $(q-1)\left\|A_{1}\left(q^{3}\right)\right\|$ |  |
|  |  |  |  |

Since $x$ is centralized by no graph or field automorphisms, it follows that $\left|C_{G}(x)\right|<N$, where $N$ is as in Table 2.2.2. This completes the proof.

Lemma 2.2.16. Let $G$ contain a subgroup $S$ isomorphic to ${ }^{2} F_{4}(q)(q>2),{ }^{2} G_{2}(q)(q>3)$ or ${ }^{2} B_{2}(q)$ ( $q>2$ ). Then there exists an element $x \in S$ of order $q-1$ such that, if $x \in M<G$, then one of the following holds:

1. $G$ contains a section isomorphic to $\operatorname{Sym}(q)$;
2. $M$ contains $S$;
3. the action of $G$ on $(G: M)$ is not binary.

In particular if $G$ is almost simple, then we can choose $x$ to have the following properties:
(i) If $M$ is any core-free subgroup of $G$ that contains $x$, then the action of $G$ on $(G: M)$ is not binary.
(ii) $\left|C_{G}(x)\right|=(q-1)^{2}, q-1$ or $q-1$, according as $S={ }^{2} F_{4}(q),{ }^{2} G_{2}(q)$ or ${ }^{2} B_{2}(q)$, respectively.

Proof. Suppose first that $S={ }^{2} F_{4}(q)$. Let $T$ be a maximal torus of $S$ of order $(q-1)^{2}$, and choose $x \in T$ of order $q-1$ such that $C_{\operatorname{Aut}(S)}(x)=T$ (such an element exists by [96]). Let $M$ be a subgroup of $G$ containing $x$ and assume that the action of $G$ on $(G: M)$ is binary.

The structure of the root subgroups of $S$ with respect to $T$ can be found in [47, Theorem 2.4.5(d)]. If $U_{1}$ is a root subgroup of type $A_{1}^{2}$ with respect to $T$, then either $U_{1}$ is contained in $M$ or else $U_{1} \rtimes\langle x\rangle$ acts 2-transitively on $\Lambda=\left\{M u \mid u \in U_{1}\right\}$, a set of size $q \geq 8$. In the latter case Lemma 1.6 .8 implies that $G$ contains a section isomorphic to $\operatorname{Sym}(q)$ and the result follows. Suppose, then that $U_{1} \leq M$. The same argument applies to $U_{1}^{-}$, the "opposite" root group of type $A_{1}^{2}$. We can apply the same argument to a root group $U_{2}$, of type $B_{2}$, although in this case we consider $Z\left(U_{2}\right) \rtimes\langle x\rangle$, and we conclude that $Z\left(U_{2}\right) \leq M$. The same argument applies to $U_{2}^{-}$, the "opposite" root group of type $B_{2}$. Since $\left\langle U_{1}^{ \pm}, Z\left(U_{2}^{ \pm}\right)\right\rangle=S$, it follows that $M \geq S$ and the result follows.

In the special case where $G$ is almost simple and $M$ is core-free, Lemma 2.1.2 implies that $G$ does not contain a section isomorphic to $\operatorname{Alt}(q)$ and the result follows.

Now suppose that $S={ }^{2} G_{2}(q)$. We refer to the main theorem of 102 for basic information about this group. We may choose an element $x \in S$ of order $q-1$ such that $C_{\operatorname{Aut}(S)}(x)=\langle x\rangle$ and $x$ normalizes a Sylow 3-subgroup $P$ of $S$. If $Z=Z(P)$, then $|Z|=q$ and [102, item (3)] implies that $\langle x\rangle$ acts fixed-point-freely on $Z$. Let $M$ be a subgroup of $G$ containing $x$ and assume that the action of $G$ on $(G: M)$ is binary.

Suppose that $Z \not \leq M$. Then $Z \cap M=\{1\}$ and, identifying $(G: M)$ with the cosets of $M$ we can set $\Lambda=\{M z \mid z \in Z\}$, a subset of $(G: M)$ of size $q$. Then $Z \rtimes\langle g\rangle$ acts 2-transitively on $\Lambda$. Lemma 1.6.8 implies that $G$ contains a section isomorphic to $\operatorname{Sym}(q)$ and the result follows.

We may suppose, then, that $Z \leq M$. The same argument applies to the "opposite" Sylow 3-subgroup $P^{-}$of $S$ and so we obtain a second subgroup, $Z^{-}$, on which $\langle x\rangle$ acts fixed-point-freely, and which is contained in $M$. Thus $\left\langle Z, Z^{-}, x\right\rangle \leq M$. From the list of maximal subgroups of ${ }^{2} G_{2}(q)$ in [86], we see that either $\left\langle Z, Z^{-}, x\right\rangle=S$ or $\left\langle Z, Z^{-}, x\right\rangle \leq 2 \times \mathrm{PSL}_{2}(q)$. The latter is not possible since $\langle x\rangle$ is fixed point free on $Z$. Hence $\left\langle Z, Z^{-}, x\right\rangle=S$. Therefore, $M \geq S$ and the result follows.

In the special case where $G$ is almost simple and $M$ is core-free, we note that $G$ has no section isomorphic to $\operatorname{Alt}(5)$ (because 5 does not divide $|S|$ ) and the result follows.

Suppose finally that $S={ }^{2} B_{2}(q)$. We refer to 98 . The first part of the argument for ${ }^{2} G_{2}(q)$ applies here word-for-word, except that this time $P$ is a Sylow 2-subgroup of $S$. As in that previous case, we obtain that $M$ contains $\left\langle Z_{1}, Z_{2}\right\rangle$, where $Z_{1}$ and $Z_{2}$ are the centres of two distinct Sylow 2-subgroups of $S$. From the list of the maximal subgroups of $S$ in [98, Theorem 9] we obtain $\left\langle Z_{1}, Z_{2}\right\rangle=S$, completing the proof as before. In the special case where $G$ is almost simple and $M$ is core-free, we note that $G$ has no section isomorphic to $\operatorname{Alt}(3)$ (because 3 does not divide $|S|$ ) and the result follows.

### 2.3 Results on odd-degree actions

In this section we present two results, both proved using magma. Our methods are described in full in $\$ 1.8$.

Lemma 2.3.1. Let $M_{0}$ be one of the following groups

$$
\begin{aligned}
& \mathrm{PSL}_{2}(r)(r \leq 31), \mathrm{PSL}_{3}(r)(r \in\{2,3,4,5\}), \mathrm{PSL}_{4}(r)(r \in\{2,3,5\}), \\
& \mathrm{PSU}_{3}(r)(r \in\{3,4,5,8\}), \mathrm{PSU}_{4}(r)(r \in\{2,3,4,5,7\}), \mathrm{PSU}_{5}(r)(r \in\{2,3,4,5,7\}), \mathrm{PSU}_{6}(2), \\
& \operatorname{PSp}_{4}(r)(r \in\{2,3,4,5,7\}), \mathrm{PSp}_{6}(r)(r \in\{2,3\}), \mathrm{PSp}_{8}(2), \\
& \Omega_{7}(r)(r \in\{3,5,7,9\}), \mathrm{P}_{8}^{-}(r)(r \in\{2,3,4\}), \Omega_{8}^{+}(2), \Omega_{10}^{ \pm}(2), \Omega_{10}^{-}(3), \\
& { }^{2} B_{2}(8),{ }^{2} B_{2}(32), G_{2}(r)(r \in\{3,4,5\}),{ }^{3} D_{4}(r)(r \in\{2,3\}), F_{4}(r)(r \in\{2,3\}),{ }^{2} F_{4}(2)^{\prime},{ }^{2} E_{6}(2) .
\end{aligned}
$$

Let $M$ be an almost simple group with socle $M_{0}$ and let $H$ be a core-free subgroup of $M$ with $|M: H|$ odd. Then either the action of $M$ on $(M: H)$ is not binary or $M, M_{0}$ and $H$ are as in Table 2.3.1.

Table 2.3.1: Some odd-degree binary actions

| $M_{0}$ | $M$ | $\|M: H\|$ |
| :---: | :---: | :---: |
| $\operatorname{Alt}(5)$ | $\operatorname{Sym}(5)$ | 5 |
| $\operatorname{Alt}(5)$ | $\operatorname{Alt}(5)$ | 15 |
| $\mathrm{PSL}_{2}(8)$ | $\mathrm{PSL}_{2}(8)$ | 63 |
| $\mathrm{PSL}_{2}(8)$ | $\mathrm{PSL}_{2}(8) .3$ | 189 |
| $\mathrm{PSL}_{2}(16)$ | $\mathrm{PSL}_{2}(16)$ | 255 |
| $\mathrm{PSL}_{2}(16)$ | $\mathrm{PSL}_{2}(16) .2$ | 51 |

Proof. Suppose first that $M_{0} \notin\left\{F_{4}(2), F_{4}(3),{ }^{3} D_{4}(3),{ }^{2} E_{6}(2)\right\}$. We have constructed all the groups $M$ under consideration and all odd index subgroups $H$ of $M$. The construction of $H$ can be done quite efficiently working recursively: for each group $M$ under consideration, the list of the maximal core-free subgroups $X$ of $M$ is either already available in magma, or it can be constructed. Then, we can simply select the subgroups $X$ with $|M: X|$ odd. In all cases, $X$ is considerably smaller than $M$ and we can directly compute the odd index subgroups of $X$. Thus, we obtain all odd index subgroups of $M$.

We then check that the action of $M$ on $(M: H)$ is not binary with a combination of techniques. First, we have checked the permutation character bound, see Lemma 1.8.1, then we have tried to apply Lemma 1.8 .4 and finally Lemma 1.6 .15 . For permutation groups failing this method, the degree of the action was less than $10^{7}$ and hence we simply searched for non-binary $t$-tuples (with $t$ relatively small: except when $M_{0}=\operatorname{PSU}_{4}(2)$, it was sufficient to consider $\left.t \in\{3,4\}\right)$.

Suppose now that $M_{0}=F_{4}(2)$. In particular, either $M=F_{4}(2)$ or $M=F_{4}(2) .2$. Let $H$ be a subgroup of $M$ with $|M: H|$ odd and let $K$ be a maximal core-free subgroup of $M$ with $H \leq K$. We have reported in Table 3.2.1 the maximal subgroups of $M$. We have proved in Proposition 3.2.1 that the action of $M$ on $(M: K)$ is not binary and hence we may suppose that $H<K$. Using the information on $K$ in Table 3.2.1, we have computed the odd index subgroups of $K$ and we have checked that, except when $M=F_{4}(2) \cdot 2$, $K=\left[2^{22}\right](\operatorname{Sym}(2) \mathrm{wr} \operatorname{Sym}(2))$ and $H$ is a Sylow 2-subgroup of $K$, the action of $K$ on $(K: H)$ is not binary. In particular, we may suppose that $M=F_{4}(2) .2, K=\left[2^{22}\right](\operatorname{Sym}(2) \mathrm{wr} \operatorname{Sym}(2))$. Now, let $T$ be a maximal subgroup of $M$ with $H \leq T$ and with $T \cong\left[2^{20}\right]$.Alt ( 6$) \cdot 2^{2}$ (clearly, this is possible from Table 3.2.1 and from Sylow's theorem). Now, the action of $T$ on $(T: H)$ is not binary and hence so is the action of $M$ on ( $M: H$ ).

Suppose now that $M_{0}=F_{4}(3)$. In particular, $M=F_{4}(3)=M_{0}$. Let $H$ be a subgroup of $M$ with $|M: H|$ odd and let $K$ be a maximal subgroup of $M$ with $H \leq M$. From [73], we see that $K$ is isomorphic to either $2 . \Omega_{9}(3)$ or to $2^{2} . \mathrm{P} \Omega_{8}^{+}(3) . \operatorname{Sym}(3)$. For each of these two groups, we have computed all the subgroups $H$ with $|K: H|$ odd and we have checked that either $K$ is not binary on $(K: H)$, or $K=H$, or $K=2^{2} \cdot \mathrm{P}_{8}^{+}(3) \cdot \operatorname{Sym}(3)$ and $|K: H|=3$. In the first case, we deduce that the action of $M$ on $(M: H)$ is not binary and hence we may consider one of the remaining cases. In all cases, 7 is a divisor of both $|M: H|$ and $|H|$ and also $7^{2}$ is the largest power of 7 dividing $|M|$. Let $V$ be a Sylow 7 -subgroup
of $M$. Now, $N_{M}(V)$ lies in the maximal rank subgroup ${ }^{3} D_{4}(3) .3$, since the normalizer of $V$ in ${ }^{3} D_{4}(3)$ is $V . \mathrm{SL}_{2}(3)$, see 57 for example. Therefore, we have $N_{M}(V) \cong V .\left(3 \times \mathrm{SL}_{2}(3)\right)$, where the action of $3 \times \mathrm{SL}_{2}(3)$ by conjugation on $V$ has two orbits of cardinality 4 on the subgroups of $V$ having order 7 . This shows that we are in the position to apply Lemma 1.6 .15 with the prime 7 and we deduce that also the action of $M$ on the remaining cases is not binary.

Suppose now that $M_{0}={ }^{3} D_{4}(3)$. In this case, the maximal subgroups of $M$ are not available in magma. However, using [73], we see that if $H$ is a core-free maximal subgroup of $M$ and $|M: H|$ is odd, then $H \cap M_{0}$ is either $G_{2}(3)$, or $\left(7 \times \mathrm{SU}_{3}(3)\right) .2$, or $\left(\mathrm{SL}_{2}(27) \circ \mathrm{SL}_{2}(3)\right) .2$. Now, using the structure of these groups, we may construct them as subgroups of $M$ (for instance, when $H \cap M_{0} \cong 7 \times \mathrm{SU}_{3}(3) .2$, we may construct $H$ by computing the normalizer of a suitable subgroup of $M$ of order 7 ). Then we have checked that the action was not binary using the permutation character method. Then, we worked recursively on the subgroups of $H$, as explained in the first part of this proof.

Suppose now that $M_{0}={ }^{2} E_{6}(2)$. We use the information in 105 . In this case, the maximal subgroups of $M$ are not available in magma. However, using the information in the work of Wilson [105], we see that if $|M: H|$ is odd, then $X$ is contained in a parabolic subgroup $P$ of $M$. The information in [105] is also enough to construct the (abstract) group $P$ explicitly using magma. At this point, we have constructed for each parabolic subgroup $P$ of $M$, all the subgroups $H$ of $P$ with $|P: H|$ odd. We have checked that the action of $P$ on $(P: H)$ is not binary (by witnessing non-binary triples or quadruples), unless $|P: H| \in\{1,3,9,15,45\}$. At this point, the only actions that we need to discuss are the actions of $M$ on $(M: H)$, where $H \leq P$ for some maximal parabolic subgroup $P$ of $M$ and for some subgroup $H$ of $P$ with $|P: H| \in\{1,3,9,15,45\}$.

Using the structure of $P$, we deduce that 7 divides both $|M: H|$ and $|H|$. Now, a Sylow 7-subgroup $V$ of $M$ has order $49=7^{2}$ and $M$ has two conjugacy classes of elements of order 7 , which are referred to as type 7 A and 7 B . The group $V$ contains 8 subgroups of order 7 , where 4 of these subgroups consist only of 7 A elements and 4 of these subgroups consist only of 7 B elements. Using this information, it is readily seen that we may use Lemma 1.6 .15 to show that $M$ is not binary on $(M: H)$.

Lemma 2.3.2. Let $M$ be an almost simple group with socle $M_{0}$ a sporadic simple group. Then every faithful odd degree action of $M$ is not binary.

Proof. We use magma to verify the statement of the lemma. We divide the proof into three cases.
(1) Suppose that $M_{0}$ is one of the following groups:

$$
\begin{gathered}
M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_{1}, J_{2}, J_{3}, H S, M c L, H e, R u, S u z, C o_{1}, C o_{2}, C o_{3} \\
F i_{22}, F i_{23}, F i_{24}^{\prime}, H N \text { or } O^{\prime} N
\end{gathered}
$$

Let $M$ be an almost simple group with socle $M_{0}$. We use magma to construct all odd index subgroups $H$ of $M$, using a recursive routine as described at the start of the previous proof. For the groups $C o_{1}$ and $F i_{24}^{\prime}$, we used some extra information at the start of the recursion: in these cases, rather than computing all maximal subgroups of $M$ and selecting those of odd index, we explicitly constructed only the maximal subgroups of $M$ having odd index using the information in [28] or in the online atlas of finite group representations. Then, to determine all the other odd index subgroups of $M$ we can simply run the procedure above, for each of the maximal subgroups that we have constructed.

Given $(M, H)$ as above, and using the terminology in $\$ 1.8$, either Test 3 (direct analysis) or Test 1 (using the permutation character) is enough to complete the magma calculations. When $M_{0}$ is one of $F i_{23}$, $F i_{24}^{\prime}, H N, O^{\prime} N$ and $C o_{3}$, we used Test 1 as well as Lemma 1.6.15,
(2) Suppose now that $M_{0}$ is one of the following groups:

$$
J_{4}, L y, T h, B
$$

| $M_{0}$ | $K$ | $H$ | $\|H: K\|$ |
| :---: | :---: | :---: | :---: |
| $J_{4}$ | $2^{3+12} \cdot\left(\operatorname{Sym}(5) \times \operatorname{PSL}_{3}(2)\right)$ | $2^{3+12} \cdot\left(\operatorname{Sym}(4) \times \operatorname{PSL}_{3}(2)\right)$ | 5 |
| $B$ | $2^{2+10+20} \cdot\left(M_{22}: 2 \times \operatorname{Sym}^{2}(3)\right)$ | $2^{2+10+20} \cdot\left(M_{22}: 2 \times 2\right)$ | 3 |
| $B$ | $\left[2^{35}\right] \cdot\left(\operatorname{Sym}(5) \times \operatorname{PSL}_{3}(2)\right)$ | $\left[2^{32}\right] \cdot\left(\operatorname{Sym}(4) \times \operatorname{PSL}_{3}(2)\right)$ | 5 |
| $M$ | $2^{2+11_{+22}} \cdot\left(M_{24} \times \operatorname{Sym}_{2}(3)\right)$ | $2^{2+11+22} \cdot\left(M_{24} \times 2\right)$ | 3 |
| $M$ | $2^{5+10+20} \cdot\left(\operatorname{Sym}(3) \times \operatorname{PSL}_{5}(2)\right)$ | $2^{5+10+20} \cdot\left(2 \times \operatorname{PSL}_{5}(2)\right)$ | 3 |

Table 2.3.2

In this case we proceeded similarly at first, by constructing all core-free odd index subgroups $H$ of $M$. However in some cases the index $|H: M|$ was too large to prove directly that the action of $M$ on $(M: H)$ is not binary. Thus we took a different approach as follows.

Let $M$ be an almost simple group with socle $M_{0}$ in the given list, let $H$ be a core-free subgroup of $M$ with $|M: H|$ odd and let $K$ be a maximal core-free subgroup of $M$ with $H \leq K$. Recall that Conjecture 1.2 has been verified for primitive actions of almost simple groups having socle a sporadic simple group [34]. Therefore, we may suppose that $H<K$. Now, rather than studying the action of $M$ on $(M: H)$, we study the action of $K$ on $(K: H)$. We use magma to confirm that, except when $\left(M, M_{0}, K, H\right)$ is in Table 2.3.2, the action of $K$ on the right cosets of $H$ is not binary (by witnessing a non-binary triple or a non-binary 4-tuple). Now Lemma 1.6 .2 implies that the action of $M$ on $(M: H)$ is not binary.

For the remaining cases in Table [2.3.2, we have computed the permutation character for the action of $M$ on the right cosets of $H$ and we have checked that this action is not binary using the permutation character bound (Test 1 in §1.8).
(3) Finally suppose that $M$ is the Monster group. The maximal subgroups $K$ of $M$ having odd index can be deduced from [104 and are:

$$
\begin{gathered}
2^{1+24} \cdot \mathrm{Co}_{1}, \quad 2^{10+16} \cdot \mathrm{O}_{10}^{+}(2), \quad 2^{2+11+22} \cdot\left(M_{24} \times \operatorname{Sym}(3)\right), \\
2^{5+10+20} \cdot\left(\operatorname{Sym}(3) \times \mathrm{PSL}_{5}(2)\right), \quad 2^{3+6+12+18} \cdot\left(\mathrm{PSL}_{3}(2) \times 3 . \operatorname{Sym}(6)\right) .
\end{gathered}
$$

Let $K$ be one of these groups and let $H$ be a subgroup of $K$ with $|M: H|$ odd. We show that the action of $M$ on the right cosets of $H$ is not binary. If $K=H$, then this follows from 34. Suppose then $H<K$. Observe that, when $K \cong 2^{1+24} . C o_{1}$, we have $O_{2}(K) \leq H \leq K$ and $K / O_{2}(K) \cong C o_{1}$. Therefore, in this case, the proof follows from the fact that the faithful transitive actions of $C o_{1}$ of odd degree are not binary.

For the remaining three groups $K$, we have constructed all odd index subgroups $H$ of $K$. Except when ( $M_{0}, K, H$ ) is in the last two lines of Table [2.3.2, we have verified that the action of $K$ on the right cosets of $H$ is not binary (by using three techniques: via the permutation character method, or via Lemma 1.8.4, or when the degree of the action is not very large via witnessing a non-binary triple or a non-binary 4 -tuple). In particular, the action of $M$ on the right cosets of $H$ is not binary in these cases.

It remains to deal with the cases in Table 2.3.2 here we cannot argue as in the paragraph above, because the information in the character table stored in magma is not enough to construct the permutation character under consideration. When $K=2^{5+10+20} .\left(\operatorname{Sym}(3) \times L_{5}(2)\right)$ and $H=2^{5+10+20} .\left(2 \times L_{5}(2)\right)$, the action of $M$ on the right cosets of $H$ is not binary by using Lemma 1.6.15 applied with the prime $p=7$ (for details see [34, Lemma 5.1 and 5.2]). When $K=2^{2+11+22} .\left(M_{24} \times \operatorname{Sym}(3)\right)$ and $H=2^{2+11+22} .\left(M_{24} \times 2\right)$, the action of $M$ on the right cosets of $H$ is not binary by using Lemma 1.6.15 applied with the prime $p=7$ (for details see [34, Lemma 5.1 and 5.2]).

### 2.4 Results on centralizers

The first result in this subsection is taken from [41, §6].
Proposition 2.4.1. Let $G=\mathrm{Cl}_{n}(q)$ be a simple classical group, and let $1 \neq g \in H$.
(i) Then $\left|C_{G}(g)\right|>f(n, q)$, where $f(n, q)$ is as in Table 2.4.1.
(ii) In particular, for any $G=\mathrm{Cl}_{n}(q)$ we have

$$
\left|C_{G}(g)\right|>\frac{q^{\lceil(n-1) / 2\rceil}}{4}\left(\frac{q-1}{2 q e\left(\log _{q}(2 n)+4\right)}\right)^{1 / 2}
$$

Table 2.4.1: Lower bounds for centralizers in classical groups

| $H$ | $f(n, q)$ |
| :---: | :---: |
| $\operatorname{PSL}_{n}(q)$ | $\frac{q^{n-1}}{e\left(1+\log _{q}(n+1)\right)(n, q-1)}$ |
| $\operatorname{PSU}_{n}(q)$ | $\frac{q^{n-1}}{(n, q+1)} \cdot\left(\frac{q-1}{e(q+1)\left(2+\log _{q}(n+1)\right)}\right)^{1 / 2}$ |
| $\operatorname{PSp}_{n}(q), \operatorname{P\Omega }_{n}^{\epsilon}(q)$ | $\frac{q^{\lceil(n-1) / 2\rceil}}{4}\left(\frac{q-1}{2 q e\left(\log _{q}(2 n)+4\right)}\right)^{1 / 2}$ |

The next result is Lemma 5.7 of [84].
Lemma 2.4.2. Let $S=G(q)$ be a be a simple group of Lie type, let $d$ be the untwisted rank of $S$, and let $g$ be an element of S. Then

$$
\left|C_{S}(g)\right| \geq \frac{(q-1)^{d}}{|\operatorname{Inndiag}(S)|}
$$

### 2.5 Outer automorphisms of groups of Lie type

Here we record a well-known result which classifies all outer automorphisms of prime order of finite groups of Lie type. In the terminology of [47, Defn. 2.5.13], all such are diagonal, field, graph-field or graph automorphisms. A proof can be found in [68, Prop. 1.1].

Proposition 2.5.1. Let $L=L(q)$ be a simple group of Lie type over $\mathbb{F}_{q}$, and let $\alpha$ be an automorphism of $L$ of prime order. If $L$ is classical with natural module $V$, suppose that $\alpha$ does not lie in PGL $(V)$; and if $L$ is exceptional, suppose that $\alpha \notin \operatorname{Inndiag}(L)$. Then one of the following holds:
(i) $\alpha$ is a field or graph-field automorphism, and $C_{L}(\alpha)$ is of type $L\left(q^{1 /|\alpha|}\right)$ or ${ }^{2} L\left(q^{1 / 2}\right)$ (or ${ }^{3} D_{4}\left(q^{1 / 3}\right)$ when $\left.L=D_{4}(q)\right)$;
(ii) $\alpha$ is a graph automorphism and the possibilities are as in Table 2.5.1. (In the last column of the table, $t$ denotes a long root element.)

Table 2.5.1

| $L$ | $\|\alpha\|$ | possible types for $C_{L}(\alpha)$ |
| :--- | :--- | :--- |
| $\operatorname{PSL}_{n}^{\epsilon}(q)$ | 2 | $\operatorname{PSO}_{n}(q)(n$ odd $)$ |
|  |  | $\operatorname{PSO}_{n}^{ \pm}(q), \operatorname{PSp}_{n}(q)(n$ even, $q$ odd $)$ |
| $D_{4}(q),{ }^{3} D_{4}(q)$ | 3 | $\operatorname{Sp}_{n}(q), C_{\mathrm{Sp}_{n}(q)}(t)(n$ even, $q$ even $)$ |
| $E_{2}(q), A_{2}^{\epsilon}(q)$ if $(3, q)=1$ |  |  |
| $E_{6}^{\epsilon}(q)$ | $G_{2}(q), C_{G_{2}(q)}(t)$ if 3 divides $q$ | 2 |
|  |  |  |
| $F_{4}(q), C_{F_{4}(q)}(t)(q$ even $)$ |  |

### 2.6 On fusion and factorization

Before working our way through the families of maximal subgroups given in Theorem 3.1.1 we record a few useful lemmas.

In the next lemma, given a group $G$ and two subgroups $X$ and $Y$ with $X<Y<G$, we say that $Y$ controls fusion of $X$ in $G$ if, whenever $X^{g}<Y$ for some $g \in G$, there exists $y \in Y$ such that $X^{g}=X^{y}$.

Lemma 2.6.1. Let $G$ be a finite group, and let $A<S<H$ be subgroups of $G$ with the following properties:
(i) $S$ controls fusion of $A$ in $G$;
(ii) $H$ controls fusion of $S$ in $G$;
(iii) $S^{x} \leq H$ for all $x \in N_{G}(A)$.

Then $N_{G}(A)=N_{H}(A)\left(N_{G}(S) \cap N_{G}(A)\right)$.
Proof. Let $x \in N_{G}(A)$. Then $S^{x} \leq H$ by (iii), so by (ii) there exists $h \in H$ such that $S^{x}=S^{h}$. Hence $x^{-1} \in H N_{G}(S)$, so

$$
\begin{equation*}
N_{G}(A) \subseteq H N_{G}(S) . \tag{2.6.1}
\end{equation*}
$$

Now let $y \in N_{G}(S)$. Then $A^{y} \leq S$, so by (i) there exists $s \in S$ such that $A^{y}=A^{s}$. Hence $N_{G}(S) \subseteq S N_{G}(A)$, and, by intersecting both sides of this inclusion by $N_{G}(S)$, it follows that

$$
\begin{equation*}
N_{G}(S)=S\left(N_{G}(S) \cap N_{G}(A)\right) . \tag{2.6.2}
\end{equation*}
$$

From (2.6.1) and (2.6.2), we deduce

$$
N_{G}(A) \subseteq H\left(N_{G}(S) \cap N_{G}(A)\right)
$$

and the proof follows by intersecting both sides of this inclusion by $N_{G}(A)$.
In our application of the above lemma we will also need the following result on factorizations of simple groups, which is a consequence of Theorem A of 69.

Lemma 2.6.2. Let $G$ be an almost simple group with socle $G_{0}=\operatorname{PSL}_{n}\left(q^{a}\right)$, where $a \geq 2$. Suppose $G$ has a factorization $G=A B$, where $A, B$ are core-free subgroups and $A$ is contained in a subfield subgroup $N_{G}\left(\operatorname{PSL}_{n}\left(q^{b}\right)\right)$, where $\mathbb{F}_{q^{b}} \subset \mathbb{F}_{q^{a}}$. Then $\left(n, q^{a}\right)=(2,4),(2,9)$, $(2,16)$ or $(3,4)$, and the possibilities for $A, B$ are as follows:

| $G_{0}$ | $A \cap G_{0}$ | $B \cap G_{0}$ |
| :---: | :---: | :---: |
| $\mathrm{PSL}_{2}(4)$ | $\operatorname{Sym}(3), C_{3}$ | $D_{10}$ |
| $\operatorname{PSL}_{2}(9)$ | $\operatorname{Sym}(4), \operatorname{Alt}(4), C_{3}$ | $\operatorname{Alt}(5)$ |
| $\mathrm{PSL}_{2}(16)$ | $\operatorname{PSL}_{2}(4)$ | $D_{34}$ |
| $\mathrm{PSL}_{3}(4)$ | $\operatorname{PSL}_{3}(2)$ | $\operatorname{Alt}(6)$ |

Proof. Theorem A of [69, together with [72] imply the listed restrictions on the pairs ( $n, q^{a}$ ). What is more we know the maximal subgroups of $G_{0}$ which contain $A \cap G_{0}$ and $B \cap G_{0}$; these are listed as the first entry in each column in the tables in [72]. We then check directly whether it is possible for $A \cap G_{0}$ or $B \cap G_{0}$ to be non-maximal. The proof follows with a case-by-case analysis or with a magma computation.

## Chapter 3

## Exceptional Groups

In this chapter we prove the following theorem.
Theorem 3.1. Let $G$ be an almost simple primitive permutation group with socle an exceptional group of Lie type. Then $G$ is not binary.

Note that the Suzuki and Ree groups ${ }^{2} B_{2}(q)$ and ${ }^{2} G_{2}(q)$ have been dealt with in 45], so we do not consider them here. Note, too, that the groups with socle ${ }^{2} F_{4}(2)^{\prime}$ were dealt with in [34], hence these too are excluded from what follows.

Our notation for finite groups of Lie type is in line with standard references such as 47]. Dynkin diagrams are labelled as in 9 .

### 3.1 Maximal subgroups of exceptional groups of Lie type

We shall need a substantial amount of information about maximal subgroups of finite exceptional groups of Lie type, taken from many sources. A summary follows; note that we write Lie ( $p$ ) to mean the set of simple groups of Lie type that are defined over a field of characteristic $p$. By the rank of a finite group of Lie type $G(q)$, we mean the Lie rank of the corresponding simple algebraic group.

Theorem 3.1.1. ([78, Theorem 8]) Let $G$ be an almost simple group with socle $G(q)$, an exceptional group of Lie type over $\mathbb{F}_{q}, q=p^{a}$, and let $H$ be a maximal subgroup of $G$. Then one of the following holds:
(I) $H$ is a parabolic subgroup;
(II) $H$ is reductive of maximal rank: the possibilities for $H$ are determined in [74, Tables 5.1,5.2];
(III) $G(q)=E_{7}(q), p>2$ and $H \cap G(q)=\left(2^{2} \times \mathrm{P} \Omega_{8}^{+}(q) \cdot 2^{2}\right)$. $\operatorname{Sym}(3)$ or $H \cap G(q)={ }^{3} D_{4}(q) .3$;
(IV) $G(q)=E_{8}(q), p>5$ and $H \cap G(q)=\mathrm{PGL}_{2}(q) \times \operatorname{Sym}(5)$;
(V) $H \cap G(q)$ is as in Table 3.1.1 below;
(VI) $H$ is of the same type as $G$ - that is, $H^{\prime} \cap G(q)=G\left(q_{0}\right)$ or a twisted version, where $\mathbb{F}_{q_{0}}$ is a subfield of $\mathbb{F}_{q}$;
(VII) $H$ is an exotic local subgroup, as in Table 3.1.2;
(VIII) $G(q)=E_{8}(q), p>5$ and $H=(\operatorname{Alt}(5) \times \operatorname{Alt}(6)) \cdot 2^{2}$;
(IX) $F^{*}(H)=H_{0}$ is simple, and not in $\operatorname{Lie}(p)$ : the possibilities for $H_{0}$ are given up to isomorphism by [77;
(X) $F^{*}(H)=H\left(q_{0}\right)$ is simple and in $\operatorname{Lie}(p)$; moreover $\operatorname{rank}\left(H\left(q_{0}\right)\right) \leq \frac{1}{2} \operatorname{rank}(G)$, and one of the following holds:
(a) $q_{0} \leq 9$;
(b) $H\left(q_{0}\right)=A_{2}^{\epsilon}(16)$;
(c) $H\left(q_{0}\right)=A_{1}\left(q_{0}\right),{ }^{2} B_{2}\left(q_{0}\right)$ or ${ }^{2} G_{2}\left(q_{0}\right)$, and $q_{0} \leq t(G)$ where $t(G)$ is as in Table 3.1.3 (given by [67]).

In cases (I)-(VIII), $H$ is determined up to $G(q)$-conjugacy.
Note that Table 3.1.1 includes the subgroups $F_{4}(q)<E_{8}(q)$ for $q=3^{a}$; these were omitted from the list in [78, but discovered later in [30.

Recent work of Craven has eliminated many of the possibilities left in parts (IX) and (X) of the above theorem:

Theorem 3.1.2. ([31, 32, 33]) Let $G$ be as in Theorem 3.1.1, and let $H$ be a maximal subgroup of $G$.
(i) Suppose $F^{*}(H)$ is an alternating group $\operatorname{Alt}(n)$. Then $n \in\{6,7\}$. Moreover, if $n=7$, then $G$ is of type $E_{7}$ or $E_{8}$.
(ii) Suppose $H$ is as in part (X) of Theorem 3.1.1 (and not in any of the other parts), and $H\left(q_{0}\right) \neq A_{1}\left(q_{0}\right)$. Then one of the following holds:
(a) $G(q)=E_{8}(q), q=3^{a}$, and $H\left(q_{0}\right)=\mathrm{PSL}_{3}(3)$ or $\mathrm{PSU}_{3}(3)$;
(b) $G(q)=E_{8}(q), q=2^{a}$ and $H\left(q_{0}\right)=\mathrm{PSL}_{3}(4), \mathrm{PSU}_{3}(4), \mathrm{PSU}_{3}(8), \mathrm{PSU}_{4}(2)$ or ${ }^{2} B_{2}(8)$.
(iii) Suppose $H$ is as in part (X) of Theorem3.1.1 (and not in any of the other parts), and H $\left.q_{0}\right)=A_{1}\left(q_{0}\right)$. Then one of the following holds:
(a) $q_{0}=q$;
(b) $G(q)=E_{7}(q)$ and $q_{0}=7,8$ or 25 ;
(c) $G(q)=E_{8}(q)$.

Table 3.1.1: Possibilities for $H$ in (V) of Theorem 3.1.1

| $G(q)$ | possibilities for $F^{*}(H \cap G(q))$ |
| :--- | :--- |
| $G_{2}(q)$ | $A_{1}(q)(p \geq 7)$ |
| ${ }^{3} D_{4}(q)$ | $G_{2}(q)^{\prime}, A_{2}^{ \pm}(q)$ |
| $F_{4}(q)$ | $A_{1}(q)(p \geq 13), G_{2}(q)(p=7), A_{1}(q) G_{2}(q)(p \geq 3, q \geq 5)$ |
| $E_{6}^{\epsilon}(q)$ | $A_{2}^{ \pm}(q)($ only for $\epsilon=+$ and $p \geq 5), G_{2}(q)^{\prime}(p \neq 7,(q, \epsilon) \neq(2,-))$, <br>  <br> $C_{4}(q)(p \geq 3), F_{4}(q), A_{2}^{\epsilon}(q) G_{2}(q)^{\prime}$ |
| $E_{7}(q)$ | $A_{1}(q)(2$ classes, $p \geq 17,19), A_{2}^{\epsilon}(q)(p \geq 5), A_{1}(q) A_{1}(q)(p \geq 5)$, <br>  <br> $A_{1}(q) G_{2}(q)(p \geq 3, q \geq 5), A_{1}(q) F_{4}(q)(q \geq 4), G_{2}(q)^{\prime} C_{3}(q)$ |
| $E_{8}(q)$ | $A_{1}(q)(3$ classes, $p \geq 23,29,31), B_{2}(q)(p \geq 5), F_{4}(q)(p=3), A_{1}(q) A_{2}^{\epsilon}(q)(p \geq 5)$, <br>  <br> $G_{2}(q)^{\prime} F_{4}(q), A_{1}(q) G_{2}(q) G_{2}(q)(p \geq 3, q \geq 5)$, <br> $A_{1}(q) G_{2}\left(q^{2}\right)(p \geq 3, q \geq 5)$ |

Note that Table 3.1.1 contains a small refinement of the corresponding table in [78, Theorem 8] for $G(q)=E_{6}^{\epsilon}(q)$ and the $A_{2}^{ \pm}(q)$ and $A_{2}^{\epsilon}(q) G_{2}(q)^{\prime}$ entries. This refinement is justified in Remark 5.2 of [16]. Note also that in Table 3.1.1, we write $G_{2}(q)^{\prime}$ rather than $G_{2}(q)$ whenever $q=2$ is allowed; this is because $G_{2}(2)$ is not itself simple, but its derived subgroup is. There is another fact, concerning $A_{2}^{-}(2)$, we need to clarify in Table 3.1.1? there are two embeddings involving $A_{2}^{-}(q)$ with $q=2$, namely $A_{2}^{-}(q)$ in ${ }^{3} D_{4}(q)$,

Table 3.1.2: Exotic local subgroups in (VII) of Theorem 3.1.1

| $2^{3} \cdot \mathrm{SL}_{3}(2)$ | $<G_{2}(p)$ | $p>2$ |
| :--- | :--- | :--- |
| $3^{3} \cdot \mathrm{SL}_{3}(3)$ | $<F_{4}(p)$ | $p \geq 5$ |
| $3^{3+3} \cdot \mathrm{SL}_{3}(3)$ | $<E_{6}^{\epsilon}(p)$ | $p \equiv \epsilon \bmod 3, p \geq 5$ |
| $5^{3} \cdot \mathrm{SL}_{3}(5)$ | $<E_{8}\left(p^{a}\right)$ | $p \neq 2,5 ; a=1$ or 2, |
|  |  | as $p^{2} \equiv 1$ or $-1 \bmod 5$ |
| $2^{5+10} \cdot \operatorname{SL}_{5}(2)$ | $<E_{8}(p)$ | $p>2$ |

Table 3.1.3: Values of $t(G)$ in (X)(c) of Theorem 3.1.1] notation $d=(2, p-1)$

| $G$ | $G_{2}(q)$ | $F_{4}(q)$ | $E_{6}^{\epsilon}(q)$ | $E_{7}(q)$ | $E_{8}(q)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $t(G)$ | $12 d$ | $68 d$ | $124 d$ | $388 d$ | $1312 d$ |

and $A_{2}^{-}(q) G_{2}(q)^{\prime}$ in $E_{6}^{-}(q)$. Since $A_{2}^{-}(2)$ is not simple and not nilpotent for $q=2$, the listed groups are not equal to $F^{*}(H \cap G(q))$ in these cases; instead one should replace $A_{2}^{-}(2)$ by $3^{2}$.

We shall divide the proof of Theorem 3.1 according to the various parts of Theorem 3.1.1. Note for future reference that by Proposition 2.5.1, the maximal subgroups in the theorem that centralize field, graph-field or graph automorphisms of $G(q)$ are as follows:
(i) subfield or twisted subgroups as in part (VI);
(ii) the following subgroups in part (V):

$$
\begin{aligned}
& C_{4}(q), F_{4}(q)<G(q)=E_{6}^{\epsilon}(q), \\
& G_{2}(q), A_{2}^{\epsilon}(q)<G(q)={ }^{3} D_{4}(q) .
\end{aligned}
$$

### 3.2 Small exceptional groups of Lie type

In this section, we deal with some small exceptional groups of Lie type; this will allow us to avoid some degeneracies in later arguments.

Proposition 3.2.1. Theorem 3.1 holds when the socle of $G$ is one of the following exceptional groups of Lie type:

$$
\begin{aligned}
& { }^{2} B_{2}(q),{ }^{2} G_{2}(q), \\
& { }^{2} F_{4}(2)^{\prime},{ }^{3} D_{4}(2), F_{4}(2), \\
& G_{2}(3), G_{2}(4), G_{2}(5) .
\end{aligned}
$$

Proof. The groups with socle ${ }^{2} B_{2}(q),{ }^{2} G_{2}(q)$ were dealt with in 45]; groups with socle ${ }^{2} F_{4}(2)^{\prime}$ were dealt with in [34. The other possibilities have been handled using computational methods, and we describe these in turn.
Socle ${ }^{3} D_{4}(2)$. Let $G$ be an almost simple group with socle ${ }^{3} D_{4}(2)$. We have computed all the core-free maximal subgroups $M$ of $G$ and we have checked that the action of $G$ on $(G: M)$ is not binary. Except when $M=3^{2}: 2 \operatorname{Alt}(4)$ or $M=13: 4$ and $G={ }^{3} D_{4}(2)$, or $M=3^{2}: 2 \operatorname{Alt}(4) \times 3$ or $M=13: 12$ and $G={ }^{3} D_{4}(2): 3$, we have used the permutation character method, a.k.a. Lemma 1.8.1, In the remaining cases, where the permutation character method does not work, we have used Lemma 1.8.4,
Socle $F_{4}(2)$. Note that $\left|F_{4}(2)\right|=2^{24} \cdot 3^{6} \cdot 5^{2} \cdot 7^{2} \cdot 13 \cdot 17$ and $G=F_{4}(2)$ or $F_{4}(2) .2$. In Table 3.2.1 we list the maximal subgroups of $G$ and their indices in $G$, as given in [85]. Let $M$ be a core-free maximal subgroup of $G$.

Observe that $G$ has a unique conjugacy class of elements of order 5. Moreover, $F_{4}(2) .2$ has a unique conjugacy class of elements of order 7 . In $F_{4}(2)$ this conjugacy class of 7 -elements splits into two distinct

| Line | Max. subgroups $F_{4}(2)$ | Index | Max. subgroups of $F_{4}(2) .2$ | Index |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $\left(2_{+}^{1+8} \times 2^{6}\right): \operatorname{Sp}_{6}(2)$ | $3^{2} \cdot 5 \cdot 7 \cdot 13 \cdot 17$ | $\left[2^{20}\right]: \operatorname{Alt}(6) \cdot 2^{2}$ | $3^{4} \cdot 5 \cdot 7^{2} \cdot 13 \cdot 17$ |
| 2 | $\operatorname{Sp}_{8}(2)$ | $2^{8} \cdot 3 \cdot 7 \cdot 13$ | $\operatorname{Sp}_{4}(4): 4$ | $2^{15} \cdot 3^{4} \cdot 7^{2} \cdot 13$ |
| 3 | $\left[2^{20}\right]:\left(\operatorname{Sym}(3) \times \operatorname{PSL}_{3}(2)\right)$ | $3^{4} \cdot 5^{2} \cdot 7 \cdot 13 \cdot 17$ | $(\operatorname{Sym}(6) \mathrm{wr} 2) .2$ | $2^{15} \cdot 3^{2} \cdot 7^{2} \cdot 13 \cdot 17$ |
| 4 | $\mathrm{O}_{8}^{+}(2): \operatorname{Sym}(3)$ | $2^{11} \cdot 7 \cdot 13 \cdot 17$ | $\left[2^{22}\right](\operatorname{Sym}(3) \mathrm{wr} 2)$ | $3^{4} \cdot 5^{2} \cdot 7^{2} \cdot 13 \cdot 17$ |
| 5 | ${ }^{3} D_{4}(2): 3$ | $2^{12} \cdot 3 \cdot 5^{2} \cdot 17$ | $7^{2}:(3 \times 2 \operatorname{Sym}(4))$ | $2^{21} \cdot 3^{4} \cdot 5^{2} \cdot 13 \cdot 17$ |
| 6 | ${ }^{2} F_{4}(2)$ | $2^{12} \cdot 3^{3} \cdot 7^{2} \cdot 17$ | $F_{4}(2) \times 2$ | $2^{12} \cdot 3^{3} \cdot 7^{2} \cdot 17$ |
| 7 | $\operatorname{PSL}_{4}(3) .2$ | $2^{16} \cdot 5 \cdot 7^{2} \cdot 17$ | $\left[\operatorname{PSL}_{4}(3) .2\right] \cdot 2$ | $2^{16} \cdot 5 \cdot 7^{2} \cdot 17$ |
| 8 | $\left(\mathrm{PSL}_{3}(2) \times L_{3}(2)\right): 2$ | $2^{17} \cdot 3^{4} \cdot 5^{2} \cdot 13 \cdot 17$ | $\left[\left(\operatorname{PSL}_{3}(2) \times \operatorname{PSL}_{3}(2)\right): 2\right] .2$ | $2^{17} \cdot 3^{4} \cdot 5^{2} \cdot 13 \cdot 17$ |
| 9 | $3 .\left(3^{2}: Q_{8} \times 3^{2}: Q_{8}\right) . \operatorname{Sym}(3)$ | $2^{17} \cdot 5^{2} \cdot 7^{2} \cdot 13 \cdot 17$ | $\left[3 .\left(3^{2}: Q_{8} \times 3^{2}: Q_{8}\right) \cdot \operatorname{Sym}(3)\right] .2$ | $2^{17} \cdot 5^{2} \cdot 7^{2} \cdot 13 \cdot 17$ |

Table 3.2.1: Maximal subgroups of $F_{4}(2)$ and $F_{4}(2) .2$
$F_{4}(2)$-conjugacy classes; furthermore, $F_{4}(2)$ has two conjugacy classes of cyclic subgroups of order 7 . This information can be deduced from [85].

Let $p \in\{5,7\}$. From the information in the previous paragraph and from Lemma 1.6.15, we deduce that, if $p||M|$ and $p||G: M|$, then the action of $G$ on the cosets of $M$ is not binary. In particular, it remains to consider the case that, for each $p \in\{5,7\}, p^{2}$ divides $|M|$ or $p^{2}$ divides $|G: M|$.

When $M \in\left\{{ }^{3} D_{4}(2): 3,{ }^{2} F_{4}(2),\left(\mathrm{PSL}_{3}(2) \times \mathrm{PSL}_{3}(2)\right): 2\right\}$ and $G=F_{4}(2)$, or when $M \in\left\{\operatorname{Sp}_{4}(4):\right.$ $4,(\operatorname{Sym}(6)$ wr 2$\left.) \cdot 2,{ }^{2} F_{4}(2) \times 2,\left[\left(\operatorname{PSL}_{3}(2) \times \mathrm{PSL}_{3}(2)\right): 2\right] \cdot 2\right\}$ and $G=F_{4}(2) .2$, we have verified that the hypothesis of Lemma 1.8 .3 with $d=2$ holds true (by computing all proper subgroups $X$ of $M$ with $|M: X|$ odd). Thus, we deduce that either $G$ is not binary in its action on $(G: M)$ or 2 divides $|G: M|-1$. However, the second possibility yields a contradiction (in each case under consideration $|G: M|-1$ is odd). Therefore, $G$ is not binary on $(G: M)$. (Observe that for this computation we only need $M$ as an abstract group and we do not require the embedding of $M$ in $G$.)

When $M=3 .\left(3^{2}: Q_{8} \times 3^{2}: Q_{8}\right) . \operatorname{Sym}(3)$ and $G=F_{4}(2)$, or when $M=\left[3 .\left(3^{2}: Q_{8} \times 3^{2}: Q_{8}\right) . \operatorname{Sym}(3)\right] .2$ and $G=F_{4}(2) .2$, since there is not enough information in Table 3.2.1 to determine the isomorphism class of $M$, we have used magma to construct $M$ inside $G$. For this we used the fact that $M$ is the normalizer of a cyclic group of order 3 generated by an element in the conjugacy class $3 C$. This was possible because generators of $G$ and an element in the class $3 C$ are available in the online atlas webpage. Then, we have argued as in the previous paragraph applying Lemma 1.8.3. The group $M$ contains a unique subgroup $X$ (up to conjugacy), such that

- $|M: X|$ is odd,
- the permutation group $M_{X}$ induced by $M$ on $(M: X)$ is binary and
- every section of $M$ is isomorphic to some section of $M_{X}$.

This subgroup $X$ has index 3 in $M$ and $M_{X} \cong \operatorname{Sym}(3)$. As $M$ is maximal in $G$, we obtain that $G$ in its action on $(G: M)$ is primitive. If $G$ acting on $(G: M)$ has a suborbit of cardinality 3 , then it follows from [97] that $|M|$ divides 48 , which is clearly a contradiction. Therefore, $G$ in its action on $(G: M)$ has no suborbits of cardinality 3 . Thus, if $G$ is binary in its action on $(G: M)$, then, from the magma computation above, $G$ has no non-trivial suborbits of odd size in its action on $(G: M)$. However, this implies that $|G: M|-1$ is even, which is clearly a contradiction.

Using Table 3.2.1, we see that it remains to deal with the action of $G=F_{4}(2) .2$ on the right cosets of $M=\left[2^{22}\right]:(\operatorname{Sym}(3) \mathrm{wr} 2)$. First, we work with the restriction of this action to $G^{\prime}:=F_{4}(2)$. Using the generators of $G^{\prime}$, we may construct $M \cap G^{\prime}$ using the fact that it is a local subgroup (first by finding a Sylow 2-subgroup $P$ of $G^{\prime}$ and then by computing the normalizer of a suitable subgroup of $P$ having index 4). Let $K$ be a Sylow 3 -subgroup of $M \cap G^{\prime}$. We see that $K$ contains four 3 -elements in the class $3 C$, two 3 -elements in the class $3 A$ and two more 3 -elements in the class $3 B$. Thus we may write $K=\langle g, h\rangle$, where $g$ and $h$ are $3 A$ and $3 B$ elements (respectively) and $g h$ is a $3 C$ element. Using the formula $\left|x^{G} \cap M\right| /\left|x^{G}\right|$ we
may compute the number of fixed points of $x \in M$ without constructing the permutation representation explicitly. We see that $g$ and $h$ both fix 945 points, $g h$ fixes 81 points and $K$ fixes 9 points. Using this information, we see that there exists a $K$-invariant subset $\Lambda \subseteq(G: M)$ having cardinality 10 , say $\Lambda=\left\{\lambda_{0}, \lambda_{1}, \ldots, \lambda_{9}\right\}$, such that

$$
\begin{aligned}
& g^{\Lambda}:=\left(\lambda_{0}\right)\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)\left(\lambda_{4}, \lambda_{5}, \lambda_{6}\right)\left(\lambda_{7}\right)\left(\lambda_{8}\right)\left(\lambda_{9}\right), \\
& h^{\Lambda}:=\left(\lambda_{0}\right)\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)\left(\lambda_{4}\right)\left(\lambda_{5}\right)\left(\lambda_{6}\right)\left(\lambda_{7}, \lambda_{8}, \lambda_{9}\right),
\end{aligned}
$$

where $g^{\Lambda}$ and $h^{\Lambda}$ are the restrictions of $g$ and $h$ to $\Lambda$. It is now easy to verify that the two 10 -tuples

$$
\left(\lambda_{0}, \lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}, \lambda_{5}, \lambda_{6}, \lambda_{7}, \lambda_{8}, \lambda_{9}\right) \text { and }\left(\lambda_{0}, \lambda_{2}, \lambda_{3}, \lambda_{1}, \lambda_{4}, \lambda_{5}, \lambda_{6}, \lambda_{7}, \lambda_{8}, \lambda_{9}\right)
$$

are 2-subtuple complete for the action of $G$ on $(G: M)$. If the action of $G$ on $(G: M)$ is binary, then there exists $a \in G$ mapping the first 10 -tuple into the second 10-tuple. As $a$ fixes $\lambda_{0}$, we get $a \in G_{\lambda_{0}}=M$. Moreover, $a$ fixes set-wise $\Lambda$ and $a^{\Lambda}=\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$. Therefore, $G_{\Lambda} \cap G_{\lambda_{0}}$ has a Sylow 3 -subgroup of cardinality divisible by $3^{3}$, but this contradicts the fact that a Sylow 3 -subgroup of $M$ has cardinality $|K|=9$. (This construction is inspired from Example 2.2 in [45].)
Socle $G_{2}(q)(q \leq 5)$. These groups (and their automorphism groups) are available in magma . For each possible group $G$ we have computed its maximal subgroups. When $q=3$, we have then constructed the permutation representations and checked that the group is not binary by witnessing non-binary triples. When $q \in\{4,5\}$, we have computed the permutation characters and used Lemma 1.8.1; this test was always successful for proving that the action was not binary except when $q=5$ and $M \cong 2^{3}$. $\mathrm{PSL}_{3}(2)$. In this last case we generated, for $10^{6}$ times, two cosets $M g_{1}$ and $M g_{2}$ of $M$ in $G$, and we tested whether Lemma 1.8.4 applies with $\omega_{0}:=M, \omega_{1}:=M g_{1}$ and $\omega_{2}:=M g_{2}$. After a few iterations we have found a suitable $g_{1}$ and $g_{2}$ and hence the action of $G$ on $(G: M)$ is not binary.

In light of Proposition 3.2.1, we assume for the remainder of this section that the socle of $G$ is not one of the groups listed in the proposition.

### 3.3 Parabolic subgroups

In this section we prove Theorem 3.1 for parabolic actions of exceptional groups of Lie type. We use the notation $P_{i}$ (resp. $P_{i j}$ ) for a parabolic subgroup which corresponds to deleting node $i$ (resp. nodes $i, j$ etc.) from the Dynkin diagram. For twisted groups we shall adopt a similar convention using the untwisted Dynkin diagram: for example for ${ }^{3} D_{4}(q)$ the maximal parabolic subgroups are denoted by $P_{2}$ and $P_{134}$, and so on.

Here is the main result of the section. The cases excluded in the proposition (those in Table 3.3.1) will be dealt with in Lemma 3.3.2,

Proposition 3.3.1. Assume $G$ is almost simple with socle $G(q)$, an exceptional group of Lie type over $\mathbb{F}_{q}$, and suppose $G(q)$ is not as in Proposition 3.2.1. Let $H$ be a maximal parabolic subgroup of $G$, and $\Omega=(G: H)$. Suppose further that $(G(q), H)$ is not as in Table 3.3.1. Then $(G, \Omega)$ is not binary.

Table 3.3.1: Exceptions in Prop. 3.3.1

| $G(q)$ | $H$ |
| :---: | :---: |
| ${ }^{3} D_{4}(q), q \in\{3,4,5\}$ | $P_{2}$ |
| $E_{6}(2)$ | $P_{2}$ |
| ${ }^{2} E_{6}(2)$ | $P_{2}, P_{4}, P_{16}$ |

Proof. Let $H=P_{i}$ (or $P_{i j}$ in cases where $G$ contains a graph automorphism of $G(q)$ ). Inspection of extended Dynkin diagrams shows that there exist subgroups $A \cong \mathrm{SL}_{r}(q)$ in $H$, and $S \cong \mathrm{SL}_{r+1}(q) / Z$ in $G$ (where $Z$ is central), such that $A \leq S$ and $S \not \leq H$, where $r$ is as in Table 3.3.2. Hence Lemma 1.6.10 produces a subset $\Delta$ of $\Omega$ for which $G^{\Delta}$ contains the 2-transitive group $\mathrm{ASL}_{r}(q)$ of degree $q^{r}$. If $G^{\Delta}$ does not contain $\operatorname{Alt}(\Delta)$, this implies that $G$ is not binary by Lemma 1.6.12, as required. So assume that $G^{\Delta} \geq \operatorname{Alt}(\Delta)$. Then $q^{r} \leq N_{G}$, where $N_{G}$ is as defined in Lemma 2.1.2. Also Alt $\left(q^{r}-1\right)$ must be a section of $H$. This implies that $(G, H, q)$ is either as in Table 3.3.1, or is one of the following:

$$
\begin{array}{r|ccc}
G(q) & G_{2}(q) & { }^{3} D_{4}(q) & { }^{2} F_{4}(q) \\
\hline H & P_{1}, \text { Borel } & P_{2} & \text { any parabolic }
\end{array}
$$

Consider $G_{2}(q)$. Here $H$ contains $T_{1}=\left\{h_{\alpha_{1}}(c): c \in \mathbb{F}_{q}\right\} \cong C_{q-1}$, and this acts fixed-point-freely on the root group $U=U_{-\alpha_{0}}$, where $\alpha_{0}$ is the longest root. Observe that $T_{1} U \cap H=T_{1}$. Hence, if we set $\Delta=\{H u: u \in U\} \subseteq \Omega$, then $|\Delta|=q$ and $G^{\Delta} \geq\left(T_{1} U\right)^{\Delta}=\mathrm{AGL}_{1}(q)$. Hence, if $q>N_{G}=6+\delta_{p, 5}$ (which is the case, as $q \neq 3,4,5$ by hypothesis), then as above, $G$ is not binary. A similar proof applies to the case $G(q)={ }^{3} D_{4}(q)$ : here $q=3,4,5$ are not excluded in the hypothesis, so these cases are included in Table 3.3.1.

Finally, consider $G(q)={ }^{2} F_{4}(q)$, and note that $q>2$ here, by hypothesis. In this case, the maximal parabolics are $P_{i}=Q_{i} L_{i}$ for $i=1,2$, where $Q_{i}$ is the unipotent radical and

$$
L_{1}=\mathrm{GL}_{2}(q), L_{2}={ }^{2} B_{2}(q) \times(q-1) .
$$

Let $H=P_{i}$ and $\Omega=(G: H)$, and suppose $(G, \Omega)$ is binary. Let $S \cong \mathbb{F}_{q}$ be the root subgroup corresponding to the highest root, and $S^{-}$its negative. For $i=1,2$ there is a torus $T_{1}<L_{i}$ of order $q-1$ acting fixed-point-freely on both $S$ and $S^{-}$. Since $S^{-} \not \leq P_{i}$, the Frobenius group $F=S^{-} T_{1}$ satisfies $F \cap P_{i}=T_{1}$, and so we obtain in the usual way a subset $\Delta$ of $\Omega$ with $G^{\Delta} \geq \operatorname{AGL}_{1}(q)$, forcing $q \leq 8$ by Lemma 2.1.2, If $q=8$ and $G^{\Delta} \geq \operatorname{Alt}(8)$, then $H=P_{i}$ must contain a section isomorphic to $\operatorname{Alt}(7)$, which is not the case. This final contradiction completes the proof.

Table 3.3.2: Values of $r$ in proof of Prop. 3.3.1
$\left.\left.\begin{array}{rl}\hline G(q)=E_{8}(q) & H=P_{i}, i \\ & r \\ & r\end{array} \right\rvert\, \begin{array}{cccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline\end{array}\right)$

The remaining cases are resolved by magma computations:
Lemma 3.3.2. Let $G$ be as in Proposition 3.3.1, and let $H$ be a maximal parabolic subgroup of $G$ as listed in Table 3.3.1. Let $\Omega=(G: H)$. Then $(G, \Omega)$ is not binary.

Proof. Suppose first that $G(q)={ }^{3} D_{4}(q)$; we refer to 43] for a description of the parabolic subgroup $H$ here (note that, although [43] assumes that $q$ is odd, [49] confirms that the same description applies for $q$ even). Let $T$ be a maximal torus contained in $H \cap G(q)$ that is isomorphic to $C_{q-1} \times C_{q^{3}-1}$. We assume that $H \cap G(q)$ contains the Borel subgroup generated by all positive root subgroups. Let $\alpha$ (resp. $\beta$ ) be the short (resp. long) fundamental root, and let $U$ be the short root group $X_{-2 \alpha-\beta}$; then $|U|=q^{3}$ and $U$ is not contained in $H$. What is more, [43, Table 2.3] confirms that $T$ acts transitively on the non-identity elements of $U$. Define $\Gamma=\{H u: u \in U\}$, a subset of $\Omega$ of order $q^{3}$. Then $U \rtimes T$ stabilizes $\Gamma$ and acts 2-transitively on $\Gamma$. Since $q \geq 3, G(q)$ does not contain a section isomorphic to $\operatorname{Alt}\left(q^{3}\right)$ by Lemma 2.1.1. Therefore, $\Gamma$ is a beautiful subset and Lemma 1.6 .12 yields the conclusion.

In the case where $G=E_{6}(2)$ or $E_{6}(2) .2$ and $H=P_{2}$, we compute the index $|G: H|$ and we select the complex irreducible characters of $G$ having degree at most $|G: H|$. Then we find all non-negative integer linear combinations of these irreducible characters having degree $|G: H|$. These combinations are our putative permutation characters. Then, for each of these characters, we use Lemma 1.8.1 to prove that the action under consideration is not binary.

Finally, for $G(q):={ }^{2} E_{6}(2)$, let $H$ be one of the parabolic subgroups in Table 3.3.1. Then we see that 5 divides both $|G: H|$ and $|H|$, but $5^{2}$ does not divide $|H|$. Moreover, $G$ contains a unique conjugacy class of elements of order 5 (see [28]). Therefore Lemma 1.6 .15 implies that the action of $G$ on $(G: H)$ is not binary.

### 3.4 Maximal rank subgroups

In this section we prove Theorem 3.1 in the case where the point stabilizer $H$ is a subgroup of maximal rank that is not the normalizer of a maximal torus in $G$. Such maximal subgroups are listed in Table 5.1 of [74. They will be listed in Tables 3.4.2-3.4.8 below, where for notational convenience we list each possibility for $H$ as a "type", which is a subgroup (usually equal to $H^{(\infty)}$ ) of small index in $H$.

Here is the main result of this section. The cases excluded in the proposition (those in Table 3.4.1 and also the case of socle ${ }^{2} F_{4}(q)$ ) will be handled later in Lemmas 3.4.2, 3.4.3 and 3.4.4,

Proposition 3.4.1. Assume $G$ is almost simple with socle $G(q)$, an exceptional group of Lie type over $\mathbb{F}_{q}$. Suppose $G(q)$ is not as in Proposition 3.2.1, and suppose also that $G(q) \neq{ }^{2} F_{4}(q)$. Let $H$ be a maximal subgroup of maximal rank in $G$, as in [74, Table 5.1], and let $\Omega=(G: H)$. Then either $(G, \Omega)$ is not binary, or $(G(q), H)$ is as in Table 3.4.1.

Table 3.4.1: Exceptions in Prop. 3.4.1

| $G(q)$ | type of $H$ |
| :---: | :---: |
| $E_{7}(q)$ | $A_{1}\left(q^{7}\right)$ |
| $E_{6}^{\epsilon}(q)$ | $A_{2}^{\epsilon}\left(q^{3}\right)$ |
| $E_{8}(2)$ | $A_{2}^{-}(2)^{4}, A_{2}^{-}\left(2^{4}\right)$ |
| ${ }^{2} E_{6}(2)$ | $A_{2}^{-}(2)^{3}, D_{4}(2) T_{2}$ |

Proof. We adopt the same method as in the previous section, using Lemma 1.6.10, In Tables 3.4.2-3.4.8 we have listed the possibilities for $H$, together with a subgroup $A \cong \operatorname{SL}_{r}\left(q^{u}\right)$ of $H$ (where $u=1$ or 2), such that $A$ is contained in a subgroup $S \cong \mathrm{SL}_{r+1}\left(q^{u}\right) / Z$ of $G$ that does not lie in $H$. We shall justify these assertions below.

Given the assertions on the tables, the argument proceeds as in the proof of Proposition 3.3.1; Lemma 1.6.10 produces a subset $\Delta$ of $\Omega$ of size $q^{r u}$, for which $G^{\Delta}$ contains $\operatorname{ASL}_{r}\left(q^{u}\right)$. If $G^{\Delta} \geq \operatorname{Alt}(\Delta)$, then $q^{r u} \leq N_{G}$

Table 3.4.2: Subgroups $H$ and $A$ for $E_{8}(q)$

| type of $H$ | $D_{8}(q)$ | $A_{1}(q) E_{7}(q)$ | $A_{8}(q)$ | $A_{2}(q) E_{6}(q)$ | $A_{4}(q)^{2}$ | $D_{4}(q)^{2}$ | $A_{2}(q)^{4}$ |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $A$ | $\mathrm{SL}_{8}(q)$ | $\mathrm{SL}_{7}(q)$ | $\mathrm{SL}_{7}(q)$ | $\mathrm{SL}_{6}(q)$ | $\mathrm{SL}_{5}(q)$ | $\mathrm{SL}_{4}(q)$ | $\mathrm{SL}_{3}(q)$ |
| type of $H$ | $A_{1}(q)^{8}(q>2)$ | $A_{8}^{-}(q)$ | $A_{2}^{-}(q) E_{6}^{-}(q)$ | $A_{4}^{-}(q)^{2}$ | $A_{4}^{-}\left(q^{2}\right)$ | $D_{4}\left(q^{2}\right)$ | ${ }^{3} D_{4}(q)^{2}$ |
| $A$ | $\mathrm{SL}_{2}(q)$ | $\mathrm{SL}_{3}\left(q^{2}\right)$ | $\mathrm{SL}_{3}\left(q^{2}\right)$ | $\mathrm{SL}_{2}\left(q^{2}\right)$ | $\mathrm{SL}_{2}\left(q^{2}\right)$ | $\mathrm{SL}_{2}\left(q^{2}\right)$ | $\mathrm{SL}_{3}(q)$ |
| type of $H$ | ${ }^{3} D_{4}\left(q^{2}\right)$ | $A_{2}^{-}(q)^{4}$ | $A_{2}^{-}\left(q^{2}\right)^{2}$ | $A_{2}^{-}\left(q^{4}\right)$ |  |  |  |
| $A$ | $\mathrm{SL}_{3}\left(q^{2}\right)$ | $\mathrm{SL}_{2}(q)$ | $\mathrm{SL}_{2}\left(q^{2}\right)$ | $\mathrm{SL}_{2}(q)$ |  |  |  |

Table 3.4.3: Subgroups $H$ and $A$ for $E_{7}(q)$

| type of $H$ | $A_{1}(q) D_{6}(q)$ | $A_{7}(q)$ | $A_{2}(q) A_{5}(q)$ | $A_{1}(q)^{3} D_{4}(q)$ | $A_{1}(q)^{7}(q>2)$ | $E_{6}(q) T_{1}$ |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $A$ | $\mathrm{SL}_{6}(q)$ | $\mathrm{SL}_{5}(q)$ | $\mathrm{SL}_{5}(q)$ | $\mathrm{SL}_{4}(q)$ | $\mathrm{SL}_{2}(q)$ | $\mathrm{SL}_{6}(q)$ |
| type of $H$ | $A_{7}^{-}(q)$ | $A_{2}^{-}(q) A_{5}^{-}(q)$ | $A_{1}\left(q^{3}\right)^{3} D_{4}(q)$ | $E_{6}^{-}(q) T_{1}$ |  |  |
| $A$ | $\mathrm{SL}_{2}\left(q^{2}\right)$ | $\mathrm{SL}_{3}\left(q^{2}\right)$ | $\mathrm{SL}_{3}(q)$ | $\mathrm{SL}_{3}\left(q^{2}\right)$ | - |  |

Table 3.4.4: Subgroups $H$ and $A$ for $E_{6}(q)$

| type of $H$ | $A_{1}(q) A_{5}(q)$ | $A_{2}(q)^{3}$ | $A_{2}\left(q^{2}\right) A_{2}^{-}(q)$ | $D_{4}(q) T_{2}$ | ${ }^{3} D_{4}(q) T_{2}$ | $D_{5}(q) T_{1}$ |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $A$ | $\mathrm{SL}_{4}(q)$ | $\mathrm{SL}_{3}(q)$ | $\mathrm{SL}_{2}\left(q^{2}\right)$ | $\mathrm{SL}_{4}(q)$ | $\mathrm{SL}_{3}(q)$ | $\mathrm{SL}_{5}(q)$ |

Table 3.4.5: Subgroups $H$ and $A$ for ${ }^{2} E_{6}(q)$

| type of $H$ | $A_{1}(q) A_{5}^{-}(q)$ | $A_{2}^{-}(q)^{3}$ | $A_{2}\left(q^{2}\right) A_{2}(q)$ | $D_{4}(q) T_{2}$ | $D_{5}^{-}(q) T_{1}$ |
| ---: | :---: | :---: | :---: | :---: | :---: |
| $A$ | $\mathrm{SL}_{2}\left(q^{2}\right)$ | $\mathrm{SL}_{2}(q)$ | $\mathrm{SL}_{3}(q)$ | $\mathrm{SL}_{3}(q)$ | $\mathrm{SL}_{2}\left(q^{2}\right)$ |

Table 3.4.6: Subgroups $H$ and $A$ for $F_{4}(q)$

| type of $H$ | $A_{1}(q) C_{3}(q)$ | $B_{4}(q)$ | $D_{4}(q)$ | ${ }^{3} D_{4}(q)$ | $A_{2}(q)^{2}$ | $A_{2}^{-}(q)^{2}$ |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $A$ | $\mathrm{SL}_{2}(q)$ | $\mathrm{SL}_{2}(q)$ | $\mathrm{SL}_{2}(q)$ | $\mathrm{SL}_{3}(q)$ | $\mathrm{SL}_{3}(q)$ | $\mathrm{SL}_{2}(q)$ |
| type of $H$ | $B_{2}(q)^{2}$ | $B_{2}\left(q^{2}\right)$ |  |  |  |  |
| $A$ | $\mathrm{SL}_{2}(q)$ | $\mathrm{SL}_{2}(q)$ |  |  |  |  |

Table 3.4.7: Subgroups $H$ and $A$ for $G_{2}(q)$

| type of $H$ | $A_{1}(q)^{2}$ | $A_{2}(q)$ | $A_{2}^{-}(q)$ |
| ---: | :---: | :---: | :---: |
| $A$ | $\mathrm{SL}_{2}(q)$ | $\mathrm{SL}_{2}(q)$ | $\mathrm{SL}_{2}(q)$ |

Table 3.4.8: Subgroups $H$ and $A$ for ${ }^{3} D_{4}(q)$

| type of $H$ | $A_{1}(q) A_{1}\left(q^{3}\right)$ | $A_{2}(q)$ | $A_{2}^{-}(q)$ |
| ---: | :---: | :---: | :---: |
| $A$ | $\mathrm{SL}_{2}(q)$ | $\mathrm{SL}_{2}(q)$ | $\mathrm{SL}_{2}(q)$ |

(as defined in Lemma 2.1.2), and also $\operatorname{Alt}\left(q^{r u}-1\right)$ must be a section of $H$. By Lemmas 2.1.1 and 2.1.2,
this eliminates all possibilities except for the list in Table 3.4.1, together with the following cases:

$$
\begin{array}{c|c}
G & F_{4}(q), q=3  \tag{3.4.1}\\
\hline H & B_{4}(q), D_{4}(q)
\end{array}
$$

We shall handle the cases in (3.4.1) after first justifying the assertions in Tables 3.4.2-3.4.8, For the cases in the tables where the maximal rank subgroup $H$ is just an untwisted subsystem subgroup over $\mathbb{F}_{q}$, the existence of the subgroups $A<S$ is clear from inspection of the extended Dynkin diagram of $G$. (The cases $(A, H)=\left(\mathrm{SL}_{2}(q), D_{4}(q)\right)$ for $F_{4}(q)$, and also $(A, H)=\left(\mathrm{SL}_{2}(q), A_{2}(q)\right)$ for $G_{2}(q)$ and ${ }^{3} D_{4}(q)$ require some small additional observations: in the first case, $D_{4}(q)$ contains a subgroup $A=\mathrm{SL}_{2}(q)$ corresponding to a short root in the $F_{4}$-system, and this lies in a short root $S=\mathrm{SL}_{3}(q)$ which is not contained in $D_{4}(q)$; and in the second case, there exists $x \in C_{G}(A) \backslash H$, and we can take $S=H^{x}$.)

Now consider cases in Tables 3.4.2-3.4.8 where $H$ involves a twisted group, or a group over a proper extension field of $\mathbb{F}_{q}$.

Consider first Table 3.4.2, where $G(q)=E_{8}(q)$. In the cases where $H$ is of type ${ }^{3} D_{4}(q)^{2}$ or $A_{2}^{-}(q)^{4}$, we choose $A$ to be a subsystem subgroup $\mathrm{SL}_{3}(q)$ or $\mathrm{SL}_{2}(q)$ of one of the factors. Now suppose $H$ is of type $A_{8}^{-}(q)$. Then $H$ has a Levi subgroup $S=\mathrm{SL}_{4}\left(q^{2}\right)$, and we let $A$ be a natural subgroup $\mathrm{SL}_{3}\left(q^{2}\right)$ of this. We use Lemma [2.6.1] to show that there is a conjugate $S^{x}$ such that $A<S^{x} \notin H$. First observe that the fusion control hypotheses of the lemma for $A<S<H$ clearly hold. Now $N_{G}(A)$ contains a subgroup $A_{2}(q) A_{2}^{-}(q)$ (a subgroup $A_{2}\left(q^{2}\right) A_{2}(q) A_{2}^{-}(q)$ can be seen inside a subsystem subgroup of type $E_{6} A_{2}$ ), whereas $N_{H}(A)$ normalizes a subgroup $A_{2}^{-}(q) T_{2} A$ of $H$, where $T_{2}$ is a torus of order $q^{2}-1$. The factor $A_{2}(q)$ of $N_{G}(A)$ does not have a factorization with one of the factors being $N\left(T_{2}\right)$ (see [69); hence $N_{G}(A) \neq N_{H}(A)\left(N_{G}(S) \cap N_{G}(A)\right)$ and the required conjugate of $S$ exists by Lemma 2.6.1.

Next suppose $H$ is of type $A_{2}^{-}(q) E_{6}^{-}(q)$. Here we choose $A$ to be a subgroup $\mathrm{SL}_{3}\left(q^{2}\right)$ of a Levi subgroup $\mathrm{SU}_{6}(q)$ of the $E_{6}^{-}(q)$ factor; this is contained in a subgroup $S=\mathrm{SL}_{4}\left(q^{2}\right)$ as defined in the previous paragraph, and $S \not \leq H$. A similar argument applies to produce a suitable subgroup $A=\operatorname{SL}_{2}\left(q^{2}\right)$ when $H$ has type $A_{4}^{-}(q)^{2}$, and also a subgroup $A=\mathrm{SL}_{3}\left(q^{2}\right)$ when $H$ has type ${ }^{3} D_{4}\left(q^{2}\right)$. In the case where $H$ is of type $A_{4}^{-}\left(q^{2}\right)$, we choose $A$ to be a subgroup $\mathrm{SL}_{2}\left(q^{2}\right)$ corresponding to a natural subgroup $\mathrm{SU}_{2}\left(q^{2}\right)$ of the unitary group; this arises from a subsystem $A_{1} A_{1}$ of the ambient algebraic group, and is conjugate to the subgroup $\mathrm{SL}_{2}\left(q^{2}\right)$ of the previous case. The same subgroup $A=\mathrm{SL}_{2}\left(q^{2}\right)$ pertains when $H$ is of type $A_{2}^{-}\left(q^{2}\right)^{2}$ or $D_{4}\left(q^{2}\right)$. In the latter case, we also need to apply Lemma 2.6.1 to produce a subgroup $S=\mathrm{SL}_{3}\left(q^{2}\right)$ such that $A<S \not \leq H$ : here $N_{G}(A)$ contains $D_{6}^{-}(q)$, which does not factorize as $N_{H}(A)\left(N_{G}(S) \cap N_{G}(A)\right)$.

Finally, for $H$ of type $A_{2}^{-}\left(q^{4}\right)$, let $A$ be a natural subgroup $\mathrm{SL}_{2}(q)$ of $A_{2}^{-}\left(q^{4}\right)$ (acting as $2 \oplus 1$ on the associated 3 -dimensional unitary module). Then $A$ is a diagonal subgroup of a subsystem subgroup of type $A_{1}(q)^{4}$ which lies in a subsystem $A_{2}(q)^{4}$, and hence $A$ lies in a diagonal $A_{2}(q)$ in the latter. This completes the justification for Table 3.4.2.

For $G(q)=E_{7}(q), E_{6}^{\epsilon}(q), F_{4}(q)$ or $G_{2}(q)$ the justification for the existence of the subgroups $A<S$ uses the same arguments as above. Extra argument using Lemma 2.6.1 is needed just for the cases

$$
(G(q), H)=\left(E_{7}(q), A_{7}^{-}(q)\right),\left(E_{6}(q), A_{2}\left(q^{2}\right) A_{2}^{-}(q)\right) \text { and }\left({ }^{2} E_{6}(q), A_{1}(q) A_{5}^{-}(q)\right) ;
$$

observe that $C_{G}(A)$ contains ${ }^{2} D_{4}(q) A_{1}(q),{ }^{2} A_{3}(q)$ or $A_{3}(q)$ in the respective cases, from which it can be seen that $N_{G}(A)$ does not factorize as $N_{H}(A)\left(N_{G}(S) \cap N_{G}(A)\right)$, so that Lemma 2.6.1 applies.

We have now justified all the assertions in Tables 3.4.2-3.4.8,
It remains to handle the cases in (3.4.1). Let $G=F_{4}(3)$, and let $H$ be a maximal rank subgroup $B_{4}(3)$ or $D_{4}(3) . \operatorname{Sym}(3)$. First consider the case where $H=D_{4}(3) . \operatorname{Sym}(3)$. Let $S=\mathrm{SL}_{4}(3)<H$ be generated by root subgroups, and $A=\mathrm{SL}_{3}(3)<S$. Then $A<S<H$, and $H$ controls fusion of $S$ in $G$ (as all subgroups $\mathrm{SL}_{4}(3)$ generated by root groups in $H$ are $H$-conjugate). We claim that $N_{G}(A)$ does not factorize as $N_{H}(A)\left(N_{G}(S) \cap N_{G}(A)\right)$. To see this, observe that $N_{G}(A) / A$ contains $\tilde{A}_{2}(3)$ (generated by short root groups); while $\left|N_{H}(A) / A\right|_{3}=3$ and $\left|\left(N_{G}(S) \cap N_{G}(A)\right) / A\right|_{3}=\left|N_{A_{1}(3) S}(A) / A\right|_{3}=3$, proving the claim. It then follows from Lemma 2.6.1 that there is a conjugate $S^{g}$ such that $A<S^{g} \notin H$, so as usual there is a subset $\Delta$ with $G^{\Delta} \geq \operatorname{ASL}_{3}(3)$, showing that $(G,(G: H))$ is not binary.

Now consider the case $H=B_{4}(3)$. Again take $A<S<H$ with $A=\mathrm{SL}_{3}(3), S=\mathrm{SL}_{4}(3)$ generated by root subgroups. This time $H$ does not control fusion of $S$ in $G$, as there are two classes of subgroups isomorphic to $\mathrm{SL}_{4}(3)$ in $B_{4}(3)$ with representatives $S_{1}, S_{2}$ of types $\mathrm{SL}_{4}(3)$ and $\Omega_{6}^{+}(3)$, respectively. We again aim to find a conjugate $S^{g}$ such that $A<S^{g} \nsubseteq H$, but we need to do this a little differently. Define

$$
\begin{aligned}
& \Lambda=\{R<G: A \leq R, R \text { conjugate to } S \text { in } G\}, \\
& \Phi=\{R \in \Lambda: R<H\} .
\end{aligned}
$$

We shall show that $|\Lambda|>|\Phi|$, which will achieve our aim, completing the proof that the action of $G$ on $(G: H)$ is not binary.

First observe that $N_{G}(A)$ acts transitively on $\Lambda$, since

$$
\begin{aligned}
R \in \Lambda & \Rightarrow R=S^{g}(g \in G) \\
& \Rightarrow A, A^{g^{-1}}<S \\
& \Rightarrow A^{g^{-1}}=A^{s}(s \in S) \\
& \Rightarrow R=S^{s g}, s g \in N_{G}(A) .
\end{aligned}
$$

Hence $|\Lambda|=\left|N_{G}(A): N_{G}(A) \cap N_{G}(S)\right|=\left|\tilde{A}_{2}(3) .2: T_{1} A_{1}(3) .2\right|$, which is divisible by $3^{2} .13$.
In similar fashion, we see that $N_{H}(A)$ has two orbits $\Phi_{1}, \Phi_{2}$ on $\Phi$, with orbit representatives $S_{1}$ and $S_{2}$. The orbit sizes are $\left|\Phi_{i}\right|=\left|N_{H}(A): N_{H}(A) \cap N_{H}\left(S_{i}\right)\right|$ for $i=1,2$. Hence $\left|\Phi_{1}\right|=\left|T_{1} A_{1}(3) .2: T_{2} \cdot 2\right|$ divides 24, while $\left|\Phi_{2}\right|=1$. Therefore $|\Lambda|>|\Phi|$, as required.

The next three results deal with the cases not covered by Proposition 3.4.1 (those in Table 3.4.1 and also the ${ }^{2} F_{4}(q)$ case).

Lemma 3.4.2. Let $G$ be as in Proposition 3.4.1, and suppose that $(G(q), H)$ is as in line 1 or 2 of Table 3.4.1. If $\Omega=(G: H)$, then $(G, \Omega)$ is not binary.

Proof. In these cases $G(q)=E_{7}(q)$ or $E_{6}^{\epsilon}(q)(\epsilon= \pm)$, and $H$ is of type $A_{1}\left(q^{7}\right)$ or $A_{2}^{\epsilon}\left(q^{3}\right)$, respectively. In the first case, $H \cap G(q)=A_{1}\left(q^{7}\right) .7$, and we choose a subfield subgroup $X=A_{1}(q) \times 7=X_{0} \times 7$ of $H$. The factor $X_{0}=A_{1}(q)$ is contained diagonally in a subsystem subgroup $A_{1}(q)^{7}$ of $E_{7}(q)$, which has normalizer acting as $L_{3}(2)$ on the 7 factors (see [74, Table 5.1]). Hence $N_{G}(X)$ is not contained in $H$, and so there is a suborbit here on which $H$ acts as $(H: Y)$, where $X \leq Y \leq N_{H}\left(X_{0}\right)$. If $q>2$, then $Y$ is maximal in $\langle H \cap G(q), Y\rangle$, and we know, by [45] that the associated action on cosets is not binary. Appealing to Lemma 1.6 .2 if necessary, we conclude that the action of $H$ on $(H: Y)$ is not binary, and now Lemma 1.6.1 implies that the action of $G$ on $(G: H)$ is not binary. Suppose now $q=2$. Let $x$ be an element of order 7 in $H$. From [6, Table 2], we see that there exists $g \in C_{G}(x) \backslash H$ and hence $H \cap H^{g}$ is a proper subgroup of $H$ containing $x$. We have calculated with magma the faithful transitive actions of $H$ having point stabiliser of order divisible by 7 . We find that all such actions are not binary. Therefore, the action of $H$ on $\left(H: H \cap H^{g}\right)$ is not binary, and hence by Lemma 1.6.1 so is the action of $G$ on $(G: \Omega)$.

For the $E_{6}^{\epsilon}(q)$ cases, we argue similarly. Choose a subfield subgroup $X=A_{2}^{\epsilon}(q) \times 3$ of $H$. The $A_{2}^{\epsilon}(q)$ factor is contained diagonally in a subsystem $A_{2}^{\epsilon}(q)^{3}$, which has normalizer acting as $\operatorname{Sym}(3)$ on the factors. Hence again $N_{G}(X) \not \leq H$ and so there is a suborbit here on which $H$ acts as $(H: Y)$ where $X \leq Y \leq N_{H}\left(A_{2}^{\epsilon}(q)\right)$. It will be sufficient to show that this action is not binary.

If $\epsilon=+$, then we take a subgroup $S=\mathrm{SL}_{2}(q)$ of $Y$, and it is easy to verify that $S$ normalizes and acts transitively on an elementary abelian subgroup $E=E_{q^{2}}$ of $H$ that is not contained in $Y$. Defining $\Delta=\{Y e: e \in E\}$, we obtain, in the usual way, that either $\Delta$ is a beautiful subset in the action of $H$ on ( $H: Y$ ) (and we are done), or else $H^{\Delta} \geq \operatorname{Alt}(\Delta)$. But by Lemma 2.1.2, this is not possible unless $q=2$. When $q=2$, for $G$ with $E_{6}(2) \leq G \leq \operatorname{Aut}\left(E_{6}(2)\right)$ and for $H:=N_{G}\left(A_{2}(8)\right)$, we have computed the subgroups $K$ of $H$ with $|H: K|$ odd. For each such pair $(H, K)$, we have checked that, if the action of $H$ on $(H: K)$ is binary, then $K$ contains $A_{2}(8)$. Hence there is no binary action of $H$ of odd degree with $A_{2}(8)$ acting non-trivially. From this we deduce that, if the action of $G$ on $(G: H)$ is binary, then each non-trivial subdegree of $G$ must be even, which implies that $|G: H|$ is odd, a contradiction.

If $\epsilon=-$, then we argue similarly with a cyclic subgroup $K$ of order $q-1$ in $Y$. There is a subgroup $E=E_{q}$ that $K$ normalizes and upon which it acts fixed-point-freely. The same line of argument rules out all cases with $q>8$. Also, using Lemma 2.1.1, we can rule out $q=8$ (since Alt(7) is not a section of $\left.\mathrm{SU}_{3}(8)\right)$ and $q=7$ (since $\operatorname{Alt}(6)$ is not a section of $\left.\mathrm{SU}_{3}(7)\right)$. To deal with $q<7$, we use magma as follows.

Let $M$ be a maximal subgroup of $G$ with socle ${ }^{2} A_{2}\left(q^{3}\right)$. We consider the permutation action of $G$ on the right cosets $\Omega$ of $M$ in $G$. Observe that $q$ divides $|\Omega|=|G: M|$ and hence $G$ has a suborbit of cardinality relatively prime to $q$. Using magma, we have verified that all faithful transitive actions of $M$ on a set of cardinality relatively prime to $q$ are non binary. Therefore, the action of $M$ on each non-trivial suborbit is non binary. From this, it follows that $G$ is not binary on $\Omega$.

Lemma 3.4.3. Let $G$ be as in Proposition 3.4.1, and suppose that $(G(q), H)$ is as in line 3 or 4 of Table 3.4.1. If $\Omega=(G: H)$, then $(G, \Omega)$ is not binary.

Proof. Adopt the hypothesis of the lemma, and assume that $(G, \Omega)$ is binary.
Suppose, first, that $G(q)={ }^{2} E_{6}(2)$ and that $H$ is of type $A_{2}^{-}(2)^{3}$. An inspection of extended Dynkin diagrams confirms that $H$ contains a subgroup $K$ of type $A_{2}^{-}(2)^{2}$ that, in turn, is embedded in the natural way inside a subgroup $L=\operatorname{PSU}_{6}(2)$ in $G(q)$. Now $K$ has a "diagonal" subgroup $Q \cong Q_{8}$ which normalizes, and acts fixed-point-freely upon an elementary abelian subgroup, $E_{9}$, of $L$. Now we study the conjugates of $E_{9}$ under $N_{L}(Q)$ and observe that $Q$ normalizes each of these conjugates. There are two possibilities: first, one of these conjugate subgroups, say $E$, does not lie in $H$. In this case, $E \cap H=\{1\}$ and setting $\Delta=H^{E}$ we see that the set-wise-stabilizer of $\Delta$ acts 2-transitively on $\Delta$. Hence, as $(G, \Omega)$ is binary, we have $G^{\Delta} \geq \operatorname{Alt}(\Delta)$. However $|\Delta|=9$, and $H$ does not have a section isomorphic to $\operatorname{Alt}(8)$, so this is impossible. This possibility is, therefore, excluded, and we conclude that all of the conjugate subgroups lie in $H$. But now direct calculation, using for instance GAP, confirms that $\left\langle E_{g}^{g} \mid g \in N_{L}(Q)\right\rangle=L$, which is a contradiction. Thus this possibility is also excluded.

Consider, next, the situation where $G(q)=E_{8}(2)$ and $H$ is of type $A_{2}^{-}(2)^{4}$. In this case a version of the previous argument yields a contradiction, this time using a subgroup $K$ of type $A_{2}^{-}(2)^{3}$ embedded in a subgroup isomorphic to $L=\mathrm{PSU}_{9}(2)$ in $G(q)$.

Suppose, next, that $G(q)=E_{8}(2)$ and $H$ has type $A_{2}^{-}\left(2^{4}\right)$. In this case, $G=E_{8}(2)$ and $H \cong$ $\operatorname{Aut}\left(\mathrm{PSU}_{3}(16)\right)=\mathrm{PSU}_{3}(16) .8$ (see [74]). We have computed all the core-free subgroups $K$ of $H$ with $|H: K|$ odd and shown that, for each of these subgroups, the action of $H$ on $(H: K)$ is not binary, by witnessing a non-binary triple. Hence, as $(G, \Omega)$ is binary (by assumption), $|G: H|$ must be odd, which is clearly a contradiction.

Suppose, finally, that $G(q)={ }^{2} E_{6}(2)$ and that $H$ is of type $D_{4}(2) T_{2}$. Consulting [105], we see that $H \cap G(q)$ has shape $\left(3 \times \Omega_{8}^{+}(2): 3\right): 2$, extending to $\left(3^{2}: 2 \times \Omega_{8}^{+}(2)\right): \operatorname{Sym}(3)$ in $G(q) . \operatorname{Sym}(3)=\operatorname{Aut}(G(q))$. We have calculated the transitive actions of all groups of the relevant shapes on sets of odd cardinality using magma. We find that the only binary actions for such groups occur when the set is of size 1,3 or 9 , in which case the kernel of the action contains $\Omega_{8}^{+}(2)$. As $(G, \Omega)$ is binary, we conclude, therefore, that all non-trivial subdegrees must be even. This contradicts the fact that $|G: H|$ is even.

Lemma 3.4.4. Let $G$ be as in Proposition 3.4.1, and suppose that $G(q)={ }^{2} F_{4}(q)(q>2)$. If $\Omega=(G: H)$, then $(G, \Omega)$ is not binary.

Proof. Here the possibilities for the maximal rank subgroup $H(q):=H \cap G(q)$ are:

$$
\mathrm{SU}_{3}(q) \cdot 2, \mathrm{PGU}_{3}(q) \cdot 2,(S z(q) \times S z(q)) \cdot 2, \mathrm{Sp}_{4}(q) \cdot 2 .
$$

We write $G=G(q)\langle\phi\rangle$, where $\phi$ is a field automorphism of odd order $f$ (possibly $f=1$ ).
In the first two cases, let $T<H(q)$ be a maximal torus of order $(q+1)^{2}$. Then $N_{H(q)}(T)=T .(\operatorname{Sym}(3) \times$ 2), while $N_{G(q)}(T)=T . \mathrm{GL}_{2}(3)$. Hence $N_{H(q)}\left(T .2^{2}\right)=T .2^{2}$ and $N_{G(q)}\left(T .2^{2}\right)=T . D_{8}$. It follows that there exists $x \in G \backslash H$ such that $H \cap H^{x}$ contains $T .2^{2}$ but does not contain $H$. We can chose $\phi$ to normalize all
of the above subgroups, so that we also have $H \cap H^{x} \geq T .\left(2^{2} \times f\right)$. Now Lemma 2.2 .13 implies that the action of $H$ on $\left(H: H \cap H^{x}\right)$ is not binary, and the result follows from Lemma 1.6.1.

Next consider the possibility that $H(q)=(S z(q) \times S z(q)) .2$. Here, as we shall see, when we study suborbits we find a primitive group via a product action hence we could appeal to [106]; nonetheless we give a direct argument. First, let $T<H(q)$ be the direct product of two maximal tori of $S z(q)$ of order $q-1$. Then $N_{H(q)}(T)=T$.[8], while $N_{G(T)}(T) \geq T$.[16] (one can see this, for instance, by using the fact that $T$ is also a subgroup of $\left.\mathrm{Sp}_{4}(q) .2\right)$. Again we obtain an $x \in G \backslash H$ such that $H \cap H^{x}$ contains $T .2^{2}$ but does not contain $H$. Choosing $\phi$ appropriately we have $H \cap H^{x} \geq T .([8] \times f)$. Then it must be the case that $H \cap H^{x}=T .([8] \times f)$. In particular, we can write $H \cap H^{x}=(M \times M) .(2 \times f)$, where $M$ is a maximal subgroup of $S z(q)$ of order $2(q-1)$.

Identify $(S z(q): M)$ with the set of conjugates of $M$ in $S z(q)$ and identify ( $H: H \cap H^{x}$ ) with the set

$$
\Gamma:=\left\{\left(M_{1}, M_{2}\right) \mid M_{1}, M_{2} \in(S z(q): M)\right\} .
$$

Now we fix $M$ and define

$$
\Gamma_{0}:=\left\{\left(M, M_{1}\right) \mid M_{1} \in(S z(q): M)\right\} .
$$

Clearly $H_{\Gamma_{0}} \cong(M \times S) . f$ and $H^{\Gamma_{0}}$ is almost simple and isomorphic to $S z(q) . f$, with the action on $\Gamma$ being isomorphic to the action of $S z(q) . f$ on $(S z(q): M)$. We know that this action is not binary by [45, Theorem 1.3]; thus there exist $k$-tuples $I=\left(M_{1}, \ldots, M_{k}\right)$ and $J=\left(M_{1}^{\prime}, \ldots, M_{k}^{\prime}\right)$ such that $(I, J)$ is 2 -subtuple complete but not $k$-subtuple complete with respect to the action of $S z(q) \cdot f$. Now the same is true for the pair of elements of $\Gamma^{k}$,

$$
\left(\left(\left(M, M_{1}\right),\left(M, M_{2}\right), \ldots,\left(M, M_{k}\right)\right),\left(\left(M, M_{1}^{\prime}\right),\left(M, M_{2}^{\prime}\right), \ldots,\left(M, M_{k}^{\prime}\right)\right)\right)
$$

with respect to the action of $H$. Now, the result follows from Lemma 1.6.1,
Consider, finally, the possibility that $H(q)=\mathrm{Sp}_{4}(q) .2$. In this case we use the fact that $H(q)$ contains an element $g$ of order $q-1$ that is centralized in $G(q)$ by a subgroup isomorphic to ${ }^{2} B_{2}(q)$. In [96, Table IV] a parametrization of such elements $g$ is given: they are conjugate to the element $t_{1}$ in the table. Working in the $F_{4}$ root system, it can be seen that there is a conjugate $g$ of $t_{1}$ that can be written in $H^{\prime}=\operatorname{Sp}_{4}(q)$ as a diagonal matrix with all its eigenvalues of order $q-1$. In particular, there is an element $x \in G \backslash H$ that centralizes $g$ and so we conclude that there is a suborbit of $G$ on which the action of $H$ is isomorphic to the action of $H$ on $(H: M)$, where $M$ is a subgroup of $H$ containing $g$. Since, by assumption, $q \geq 8$, Lemma 2.2.12 implies that this action is not binary, and the result follows by Lemma 1.6.1,

### 3.5 Maximal torus normalizers

In this section we prove Theorem 3.1 in the case where the point stabilizer $H$ is the normalizer of a maximal torus. Such maximal subgroups are listed in Table 5.2 of [74]. The main result of the section follows. The cases excluded in the proposition (those in Table 3.5.1) will be dealt with in Lemma 3.5.3,

Proposition 3.5.1. Assume $G$ is almost simple with socle $G(q)$, an exceptional group of Lie type over $\mathbb{F}_{q}$, and suppose $G(q)$ is not as in Proposition 3.2.1. Let $H$ be a maximal subgroup of $G$ that is the normalizer of a maximal torus $T$, as in [74, Table 5.2], and let $\Omega=(G: H)$. Then either $(G, \Omega)$ is not binary, or $(G(q),|T \cap G(q)|)$ is as in Table 3.5.1.

For the proof we need the following lemma. In the statement, by a semisimple group we mean a perfect group that is a central product of quasisimple groups.

Lemma 3.5.2. Let $G$ be an almost simple group with socle of Lie type, and let $H=N_{G}(T)$ be a maximal subgroup of $G$ that is the normalizer of a maximal torus $T$. Write $\Omega=(G: H)$.
(A) Suppose there exist subgroups $A, D$ of $G$ with the following properties:

Table 3.5.1: Exceptions in Prop. 3.5.1

| $G(q)$ | $\|T \cap G(q)\|$ |
| :---: | :---: |
| $E_{7}(2)$ | $3^{7}$ |
| ${ }^{2} E_{6}(2)$ | $3^{5}$ |

(i) $A$ is quasisimple, $D$ is either semisimple or a torus containing $Z(A),[A, D]=1$ and $C_{G}(A)=$ $D Z(A)$;
(ii) $T \leq N_{G}(A)$ and $T \cap A D=T_{1} T_{0}$, where $T_{1}=T \cap A, T_{0}=T \cap D$;
(iii) $C_{G}\left(T_{0}\right)^{\prime}=A$.

Define

$$
\Delta=\left\{T^{g}: g \in N_{D}\left(T_{0}\right) A\right\} \subseteq \Omega .
$$

Then $G^{\Delta}$ has socle $A / Z(A)$ acting on $\left(A: N_{A}\left(T_{1}\right)\right)$.
(B) Suppose that in addition to (i)-(iii) above, the following hold:
(iv) $C_{G}\left(T_{1} T_{0}\right)=T$;
(v) for any distinct $T^{\prime}, T^{\prime \prime} \in \Delta$ we have $T^{\prime} \cap T^{\prime \prime}=\bigcap_{a \in A} T^{a}$;
(vi) for any $g \in N_{G}(A)$, there exists $a \in A$ such that $T_{1}^{g}=T_{1}^{a}$;
(vii) the action of $G^{\Delta}$ on $\Delta$ is not binary.

Then the action of $G$ on $\Omega=(G: H)$ is not binary.
Proof. (A) Write $K=G_{(\Delta)}$, the point-wise stabilizer of $\Delta$. We claim first that $K$ normalizes $A$. To see this, observe first that $K$ normalizes $X:=\cap_{a \in A} T^{a}$. Since $X$ is $A$-invariant and $A$ is quasisimple, $X \cap A=Z(A)$. Also, $X \leq T$ and so $X$ normalizes $A$ by (ii). Hence $[X, A] \leq X \cap A \leq Z(A)$. As $A$ is perfect, this implies that $[X, A]=1$, and hence $X \leq D Z(A)$ by (i). It follows that $X=T_{0} Z(A)$. By (iii) we have $C_{G}(X)^{\prime}=A$, and hence $K$ normalizes $A$, as claimed.

Next, we claim that

$$
\begin{equation*}
C_{G}(K)^{\prime}=A . \tag{3.5.1}
\end{equation*}
$$

Clearly $T_{0} \leq K$, so $C_{G}(K)^{\prime} \leq C_{G}\left(T_{0}\right)^{\prime}=A$, by (iv). For the reverse containment, let $x \in K$. Then $T^{a x}=T^{a}$ for all $a \in A$. Now $x$ normalizes $A$, hence normalizes $T^{a} \cap A=T_{1}^{a}$ for all $a \in A$. In other words, $x$ induces an automorphism of $A$ that lies in the kernel, $L$ say, of the action on the set of $A$-conjugates of $T_{1}$. As $A$ is quasisimple, either $L \leq Z(A)$ or $A \leq L$. If $L \leq Z(A)$, then $x$ commutes with $A$ and hence (3.5.1) holds. So assume the latter. Since the action in question is on $A$-conjugates of $T_{1}$ and $A \leq L$, we get $T_{1} \unlhd A$. As $A$ is quasisimple, this means that $T_{1} \leq Z(A)$. Therefore, $A$ centralizes $T$, a maximal torus of $G$. But then $A$ must be in the centre of $G$ which is a contradiction to the fact that $A$ is quasisimple. Summing up, in all cases $x$ commutes with $A$ and hence (3.5.1) holds.

Now $G_{\Delta}$ normalizes $K=G_{(\Delta)}$, hence normalizes $A$, by (3.5.1). Therefore by (i), $G_{\Delta}$ also normalizes $D A$. Let $g \in G_{\Delta}$. Then $T^{g} \in \Delta$, so by definition of $\Delta$, intersecting with $D A$ gives $\left(T_{0} T_{1}\right)^{g}=T_{0} T_{1}^{a}$ for some $a \in A$, and since $g \in N_{G}(D A)$ this implies that $T_{0}^{g}=T_{0}$. Hence $G_{\Delta} \leq N_{G}\left(T_{0}\right)$ and $G_{\Delta} \cap D A=N_{D}\left(T_{0}\right) A$. As $K \geq N_{D}\left(T_{0}\right) Z(A)$, it follows that $G^{\Delta}=G_{\Delta} / K$ has socle $A / Z(A)$ acting on the conjugates of $T_{1}$, as required (note that $N_{A}\left(T_{1}\right) \leq N_{A}(T)$ since $N_{A}\left(T_{1}\right)$ normalizes $T_{1} T_{0}$, hence normalizes $C_{G}\left(T_{1} T_{0}\right)=T$ ).
(B) By condition (vii), there is a non-binary witness $(\delta, \lambda)$ for $G^{\Delta}$, where $\delta=\left(\delta_{1}, \delta_{2}, \ldots\right), \lambda=$ $\left(\lambda_{1}, \lambda_{2}, \ldots\right)$. Suppose there exists $g \in G$ sending $\delta \rightarrow \lambda$. Then $g$ sends $\delta_{1} \cap \delta_{2} \rightarrow \lambda_{1} \cap \lambda_{2}$, and so by condition (v), $g$ normalizes the group $X=T_{0} Z(A)$. Hence as above, $g$ normalizes $A$, hence also $D$ and $T_{0}$. Now for $x \in N_{D}\left(T_{0}\right) A$ we have $T^{x} \cap D A=T_{0} T_{1}^{a}$ for some $a \in A$, and hence using (vi),

$$
T^{x g} \cap D A=\left(T^{x} \cap D A\right)^{g}=T_{0} T_{1}^{a g}=T_{0} T_{1}^{a^{\prime}}
$$

for some $a^{\prime} \in A$. Hence by (iv) we see that $T^{x g} \in \Delta$. This shows that $g \in G_{\Delta}$, contradicting the fact that $(\delta, \lambda)$ is a non-binary witness for $G^{\Delta}$. Hence $\delta$ and $\lambda$ are in different $G$-orbits, showing that $(G, \Omega)$ is not binary.

Remark The proof shows that condition (v) could be replaced by
( $\mathrm{v}^{\prime}$ ) there exists a non-binary witness $(\delta, \lambda)$ for $G^{\Delta}$ such that

$$
\bigcap_{i=1}^{k} \delta_{i}=\bigcap_{i=1}^{k} \lambda_{i}=\bigcap_{a \in A} T^{a}
$$

Table 3.5.2: Possibilities for $T, A, D$

| $G(q)$ | $\|T\|$ | $A$ | $D$ | Maximality <br> condition | Comment |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $E_{8}(q)$ | $(q-1)^{8}$ | $A_{1}(q)$ | $E_{7}(q)$ | $q \geq 5$ |  |
|  | $(q+1)^{8}$ | $A_{1}(q)$ | $E_{7}(q)$ |  | $q>3$ |
|  | $(q+1)^{8}$ | $A_{4}^{-}(q)$ | $A_{4}^{-}(q)$ |  | $q \leq 3$ |
|  | $\left(q^{2}+\epsilon q+1\right)^{4}$ | $A_{2}^{\epsilon}(q)$ | $E_{6}^{\epsilon}(q)$ | $(q, \epsilon) \neq(2,-)$ | $A D<D_{8}(q)$ |
|  | $\left(q^{2}+1\right)^{4}$ | $A_{1}\left(q^{2}\right)$ | $D_{6}^{-}(q)$ |  |  |
|  | $\left(q^{4}+\epsilon q^{3}+q^{2}+\epsilon q+1\right)^{2}$ | $A_{4}^{\epsilon}(q)$ | $A_{4}^{\epsilon}(q)$ |  | $q>3$ |
| $E_{7}(q)$ | $(q-1)^{7}$ | $A_{1}(q)$ | $D_{6}(q)$ | $q \geq 5$ | $q>3$ |
|  | $(q+1)^{7}$ | $A_{1}(q)$ | $D_{6}(q)$ |  | $q \leq 3$ |
|  | $(q+1)^{7}$ | $A_{2}^{-}(q)$ | $A_{5}^{-}(q)$ |  | $q>3$ |
| $E_{6}^{\epsilon}(q)$ | $(q-1)^{6}(\epsilon=+)$ | $A_{1}(q)$ | $A_{5}(q)$ | $q \geq 5$ | $q \leq 3$ |
|  | $(q+1)^{6}(\epsilon=-)$ | $A_{1}(q)$ | $A_{5}^{-}(q)$ |  |  |
|  | $(q+1)^{6}(\epsilon=-)$ | $A_{2}^{-}(q)$ | $A_{2}^{-}(q)^{2}$ |  |  |
| $F_{4}(q)$, | $\left(q^{2}+\epsilon q+1\right)^{3}$ | $A_{2}^{\epsilon}(q)$ | $A_{2}^{\epsilon}(q)^{2}$ | $(q, \epsilon) \neq(2,-)$ |  |
| $q$ even | $\left.\left(q^{2}+\epsilon q\right)^{4}+1\right)^{2}$ | $A_{1}(q)$ | $C_{3}(q)$ | $A_{2}^{\epsilon}(q)$ | $A_{2}^{\epsilon}(q)$ |
|  | $\left(q^{2}+1\right)^{2}$ | $A_{1}\left(q^{2}\right)$ | $B_{2}(q)$ |  |  |
| $G_{2}(q)$, | $(q-\epsilon)^{2}$ | $A_{1}(q)$ | $A_{1}(q)$ | $q \geq 9,-)$ | $A D<B_{4}(q)$ |
| $q=3^{a}$ |  |  |  |  |  |
| ${ }^{2} F_{4}(q)^{\prime}$, | $(q+1)^{2}$ | $A_{1}(q)$ | $A_{1}(q)$ | $q \geq 8$ |  |
| ${ }^{3} D_{4}(q)$ | $(q+\epsilon \sqrt{2 q}+1)^{2}$ | ${ }^{2} B_{2}(q)$ | ${ }^{2} B_{2}(q)$ | $(q, \epsilon) \neq(2,-)$ |  |

Table 3.5.3: Remaining possibilities for $T$

| $G(q)$ | $\|T\|$ | $N_{G(q)}(T) / T$ |
| :---: | :---: | :---: |
| $E_{8}(q)$ | $q^{8}+\epsilon q^{7}-\epsilon q^{5}-q^{4}-\epsilon q^{3}+\epsilon q+1$ | $Z_{30}$ |
| $F_{4}(q), q=2^{a}>2$ | $q^{4}-q^{2}+1$ | $Z_{12}$ |
| $G_{2}(q), q=3^{a}>3$ | $q^{2}+\epsilon q+1$ | $Z_{6}$ |
| ${ }^{2} F_{4}(q)^{\prime}$ | $q^{2}+\epsilon \sqrt{2 q^{3}}+q+\epsilon \sqrt{2 q}+1$ | $Z_{12}$ |
| ${ }^{3} D_{4}(q)$ | $q^{4}-q^{2}+1$ | $Z_{4}$ |

Proof of Proposition 3.5.1. Let $G$ be almost simple with socle $G(q)$ an exceptional group of Lie type, and let $H=N_{G}(T)$ be a maximal subgroup of $G$ normalizing a maximal torus $T$, as in [74, Table 5.2]. We aim
to apply Lemma 3.5.2. Tables 3.5 .2 and 3.5 .3 together list all possibilities for $T$, and the first table also lists a pair of subgroups $A, D$ that, as we shall see, satisfy the hypotheses of Lemma 3.5.2 (there are no such subgroups for the cases in Table (3.5.3). Note that in the tables, the values for $|T|$ are those for the relevant maximal torus in the inner-diagonal group $\operatorname{InnDiag}(G(q))$ rather than in the simple group $G(q)$ itself.

Suppose $T, A, D$ are as in Table 3.5.2. The cases where $A$ is not quasisimple are those listed in Table 3.5.1, and so are excluded from further consideration here. Thus $A$ is quasisimple, and we must check that $A, D$ satisfy the first three hypotheses of Lemma 3.5.2. In all cases except the two with entries in the "Comment" column of the table, $A D$ is a subsystem subgroup with maximal normalizer in $G(q)$ as in [74, Table 5.1], so condition (i) holds. Moreover, $N_{G}(A D)$ contains a maximal torus $T=T_{1} T_{0}$ of order as in column 2 of the table, giving (ii). Finally, we can check that condition (iii) holds by computing the action of $T_{0}$ on the Lie algebra $L(\bar{G})$ (where $\bar{G}$ is the ambient algebraic group) and seeing that the zero-weight space has dimension equal to that of $A$. Hence, by Lemma 3.5.2, there is a subset $\Delta$ of $\Omega=(G: H)$ such that $G^{\Delta}$ has socle $A / Z(A)$ acting on $\left(A: N_{A}\left(T_{1}\right)\right)$, where $T_{1}=T \cap A$.

Suppose $A$ is of type $A_{1}$. We check that the further conditions (iv) - (vii) of Lemma 3.5.2hold. Condition (vii) holds, since the group $G^{\Delta}$ is not binary, by [45]; and to verify (iv), we compute the action of $T_{1} T_{0}$ on $L(\bar{G})$ again to see that $C_{G}\left(T_{1} T_{0}\right)$ is a maximal torus, which must be $T$. For (v), let $T^{\prime}, T^{\prime \prime} \in \Delta$. Then $T^{\prime} \cap D A=T_{0} T_{1}^{a^{\prime}}$ and $T^{\prime \prime} \cap D A=T_{0} T_{1}^{a^{\prime \prime}}$, for some $a^{\prime}, a^{\prime \prime} \in A$. Since $A=\mathrm{SL}_{2}(q)$, we have $T_{1}^{a^{\prime}} \cap T_{1}^{a^{\prime \prime}}=Z(A)$, and so $T^{\prime} \cap T^{\prime \prime} \cap D A=T_{0} Z(A)$. As in the proof of Lemma 3.5.2(A), it follows that $T^{\prime} \cap T^{\prime \prime}=T_{0} Z(A)$, and this is equal to $\cap_{a \in A} T^{a}$, giving (v). Finally, (vi) is a standard property of tori in $\mathrm{SL}_{2}(q)$. Hence conditions (iv) - (vii) in Lemma 3.5 .2 hold, and so the lemma shows that $G$ is not binary.

If $A$ is not of type $A_{1}$, then we have three families of examples and three sporadic examples. Let us consider the families first: we find that

$$
\left(A,\left|T_{1}\right|\right)=\left(A_{2}^{\epsilon}(q), q^{2}+\epsilon q+1\right),\left(A_{4}^{\epsilon}(q), q^{4}+\epsilon q^{3}+q^{2}+\epsilon q+1\right) \text { or }\left({ }^{2} B_{2}(q), q+\epsilon \sqrt{2 q}+1\right) .
$$

Lemma 1.7 .2 implies that $G^{\Delta}$ is not binary and we again check that Lemma 3.5.2 applies to show that $G$ is not binary.

Finally we must deal with the remaining sporadic examples: here

$$
\left(A,\left|T_{1}\right|\right)=\left(A_{4}^{-}(q),(q+1)^{4}\right)(q=2,3) \text { or }\left(A_{2}^{-}(q),(q+1)^{2}\right)(q=3) .
$$

A magma calculation verifies that, in each case, the action of an almost simple group $X$ with socle $A$ on $\Delta=\left(X: N_{X}\left(T_{1}\right)\right)$ is not binary and, what is more, there exists a non-binary witness $(\delta, \lambda)=$ $\left(\left(\delta_{1}, \delta_{2}, \delta_{3}, \delta_{4}\right),\left(\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right)\right)$ of length 4 for $X^{\Delta}$ such that

$$
\bigcap_{i=1}^{4} \delta_{i}=\bigcap_{i=1}^{4} \lambda_{i}=\bigcap_{a \in A} T^{a} .
$$

Note that the computation here is straightforward: we have constructed the permutation representations under consideration and then we have checked 4 -tuples until we found one satisfying the required property. Thus condition (vii), and also condition (v') of the Remark following Lemma 3.5.2 hold. Conditions (iv) is verified as before, and (vi) is straightforward, as $T_{1}$ is the unique maximal torus of its order up to conjugacy in $A$. Hence Lemma 3.5 .2 gives the conclusion in these cases also.

Suppose finally that $T$ is as in Table 3.5.3. In these cases $T$ is cyclic. One can check that for each prime $t$ dividing $|T|, T$ contains a Sylow $t$-subgroup of $G$. Now [35, 36, 39, 95] imply that, for every $g \in T \backslash\{1\}$, $C_{G}(g)=T$, and we conclude that $N=N_{G(q)}(T)$ is a Frobenius group, with $T$ the Frobenius kernel. Let $C$ be a Frobenius complement; observe that $C$ is cyclic, and let $c$ be a generator of $C$. Now Lemma 2.4.2 implies that $C_{G}(c)>C$ and so we can choose an element $x \in C_{G}(c) \backslash N_{G}(T)$. Then the action of $N$ on ( $N: N \cap N^{x}$ ) is a Frobenius action and, since $|C|>2$ in every case, and, since $N \cap N^{x}=N \cap H \cap H^{x}$, Lemma 1.7 .2 implies that the action of $H$ on $\left(H: H \cap H^{x}\right)$ is not binary; hence $G$ is not binary by Lemma 1.6.1.

The remaining cases are resolved by calculations with magma:
Lemma 3.5.3. Let $G$ be as in Proposition 3.5.1, and suppose that $H=N_{G}(T)$ where $(G(q),|T|)$ is as listed in Table 3.5.1. If $\Omega=(G: H)$, then $(G, \Omega)$ is not binary.

Proof. For the group $G(q):={ }^{2} E_{6}(2)$, observe that $N_{G(q)}(T) \cong 3^{5} . \mathrm{SO}_{5}(3)$ and also that $G(q)$ has a unique conjugacy class of elements of order 5 (see [28]). Hence Lemma 1.6.15, applied with the prime $p:=5$, gives the conclusion.

When $G(q):=E_{7}(2)$, we gain use Lemma 1.6 .15 with the prime $p:=7$. Using information from [81, we see that there exists a unique conjugacy class of elements of order 7. Furthermore, from [74], we have $H \cong 3^{7} \cdot 2 \cdot \operatorname{Sp}_{6}(2)$. Since the Sylow 7 -subgroup of $G(q)$ is elementary-abelian of order $7^{3}$ and a Sylow 7-subgroup of $H$ is of order 7, Lemma 1.6.15 implies that $G(q)$ is not binary.

### 3.6 Maximal subgroups in (V) of Theorem 3.1.1

The main result of this section is the following proposition. The cases excluded in the proposition (those in Table 3.6.1) will be dealt with in Lemma 3.6.2.

Proposition 3.6.1. Assume $G$ is almost simple with socle $G(q)$, an exceptional group of Lie type over $\mathbb{F}_{q}$, and suppose $G(q)$ is not as in Proposition 3.2.1. Let $H$ be a maximal subgroup of $G$ as in part ( $V$ ) of Theorem 3.1.1. Let $\Omega=(G: H)$. Then either $(G, \Omega)$ is not binary, or $(G, H)$ is as in Table 3.6.1.

Table 3.6.1: Exceptions in Prop. 3.6.1

| $G(q)$ | $H \cap G(q)$ |
| :---: | :---: |
| $E_{6}(2)$ | $G_{2}(2)$ |
| ${ }^{2} E_{6}^{-}(2)$ | $F_{4}(2)$ |
| $E_{7}(2)$ | $G_{2}(q) C_{3}(2)$ |

Proof. Here $H$ is one of the subgroups given in Table 3.1.1.
If $H$ has socle $A_{1}(q)$, then $q>5$ and we consider an element $x \in H$ of order $\frac{q-1}{(2, q-1)}$, as given in Table 2.2.1. As $C_{G}(x)$ contains a maximal torus of $G(q)$, there exists $g \in C_{G}(x) \backslash H$. Now Lemmas [2.2.5 and 2.2.6 imply that the action of $H$ on $\left(H: H \cap H^{g}\right)$ is not binary. The result then follows by Lemma1.6.1.

If $H$ has socle $B_{2}(q)$ then $q \geq 5$ and we proceed in the same way, using an element of order $q-1$ together with Lemma 2.2.8 in place of Lemma 2.2.5. A similar argument applies in the case where $F^{*}(H \cap G(q))=$ $C_{4}(q) \cong \operatorname{PSp}_{8}(q)$.

Now consider the cases listed in Table 3.6.2. In each of these cases $F^{*}(H)$ has a factor that is generated by long root subgroups of $G$, from which it follows easily that there are subgroups $A \cong \operatorname{SL}_{r}(q)$ of $H$, and $S \cong \mathrm{SL}_{r+1}(q) / Z$ of $G$, satisfying the hypotheses of Lemma 1.6.10, where $r$ is as indicated in Table 3.6.2. Thus Lemmas 1.6 .10 provides a subset $\Delta$ of $\Omega$ of size $q^{r}$, and this is a beautiful subset unless $\operatorname{Alt}\left(q^{r}\right)$ and Alt $\left(q^{r}-1\right)$ are sections of $G(q)$ and $H$, respectively. Hence Lemmas 2.1.1 and 2.1.2 show that if $(G, \Omega)$ is binary, the only possibility for $(G(q), H)$ is $\left(E_{7}(2), G_{2}(2) C_{3}(2)\right)$, as in Table 3.6.1.

Similarly, the subgroup $A_{1}(q) G_{2}\left(q^{2}\right)$ of $E_{8}(q)$ in Table 3.1.1] is a twisted version of the subgroup $A_{1} G_{2} G_{2}$ in the algebraic group $E_{8}$; hence this contains a subgroup $A \cong \mathrm{SL}_{3}\left(q^{2}\right)$ which lies in a subgroup $S \cong \mathrm{SL}_{4}\left(q^{2}\right)$ of $G$, and again Lemmas 1.6.10 and 2.1.2 give the conclusion.

It remains to deal with the following subgroups from Table 3.1.1.
(1) $E_{8}(q): F^{*}(H)=F_{4}(q)(p=3)$ or $A_{1}(q) A_{2}^{\epsilon}(q)(p \geq 5)$
(2) $E_{7}(q): F^{*}(H)=A_{2}^{\epsilon}(q)(p \geq 5), A_{1}(q) A_{1}(q)(p \geq 5)$, or $A_{1}(q) G_{2}(q)(p \geq 3, q \geq 5)$

Table 3.6.2: Subgroups in Table 3.1.1 with a long root factor

| $G(q)$ | $H$ | $A$ |
| :---: | :---: | :---: |
| $E_{8}(q)$ | $G_{2}(q) F_{4}(q)$ | $\mathrm{SL}_{4}(q)$ |
|  | $A_{1}(q) G_{2}(q) G_{2}(q)$ | $\mathrm{SL}_{3}(q)$ |
| $E_{7}(q)$ | $G_{2}(q) C_{3}(q)$ | $\mathrm{SL}_{3}(q)$ |
|  | $A_{1}(q) F_{4}(q)$ | $\mathrm{SL}_{4}(q)$ |
| $E_{6}(q)$ | $F_{4}(q)$ | $\mathrm{SL}_{4}(q)$ |
|  | $A_{2}(q) G_{2}(q)$ | $\mathrm{SL}_{3}(q)$ |
| $E_{6}^{-}(q)$ | $A_{2}^{-}(q) G_{2}(q)$ | $\mathrm{SL}_{3}(q)$ |
| $F_{4}(q)$ | $A_{1}(q) G_{2}(q)$ | $\mathrm{SL}_{3}(q)$ |

(3) $E_{6}^{\epsilon}(q): F^{*}(H)=A_{2}^{ \pm}(q)(\epsilon=+, p \geq 5)$ or $G_{2}(q)(p \neq 7)$
(4) $E_{6}^{-}(q): F^{*}(H)=F_{4}(q)$
(5) $F_{4}(q): F^{*}(H)=G_{2}(q)(p=7)$
(6) ${ }^{3} D_{4}(q): F^{*}(H)=G_{2}(q)$ or $A_{2}^{\epsilon}(q)$.

Case (1) Here $G(q)=E_{8}(q)$. First consider $F^{*}(H)=F_{4}(q)$ with $q=3^{a}$. If $q>3$, let $x \in F^{*}(H)$ be the semisimple element defined in Lemma 2.2.15. Then $C_{G}(x)$ contains a maximal torus of $G(q)$, hence there exists $g \in C_{G}(x) \backslash H$, and so $x \in H \cap H^{g}$, a core-free subgroup of $H$. By Lemma 2.2.15, the action of $H$ on ( $H: H \cap H^{g}$ ) is not binary, giving the conclusion. If $q=3$, we use the result of Lemma 2.3.1 for the group $H=F_{4}(3)$ : since $|G: H|$ is even, there exists a non-trivial orbit of $H$ on $\Omega$ of odd size, and the action of $H$ on this orbit is not binary by Lemma 2.3.1, giving the conclusion.

Now consider the other possibility $F^{*}(H)=A_{1}(q) A_{2}^{\epsilon}(q)(p \geq 5)$. Let $R$ be the $A_{2}^{\epsilon}(q)$ factor of $F^{*}(H)$. From the construction of the corresponding maximal subgroup $A_{1} A_{2}$ in the algebraic group $E_{8}$ given in [93, p.46], we see that $R$ lies in a Levi subgroup $L=A_{7}(q)$ of $G$, with embedding given by the adjoint representation. Let $A$ be a natural subgroup $\mathrm{SL}_{2}(q)$ of $R$ (i.e. acting as $1 \oplus 0$ on the natural 3-dimensional module, where we denote by a non-negative integer $r$ the irreducible $\mathbb{F}_{q} A$-module of highest weight $r$ ). The restriction of the natural 8-dimensional $L$-module to $A$ is $2 \oplus 1^{2} \oplus 0$, so in particular $C_{L}(A)$ contains a subgroup $\mathrm{SL}_{2}(q)$ and also $A$ lies in a subgroup $S=A_{2}(q)$ of $L$. Hence there exists $x \in C_{L}(A) \backslash H$ such that $A<S^{x} \not \leq H$. Now an application of Lemma 1.6.10 yields a subset $\Delta$ of $\Omega$ such that $G^{\Delta} \geq \operatorname{ASL}_{2}(q)$, giving the conclusion in the usual way using Lemma 2.1.2.
Case (2) Here $G(q)=E_{7}(q)$. First consider $F^{*}(H)=A_{2}^{\epsilon}(q)$. Again let $A$ be a natural $\mathrm{SL}_{2}(q)$ in $F^{*}(H)$. Since maximal subgroups $A_{2}^{\epsilon}(q)$ exist for both $\epsilon=+$ and $\epsilon=-$, and each of these arises from a fixed maximal $A_{2}$ in the algebraic group $E_{7}$, it follows that there is a subgroup $S \cong A_{2}(q)$ of $G$ containing $A$ (it could be that $S=F^{*}(H)$ ). From [93, p.83], it follows that $A$ is contained in a Levi subgroup $L$ of $G$ of type $A_{1} A_{4} T_{2}$. The torus $T_{2}$ centralizes $A$, and so there exists $x \in C_{G}(A) \backslash H$. Then $A<S^{x} \not \leq H$, and the conclusion follows as in Case (1) above.

Next consider $F^{*}(H)=A_{1}(q) G_{2}(q)$. Let $A$ be an $\mathrm{SL}_{3}(q)$ subgroup of the $G_{2}(q)$ factor. From [93, 3.12] we see that the $G_{2}(q)$ factor lies in a Levi subgroup $L=A_{6}(q)$ of $G$, acting irreducibly on the natural 7 -dimensional $L$-module $V_{7}$. Then $V_{7} \downarrow A=10 \oplus 01 \oplus 00$. Now $L$ lies in a subsystem subgroup $M=A_{7}(q)$ of $G$, and so we see that $A$ is contained in a subgroup $S \cong A_{3}(q)$ of $M$ acting on the natural $M$-module as $100 \oplus 001$. Hence $A<S \not \leq H$, and now Lemmas 1.6.10 and 2.1.2 give the conclusion.

Finally consider $F^{*}(H)=A_{1}(q) A_{1}(q)=A_{1}^{(1)} A_{1}^{(2)} \cong \operatorname{PSL}_{2}(q)^{2}(p \geq 5)$. We assume that the action is binary, and for $i=1,2$ let $T^{(i)}$ be a torus in $A_{1}^{(i)}$ of order $\frac{q-1}{2}$. Write $T=T_{1}^{(1)} T_{1}^{(2)}$ and observe that $N_{F^{*}(H)}(T)$ contains $D_{q-1} \times D_{q-1}$. From [93, p.37], we see that (re-labelling the $A_{1}^{(i)}$ if necessary), we have $C_{G}\left(T^{(1)}\right) \geq T^{(1)} A_{2}(q) A_{4}(q)$, and $A_{1}^{(2)}$ is embedded in this via the irreducibles of highest weight 2 and
4. Hence $T_{1}^{(2)}$ is diagonal in $A_{2}(q) A_{4}(q)$, and so $C_{G(q)}(T)$ contains a maximal torus of order $(q-1)^{7} / 2$. Hence a Sylow 2-subgroup of $C_{G(q)}(T)$ is strictly larger than the Sylow 2-subgroup of $T$. This means, in particular, that the group $\left(D_{q-1} \times D_{q-1}\right) / T$, which is a Klein 4 -group, is a proper subgroup of a Sylow 2-subgroup of $N_{G(q)}(T) / T$. The group $H \cap G(q)$ is $\left(\mathrm{PSL}_{2}(q) \times \mathrm{PSL}_{2}(q)\right)$. , of index 2 in $\mathrm{PGL}_{2}(q) \times \mathrm{PGL}_{2}(q)$. Now, observe that $\left|C_{G(q)}(T)\right|_{2}$ is strictly larger than $\left|C_{H \cap G(q)}(T)\right|_{2}$. We conclude that there exists $x$ in $G(q) \backslash H$ such that $H^{x} \cap H$ contains $D_{q-1} \times D_{q-1}$. We therefore obtain a suborbit of $G$ on which the action of the stabiliser $H$ is isomorphic to the action of $H$ on $(H: M)$ where $M$ is a subgroup of $H$ containing $D_{q-1} \times D_{q-1}$. But now, when $q>5$, Lemma 1.6.1 and Lemma 2.2.2 imply that if $(G, \Omega)$ is binary then $M$ must contain $F^{*}(H)$; this would mean that $x \in H$, a contradiction, as required. Suppose now $q=5$. We have $F^{*}(H) \cong \operatorname{Alt}(5) \times \operatorname{Alt}(5)$. Write $F^{*}(H)=A \times B$, with $A \cong \operatorname{Alt}(5) \cong B$. Pick $F<H$ with $F$ elementary abelian of order $5^{2}$. Clearly, there exists $x$ in $N_{G}(F) \backslash H$, so $F \leq H^{x} \cap H$. Suppose one of the factors, say $A$, of $F^{*}(H)$ is contained in $H^{x} \cap H$. Then $A$ and $A^{x^{-1}}$ are contained in $H$. Hence $A^{x^{-1}}$ is equal to $A$, to $B$ or to a diagonal subgroup of $A \times B$. Unipotent elements of order 5 in $B$ or a diagonal subgroup are in different classes to those in $A$ (see [66, Table 34]). Hence $A^{x^{-1}}=A$ and so $x \in N_{G}(A)=H$, a contradiction. We conclude that neither factor $A$ or $B$ is contained in $H^{x} \cap H$. Now, the proof follows with a magma computation: we have verified that, for every group $H$ with $F^{*}(H)=\operatorname{Alt}(5) \times \operatorname{Alt}(5)$ and for every subgroup $X$ of $H$, the action of $H$ on $(H: X)$ is binary only when $X$ contains the whole of $F^{*}(H)$ or one of the two simple factors $\operatorname{Alt}(5)$ of $F^{*}(H)$.
Case (3) Let $G(q)=E_{6}^{\epsilon}(q)$. First consider $F^{*}(H)=A_{2}^{ \pm}(q)$. Here $\epsilon=+$. Let $A$ be a natural $\mathrm{SL}_{2}(q)$ in $F^{*}(H)$. Since maximal subgroups $A_{2}^{+}(q)$ and $A_{2}^{-}(q)$ both exist (actually just for $q \equiv \epsilon \bmod 4$ ), and each of these arises from a fixed maximal $A_{2}$ in the algebraic group $E_{6}$, it follows that there is a subgroup $S \cong A_{2}(q)$ of $G$ containing $A$ (it could be that $S=F^{*}(H)$ ). From [93, 5.5] we know that $L\left(E_{6}\right) \downarrow A_{2}=11 \oplus 41 \oplus 14$. Hence we can work out $L\left(E_{6}\right) \downarrow A$, and in particular compute that, if $t$ denotes the central involution of $A$, then $\operatorname{dim} C_{L\left(E_{6}\right)}(t)=38$, whence $C_{E_{6}}(t)=A_{1} A_{5}$. Also using $L\left(E_{6}\right) \downarrow A$, the only possible embedding of $A$ in $A_{1} A_{5}$ is via the representations $1,2^{2}$. Hence $C_{A_{5}}(A)=A_{1}$, and so $C_{G}(A) \not \leq H$. If we pick $x \in C_{G}(A) \backslash H$, then $A<S^{x} \not \leq H$, and the conclusion follows in the usual way.

Now consider $F^{*}(H)=G_{2}(q)(p \neq 7)$. First suppose that $p \neq 2$. Then $G_{2}(q)$ has an involution $t$ with centralizer $A \tilde{A}$, where $A$ and $\tilde{A}$ are long and short $\mathrm{SL}_{2}(q)$ subgroups (respectively) in $G_{2}(q)$. Arguing as above using $L\left(E_{6}\right) \downarrow G_{2}$ (given in [79, Table 10.1]), we see that $A \tilde{A}<C_{E_{6}}(t)=A_{1} A_{5}$, with embedding given by $0 \otimes 1,1 \otimes 2$. Hence $C_{A_{5}}(A)=A_{2}$, and so $N_{G}(A)$ contains a subgroup $A_{2}^{\epsilon}(q) A$.

Now $G_{2}(q)$ has a subgroup $S \cong \mathrm{SL}_{3}(q)$ containing $A$. The composition factors of $S$ on $L\left(E_{6}\right)$ are given in [99, Table 5 and Lemma 5.5], from which we can deduce that the only subsystem subgroup containing $S$ is $A_{2}^{3}$ (this is $C_{E_{6}}(Z(S))$ unless $p=3$ ). It follows that $C_{G}(S)=Z(S)$, and so $N_{G(q)}(S)=S .2<H$. In particular, it follows that $N_{G}(A) \neq N_{H}(A)\left(N_{G}(S) \cap N_{G}(A)\right)$. Therefore Lemma 2.6.1 implies that there exists $x \in N_{G}(A)$ such that $S^{x} \notin H$. Hence $A<S^{x} \not \leq H$, giving the conclusion in the usual way.

It remains to consider the case where $p=2$. Again let $\mathrm{SL}_{3}(q) \cong S<G_{2}(q)$, let $A$ be a natural subgroup $\mathrm{SL}_{2}(q)$ of $S$, and let $\tilde{A}=C_{G_{2}(q)}(A) \cong \mathrm{SL}_{2}(q)$. Now [99, 5.5] shows that $S$ lies in a subsystem subgroup $A_{2}^{3}$ of the algebraic group $E_{6}$, and so $A<A_{1}^{3}<A_{5}$. Therefore $C_{A_{5}}(A)$ contains an $A_{1}$ subgroup, and so $C_{G}(A)$ contains $A_{1}(q)^{2}$. Hence we see as in the previous paragraph that $N_{G}(A) \neq N_{H}(A)\left(N_{G}(S) \cap N_{G}(A)\right)$, and now the argument goes through as before, the only difference being that this time Lemma 2.1.2 does not give a contradiction when $q=2$, leaving that possibility in Table 3.6.1.

Case (4) Let $G(q)=E_{6}^{-}(q)$ and $F^{*}(H)=F_{4}(q)$. There are long root subsystem subgroups $A<S<H$ with $A \cong \mathrm{SL}_{3}(q), S \cong \mathrm{SL}_{4}(q) / Z$. Their centralizers can be read off using [75, Sec. 4], and we have $C_{G}(A)=A_{2}\left(q^{2}\right), C_{G}(S)=A_{1}\left(q^{2}\right) T_{1}$ and $C_{H}(A)=A_{1}(q) T_{1}$. Hence $N_{G}(A) \neq N_{H}(A)\left(N_{G}(S) \cap N_{G}(A)\right)$, and so Lemma 2.6.1 yields an element $x \in G$ such that $A<S^{x} \notin H$. Now Lemmas 1.6.10 and [2.1.2 give a contradiction, except when $q=2$, leaving that possibility in Table 3.6.1.

Case (5) Let $G(q)=F_{4}(q)$ and $F^{*}(H)=G_{2}(q)$ with $p=7$. We argue as for $G_{2}(q)$ in case (3) above. For an involution $t \in G_{2}(q)$ we have $C_{G_{2}(q)}(t)=A \tilde{A}$ where $A$ and $\tilde{A}$ are long and short $\mathrm{SL}_{2}(q)$ subgroups, and also $A<S<G_{2}(q)$ with $S \cong \operatorname{SL}_{3}(q)$. Then $A \tilde{A}<C_{F_{4}}(t)=A_{1} C_{3}$ with embedding $0 \otimes 1,1 \otimes 2$, and
so $C_{C_{3}}(A)=A_{1}$. Again $N_{G}(S)<H$, and hence $N_{G}(A) \neq N_{H}(A)\left(N_{G}(S) \cap N_{G}(A)\right)$. Now Lemma 2.6.1 yields an element $x \in G$ such that $A<S^{x} \not \leq H$ and we proceed as before.

Case (6) Here $G(q)={ }^{3} D_{4}(q)$ and $F^{*}(H)=G_{2}(q)$ or $A_{2}^{\epsilon}(q)$. Consider the first case. Let $A<S<G_{2}(q)$ with $A \cong \mathrm{SL}_{2}(q), S \cong \mathrm{SL}_{3}(q)$ generated by long root subgroups of $H$ (and of $G$ ). Then $C_{G}(A)=A_{1}\left(q^{3}\right)$, $C_{H}(A)=A_{1}(q)$ and $N_{G(q)}(S)=S \cdot\left(q^{2}+q+1\right) \cdot 2$. There is no factorization of a group with socle $A_{1}\left(q^{3}\right)$ with one of the factors being $N\left(A_{1}(q)\right)$ (see Lemma 2.6.2), and so Lemma 2.6.1 applies to give an element $x \in G$ such that $A<S^{x} \not \leq H$. Now Lemmas 1.6 .10 and 2.1 .2 give a contradiction (except when $q=2$, in which case $G(q)={ }^{3} D_{4}(2)$, excluded by hypothesis).

Now let $F^{*}(H)=A_{2}^{\epsilon}(q)$. From [57], we see that $H \cap G(q)=\operatorname{PGL}_{3}^{\epsilon}(q)$ with $q \equiv \epsilon \bmod 3$ and $q>2$. First assume that $\epsilon=+$, and let $A$ be a natural $\mathrm{SL}_{2}(q)$ subgroup of $H$. Then $A$ centralizes an element $g$ of order $q-1$ in $H$, and from the list of centralizers in $G$ (see for example [57, p.184]), we see that $C_{G}(g)^{\prime}$ must be $\mathrm{SL}_{2}\left(q^{3}\right)$, or possibly $\mathrm{SL}_{3}(q)$ when $q=4$. Excluding the latter possibility, it follows that $C_{G}(A)$ contains the centralizer of $\mathrm{SL}_{2}\left(q^{3}\right)$, which is a root subgroup $\mathrm{SL}_{2}(q)$. Hence in any case (including the extra $q=4$ possibility), there is a group $S \cong A_{2}(q)$ such that $A<S \not 又 H$, giving the result in the usual way.

This leaves the case where $\epsilon=-$, so that $H \cap G(q)=\operatorname{PGU}_{3}(q)$ with $q \equiv-1 \bmod 3$. We refer to Lemma 2.2.10, and let $g$ be the element of $H \cap G(q)$ defined in that lemma. Observe that $g$ is semisimple in $G(q)$, hence, using the list of maximal tori of $G(q)$ given in 53 , we can conclude that there exists $x \in G(q) \backslash H$ such that $x \in C_{G}(g)$. Now consider the action of $H$ on the cosets of $H \cap H^{x}$, a subgroup containing the element $g$ and not containing $\mathrm{PSU}_{3}(q)$. If $q \leq 5$, we use magma to show that this action is not binary, giving the conclusion. And if $q \geq 7$, Lemma 2.1.1 shows that $H$ has no section $\operatorname{Sym}(q)$, and so Lemma 2.2.10 shows that the action $\left(H,\left(H: H \cap H^{x}\right)\right)$ is not binary, again giving the conclusion.

The remaining cases are resolved with the aid of magma:
Lemma 3.6.2. Let $G$ be as in Proposition 3.6.1, and suppose that $(G(q), H)$ is listed in Table 3.6.1. Then $(G, \Omega)$ is not binary.

Proof. Suppose that $G(q)=E_{6}(2)$ and $H \cap G(q)=G_{2}(2)$. Referring to [58], we see that $H$ is maximal in $G$ only when $G=G(q)=E_{6}(2)$, thus we assume this from here on. Now, using magma, we have computed all the binary transitive actions of $G_{2}(2)$, and we have found that these have degree $1,2,4032,6048$ and 12096. Now Lemma 1.6 .1 implies that, if the action of $G$ on $(G: H)$ is binary, then the action of $H$ on each of its suborbits must be binary - thus all suborbits must have size one of the five listed numbers. There is precisely one suborbit of size 1 (by maximality), the other suborbits are of even size, hence $\left|E_{6}(2): G_{2}(2)\right|$ is odd, a contradiction.

Next assume that $G(q)={ }^{2} E_{6}(2)$ and $H \cap G(q)=F_{4}(2)$. Here $H$ is either $F_{4}(2)$ or $F_{4}(2) \times 2$. Now $F_{4}(2)$ has a maximal subgroup isomorphic to $D_{4}(2) . \operatorname{Sym}(3)$; let $X$ be the subgroup $D_{4}(2)$ of this. Then $X$ is centralized by an element $g$ of order 3 in $G(q) \backslash H$ (see [28]). Hence $X \triangleleft H \cap H^{g}$. At this point we can argue as in the proof of Proposition 3.2 .1 (the $F_{4}(2)$ case); indeed, Lemma 1.6.15 applied with $p=7$ shows that $\left(H,\left(H: H \cap H^{g}\right)\right)$ is not binary. The conclusion follows.

Finally, assume that $G=E_{7}(2)$ and $H=G_{2}(2) C_{3}(2)$. Choose $x \in G \backslash H$ normalizing a Sylow 2subgroup of $H$, so that $\left|H: H \cap H^{x}\right|$ is odd and greater than 1. A magma computation show that all transitive actions of $H$ of odd degree greater than 1 are not binary, so the conclusion follows.

### 3.7 Maximal subgroups in (VI) of Theorem 3.1.1

In this section we prove
Proposition 3.7.1. Assume $G$ is almost simple with socle $G(q)$, an exceptional group of Lie type over $\mathbb{F}_{q}$, and suppose $G(q)$ is not as in Proposition 3.2.1. Let $H$ be a maximal subgroup of $G$ as in part (VI) of Theorem 3.1.1. Let $\Omega=(G: H)$. Then either $(G, \Omega)$ is not binary, or $(G, H)$ is as in Table 3.7.1.

Table 3.7.1: Exceptions in Prop. 3.7.1

| $G(q)$ | $H \cap G(q)$ |
| :---: | :---: |
| $G_{2}\left(2^{e}\right), e$ prime | $G_{2}(2)$ |
| $F_{4}\left(2^{e}\right), e$ prime | $F_{4}(2)$ |
| $E_{6}^{\epsilon}\left(2^{e}\right), e$ prime | $E_{6}^{\epsilon}(2)$ |

Proof. Here $H$ is of the same type as $G$ - that is, one of the following holds:
(i) $H \cap G(q)=G\left(q_{0}\right)$, where $\mathbb{F}_{q_{0}} \subset \mathbb{F}_{q}$;
(ii) $H \cap G(q)<G(q)$ is a twisted subgroup, namely one of

$$
\begin{aligned}
& { }^{2} E_{6}\left(q^{1 / 2}\right)<E_{6}(q), \\
& { }^{2} F_{4}(q)<F_{4}(q), \\
& { }^{2} G_{2}(q)<G_{2}(q) .
\end{aligned}
$$

Consider first case (i). Here for each possible $G(q)$, we define subgroups $A<S<G\left(q_{0}\right)$ with $A \cong$ $\mathrm{SL}_{r}\left(q_{0}\right)$ and $S \cong \mathrm{SL}_{r+1}\left(q_{0}\right)$, both subsystem subgroups of $G\left(q_{0}\right)$, as in Table 3.7.2. In each case $C_{G}(A)^{\prime}$ and $C_{G}(S)^{\prime}$ are as indicated in Table 3.7 .2 and $C_{H}(A)$ is of the same type as $C_{G}(A)$ over the subfield $\mathbb{F}_{q_{0}}$. It then follows from Lemma 2.6.2 that $N_{G}(A) \neq N_{H}(A)\left(N_{G}(S) \cap N_{G}(A)\right)$, and so Lemma 2.6.1 yields an element $x \in G$ such that $A<S^{x} \not \leq H$. Now Lemmas 1.6.10 and 2.1.2 give a contradiction, except in the cases with $q_{0}=2$ in Table 3.7.1.

Table 3.7.2

| $G(q)$ | $r$ | $C_{G}(A)$ | $C_{H}(A)$ | $C_{G}(S)$ |
| :---: | :---: | :---: | :---: | :---: |
| $G_{2}(q)$ | 2 | $A_{1}(q)$ | $A_{1}\left(q_{0}\right)$ | $\left(3, q_{0}-1\right)$ |
| $F_{4}(q)$ | 3 | $A_{2}(q)$ | $A_{2}\left(q_{0}\right)$ | $A_{1}(q)$ |
| $E_{6}(q)$ | 3 | $A_{2}(q)^{2}$ | $A_{2}\left(q_{0}\right)^{2}$ | $A_{1}(q)^{2}$ |
| ${ }^{2} E_{6}(q)$ | 3 | $A_{2}\left(q^{2}\right)$ | $A_{2}\left(q_{0}^{2}\right)$ | $A_{1}\left(q^{2}\right)$ |
| $E_{7}(q)$ | 4 | $A_{3}(q) A_{1}(q)$ | $A_{3}\left(q_{0}\right) A_{1}\left(q_{0}\right)$ | $A_{2}(q) T_{1}$ |
| $E_{8}(q)$ | 5 | $A_{4}(q)$ | $A_{4}\left(q_{0}\right)$ | $A_{2}(q) A_{1}(q)$ |

Now consider case (ii). In the first case, $F^{*}(H)={ }^{2} E_{6}\left(q^{1 / 2}\right)<E_{6}(q)$, and as above we pick $A<S$ with $A \cong \mathrm{SL}_{4}\left(q^{1 / 2}\right)$ and $S \cong \operatorname{SL}_{5}\left(q^{1 / 2}\right)$. Then $S \not \leq H$ as $H$ has no subgroup of type $A_{4}\left(q^{1 / 2}\right)$, and the conclusion follows as usual.

Next let $F^{*}(H)={ }^{2} F_{4}(q)<F_{4}(q)$ with $q=2^{2 a+1}$, and note that $q>2$ by hypothesis. Regard $F^{*}(H)$ as the centralizer in $F_{4}(q)$ of a graph automorphism $\tau$. Then $H$ has a subgroup $A \cong \mathrm{SL}_{2}(q)$ arising as the fixed point group of $\tau$ on a subsystem subgroup $A_{1}(q) \tilde{A}_{1}(q)$ in $F_{4}(q)$, and this lies in a subgroup $S=A_{2}(q)$ of $F_{4}(q)$ that is a diagonal subgroup of a subsystem $A_{2}(q) \tilde{A}_{2}(q)$. As $H$ has no subgroup $A_{2}(q)$, we have $A<S \not \leq H$, giving the conclusion.

Now consider the case where $H \cap G(q)={ }^{2} G_{2}(q)<G_{2}(q)$, and note that $q>3$ by hypothesis. Choose $x \in H \cap G(q)$ of order $q-1$, as in Lemma 2.2.16. Since $C_{G(q)}(x)$ is a torus of order $(q-1)^{2}$, there exists $g \in C_{G}(x) \backslash H$. Then $x \in H \cap H^{g}$, and the action of $H$ on $\left(H: H \cap H^{g}\right)$ is not binary by Lemma 2.2.16, giving the conclusion.

The treatment of groups of type (VI) is completed with the following result.
Lemma 3.7.2. Let $G$ be as in Proposition 3.7.1, and suppose that $(G(q), H)$ is listed in Table 3.7.1. Then $(G, \Omega)$ is not binary.

Proof. Consider the action in Line 1 of the table. Here $G=G_{2}\left(2^{e}\right)\langle\phi\rangle$ where $\phi$ is a field automorphism of order 1 or $e$, and $H=G_{2}(2) \times\langle\phi\rangle$. Choose $x \in G \backslash H$ normalizing a Sylow 2-subgroup $P$ of $H \cap G_{2}(2)$, and let $X=H \cap H^{x} \cap G_{2}(2)$, so that $P \leq X$. The subgroups of $H \cap G(q)=G_{2}(2)$ containing $P$ are the Borel subgroup $P$ itself, and two maximal parabolics of shape $\left[2^{5}\right]$. $\operatorname{Sym}(3)$. These are all self-normalizing in $G_{2}(2)$, so it follows that $H \cap H^{x}=X \times\langle\sigma\rangle$, where $\sigma=\phi$ or 1 . Note that $\sigma$ is in the kernel of the action of $H$ on $\left(H: H \cap H^{x}\right)$. Thus the latter action is either $\left(G_{2}(2),\left(G_{2}(2): X\right)\right)$ or $\left(G_{2}(2) \times e,\left(G_{2}(2) \times e: X\right)\right)$. Using magma we check that the action $\left(G_{2}(2),\left(G_{2}(2): X\right)\right)$ is not binary for each of the three possibilities for $X$. Hence also $\left(G_{2}(2) \times e,\left(G_{2}(2) \times e: X\right)\right)$ is not binary, by Lemma 1.6.2. It follows that the action of $H$ on ( $H: H \cap H^{x}$ ) is not binary, giving the conclusion.

Now consider Line 3 of Table 3.7.1. First suppose $H \cap G(q)={ }^{2} E_{6}(2)$, with $G(q)={ }^{2} E_{6}\left(2^{e}\right)$. Let $D_{0}={ }^{2} D_{5}(2)$ be a subsystem subgroup of $H \cap G(q)$. Then $D<{ }^{2} D_{5}(q)<G(q)$, a subgroup centralized by a torus of order $\frac{q+1}{3}$. Choosing $g \in C_{G(q)}\left(D_{0}\right) \backslash H$, we have $H \cap H^{g} \triangleright D$. As in the proof of Proposition 3.4.1, there is a subgroup $A=\mathrm{SL}_{2}(4)$ of $D$ and a subgroup $S=\mathrm{PSL}_{3}(4)$ of $H$ such that $A<S \notin H \cap H^{g}$. Hence it follows in the usual way using Lemmas 1.6 .10 and 2.1.2 that the action of $H$ on $\left(H: H \cap H^{g}\right)$ is not binary, giving the conclusion in this case. A similar argument handles the case where $H=E_{6}(2)$ : here we take $D=A_{5}(2)$ and again choose $g \in C_{G(q)}\left(D_{0}\right) \backslash H$, so that $H \cap H^{g} \triangleright D$. There is a subgroup $A=\operatorname{SL}_{4}(2)$ of $D$ and a subgroup $S=\mathrm{SL}_{5}(2)$ of $H$ such that $A<S \not \leq H$, and the conclusion again follows.

Finally, consider $H \cap G(q)=F_{4}(2)$ with $G(q)=F_{4}\left(2^{e}\right)$. Choose a subgroup $D=A_{2}(2) \times 7$ lying in a subsystem subgroup $A_{2}(2) \times \tilde{A}_{2}(2)$ of $H$, where the factor $\tilde{A}_{2}(2)$ is generated by short root elements. There is an element $x \in N_{G}(D) \backslash H$, and so $D \leq H \cap H^{x}<H$. From [85], it follows that $H \cap H^{x}$ is contained in a subsystem subgroup $\left(A_{2}(2) \times \tilde{A}_{2}(2)\right) .2$ of $H$. The factor $A_{2}(2)$ of $D$ lies in a subgroup $A_{3}(2)$ of $H$ that is not contained in $H \cap H^{x}$, and so it follows, using Lemma 1.6.10 in the usual way, that the action of $H$ on the suborbit ( $H: H \cap H^{x}$ ) is not binary. This completes the proof.

### 3.8 The remaining families in Theorem 3.1.1

We proceed family by family.

### 3.8.1 Type (III)

Lemma 3.8.1. Assume $G$ is almost simple with socle $G(q)$, an exceptional group of Lie type over $\mathbb{F}_{q}$, and let $H$ be a maximal subgroup of $G$ as in part (III) of Theorem 3.1.1. If $\Omega=(G: H)$, then $(G, \Omega)$ is not binary.

Proof. Here $G(q)=E_{7}(q), p>2$ and $H \cap G(q)=\left(2^{2} \times D_{4}(q) \cdot 2^{2}\right) . \operatorname{Sym}(3)$ or ${ }^{3} D_{4}(q) .3$. Let $D:=D_{4}(q)$ or ${ }^{3} D_{4}(q)$ in $H$, and let $A$ be a subsystem subgroup $\mathrm{SL}_{3}(q)$ of $D$. Here $D$ arises from a subgroup $D_{4}$ of the algebraic group $E_{7}\left(\overline{\mathbb{F}}_{q}\right)$ that lies in a subsystem $A_{7}$ (see the discussion after [78, Theorems 1,7$]$ ), and we see that $A$ lies in a subgroup $A_{7}(q)$ of $G(q)$, acting on the natural 8-dimensional module as $10 \oplus 01 \oplus 00^{2}$. Then $A$ lies in a subgroup $A_{3}(q)$ of this $A_{7}(q)$ that does not lie in $D$. At this point we can apply Lemma 1.6.10 to see that there is a subset $\Delta$ of $\Omega$ such that $G^{\Delta} \geq \operatorname{ASL}_{3}(q)$. This shows that $G$ is not binary in the usual way using Lemma 2.1.2,

### 3.8.2 Type (IV)

Lemma 3.8.2. Assume $G$ is almost simple with socle $G(q)$, an exceptional group of Lie type over $\mathbb{F}_{q}$, and let $H$ be a maximal subgroup of $G$ as in part (IV) of Theorem 3.1.1. If $\Omega=(G: H)$, then $(G, \Omega)$ is not binary.

Proof. In this case $G(q)=E_{8}(q)$ with $p>5$ and $H \cap G(q)=\operatorname{PGL}_{2}(q) \times \operatorname{Sym}(5)$. Let $L$ be the factor $\mathrm{PGL}_{2}(q)$ and let $g$ be an element of order $q-1$ in $L$. A consideration of the centralizers of semisimple elements in $E_{8}(q)$ implies that there exists $x \in C_{G(q)}(g) \backslash H$. Note that $x \notin N_{G}(L)$ because the maximality
of $H \cap G(q)$ requires that $N_{G(q)}(L)=H \cap G(q)$. Then $H \cap H^{x}$ contains the element $g$ but does not contain the subgroup $L$, and now Lemma 2.2.1 implies that the action of $H$ on $\left(H: H \cap H^{x}\right)$ is not binary. The result follows by Lemma 1.6.1.

### 3.8.3 Type (VII)

Lemma 3.8.3. Assume $G$ is almost simple with socle $G(q)$, an exceptional group of Lie type over $\mathbb{F}_{q}$, and let $H$ be a maximal subgroup of $G$ as in part (VII) of Theorem 3.1.1. If $\Omega=(G: H)$, then $(G, \Omega)$ is not binary.

Proof. Here $H$ is an exotic local subgroup as listed in Table 3.1.2, When $H \cap G(q)=2^{3} . \mathrm{SL}_{3}(2)$, let $r=3$; and when $H \cap G(q) \in\left\{3^{3} . \mathrm{SL}_{3}(3), 3^{3+3} . \mathrm{SL}_{3}(3), 5^{3} . \mathrm{SL}_{3}(5), 2^{5+10} . \mathrm{SL}_{5}(2)\right\}$, let $r=2$. We have verified with magma that every non-trivial transitive action of $H$ of degree coprime to $r$ is not binary. In particular, if the action of $G$ on $(G: H)$ is binary, then every non-trivial suborbit of $G$ has cardinality divisible by $r$ and hence $r$ divides $|G: H|-1$. However in all cases $r$ divides $|G: H|$, and hence we reach a contradiction.

### 3.8.4 Type (VIII)

Lemma 3.8.4. Assume $G$ is almost simple with socle $G(q)$, an exceptional group of Lie type over $\mathbb{F}_{q}$, and let $H$ be a maximal subgroup of $G$ as in part (VIII) of Theorem 3.1.1. If $\Omega=(G: H)$, then $(G, \Omega)$ is not binary.

Proof. Here $H=(\operatorname{Alt}(5) \times \operatorname{Alt}(6)) \cdot 2^{2}<E_{8}(q)$, where the Klein 4-group acts faithfully on $F^{*}(H)=$ $\operatorname{Alt}(5) \times \operatorname{Alt}(6)$. There are several non-isomorphic groups having this shape, but a magma calculation confirms that if $H$ is any such group, and $M$ is a subgroup of $H$ of odd index, then either the action of $H$ on cosets of $M$ is not binary or $M$ contains the simple factor Alt(6) of $H$. Now let $x$ be any member of $G \backslash H$ that normalizes a Sylow 2-subgroup of $H$. If the action of $H$ on $\left(H: H \cap H^{x}\right)$ is binary, then by the previous sentence, $H \cap H^{x}$ contains $\operatorname{Alt}(6)$, and hence $x \in N_{G}(\operatorname{Alt}(6))=H$, a contradiction.

### 3.8.5 Type (IX)

Lemma 3.8.5. Assume $G$ is almost simple with socle $G(q)$, an exceptional group of Lie type over $\mathbb{F}_{q}$, and let $H$ be a maximal subgroup of $G$ as in part (IX) of Theorem 3.1.1. If $\Omega=(G: H)$, then $(G, \Omega)$ is not binary.

Proof. Here $F^{*}(H)$ is a simple group not in $\operatorname{Lie}(p)$, as listed in Tables 10.1-10.4 of [77]. Using also Theorem 3.1.2(i), we see that the possibilities for $F^{*}(H)$ are:
(1) $\operatorname{Alt}(6), \operatorname{Alt}(7)$;
(2) $M_{11}, M_{12}, M_{22}, J_{1}, J_{2}, J_{3}, R u, F i_{22}, H S, T h ;$
(3) $\mathrm{PSL}_{2}(r)$ for $r \leq 61$;
(4) $\mathrm{PSL}_{3}(3), \mathrm{PSL}_{3}(4), \mathrm{PSL}_{3}(5), \mathrm{PSL}_{4}(3), \mathrm{PSL}_{4}(5), \mathrm{PSU}_{3}(3), \mathrm{PSU}_{3}(8), \mathrm{PSU}_{4}(2), \mathrm{PSU}_{4}(3), \mathrm{PSp}_{4}(5)$, $\mathrm{Sp}_{6}(2), \Omega_{7}(3), \Omega_{8}^{+}(2), G_{2}(3),{ }^{3} D_{4}(2),{ }^{2} F_{4}(2)^{\prime},{ }^{2} B_{2}(8),{ }^{2} B_{2}(32)$.

Suppose first that $F^{*}(H)$ is not $\operatorname{Alt}(6), \operatorname{Alt}(7)$ or $\mathrm{PSL}_{2}(r)$, so that $H$ is as in (2) or (4). Observe that $|G: H|$ is even (see [73]), so there must be a non-trivial odd subdegree. However Lemmas 2.3.1 and 2.3.2 imply that if $M$ is any core-free subgroup of $H$ of odd index, then the action of $H$ on cosets of $M$ is not binary. Now Lemma 1.6.1 implies that $(G, \Omega)$ is not binary.

Suppose next that $F^{*}(H) \cong \operatorname{Alt}(7)$. Then $G(q)=E_{7}(q)$ or $E_{8}(q)$ by Theorem 3.1.2(i), and hence $|G|$ is divisible by $7^{2}$. Therefore there is an element $g \in G \backslash H$ such that $H \cap H^{g}$ has order divisible by 7 . However a magma computation shows that all faithful transitive actions of $H$ of degree coprime to 7 are not binary, completing the proof in this case.

Suppose next that $F^{*}(H) \cong \mathrm{PSL}_{2}(r)$ for some $r \leq 61$. If $r=4$ or 5 then $F^{*}(H) \cong \operatorname{Alt}(5)$, contrary to Theorem 3.1.2(i). Hence $r \geq 7$. Let $g$ be an element of $H$ of order $\frac{r-1}{(2, r-1)}$. Note that $g$ has order at most 31. We claim that $C_{G}(g) \not \leq H$ : for if $C_{G}(g) \leq H$, then $C_{G}(g)=C_{H}(g)$ is a cyclic maximal torus of $G(q)$ of order either $\frac{r-1}{(2, r-1)}$ or $r-1$ (the latter only if $H$ contains $\left.\mathrm{PGL}_{2}(r)\right)$. The orders of cyclic maximal tori are given in [53, Sec. 2]. Recalling that $G(q)$ is not as in Proposition 3.2.1, we see that the only possibility is that $G(q)={ }^{2} E_{6}(2)$ and $g$ has order 13,19 or 21 . However, $\mathrm{PSL}_{2}(r)$ is not a subgroup of ${ }^{2} E_{6}(2)$ for $r=27$, 39 or 43, as shown in [105, Sec. 12]. Thus $C_{G}(g) \not \leq H$, as claimed. Hence there exists $x \in C_{G}(g) \backslash H$, and so $H \cap H^{x}$ is a core-free subgroup of $H$ containing $g$. Now Lemma 2.2.3 implies that ( $H,\left(H: H \cap H^{x}\right)$ ) is not binary and the conclusion follows from Lemma 1.6.1.

### 3.8.6 Type (X)

Lemma 3.8.6. Assume $G$ is almost simple with socle $G(q)$, an exceptional group of Lie type over $\mathbb{F}_{q}$, and let $H$ be a maximal subgroup of $G$ as in part ( $X$ ) of Theorem 3.1.1. If $\Omega=(G: H)$, then $(G, \Omega)$ is not binary.

Proof. Here $F^{*}(H)$ is a simple group in Lie $(p)$. By Theorem 3.1.2(ii),(iii), the possibilities for $F^{*}(H)$ are
(1) $\operatorname{PSL}_{2}\left(q_{0}\right), q_{0} \leq t(G)$ and as in Theorem 3.1.2(iii);
(2) $\mathrm{PSL}_{3}(3), \mathrm{PSU}_{3}(3)\left(\right.$ with $\left.G(q)=E_{8}(q), q=3^{a}\right)$;
(3) $\mathrm{PSL}_{3}(4), \mathrm{PSU}_{3}(4), \mathrm{PSU}_{3}(8), \mathrm{PSU}_{4}(2),{ }^{2} B_{2}(8)\left(\right.$ with $\left.G(q)=E_{8}(q), q=2^{a}\right)$.

Suppose first that $F^{*}(H)$ is as in (2) or (3). Using similar magma computations to those described in the proof of Lemma 3.8.5, we verify that if $M$ is any core-free subgroup of $H$ of index coprime to $p$, then the action of $H$ on $(H: M)$ is not binary. Since $|G: H|$ is divisible by $p$, there exists $x \in G \backslash H$ normalizing a Sylow $p$-subgroup of $H$. Hence $H \cap H^{x}$ is a core-free subgroup of $H$ of index coprime to $p$, and so $\left(H,\left(H: H \cap H^{x}\right)\right)$ is not binary, completing the proof in cases (2) and (3).

Suppose finally that $F^{*}(H)$ is isomorphic to $\operatorname{PSL}_{2}\left(q_{0}\right)$ as in (1), and note that $q_{0} \neq 4,5$ by Theorem3.1.2(i). Let $g$ be an element of $H$ of order $\frac{q_{0}-1}{\left(2, q_{0}-1\right)}$. As in the last paragraph of the proof of Lemma 3.8.5, it is enough to show that $C_{G}(g) \notin H$. So assume that $C_{G}(g) \leq H$, in which case $C_{G}(g)=C_{H}(g)$ is a cyclic maximal torus of $G$ of order $\frac{q_{0}-1}{\left(2, q_{0}-1\right)}$ or $q_{0}-1$. Also, by Theorem 3.1.2(iii), if $G(q) \neq E_{8}(q)$, then either $q_{0}=q$ or $G(q)=E_{7}(q)$ and $q_{0}=7,8$ or 25 . The orders of cyclic maximal tori of $G(q)$ are given in [53, Sec. 2], and there are none of order $q_{0}-1$ or $\left(q_{0}-1\right) / 2$ with $q_{0}$ as in the previous sentence. Hence we may assume that $G(q)=E_{8}(q)$. Here the only possible cyclic maximal tori of order $q_{0}-1$ or ( $q_{0}-1$ )/2 (and also with $\left.q_{0} \leq t(G)\right)$ have $q=2$ and $q_{0}=2^{7}$ or $2^{8}$. However, $\mathrm{PSL}_{2}\left(2^{8}\right) \not \leq E_{8}(2)$, as $E_{8}(2)$ has no torus of order $2^{8}+1$. And if $F^{*}(H)=\mathrm{PSL}_{2}\left(2^{7}\right)$, then the element $g \in H$ of order $2^{7}-1$ lies in a subgroup $E_{7}(2)$ of $G$, and is centralized by an element of order 3 in $G \backslash H$, so $C_{G}(g) \not 又 H$ and the conclusion follows.

This completes the proof of Theorem 3.1.

## Chapter 4

## Classical Groups

In this chapter we prove Theorem 1.3 for classical groups:
Theorem 4.1. Let $G$ be an almost simple group with socle a classical group, and assume that $G$ has a primitive and binary action on $a$ set $\Omega$. Then $|\Omega| \in\{5,6,8\}$ and $G \cong \operatorname{Sym}(\Omega)$.

The examples with $|\Omega| \in\{5,6,8\}$ arise via the isomorphisms listed after the statement of Theorem 1.3.
The case where $G$ has socle isomorphic to $\mathrm{PSL}_{2}(q)$ or $\mathrm{PSU}_{3}(q)$ has been dealt with in [45], so Theorem 4.1 is already proved in this case.

### 4.1 Background on classical groups

Let us set up the group-theoretic notation that we need to prove Theorem 4.1. We assume throughout that our group $G$ is almost simple with socle a finite simple classical group. We write $M$ for the stabilizer in $G$ of a point in the action on $\Omega$. Since the action is primitive, $M$ is a maximal subgroup of $G$, and so we can use the classification of the maximal subgroups of the almost simple finite classical groups due to Aschbacher [1]. This classification divides the maximal subgroups into nine families, labelled $\mathcal{C}_{1}-\mathcal{C}_{8}$ and $\mathcal{S}$. We shall give rough descriptions of these families at the beginning of each section of this chapter; full details can be found in [54, Chapter 4], to which we will often refer. The case where $M$ is in family $\mathcal{C}_{1}$ has been handled in [46]. In this chapter we deal with the families $\mathcal{C}_{2}-\mathcal{C}_{8}$ and $\mathcal{S}$, in Sections 4.2- 4.9. Some almost simple groups with socle $\mathrm{P} \Omega_{8}^{+}(q)$ or $\mathrm{Sp}_{4}\left(2^{a}\right)$ have extra families of maximal subgroups, and these are handled in the last Section 4.10.

In what follows we shall take $S$ to be a certain quasisimple classical group for which $S / Z(S)$ is isomorphic to the socle of $G$ : namely, $S$ will be one of $\mathrm{SL}_{n}(q), \mathrm{Sp}_{n}(q), \mathrm{SU}_{n}(q), \Omega_{n}(q)$ (with $n q$ odd) or $\Omega_{n}^{\varepsilon}(q)$ (with $n$ even and $\varepsilon \in\{+,-\}$ ). As in [54], we denote these cases by L, S, U and O. Sometimes, for uniformity of notation, we shall allow ourselves to write $\Omega_{n}^{\varepsilon}(q)$ also in the case where $n$ is odd - in which case it just denotes $\Omega_{n}(q)$. Note that we can think of $S$ as acting on the set $\Omega$ - although we emphasise that this action is not necessarily primitive, and not necessarily faithful. We shall always write $\bar{S}$ for the simple group $S / Z(S)$.

The group $S$ is a subgroup of the group of isometries of some fixed bilinear, quadratic or sesquilinear form $\varphi$. We will write $V$ for the associated vector space of dimension $n$ over the field $\mathbb{K}$ where $\mathbb{K}=\mathbb{F}_{q^{u}}$ with $u=2$ in case U , and $u=1$ otherwise. The form $\varphi$ is either non-degenerate or the zero form (in the case $\left.S=\mathrm{SL}_{n}(q)\right)$.

When $\varphi$ is non-degenerate, we will make use of a hyperbolic basis $\mathcal{B}$ of $V$ of form

$$
\left\{e_{1}, \ldots, e_{k}, f_{1}, \ldots, f_{k}\right\} \cup \mathcal{A},
$$

where $k$ is the Witt index of $\varphi,\left\langle e_{i}, f_{i}\right\rangle$ are hyperbolic lines for $i=1, \ldots, k$ and either $\mathcal{A}$ is empty, or $S$ is orthogonal and $\mathcal{A}$ has size at most 2 and spans an anisotropic subspace of $V$.

### 4.1.1 Basic assumptions

We make use of isomorphisms between classical groups of small dimension, as well as known results on Cherlin's conjecture to make the following assumptions.

1. If $S=\mathrm{SL}_{n}(q)$, then $n \geq 3$ (using the main result of [45]).
2. If $S=\mathrm{SU}_{n}(q)$, then $n \geq 4$ (using the main result of (45).
3. If $S=\operatorname{Sp}_{n}(q)$, then $n \geq 4$.
4. If $S=\Omega_{n}(q)$ with $n$ odd, then $q$ is odd and $n \geq 7$.
5. If $S=\Omega_{n}^{\varepsilon}(q)$ with $n$ even and with $\varepsilon \in\{+,-\}$, then $n \geq 8$.

Notice that, under these assumptions, $S$ is quasisimple, unless $S=\operatorname{Sp}_{4}(2)$, in which case $S \cong \operatorname{Sym}(6)$.
In addition, by [46, we can assume that $M$ does not lie in Aschbacher's family $\mathcal{C}_{1}$. We also use magma to exclude some small cases:

Lemma 4.1.1. Let $G$ be an almost simple primitive group with socle one of the following groups

1. $\operatorname{PSL}_{3}(q)$ with $q \leq 25, \operatorname{PSL}_{4}(q)$ with $2<q \leq 9$ or $q \in\{16,25\}, \operatorname{PSL}_{5}(q)$ with $q \leq 7, \operatorname{PSL}_{6}(q)$ with $q \leq 4, \operatorname{PSL}_{7}(3), \operatorname{PSL}_{8}(q)$ with $q \leq 3$;
2. $\operatorname{PSU}_{4}(q)$ with $q \leq 7, \operatorname{PSU}_{5}(q)$ with $q \leq 5, \operatorname{PSU}_{6}(q)$ with $q \leq 3, \operatorname{PSU}_{7}(q)$ with $q \leq 3, \operatorname{PSU}_{8}(2)$;
3. $\mathrm{PSp}_{4}(q)$ with $q \in\{4,5,8,16\}, \mathrm{PSp}_{6}(q)$ with $q \leq 5, \mathrm{PSp}_{8}(q)$ with $q \leq 3$;
4. $\mathrm{P} \Omega_{7}(3), \mathrm{P} \Omega_{8}^{-}(2), \mathrm{P} \Omega_{8}^{+}(2), \mathrm{P} \Omega_{8}^{+}(3), \mathrm{P} \Omega_{8}^{+}(4), \mathrm{P} \Omega_{9}(5), \mathrm{P} \Omega_{10}^{-}(2), \mathrm{P} \Omega_{12}^{+}(2)$.

## Then the action of $G$ is not binary.

Proof. The magma computations here are all rather similar. We give an indication of what we have done in the unitary case only.

We have computed all the possible almost simple groups $G$ and all of their (faithful) primitive actions on a set $\Omega$. We have tested that each of these actions is not binary. Indeed, except when $S=\operatorname{SU}_{4}(2)$ and $|\Omega|=27$, or $S=\mathrm{SU}_{4}(3)$ and $|\Omega|=112$, or $S=\mathrm{SU}_{4}(4)$ and $|\Omega|=325$, we can witness that $G$ is non binary by applying Lemmas 1.6.15, 1.6.16, 1.8.1, 1.8.4, or by finding a suitable non-binary triple. When $S=\mathrm{SU}_{4}(3)$ and $|\Omega|=112$, or $S=\mathrm{SU}_{4}(4)$ and $|\Omega|=325$, we can witness that $G$ is not binary by finding a suitable non-binary 4 -tuple. The case $S=\mathrm{SU}_{4}(2)$ and $|\Omega|=27$ requires a little more care because triples and 4 -tuples are not enough to witness that $G$ is not binary. We have proved that this group is non-binary using longer tuples (of length 7).

Finally, from here on, except for the final two sections ( $\$ 4.9$ and $\S 4.10$ ), we will assume that

- if $S=\operatorname{Sp}_{4}\left(2^{a}\right)$, then $G \leq \Gamma \operatorname{Sp}_{4}\left(2^{a}\right)$ (and so does not contain a graph automorphism); and
- if $S=\Omega_{8}^{+}(q)$, then $G \leq \mathrm{P}_{8}^{+}(q)$ (and so does not contain a triality automorphism).

These assumptions ensure that if $V$ denotes the natural $n$-dimensional module for $S$, then $G \leq \mathrm{P} \Gamma \mathrm{L}(V)$, except for the case where $S=\mathrm{SL}(V)$, in which case $G \leq \mathrm{P} \Gamma \mathrm{L}(V) .2$ (where the .2 denotes a graph automorphism).

Note also, for future reference, that by Proposition 2.5.1, the maximal subgroups of $G$ that centralize field, graph-field or graph automorphisms are in families $\mathcal{C}_{5}$ (subfield subgroups) and $\mathcal{C}_{8}$ (classical subgroups).

### 4.2 Family $\mathcal{C}_{2}$

In this case $M$ is the projective image of the stabilizer of a direct sum decomposition $D$ of $V$ into $t$ subspaces $W_{1}, \ldots, W_{t}$, each of dimension $m$, as described in [54, §4.2]. In particular, $n=m t$. The possibilities are summarized in Table 4.2.1.

| case | type | conditions |
| :---: | :---: | :---: |
| L | $\mathrm{GL}_{m}(q) \mathrm{wr} \operatorname{Sym}(t)$ |  |
| U | $\mathrm{GU}_{m}(q) \mathrm{wr} \operatorname{Sym}(t)$ | $W_{i}$ non-degenerate |
| S | $\operatorname{Sp}_{m}(q) \mathrm{wr} \operatorname{Sym}(t)$ | $W_{i}$ non-degenerate |
| O | $\mathrm{O}_{m}^{\delta}(q) \mathrm{wr} \operatorname{Sym}(t)$ | $W_{i}$ non-degenerate |
| $\mathrm{U}, \mathrm{S}, \mathrm{O}^{+}$ | $\mathrm{GL}_{n / 2}\left(q^{u}\right) .2$ | $W_{i}$ totally singular, |
|  |  | $q$ odd in case S |

Table 4.2.1: Maximal subgroups in family $\mathcal{C}_{2}$
The main result of this section is the following. The result will be proved in a series of lemmas.
Proposition 4.2.1. Suppose that $G$ is an almost simple group with socle $\bar{S}=\mathrm{Cl}_{n}(q)$, and assume that
(i) $n \geq 3,4,4,7$ in cases $L, U, S, O$ respectively, and
(ii) $\mathrm{Cl}_{n}(q)$ is not one of the groups listed in Lemma 4.1.1.

Let $M$ be a maximal subgroup of $G$ in the family $\mathcal{C}_{2}$. Then the action of $G$ on $(G: M)$ is not binary.

### 4.2.1 Case $S=\mathrm{SL}_{n}(q)$

Assume that $S=\operatorname{SL}_{n}(q)$ with $n \geq 3$, and the socle of $G$ is not as in Lemma 4.1.1(1). Assume also that $\Omega=(G: M)$, where $M$ is in the family $\mathcal{C}_{2}$ (as in the first row of Table 4.2.1).

Lemma 4.2.2. In this case, $\Omega$ contains a beautiful subset.
Proof. There is a basis $\mathcal{B}=\left\{v_{1}, \ldots, v_{m t}\right\}$ of $V$ such that $M$ stabilizes the deomposition $V=W_{1} \oplus \cdots \oplus W_{t}$, where

$$
W_{i}=\left\langle v_{m(i-1)+1}, v_{m(i-1)+2}, \ldots, v_{m i}\right\rangle .
$$

First, assume that $q \geq 5$. We let $U$ be the subgroup whose elements fix all elements of $\mathcal{B}$ except $v_{1}$ and satisfy

$$
v_{1} \mapsto v_{1}+k_{1} v_{m+1},
$$

for some $k_{1} \in \mathbb{F}_{q}$, and we define $\Lambda=D^{U}$. For $k \in \mathbb{F}_{q}$, we define

$$
W_{1}(k)=\left\langle v_{1}+k v_{m+1}, v_{2}, \ldots, v_{m}\right\rangle,
$$

and observe that $\Lambda=\left\{D(k) \mid k \in \mathbb{F}_{q}\right\}$, where

$$
D(k)=W_{1}(k) \oplus W_{2} \oplus \cdots \oplus W_{t} .
$$

Note, in particular, that $D(0)=D$.
Let $T$ be the maximal split torus whose elements are diagonal when written with respect to $\mathcal{B}$. Then $U \rtimes T$ is 2-transitive on $\Lambda=D^{U}$.

Now suppose that $g \in S_{\Lambda}$ and suppose that $g$ maps $W_{i}$ to $W_{1}(k)$ for some $i>1$ and some $k \in \mathbb{F}_{q}$. This implies that there exists $v \in W_{i}$ such that $v^{g}=v_{1}+k v_{m+1}$. But $v_{1}+k v_{m+1}$ lies in $W_{1}(k)$ and not in
$W_{1}(\ell)$ for all $\ell \neq k$ and, similarly, $v_{1}+k v_{m+1}$ does not lie in $W_{j}$ for all $j>1$. Thus $g$ does not preserve $\Lambda$, a contradiction. We conclude that $g$ preserves $\left\{W_{1}(k) \mid k \in \mathbb{F}_{q}\right\}$ set-wise, and preserves $\left\{W_{2}, \ldots, W_{t}\right\}$ set-wise.

Our aim now is to show that $\Lambda$ is an $S$-beautiful subset; to do this, we will show that $S^{\Lambda} \cong U \rtimes T$. For this, we suppose that $g \in S_{\Lambda}$ fixes both $D(0)$ and $D(1)$ and we will show that $g \in S_{(\Lambda)}$. Observe first that, since $g$ fixes $D(0)$, it follows that $g$ fixes $W_{1}$ and so

$$
v_{1}^{g} \in\left\langle v_{1}, v_{2}, \ldots, v_{m}\right\rangle
$$

Similarly, since $g$ fixes $D(1)$, we conclude that $g$ fixes $W_{1}(1)$ and so

$$
\left(v_{1}+v_{m+1}\right)^{g} \in\left\langle v_{1}+v_{m+1}, v_{2}, \ldots, v_{m}\right\rangle .
$$

Hence $v_{m+1}^{g} \in\left\langle v_{1}, v_{2}, \ldots, v_{m}, v_{m+1}\right\rangle$.
On the other hand $g$ preserves $\left\{W_{2}, \ldots, W_{t}\right\}$ set-wise, and so

$$
v_{m+1}^{g} \in\left\langle v_{m+1}, \ldots, v_{m t}\right\rangle
$$

which implies that $v_{m+1}^{g} \in\left\langle v_{m+1}\right\rangle$. In other words, for some $\ell_{1} \in \mathbb{F}_{q}$, we have

$$
\begin{equation*}
v_{m+1}^{g}=\ell_{1} v_{m+1} . \tag{4.2.1}
\end{equation*}
$$

Now, since $g$ fixes $W_{1}$, we conclude that there exist $\ell_{2}, c_{2}, \ldots, c_{m}$ such that

$$
\begin{equation*}
v_{1}^{g}=\ell_{2} v_{1}+\sum_{i=2}^{m} c_{i} v_{i} \tag{4.2.2}
\end{equation*}
$$

Finally, since $g$ fixes $W_{1}(1)$, there exist $\ell_{3}, d_{2}, \ldots, d_{m}$ such that

$$
\begin{equation*}
\left(v_{1}+v_{m+1}\right)^{g}=\ell_{3}\left(v_{1}+v_{m+1}\right)+\sum_{i=2}^{m} d_{i} v_{i} \tag{4.2.3}
\end{equation*}
$$

From, (4.2.1), (4.2.2) and (4.2.3), we conclude that $c_{i}=d_{i}$ for all $i=2, \ldots, m$ and that $\ell_{1}=\ell_{2}=\ell_{3}=\ell$.
We finally obtain that, for each $k \in \mathbb{F}_{q}$,

$$
\begin{aligned}
\left(v_{1}+k v_{m+1}\right)^{g} & =\left(v_{1}+v_{m+1}\right)^{g}+(k-1) v_{m+1}^{g} \\
& =\ell\left(v_{1}+v_{m+1}\right)+\sum_{i=2}^{m} c_{i} v_{i}+(k-1) \ell v_{m+1} \\
& =\ell\left(v_{1}+k v_{m+1}\right)+\sum_{i=2}^{m} c_{i} v_{i} \in W_{1}(k)
\end{aligned}
$$

Thus $g$ fixes $W_{1}(k)$ for each $k \in \mathbb{F}_{q}$, and so $g$ fixes $D(k)$ for each $k \in \mathbb{F}_{q}$. We conclude that $g \in S_{(\Lambda)}$, as required.

Next, assume that $q \in\{3,4\}$; then [54, Tables 3.5.A and 3.5.H] allows us to assume that $m \geq 2$. We let $U$ be the subgroup whose elements fix all elements of $\mathcal{B}$ except $v_{1}$ and satisfy

$$
v_{1} \mapsto v_{1}+k_{1} v_{m+1}+k_{2} v_{m+2},
$$

for some $k_{1}, k_{2} \in \mathbb{F}_{q}$, and we define $\Lambda=D^{U}$. For $k_{1}, k_{2} \in \mathbb{F}_{q}$, we define

$$
W_{1}\left(k_{1}, k_{2}\right)=\left\langle v_{1}+k_{1} v_{m+1}+k_{2} v_{m+2}, v_{2}, \ldots, v_{m}\right\rangle,
$$

and observe that $\Lambda=\left\{D\left(k_{1}, k_{2}\right) \mid k_{1}, k_{2} \in \mathbb{F}_{q}\right\}$, where

$$
D\left(k_{1}, k_{2}\right)=W_{1}\left(k_{1}, k_{2}\right) \oplus W_{2} \oplus \cdots \oplus W_{t} .
$$

Note, in particular, that $D(0,0)=D$. Let $X$ be the stabilizer of the subspaces

$$
\left\langle v_{1}\right\rangle, \ldots,\left\langle v_{m}\right\rangle,\left\langle v_{m+1}, v_{m+2}\right\rangle,\left\langle v_{m+3}\right\rangle, \ldots,\left\langle v_{m t}\right\rangle .
$$

Then $U \rtimes X$ is 2-transitive on $\Lambda=D^{U}$. Our aim now is to show that $\Lambda$ is a beautiful subset.
Take $g \in S_{\Lambda}$ and suppose that $\Lambda$ is not beautiful. An analogous argument to the previous case allows us to conclude that $g$ preserves $\left\{W_{1}\left(k_{1}, k_{2}\right) \mid k_{1}, k_{2} \in \mathbb{F}_{q}\right\}$ set-wise, and preserves $\left\{W_{2}, \ldots, W_{t}\right\}$ set-wise. This implies, moreover, that $g$ preserves the subspaces

$$
Y_{1}:=\operatorname{span}_{\mathbb{F}_{q}}\left\{W_{1}\left(k_{1}, k_{2}\right) \mid k_{1}, k_{2} \in \mathbb{F}_{q}\right\} \text { and } Y_{0}:=\bigcap_{k_{1}, k_{2} \in \mathbb{F}_{q}} W_{1}\left(k_{1}, k_{2}\right) .
$$

Thus there is a homomorphism $\theta: S_{\Lambda} \rightarrow \mathrm{GL}\left(Y_{1} / Y_{0}\right) \cong \mathrm{GL}_{3}(q)$. Since $\mathrm{GL}_{3}(q)$ does not contain a subgroup with a composition factor isomorphic to $\operatorname{Alt}(s)$ for $s \geq 8$, we conclude that the action of $\operatorname{ker}(\theta)$ on $\Lambda$ must induce $\operatorname{Alt}(\Lambda)$ or $\operatorname{Sym}(\Lambda)$. However $\operatorname{ker}(\theta)$ is not transitive on $\Lambda$ so we have a contradiction.

Next, assume that $q=2$; then [54, Tables 3.5.A and 3.5.H] allows us to assume that $m \geq 3$. We let $U$ be the subgroup whose elements fix all elements of $\mathcal{B}$ except $v_{1}$ and satisfy

$$
v_{1} \mapsto v_{1}+k_{1} v_{m+1}+k_{2} v_{m+2}+k_{3} v_{m+3},
$$

for some $k_{1}, k_{2}, k_{3} \in \mathbb{F}_{q}$, and we define $\Lambda=D^{U}$. For $k_{1}, k_{2}, k_{3} \in \mathbb{F}_{q}$, we define

$$
W_{1}\left(k_{1}, k_{2}, k_{3}\right)=\left\langle v_{1}+k_{1} v_{m+1}+k_{2} v_{m+2}+k_{3} v_{m+3}, v_{2}, \ldots, v_{m}\right\rangle,
$$

and observe that $\Lambda=\left\{D\left(k_{1}, k_{2}, k_{3}\right) \mid k \in \mathbb{F}_{q}\right\}$, where

$$
D\left(k_{1}, k_{2}, k_{3}\right)=W_{1}\left(k_{1}, k_{2}, k_{3}\right) \oplus W_{2} \oplus \cdots \oplus W_{t} .
$$

Note, in particular, that $D(0,0,0)=D$. Let $X$ be the stabilizer of the subspaces

$$
\left\langle v_{1}\right\rangle, \ldots,\left\langle v_{m}\right\rangle,\left\langle v_{m+1}, v_{m+2}, v_{m+3}\right\rangle,\left\langle v_{m+4}\right\rangle, \ldots,\left\langle v_{m t}\right\rangle .
$$

Then $U \rtimes X$ is 2-transitive on $\Lambda=D^{U}$. Suppose that $g \in S_{\Lambda}$ fixes both $D(0,0,0)$ and $D(1,0,0)$. Observe first that, since $g$ fixes $D(0,0,0)$, it follows that $g$ fixes $W_{1}$ and so

$$
v_{1}^{g} \in\left\langle v_{1}, v_{2}, \ldots, v_{m}\right\rangle .
$$

Similarly, since $g$ fixes $D(1,0,0)$, it also fixes $W_{1}(1,0,0)$ and so

$$
\left(v_{1}+v_{m+1}\right)^{g} \in\left\langle v_{1}+v_{m+1}, v_{2}, \ldots, v_{m}\right\rangle .
$$

We conclude that $v_{m+1}^{g} \in\left\langle v_{1}, v_{2}, \ldots, v_{m}, v_{m+1}\right\rangle$.
On the other hand $g$ preserves $\left\{W_{2}, \ldots, W_{t}\right\}$ set-wise, and so

$$
v_{m+1}^{g} \in\left\langle v_{m+1}, \ldots, v_{m t}\right\rangle
$$

which implies that $v_{m+1}^{g} \in\left\langle v_{m+1}\right\rangle$. Since we are working over $\mathbb{F}_{2}$, we conclude that $v_{m+1}^{g}=v_{m+1}$. Now, we can repeat this same argument assuming that $g$ also fixes $D(0,1,0)$ and $D(0,0,1)$ and we see that $v_{m+2}^{g}=v_{m+2}$ and $v_{m+3}^{g}=v_{m+3}$. But in this case $g$ clearly fixes $W_{1}\left(k_{1}, k_{2}, k_{3}\right)$ for all $k_{1}, k_{2}, k_{3} \in \mathbb{F}_{2}$, and so $g$ fixes $D\left(k_{1}, k_{2}, k_{3}\right)$ for all $k_{1}, k_{2}, k_{3} \in \mathbb{F}_{2}$. Thus if $g \in S_{\Lambda}$ and fixes the four points $D(0,0,0), D(1,0,0)$, $D(0,1,0)$ and $D(0,0,1)$ of $\Lambda$, then $g$ fixes all of $\Lambda$. Since $|\Lambda|=8$, we conclude that $S^{\Lambda}$ does not contain $\operatorname{Alt}(\Lambda)$, and so $\Lambda$ is a beautiful subset.

### 4.2.2 The totally singular case

In this section we deal with the case when $S$ preserves a non-degenerate form on $V, t=2$ and $W_{1}$ and $W_{2}$ are both totally singular. This occurs when $S$ is unitary, symplectic with $q$ odd, or of type $\mathrm{O}^{+}$(as in the last row of Table 4.2.1). So assume that $S$ is one of these types, and also that the socle of $G$ is not as in Lemma 4.1.1.

Lemma 4.2.3. In this case $\Omega$ contains a beautiful subset.
Proof. We can assume that $W_{1}=\left\langle e_{1}, \ldots, e_{m}\right\rangle$ and $W_{2}=\left\langle f_{1}, \ldots, f_{m}\right\rangle$, where $\mathcal{B}=\left\{e_{1}, \ldots, e_{m}, f_{1}, \ldots, f_{m}\right\}$ is a hyperbolic basis for $V$ (and $m \geq 2$ ). Note that, with respect to the basis $\mathcal{B}, M$ contains the group of matrices

$$
\left\{\left.\left(\begin{array}{cc}
A &  \tag{4.2.4}\\
& A^{-T}
\end{array}\right) \right\rvert\, A \in \mathrm{GL}_{m}\left(\mathbb{F}_{q}\right)\right\}
$$

except in the $\mathrm{O}^{+}$case with $q$ odd, when we need to add the requirement that $\operatorname{det}(A)$ is a square in $\mathbb{F}_{q}$ for such matrices. Notice that in the unitary case we have written $A^{-T}$ rather than $A^{-T \sigma}$ (where $\sigma$ is the involutory automorphism of the field $\mathbb{F}_{q^{2}}$ ) since we are only considering matrices with entries from the field $\mathbb{F}_{q}$.

First, assume that $q \geq 5$. We let $U$ be the subgroup whose elements fix all elements of $\mathcal{B}$ except $e_{1}$ and $e_{2}$ and satisfy

$$
e_{1} \mapsto e_{1}+k f_{2} \text { and } e_{2} \mapsto e_{2} \pm k f_{1}
$$

for some $k \in \mathbb{F}_{q}$. (The choice of sign for the image of $e_{2}$ will depend on the type of form preserved by $S$.) We define $\Lambda=D^{U}$ and, for $k \in \mathbb{F}_{q}$, we define

$$
W_{1}(k)=\left\langle e_{1}+k f_{2}, e_{2} \pm k f_{1}, e_{3}, \ldots, e_{m}\right\rangle .
$$

Observe that $\Lambda=\left\{D(k) \mid k \in \mathbb{F}_{q}\right\}$, where

$$
D(k)=W_{1}(k) \oplus W_{2} .
$$

Note, in particular, that $D(0)=D$.
Let $T$ be the maximal split torus whose elements are diagonal when written with respect to $\mathcal{B}$. Then $U \rtimes T$ is 2-transitive on $\Lambda=D^{U}$. Our aim now is to show that $\Lambda$ is an $S$-beautiful subset; to do this, we will show that $S^{\Lambda} \cong U \rtimes T$. For this, we suppose that $g \in S_{\Lambda}$ fixes both $D(0)$ and $D(1)$ and we will show that $g \in S_{(\Lambda)}$.

Observe that, since $g$ fixes $D(0)$ and $D(1), g$ must fix $W_{2}$ and hence must fix $W_{1}$ and $W_{1}(1)$. The fact that $g$ fixes $W_{1}$ implies that

$$
e_{1}^{g} \in\left\langle e_{1}, e_{2}, \ldots, e_{m}\right\rangle
$$

Similarly, the fact that $g$ fixes $W_{1}(1)$ implies that

$$
\left(e_{1}+f_{2}\right)^{g} \in\left\langle e_{1}+f_{2}, e_{2} \pm f_{1}, e_{3}, \ldots, e_{m}\right\rangle
$$

We conclude that $f_{2}^{g} \in\left\langle e_{1}, e_{2}, \ldots, e_{m}, f_{1}, f_{2}\right\rangle$.
On the other hand $g$ also fixes $W_{2}$, and so $f_{2}^{g} \in W_{2}$ and we conclude that $f_{2}^{g} \in\left\langle f_{1}, f_{2}\right\rangle$; in other words, there exist $\ell_{1}, \ell_{1}^{\prime} \in \mathbb{F}_{q}$ such that

$$
\begin{equation*}
f_{2}^{g}=\ell_{1}^{\prime} f_{1} \pm \ell_{1} f_{2} \tag{4.2.5}
\end{equation*}
$$

Now, since $g$ fixes $W_{1}$, we conclude that there exist $\ell_{2}, \ell_{2}^{\prime}, c_{3}, \ldots, c_{m}$ such that

$$
\begin{equation*}
e_{1}^{g}=\ell_{2} e_{1}+\ell_{2}^{\prime} e_{2}+\sum_{i=3}^{m} c_{i} e_{i} . \tag{4.2.6}
\end{equation*}
$$

Finally, since $g$ fixes $W_{1}(1)$, we conclude that there exist $\ell_{3}, \ell_{3}^{\prime}, d_{3}, \ldots, d_{m}$ such that

$$
\begin{equation*}
\left(e_{1}+f_{2}\right)^{g}=\ell_{3}\left(e_{1}+f_{2}\right)+\ell_{3}^{\prime}\left(e_{2} \pm f_{1}\right)+\sum_{i=3}^{m} d_{i} e_{i} \tag{4.2.7}
\end{equation*}
$$

From (4.2.5), (4.2.6) and (4.2.7), we conclude that $c_{i}=d_{i}$ for all $i=3, \ldots, m, \ell_{1}=\ell_{2}=\ell_{3}$ and $\ell_{1}^{\prime}=\ell_{2}^{\prime}=\ell_{3}^{\prime}$. Set $\ell:=\ell_{1}$ and $\ell^{\prime}:=\ell_{1}^{\prime}$.

We finally obtain that, for each $k \in \mathbb{F}_{q}$,

$$
\begin{aligned}
\left(e_{1}+k f_{2}\right)^{g} & =\left(e_{1}+f_{2}\right)^{g}+(k-1) f_{2}^{g} \\
& =\ell\left(e_{1}+f_{2}\right)+\ell^{\prime}\left(e_{2} \pm f_{1}\right)+\sum_{i=3}^{m} c_{i} e_{i}+(k-1)\left( \pm \ell^{\prime} f_{1}+\ell f_{2}\right) \\
& =\ell\left(e_{1}+k f_{2}\right)+\ell^{\prime}\left(e_{2} \pm k f_{1}\right)+\sum_{i=3}^{m} c_{i} e_{i} \in W_{1}(k) .
\end{aligned}
$$

Thus $g$ fixes $W_{1}(k)$ for each $k \in \mathbb{F}_{q}$, and so $g$ fixes $D(k)$ for each $k \in \mathbb{F}_{q}$. We conclude that $g \in S_{(\Lambda)}$, as required.

Next assume that $q \in\{3,4\}$. Since $\mathrm{SU}_{4}(3), \mathrm{SU}_{4}(4)$ and $\mathrm{Sp}_{4}(3)$ were dealt with at the start (recall that in the symplectic case, $q$ is odd), we require $n \geq 6$. Our argument is similar to before. We let $U$ be the subgroup whose elements fix all elements of $\mathcal{B}$ except $e_{1}, e_{2}$ and $e_{3}$ and satisfy

$$
e_{1} \mapsto e_{1}+k_{2} f_{2}+k_{3} f_{3}
$$

for some $k_{2}, k_{3} \in \mathbb{F}_{q}$, and we define $\Lambda=D^{U}$. For $k_{2}, k_{3} \in \mathbb{F}_{q}$, we define

$$
W_{1}\left(k_{2}, k_{3}\right)=\left\langle e_{1}+k_{2} f_{2}+k_{3} f_{3}, e_{2}, \ldots, e_{m}\right\rangle
$$

and observe that $\Lambda=\left\{D\left(k_{2}, k_{3}\right) \mid k_{2}, k_{3} \in \mathbb{F}_{q}\right\}$ where

$$
D\left(k_{2}, k_{3}\right)=W_{1}\left(k_{2}, k_{3}\right) \oplus W_{2} .
$$

Note, in particular, that $D(0,0)=D$. Let $X$ be the stabilizer of the subspaces

$$
\left\langle e_{1}\right\rangle,\left\langle e_{2}, e_{3}\right\rangle,\left\langle e_{4}\right\rangle, \ldots,\left\langle e_{m}\right\rangle,\left\langle f_{1}\right\rangle,\left\langle f_{2}, f_{3}\right\rangle,\left\langle f_{4}\right\rangle, \ldots,\left\langle f_{m}\right\rangle .
$$

Then $U \rtimes X$ is 2-transitive on $\Lambda=D^{U}$ (making use of Lemma 1.6 .9 and the fact that $M$ contains the matrices in (4.2.4)). Our aim now is to show that $\Lambda$ is a beautiful subset of size $q^{2}$.

Take $g \in S_{\Lambda}$ and suppose that $\Lambda$ is not beautiful. An analogous argument to the previous case allows us to conclude that $g$ preserves $\left\{W_{1}\left(k_{2}, k_{3}\right) \mid k_{2}, k_{3} \in \mathbb{F}_{q}\right\}$ set-wise. This implies that $g$ preserves the subspaces

$$
Y_{1}:=\operatorname{span}_{\mathbb{K}}\left\{W_{1}\left(k_{2}, k_{3}\right) \mid k_{2}, k_{3} \in \mathbb{F}_{q}\right\} \text { and } Y_{0}:=\bigcap_{k_{2}, k_{3} \in \mathbb{F}_{q}} W_{1}\left(k_{2}, k_{3}\right) .
$$

Thus there is a homomorphism $\theta: S_{\Lambda} \rightarrow \mathrm{GL}\left(Y_{1} / Y_{0}\right) \cong \mathrm{GL}_{3}(\mathbb{K})$. Since $\mathrm{GL}_{3}(\mathbb{K})$ does not contain a subgroup with a composition factor isomorphic to $\operatorname{Alt}(s)$ for $s>7$, we conclude that the action of $\operatorname{ker}(\theta)$ on $\Lambda$ must induce $\operatorname{Alt}(\Lambda)$ or $\operatorname{Sym}(\Lambda)$. However $\operatorname{ker}(\theta)$ is not transitive on $\Lambda$ so we have a contradiction.

Finally, assume that $q=2$. In this case, we require that $n \geq 8$. This requirement excludes the groups $\mathrm{SU}_{4}(2)$ and $\mathrm{SU}_{6}(2)$ which were dealt with at the start. In addition Lemma 4.1.1 allows us to exclude the case when $S=\Omega_{8}^{+}(2)$ (notice that, in the proof, we prove that there is a beautiful subset). If $n \geq 10$, then our method here is similar to the previous case, but we start with a subgroup $U$ whose elements fix all elements of $\mathcal{B}$ except $e_{1}, e_{2}, e_{3}, e_{4}$ and $e_{5}$ and satisfy

$$
e_{1} \mapsto e_{1}+k_{2} f_{2}+k_{3} f_{3}+k_{4} f_{4}+k_{5} f_{5},
$$

for some $k_{2}, k_{3}, k_{4}, k_{5} \in \mathbb{F}_{q}$. We obtain a beautiful subset of cardinality 16 ; we leave the details to the reader. For the case when $S=\mathrm{SU}_{8}(2)$, we use a subgroup $U$ whose elements fix all elements of $\mathcal{B}$ except $e_{1}, e_{2}$ and $e_{3}$ and satisfy

$$
e_{1} \mapsto e_{1}+k_{2} f_{2}+k_{3} f_{3},
$$

for some $k_{2}, k_{3} \in \mathbb{F}_{4}$. Again we obtain a beautiful subset of cardinality 16 .

### 4.2.3 A general reduction

In light of the previous subsections, we can now assume that we are in the case when $S$ preserves a nondegenerate form on $V$ and $W_{1}, \ldots, W_{t}$ are all non-degenerate subspaces of $V$ of dimension $m$. In the end we will need to split into separate cases, depending on the type of the form, but before we do that we give three general lemmas that significantly reduce the subsequent case work.

Lemma 4.2.4. If $q \geq 5$, then either $\Omega$ contains a beautiful subset or else one of the following holds:

1. $m=1$;
2. $m=2, S=\Omega_{n}^{\varepsilon}(q)$ for some $\varepsilon \in\{+,-\}$, and $W_{1}, \ldots, W_{t}$ are all of type $\mathrm{O}_{2}^{-}$.

Proof. Suppose that neither of the listed outcomes occurs - we must show that $\Omega$ contains a beautiful subset. Choose a hyperbolic basis for each of $W_{1}, \ldots, W_{t}$ and let $\mathcal{B}$ be the union of these bases.

Since we have excluded the two listed outcomes, we can let ( $e_{1}, f_{1}$ ) (resp. $\left(e_{2}, f_{2}\right)$ ) be hyperbolic pairs whose elements are in $\mathcal{B} \cap W_{1}$ (resp. $\mathcal{B} \cap W_{2}$ ). Let $U$ be the subgroup whose elements fix all elements of $\mathcal{B}$ except $e_{1}$ and $f_{2}$ and satisfy

$$
e_{1} \mapsto e_{1}+k e_{2}, \quad f_{2} \mapsto f_{2} \pm k f_{1},
$$

for some $k \in \mathbb{F}_{q}$, and we let $\Lambda=M^{U}$. (The choice of sign for the image of $f_{2}$ will depend on the type of form preserved by $S$.) For $k \in \mathbb{F}_{q}$, we define

$$
\begin{aligned}
& W_{1}(k)=\left\langle e_{1}+k e_{2}, f_{1}, x_{1} \ldots, x_{m-2}\right\rangle, \\
& W_{2}(k)=\left\langle e_{2}, f_{2} \pm k f_{1}, y_{1}, \ldots, y_{m-2}\right\rangle,
\end{aligned}
$$

where $\mathcal{B} \cap W_{1}=\left\{e_{1}, f_{1}, x_{1}, \ldots, x_{m-2}\right\}$ and $\mathcal{B} \cap W_{2}=\left\{e_{2}, f_{2}, y_{1}, \ldots, y_{m-2}\right\}$. Observe that $\Lambda=\{D(k) \mid k \in$ $\left.\mathbb{F}_{q}\right\}$, where

$$
D(k)=W_{1}(k) \oplus W_{2}(k) \oplus W_{3} \oplus \cdots \oplus W_{t} .
$$

Note, in particular, that $D(0)=D$. Now we follow the argument of Lemma 4.2.3 with some slight adjustments. As before, we are able to conclude that, if $g \in S_{\Lambda}$, then $g$ preserves the set $\left\{W_{3}, \ldots, W_{t}\right\}$ as well as the set

$$
\left\{W_{1}(k) \mid k \in \mathbb{F}_{q}\right\} \cup\left\{W_{2}(k) \mid k \in \mathbb{F}_{q}\right\} .
$$

Now suppose that $g \in S_{\Lambda}$ and $g$ fixes both $D(0)$ and $D(1)$. We study the image of the spaces $W_{1}, W_{2}, W_{1}(1)$ and $W_{2}(1)$.

Suppose that $W_{1}^{g}=W_{1}$ and $W_{1}(1)^{g}=W_{2}(1)$. This implies that $f_{1}^{g} \in W_{1} \cap W_{2}(1)=\{0\}$, a contradiction. Similarly, we cannot have $W_{1}^{g}=W_{2}$ and $W_{1}(1)^{g}=W_{1}(1)$.

Suppose next that $W_{1}^{g}=W_{1}$ and $W_{1}(1)^{g}=W_{1}(1)$. Then there exist $a, b, c_{1}, \ldots, c_{m-2}$ such that

$$
e_{1}^{g}=a e_{1}+b f_{1}+\sum_{i=1}^{m-2} c_{i} x_{i} .
$$

Similarly there exist $a^{\prime}, b^{\prime}, c_{1}^{\prime}, \ldots, c_{m-2}^{\prime}$ such that

$$
\left(e_{1}+e_{2}\right)^{g}=a^{\prime}\left(e_{1}+e_{2}\right)+b^{\prime} f_{1}+\sum_{i=1}^{m-2} c_{i}^{\prime} x_{i}
$$

We obtain that

$$
e_{2}^{g}=\left(a^{\prime}-a\right) e_{1}+a^{\prime} e_{2}+\left(b^{\prime}-b\right) f_{1}+\sum_{i=1}^{m-2}\left(c_{i}^{\prime}-c_{i}\right) x_{i} .
$$

But, since $e_{2} \in W_{2} \cap W_{2}(1)$ and since $g$ fixes $D(0)$ and $D(1)$, we deduce $e_{2}^{g} \in W_{2} \cap W_{2}(1)=\operatorname{span}_{\mathbb{K}}\left\{e_{2}, y_{1}, \ldots, y_{m-2}\right\}$. Now, we conclude $e_{2}^{g} \in \operatorname{span}_{\mathbb{K}}\left\{e_{2}\right\}$. This implies, in particular, that $a^{\prime}=a, b^{\prime}=b, c_{1}^{\prime}=c_{1}, \ldots, c_{m-2}^{\prime}=c_{m-2}$ and, in particular $e_{2}^{g}=a e_{2}$.

But now observe that

$$
\begin{aligned}
\left(e_{1}+k e_{2}\right)^{g} & =\left(e_{1}+e_{2}\right)^{g}+(k-1) e_{2}^{g} \\
& =a\left(e_{1}+e_{2}\right)+b f_{1}+\sum_{i=1}^{m-2} c_{i} x_{i}+(k-1) a e_{2} \\
& =a\left(e_{1}+k e_{2}\right)+b f_{1}+\sum_{i=1}^{m-2} c_{i} x_{i} \in W_{1}(k) .
\end{aligned}
$$

This shows that $W_{1}(k)^{g}=W_{1}(k)$, for every $k \in \mathbb{F}_{q}$. We conclude that $D(k)^{g}=D(k)$ for all $k \in \mathbb{F}_{q}$.
So let us consider the remaining case, when $W_{1}^{g}=W_{2}$ and $W_{1}(1)^{g}=W_{2}(1)$. Then there exist $a, b, c_{1}, \ldots, c_{m-2}$ such that

$$
e_{1}^{g}=a e_{2}+b f_{2}+\sum_{i=1}^{m-2} c_{i} y_{i} .
$$

Similarly there exist $a^{\prime}, b^{\prime}, c_{1}^{\prime}, \ldots, c_{m-2}^{\prime}$ such that

$$
\left(e_{1}+e_{2}\right)^{g}=a^{\prime} e_{2}+b^{\prime}\left(f_{2} \pm f_{1}\right)+\sum_{i=1}^{m-2} c_{i}^{\prime} y_{i}
$$

We obtain that

$$
e_{2}^{g}=\left(a^{\prime}-a\right) e_{2}+b^{\prime}\left(f_{2} \pm f_{1}\right)-b f_{2}+\sum_{i=1}^{m-2}\left(c_{i}^{\prime}-c_{i}\right) y_{i}
$$

But, since $e_{2} \in W_{2} \cap W_{2}(1)$ and since $g$ fixes $D(0)$ and $D(1)$, we deduce $e_{2}^{g} \in W_{1} \cap W_{1}(1)=\operatorname{span}_{\mathbb{K}}\left\{f_{1}, x_{1}, \ldots, x_{m-2}\right\}$, and we conclude that $e_{2}^{g} \in \operatorname{span}_{\mathbb{K}}\left\{f_{1}\right\}$. This implies, in particular, that $a^{\prime}=a, b^{\prime}=b, c_{1}^{\prime}=c_{1}, \ldots, c_{m-2}^{\prime}=$ $c_{m-2}$ and, in particular $e_{2}^{g}= \pm b f_{1}$.

But now observe that

$$
\begin{aligned}
\left(e_{1}+k e_{2}\right)^{g} & =\left(e_{1}+e_{2}\right)^{g}+(k-1) e_{2}^{g} \\
& =a e_{2}+b\left(f_{2} \pm f_{1}\right)+\sum_{i=1}^{m-2} c_{i} y_{i}+(k-1)( \pm b) f_{1} \\
& =a e_{2}+b\left(f_{2} \pm k f_{1}\right)+\sum_{i=1}^{m-2} c_{i} y_{i} \in W_{2}(k) .
\end{aligned}
$$

This shows that $W_{1}(k)^{g}=W_{2}(k)$, for every $k \in \mathbb{F}_{q}$. We conclude that $D(k)^{g}=D(k)$ for all $k \in \mathbb{F}_{q}$.
In all cases, then, we conclude that, if $g \in S_{\Lambda}$ and $g$ fixes the two points $D(0)$ and $D(1)$ of $\Lambda$, then $g$ fixes all elements of $\Lambda$. But this implies that $S^{\Lambda}$ does not contain $\operatorname{Alt}(\Lambda)$ and we are done.

The next case deals with the first outcome of the preceding lemma, but also applies when $q=4$.
Lemma 4.2.5. If $m=1$ and $q \geq 4$, then either the action is not binary or else $S$ is orthogonal and $q=5$.

Proof. Our method is based on the treatment of this case for $\mathrm{PSU}_{3}(q)$ in 45]. Note that $m=1$ cannot occur if $S$ is symplectic - thus we may assume that $S$ is either unitary or orthogonal. In the orthogonal case, $m=1$ implies that $q$ is an odd prime number by [54, §4.2].

The action of $G$ on $\Omega$ is permutation equivalent to the natural action of $G$ on

$$
\left\{\left\{X_{1}, X_{2}, \ldots, X_{n}\right\} \left\lvert\, \begin{array}{c}
\operatorname{dim}_{\mathbb{K}}\left(X_{1}\right)=\operatorname{dim}_{\mathbb{K}}\left(X_{2}\right)=\cdots=\operatorname{dim}_{\mathbb{K}}\left(X_{n}\right)=1 ; \\
V=X_{1} \perp X_{2} \perp \cdots \perp X_{n} ; X_{1}, X_{2}, \ldots, X_{n} \text { non-degenerate }
\end{array}\right.\right\} .
$$

Let $\left\{v_{1}, \ldots, v_{n}\right\}$ be an orthonormal basis for $V$; thus $\omega_{0}:=\left\{\left\langle v_{1}\right\rangle,\left\langle v_{2}\right\rangle, \ldots,\left\langle v_{n}\right\rangle\right\} \in \Omega$. If $S$ is unitary, then define $n_{0}=3$; if $S$ is orthogonal, then define $n_{0}=4$.

Now consider

$$
\Lambda:=\left\{\left\{X_{1}, X_{2}, \ldots, X_{n}\right\} \in \Omega \mid X_{i}=\left\langle v_{i}\right\rangle \text { for } i=n_{0}, \ldots, n\right\} .
$$

Then $G_{\Lambda}$ is equal to the stabilizer of $\left\{\left\langle v_{n_{0}}\right\rangle, \ldots,\left\langle v_{n}\right\rangle\right\}$. In the unitary case $G^{\Lambda}$ is almost simple with socle $\operatorname{PSU}_{2}(q)$; in the orthogonal case, $G^{\Lambda}$ is almost simple with socle $\Omega_{3}(q)$ (here we are using $q \geq 4$ ). In both cases the socle is isomorphic to $\operatorname{PSL}_{2}(q)$, and the action of $G_{\Lambda}$ on $\Lambda$ is permutation equivalent to the action of $G_{\left\{\left\langle v_{n_{0}}\right\rangle, \ldots,\left\langle v_{n}\right\rangle\right\}}$ on

$$
\begin{aligned}
\Lambda^{\prime}:=\left\{\left\{X_{1}, \cdots, X_{n_{0}-1}\right\}\right. & \mid \operatorname{dim}\left(X_{1}\right)=\cdots=\operatorname{dim}\left(X_{n_{0}-1}\right), \\
& \left.\left\langle v_{n_{0}}, \ldots, v_{n}\right\rangle^{\perp}=X_{1} \perp \cdots \perp X_{n_{0}-1}, X_{1}, \ldots, X_{n_{0}-1} \text { non-degenerate }\right\} .
\end{aligned}
$$

Suppose first that $S$ is unitary. Then, this action of $G^{\Lambda}$ has degree $|\Lambda|=q(q-1) / 2$. By consulting the table of the maximal subgroups of almost simple groups with socle $\operatorname{PSL}_{2}(q)$ in [10, we see that provided $q \notin\{7,9\}$ this action is primitive and hence by applying [45, Theorem 1.1] to $G^{\Lambda}$, we obtain that $G^{\Lambda}$ is not binary. Moreover, when $q \in\{7,9\}$, it can be easily checked with magma that the action of $G^{\Lambda}$ is not binary.

Suppose now that $S$ is orthogonal. (Recall that $q$ is a prime number.) In particular, $G^{\Lambda} \cong \operatorname{PSL}_{2}(q)$ or $G^{\Lambda} \cong \mathrm{PGL}_{2}(q)$. Let us denote by $X$ the socle of $G^{\Lambda}$ and by $Y$ the stabilizer in $G^{\Lambda}$ of an element of $\Lambda$. By consulting the table of the maximal subgroups of almost simple groups with socle $\mathrm{PSL}_{2}(q)$ in [10, Table 8.7], we have

$$
X \cap Y \cong \begin{cases}\operatorname{Sym}(4), & \text { when } q \equiv \pm 1 \quad(\bmod 8) \\ \operatorname{Alt}(4), & \text { when } q \equiv \pm 3,5, \pm 11, \pm 13, \pm 19 \quad(\bmod 40)\end{cases}
$$

From the same table we infer that $X \cap Y$ is a maximal subgroup of $X$ unless

$$
G^{\Lambda}=X \text { and } q \equiv \pm 11, \pm 19 \quad(\bmod 40) .
$$

Therefore, except for the cases where $q=p \equiv \pm 11, \pm 19(\bmod 40)$ and $G^{\Lambda}=X \cong \operatorname{PSL}_{2}(q)$, by applying [45, Theorem 1.1] to $G^{\Lambda}$ we obtain that $G^{\Lambda}$ is not binary for $q \neq 5$. (Observe that $q=5$ is the exception listed in the statement of the lemma.) We claim that this is the case also when $q=p \equiv \pm 11, \pm 19(\bmod 40)$ and $G^{\Lambda}=X \cong \operatorname{PSL}_{2}(q)$. To see this, observe that from [10, Table 8.7], there exists a subgroup $Z$ of $X=G^{\Lambda}$ with $Y<Z, Z \cong \operatorname{Alt}(5)$ and with $Z$ maximal in $X=G^{\Lambda}$. Now, the action of $Z$ on $(Z: Y)$ is not binary because it is permutation isomorphic to the non-binary degree 5 action of $\operatorname{Alt}(5)$. Hence $G^{\Lambda}$ is not binary by Lemma 1.6.2, as claimed.

Summing up, for the rest of the proof we may suppose that $G^{\Lambda}$ is not binary. In particular there exist two $\ell$-tuples

$$
\left(\left\{W_{1,1}, \ldots, W_{1, n_{0}-1}\right\}, \ldots,\left\{W_{\ell, 1}, \ldots, W_{\ell, n_{0}-1}\right\}\right)
$$

and

$$
\left(\left\{W_{1,1}^{\prime}, \ldots, W_{1, n_{0}-1}^{\prime}\right\}, \ldots,\left\{W_{\ell, 1}^{\prime}, \ldots, W_{\ell, n_{0}-1}^{\prime}\right\}\right)
$$

in $\Lambda^{\ell \ell}$ which are 2-subtuple complete for the action of $G_{\Lambda}$ but not in the same $G_{\Lambda}$-orbit. By construction the two $\ell$-tuples

$$
\begin{aligned}
I:= & \left\{W_{1,1}, \ldots, W_{1, n_{0}-1},\left\langle v_{n_{0}}\right\rangle, \ldots,\left\langle v_{n}\right\rangle\right\},\left\{W_{2,1}, \ldots, W_{2, n_{0}-1},\left\langle v_{n_{0}}\right\rangle, \ldots,\left\langle v_{n}\right\rangle\right\}, \ldots, \\
& \left.\left\{W_{\ell, 1}, \ldots, W_{\ell, n_{0}-1},\left\langle v_{n_{0}}\right\rangle, \ldots,\left\langle v_{n}\right\rangle\right\}\right) \\
J:= & \left\{W_{1,1}^{\prime}, \ldots, W_{1, n_{0}-1}^{\prime},\left\langle v_{n_{0}}\right\rangle, \ldots,\left\langle v_{n}\right\rangle\right\},\left\{W_{2,1}^{\prime}, \ldots, W_{2, n_{0}-1}^{\prime},\left\langle v_{n_{0}}\right\rangle, \ldots,\left\langle v_{n}\right\rangle\right\}, \ldots, \\
& \left.\left\{W_{\ell, 1}^{\prime}, \ldots, W_{\ell, n_{0}-1}^{\prime},\left\langle v_{n_{0}}\right\rangle, \ldots,\left\langle v_{n}\right\rangle\right\}\right)
\end{aligned}
$$

are in $\Omega^{\ell}$ and are 2-subtuple complete. Furthermore it is easy to see that $I$ and $J$ are not in the same $G$-orbit. Thus $G$ is not binary.

In the next lemma we write $X_{n}(q)$ to represent one of the three families of classical groups associated with non-degenerate forms. So, for instance, to get the result in the unitary case, the reader should read "PSU" wherever $X$ occurs.

Lemma 4.2.6. Let $m$ be a fixed positive integer, and let $n_{0}$ be a multiple of $m$ such that $X_{n_{0}}(q)$ is almost simple. If the primitive $\mathcal{C}_{2}$-action of $X_{n_{0}}(q)$ on $m$-decompositions of fixed type is not binary, then the same is true of $X_{n}(q)$, for all $n$ that are multiples of $m$ and that exceed $n_{0}$.

Note that the caveat "of fixed type" is included to account for the orthogonal case with $m$ even, where we have decompositions of type $\mathrm{O}^{+}$and $\mathrm{O}^{-}$.

Proof. We assume that $n>n_{0}$ and we proceed similarly to the previous lemma. First we identify $\Omega$ with the set

$$
\left\{\left\{X_{1}, X_{2}, \ldots, X_{t}\right\} \left\lvert\, \begin{array}{c}
\operatorname{dim}_{\mathbb{K}}\left(X_{1}\right)=\operatorname{dim}_{\mathbb{K}}\left(X_{2}\right)=\cdots=\operatorname{dim}_{\mathbb{K}}\left(X_{t}\right)=m ; \\
V=X_{1} \perp X_{2} \perp \cdots \perp X_{t} ; X_{1}, X_{2}, \ldots, X_{t} \text { non-degenerate }
\end{array}\right.\right\} .
$$

Now, fix an element $\left\{W_{1}, \ldots, W_{t}\right\}$ of $\Omega$ and consider

$$
\Lambda:=\left\{\left\{X_{1}, X_{2}, \ldots, X_{t}\right\} \in \Omega \mid X_{i}=W_{i} \text { for } i=\frac{n_{0}}{m}+1, \ldots, t\right\} .
$$

Clearly, $G_{\Lambda}$ is equal to the stabilizer of $\left\{W_{n_{0} / m+1}, \ldots, W_{t}\right\}$ and $G^{\Lambda}$ is almost simple with socle isomorphic to $X_{n_{0}}(q)$, and the action of $G_{\Lambda}$ on $\Lambda$ is permutation equivalent to the action of $G_{\left\{W_{n_{0} / m+1}, \ldots, W_{t}\right\}}$ on

$$
\Lambda^{\prime}:=\left\{\left\{W_{1}, \ldots, W_{n_{0} / m}\right\} \left\lvert\, \begin{array}{c}
\operatorname{dim}\left(W_{1}\right)=\cdots=\operatorname{dim}\left(W_{n_{0} / m}\right) ; W_{1}, \ldots, W_{n_{0} / m} \text { non-degenerate; } \\
\left\langle W_{n_{0} / m+1}, \ldots, W_{t}\right\rangle^{\perp}=W_{1} \perp \cdots \perp W_{n_{0} / m}
\end{array}\right.\right\} .
$$

Therefore $G^{\Lambda}$ is an almost simple primitive group with socle isomorphic to $X_{n_{0}}(q)$ in a $\mathcal{C}_{2}$-action on $m$-decompositions of given type. By assumption, $G^{\Lambda}$ is not binary and hence there exist two $\ell$-tuples $\left(\left\{W_{1,1}, \ldots, W_{1, n_{0} / m}\right\}, \ldots,\left\{W_{\ell, 1}, \ldots W_{\ell, n_{0} / m}\right\}\right)$ and $\left(\left\{W_{1,1}^{\prime}, \ldots W_{1, n_{0} / m}^{\prime}\right\}, \ldots,\left\{W_{\ell, 1}^{\prime}, \ldots W_{\ell, n_{0} / m}^{\prime}\right\}\right)$ in $\Lambda^{\ell}$ which are 2-subtuple complete for the action of $G_{\Lambda}$ but not in the same $G_{\Lambda}$-orbit. By construction the two $\ell$-tuples

$$
\begin{aligned}
I & :=\left(\left\{W_{1,1}, \ldots, W_{1, n_{0} / m}, W_{n_{0} / m+1}, \ldots, W_{t}\right\},\left\{W_{2,1}, \ldots, W_{2, n_{0} / m}, W_{n_{0} / m+1}, \ldots, W_{t}\right\}, \ldots,\right. \\
& \left.\left\{W_{\ell, 1}, \ldots, W_{\ell, n_{0} / m}, W_{n_{0} / m+1}, \ldots, W_{t}\right\}\right) \\
J & :=\left(\left\{W_{1,1}^{\prime}, \ldots, W_{1, n_{0} / m}^{\prime}, W_{n_{0} / m+1}, \ldots, W_{t}\right\},\left\{W_{2,1}^{\prime}, \ldots, W_{2, n_{0} / m}^{\prime}, W_{n_{0} / m+1}, \ldots, W_{t}\right\}, \ldots,\right. \\
& \left.\left\{W_{\ell, 1}^{\prime}, \ldots, W_{\ell, n_{0} / m}^{\prime}, W_{n_{0} / m+1}, \ldots, W_{t}\right\}\right)
\end{aligned}
$$

are in $\Omega^{\ell}$ and are 2-subtuple complete. As before, $I$ and $J$ are not in the same $G$-orbit. Thus $G$ is not binary.

| Group | Details of action |
| :---: | :---: |
| $\operatorname{SU}_{n}(q)$ | $m=1$ |
| $\operatorname{SU}_{4}(3), \mathrm{SU}_{4}(4)$ | $m=2$ |
| $\mathrm{SU}_{n}(2)$ | $m=3$ |

Table 4.2.2: $\mathcal{C}_{2}-S=\mathrm{SU}_{n}(q)$ and the $W_{i}$ are non-degenerate.

### 4.2.4 Case where $S=\mathrm{SU}_{n}(q)$ and the $W_{i}$ are non-degenerate

Assume that $S=\mathrm{SU}_{n}(q)$, the $W_{i}$ are non-degenerate, and the socle of $G$ is not as in Lemma 4.1.1. Here $V=V_{n}(\mathbb{K})$ where $\mathbb{K}=\mathbb{F}_{q^{2}}$, and we denote by $\sigma$ the involutory field automorphism of $\mathbb{K}$.

Lemma 4.2.7. In this case either $\Omega$ contains a beautiful subset or else $S$ is listed in Table 4.2.2.

Proof. Lemma 4.2 .4 implies that when $q \geq 5$, either $\Omega$ contains a beautiful subset or else we obtain the first line of Table 4.2.2. Now assume that $q \leq 4$ and $m \geq 2$.

If $q \in\{3,4\}$, then we repeat the same set-up as Lemma 4.2.4, except that this time $U$ is the subgroup whose elements fix all elements of $\mathcal{B}$ except $e_{1}$ and $f_{2}$ and satisfy

$$
\begin{aligned}
& e_{1} \mapsto e_{1}+k e_{2}, \\
& f_{2} \mapsto f_{2}-k^{\sigma} f_{1},
\end{aligned}
$$

for some $k \in \mathbb{K}=\mathbb{F}_{q^{2}}$, and we let $\Lambda=M^{U}$. Notice that $\Lambda$ is of size $q^{2}$ rather than $q$ as in Lemma 4.2.4. Now the same argument as before allows us to conclude that $\Lambda$ is a beautiful subset of order $q^{2}$, provided that $n>4$. (When $n=4$, we cannot conclude that $G_{\Lambda}$ acts 2 -transitively on $\Lambda$; notice that the groups $\mathrm{SU}_{4}(3)$ and $\mathrm{SU}_{4}(4)$ are listed as exceptions in Table 4.2.2.) Now the argument of Lemma 4.2.4implies that $\Lambda$ is a beautiful subset, as required.

If $q=2$ and $m \geq 2$, then [54, Table 3.5.H] implies that $m \geq 3$; if $m=3$, the action is listed in Table 4.2.2, hence we assume that $m \geq 4$. We consider hyperbolic pairs from $\mathcal{B}$ as before; this time assume that $e_{1}, f_{1}, e_{2}, f_{2} \in W_{1}$ and $e_{3}, f_{3}, e_{4}, f_{4} \in W_{2}$. Let $x_{1}, \ldots, x_{m-4}, y_{1}, \ldots, y_{m-4} \in \mathcal{B}$ be such that

$$
W_{1}=\operatorname{span}_{\mathbb{K}}\left\{e_{1}, e_{2}, f_{1}, f_{2}, x_{1}, \ldots, x_{m-4}\right\} \text { and } W_{2}=\operatorname{span}_{\mathbb{K}}\left\{e_{3}, e_{4}, f_{3}, f_{4}, y_{1}, \ldots, y_{m-4}\right\}
$$

We let $U$ be the subgroup whose elements fix all elements of $\mathcal{B}$ except $e_{1}, f_{3}$ and $f_{4}$ and satisfy

$$
\begin{aligned}
& e_{1} \mapsto e_{1}+k_{3} e_{3}+k_{4} e_{4}, \\
& f_{3} \mapsto f_{3}-k_{3}^{\sigma} f_{1}, \\
& f_{4} \mapsto f_{4}-k_{4}^{\sigma} f_{1},
\end{aligned}
$$

for some $k_{3}, k_{4} \in \mathbb{K}$, and we define $\Lambda=D^{U}$. For $k_{1}, k_{2} \in \mathbb{K}$, we define

$$
\begin{aligned}
& W_{1}\left(k_{1}, k_{2}\right)=\left\langle e_{1}+k_{1} e_{3}+k_{2} e_{4}, e_{2}, f_{1}, f_{2}, x_{1}, \ldots, x_{m-4}\right\rangle \text { and } \\
& W_{2}\left(k_{1}, k_{2}\right)=\left\langle e_{3}, e_{4}, f_{3}-k_{1}^{\sigma} f_{1}, f_{4}-k_{2}^{\sigma} f_{1}, y_{1}, \ldots, y_{m-4}\right\rangle .
\end{aligned}
$$

Observe that $\Lambda=\left\{D\left(k_{1}, k_{2}\right) \mid k_{1}, k_{2} \in \mathbb{K}\right\}$, where

$$
D\left(k_{1}, k_{2}\right)=W_{1}\left(k_{1}, k_{2}\right) \oplus W_{2}\left(k_{1}, k_{2}\right) \oplus W_{3} \oplus \cdots \oplus W_{t} .
$$

Note, in particular, that $D(0,0)=D$. Let $X$ be the stabilizer of the subspaces

$$
\left\langle e_{1}\right\rangle,\left\langle e_{2}\right\rangle,\left\langle e_{3}, e_{4}\right\rangle,\left\langle f_{1}\right\rangle,\left\langle f_{2}\right\rangle,\left\langle f_{3}, f_{4}\right\rangle,\left\langle x_{1}, \ldots, x_{m-4}\right\rangle,\left\langle y_{1}, \ldots, y_{m-4}\right\rangle, W_{3}, \ldots, W_{t} .
$$

| Group | Details of action |
| :---: | :---: |
| $\mathrm{Sp}_{n}(2), \mathrm{Sp}_{n}(3), \mathrm{Sp}_{n}(4)$ | $m=2$ |

Table 4.2.3: $\mathcal{C}_{2}-S=\operatorname{Sp}_{n}(q)$ and the $W_{i}$ are non-degenerate.

Then $U \rtimes X$ is 2 -transitive on $\Lambda=D^{U}$, a set of size 16 . Our aim now is to show that $\Lambda$ is a beautiful subset. Let $g \in S_{\Lambda}$. As in Lemma 4.2.4 we can see that $g$ preserves the set

$$
\left\{W_{1}\left(k_{1}, k_{2}\right) \mid k_{1}, k_{2} \in \mathbb{K}\right\} \cup\left\{W_{2}\left(k_{1}, k_{2}\right) \mid k_{1}, k_{2} \in \mathbb{K}\right\} .
$$

Now suppose that there exist $k_{1}, k_{2}, k_{1}^{\prime}, k_{2}^{\prime} \in \mathbb{K}$ such that $W_{1}\left(k_{1}, k_{2}\right)^{g}=W_{2}\left(k_{1}^{\prime}, k_{2}^{\prime}\right)$; then, by considering the vectors $e_{2}^{g}, f_{1}^{g}, f_{2}^{g}$, it is clear that for all $k_{1}, k_{2} \in \mathbb{K}$, there exist $k_{1}^{\prime}, k_{2}^{\prime} \in \mathbb{K}$ such that $W_{1}\left(k_{1}, k_{2}\right)^{g}=W_{2}\left(k_{1}^{\prime}, k_{2}^{\prime}\right)$. We conclude that $S_{\Lambda}$ has a subgroup $H$ of index at most 2 such that, if $g \in H$, then for all $k_{1}, k_{2} \in \mathbb{K}$ there exist $k_{1}^{\prime}, k_{2}^{\prime} \in \mathbb{K}$ (which may depend upon $g, k_{1}, k_{2}$ ) such that $W_{1}\left(k_{1}, k_{2}\right)^{g}=W_{1}\left(k_{1}^{\prime}, k_{2}^{\prime}\right)$.

This implies that $H$ preserves the subspaces

$$
Y_{1}:=\operatorname{span}_{\mathbb{K}}\left\{W_{1}\left(k_{1}, k_{2}\right) \mid k_{1}, k_{2} \in \mathbb{K}\right\} \text { and } Y_{0}:=\bigcap_{k_{1}, k_{2} \in \mathbb{K}} W_{1}\left(k_{1}, k_{2}\right) .
$$

Thus there is a homomorphism $\theta: H \rightarrow \mathrm{GL}\left(Y_{1} / Y_{0}\right) \cong \mathrm{GL}_{3}(\mathbb{K})$. Since $\mathrm{GL}_{3}(\mathbb{K})$ does not contain a subgroup with a composition factor isomorphic to $\operatorname{Alt}(s)$ for $s>7$, we conclude that either $\Lambda$ is beautiful or the action of $\operatorname{ker}(\theta)$ on $\Lambda$ must induce $\operatorname{Alt}(\Lambda)$ or $\operatorname{Sym}(\Lambda)$. However $\operatorname{ker}(\theta)$ is not transitive on $\Lambda$ and the result follows.

We need to deal with the cases listed in Table 4.2.2, Lemma 4.1.1 deals with the second line of the table. Now Lemma 4.2.5 means that, to deal with the first line of Table 4.2.2, we may assume that $q \in\{2,3\}$. Thus the next lemma deals with what remains.

Lemma 4.2.8. Suppose that $(q, m)$ is one of $(2,3),(2,1)$ or $(3,1)$. Then the action is not binary.
Proof. By Lemma 4.1.1 we have $n \geq 7$. Now Lemma 4.2.6 implies that the result holds for $n \geq 7$.

### 4.2.5 Case where $S=\operatorname{Sp}_{n}(q)$ and the $W_{i}$ are non-degenerate

Assume that $S=\operatorname{Sp}_{n}(q)$ with $n \geq 4$, the $W_{i}$ are non-degenerate, and the socle of $G$ is not as in Lemma4.1.1.
Lemma 4.2.9. In this case either $\Omega$ contains a beautiful subset or else $S$ is listed in Table 4.2.3.
Proof. Lemma 4.2.4 implies that, when $q \geq 5, \Omega$ contains a beautiful subset. Now assume that $q \leq 4$. Choose a hyperbolic basis for each of $W_{1}, \ldots, W_{t}$ and let $\mathcal{B}$ be the union of these bases. Write $m=2 \ell$ and order the hyperbolic basis so that $e_{1}, f_{1}, \ldots, e_{\ell}, f_{\ell} \in W_{1} ; e_{\ell+1}, f_{\ell+1}, \ldots, e_{2 \ell}, f_{2 \ell} \in W_{2}$ and so on.

We exclude the case $m=2$, since this is listed in Table 4.2.3 and we assume that $m \geq 4$. Now let $U$ be the subgroup whose elements fix all elements of $\mathcal{B}$ except $e_{1}, e_{\ell+1}, e_{\ell+2}, f_{\ell+1}$ and $f_{\ell+2}$ and satisfy

$$
\begin{aligned}
& e_{1} \mapsto e_{1}+k_{1} e_{\ell+1}+k_{2} e_{\ell+2}+k_{3} f_{\ell+1}+k_{4} f_{\ell+2}, \\
& e_{\ell+1} \mapsto e_{\ell+1}+k_{3} f_{1}, \\
& e_{\ell+2} \mapsto e_{\ell+2}+k_{4} f_{1}, \\
& f_{\ell+1} \mapsto f_{\ell+1}-k_{1} f_{1}, \\
& f_{\ell+2} \mapsto f_{\ell+2}-k_{2} f_{1},
\end{aligned}
$$

for some $k_{1}, k_{2}, k_{3}, k_{4} \in \mathbb{F}_{q}$, and we define $\Lambda=D^{U}$. For $k_{1}, k_{2}, k_{3}, k_{4} \in \mathbb{F}_{q}$, we define

$$
\begin{aligned}
W_{1}\left(k_{1}, k_{2}, k_{3}, k_{4}\right) & =\left\langle e_{1}+k_{1} e_{\ell+1}+k_{2} e_{\ell+2}+k_{3} f_{\ell+1}+k_{4} f_{\ell+2}, e_{2}, \ldots, e_{\ell}, f_{1}, \ldots, f_{\ell}\right\rangle \text { and } \\
W_{2}\left(k_{1}, k_{2}, k_{3}, k_{4}\right) & =\left\langle e_{\ell+1}+k_{3} f_{1}, e_{\ell+2}+k_{4} f_{1}, e_{\ell+3}, \ldots, e_{2 \ell}, f_{\ell+1}-k_{1} f_{1}, f_{\ell+2}-k_{2} f_{1}, f_{\ell+3}, \ldots, f_{2 \ell}\right\rangle,
\end{aligned}
$$

| Group | Details of action |
| :---: | :---: |
| $\Omega_{n}(p)$ | $m=1$ |
| $\Omega_{n}(3)$ | $m=3$ |

Table 4.2.4: $\mathcal{C}_{2}-S=\Omega_{n}(q)$ with $n q$ odd and the $W_{i}$ are non-degenerate.
and observe that $\Lambda=\left\{D\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \mid k_{1}, k_{2}, k_{3}, k_{4} \in \mathbb{F}_{q}\right\}$, where

$$
D\left(k_{1}, k_{2}, k_{3}, k_{4}\right)=W_{1}\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \oplus W_{2}\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \oplus W_{3} \oplus \cdots \oplus W_{t}
$$

Note, in particular, that $D(0,0,0,0)=D$. Let $X$ be the stabilizer of the subspaces

$$
\left\langle e_{\ell+1}, e_{\ell+2}, f_{\ell+1}, f_{\ell+2}\right\rangle,\left\langle e_{1}\right\rangle, \ldots,\left\langle e_{\ell}\right\rangle,\left\langle e_{\ell+3}\right\rangle, \ldots,\left\langle e_{\ell t}\right\rangle,\left\langle f_{1}\right\rangle, \ldots,\left\langle f_{\ell}\right\rangle,\left\langle f_{\ell+3}\right\rangle, \ldots,\left\langle f_{\ell t}\right\rangle .
$$

Then $U \rtimes X$ is 2-transitive on $\Lambda=D^{U}$ (making use of Lemma 1.6.9 and the fact that $\mathrm{Sp}_{4}(q)$ acts transitively on the set of non-zero vectors in the natural $\mathbb{F}_{q}$-module), a set of size $q^{4}$. Our aim now is to show that $\Lambda$ is a beautiful subset. Let $g \in S_{\Lambda}$. As before we can see that $g$ preserves the set

$$
\left\{W_{1}\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \mid k_{1}, k_{2}, k_{3}, k_{4} \in \mathbb{F}_{q}\right\} \cup\left\{W_{1}\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \mid k_{1}, k_{2}, k_{3}, k_{4} \in \mathbb{F}_{q}\right\} .
$$

Now suppose that there exist $k_{1}, k_{2}, k_{3}, k_{4}, k_{1}^{\prime}, k_{2}^{\prime}, k_{3}^{\prime}, k_{4}^{\prime} \in \mathbb{F}_{q}$ such that $W_{1}\left(k_{1}, k_{2}, k_{3}, k_{4}\right)^{g}=W_{2}\left(k_{1}^{\prime}, k_{2}^{\prime}, k_{3}^{\prime}, k_{4}^{\prime}\right)$; then, by considering the vectors $e_{2}^{g}, f_{1}^{g}, f_{2}^{g}$, it is clear that, for all $k_{1}, k_{2}, k_{3}, k_{4} \in \mathbb{F}_{q}$, there exist $k_{1}^{\prime}, k_{2}^{\prime}, k_{3}^{\prime}, k_{4}^{\prime} \in$ $\mathbb{F}_{q}$ such that $W_{1}\left(k_{1}, k_{2}, k_{3}, k_{4}\right)^{g}=W_{2}\left(k_{1}^{\prime}, k_{2}^{\prime}, k_{3}^{\prime}, k_{4}^{\prime}\right)$. We conclude that $S_{\Lambda}$ has a subgroup $H$ of index at most 2 such that, if $h \in H$, then for all $k_{1}, k_{2}, k_{3}, k_{4} \in \mathbb{F}_{q}$ there exist $k_{1}^{\prime}, k_{2}^{\prime}, k_{3}^{\prime}, k_{4}^{\prime} \in \mathbb{F}_{q}$ with $W_{1}\left(k_{1}, k_{2}, k_{3}, k_{4}\right)^{g}=W_{1}\left(k_{1}^{\prime}, k_{2}^{\prime}, k_{3}^{\prime}, k_{4}^{\prime}\right)$.

We conclude that $H$ preserves the subspaces

$$
Y_{1}:=\operatorname{span}_{\mathbb{K}}\left\{W_{1}\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \mid k_{1}, k_{2}, k_{3}, k_{4} \in \mathbb{F}_{q}\right\} \text { and } Y_{0}:=\bigcap_{k_{1}, k_{2}, k_{3}, k_{4} \in \mathbb{F}_{q}} W_{1}\left(k_{1}, k_{2}, k_{3}, k_{4}\right) .
$$

Thus there is a homomorphism $\theta: S_{\Lambda} \rightarrow \mathrm{GL}\left(Y_{1} / Y_{0}\right) \cong \mathrm{GL}_{5}\left(\mathbb{F}_{q}\right)$. By Lemma 2.1.1, $\mathrm{GL}_{5}\left(\mathbb{F}_{q}\right)$ does not have a section isomorphic to $\operatorname{Alt}(s)$ for $s>8$, so we conclude that either $\Lambda$ is a beautiful set, or the action of $\operatorname{ker}(\theta)$ on $\Lambda$ must induce $\operatorname{Alt}(\Lambda)$ or $\operatorname{Sym}(\Lambda)$. However $\operatorname{ker}(\theta)$ is not transitive on $\Lambda$, and we conclude that $\Lambda$ is beautiful as required.

We must show that the actions listed in Table 4.2 .3 are not binary; the next lemma does the job.
Lemma 4.2.10. Suppose that $(q, m)$ is one of $(2,2),(3,2)$ or $(4,2)$. Then the action is not binary.
Proof. Lemma4.1.1 gives the result for $n=4$. Now Lemma4.2.6 implies that the result holds for $n>4$.

### 4.2.6 Case where $S=\Omega_{n}(q)$ for $n q$ odd, and the $W_{i}$ are non-degenerate

Lemma 4.2.11. In this case either $\Omega$ contains a beautiful subset or else $S$ is listed in Table 4.2.4.
Proof. Lemma 4.2.4 implies that when $q \geq 5$, either $\Omega$ contains a beautiful subset or else $m=1$; in the latter case, [54, Table 3.5.D] implies that $q=p$, a prime, and we obtain the first line of Table 4.2.4. If $q=3$ and $m=3$, then we obtain the second line of Table 4.2.4.

Assume, then, that $q=3$ and $m \geq 5$. The assumption on $m$ means that each $W_{i}$ contains at least two hyperbolic pairs. Now the argument of Lemma 4.2 .7 for $q=2$ carries over here and we obtain a beautiful subset of size 9 .

We must show that the actions listed in Table 4.2 .4 are not binary. Lemma 4.2.5 deals with the first line, provided $q>5$; the next lemma deals with what remains.

| Group | Details of action |
| :--- | :--- |
| $\Omega_{n}^{+}(p)$ | $p \geq 3, m=1$ |
| $\Omega_{n}^{+}(q)$ | $W_{i}$ of type $\mathrm{O}_{2}^{-}$ |
| $\Omega_{n}^{+}(4)$ | $W_{i}$ of type $\mathrm{O}_{2}^{ \pm}$or $\mathrm{O}_{4}^{-}$ |
| $\Omega_{n}^{+}(3)$ | $W_{i}$ of type $\mathrm{O}_{2}^{ \pm}, \mathrm{O}_{3}$ or $\mathrm{O}_{4}^{-}$ |
| $\Omega_{n}^{+}(2)$ | $W_{i}$ of type $\mathrm{O}_{2}^{ \pm}, \mathrm{O}_{4}^{ \pm}$or $\mathrm{O}_{6}^{-}$ |

Table 4.2.5: $\mathcal{C}_{2}-S=\Omega_{n}^{+}(q)-$ and the $W_{i}$ are non-degenerate.

| Group | Details of action |
| :--- | :--- |
| $\Omega_{n}^{-}(p)$ | $p \geq 3, m=1$ |
| $\Omega_{n}^{-}(q)$ | $W_{i}$ of type $\mathrm{O}_{2}^{-}$ |
| $\Omega_{n}^{-}(4)$ | $W_{i}$ of type $\mathrm{O}_{2}^{-}$or $\mathrm{O}_{4}^{-}$ |
| $\Omega_{n}^{-}(3)$ | $W_{i}$ of type $\mathrm{O}_{2}^{-}, \mathrm{O}_{3}$ or $\mathrm{O}_{4}^{-}$ |
| $\Omega_{n}^{-}(2)$ | $W_{i}$ of type $\mathrm{O}_{2}^{-}, \mathrm{O}_{4}^{-}$or $\mathrm{O}_{6}^{-}$ |

Table 4.2.6: $\mathcal{C}_{2}-S=\Omega_{n}^{-}(q)-$ and the $W_{i}$ are non-degenerate.

Lemma 4.2.12. Suppose that $(q, m)$ is one of $(3,1),(5,1),(3,3)$. Then the action is not binary.
Proof. We begin by checking the truth of this statement for $S \in\left\{\Omega_{5}(3), \Omega_{5}(5), \Omega_{9}(3)\right\}$. For the first two cases it follows from Lemma 4.1.1. And when $S=\Omega_{9}(3)$, we use the permutation character method. Let $M$ be a maximal subgroup of $S$ in the Aschbacher class $\mathcal{C}_{2}$, let $1_{S}$ be the principal character of $S$ and let $\pi_{M}$ be the permutation character for the action of $S$ on the right cosets of $M$. We have verified that in all cases

$$
\left\langle\pi\left(\pi-1_{S}\right)\left(\pi-2 \cdot 1_{S}\right), 1_{S}\right\rangle>\left(|\operatorname{Out}(S)|\left\langle\pi\left(\pi-1_{S}\right), 1_{S}\right\rangle\right)^{3} .
$$

In particular, all actions under consideration are not binary in view of Lemma 1.8.1.
Now Lemma 4.2.6 implies that the result holds for all $n \geq 7$, as required.

### 4.2.7 Case where $S=\Omega_{n}^{ \pm}(q)$ and the $W_{i}$ are non-degenerate

Lemma 4.2.13. In this case either $\Omega$ contains a beautiful subset or else $S$ is listed in Table 4.2.5 or Table 4.2.6.

Proof. If $q \geq 5$, then Lemma 4.2 .4 yields the first two lines of each table. We also use the fact, from 54, Tables 3.5.E and 3.5.F], that if $m=1$, then $q=p \geq 3$, where $p$ is prime. Assume, then, that $q \leq 4$. Recall that if $q$ is even, then $m$ is even. We consider the case where $q \in\{3,4\}$ first. We require that $W_{1}$ contains at least two orthogonal hyperbolic lines. All cases that do not satisfy this requirement are listed in the tables.

Now we let $e_{1}, f_{1}$ be a hyperbolic pair in $W_{1}$, and $e_{\ell+1}, f_{\ell+1}, e_{\ell+2}, f_{\ell+2}$ two hyperbolic pairs in $W_{2}$. We let $U$ be the subgroup whose elements fix all elements of $\mathcal{B}$ except $e_{1}, f_{\ell+1}$ and $f_{\ell+2}$ and satisfy

$$
\begin{aligned}
& e_{1} \mapsto e_{1}+k_{1} e_{\ell+1}+k_{2} e_{\ell+2}, \\
& f_{\ell+1} \mapsto f_{\ell+1}-k_{1} f_{1}, \\
& f_{\ell+2} \mapsto f_{\ell+2}-k_{2} f_{1},
\end{aligned}
$$

for some $k_{1}, k_{2} \in \mathbb{F}_{q}$, and we define $\Lambda=D^{U}$. We now proceed using the argument for $q=2$ in Lemma 4.2.7 to conclude that we have a beautiful subset of size $q^{2}$. (Note that $S^{\Lambda}$ contains $\mathrm{ASL}_{2}(q)$, hence the 2transitivity of $S^{\Lambda}$ is immediate.)

If $q=2$, then the argument is similar, but we require that $W_{1}$ contains at least three orthogonal hyperbolic lines. All cases that do not satisfy this requirement are listed in the tables. We let $U$ be the subgroup whose elements fix all elements of $\mathcal{B}$ except $e_{1}, f_{\ell+1}, f_{\ell+2}$ and $f_{\ell+3}$ and satisfy

$$
\begin{aligned}
& e_{1} \mapsto e_{1}+k_{1} e_{\ell+1}+k_{2} e_{\ell+2}+k_{3} e_{\ell+3}, \\
& f_{\ell+1} \mapsto f_{\ell+1}-k_{1} f_{1}, \\
& f_{\ell+2} \mapsto f_{\ell+2}-k_{2} f_{1}, \\
& f_{\ell+3} \mapsto f_{\ell+3}-k_{3} f_{1},
\end{aligned}
$$

for some $k_{1}, k_{2}, k_{3} \in \mathbb{F}_{q}$, and we define $\Lambda=D^{U}$. As before we get a homomorphism $\theta: S_{\Lambda} \mapsto \mathrm{GL}\left(Y_{1} / Y_{0}\right) \cong$ $\mathrm{GL}_{4}(2)$. Moreover, $\Lambda$ corresponds to a set of 8 vectors in $Y_{1} / Y_{0}$, namely the set of vectors $e_{1}+k_{1} e_{\ell+1}+$ $k_{2} e_{\ell+2}+k_{3} e_{\ell+3}+Y_{0}$. Since the stabilizer in $\mathrm{GL}_{4}(2)$ of this set does not induce Alt(8), it follows as before that $\Lambda$ is a beautiful subset of size 8 , completing the proof.

We must show that the actions listed in Tables 4.2 .5 and 4.2 .6 are not binary. Lemma 4.2.5 deals with the first line of each table, provided $q>5$. The next lemma deals with the second line of each table.

Lemma 4.2.14. If the $W_{i}$ are of type $\mathrm{O}_{2}^{-}$, then the action is not binary.
Proof. The proof is similar to that of Lemma 4.2.5.
Note that the action of $G$ on $\Omega$ is permutation equivalent to the natural action of $G$ on

$$
\left\{\left\{X_{1}, X_{2}, \ldots, X_{n / 2}\right\} \left\lvert\, \begin{array}{c}
X_{1}, \ldots, X_{n / 2} \text { of type } \mathrm{O}_{2}^{-} ; \\
V=X_{1} \perp X_{2} \perp \cdots \perp X_{n / 2} ; X_{1}, X_{2}, \ldots, X_{n / 2} \text { non-degenerate }
\end{array}\right.\right\} .
$$

Now consider

$$
\Lambda:=\left\{\left\{X_{1}, X_{2}, \ldots, X_{n / 2}\right\} \in \Omega \mid X_{i}=W_{i} \text { for } i \in\{4, \ldots, n / 2\}\right\} .
$$

Then $G_{\Lambda}$ is equal to the stabilizer of $\left\{W_{4}, \ldots, W_{n / 2}\right\}$ and $G^{\Lambda}$ is almost simple with socle $\mathrm{P} \Omega_{6}^{-}(q)$; therefore, the socle of $G^{\Lambda}$ is isomorphic to $\operatorname{PSU}_{4}(q)$ and the action is isomorphic to a $\mathcal{C}_{2}$-action of an almost simple group with socle $\mathrm{PSU}_{4}(q)$ on non-degenerate 1 -spaces of $\mathbb{F}_{q^{2}}^{4}$. We saw in 84.2 .4 that this action is not binary, thus there exist two $\ell$-tuples $\left(\left\{W_{1,1}, W_{1,2}, W_{1,3}\right\}, \ldots,\left\{W_{\ell, 1}, W_{\ell, 2}, W_{\ell, 3}\right\}\right)$ and ( $\left\{W_{1,1}^{\prime}, W_{1,2}^{\prime}, W_{1,3}^{\prime}\right\}, \ldots$ ,$\left.\left\{W_{\ell, 1}^{\prime}, W_{\ell, 2}^{\prime}, W_{\ell, 3}^{\prime}\right\}\right)$ in $\Lambda^{\ell}$ which are 2-subtuple complete for the action of $G_{\Lambda}$ but not in the same $G_{\Lambda}$-orbit. By construction the two $\ell$-tuples

$$
\begin{aligned}
I:= & \left(\left\{W_{1,1}, W_{1,2}, W_{1,3}, W_{4}, \ldots, W_{n / 2}\right\},\left\{W_{2,1}, W_{2,3}, W_{2,3}, W_{4}, \ldots, W_{n / 2}\right\}, \ldots,\right. \\
& \left.\left\{W_{\ell, 1}, W_{\ell, 2}, W_{\ell, 3}, W_{4}, \ldots, W_{n / 2}\right\}\right) \\
J:= & \left(\left\{W_{1,1}^{\prime}, W_{1,2}^{\prime}, W_{1,3}^{\prime}, W_{4}, \ldots, W_{n / 2}\right\},\left\{W_{2,1}^{\prime}, W_{2,2}^{\prime}, W_{2,3}^{\prime}, W_{4}, \ldots, W_{n / 2}\right\}, \ldots,\right. \\
& \left.\left\{W_{\ell, 1}^{\prime}, W_{\ell, 2}^{\prime}, W_{\ell, 3}^{\prime}, W_{4}, \ldots, W_{n / 2}\right\}\right)
\end{aligned}
$$

are in $\Omega^{\ell}$ and are 2-subtuple complete. Moreover, $I$ and $J$ are not in the same $G$-orbit. Thus $G$ is not binary.

The next lemma deals with the remaining lines of Tables 4.2.5 and 4.2.6.
Lemma 4.2.15. Suppose that $(q, m)$ is in

$$
\{(3,1),(5,1),(2,2),(3,2),(4,2),(3,3),(2,4),(3,4),(4,4),(2,6)\} .
$$

Then the action is not binary.

Proof. If $m=1$, then we use the fact that we have already studied all $\mathcal{C}_{2}$ actions for $n$ odd. In particular Lemma 4.2.12 attends to the case where $(m, n, q) \in\{(1,7,3),(1,7,5)\}$. Now Lemma 4.2.6 yields the result for $(q, m) \in\{(3,1),(5,1)\}$ and $n \geq 8$.

If $m=3$, a similar argument works, using the fact that Lemma 4.2.12 attends to the case where $(m, n, q)=(3,9,3)$.

If $m=2$ or 4 , then we must deal with the cases where $q \in\{2,3,4\}$ (note that when $m=2$, Lemma 4.2.14 allows us to assume that the $W_{i}$ are of type $\mathrm{O}_{2}^{+}$). When $n=8$, Lemma 4.1.1 gives the result for $q=2$. We use magma to verify that, when $n=8$ and $q \in\{3,4\}$, then the corresponding $\mathcal{C}_{2}$ actions are not binary. Lemma 4.2.6 then implies the result for $n>8$.

If $m=6$, then we must deal with the case $q=2$ and the situation where the $W_{i}$ are of type $\mathrm{O}_{6}^{-}$. We consider first what happens when $n=12$ : note that, in this case, $S=\Omega_{12}^{+}(2)$, since $n / m$ is even. Now we use magma, with the permutation character method (using Lemma 1.8.1), to confirm that, in the case $S=\Omega_{12}^{+}(2)$, the action is not binary. Now Lemma 4.2.6 implies the result for $n>12$.

### 4.3 Family $\mathcal{C}_{3}$

In this section, the subgroup $M$ is a "field extension subgroup". Such subgroups are described in [54, Section 4.3], and are listed in Table 4.3.1. In every case we start with a field extension $\mathbb{K}_{\#}$ of $\mathbb{K}$ of prime degree. We will usually denote this degree by the letter " $r$ ", although in a few subfamilies, the degree is always equal to 2 . We set $m=n / r$.

| case | type | conditions |
| :---: | :---: | :---: |
| L | $\mathrm{GL}_{m}\left(q^{r}\right)$ |  |
| U | $\mathrm{GU}_{m}\left(q^{r}\right)$ | $r$ odd |
| S | $\mathrm{Sp}_{m}\left(q^{r}\right)$ |  |
| $\mathrm{O}^{\epsilon}$ | $\mathrm{O}_{m}^{\epsilon}\left(q^{r}\right)$ | $m \geq 3$ |
| $\mathrm{~S}, \mathrm{O}^{\epsilon}$ | $\mathrm{GU}_{n / 2}(q)$ | $r=2, q$ odd in case S |

Table 4.3.1: Maximal subgroups in family $\mathcal{C}_{3}$
In the case $S=\mathrm{SL}_{n}(q)$, the group $M$ is embedded in $G$ by considering the group $\Gamma \mathrm{L}_{m}\left(\mathbb{K}_{\#}\right)$ acting on an $m$-dimensional vector space $V_{\#}$ over $\mathbb{K}_{\#}$ and then considering those $\mathbb{K}_{\#}$-semilinear transformations of $V_{\#}$ that induce $\mathbb{K}$-linear transformations on $V$, where $V$ is simply $V_{\#}$ viewed as a $\mathbb{K}$-vector space. For the other cases, one must also consider $\mathbb{K}_{\#}$-forms defined on $V_{\#}$; full details are given in [54].

It is convenient to give a geometrical interpretation for the set of right cosets of $M$ (which is the set on which we are acting). To do this, we take $\mathbb{K}_{\#}$ to be a field extension of our field $\mathbb{K}$ and we wish to define a $\mathbb{K}_{\#}$-structure on $V$.

We start by considering a $\mathbb{K}$-linear isomorphism $\phi: V_{\#} \rightarrow V$. Let $\Sigma$ be the set of all such isomorphisms, and we observe that two groups act naturally on $\Sigma$ :

1. $\mathrm{GL}_{m}\left(\mathbb{K}_{\#}\right)$ acts on $\Sigma$ via $\phi^{g}(\mathbf{v})=\phi\left(\mathbf{v}^{g^{-1}}\right)$;
2. $\mathrm{GL}_{n}(\mathbb{K})$ acts on $\Sigma$ via $\phi^{h}(\mathbf{v})=(\phi(\mathbf{v}))^{h}$.

Clearly these two actions commute. Thus we define a $\mathbb{K}_{\#}$-structure on $V$ to be an orbit of the group $\mathrm{GL}_{m}\left(\mathbb{K}_{\#}\right)$ on $\Sigma$, and (using commutativity of the actions) we observe that $\mathrm{GL}_{n}(\mathbb{K})$ acts on the set of all $\mathbb{K}_{\#}$-structures on $V$. What is more, the stabilizer of such an action is a field extension subgroup $M$, hence we have the geometrical interpretation that we require.

Note that we can replace the word "linear" with the word "semilinear" in the previous paragraph to extend this geometrical interpretation to subgroups of $\Gamma L_{n}(\mathbb{K})$.

The main result of this section is the following. The result will be proved in a series of lemmas.

Proposition 4.3.1. Suppose that $G$ is an almost simple group with socle $\bar{S}=\mathrm{Cl}_{n}(q)$, and assume that
(i) $n \geq 3,4,4,7$ in cases $L, U, S, O$ respectively, and
(ii) $\mathrm{Cl}_{n}(q)$ is not one of the groups listed in Lemma 4.1.1,

Let $M$ be a maximal subgroup of $G$ in the family $\mathcal{C}_{3}$. Then the action of $G$ on $(G: M)$ is not binary.

### 4.3.1 Case $S=\operatorname{SL}_{n}(q)$

Lemma 4.3.2. Suppose that $G$ is almost simple with socle equal to $\operatorname{PSL}_{n}(q)$. Let $M$ be a $\mathcal{C}_{3}$-maximal subgroup such that $F^{*}(M)$ contains $M_{1}$, a quasisimple cover of $\mathrm{PSL}_{m}\left(q^{r}\right)$, where $n=m r, r$ is prime and $m \geq 3$. Then the action of $G$ on $(G: M)$ is not binary.

Proof. We define

$$
\tilde{x}=\left(\begin{array}{ccc}
1 & & \\
& A & \\
& & a^{-1}
\end{array}\right) \in \operatorname{SL}_{m}\left(q^{r}\right)
$$

where $A$ is an element of $\mathrm{GL}_{m-2}\left(q^{r}\right)$ of order $\left(q^{r}\right)^{m-2}-1$ and $a=\operatorname{det}(A)$. We let $x$ be the element in $F^{*}(M)$ which is the projective image of $\tilde{x}$ in $M_{1}$. Observe that $\tilde{x}$ has a 1 -eigenspace over $\mathbb{F}_{q^{r}}$ of dimension 1 , and so has a 1-eigenspace over $\mathbb{F}_{q}$ of dimension $r$; we conclude that $C_{M}(x)$ is a proper subgroup of $C_{G}(x)$. Thus there is a suborbit, $\Delta$, on which the action of $M$ is isomorphic to the action of $M$ on $(M: H)$, where $H$ is a subgroup of $M$ that does not contain $M_{1}$ and does contain $x$.

We now refer to Lemma 2.2.5. This shows that either the action of $M$ on $\Delta$ is not binary, or $M$ has a section $\operatorname{Alt}\left(q^{r(m-2)}\right)$. In the former case, the conclusion of the lemma holds. In the latter case, Lemma 2.1.1 implies that the only possibility is $m=3, q=2, r=2$. But then $S=\mathrm{SL}_{6}(2)$, a case covered by Lemma 4.1.1

The remaining lemmas deal with the case when $m \leq 2$.
Lemma 4.3.3. Suppose that $G$ is an almost simple group with socle $\mathrm{PSL}_{n}(q)$, where $n=2 r$ for $r$ a prime. Suppose that $M$ is a $\mathcal{C}_{3}$-maximal subgroup such that $M \triangleright \mathrm{PSL}_{2}\left(q^{r}\right)$. Then the action of $G$ on $(G: M)$ is not binary.

Proof. Let $\mathbb{K}_{\#}$ be the field $\mathbb{F}_{q^{r}}$, and let $\left\{v_{1}, v_{2}\right\}$ be a $\mathbb{K}_{\#}$-basis for $V$. Let $\lambda$ be an element of $\mathbb{K}_{\#}$ such that $\left\{\lambda, \lambda^{q}, \ldots, \lambda^{q^{r-1}}\right\}$ is a basis for $\mathbb{K}_{\#}$ over $\mathbb{K}=\mathbb{F}_{q}$ and observe that

$$
\mathcal{B}=\left\{v_{1} \lambda, v_{1} \lambda^{q}, \ldots, v_{1} \lambda^{q^{r-1}}, v_{2} \lambda, v_{2} \lambda^{q}, \ldots, v_{2} \lambda^{q^{r-1}}\right\}
$$

is an $\mathbb{F}_{q}$-basis for $V$. We take $M \cap \operatorname{PGL}(V)$ to be the subgroup that preserves the (semilinear) $\mathbb{K}_{\#}$-vector space structure of $V$.

Suppose first that $r>2$ and $q>2$. Then $M \cap \operatorname{PSL}(V)$ has the structure $\left(C \times \operatorname{PSL}_{2}\left(q^{r}\right)\right) . r$ with $C$ cyclic (see [54, 4.3.6]). Let $M_{0}$ be a subgroup of $M \cap \operatorname{PSL}_{n}(q)$ isomorphic to $\mathrm{PSL}_{2}\left(q^{r}\right) . r$. Then $M_{0}$ has a subgroup $H=H_{0} \times\langle\sigma\rangle$, where $H_{0} \cong \operatorname{PSL}_{2}(q)$ and $\sigma$ is a field automorphism of order $r$. Moreover $H$ is maximal in $M_{0}$ (see [10, Table 8.1]).

Consider the direct sum $\mathbb{F}_{q}$-decomposition

$$
\begin{equation*}
V=\left\langle v_{1} \lambda, v_{2} \lambda\right\rangle \oplus\left\langle v_{1} \lambda^{q}, v_{2} \lambda^{q}\right\rangle \oplus \cdots \oplus\left\langle v_{1} \lambda^{q^{r-1}}, v_{2} \lambda^{q^{r-1}}\right\rangle . \tag{4.3.1}
\end{equation*}
$$

Observe that $H_{0}$ stabilizes each subspace in the decomposition, while the field automorphism $\sigma: x \mapsto x^{q}$ induces an $r$-cycle on the $r$ subspaces in the decomposition. Thus $H$ stabilizes the decomposition. Since $r>2$, there is an element $g$ of order $r-1$ stabilizing the decomposition that is not in $M$, centralizes $H_{0}$ and normalizes $\langle\sigma\rangle$. Then $H \leq M \cap M^{g}$, and so $M \cap M^{g}$ is a maximal subgroup of $M_{1}=\left(M \cap M^{g}\right) M_{0}^{\prime}$. Hence the action of $M_{1}$ on ( $M_{1}: M \cap M^{g}$ ) is isomorphic to the action of an almost simple group with
socle $\mathrm{PSL}_{2}\left(q^{r}\right)$ on a maximal $\mathcal{C}_{5}$-subgroup containing $\mathrm{PSL}_{2}(q)$. By [45], this action is not binary, and now Lemma 1.6 .2 implies that the action of $M$ on $\left(M: M \cap M^{g}\right)$ is not binary. Then Lemma 1.6.1 implies that the action of $G$ on $(G: M)$ is not binary.

Suppose, next, that $r=2$. Again $M$ has a subgroup $M_{0} \cong \operatorname{PSL}_{2}\left(q^{2}\right)$ preserving the decomposition (4.3.1), and $M_{0}$ has a subgroup $H_{0} \cong \mathrm{PGL}_{2}(q)$. Define $U$ to be the set of elements in $S$ that fix all elements of $\mathcal{B}$ except $v_{1} \lambda$ and which satisfy

$$
v_{1} \lambda \mapsto v_{1}+\alpha v_{2} \lambda \quad\left(\alpha \in \mathbb{F}_{q}\right) .
$$

Then $U$ is a group of order $q$ with $U \cap M=1$, and $U$ is normalized by a torus $T<H_{0}$ of order $q-1$, acting fixed-point-freely. In the usual way we obtain a subset $\Delta$ of $\Omega$ of size $q$ for which $G^{\Delta}$ is 2 -transitive. Now, by Lemma 2.1.1, $M$ does not have a section isomorphic to $\operatorname{Alt}(t)$ for $t \geq 7$ and, by Lemma 1.6.12 the conclusion of the lemma follows for $q>5$. If $q=2$, then $S=\operatorname{SL}_{4}(2) \cong \operatorname{Alt}(8)$ and the result follows from [46]. And if $q \in\{3,4,5\}$, then the result follows from Lemma 4.1.1.

Finally assume that $q=2$ and $r>2$. In this case, writing matrices with respect to $\mathcal{B}, M$ contains an element

$$
g=\left(\begin{array}{ll}
I_{r} & \\
& A
\end{array}\right)
$$

where $A \in \operatorname{GL}_{r}(2)$ is an element of order $2^{r}-1$, and we let $T=\langle g\rangle$. Now let $U$ be the subgroup of $S$ consisting of elements $u$ that fix all elements of $\mathcal{B}$ except $v_{1} \lambda$ and for which

$$
v_{1} \lambda^{u}-v_{1} \lambda \in \operatorname{span}_{\mathbb{F}_{q}}\left\{v_{2} \lambda, v_{2} \lambda^{q}, \ldots, v_{2} \lambda^{q^{r-1}}\right\} .
$$

Then $U$ is a group of order $q^{r}$ and $U \rtimes T$ is a Frobenius group. Then the set $\Lambda=M^{U}$ is a set of size $q^{r}$ on which $G^{\Lambda}$ acts 2 -transitively. By Lemma 2.1.1, $M$ does not contain a section isomorphic to $\operatorname{Alt}(t)$ for $t>6$. Thus, we conclude that $\Lambda$ is a beautiful set, and Lemma 1.6 .12 yields the result.

Lemma 4.3.4. Suppose that $G$ is almost simple with socle equal to $\operatorname{PSL}_{n}(q)$ and $n$ is an odd prime. Let $M$ be the normalizer of a Singer cycle in $G$. Then the action of $G$ on cosets of $M$ is not binary.

Proof. We can write the group $F=M \cap \operatorname{PSL}_{n}(q)$ as a semidirect product $T \rtimes C$ where $T$ is cyclic of order $\frac{q^{n}-1}{(q-1)(q-1, n)}$ and $C$ is cyclic of order $n$, and acts fixed-point-freely on $T$. Choosing an appropriate basis we may take $C$ to be generated by a permutation matrix $c$ corresponding to an $n$-cycle, and one sees immediately that $C_{\mathrm{PSL}_{n}(q)}(c)>\langle c\rangle$. Let $x \in C_{\mathrm{PSL}_{n}(q)}(c) \backslash\langle c\rangle$ and observe that the group $F$ acts as a Frobenius group on the set ( $F: F \cap F^{x}$ ).

Since $n>2$, Lemma 1.7 .2 implies that the action of $M$ on $\left(M: M \cap M^{x}\right)$ is not binary. Now Lemma 1.6.1 yields the result.

### 4.3.2 Case $S=\mathrm{SU}_{n}(q)$

Lemma 4.3.5. Suppose that $G$ is almost simple with socle equal to $\operatorname{PSU}_{n}(q)$. Let $M$ be a $\mathcal{C}_{3}$-subgroup. Then the action of $G$ on $(G: M)$ is not binary.

Proof. Note that $F^{*}(M)$ contains a normal subgroup $M_{1}$ which is a quasisimple cover of $\operatorname{PSU}_{m}\left(q^{r}\right)$ and, by [54, Table 3.5.B], $r \geq 3$.

First suppose that $m \geq 5$. Here we refer to Lemma 2.2 .8 and we write elements of $M_{1}$ with respect to the basis $\mathcal{B}$ of $V_{m}\left(q^{2 r}\right)$ in that lemma. Define $j=\left\lfloor\frac{m-1}{2}\right\rfloor$ and $y=(m, 2)$, so that $m=2 j+y$. Then let

$$
\tilde{x}=\left(\begin{array}{ccccc}
1 & & & & \\
& A & & & \\
& & 1 & & \\
& & & \overline{A^{-T}} & \\
& & & & J_{y}
\end{array}\right) \text {, }
$$

where $A$ is an element of $\mathrm{GL}_{j-1}\left(q^{2 r}\right)$ of order $\left(q^{2 r}\right)^{j-1}-1$ and $J_{y}$ is a $y$-by- $y$ matrix chosen so that $\tilde{x}$ is an element of $\mathrm{SU}_{m}\left(q^{r}\right)$. Observe that 1 is not an eigenvalue for $J_{y}$. Now we let $x$ be the element of $M_{1}$ that is an image of $\tilde{x}$.

Observe that $\tilde{x}$ has a 1 -eigenspace over $\mathbb{F}_{q^{r}}$ of dimension 2, and so has a 1-eigenspace over $\mathbb{F}_{q}$ of dimensions $2 r$. From this we conclude that $C_{M}(x)$ is a proper subgroup of $C_{G}(x)$. Let $g \in C_{G}(x) \backslash C_{M}(x)$ and set $H:=M \cap M^{g}$. Then $H$ contains $x$ but does not contain $M_{1}$, and there is a suborbit, $\Delta$, on which the action of $M$ is isomorphic to the action of $M$ on $(M: H)$.

We now refer to Lemma 2.2.8, This shows that $(M: H)$ has a subset $\Delta$ of size $q^{2 r(j-1)}$ such that $M^{\Delta}$ is 2-transitive. Since $M$ does not have a section isomorphic to $\operatorname{Alt}\left(q^{2 r(j-1)}\right)$ by Lemma 2.1.1, it follows that $\Delta$ is a beautiful subset, and the conclusion holds by Lemma 1.6.1.

If $m=4$, then we use the same argument with Lemma 2.2.11 in place of Lemma 2.2.8. Now suppose that $m=3$. Choose a hyperbolic basis, $\mathcal{B}_{0}=\left\{e_{1}, x, f_{1}\right\}$, for a 3 -dimensional Hermitian space $V$ over $\mathbb{K}_{\#}=\mathbb{F}_{q^{2 r}}$ associated with a form $\varphi$. We will use the fact that the isometry group of $\varphi$ contains an element

$$
g:=\left(\begin{array}{lll}
a & & \\
& 1 & \\
& & a^{-1}
\end{array}\right)
$$

where $a$ is an element of $\mathbb{F}_{q^{r}}$ of order $q^{r}-1$. Now we can take $M$ to contain a projective image of the special isometry group of $\varphi$, and let $S$ be the special isometry group of the form $\operatorname{Tr}_{\mathbb{K}_{\#} / \mathbb{K}}(\varphi)$ on $V$, considered as an $\mathbb{F}_{q}$-space.

Set $E=\operatorname{span}_{\mathbb{K}_{\#}}\left\{e_{1}\right\}, F=\operatorname{span}_{\mathbb{K}_{\#}}\left\{f_{1}\right\}$ and $X=\operatorname{span}_{\mathbb{K}_{\#}}\{x\}$ and observe that $\langle E, F\rangle$ is a non-degenerate $2 r$-dimensional $\mathbb{F}_{q}$-subspace of $V$, while $X$ is a non-degenerate $r$-dimensional $\mathbb{F}_{q}$-subspace of $V$. Choose a hyperbolic $\mathbb{F}_{q}$-basis for $\langle E, F\rangle, \mathcal{B}_{1}=\left\{e_{1}, \ldots, e_{r}, f_{1}, \ldots, f_{r}\right\}$, where $e_{1}, \ldots, e_{r} \in E$ and $f_{1}, \ldots, f_{r} \in F$. Let $\mathcal{B}_{2}$ be a hyperbolic basis $\mathbb{F}_{q}$-basis for $X$ and assume that $\mathcal{B}_{2}$ contains elements $e_{r+1}, f_{r+1}$ such that $\left(e_{r+1}, f_{r+1}\right)$ is a hyperbolic pair. Observe that $\mathcal{B}=\mathcal{B}_{1} \cup \mathcal{B}_{2}$ is a hyperbolic $\mathbb{F}_{q}$-basis for $V$.

Define a group $U$ in $S$ whose elements fix all elements in $\mathcal{B}$ except $e_{1}, \ldots, e_{r}$ and $f_{r+1}$, and which satisfy

$$
\begin{aligned}
& e_{i} \mapsto e_{i}+c_{i} e_{r+1} \text { for } 1 \leq i \leq r, \\
& f_{r+1} \mapsto f_{r+1}-c_{1} f_{1}-c_{2} f_{2}-\cdots-c_{r} f_{r} \quad\left(c_{1}, \ldots, c_{r} \in \mathbb{F}_{q}\right) .
\end{aligned}
$$

Now $U$ is of order $q^{r}$ and $\langle g\rangle$ normalizes and acts fixed-point-freely on $U$. What is more, $U$ is not in $M$ (since it contains non-trivial elements with a 1 -eigenspace of dimension at least $2 r+1$ over $\mathbb{F}_{q}$ ). Thus, in the usual way, we obtain $\Delta$, a set of size $q^{r}$ whose stabilizer is 2 -transitive. By Lemma 4.1.1, $M$ does not have a section isomorphic to $\operatorname{Alt}\left(q^{r}\right)$ (recall that $r \geq 3$ here), so $\Delta$ is a beautiful set and Lemma 1.6.12 yields the result.

Suppose now that $m=2$ and $q>2$. Here we proceed in a similar fashion to Lemma 4.3.3, In this case we start with $\mathcal{B}_{1}=\{e, f\}$, a hyperbolic $\mathbb{K}_{\#}$-basis for $V$ with respect to a unitary form $\varphi_{\#}$. Let $\lambda$ be an element of $\mathbb{K}_{\#}$ such that $\mathcal{B}_{2}=\left\{\lambda, \lambda^{q}, \ldots, \lambda^{q^{r-1}}\right\}$ is a basis for $\mathbb{K}_{\#}$ over $\mathbb{K}$. Then taking tensor products of elements of $\mathcal{B}_{1}$ and $\mathcal{B}_{2}$ we obtain a $\mathbb{K}$-basis for $V$. We write $V_{1}$ for the $\mathbb{K}$-span of $\mathcal{B}_{1}$ and $V_{2}$ for the $\mathbb{K}$-span of $\mathcal{B}_{2}$.

Now we take $S$ to preserve the form $\varphi=\operatorname{Tr}_{\mathbb{K}_{\#} / \mathbb{K}}\left(\varphi_{\#}\right)$, and $M$ to preserve $\varphi_{\#}$, so that $M$ is a subgroup of $G$. What is more we observe that

$$
\begin{aligned}
\varphi\left(e \otimes \lambda^{q^{i}}, e \otimes \lambda^{q^{j}}\right) & =0, \\
\varphi\left(f \otimes \lambda^{q^{i}}, f \otimes \lambda^{q^{j}}\right) & =0, \\
\varphi\left(e \otimes \lambda^{q^{i}}, f \otimes \lambda^{q^{j}}\right) & =\operatorname{Tr}_{\mathbb{K}_{\#} / \mathbb{K}}\left(\lambda^{q^{i}+q^{r+j}}\right) .
\end{aligned}
$$

In particular this means that $\varphi$ can be written as a product $\varphi=\varphi_{1} \varphi_{2}$, where $\varphi\left(u_{1} \otimes u_{2}, v_{1} \otimes v_{2}\right)=$ $\varphi_{1}\left(u_{1}, v_{1}\right) \varphi_{2}\left(u_{2}, v_{2}\right)$; here $u_{1}, v_{1} \in V_{1}, u_{2}, v_{2} \in V_{2}, \varphi_{1}=\left.\varphi\right|_{V_{1}}$ and $\varphi_{2}$ is defined by setting $\varphi_{2}\left(\lambda^{q^{i}}, \lambda^{q^{j}}\right)=$ $\operatorname{Tr}_{\mathbb{K}_{\#} / \mathbb{K}}\left(\lambda^{q^{i}+q^{r+j}}\right)$ and extending linearly on $V_{2}$.

With this set-up we see that the group $M$ contains a subfield subgroup isomorphic to the projective image of a group $H=\mathrm{SU}_{2}(q) \times r$ which preserves this tensor product structure (observe that the Frobenius automorphism on $V_{2}$ preserves the form $\left.\varphi_{2}\right)$. Then $H$ lies in a group of the form $K=\mathrm{SU}_{2}(q) \times \mathrm{SU}_{r}(q)$, and it is clear that $H$ is not self-normalizing in $K$.

Now we proceed as before: we obtain a suborbit of $M$ whose action is isomorphic to the action of an almost simple group with socle $\mathrm{PSL}_{2}\left(q^{r}\right)$ on a maximal $\mathcal{C}_{5}$-subgroup containing $\mathrm{PSL}_{2}(q)$; by [45], this action is not binary, and now Lemma 1.6 .2 implies that the action of $M$ on ( $M: M \cap M^{g}$ ) is not binary. Then Lemma 1.6.1 implies that the action of $G$ on $(G: M)$ is not binary, as required.

Suppose next that $m=2$ and $q=2$. Then $S=\operatorname{SU}_{2 r}(2)$. If $r=3$, then the result follows from Lemma 4.1.1. Assume, then, that $r>3$ and notice that $S$ is simple. For convenience we shall work with the group $X=\mathrm{GU}_{2 r}(2)=Z \times S$, where the centre $Z$ has order 3 , and replace $M$ by $Z M$; the centre $Z$ will act trivially on all the sets we consider in the rest of the proof.

We have $N_{X}(M \cap X)=\mathrm{GU}_{2}\left(q_{0}\right) \cdot r$, where $q_{0}=2^{r}$, and this contains a maximal torus $T$ of order $\left(q_{0}+1\right)^{2}$. Then

$$
N_{X}(T) \geq T .(r \times r) \text { and } T .(r \times r) \not \leq M,
$$

while $N_{M}(T) \geq T . r$. Hence there exists $g \in N_{X}(T) \backslash M$ such that $T . r \leq M \cap M^{g}$. Note also that $N:=N_{M}(T) \cap X=T .2 r$. Since $N_{X}(N) \leq M$, it follows that $N \not \subset M \cap M^{g}$, and hence $M \cap M^{g} \cap X=T$.r.

Let $H$ be the subgroup $\mathrm{GU}_{2}\left(q_{0}\right)$ of $M$, so that $H \cap M \cap M^{g}=T$. We consider the action of $H$ on ( $H: T$ ); the kernel of this action is the centre $Z_{0}$ of $H$, of order $q_{0}+1$. Let $V$ be a 2 -space over $\mathbb{F}_{q_{0}^{2}}$ with unitary form (, ) preserved by $H$, and let $v_{1}, v_{2}$ be an orthonormal basis of $V$. Replacing $T$ by a conjugate if necessary, we may take $T$ to be the stabilizer in $H$ of $\left\langle v_{1}\right\rangle$ (hence also of $\left\langle v_{2}\right\rangle$ ). So the action of $H$ on ( $H: T$ ) is equivalent to the action on the set $\Lambda=\{\langle v\rangle:(v, v)=1\}$.

Write $\alpha \rightarrow \bar{\alpha}$ for the involutory automorphism of $\mathbb{F}_{q_{0}^{2}}$. The orbits of the point-stabilizer $H_{\left\langle v_{1}\right\rangle}$ on $\Lambda$ are the singletons $\left\langle v_{1}\right\rangle,\left\langle v_{2}\right\rangle$ and the sets $\Lambda_{\lambda}$, for $\lambda \bar{\lambda} \notin\{0,1\}$, where $\lambda \in \mathbb{F}_{q_{0}^{2}}$ and

$$
\Lambda_{\lambda}=\left\{\langle w\rangle \in \Lambda:(w, w)=1,\left(v_{1}, w\right)=\lambda\right\} .
$$

Note that $\Delta_{\lambda}=\Delta_{\alpha \lambda}$ if and only if $\alpha \bar{\alpha}=1$, so there are precisely $q_{0}-2$ suborbits $\Lambda_{\lambda}$, all of size $q_{0}+1$. In particular, $H_{a b}=Z_{0}$, for any distinct $a, b \in \Lambda$ such that $b \neq a^{\perp}$.

We claim that there exist scalars $\lambda_{1}, \lambda_{2}, \beta \in \mathbb{F}_{q_{0}^{2}}$ with the following properties:
(i) $\lambda_{1} \bar{\lambda}_{1} \neq \lambda_{2} \bar{\lambda}_{2}$ and $\lambda_{i} \bar{\lambda}_{i} \neq 0,1$ for $i=1,2$,
(ii) $\beta \bar{\beta}=1+\lambda_{1} \bar{\lambda}_{1}$,
(iii) $\lambda_{1} \bar{\lambda}_{2} \bar{\beta}=1$.

To see this, first choose $\lambda_{1}$ and $\beta$ such that $\lambda_{1} \bar{\lambda}_{1} \neq 0,1$ and $\beta \bar{\beta}=1+\lambda_{1} \bar{\lambda}_{1}$. Define $\lambda_{2}=\beta^{-1} \bar{\lambda}_{1}^{-1}$. Then setting $y=\lambda_{1} \bar{\lambda}_{1}$, we have

$$
\lambda_{2} \bar{\lambda}_{2}=(\beta \bar{\beta})^{-1}\left(\lambda_{1} \bar{\lambda}_{1}\right)^{-1}=\frac{y^{-1}}{1+y}=\frac{1}{y+y^{2}} .
$$

If this is equal to $y$, then $y^{3}+y^{2}+1=0$; but there is no such $y \in \mathbb{F}_{q_{0}}=\mathbb{F}_{2^{r}}$, as $r \geq 5$ by assumption. Similarly, $\lambda_{2} \bar{\lambda}_{2}$ is not equal to 1 or 0 . Thus (i)-(iii) hold.

Now choose $\gamma \in \mathbb{F}_{q_{0}^{2}} \backslash \mathbb{F}_{q_{0}}$ such that $\gamma \bar{\gamma}=1+\lambda_{2} \bar{\lambda}_{2}$, and define the following four points $a, b, c, d \in \Lambda$ :

$$
\begin{aligned}
a & =\left\langle v_{1}\right\rangle, \\
b & =\left\langle\lambda_{1} v_{1}+\beta v_{2}\right\rangle, \\
c & =\left\langle\lambda_{2} v_{1}+\gamma v_{2}\right\rangle, \\
d & =\left\langle\lambda_{2} v_{1}+\bar{\gamma} v_{2}\right\rangle .
\end{aligned}
$$

We shall show that the triples $(a, b, c)$ and $(a, b, d)$ are 2 -subtuble complete, but not 3 -subtuple complete under the action of $H$.

Since $c, d \in \Lambda_{\lambda_{2}}$, we have $(a, c) \sim(a, d)$. Also $(b, c) \sim(b, d)$ if and only if

$$
\left(\lambda_{1} v_{1}+\beta v_{2}, \lambda_{2} v_{1}+\gamma v_{2}\right)=\nu\left(\lambda_{1} v_{1}+\beta v_{2}, \lambda_{2} v_{1}+\bar{\gamma} v_{2}\right)
$$

for some $\nu \in \mathbb{F}_{q_{0}^{2}}$ satisfying $\nu \bar{\nu}=1$. This is equivalent to the equation

$$
\left(\lambda_{1} \bar{\lambda}_{2}+\beta \bar{\gamma}\right)\left(\bar{\lambda}_{1} \lambda_{2}+\bar{\beta} \gamma\right)=\left(\lambda_{1} \bar{\lambda}_{2}+\beta \gamma\right)\left(\bar{\lambda}_{1} \lambda_{2}+\bar{\beta} \bar{\gamma}\right)
$$

which boils down to $\lambda_{1} \bar{\lambda}_{2} \bar{b}(\gamma+\bar{\gamma})=\bar{\lambda}_{1} \lambda_{2} \beta(\gamma+\bar{\gamma})$. This holds, since $\lambda_{1} \bar{\lambda}_{2} \bar{\beta}=1$ by (iii).
Hence $(a, b, c)$ and $(a, b, d)$ are 2 -subtuple complete. They are clearly not 3 -subtuple complete under the action of $H$, since $H_{a b}=Z_{0}$ which is the kernel of the action on $\Lambda$.

Thus the action of $H$ on $\Lambda$ is non-binary. The same is true when we add field automorphisms to get the group $M=H$.r or $H .(2 r)$ acting on $\Lambda$ : for any non-trivial field automorphism does not fix any of the suborbits $\Lambda_{\lambda}$, and hence $M_{a b}=H_{a b}=Z_{0}$, so $(a, b, c)$ and $(a, b, d)$ are not in the same orbit under the action of $M$.

We have now established that the action of $M$ on $\left(M: M \cap M^{g}\right)$ is not binary. Hence the result follows by Lemma 1.6.1.

Suppose finally that $m=1$. Here we use the method of Lemma 4.3.4; first we write the group $F=M \cap \operatorname{PSU}_{n}(q)$ as a semidirect product $T \rtimes C$, where $T$ is cyclic of order $\frac{q^{n}+1}{(q+1)(q+1, n)}$ and $C$ is cyclic of order $n$, and acts fixed-point-freely on $T$. Proposition 2.4.1 implies that $C_{\mathrm{PSU}_{n}(q)}(c)>\langle c\rangle$ unless $(n, q)=(5,2)$, but this case can be excluded since $M$ is not maximal, see [10, Table 8.20]. Let $x \in C_{\mathrm{PSU}_{n}(q)}(c) \backslash\langle c\rangle$ and observe that the group $F$ acts as a Frobenius group on the set $\left(F: F \cap F^{x}\right)$. Since $n>2$, Lemma 1.7.2 implies that the action of $M$ on $\left(M: M \cap M^{x}\right)$ is not binary. Now Lemma 1.6.1 yields the result.

### 4.3.3 Case $S=\operatorname{Sp}_{n}(q)$

Lemma 4.3.6. Suppose that $G$ is almost simple with socle equal to $\operatorname{PSp}_{n}(q)$ with $n \geq 4$ and $(n, q) \neq(4,2)$, and let $M$ be a $\mathcal{C}_{3}$-subgroup. Then the action of $G$ on $(G: M)$ is not binary.

Proof. There are two cases to consider here, namely $M \triangleright \operatorname{PSp}_{m}\left(q^{r}\right)$ with $m r=n$, and $M \triangleright \operatorname{SU}_{n / 2}(q) / Z(q$ odd).

Consider the first case, where $M$ is almost simple with socle $\operatorname{PSp}_{m}\left(q^{r}\right)$ and $m$ is even, $r$ is prime and $n=m r$. If $m \geq 4$, then we let $x$ be the element in Lemma 2.2.8 (applied to the group $M$ rather than $G)$. Since $x$ has a 1-eigenspace of dimension 2 over $\mathbb{F}_{q^{r}}$, it is easy to see that $C_{G}(x) \backslash C_{M}(x)$ is non-empty, and so we take an element $g$ in $C_{G}(x) \backslash C_{M}(x)$ and appeal to Lemma 2.2.8 to see that the action of $M$ on ( $M: M \cap M^{g}$ ) is not binary. Then Lemma 1.6.1 yields the result.

Now suppose that $m=2$ and $r>2$ (we will deal with the case where $m=r=2$ in the last paragraph of the proof). Here $M \cap S / Z(S)$ is the projective image of $M_{0} \cong \operatorname{Sp}_{2}\left(q^{r}\right) \cdot r$, and we let $H_{0} \cong \mathrm{SL}_{2}(q) \times\langle\sigma\rangle$ be a subfield subgroup of $M_{0}$, where $\sigma$ is a field automorphism of order $r$; this is maximal in $M_{0}$ for $q>2$, see for instance [10, Table 8.1]. We claim that there exists $g \in S \backslash M_{0}$ normalizing $H_{0}$. Once we have shown this, then in a similar manner to the previous paragraph, we obtain a suborbit for which the action is isomorphic to the action of an almost simple group with socle $\mathrm{PSL}_{2}\left(q^{r}\right)$ on a maximal subfield subgroup; then [45] yields the result for $q>2$.

To see the existence of the element $g$ we note that the subfield subgroup $H_{0}$ preserves a tensor product structure on $V$ and so lies in a maximal subgroup $\mathrm{Sp}_{2}(q) \circ I_{r}(q)$, where $I_{r}(q)$ is the isometry group of a symmetric bilinear form having matrix $I_{r}$, the identity. We can choose $g$ in $I_{r}(q)$ of order $r-1$ normalizing the subgroup $\langle\sigma\rangle$ of order $r$ in $I_{r}(q)$. The claim follows.

We also need to deal with the case where $m=2, r>2$ and $q=2$. In this case an element $g$ as above exists, but this time the group $M \cap M^{g}$ is not necessarily maximal in $M$. However, in this case $M \cap M^{g}$ contains $\mathrm{Sp}_{2}(2)$ and has order not divisible by 4 , so the conditions for applying Lemma 1.6.15 to the action $\left(M,\left(M: M \cap M^{g}\right)\right)$ are met with the prime 2 , and the result follows.

Now consider the second case, where $M \triangleright \mathrm{SU}_{n / 2}(q) / Z$ with $q$ odd. For $n \geq 6$, we proceed as in Lemma 4.3.5, taking $x$ to be the element given in Lemma 2.2.8 (for $n / 2 \geq 5$ ), in Lemma 2.2.11 (for $n / 2=4$ ) and in Lemma 2.2.10 (for $n / 2=3$ ). Lemma 1.6.1 implies that, choosing (as we may) $g \in C_{G}(x) \backslash M$, the action of $M$ on $\left(M: M \cap M^{g}\right)$ is not binary and thus the listed lemmas imply that $M$ must contain a section isomorphic to $\operatorname{Sym}(s)$ where

$$
s=\left\{\begin{array}{l}
q, \text { if } n / 2=3 \text { or } 4 ; \\
q^{2\lfloor(n-6) / 4\rfloor, ~ i f ~} n / 2 \geq 5 .
\end{array}\right.
$$

By Lemma 2.1.1, this implies that one of the following holds:
(i) $q=7, n / 2=3$ or 4 ,
(ii) $q \leq 5, n / 2=3$ or 4 .

Using [10, Table 8.5], we see that $\operatorname{Alt}(7)$ is not a section of $\mathrm{PSU}_{3}(7)$. Hence the socle of $G$ is $\mathrm{PSp}_{6}(q)(q \leq 5)$ or $\mathrm{PSp}_{8}(q)(q \leq 7)$. All of these groups apart from $\mathrm{PSp}_{8}(q)(5 \leq q \leq 7)$ are covered by Lemma4.1.1, and the $C_{3}$-actions of the remaining groups were shown to be not binary by a computation using the permutation character method of Lemma 1.8.1.

We are left with the situation when $S=\operatorname{Sp}_{4}(q)$ and $r=2$. Here we need to consider both the case where $M$ is almost simple with socle $\operatorname{PSp}_{2}\left(q^{2}\right)$ (the situation we have postponed above) and the case where $q$ is odd and $M$ contains a subgroup isomorphic to $\mathrm{GU}_{2}(q) .2$. Lemma 4.1.1 and [46] imply that we can assume that $q \geq 7$.

For the case where $M$ has socle $\operatorname{PSp}_{2}\left(q^{2}\right)$, we choose a hyperbolic basis, $\mathcal{B}_{0}=\left\{e_{1}, f_{1}\right\}$, for a 2 dimensional space $V$ over $\mathbb{K}_{\#}=\mathbb{F}_{q^{2}}$ associated with an alternating form $\varphi$. We will use the fact that the isometry group of $\varphi$ contains an element

$$
g:=\left(\begin{array}{ll}
a & \\
& a^{-1}
\end{array}\right),
$$

where $a$ is an element of $\mathbb{F}_{q^{2}}$ of order $(q, 2)(q-1)$. Now we can take $M$ to contain a projective image of
 considered as an $\mathbb{F}_{q}$-space.

Set $E=\operatorname{span}_{\mathbb{K}_{\#}}\left\{e_{1}\right\}, F=\operatorname{span}_{\mathbb{K}_{\#}}\left\{f_{1}\right\}$. Choose a hyperbolic $\mathbb{F}_{q^{-}}$-basis for $V, \mathcal{B}=\left\{e_{1}, e_{2}, f_{2}, f_{1}\right\}$, where $e_{1}, e_{2} \in E$ and $f_{1}, f_{2} \in F$.

Define a group $U$ in $S$ whose elements can be written with respect to $\mathcal{B}$ as

$$
\left(\begin{array}{llll}
1 & & & \\
& 1 & b & \\
& & 1 & \\
& & & 1
\end{array}\right)
$$

for some $b \in \mathbb{F}_{q}$. Now $U$ is of size $q$ and $\langle g\rangle$ normalizes, and acts transitively on the set of non-trivial elements of, $U$. What is more, $U$ is not in $M$ (since it contains non-trivial elements with a 1-eigenspace of dimension 3 over $\mathbb{F}_{q}$ ). We therefore obtain $\Delta$, a set of size $q$ whose stabilizer is 2-transitive. Note that $M$ does not contain a section isomorphic to $\operatorname{Alt}(q-1)$ for $q \geq 7$, thus $\Delta$ is a beautiful set and Lemma 1.6.12 yields the result.

The other case is handled very similarly: here $q$ is odd and $M$ contains $\mathrm{GU}_{2}(q)$. As before we start with a hyperbolic basis $\mathcal{B}_{0}=\left\{e_{1}, f_{1}\right\}$, but this time for a 2-dimensional space $V$ over $\mathbb{K}_{\#}=\mathbb{F}_{q^{2}}$ associated with a unitary $\sigma$-form $\varphi$. We use the same element $g$, and we let $S$ be the special isometry group of the form $\operatorname{Tr}_{\mathbb{K}_{\#} / \mathbb{K}}(\zeta \varphi)$ on $V$, considered as an $\mathbb{F}_{q^{-}}$-space; here $\zeta$ is an element of $\mathbb{K}_{\#}$ that satisfies $\zeta^{\sigma}=-\zeta$. Now the proof proceeds as before.

### 4.3.4 $\quad$ Case $S=\Omega_{n}^{\varepsilon}(q)$

Lemma 4.3.7. Suppose that $G$ is almost simple with socle equal to $\mathrm{P} \Omega_{n}^{\varepsilon}(q)(n \geq 7)$, and let $M$ be a maximal $\mathcal{C}_{3}$-subgroup. Then the action of $G$ on $(G: M)$ is not binary.

Proof. First assume that $n$ is odd, so $q$ is odd. In this case $M$ is almost simple with socle $\Omega_{m}\left(q^{r}\right)$ where $n=m r$. For $m \geq 7$, let $x \in M$ be as in Lemma 2.2.9, then there exists $g \in C_{G}(x) \backslash C_{M}(x)$, and the lemma shows that the action of $M$ on ( $M: M \cap M^{g}$ ) is not binary, giving the conclusion. If $m=5$ we use the isomorphism $\Omega_{5}\left(q^{r}\right) \cong \operatorname{PSp}_{4}\left(q^{r}\right)$ : the element $x \in \operatorname{PSp}_{4}\left(q^{r}\right)$ defined in Lemma 2.2.8 acts as $\operatorname{diag}\left(1, a, a, a^{-1}, a^{-1}\right)$ in $\Omega_{5}\left(q^{r}\right)$, so again there exists $g \in C_{G}(x) \backslash C_{M}(x)$, and the action $\left(M,\left(M: M \cap M^{g}\right)\right)$ is not binary by the lemma. Finally, if $m=3$, the element $x \in \operatorname{PSL}_{2}\left(q^{r}\right)$ defined in Lemma 2.2.3 acts as $\operatorname{diag}\left(1, a^{2}, a^{-2}\right) \in \Omega_{3}\left(q^{r}\right)$, and so again there exists $g$ for which $\left(M,\left(M: M \cap M^{g}\right)\right)$ is not binary.

Next assume that $n$ is even and $S=\Omega_{n}^{\varepsilon}(q)$, where $\varepsilon \in\{+,-\}$. We refer to [54, Tables 3.5.E and 3.5.F] and split into three cases:
(1) $m=n / r \geq 4$ is even and $M$ is of type $\mathrm{O}_{m}^{\varepsilon}\left(q^{r}\right)$;
(2) $q m=q n / 2$ is odd, $r=2$ and $M$ is of type $\mathrm{O}_{n / 2}\left(q^{2}\right)$;
(3) $m=n / 2 \geq 4, r=2$ and $M$ is of type $\mathrm{SU}_{m}(q)$.

Case (1) Assume that $m=n / r \geq 4$ is even and $M$ is of type $\mathrm{O}_{m}^{\varepsilon}\left(q^{r}\right)$. Observe that provided $(m, \varepsilon) \neq(4,+), M$ is almost simple with socle $\mathrm{P} \Omega_{m}^{\varepsilon}\left(q^{r}\right)$.

Proceeding as before, using Lemma 2.2.9, the conclusion follows directly for $m \geq 8$, except for $M$ of type $\Omega_{8}^{-}(9)$ in $S=\Omega_{16}^{-}(3)$. Therefore, we look closer at this embedding. Choose Sylow 13 -subgroups $Q$ of $M$ and $P$ of $S$ such that $Q<P$. Then $|Q|=13, P \cong 13^{2}$. Observe that a Sylow 13-subgroup of the $\mathcal{C}_{1}$-subgroup $\Omega_{8}^{-}(3) \times \Omega_{8}^{+}(3)$ has order $13^{2}$; therefore, we may assume that $P \leq \Omega_{8}^{-}(3) \times \Omega_{8}^{+}(3)$ and $P=P_{-} \times P_{+}$, where $P_{-}$is a Sylow 13 -subgroup of $\Omega_{8}^{-}(3)$ and $P_{+}$is a Sylow 13-subgroup of $\Omega_{8}^{+}(3)$. Thus $13=\left|P_{-}\right|=\left|P_{+}\right|$. Write $P_{-}=\left\langle g_{-}\right\rangle$and $P_{+}=\left\langle g_{+}\right\rangle$. Now, as $Q \leq P$, we may write $Q=\left\langle g_{-}^{i} g_{+}^{j}\right\rangle$, for some positive integers $i$ and $j$. Replacing the original generators of $P_{-}$and $P_{+}$if necessary, we may suppose that $i=j=1$ : observe that neither $i$ nor $j$ cannot be 0 , because a Sylow 13 -subgroup of $\Omega_{8}^{-}(9)$ cannot fix an 8 -dimensional subspace of the underlying vector space for $\Omega_{16}^{-}(3)$. It can be easily verified (for instance with [28] or with magma) that $g_{+}^{4}$ and $g_{+}$are in the same $\Omega_{8}^{+}(3)$-conjgacy class. Therefore, $h:=g_{-} g_{+}^{4}$ and $g:=g_{-} g_{+}$are conjugate in $\Omega_{8}^{-}(3) \times \Omega_{8}^{+}(3)$ and so also in $S$. Moreover, $P=\langle g, h\rangle$. Now, $g h=g_{-}^{2} g_{+}^{5}$ and $\langle g h\rangle=\left\langle(g h)^{2}\right\rangle=\left\langle g_{-}^{4} g_{+}^{10}\right\rangle$. Again, it is easy to verify that $g_{-}$and $g_{-}^{4}$ are in the same $\Omega_{8}^{-}(3)$-conjugacy class and $g_{+}$and $g_{+}^{10}$ are in the same $\Omega_{8}^{+}(3)$-conjugacy class. Therefore, $\langle h\rangle$ and $\langle g h\rangle$ are conjugate in $\Omega_{8}^{-}(3) \times \Omega_{8}^{+}(3)$ and so also in $S$. Summing up, $P=\langle g, h\rangle, Q=\langle g\rangle$ and $\langle g\rangle,\langle h\rangle$ and $\langle g h\rangle$ are $S$-conjugate. Hence $(G, \Omega)$ is not binary in this case by Lemma 1.6.15,

Now consider $m=6$. Here $M$ has socle $P \Omega_{6}^{\varepsilon}\left(q^{r}\right) \cong \operatorname{PSL}_{4}^{\varepsilon}\left(q^{r}\right)$, and we use the usual argument using the elements given in Lemma 2.2.7 (for $\varepsilon=+$ ) and Lemma 2.2.11 for $\varepsilon=-$. These give the conclusion unless $\varepsilon=+$ and $q^{r}=4$. In the latter case, $S=\Omega_{12}^{+}(2)$, which is covered by Lemma 4.1.1.

If $m=4$ and $M$ has socle $\mathrm{P}_{4}^{-}\left(q^{r}\right) \cong \mathrm{PSL}_{2}\left(q^{2 r}\right)$, then we use Lemma 2.2.4. The element $x$ given in that lemma acts in $\Omega_{4}^{-}\left(q^{r}\right)$ as $\operatorname{diag}\left(b, b^{-1},-1,-1\right)$ for some $b$, so has a larger centralizer in $G$ than in $M$, and so the result follows as usual, unless $q^{r}=4$, in which case $S=\Omega_{8}^{-}(2)$, which is covered by Lemma 4.1.1.

Suppose now that $m=4$ and $M$ has socle $\mathrm{P}_{4}^{+}\left(q^{r}\right) \cong \operatorname{PSL}_{2}\left(q^{r}\right) \times \operatorname{PSL}_{2}\left(q^{r}\right)$. Thus $S \cong \Omega_{4 r}^{+}(q)$. Note that, if $r=2$, then [10] implies that $M$ is conjugate by a triality automorphism to a maximal subgroup of the $\mathcal{C}_{2}$-class, hence we know already that the action in this case is not binary. Assume, from here on, that $r \geq 3$. First assume that $q$ is odd. Here $M \cap S=\mathrm{P}_{4}^{+}\left(q^{r}\right) . r \cong\left(\mathrm{PSL}_{2}\left(q^{r}\right) \times \mathrm{PSL}_{2}\left(q^{r}\right)\right) . r$ (see [54, 4.3.14]). Write $M_{0}=M \cap S$, and let $H$ be a maximal subgroup $\Omega_{3}\left(q^{r}\right)$.r of $M_{0}$. Then $H<\Omega_{3 r}(q)<S$, and $C_{S}(H)$ contains a subgroup $\Omega_{r}(q)$. Picking $g \in C_{S}(H) \backslash M$, we have $M_{0}^{g} \cap M_{0}=H$, a maximal subgroup of $M_{0}$. The action of $M_{0}$ on ( $M_{0}: H$ ) is a primitive permutation action of diagonal type, and is not binary by
[106, Proposition 4.1]. Hence the action of $S$ on $\left(S: M_{0}\right)$ is also not binary. To prove the same assertion for $(G: M)$, let $G_{1}=\left\langle M^{g} \cap M, S\right\rangle$, and $M_{1}=M \cap G_{1}$. Then $M^{g} \cap M$ is maximal in $M_{1}$, and the action of $M_{1}$ on $\left(M_{1}: M^{g} \cap M\right)$ is not binary, again by [106, Proposition 4.1]. Hence ( $M,\left(M: M^{g} \cap M\right)$ ) is also not binary, by Lemma 1.6.2, and so $(G,(G: M))$ is not binary by Lemma 1.6.1.

Next consider the case where $q$ is even. Again let $H=\Omega_{3}\left(q^{r}\right) \cong \mathrm{PSL}_{2}\left(q^{r}\right)$ be a diagonal subgroup of $M_{0}=\operatorname{soc}(M) \cong \operatorname{PSL}_{2}\left(q^{r}\right) \times \mathrm{PSL}_{2}\left(q^{r}\right)$, and let $T$ be a cyclic torus of order $q^{r}-1$ in $H$. Then $T$ lies in a subgroup $\Omega_{2}^{+}\left(q^{r}\right)$ of $H$, so is centralized by a subgroup $\Omega_{2 r}^{+}(q)$ of $S$. Pick $g \in C_{S}(T) \backslash M$, so that $T<M^{g} \neq M$. As $T$ centralizes a $2 r$-subspace of $V=V_{4 r}(q)$, it must act on the natural module $V_{4}\left(q^{r}\right)$ for $M^{g}$ with eigenvalues $\left(\lambda, \lambda^{-1}, 1,1\right)$ for some $\lambda \in G F\left(q^{r}\right)$, and so $T$ lies in $q^{r}-1$ nonsingular point-stabilizers $\Omega_{3}\left(q^{r}\right)$ in $M^{g}$. These generate $M^{g}$, so not all of them can lie in $M$. Hence there exists a subgroup $H_{1}=\Omega_{3}\left(q^{r}\right)$ of $M^{g}$ such that $T<H_{1} \not \leq M$. Hence there is a Frobenius group $U T<H_{1}$ of order $q^{r}\left(q^{r}-1\right)$ with $U \not 又 M$, and so in the usual way we obtain a subset $\Delta$ of $\Omega=(G: M)$ such that $G^{\Delta}$ contains the 2-transitive group $\mathrm{AGL}_{1}\left(q^{r}\right)$. Then $\Delta$ is a beautiful subset, unless possibly $\operatorname{Alt}\left(q^{r}-1\right)$ is a section of $M$. Lemma 2.1.1]implies that $q^{r}-1 \leq 6$ and, since $r>2$, we obtain a contradiction as required.

Case (2) Next assume that $r=2$ and $q m=q n / 2$ is odd, and that $M$ has socle $\Omega_{n / 2}\left(q^{2}\right)$. If $m \geq 7$, then we proceed as before using Lemma 2.2 .9 , the result follows except for the embedding $\Omega_{7}(9)$ in $\mathrm{P} \Omega_{14}^{\varepsilon}(3)$. For this embedding we observe that $|G: M|$ is even and $M$ is almost simple. Let $P$ be a Sylow 2-subgroup of $M$, let $Q$ be a Sylow 2-subgroup of $G$ that contains $P$, and let $x$ be an element in $G \backslash M$ that normalizes $P$. Then $\left|M: M \cap M^{x}\right|$ is odd and $M \cap M^{x}$ is core-free. Now Lemma 2.3.1 implies that the action of $M$ on ( $M: M \cap M^{x}$ ) is not binary, and Lemma 1.6 .1 yields the result. If $m=5$, then we use the fact that $\Omega_{5}\left(q^{2}\right) \cong \operatorname{Sp}_{4}\left(q^{2}\right)$ and the result follows using the same method replacing Lemma 2.2.9 with Lemma 2.2.8,

Case (3) Finally assume that $r=2$ and $m=n / 2 \geq 4$, and that $M$ contains a normal subgroup that is a quotient of $\mathrm{SU}_{m}(q)$. Note that when $n=8$, [54] implies that $\varepsilon=+$ while [10] implies that these $\mathcal{C}_{3}$-subgroups of $\mathrm{P} \Omega_{8}^{+}(q)$ are conjugate, via a triality automorphism, to $\mathcal{C}_{1}$-subgroups, hence are already dealt with in [46]. Assume, then, that $n \geq 10$.

We proceed as before: let $x \in M$ be the element given in Lemma 2.2.8, As this has a non-trivial 1-eigenspace, there exists $g \in C_{G}(x) \backslash C_{M}(x)$, and so in the action of $M$ on $\left(M: M \cap M^{g}\right)$, the stabilizer is a subgroup of $M$ containing $x$ but not containing a homomorphic image of $\mathrm{SU}_{m}(q)$. If the action of $G$ on $(G: M)$ is binary, then Lemma 1.6 .1 implies that the action of $M$ on $\left(M: M \cap M^{g}\right)$ is binary and Lemma 2.2 .8 implies that $M$ contains a section isomorphic to $\operatorname{Sym}(s)$ where $s=q^{2(L(m-3) / 2\rfloor)}$. Then Lemma 2.1.1 implies that $(m, q)=(5,2)$ or $(6,2)$. In the former case $S=\Omega_{10}^{-}(2)$, while in the latter case $S=\Omega_{12}^{+}(2)$; both are covered by Lemma 4.1.1. Thus in all cases the action of $M$ on $\left(M: M \cap M^{g}\right)$ is non-binary, and hence the same is true of the action of $G$ on $(G: M)$, completing the proof.

### 4.4 Family $\mathcal{C}_{4}$

In this section, the subgroup $M$ preserves a tensor product. We start with two $\mathbb{K}$-vector spaces, $W_{1}$ and $W_{2}$, of dimension $n_{1}$ and $n_{2}$, respectively, and satisfying $n=n_{1} n_{2}$. Roughly speaking, we identify $V$ with the tensor product $W_{1} \otimes W_{2}$, and $M$ is the subgroup of $G$ that preserves this identification. The list of subgroups is given in Table 4.4.1; the details of their precise structure and embeddings can be found in [54, §4.4].

As in the $\mathcal{C}_{3}$-case we give a geometrical interpretation to the set of cosets of $M$ (which is the set on which we are acting); this will involve defining a tensor product structure on $V$. Let us start with the case where $S=\mathrm{SL}_{n}(q)$.

We begin with a $\mathbb{K}$-linear isomorphism $\phi: W_{1} \otimes W_{2} \rightarrow V$. Let $\Sigma$ be the set of all such isomorphisms, and we observe that two groups act naturally on $\Sigma$ :

1. $\mathrm{GL}\left(W_{1}\right) \circ \mathrm{GL}\left(W_{2}\right)$ acts on $\Sigma$ via $\phi^{g}\left(\mathbf{w}_{\mathbf{1}} \otimes \mathbf{w}_{\mathbf{2}}\right)=\phi\left(\mathbf{w}_{\mathbf{1}}{ }^{g^{-1}} \otimes \mathbf{w}_{\mathbf{2}}{ }^{g^{-1}}\right)$ (and extended linearly);
2. $\mathrm{GL}_{n}(\mathbb{K})$ acts on $\Sigma$ via $\phi^{h}\left(\mathbf{w}_{\mathbf{1}} \otimes \mathbf{w}_{\mathbf{2}}\right)=\left(\phi\left(\mathbf{w}_{\mathbf{1}} \otimes \mathbf{w}_{\mathbf{2}}\right)\right)^{h}$ (and extended linearly).

| case | type | conditions |
| :---: | :---: | :---: |
| $\mathrm{L}^{\epsilon}$ | $\mathrm{GL}_{n_{1}}^{\epsilon}(q) \otimes \mathrm{GL}_{n_{2}}^{\epsilon}(q)$ | $n_{1}<n_{2}$ |
| S | $\mathrm{Sp}_{n_{1}}(q) \otimes \mathrm{O}_{n_{2}}^{\epsilon}(q)$ | $n_{2} \geq 3, q$ odd |
| $\mathrm{O}^{+}$ | $\mathrm{Sp}_{1_{1}}(q) \otimes \mathrm{Sp}_{n_{2}}(q)$ | $n_{1}<n_{2}$ |
| O | $\mathrm{O}_{n_{1}}^{\epsilon_{1}}(q) \otimes \mathrm{O}_{n_{2}}^{\epsilon_{2}}(q)$ | $n_{i} \geq 3, q$ odd |

Table 4.4.1: Maximal subgroups in family $\mathcal{C}_{4}$

As in the $\mathcal{C}_{3}$-case, these two actions commute. Thus we define a tensor product structure on $V$ to be an orbit of the group $\mathrm{GL}\left(W_{1}\right) \circ \mathrm{GL}\left(W_{2}\right)$ on $\Sigma$, and (using commutativity of the actions) we observe that $\mathrm{GL}_{n}(\mathbb{K})$ acts on the set of all tensor product structures on $V$. What is more the stabilizer of this action is the subgroup $M$, hence we have the geometrical interpretation that we require.

Again, just as before, we can replace the word "linear" with the word "semilinear" in the previous paragraph to extend this geometrical interpretation to subgroups of $\Gamma L_{n}(\mathbb{K})$.

For the remaining classical groups, we need to clarify what is meant by a tensor product structure on a vector space equipped with a form. So let us assume that our two vector spaces, $W_{1}$ and $W_{2}$ are equipped with forms $\langle,\rangle_{1}$ and $\langle,\rangle_{2}$, respectively. Now we define

$$
\begin{aligned}
\langle\cdot, \cdot\rangle:\left(W_{1} \otimes W_{2}\right) \times\left(W_{1} \otimes W_{2}\right) & \rightarrow \mathbb{K}, \\
\left(\sum_{i} v_{1}^{i} \otimes v_{2}^{i}, \sum_{j} w_{1}^{j} \otimes w_{2}^{j}\right) & \mapsto \sum_{i, j}\left\langle v_{1}^{i}, w_{1}^{j}\right\rangle_{1}\left\langle v_{2}^{i}, w_{2}^{j}\right\rangle_{2},
\end{aligned}
$$

where $v_{1}^{i}, w_{1}^{j} \in W_{1}$ and $v_{2}^{i}, w_{2}^{j} \in W_{2}$ for all $i$ and $j$. One can check that $\langle$,$\rangle is a well-defined form on W_{1} \otimes W_{2}$. Now our map $\phi: W_{1} \otimes W_{2} \rightarrow V$ carries this form onto the vector space $V$, and we obtain a map to a formed space. Following the same approach as above, we see that there are actions of $\operatorname{Isom}\left(\langle,\rangle_{1}\right) \circ \operatorname{Isom}\left(\langle,\rangle_{2}\right)$ and of $\operatorname{Isom}(\langle\rangle$,$) acting on the set of all such maps; this yields a definition of a tensor product structure on a$ formed space, and provides an embedding of $\operatorname{Isom}\left(\langle,\rangle_{1}\right) \circ \operatorname{Isom}\left(\langle,\rangle_{2}\right)$ in the group $\operatorname{Isom}(\langle\rangle$,$) , as the stabilizer$ of such a tensor product structure. Moreover, in the case where the characteristic $p=2$ and both $\langle,\rangle_{1}$ and $\langle,\rangle_{2}$ are symplectic, the group $\operatorname{Isom}\left(\langle,\rangle_{1}\right) \circ \operatorname{Isom}\left(\langle,\rangle_{2}\right)$ also preserves a quadratic form on $W_{1} \otimes W_{2}$ with associated bilinear form $\langle$,$\rangle , yielding an embedding into O^{+}(V)$. Thus we obtain all the embeddings listed in Table 4.4.1.

In the formed space case, it is useful to observe that if we start with hyperbolic bases $e_{1}, \ldots, f_{1}, \ldots$ for $W_{1}$ and $u_{1}, \ldots, v_{1}, \ldots$ for $W_{2}$, then, by taking pure tensors, we obtain a hyperbolic basis for $W_{1} \otimes W_{2}$; the hyperbolic pairs are

$$
\left(e_{i} \otimes u_{j}, f_{i} \otimes v_{j}\right) \text { and }\left(e_{i} \otimes v_{j}, f_{i} \otimes u_{j}\right)
$$

Similarly, if $W_{1}$ contains a vector $x$ such that $\langle x, x\rangle_{1}=1$, then ( $x \otimes u_{i}, x \otimes v_{i}$ ) is a hyperbolic pair in $W_{1} \otimes W_{2}$; and also if $W_{2,0}$ is a non-degenerate subspace of $W_{2}$, then $x \otimes W_{2,0}$ is a non-degenerate subspace of the tensor product.

The main result of this section is the following. The result will be proved in a series of lemmas.
Proposition 4.4.1. Suppose that $G$ is an almost simple group with socle $\bar{S}=\mathrm{Cl}_{n}(q)$, and assume that
(i) $n \geq 3,4,4,7$ in cases $L, U, S, O$ respectively, and
(ii) $\mathrm{Cl}_{n}(q)$ is not one of the groups listed in Lemma 4.1.1,

Let $M$ be a maximal subgroup of $G$ in the family $\mathcal{C}_{4}$. Then the action of $G$ on $(G: M)$ is not binary.

| Group | Details of action |
| :---: | :--- |
| $\mathrm{SL}_{6}(q)$ | $q \in\{3,4,5\}, n_{1}=2, n_{2}=3: M \triangleright \operatorname{PSL}_{2}(q) \times \operatorname{PSL}_{3}(q)$. |
| $\mathrm{SL}_{12}(2)$ | $n_{1}=3, n_{2}=4: M \triangleright \mathrm{PSL}_{3}(2) \times \mathrm{PSL}_{4}(2)$. |

Table 4.4.2: $\mathcal{C}_{4}-\mathrm{SL}_{n}(q)$ - Cases where a beautiful subset was not found.

### 4.4.1 Case $S=\mathrm{SL}_{n}(q)$

Lemma 4.4.2. In this case either $\Omega$ contains a beautiful subset or else $S$ is listed in Table 4.4.2.
Proof. In this case $M$ contains a normal subgroup isomorphic to $\mathrm{SL}_{n_{1}}(q) \circ \mathrm{SL}_{n_{2}}(q)$ where $n=n_{1} n_{2}$ and we may assume that $2 \leq n_{1}<n_{2}$.

Let $\mathcal{B}_{1}=\left\{u_{1}, \ldots, u_{n_{1}}\right\}$ be a basis for $W_{1}, \mathcal{B}_{2}=\left\{w_{1}, \ldots, w_{n_{2}}\right\}$ a basis for $W_{2}$. Now $\mathcal{B}=\left\{u_{i} \otimes w_{j}\right.$ : all $i, j\}$ is a basis for $W_{1} \otimes W_{2}$, which is mapped to $V$ via a map $\phi$ contained in a tensor product structure $\mathcal{P}$, which is stabilized by $M$.

Assume that $q \geq 7$. Let $T_{1}$ (resp. $T_{2}$ ) be the maximal split torus in GL( $W_{1}$ ) (resp. GL $\left(W_{2}\right)$ ) that is diagonal with respect to $\mathcal{B}_{1}$ (resp. $\mathcal{B}_{2}$ ). Let $T$ be the intersection of the tensor product of $T_{1}$ and $T_{2}$ with the group $S$.

Now we let $U$ be the subgroup whose elements fix all elements of $\mathcal{B}$ except $u_{1} \otimes w_{1}$, where we require

$$
u_{1} \otimes w_{1} \mapsto u_{1} \otimes w_{1}+\alpha u_{1} \otimes w_{2},
$$

for some $\alpha \in \mathbb{F}_{q}$. Note that $U \not Z M$ (consider, for instance, the 1-eigenspace of non-trivial elements of $U$ ), and that $n_{2}$ is necessarily greater than or equal to 3 . Hence the group $H=U \rtimes T$ acts 2 -transitively on the set $\Lambda=\mathcal{P}^{U}:=\{\mathcal{P} u: u \in U\}$, and $|\Lambda|=q$.

Let $G_{1} \cong \mathrm{GL}_{n_{1}-1}(q)$ be the subgroup of GL $\left(W_{1}\right)$ fixing $u_{1}$ and $\left\langle u_{2}, \ldots, u_{n_{1}}\right\rangle$; and let $G_{2} \cong \mathrm{GL}_{n_{2}-2}(q)$ be the subgroup of GL $\left(W_{2}\right)$ fixing $w_{1}, w_{2}$ and $\left\langle w_{3}, \ldots, w_{n_{2}}\right\rangle$. Then $M_{(\Lambda)}$, the pointwise stabilizer of $\Lambda$ in $M$, contains $\left(G_{1} \otimes G_{2}\right) \cap S$ (since this subgroup commutes with $U$ ). It follows that any (non-abelian) simple section of $M^{\Lambda}=M_{\Lambda} / M_{(\Lambda)}$ is isomorphic to a section of $\mathrm{GL}_{2}(q)$. By Lemma 2.1.1, for $q \geq 7$ this precludes the possibility that $M^{\Lambda} \geq \operatorname{Alt}(q-1)$, and we obtain that $\Lambda$ is a beautiful subset; now Lemma 1.6.12 allows us to conclude that there are no such binary actions.

For $q \in\{3,4,5\}$ we exclude the case $\left(n_{1}, n_{2}\right)=(2,3)$ (since this appears on the table), and so we assume that $n_{2} \geq 4$. Now we proceed as before, this time taking $U$ to be the subgroup whose elements fix all elements of $\mathcal{B}$ except $u_{1} \otimes w_{1}$ and

$$
u_{1} \otimes w_{1} \mapsto u_{1} \otimes w_{1}+\alpha u_{1} \otimes w_{2}+\beta u_{1} \otimes w_{3}
$$

for some $\alpha, \beta \in \mathbb{F}_{q}$. Now we take $T_{2}$ to be a maximal torus of $\operatorname{GL}\left(W_{2}\right)$ that preserves the decomposition

$$
\left\langle w_{1}\right\rangle \oplus\left\langle w_{2}, w_{3}\right\rangle \oplus\left\langle w_{4}\right\rangle \oplus \cdots
$$

and induces a Singer cycle on the subspace $\left\langle w_{2}, w_{3}\right\rangle$. Defining $T$ as before, we obtain a beautiful set of size $q^{2}$ unless $\operatorname{Alt}\left(q^{2}-1\right)$ is isomorphic to a section of $\mathrm{GL}_{3}(q)$. By Lemma 2.1.1, $\operatorname{Alt}\left(q^{2}-1\right)$ is not isomorphic to a section of $\mathrm{GL}_{3}(q)$ and hence the result follows.

For $q=2$, [54, Table 3.5.A] allows us to assume that $n_{1}>2$. Now we exclude the case $\left(n_{1}, n_{2}\right)=(3,4)$ (this is in Table 4.4.2), and we conclude that $n_{2} \geq 5$. The argument proceeds as before, taking $U$ to be the subgroup whose elements fix all elements of $\mathcal{B}$ except $u_{1} \otimes w_{1}$ and

$$
u_{1} \otimes w_{1} \mapsto u_{1} \otimes w_{1}+\alpha u_{1} \otimes w_{2}+\beta u_{1} \otimes w_{3}+\gamma u_{1} \otimes w_{4}+\delta u_{2} \otimes w_{5}
$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{F}_{q}$. We obtain a beautiful set of order 16 unless Alt(15) is a isomorphic to a section of $\mathrm{GL}_{4}(2)$. Again, by Lemma 2.1.1, Alt(15) is not isomorphic to a section of $\mathrm{GL}_{4}(2)$ and hence the result follows.

| Group | Details of action |
| :---: | :--- |
| $\mathrm{SU}_{6}(q), \mathrm{SU}_{8}(q)$ | $q \in\{3,4,5\}, n_{1}=2: M \triangleright \operatorname{PSU}_{2}(q) \times \operatorname{PSU}_{n_{2}}(q)$. |
| $\mathrm{SU}_{12}(q)$, | $q \in\{3,4,5\}, n_{1}=3: M \triangleright \operatorname{PSU}_{3}(q) \times \operatorname{PSU}_{4}(q)$. |
| $\operatorname{SU}_{n}(2)$ | $3 \leq n_{1}<n_{2} \leq 5: M \triangleright \operatorname{PSU}_{n_{1}}(2) \times \operatorname{PSU}_{n_{2}}(2)$. |

Table 4.4.3: $\mathcal{C}_{4}-\mathrm{SU}_{n}(q)$ - Cases where a beautiful subset was not found.

### 4.4.2 Case $S=\mathrm{SU}_{n}(q)$

Lemma 4.4.3. In this case either $\Omega$ contains a beautiful subset or else $S$ is listed in Table 4.4.3.
Proof. In this case $M$ contains a normal subgroup isomorphic to $\mathrm{SU}_{n_{1}}(q) \circ \mathrm{SU}_{n_{2}}(q)$ where $n=n_{1} n_{2}$ and we may assume that $2 \leq n_{1}<n_{2}$.

Assume first that $q \geq 7$. Our method here exploits the existence of a Frobenius group inside $\mathrm{SU}_{3}(q)$, as follows: let $W_{2,0}=\left\langle u_{1}, x, v_{1}\right\rangle$ be a non-degenerate 3-dimensional subspace of $W_{2}$ and observe that we have two subgroups of $\mathrm{SU}_{3}(q)$ consisting of elements of the form

$$
\begin{align*}
& U=\left\{\left.\left(\begin{array}{ccc}
1 & b & c \\
& 1 & -b^{q} \\
& & 1
\end{array}\right) \right\rvert\, b, c \in \mathbb{K} \text { with } b^{q+1}+c+c^{q}=0\right\} ;  \tag{4.4.1}\\
& T=\left\{\left.\left(\begin{array}{lll}
r & r^{q-1} & \\
& & r^{-q}
\end{array}\right) \right\rvert\, r \in \mathbb{K}^{\times}\right\} . \tag{4.4.2}
\end{align*}
$$

Then $U \rtimes T$ is a Borel subgroup of $\mathrm{SU}_{3}(q)$.
Now, first, assume that $q$ is odd. Take $U_{0}$ to be the subgroup of $U$ obtained by requiring that $b \in \mathbb{F}_{q}$ and that $c=-\frac{1}{2} b^{2}$; take $y \in W_{1}$ such that $\langle y, y\rangle_{1}=1$. We now define an isomorphic group in $S$ : let $U_{1}$ consist of those elements for which there exists $b \in \mathbb{F}_{q}$ such that

$$
\begin{aligned}
y \otimes u_{1} & \mapsto y \otimes u_{1}+b y \otimes x-\frac{1}{2} b^{2} y \otimes v_{1}, \\
y \otimes x & \mapsto y \otimes x-b y \otimes v_{1}, \\
y \otimes v_{1} & \mapsto y \otimes v_{1},
\end{aligned}
$$

and all elements of $\left\langle y \otimes u_{1}, y \otimes x, y \otimes v_{1}\right\rangle^{\perp}$ are fixed. Then $U_{1}$ is a subgroup of order $q$ that is not contained in $M$.

Similarly, think of $T$ as a subgroup of $\mathrm{SU}\left(W_{2}\right)$ by requiring that it fixes all elements in $W_{2,0}^{\perp}$, and take $T_{0}$ to be the subgroup of $T$ obtained by requiring that $r \in \mathbb{F}_{q}$; let $T_{1}$ be the subgroup in $S$ obtained by tensoring elements of $T_{0}$ with $1 \in \mathrm{SU}\left(W_{1}\right)$. Then $T_{1}$ is a group of order $q-1$ that normalizes $U_{1}$ and acts transitively on the set of non-trivial elements in $U_{1}$.

In the case when $q$ is even, we do similarly; this time $U_{0}$ is the subgroup of $U$ obtained by setting $b=0$ and letting $c$ range through $\mathbb{F}_{q}$, while $T_{0}=T$. Again, $T_{0}$ acts transitively upon the non-trivial elements of $U_{0}$; the same is therefore true of $U_{1}$.

In both cases in the usual way, we set $\Lambda$ to be $\mathcal{P}^{U_{1}}$, where $\mathcal{P}$ is the tensor product structure stabilized by $M$, and we see that $S^{\Lambda}$ acts 2 -transitively upon $\Lambda$, with $\Lambda$ a set of size $q$.

We wish to show that this set is beautiful. As before, we see that $M_{(\Lambda)}$ contains $S \cap\left(\mathrm{GU}_{n_{1}-1}(q) \circ\right.$ $\mathrm{GU}_{n_{2}-3}(q)$ ), where the first factor fixes $y$ and the second fixes $u_{1}, x, v_{1}$. Hence we see that any non-abelian simple section of $M^{\Lambda}$ is isomorphic to a section of $\mathrm{GU}_{3}(q)$. Since $q \geq 7$, by Lemma 2.1.1 this precludes the possibility that $M^{\Lambda} \geq \operatorname{Alt}(q-1)$, and hence $\Lambda$ is a beautiful subset; now Lemma 1.6 .12 allows us to conclude that there are no such binary actions.

For $q \in\{3,4,5\}$ we assume that $n_{2} \geq 5$ (the first two lines of Table 4.4.3 cover $n_{2} \leq 4$ ). We proceed as for $q \geq 7$ but we use the existence of a Frobenius group in $\mathrm{SU}_{5}(q)$ this time. We let $W_{2,0}:=\left\langle u_{1}, u_{2}, x, v_{2}, v_{1}\right\rangle$ be a non-degenerate 5 -subspace of $W_{2}$, and consider the group:

$$
U \rtimes T=\left\langle\left(\begin{array}{ccccc}
1 & a & & & \\
& 1 & & & \\
& & 1 & & \\
& & & 1 & -a^{q} \\
& & & & 1
\end{array}\right), \left.\left(\begin{array}{ccccc}
r & & & & \\
& 1 & & & \\
& & r^{q-1} & & \\
& & & 1 & \\
& & & & r^{-q}
\end{array}\right) \right\rvert\, a, r \in \mathbb{K}, r \neq 0\right\rangle
$$

Now we define $U_{1}$, the subgroup for which there exists $a \in \mathbb{K}$ such that

$$
\begin{aligned}
& y \otimes u_{1} \mapsto y \otimes u_{1}+a y \otimes u_{2}, \\
& y \otimes v_{2} \mapsto y \otimes v_{2}-a^{q} y \otimes v_{1},
\end{aligned}
$$

and which fixes $y \otimes u_{1}, y \otimes v_{1}$, and the orthogonal complement of $\left\langle y \otimes u_{1}, y \otimes u_{2}, y \otimes v_{1}, y \otimes v_{2}\right\rangle$. Note that this group is not contained in $M$, so we define $\Lambda=\mathcal{P}^{U_{1}}$ as before, this time a set of size $q^{2}$.

We take $T_{1}$ to be the group obtained by tensoring elements of $T_{0}$ with $1 \in \mathrm{SU}\left(W_{1}\right)$. Then $T_{1}$ is a group of order $q^{2}-1$ that normalizes $U_{1}$ and acts transitively on $U_{1} \backslash\{1\}$, and as usual we conclude that $S^{\Lambda}$ acts 2 -transitively on $\Lambda$.

Arguing as above we see that that any simple section of $M^{\Lambda}$ must appear as a section of $\mathrm{GU}_{5}(q)$ and so $\Lambda$ is beautiful provided $\operatorname{Alt}\left(q^{2}-1\right)$ is not a section of $\mathrm{GU}_{5}(q)$ - this is true for $q \geq 3$ by Lemma 2.1.1.

Finally for $q=2$, we do as in the previous case, but we use the existence of a 2 -transitive group in $\mathrm{SU}_{6}(q)$ this time. We require that $n_{2} \geq 6$ and we let $W_{2,0}=\left\langle u_{1}, u_{2}, u_{3}, v_{1}, v_{2}, v_{3}\right\rangle$ be a non-degenerate 6 -subspace of $W_{2}$. Now consider the group

$$
U \rtimes L=\left\langle\left.\left(\begin{array}{cccccc}
1 & a & b & & & \\
& 1 & & & & \\
& & 1 & & & \\
& & & 1 & & \\
& & & -a^{q} & 1 & \\
& & & -b^{q} & & \\
& & & & \\
1 & & & \\
& A & & \\
& & 1 & \\
& & & \bar{A}^{-T}
\end{array}\right) \right\rvert\, a, b \in \mathbb{K}, A \in \mathrm{SL}_{2}(\mathbb{K})\right\rangle
$$

where we write $\bar{A}$ to denote the matrix obtained from $A$ by raising each entry to the $q$-th power. Proceeding as before we obtain a beautiful set provided $\operatorname{Alt}\left(2^{4}-1\right)$ does not appear as a section of $\mathrm{GU}_{6}(2)$ - it does not, so we are done.

### 4.4.3 Case where $S$ is symplectic or orthogonal

In all of the remaining cases the formed spaces $W_{1}$ and $W_{2}$ are either symplectic or orthogonal. These are the embeddings in the last three lines of Table 4.4.1. Our strategy is similar to the one already used, namely:

1. We identify a subspace $W_{2,0}$ in $W_{2}$, and we identify a group $U \rtimes T$ in $\operatorname{Isom}\left(W_{2,0}\right)$ for which $T$ acts transitively on the non-identity elements of $U$.
2. If $W_{1}$ is orthogonal, we choose a non-degenerate 1-space $X=\langle x\rangle \subseteq W_{1}$, and identify a subgroup $U_{1}$ of $\operatorname{Isom}(V)$ whose action on $X \otimes W_{2,0}$ is isomorphic to the action of $U$ on $W_{2,0}$, and which fixes the vectors in $\left(X \otimes W_{2,0}\right)^{\perp}$. In particular, since $\operatorname{dim}\left(W_{1}\right)>1$, this means that $U_{1}$ is not a subgroup of $M$. If $W_{1}$ is symplectic, we do soemthing similar, working with a non-degenerate 2 -space $X \subseteq W_{1}$.
3. We define $T_{1}$ to be $1 \otimes T$, and observe that $T_{1}$ normalizes $U_{1}$, and lies in $M$. This then allows us to define $\Lambda=\mathcal{P}^{U_{1}}$, where $\mathcal{P}$ is the tensor product structure stabilized by $M$, and we observe that $S^{\Lambda}$ is 2-transitive.

| Group | Details of action |
| :--- | :--- |
| $\operatorname{Sp}_{n}(q)$ | $q \in\{3,5,7\}, n_{1} \in\{2,4\}, n_{2} \in\{3,4\}: M \triangleright \operatorname{PSp}_{n_{1}}(q) \times \mathrm{P}_{n_{2}}^{\varepsilon}(q)$. |

Table 4.4.4: $\mathcal{C}_{4}-\operatorname{Sp}_{n}(q)$ - Cases where a beautiful subset was not found.

| Group | Details of action |
| :--- | :--- |
| $\Omega_{15}(q)$ | $q \in\{5,7\}, M \triangleright \Omega_{3}(q) \times \Omega_{5}(q)$ |

Table 4.4.5: $\mathcal{C}_{4}-\Omega_{n}(q)$ - Cases where a beautiful subset was not found.
4. We then identify $M_{\Lambda}$ and use this to define a monomorphism from $M^{\Lambda}$ into a small rank classical group, $H$. The proof is complete, by Lemma 1.6.12, provided $M$ does not contain a section isomorphic to $\operatorname{Alt}(|\Lambda|-1)$.

We start by considering the possibilities for $W_{2,0}$ for $q$ not too small. First, suppose that $W_{2}$ is symplectic and contains a subspace $W_{2,0}:=\left\langle u_{1}, u_{2}, v_{2}, v_{1}\right\rangle$, where $\left\langle u_{i}, v_{i}\right\rangle$ are mutually perpendicular hyperbolic pairs. Then with respect to this basis we define

$$
U \rtimes T:=\left\langle\left(\begin{array}{cccc}
1 & a & & \\
& 1 & & \\
& & 1 & -a \\
& & & 1
\end{array}\right), \left.\left(\begin{array}{cccc}
r & & & \\
& 1 & & \\
& & 1 & \\
& & & r^{-1}
\end{array}\right) \right\rvert\, a, r \in \mathbb{F}_{q}, r \neq 0\right\rangle .
$$

Observe that $U \rtimes T<\operatorname{Sp}\left(W_{2}\right)$, where we take this subgroup to fix $W_{2,0}^{\perp}$ pointwise.
Suppose next that $W_{2}$ is orthogonal, in which case $q$ is odd. In some cases we need $U \rtimes T$ to lie in $\Omega\left(W_{2}\right)$, and in such cases we will assume that $W_{2}$ has a non-degenerate subspace $W_{2,0}$ with standard basis $u_{1}, u_{2}, x, v_{2}, v_{1}$. With respect to this basis we define

$$
\left.U \rtimes T:=\left\langle\left(\begin{array}{ccccc}
1 & & a & & -\frac{1}{2} a^{2} \\
& 1 & & & \\
& & 1 & & -a \\
& & & 1 & \\
& & & & 1
\end{array}\right),\left(\begin{array}{llll}
r & & & \\
& s & & \\
& & 1 & \\
& & & s^{-1} \\
\\
& & & \\
& & r^{-1}
\end{array}\right)\right| a, r, s \in \mathbb{F}_{q}, r, s \neq 0, r s \text { a square in } \mathbb{F}_{q}\right\rangle
$$

Note that $U \rtimes T<\Omega\left(W_{2}\right)$ by [12, Lemmas 2.5.7 and 2.5.9].
If we only need $U \rtimes T$ to lie in the special isometry group $\mathrm{SO}\left(W_{2}\right)$, then we take $W_{2,0}=\left\langle u_{1}, x, v_{1}\right\rangle$, and with respect to this basis we define

$$
U \rtimes T:=\left\langle\left(\begin{array}{ccc}
1 & a & -\frac{1}{2} a^{2} \\
& 1 & -a \\
& & 1
\end{array}\right), \left.\left(\begin{array}{lll}
r & & \\
& 1 & \\
& & r^{-1}
\end{array}\right) \right\rvert\, a, r \in \mathbb{F}_{q}, r \neq 0\right\rangle .
$$

Observe that $U \rtimes T<\mathrm{SO}\left(W_{2}\right)$.
Lemma 4.4.4. In this case, if $q \geq 8$, then $\Omega$ contains a beautiful subset.
Proof. Suppose that (relabelling $W_{1}$ and $W_{2}$ if necessary) $W_{2}$ satisfies one of the following possibilities:

1. $W_{2}$ is symplectic of dimension at least 4 ;
2. $W_{2}$ is orthogonal of dimension at least 5 ;

| Group | Details of action |
| :---: | :--- |
| $\Omega_{24}^{+}(2), \Omega_{32}^{+}(2), \Omega_{48}^{+}(2)$ | $4 \leq n_{1}<n_{2} \leq 8: M \triangleright \mathrm{PSp}_{n_{1}}(2) \times \mathrm{PSp}_{n_{2}}(2)$. |
| $\Omega_{16}^{+}(q)$ | $q \in\{3,5,7\}, n_{1}=n_{2}=4: M \triangleright \mathrm{P} \Omega_{4}^{-}(q) \times \mathrm{P} \Omega_{4}^{+}(q)$. |
| $\Omega_{12}^{+}(q)$ | $q \in\{5,7\}, M \triangleright \mathrm{P} \Omega_{3}(q) \times \mathrm{P} \Omega_{4}^{+}(q)$ |
| $\Omega_{18}^{+}(q)$ | $q \in\{5,7\}, M \triangleright \Omega_{3}(q) \times \mathrm{P} \Omega_{6}^{+}(q)$ |

Table 4.4.6: $\mathcal{C}_{4}-\Omega_{n}^{+}(q)$ - Cases where a beautiful subset was not found.

| Group | Details of action |
| :--- | :--- |
| $\Omega_{12}^{-}(q)$ | $q \in\{5,7\}, M \triangleright \Omega_{3}(q) \times \mathrm{P} \Omega_{4}^{-}(q)$ |
| $\Omega_{18}^{-}(q)$ | $q \in\{5,7\}, M \triangleright \Omega_{3}(q) \circ \mathrm{P} \Omega_{6}^{-}(q)$ |

Table 4.4.7: $\mathcal{C}_{4}-\Omega_{n}^{-}(q)$ - Cases where a beautiful subset was not found.
3. $W_{2}$ is orthogonal of dimension 3 or 4 , and there exists $X \leq \operatorname{Isom}\left(W_{1}\right)$ such that $X \otimes \mathrm{SO}\left(W_{2}\right)$ embeds in $S$.

In each of these cases we take $W_{2,0}$ to be the space described above, and $U \rtimes T$ as defined above.
Suppose, first, that $W_{1}$ is not symplectic. Then [54, §4.4] confirms that $q$ is odd, and we take $x$ to be a non-isotropic element of $W_{1}$. Then we proceed as detailed above: so $U_{1}$ is a subgroup of $S$ whose action on $x \otimes W_{2,0}$ is isomorphic to the action of $U$ on $W_{2,0}$, and $T_{1}$ is the group $1 \otimes T$. Setting $\Lambda=\mathcal{P}^{U_{1}}$, we observe that $\Lambda$ is a set of size $q$ such that $S^{\Lambda}$ is 2 -transitive.

Now let $Y_{2}$ be the subgroup of $\operatorname{Isom}\left(W_{2}\right)$ that fixes point-wise the elements of $W_{2,0}$, and let $Y_{1}$ be the subgroup of Isom $\left(W_{1}\right)$ that fixes the element $x$. Then $M_{(\Lambda)}$ contains $S \cap\left(Y_{1} \otimes Y_{2}\right)$; hence any non-abelian simple section of $M^{\Lambda}$ is isomorphic to a section of $\operatorname{Isom}\left(W_{2,0}\right)$. Now $W_{2,0}$ is either symplectic of dimension 4, or orthogonal of dimension 3 or 5 . By Lemma 2.1.1, for $q \geq 9$, $\operatorname{Isom}\left(W_{2,0}\right)$ does not have a section $\operatorname{Alt}(q-1)$. Thus $\Lambda$ is a beautiful subset, and the action is not binary by Lemma 1.6.12,

This argument yields the result except when one of the following holds:
(a) both $W_{1}$ and $W_{2}$ are symplectic;
(b) both $W_{1}$ and $W_{2}$ are orthogonal, and cannot be labeled so that $W_{2}$ satisfies the restrictions stated at the start;
(c) labelling appropriately, $W_{1}$ is symplectic, and $W_{2}$ is orthogonal and does not satisfy the restrictions stated at the start.

We see that situation (b) occurs only if $n_{i}=\operatorname{dim}\left(W_{i}\right) \leq 4$ for $i=1,2$; however [54, 4.4.13] implies that $1 \otimes S O\left(W_{2}\right)<S$ for $n_{2}=3$ or 4 , so in fact case 3 above pertains and we are done. Situation (c) is similarly ruled out, except when $W_{1}$ is symplectic of dimension 2 . Suppose, then, that we are in this case: $W_{1}$ is symplectic of dimension 2, and $W_{2}$ is orthogonal. Again, $q$ is odd here, $S$ is symplectic, and [54, Lemma 4.4.11] implies that $1 \otimes \mathrm{O}\left(W_{2}\right)$ embeds in $S$. We write $W_{1}=\left\langle e_{1}, f_{1}\right\rangle$, and we take $W_{2,0}$ to be the 3 -dimensional subspace of $W_{2}$ described before the statement of the lemma. Then $W_{1} \otimes W_{2,0}$ is a 6 -dimensional non-degenerate symplectic space with a hyperbolic basis given as follows (we omit the tensor sign for clarity, and we list hyperbolic pairs together, starting with the first two):

$$
\left\{e_{1} u_{1}, f_{1} v_{1}, e_{1} v_{1}, f_{1} u_{1}, e_{1} x, f_{1} x\right\}
$$

Now if we consider the group $T_{1}=1 \otimes T$ with respect to this basis, we see that it is diagonal with entries

$$
\left[r, r^{-1}, r^{-1}, r, 1,1\right] .
$$

On the other hand, we can take $U_{0}$ to be the set of elements given with respect to this basis by

$$
\left\{\left.\left(\begin{array}{cccccc}
1 & & & & a & \\
& 1 & & & & \\
& & 1 & & & \\
& & & 1 & & \\
& & & & 1 & \\
& -a & & & & 1
\end{array}\right) \right\rvert\, a \in \mathbb{F}_{q}\right\}
$$

and we take $U_{1}$ to be the subgroup of $S$ which acts like $U_{0}$ on $W_{1} \otimes W_{0,2}$, and fixes the elements in its orthogonal complement. Then $U_{1}$ is not a subgroup of $M$, and is normalized by $T_{1}$, and so we obtain a set $\Lambda=\mathcal{P}^{U_{1}}$ of size $q$ for which $S^{\Lambda}$ is 2 -transitive. Arguing as before, we find that a simple section of $M^{\Lambda}$ is isomorphic to a section of either $\operatorname{Isom}\left(W_{2,0}\right)$ or $\operatorname{Isom}\left(W_{1}\right)$. We obtain, therefore, a beautiful subset, provided $\operatorname{Alt}(q-1)$ is not a section of $\mathrm{O}_{3}(q)$ or $\mathrm{Sp}_{2}(q)$. This is true for $q \geq 7$, and we are done.

Finally, we suppose that situation (a) holds. Here both $W_{1}$ and $W_{2}$ are symplectic, $S=\Omega_{n}^{+}(q)$ and, since $n \geq 8$, we can assume without loss of generality that $\operatorname{dim}\left(W_{2}\right) \geq 4$. In this case, we set $W_{2,0}=\left\langle u_{1}, u_{2}, v_{2}, v_{1}\right\rangle$ as detailed above, and we consider the basis of $\left\langle e_{1}, f_{1}\right\rangle \otimes W_{2,0}$ given by

$$
\left\{-f_{1} u_{1}, e_{1} v_{1},-f_{1} u_{2}, e_{1} v_{2}, f_{1} v_{1}, e_{1} u_{1}, f_{1} v_{2}, e_{1} u_{2}\right\}
$$

Again $T_{1}=1 \otimes T$ is given by the diagonal matrix with entries

$$
\left[r, r^{-1}, 1,1, r^{-1}, r, 1,1\right]
$$

On the other hand, we can take $U_{0}$ to be the set of elements given with respect to the subspace, $Y$, spanned by the first four of these elements

$$
\left\{\left.\left(\begin{array}{cccc}
1 & & a & \\
& 1 & & \\
& & 1 & \\
& -a & & 1
\end{array}\right) \right\rvert\, a \in \mathbb{F}_{q}\right\}
$$

and we take $U_{1}$ to be the subgroup of $S$ which acts like $U_{0}$ on $Y$, and fixes the elements in its orthogonal complement. Then $U_{1}<\mathrm{SO}_{n}^{+}(q)$.

Now $U_{1}$ is not a subgroup of $M$, is normalized by $T_{1}$, and so we obtain a set $\Lambda=\mathcal{P}^{U_{1}}$ of size $q$ for which $S^{\Lambda}$ is 2-transitive. Arguing as before, we find that a simple section of $M^{\Lambda}$ is isomorphic to a section of either $\operatorname{Isom}\left(W_{2,0}\right)$ or $\operatorname{Isom}\left(\left\langle e_{1}, f_{1}\right\rangle\right)$. We obtain, therefore, a beautiful subset, provided $\operatorname{Alt}(q-1)$ is not a section of $\operatorname{Sp}_{4}(q)$. This is true for $q \geq 8$, and we are done.

Lemma 4.4.5. In this case, if $q \leq 7$, then $\Omega$ contains a beautiful subset or else $S$ is listed in Tables 4.4.4, 4.4.5, 4.4.6 or 4.4.7.

Proof. Let us suppose first that $W_{1}$ and $W_{2}$ are symplectic, and so $S=\Omega_{n}^{+}(q)$; then [54, Table 4.4.A] implies that we can assume that $n_{2}>n_{1}$.

Suppose, first, that $n_{2}=4$; then $n_{1}=2$ and $n=8$. Now [10, Table 8.50] confirms that no $\mathcal{C}_{4}$-maximal subgroups exist when $q$ is even. What is more, when $q$ is odd, all $\mathcal{C}_{4}$-subgroups are conjugate, via a triality automorphism, to certain maximal $\mathcal{C}_{1}$-subgroups; then [46, Proposition 4.6] asserts that $\Omega$ contains a beautiful subset.

Suppose from here on that $n_{2} \geq 6$. If $q>2$, then the procedure is virtually identical to that in the previous lemma, but this time we build a beautiful set of size $q^{2}$. To do this we start with a 6 -dimensional
subspace of $W_{2}$ : define $W_{2,0}=\left\langle u_{1}, u_{2}, u_{3}, v_{1}, v_{2}, v_{3}\right\rangle$. Then with respect to this basis we define

$$
U \rtimes T:=\left\langle\left(\begin{array}{cccccc}
1 & a_{1} & a_{2} & & &  \tag{4.4.3}\\
& 1 & & & & \\
& & 1 & & & \\
& & & 1 & & \\
& & & -a_{1} & 1 & \\
& & & -a_{2} & & 1
\end{array}\right), \left.\left(\begin{array}{cccc}
1 & & & \\
& A & & \\
& & 1 & \\
& & & A^{-T}
\end{array}\right) \right\rvert\, a_{1}, a_{2} \in \mathbb{F}_{q}, A \in \operatorname{GL}_{2}(q)\right\rangle .
$$

Observe that $U \rtimes T<\operatorname{Sp}\left(W_{2}\right)$. We set $T_{1}=1 \otimes T$, and we set $U_{1}$ to be the set of elements given by the same matrices as $U$ above, but with respect to the basis

$$
\left\{e_{1} u_{1}, e_{1} u_{2}, e_{1} u_{3}, f_{1} v_{1}, f_{1} v_{2}, f_{1} v_{3}\right\}
$$

and fixing the elements in the orthogonal complement. As before we obtain a beautiful subset of size $q^{2}$, provided $\operatorname{Alt}\left(q^{2}-1\right)$ is not a section of $\operatorname{Sp}_{6}(q)$. This is true for $q \geq 3$.

Now assume that $q=2$, in which case [54, Table 3.5.E] implies that $n_{1}>2$. We proceed as in the previous paragraph, but this time, we assume that $n_{2} \geq 10$, and we take $T$ to be a group isomorphic to $\mathrm{GL}_{4}(q)$. We obtain a beautiful subset of size $q^{4}=16$, provided $\operatorname{Alt}\left(q^{4}-1\right)=\operatorname{Alt}(15)$ is not a section of $\mathrm{Sp}_{10}$ (2); it is not (by Lemma 2.1.1), so the result follows. The exceptions occur when $4 \leq n_{1}<n_{2} \leq 8$, and are listed in Table 4.4.6. This completes the case where both $W_{1}$ and $W_{2}$ are symplectic.

Suppose now that $W_{1}$ is orthogonal. Then $q$ is odd, so $q \in\{3,5,7\}$.
First, assume that $W_{2}$ is symplectic of dimension at least 6 . As $W_{1}$ is orthogonal, it contains a nonisotropic vector $x$. We define $U \rtimes T$ exactly as for (4.4.3). We let $T_{1}:=1 \otimes T$, and we set $U_{1}$ to be the set of elements given by the same matrices as $U$ above, but with respect to the basis

$$
\left\{x u_{1}, x u_{2}, x u_{3}, x v_{1}, x v_{2}, x v_{3}\right\},
$$

and fixing the elements in the orthogonal complement. As before we obtain a beautiful subset of size $q^{2}$, provided $\operatorname{Alt}\left(q^{2}-1\right)$ is not a section of $\mathrm{Sp}_{6}(q)$ (which is true, by Lemma 2.1.1, since $q \geq 3$ ).

Now assume that $W_{2}$ is symplectic of dimension 2 or 4 , and also that $\operatorname{dim} W_{1} \geq 5$. To keep notation consistent, relabel $W_{2}$ as $W_{1}$ and vice versa. Then $W_{2}$ is orthogonal of dimension at least 5 , and we define $W_{2,0}=\left\langle u_{1}, u_{2}, x, v_{1}, v_{2}\right\rangle$. Then with respect to this basis we define

$$
U \rtimes T:=\left\langle\left(\begin{array}{ccccc}
1 & & a_{1} & -\frac{1}{2} a_{1}^{2} &  \tag{4.4.4}\\
& 1 & a_{2} & & -\frac{1}{2} a_{2}^{2} \\
& & 1 & -a_{1} & -a_{2} \\
& & & 1 & \\
& & & & 1
\end{array}\right), \left.\left(\begin{array}{ccc}
A & & \\
& 1 & \\
& & A^{-T}
\end{array}\right) \right\rvert\, a_{1}, a_{2} \in \mathbb{F}_{q}, A \in \operatorname{GL}_{2}(q)\right\rangle .
$$

As usual we set $T_{1}=1 \otimes T$. To define $U_{1}$ we let $e_{1}, f_{1}$ be a hyperbolic pair in $W_{1}$, and we consider the space

$$
W_{2,0}^{\prime}=\left\langle e_{1} u_{1}, e_{1} u_{2}, e_{1} x, f_{1} x, f_{1} v_{1}, f_{1} v_{2}\right\rangle,
$$

which we observe is a non-degenerate symplectic 6 -space. We define $U_{1}$ to act as $1 \otimes U$ on $W_{2,0}^{\prime}$, and to fix $W_{2,0}^{\prime \perp}$. Now, proceeding as above we obtain a beautiful subset of size $q^{2}$, provided $\operatorname{Alt}\left(q^{2}-1\right)$ is not a section of $\mathrm{O}_{5}(q)$ (which is true, by Lemma 2.1.1, since $q \geq 3$ ).

The previous two paragraphs cover all cases where $S=\operatorname{Sp}_{n}(q)$, since Table4.4.4 contains the remaining cases with $n_{1}, n_{2} \leq 4$.

Finally, suppose that $W_{1}$ and $W_{2}$ are both orthogonal. Recall that $q \in\{3,5,7\}$.
Assume $n_{1}$ and $n_{2}$ are both even. Then [54, 4.4.2, 4.4.13] implies that $S=\Omega_{n}^{+}(q)$, and that $1 \otimes \mathrm{SO}\left(W_{2}\right)$ and $\mathrm{SO}\left(W_{1}\right) \otimes 1$ both embed into $S$. We suppose now that $n_{2} \geq n_{1}$ with $n_{2} \geq 6$. Then we define $W_{2,0}, U$
and $T$ via (4.4.4). We set $T_{1}=1 \otimes T$, and we set $U_{1}$ to be the set of elements given by the same matrices as $U$ above, but with respect to the basis

$$
\left\{y u_{1}, y u_{2}, y x, y v_{1}, y v_{2}\right\}
$$

(where $y$ is an anisotropic element of $W_{1}$ ), and fixing the elements in the orthogonal complement. As before we obtain a beautiful subset of size $q^{2}$, provided $\operatorname{Alt}\left(q^{2}-1\right)$ is not a section of $\mathrm{SO}_{5}(q)$. This is true, by Lemma 2.1.1, for $q \geq 3$. This leaves the case where $n_{1}=n_{2}=4$, which is in Table 4.4.6,

Notice that the same argument works if $n_{1}$ is even and $n_{2}$ is odd with $n_{2} \geq 5$ (again using the fact, given in [54, 4.4.13], that $1 \otimes \mathrm{SO}\left(W_{2}\right)$ embeds into $\left.S\right)$. Again we obtain a beautiful subset since $q \geq 3$.

We are left with the possibility that both $n_{1}$ and $n_{2}$ are odd (in which case we may assume that $3 \leq n_{1}<n_{2}$ ), or (relabelling if necessary), that $n_{1}=3$ and $n_{2}$ is even. In this case we assume now that $n_{2} \geq 7$. Now the argument of the previous paragraph works except that we cannot assume that $1 \otimes \mathrm{SO}\left(W_{2}\right) \leq S$; this means that we must adjust the definition of $W_{2,0}$ to ensure that $T \leq \Omega\left(W_{2}\right)$. To do this we define $W_{2,0}=\left\langle u_{1}, u_{2}, u_{3}, x, v_{1}, v_{2}, v_{3}\right\rangle$; we take $U$ to be identical to that given in (4.4.4), except that we prescribe that the $7 \times 7$-matrices of $U$ fix $u_{3}$ and $v_{3}$; then we define

$$
T=\left\{\left.\left(\begin{array}{ccccc}
A & & & & \\
& s & & & \\
& & 1 & & \\
& & & A^{-T} & \\
& & & & s^{-1}
\end{array}\right) \right\rvert\, s \in \mathbb{F}_{q}^{*}, A \in \mathrm{GL}_{2}(q) \text { and } s . \operatorname{det}(A) \text { is a square }\right\} .
$$

Now the argument proceeds as before.
The cases not yet covered have $n_{1}=3, n_{2}=4,5,6$, and are listed in Tables 4.4.6 and 4.4.7. Note that [54, Tables 3.5.D, 3.5.E and 3.5.F] imply that if $n_{1}=3$, then we can exclude $q=3$.

### 4.4.4 The remaining cases

The remaining cases are dealt with by the following result.
Lemma 4.4.6. If the action is listed in Tables 4.4.2, 4.4.3, 4.4.4, 4.4.5, 4.4.6 or 4.4.7, then it is not binary.
Proof. The socle of $M$ is a direct product, as given by the tables. Our method for most cases is as follows: suppose that the action of $G$ on $(G: M)$ is binary. In every case we can see that $|G: M|$ is even. Thus, given a Sylow 2-subgroup $P$ of $M$, there exists an element $x$ of order a power of 2 in $G \backslash M$ that normalizes $P$. Then $\left|M: M \cap M^{x}\right|$ is odd and $M \cap M^{x}$ is core-free in $M$. Now a magma computation shows that every faithful transitive action of odd degree of a group $M$, with socle as given in one of the tables, is not binary. Hence $\left(M,\left(M: M \cap M^{x}\right)\right)$ is not binary, and the conclusion follows by Lemma 1.6.1.

In a couple of cases where the magma computation required too much time we have, instead, found a suitable group $H<M$ with the property that $N_{G}(H)$ is not contained in $M$. This guarantees that there is a suborbit of $M$ for which the stabilizer contains $H$. Now we use magma to show that the action of $M$ on such a suborbit is not binary, and the result follows, again, by Lemma 1.6.1.

### 4.5 Family $\mathcal{C}_{5}$

In this case $M$ is a "subfield subgroup": let $\mathbb{F}_{q_{0}}$ be a subfield of $\mathbb{K}$ (with $|\mathbb{K}|=q_{0}^{r}$ for some prime $r$ ), and let $\mathcal{B}$ be a basis of $V$. Then $V_{0}=\operatorname{span}_{\mathbb{F}_{0}}(\mathcal{B})$ is an $n$-dimensional $\mathbb{F}_{q_{0}}$-vector space. The group $\mathrm{GL}_{n}(q)$ acts naturally on the set of all such vector spaces and $M$ can be taken to be a subgroup of the stabilizer of $V_{0}$ in this action.

When $G$ is not $\mathrm{SL}_{n}(q)$, the group $S$ is a set of isometries for some non-degenerate form $\varphi$ on $V$. Now we require that $M$ is also a subset of the set of isometries of the form $\varphi_{0}$ on $V_{0}$ which is the restriction
of $\varphi$ (or a scalar multiple of $\varphi$ ); full details are given in [54, §4.5]. We list the embeddings in Table 4.5.1. Note that the subfield subgroups are centralized by outer automorphisms of $S$ (see Proposition 2.5.1), so $M$ may not be almost simple. We will need to take account of this possibility in the proofs below.

| case | type | conditions |
| :---: | :---: | :---: |
| L | $\mathrm{GL}_{n}\left(q^{1 / r}\right)$ |  |
| S | $\mathrm{Sp}_{n}\left(q^{1 / r}\right)$ |  |
| $\mathrm{O}^{\epsilon}$ | $\mathrm{O}_{n}^{\delta}\left(q^{1 / r}\right)$ | $\epsilon=\delta^{r}$ |
| U | $\mathrm{GU}_{n}\left(q^{1 / r}\right)$ | $r$ odd |
| U | $\mathrm{O}_{n}^{\epsilon}(q)$ | $r=2, q$ odd |
| U | $\mathrm{Sp}_{n}(q)$ | $r=2, n$ even |

Table 4.5.1: Maximal subgroups in family $\mathcal{C}_{5}$
The main result of this section is the following. The result will be proved in a series of lemmas.
Proposition 4.5.1. Suppose that $G$ is an almost simple group with socle $\bar{S}=\mathrm{Cl}_{n}(q)$, and assume that
(i) $n \geq 3,4,4,7$ in cases $L, U, S, O$ respectively, and
(ii) $\mathrm{Cl}_{n}(q)$ is not one of the groups listed in Lemma 4.1.1,

Let $M$ be a maximal subgroup of $G$ in the family $\mathcal{C}_{5}$. Then the action of $G$ on $(G: M)$ is not binary.

### 4.5.1 Case $S=\operatorname{SL}_{n}(q)$

Lemma 4.5.2. In this case either $\Omega$ contains a beautiful subset or else $S$ is listed in Table 4.5.2.
Proof. Let $\mathcal{B}=\left\{v_{1}, \ldots, v_{n}\right\}$ be a basis for $V$, and we assume that $M$ stabilizes the $\mathbb{F}_{q_{0}}$-span of $\mathcal{B}$. We use Lemma 1.6 .10 for which we need to exhibit two subgroups, as follows.

We set $A \cong \mathrm{SL}_{n-2}\left(q_{0}\right)$ to be the subgroup of $M$ that fixes $v_{n-1}$ and $v_{n}$; we let $B_{0} \cong \mathrm{SL}_{n-1}\left(q_{0}\right)$ be the subgroup of $M$ which fixes $v_{n}$. Then let $g \in C_{G}(A)$ such that $v_{n-1}^{g}$ is not in the $\mathbb{F}_{q_{0}}$-span of $\mathcal{B}$; we set $B=B_{0}^{g}$ and note that $B \not \leq M$. Now Lemma 1.6 .10 implies that there is a subset $\Delta$ of $\Omega$ such that $|\Delta|=q_{0}^{n-2}$ and $G_{\Delta}$ acts 2-transitively on $\Delta$.

If $\Delta$ is a beautiful subset, then Lemma 1.6 .12 yields the result; if $\Delta$ is not a beautiful subset, then $\operatorname{Alt}\left(q_{0}^{n-2}\right)$ must be a section of $\mathrm{SL}_{n}(q)$. By Lemma 2.1.1, this is impossible unless $\left(n, q_{0}\right) \in\{(4,2),(5,2)\}$ or $n=3, q_{0} \leq 7$.

Consider the remaining situations, and set $A \cong \operatorname{SL}_{n-1}\left(q_{0}\right)$ to be the subgroup of $M$ that fixes $v_{n}$. Let $g$ be the diagonal matrix with entries

$$
\left(\lambda, \ldots, \lambda, \lambda^{-n+1}\right)
$$

where $\lambda$ is an element of $\mathbb{F}_{q} \backslash \mathbb{F}_{q_{0}}$ such that $\lambda^{n} \notin \mathbb{F}_{q_{0}}$; this is possible unless $(n, q)=(3,4)$. Setting $B=M^{g}$, and applying Lemma 1.6 .10 yields a set $\Delta$ as above, except that this time $|\Delta|=q_{0}^{n-1}$. Again we obtain a beautiful subset unless $\operatorname{Alt}\left(q_{0}^{n-1}\right)$ is a section of $\operatorname{SL}_{n}(q)$; we conclude that $n \leq 4$ and $q_{0}=2$.

Lemma 4.5.3. If $S$ is listed in Table 4.5.2, then the action is not binary.

| Group $S$ | Details of action |
| :--- | :--- |
| $\mathrm{SL}_{3}\left(2^{r}\right)$ | $r$ prime, $M \triangleright \mathrm{SL}_{3}(2)$. |
| $\mathrm{SL}_{4}\left(2^{r}\right)$ | $r$ prime, $M \triangleright \mathrm{SL}_{4}(2)$. |

Table 4.5.2: $\mathcal{C}_{5}-\mathrm{SL}_{n}(q)$ - Cases where a beautiful subset was not found.

| Group $S$ | Details of action |
| :--- | :--- |
| $\mathrm{SU}_{n}\left(q_{0}^{r}\right)$ | $n \in\{4,5\}, q_{0} \in\{2,3,4,5,7\}, r$ odd prime, $M \triangleright \mathrm{PSU}_{n}\left(q_{0}\right)$ |
| $\mathrm{SU}_{8}(3)$ | $M \triangleright \mathrm{P} \Omega_{8}^{ \pm}(3), \mathrm{PSp}_{8}(3)$ |
| $\mathrm{SU}_{8}(2)$ | $M \triangleright \mathrm{PSp}_{8}(2)$ |
| $\mathrm{SU}_{7}(3)$ | $M \triangleright \mathrm{P}_{7}(3)$ |
| $\mathrm{SU}_{6}(2)$ | $M \triangleright \mathrm{PSp}_{6}(2)$ |
| $\mathrm{SU}_{6}(q)$ | $q \in\{3,5,7\}, M \triangleright \mathrm{P}_{6}^{-}(q)$ |
| $\mathrm{SU}_{5}(q)$ | $q \in\{3,5,7\}, M \triangleright \Omega_{5}(q)$ |
| $\mathrm{SU}_{4}(q)$ | $q \in\{2,3,4,5,7\}, M \triangleright \operatorname{PSp}_{4}(q)$ |
| $\mathrm{SU}_{4}(q)$ | $q \in\{3,5,7\}, M \triangleright \mathrm{P} \Omega_{4}^{ \pm}(q)$ |

Table 4.5.3: $\mathcal{C}_{5}-\mathrm{SU}_{n}(q)$ - Cases where a beautiful subset was not found.

Proof. Here $n \in\{3,4\}, S=\operatorname{SL}_{n}\left(2^{r}\right)$ with $r$ a prime, and $M$ contains a normal subgroup $\mathrm{SL}_{n}(2)$. If $r \in\{2,3\}$, then Lemma 4.1.1 yields the result.

Assume from here on that $r \geq 5$. We have $M=M_{0} \times\langle\phi\rangle$, where $M_{0} \cong \operatorname{SL}_{n}(2) . a$ with $a \in\{1,2\}$, and $\phi$ is either 1 or a field automorphism of $S$ of order $r$.

Let $Q$ be a Sylow 2-subgroup of $M_{0}$. As $|G: M|$ is even, there exists $g \in N_{G}(Q) \backslash M$. Then $M_{0} \cap M_{0}^{g}$ contains $Q$, hence is a parabolic subgroup $P$ of $M_{0}$, and $N_{M_{0}}(P)=P$. It follows that $M \cap M^{g}=P \times\langle\sigma\rangle$, where $\sigma=1$ or $\phi$. In particular, $\sigma$ is in the kernel of the action of $M$ on $\left(M: M \cap M^{g}\right)$. Hence this action is isomorphic to either $\left(M_{0},\left(M_{0}: P\right)\right)$ or $\left(M_{0} \times\langle\phi\rangle,\left(M_{0} \times\langle\phi\rangle: P\right)\right)$. Lemma 2.3.1 shows that the first action is not binary, and it follows using Lemma 1.6 .2 that the second action is also not binary.

### 4.5.2 Case $S=\operatorname{SU}_{n}(q)$

Note that we are assuming that $n \geq 4$, since the case where $S=\mathrm{SU}_{3}(q)$ is covered in [45]. Note, though, that an inspection of the proof [45] shows up a missing case when $q_{0}=2$. Let us deal with that case now.

Lemma 4.5.4. Suppose that $S=\mathrm{SU}_{3}\left(2^{r}\right)$ with $r$ an odd prime, and that $M$ is a subfield subgroup of $G$ containing $\mathrm{PSU}_{3}(2)$. Then the action of $G$ on $(G: M)$ is not binary.

Proof. We use magma, first, to confirm the result when $r=3$. For the rest of the proof we suppose $r \geq 5$. We have $M=M_{0} \times\langle\phi\rangle$, where $M_{0} \in\left\{\operatorname{PSU}_{3}(2), \mathrm{PSU}_{3}(2) .2, \mathrm{PGU}_{3}(2), \mathrm{PGU}_{3}(2) .2\right\}$ and $\phi$ is either 1 or a field automorphism of $S$ of order $r$. Another magma computation confirms that all non-trivial odd-degree core-free actions of $M_{0}$ are not binary.

The proof is now similar to that of the previous lemma. Let $Q \in \operatorname{Syl}_{2}\left(M_{0}\right)$ and $g \in N_{G}(Q) \backslash M$. Then $Q \leq M_{0} \cap M_{0}^{g} \leq Q\langle h\rangle$, where $h$ has order 1 or 3 . If $M \cap M^{g} \not \leq M_{0}$, then $M \cap M^{g}$ contains an element $h^{i} \phi$ for some $i$, and hence also contains $\phi$ (as $\phi$ has order $r>3$ ). Thus $M \cap M^{g}=\left(M_{0} \cap M_{0}^{g}\right) \times\langle\sigma\rangle$, where $\sigma=1$ or $\phi$. Now we complete the proof as in Lemma 4.5.3.

Lemma 4.5.5. In this case either $\Omega$ contains a beautiful subset or else $S$ is listed in Table 4.5.3.
Proof. Our proof splits into two cases, depending on whether $r$ is odd or even. Suppose, first, that $r$ is odd. In this case we let $\mathcal{B}=\left\{e_{1}, \ldots, e_{m}, x, f_{1} \ldots, f_{m}\right\}$ be a hyperbolic basis for $V$ (for $n$ even we do not need the element $x$ ); then $\varphi_{0}$, the restriction of $\varphi$ to the $\mathbb{F}_{q_{0}^{2}}$-span of $\mathcal{B}$, is unitary.

First assume that $m \geq 3$, in which case we will use Lemma 1.6.10. We start by defining $A \cong \operatorname{SL}_{m-1}\left(q_{0}^{2}\right)$ to be the set of elements stabilizing the $\mathbb{F}_{q_{0}^{2}}$-subspaces

$$
\left\langle e_{1}, \ldots, e_{m-1}\right\rangle,\left\langle e_{m}\right\rangle,\left\langle f_{1}, \ldots, f_{m-1}\right\rangle,\left\langle f_{m}\right\rangle \text { (and }\langle x\rangle \text { if } n \text { is odd), }
$$

and acting on each as an element of determinant 1.

Now let $g$ be the diagonal element with respect to $\mathcal{B}$ whose diagonal entries are 1 except in the entries corresponding to $e_{m}$ and $f_{m}$, in which case the entries are $\mu$ and $\mu^{-1}$, respectively, where $\mu \in \mathbb{F}_{q}^{*} \backslash \mathbb{F}_{q_{0}}^{*}$. Let $B_{0} \cong \mathrm{SL}_{m}\left(q_{0}^{2}\right)$ be the subgroup of $M$ stabilizing the subspaces

$$
\left\langle e_{1}, \ldots, e_{m-1}, e_{m}\right\rangle \text { and }\left\langle f_{1}, \ldots, f_{m-1}, f_{m}\right\rangle \text { (and }\langle x\rangle \text { if } n \text { is odd), }
$$

and acting on each as an element of determinant 1 . Let $B=B_{0}^{g}$, and observe that $A<B$ and $B \not \leq M$; thus Lemma 1.6 .10 implies that we have a subset $\Lambda \subseteq \Omega$ of cardinality $q_{0}^{2(m-1)}$ such that $S^{\Lambda}$ is a 2 -transitive group. This yields a beautiful subset unless $\operatorname{Alt}\left(q_{0}^{2(m-1)}\right)$ is a section of $\mathrm{SU}_{n}(q)$; since we are assuming that $m \geq 3$, Lemma 2.1.1 eliminates the latter possibility, and we are done.

We are left with the possibility that $m=2$, in which case $n \in\{4,5\}$. We define two subgroups, $T$ and $U$, as follows. First, if $n=5$, then both subgroups fix the vector $x$. Then, in both cases, we fix an element $\zeta \in \mathbb{F}_{q}^{*} \backslash \mathbb{F}_{q_{0}}^{*}$ and describe the action of the two groups on the space $\left\langle e_{1}, e_{2}, f_{1}, f_{2}\right\rangle$ (writing elements with respect to the ordered basis $\left.\left\{e_{1}, e_{2}, f_{1}, f_{2}\right\}\right)$ :

$$
T=\left\{\left.\left(\begin{array}{cccc}
a & & &  \tag{4.5.1}\\
& 1 & & \\
& & a^{-1} & \\
& & & 1
\end{array}\right) \right\rvert\, a \in \mathbb{F}_{q_{0}}^{*}\right\}, \quad U=\left\{\left.\left(\begin{array}{cccc}
1 & \zeta x & & \\
& 1 & & \\
& & 1 & \\
& & -\zeta x & 1
\end{array}\right) \right\rvert\, x \in \mathbb{F}_{q_{0}}\right\}
$$

As usual, we can check that $T \leq M, U \not \subset M$, and $T$ normalizes $U$ and acts transitively on the set of non-identity elements of $U$. Then $\Delta=M^{U}$ is a subset of $\Omega$ on which $G_{\Delta}$ acts 2-transitively, and we have a beautiful subset unless $\operatorname{Alt}\left(q_{0}\right)$ is a section of $\mathrm{SU}_{n}(q)$, which by Lemma 2.1.1 can only occur if $q_{0} \leq 7$, as listed in the first line of Table 4.5.3.

Suppose, next, that $r=2$. In this case $\varphi_{0}$ is either symmetric (and $q$ is odd) or alternating (and $q$ can be either even or odd). These are the embeddings in the last two lines of Table 4.5.1. In the case where $\varphi_{0}$ is symmetric and not of type $\mathrm{O}^{-}$, we take $\mathcal{B}$, as before, to be a hyperbolic basis.

For the other two cases, we adjust $\mathcal{B}$ slightly in order to see more clearly the embeddings (namely, $\mathrm{SO}_{n}^{-}(q)<\mathrm{SU}_{n}(q)$ and $\mathrm{Sp}_{n}(q)<\mathrm{SU}_{n}(q)$ ). In the symplectic case we take $\mathcal{B}=\left\{e_{1}, \ldots, e_{m}, f_{1}, \ldots, f_{m}\right\}$ such that

$$
\varphi\left(e_{i}, e_{j}\right)=\varphi\left(f_{i}, f_{j}\right)=0 \text { and } \varphi\left(e_{i}, f_{j}\right)=\delta_{i j} \zeta
$$

where $\zeta \in \mathbb{F}_{q^{2}}$ satisfies $\zeta^{q}=-\zeta$. It is easy to see that the restriction $\varphi_{0}$ of $\varphi$ to the $\mathbb{F}_{q^{-}}$-span of $\mathcal{B}$ is symplectic; what is more the matrix for $\varphi_{0}$ written in block form with respect to $\mathcal{B}$ is

$$
\zeta\left(\begin{array}{cc}
0 & I \\
-I & 0
\end{array}\right)
$$

a scalar multiple of the "usual" alternating matrix; hence $\operatorname{Isom}\left(\varphi_{0}\right)$ is a symplectic group $\operatorname{Sp}_{n}(q)$.
In the $\mathrm{O}^{-}$case we take $\mathcal{B}=\left\{e_{1}, \ldots, e_{m}, f_{1}, \ldots, f_{m}, x, y\right\}$ to be a hyperbolic basis for $\varphi_{0}$ over $\mathbb{F}_{q}$, and we simply define $\varphi$ to be the Hermitian form obtained by extending $\varphi_{0}$ to include scalars over $\mathbb{F}_{q^{2}}$.

In all cases, $m$ is the Witt index of $\varphi_{0}$, and we now proceed as in the first part of the proof. First assume that $m \geq 3$ and define $A \cong \mathrm{SL}_{m-1}(q)$ to be the set of elements in $M$ stabilizing the $\mathbb{F}_{q^{-}}$-subspaces

$$
\left\langle e_{1}, \ldots, e_{m-1}\right\rangle,\left\langle e_{m}\right\rangle,\left\langle f_{1}, \ldots, f_{m-1}\right\rangle,\left\langle f_{m}\right\rangle \text { (and }\langle x\rangle \text { and }\langle y\rangle \text { if needed), }
$$

and acting on each as an element of determinant 1.
Now we define $g$ according to two cases:

1. If $\varphi_{0}$ is orthogonal with $n$ odd, or of type $\mathrm{O}^{-}$with $n$ even, then let $g$ send

$$
e_{m} \mapsto \mu e_{m}, f_{m} \mapsto \mu^{-q} f_{m}, x \mapsto \mu^{q-1} x,
$$

and fix all other elements of $\mathcal{B}$, where $\mu$ is a primitive element of $\mathbb{F}_{q^{2}}$.
2. If $\varphi_{0}$ is symplectic or of type $\mathrm{O}^{+}$, then we let $\lambda, \mu \in \mathbb{F}_{q^{2}}$ with $\lambda$ primitive, and let $g$ act as

$$
\begin{aligned}
& \lambda I \text { on }\left\langle e_{1}, \ldots, e_{m-1}\right\rangle, \\
& \mu \text { on }\left\langle e_{m}\right\rangle, \\
& \lambda^{-q} I \text { on }\left\langle f_{1}, \ldots, f_{m-1}\right\rangle, \\
& \mu^{-q} \text { on }\left\langle f_{m}\right\rangle .
\end{aligned}
$$

We require that $\lambda^{-(q-1)(m-1)}=\mu^{q-1}$ to ensure that $\operatorname{det}(g)=1$, and we require that $\lambda \mu^{-1} \notin \mathbb{F}_{q}$ (this condition ensures that $B=B_{0}^{g} \not \leq M$, see next paragraph). This can be done provided $q+1$ does not divide $m$ - we defer this remaining case for the moment.

Let $B_{0} \cong \mathrm{SL}_{m}(q)$ be the subgroup of $M$ stabilizing the subspaces

$$
\left\langle e_{1}, \ldots, e_{m-1}, e_{m}\right\rangle \text { and }\left\langle f_{1}, \ldots, f_{m-1}, f_{m}\right\rangle \text { (and }\langle x\rangle \text { and }\langle y\rangle \text { if needed), }
$$

and acting on each as an element of determinant 1 . Let $B=B_{0}^{g}$, and observe that $A<B$ and $B \not \leq M$; thus Lemma 1.6 .10 implies that we have a subset $\Lambda \subseteq \Omega$ of order $q^{m-1}$ such that $S_{\Lambda}$ is 2 -transitive. This yields a beautiful subset unless $\operatorname{Alt}\left(q^{m-1}\right)$ is a section of $\mathrm{SU}_{n}(q)$. By Lemma [2.1.1, the latter is only possible if $(n, q)$ is one of $(6,2),(7,3),(8,2),(8,3)$ (recall that we are assuming $m \geq 3$ here), and $M$ is as in Table 4.5.3.

Now let us deal with the deferred case: we suppose that $\varphi_{0}$ is symplectic or of type $\mathrm{O}^{+}$and $q+1$ divides $m=\frac{n}{2}$. Then we repeat the argument with $m$ redefined to equal $\frac{n-2}{2}$. Note, though, that for the argument to work we must have $(n-2) / 2=m \geq 3$, that is, $n \geq 8$. We set $g$ to act as

$$
\begin{aligned}
& I \text { on }\left\langle e_{1}, \ldots, e_{m-1}\right\rangle, \\
& \mu \text { on }\left\langle e_{m}\right\rangle, \\
& \mu^{-1} \text { on }\left\langle e_{m+1}\right\rangle, \\
& I \text { on }\left\langle f_{1}, \ldots, f_{m-1}\right\rangle \text {, } \\
& \mu^{-q} \text { on }\left\langle f_{m}\right\rangle, \\
& \mu^{q} \text { on }\left\langle e_{m+1}\right\rangle .
\end{aligned}
$$

We obtain the same outcome: a beautiful subset of size $q^{m-1}$ unless $\operatorname{Alt}\left(q^{m-1}\right)$ is a section of $\mathrm{SU}_{n}(q)$. The latter is only possible when $(n, q)=(8,3)$, which situation is listed in the second line of Table 4.5.3,

If we are in the deferred case with $n<8$, then $n=6$. As $q+1$ divides $m+1=3$, we have $q=2$ and $S=\mathrm{SU}_{6}(2)$, and $\bar{S}$ is listed in Lemma 4.1.1. This concludes the analysis of the deferred case.

Next, we consider the possibility that $m=2$ (defined, as it was originally, to be the Witt index of $\varphi_{0}$ ) in which case $n \in\{4,5,6\}$. In this case we proceed as at the start of this proof - defining two subgroups $U$ and $T$ as in (4.5.1) - so that we obtain a beautiful subset unless $\operatorname{Alt}(q)$ is a section of $\operatorname{SU}_{n}(q)$. Using Lemma 2.1.1, we conclude that $q \leq 7$ in the latter case, giving the cases listed in Table 4.5.3.

Finally, if $m=1$, then $n=4$ and $M$ is of type $\mathrm{O}^{-}$. We shall work with the quasisimple group $S=\mathrm{SU}_{4}(q)$ with centre $Z$ of order $d=(4, q+1)$. In this group, the corresponding maximal subgroup, which we shall also denote as $M$, has structure $\mathrm{SO}_{4}^{-}(q) . d$ (see [10, Table 8.10]). Let $X=\mathrm{SO}_{3}(q)<M$, and let $T=\left\{\left(\lambda, \lambda^{-1}, 1\right): \lambda \in \mathbb{F}_{q}^{*}\right\}$ be a maximal torus of order $q-1$ in $X$ (matrices relative to a standard basis for the $\mathrm{O}_{3}$-space). Thus

$$
\begin{equation*}
T<X<M<S \tag{4.5.2}
\end{equation*}
$$

We claim that there is an $S$-conjugate $Y$ of $X$ such that $T<Y \nsubseteq M$. Given the claim, we can complete the proof as follows. Since $Y \cong \mathrm{SO}_{3}(q) \cong \mathrm{PGL}_{2}(q)$, there are subgroups $U_{+}, U_{-}$of order $q$ in $Y$ such that $T$ acts by conjugation fixed-point-freely on both of them. These cannot both be contained in $M$, as $Y \not 又 M$. Hence, say, $U_{+} \cap M=1$. Then in the usual way, $\Delta=M^{U_{+}}$is a set of $q$ points on which
$T U_{+}$acts 2-transitively. For $q>7, \operatorname{Alt}(q)$ is not a section of $G$, and so $\Delta$ is a beautiful subset of $\Omega$, giving the conclusion; when $q \leq 7$, these case are listed in the last line of Table 4.5.3.

So it remains to prove the claim. The claim would follow by applying Lemma 2.6 .1 to the sequence (4.5.2) if we knew that $M$ controls fusion of $X$ in $S$, but this may not be the case: there are two conjugacy classes of subgroups $\mathrm{SO}_{3}(q)$ in $\mathrm{SO}_{4}^{-}(q)$, with representatives $X_{1}, X_{2}$, say; then $X_{1}$ and $X_{2}$ are $S$-conjugate, but may or may not be $M$-conjugate (this depends on certain congruences of $q$ which we do not need to state here). Therefore, our argument is different. Define

$$
\begin{aligned}
& \Lambda=\{Y<S: T<Y, Y \text { conjugate to } X \text { in } S\}, \\
& \Phi=\{Y \in \Lambda: Y<M\} .
\end{aligned}
$$

We shall compute the sizes of $\Lambda$ and $\Phi$, showing that $|\Lambda|>|\Phi|$, hence proving the claim.
First observe that $N_{S}(T)$ acts on $\Lambda$. The action of $N_{S}(T)$ on $\Lambda$ is transitive; indeed,

$$
\begin{aligned}
Y \in \Lambda & \Rightarrow Y=X^{s}(s \in S) \\
& \Rightarrow T, T^{s^{-1}}<X \\
& \Rightarrow T^{s^{-1}}=T^{x} \text { for some } x \in X \\
& \Rightarrow Y=X^{x s} \text { with } x s \in N_{S}(T) .
\end{aligned}
$$

Hence $|\Lambda|=\left|N_{S}(T): N_{S}(T) \cap N_{S}(X)\right|$. Since $N_{S}(T)$ has a subgroup of order $\left|\operatorname{GU}_{2}(q) \cdot(q-1) /|Z|\right.$, while the order of $N_{S}(T) \cap N_{S}(X)$ divides $2\left(q^{2}-1\right) /|Z|$, it follows that $|\Lambda|$ is divisible by $\frac{1}{2} q\left(q^{2}-1\right)$.

In the same way we see that $N_{M}(T)$ has at most 2 orbits on $\Phi$. The orbit $\Phi_{1}$ of $X_{1}$ has size $\mid N_{M}(T)$ : $N_{M}(T) \cap N_{M}\left(X_{1}\right) \mid$. Since $\left|N_{M}(T)\right|$ divides $4\left(q^{2}-1\right)$ and $\left|N_{M}(T) \cap N_{M}\left(X_{1}\right)\right|$ is divisible by $2(q-1)$, it follows that $\left|\Phi_{1}\right|$ divides $2(q+1)$. If there is a second orbit $\Phi_{2}$, its size also divides $2(q+1)$. Hence $|\Phi|$ divides $4(q+1)$.

As $q>7$, it is clear from the previous two paragraphs that $|\Lambda|>|\Phi|$. This yields the claim and completes the proof.

Lemma 4.5.6. If $S$ is listed in Table 4.5.3, then the action is not binary.
Proof. Suppose, first, that $r=2$ - this covers all but the first line of Table 4.5.3, Now Lemma 4.1.1 deals with all the possible groups $S$ except for $\mathrm{SU}_{8}(3), \mathrm{SU}_{6}(5), \mathrm{SU}_{6}(7)$ and $\mathrm{SU}_{5}(7)$. We handled these cases with magma computations using the permutation character method.

Suppose, next, that $r \geq 3$, so we are in the first line of the table. Here $M=M_{0} \times\langle\phi\rangle$, where $M_{0}$ has socle $\operatorname{PSU}_{4}\left(q_{0}\right)$ or $\operatorname{PSU}_{5}\left(q_{0}\right)$ with $q_{0} \leq 7$, and $\phi$ is either 1 or a field automorphism of $S$ of order $r$. We adopt the strategy of the proof of Lemma 4.5.3. Let $p$ be the characteristic of $\mathbb{F}_{q}$, let $Q \in \operatorname{Syl}_{p}\left(M_{0}\right)$, and choose $g \in N_{G}(Q) \backslash M$. Then $Q \leq M_{0} \cap M_{0}^{g}$, and so (by the well-known "Tits lemma", or by computation) there is a parabolic subgroup $P$ of $M_{0}$ such that $U L^{\prime} \leq M_{0} \cap M_{0}^{g} \leq P$, where $U$ is the unipotent radical and $L$ a Levi factor.

Write $M_{1}=M_{0} \cap M_{0}^{g}$, a core-free subgroup of $M_{0}$. A magma computation shows that any transitive action of $M_{0}$ of $p^{\prime}$-degree is not binary. Hence, if $M \cap M^{g}=M_{1} \times\langle\sigma\rangle$ with $\sigma=1$ or $\phi$, then we obtain the conclusion as in the proof of Lemma 4.5.3, Otherwise, $M \cap M^{g}=M_{1}\langle h \phi\rangle$, where $h \in N_{M_{0}}\left(M_{1}\right)$. Analysing this normalizer, we see that we can take $h$ to be diagonal of order dividing $q_{0}^{2}-1$. Since $\phi \notin M \cap M^{g}$, the order of $h$ must be divisible by $r$, and hence as $q_{0} \leq 7$, we must have $r=3$ or 5 . Hence $M=M_{0} \times r, r=3$ or 5 , and $\left|M: M \cap M^{g}\right|$ is coprime to $p$. Now a further magma computation shows that any such action $\left(M,\left(M: M \cap M^{g}\right)\right.$ ) (with $\left.\operatorname{soc}\left(M_{0}\right) \not \subset M \cap M^{g}\right)$ is not binary.

### 4.5.3 Case $S=\operatorname{Sp}_{n}(q)$

Lemma 4.5.7. In this case either $\Omega$ contains a beautiful subset or else $S$ is listed in Table 4.5.4. In all cases the action of $G$ on $\Omega$ is not binary.

| Group $S$ | Details of action |
| :--- | :--- |
| $\mathrm{Sp}_{4}\left(2^{r}\right)$ | $r$ prime, $M \triangleright \mathrm{Sp}_{4}(2)$. |

Table 4.5.4: $\mathcal{C}_{5}-\operatorname{Sp}_{n}(q)-$ Cases where a beautiful subset was not found.

Proof. Let $\mathcal{B}=\left\{e_{1}, \ldots, e_{k}, f_{k}, \ldots, f_{1}\right\}$ be a hyperbolic basis for $V$ with $k=\frac{n}{2}$. Let $M$ be the group stabilizing the $\mathbb{F}_{q_{0}}$-span of $\mathcal{B}$.

First fix an element $\zeta \in \mathbb{F}_{q} \backslash \mathbb{F}_{q_{0}}$. We define two subgroups, writing elements with respect to $\mathcal{B}$ :

$$
T=\left\{\left.\left(\begin{array}{ccc}
1 & & \\
& A & \\
& & 1
\end{array}\right) \right\rvert\, A \in \operatorname{Sp}_{n-2}\left(q_{0}\right)\right\}, \quad U=\left\{\left.\left(\begin{array}{ccc}
1 & \zeta x & \\
& I_{n-2} & \zeta x^{\prime T} \\
& & 1
\end{array}\right) \right\rvert\, x \in \mathbb{F}_{q_{0}}^{n-2}\right\}
$$

where $x^{\prime}=x J, J$ being the matrix of the form relative to the basis $\mathcal{B}$, omitting $e_{1}, f_{1}$. As usual, we can check that $T \leq M, U \not \leq M$, and $T$ normalizes $U$ and acts transitively on $U \backslash 1$. Then $\Delta=M^{U}$ is a subset of $\Omega$ of size $q_{0}^{n-2}$ on which $G_{\Delta}$ acts 2-transitively, and we have a beautiful subset unless Alt $\left(q_{0}^{n-2}\right)$ is a section of $\mathrm{Sp}_{n}(q)$. By Lemma 2.1.1, the latter is only possible if $n=4$ and $q_{0}=2$, the case listed in Table 4.5.4. Finally, the case in the table is dealt with exactly as in Lemma 4.5.3,

### 4.5.4 Case $S$ is orthogonal

In this section we deal with all of the orthogonal families in one go. Recall from Section 4.1.1 that $n \geq 7$, and also if $S=\Omega_{8}^{+}(q)$, then we are assuming that $G \leq \mathrm{P} \Gamma \Omega_{8}^{+}(q)$.

| Group $S$ | Details of action |
| ---: | :--- |
| $\Omega_{7}\left(3^{r}\right)$ | $q_{0}=3, M \triangleright \Omega_{7}(3)$ |
| $\Omega_{8}^{-}\left(q_{0}^{r}\right)$ | $q_{0} \in\{2,3\}, r$ odd, $M \triangleright \mathrm{P} \Omega_{8}^{-}\left(q_{0}\right)$ |
| $\Omega_{8}^{+}\left(q_{0}^{2}\right)$ | $q_{0} \in\{2,3\}, M \triangleright \mathrm{P} \Omega_{8}^{-}\left(q_{0}\right)$ |
| $\Omega_{8}^{+}\left(2^{r}\right)$ | $q_{0}=2, M \triangleright \Omega_{8}^{+}(2)$ |
| $\Omega_{10}^{-}\left(2^{r}\right)$ | $q_{0}=2, r$ odd, $M \triangleright \Omega_{10}^{-}(2)$ |
| $\Omega_{10}^{+}(4)$ | $q_{0}=2, M \triangleright \Omega_{10}^{-}(2)$ |

Table 4.5.5: $\mathcal{C}_{5}-\Omega_{n}^{\varepsilon}(q)$ - Cases where a beautiful subset was not found.

Lemma 4.5.8. In this case either $\Omega$ contains a beautiful subset or else $S$ is listed in Table 4.5.5.
Proof. Let $W$ be an $n$-dimensional orthogonal space over $\mathbb{F}_{q_{0}}$, with associated quadratic form $Q_{W}$, and let $\mathcal{B}$ be a hyperbolic basis for $W$. Define $V=W \otimes_{\mathbb{F}_{q_{0}}} \mathbb{F}_{q}$, with $Q_{W}$ extended to a quadratic form, $Q_{V}$, on $V$. This yields an embedding of $\operatorname{Isom}\left(Q_{W}\right) \leq \operatorname{Isom}\left(Q_{V}\right)$. The embeddings listed in row 3 of Table 4.5.1 follow immediately. Note that, in the case where $\Omega_{n}^{-}\left(q_{0}\right)$ is embedded in $\Omega_{n}^{+}\left(q_{0}^{2}\right), \mathcal{B}$ is not a hyperbolic basis for $V$.

We write $\mathcal{B}=\left\{e_{1}, \ldots, e_{k}, f_{1}, \ldots, f_{k}, x, y\right\}$ (omitting $x$ if $n$ is odd, and omitting $x$ and $y$ if $n$ is even and $\varepsilon=+)$. We write $\mathcal{A}$ for the $\mathbb{F}_{q}$-span of the anisotropic vectors in $\mathcal{B}$; so $\operatorname{dim}(\mathcal{A}) \in\{0,1,2\}$.

We define two subgroups:

$$
\begin{aligned}
A & =\left\{g \in M \mid g \text { stabilizes }\left\langle e_{k}\right\rangle,\left\langle f_{k}\right\rangle,\left\langle e_{1}, \ldots, e_{k-1}\right\rangle \text { and }\left\langle f_{1}, \ldots, f_{k-1}\right\rangle ; v^{g}=v \forall v \in \mathcal{A}\right\} ; \\
B_{0} & =\left\{g \in M \mid g \text { stabilizes }\left\langle e_{1}, \ldots, e_{k}\right\rangle \text { and }\left\langle f_{1}, \ldots, f_{k}\right\rangle ; v^{g}=v \forall v \in \mathcal{A}\right\} .
\end{aligned}
$$

Observe that $A \triangleright \mathrm{SL}_{k-1}\left(q_{0}\right)$ and $B_{0} \triangleright \mathrm{SL}_{k}\left(q_{0}\right)$. Now define $g \in G$ to send $e_{k} \mapsto \lambda e_{k}, f_{k} \mapsto \lambda^{-1} f_{k}$ and to fix the other elements of $\mathcal{B}$, where $\lambda \in \mathbb{F}_{q} \backslash \mathbb{F}_{q_{0}}$. Set $B=B_{0}^{g}$ and observe that $B$ contains $A$ but is not
contained in $M$. Then Lemma 1.6 .10 implies that there is a subset $\Delta$ of $\Omega$ such that $|\Delta|=q_{0}^{k-1}$ and $G_{\Delta}$ acts 2 -transitively on $\Delta$. Then $\Delta$ is a beautiful subset (and we are done) or else $\operatorname{Alt}\left(q_{0}^{k-1}\right)$ is a section of $\Omega_{n}^{\varepsilon}(q)$. In the latter case, Lemma 2.1.1 implies that $S, M$ are as in Table 4.5.5.

Lemma 4.5.9. If $S$ is listed in Table 4.5.5, then the action is not binary.
Proof. Suppose, first, that $r=2$. In this case, $S \in\left\{\Omega_{8}^{+}(4), \Omega_{8}^{+}(9), \Omega_{10}^{+}(4)\right\}$ and we confirm the result using magma.

Now suppose that $r \geq 3$. Then $M=M_{0} \times\langle\phi\rangle$, where $M_{0}$ has socle $\mathrm{P} \Omega_{n}^{\epsilon}\left(q_{0}\right)$ with $n \leq 10$ and $q_{0} \leq 3$, and $\phi$ is either 1 or a field automorphism $S$ of order $r$. We use the same argument as for Lemma 4.5.6. First, a magma computation shows that any transitive action of $M_{0}$ of $p^{\prime}$-degree is not binary. Then the argument shows that there exists $g \in G$ such that $\left(M,\left(M: M \cap M^{g}\right)\right)$ is not binary, unless possibly $r$ divides $q_{0}^{2}-1$. As $q_{0} \leq 3$, this forces $r=3$, and now a further magma computation shows that any transitive $p^{\prime}$-action of $M_{0} \times 3$ is not binary, completing the proof.

### 4.6 Family $\mathcal{C}_{6}$

The members in the Aschbacher class $\mathcal{C}_{6}$ arise as local subgroups; more specifically they are normalizers of certain absolutely irreducible $r$-groups $R$ of symplectic-type, where $r$ is a prime number with $r \neq p$ and $p$ is the characteristic of the defining field for the classical group. For $r$ odd, the $r$-group $R$ is extraspecial of exponent $r$, denoted by its order $r^{1+2 a}$; and for $r=2$, either $R$ is an extraspecial group $2_{ \pm}^{1+2 a}$, or is a central product $4 \circ 2^{1+2 a}$. These $r$-groups have absolutely irreducible embeddings in various classical groups of dimension $r^{a}$, and the normalizers of $R$ in these classical groups comprise the $\mathcal{C}_{6}$ subgroups; more precisely, if $G$ is an almost simple classical group and $\bar{R}$ is the projective image of $R$ in $G$, then $M=N_{G}(\bar{R})$ is in the $\mathcal{C}_{6}$ class. Full details are given in [54, §4.6], and we give a list of the embeddings in Table 4.6.1.

| case | normalizer | conditions |
| :---: | :---: | :---: |
| $\mathrm{L}^{\epsilon}$ | $r^{1+2 a} \cdot \mathrm{Sp}_{2 a}(r)<\mathrm{GL}_{r^{a}}^{\epsilon}(q)$ | $r$ odd $q \equiv \epsilon \bmod r$ |
| $\mathrm{~L}^{\epsilon}$ | $4 \circ 2^{1+2 a} \cdot \mathrm{Sp}_{2 a}(2)<\mathrm{GL}_{2^{a}}^{\epsilon}(q)$ | $q=p \equiv \epsilon \bmod 4$ |
| S | $2_{-}^{1+2 a} \cdot \mathrm{O}_{2 a}^{-}(2)<\mathrm{GSp}_{2^{a}}(q)$ | $q=p$ |
| $\mathrm{O}^{+}$ | $2_{+}^{1+2 a} \cdot \mathrm{O}_{2 a}^{+}(2)<\mathrm{O}_{2^{a}}^{+}(q)$ | $q=p$ |

Table 4.6.1: Maximal subgroups in family $\mathcal{C}_{6}$
In Line 1 of Table 4.6.1 there is a further condition on $q$ : namely, let $e$ be the smallest positive integer such that $p^{e} \equiv 1 \bmod r$. If $e$ is odd, then $\epsilon=+$ and $q=p^{e}$; and if $e$ is even, then $\epsilon=-$ and $q=p^{e / 2}$.

The main result of this section is the following. The result will be proved in a series of lemmas.
Proposition 4.6.1. Suppose that $G$ is an almost simple group with socle $\bar{S}=\mathrm{Cl}_{n}(q)$, and assume that
(i) $n \geq 3,4,4,7$ in cases $L, U, S, O$ respectively, and
(ii) $\mathrm{Cl}_{n}(q)$ is not one of the groups listed in Lemma 4.1.1,

Let $M$ be a maximal subgroup of $G$ in the family $\mathcal{C}_{6}$. Then the action of $G$ on $(G: M)$ is not binary.
Our first lemma deals with the situation when $r$ is odd, in which case $S=\operatorname{SL}_{r^{a}}^{\varepsilon}(q)$ and $q$ is as given above, so that $\mathbb{K}=\mathbb{F}_{p^{e}}$. To prove the lemma we recall the set-up described in [54, §4.6] and establish some notation.

We let $R:=\left\langle x_{1}, \ldots, x_{a}, y_{1}, \ldots, y_{a}, z\right\rangle$ be the extraspecial $r$-group with center $Z(R)=\langle z\rangle$ and where, for every $i, j \in\{1, \ldots, a\}$,

$$
x_{i}^{r}=y_{j}^{r}=\left[x_{i}, x_{j}\right]=\left[y_{i}, y_{j}\right]=1
$$

and

$$
\left[y_{i}, x_{j}\right]= \begin{cases}z & \text { when } j=i \\ 1 & \text { otherwise }\end{cases}
$$

Clearly, $\bar{R}:=R / Z(R)$ is an elementary abelian $r$-group and one can see that $\bar{R}$ embeds naturally in $C_{\operatorname{Aut}(R)}(Z(R))$. We use an additive notation for the elements of $\bar{R}$ and a multiplicative notation for the elements of $R$ and observe that the commutator function

$$
\begin{aligned}
\mathfrak{B}: \bar{R} \times \bar{R} & \longrightarrow Z(R) \\
(g Z(R), h(Z(R)) & \longmapsto[g, h]
\end{aligned}
$$

defines a non-degenerate symplectic form on $\bar{R}$, which endows $\bar{R}$ with the structure of a symplectic space over the field $\mathbb{F}_{r}$. Using the basis $\left(\bar{x}_{1}, \ldots, \bar{x}_{a}, \bar{y}_{1}, \ldots, \bar{y}_{a}\right)$ of $\bar{R}$, the symplectic form $\mathfrak{B}$ on $\bar{R}$ is represented by the skew-symmetric matrix in block form

$$
J:=\left(\begin{array}{cc}
0 & -I \\
I & 0
\end{array}\right)
$$

Observe that, under the natural projection $R \rightarrow \bar{R}$, the abelian subgroups of $R$ correspond to the totally isotropic subspaces of $\bar{R}$. We let $X:=\left\langle x_{1}, \ldots, x_{a}\right\rangle$ and $Y:=\left\langle y_{1}, \ldots, y_{a}\right\rangle$. Observe that $X$ and $Y$ are elementary abelian subgroups of $R$ of cardinality $r^{a}$ and $\bar{X}$ and $\bar{Y}$ are maximal totally isotropic subspaces of $\bar{R}$.

From the structure of $R$, it is clear that each element of $R$ can be written uniquely in the form

$$
x_{1}^{\varepsilon_{1}} \cdots x_{a}^{\varepsilon_{a}} y_{1}^{\eta_{1}} \cdots y_{a}^{\eta_{a}} z^{\nu}
$$

where $\varepsilon_{1}, \ldots, \varepsilon_{a}, \eta_{1}, \ldots, \eta_{a}, \nu$ can be taken in $\mathbb{F}_{r}$. Given $v=\sum_{i=1}^{a} \varepsilon_{i} \bar{x}_{i}+\sum_{i=1}^{a} \eta_{i} \bar{y}_{i}$, an element in $\bar{R}$, we write $\underline{v}=x_{1}^{\varepsilon_{1}} \cdots x_{a}^{\varepsilon_{a}} y_{1}^{\eta_{1}} \cdots y_{a}^{\eta_{a}}$, a corresponding element in $R$.

Given a matrix $A \in \mathrm{GL}_{2 a}(r)$ that preserves the symplectic form $\mathfrak{B}$, we find that the function

$$
\begin{aligned}
\theta_{A}: R & \longrightarrow R \\
\underline{x_{i}} & \longmapsto \underline{A x_{i}} \\
\underline{y_{i}} & \longmapsto \underline{A y_{i}},
\end{aligned}
$$

defined on the generators of $R$ extends to an automorphism of $R$ that centralizes $Z(R)$. In this way we obtain an embedding $\bar{R} . \mathrm{Sp}_{2 a}(r)$ in $C_{\mathrm{Aut}(R)}(Z(R))$ and now [54, Table 4.6.A] asserts that in fact $C_{\operatorname{Aut}(R)}(Z(R))=\bar{R} \cdot \mathrm{Sp}_{2 a}(r) \cong r^{2 a} . \mathrm{Sp}_{2 a}(r)$.

Now [54, p.151] describes an absolutely irreducible representation of $R$ over $\mathbb{K}$ of dimension $r^{a}$ that induces an embedding of $C_{\operatorname{Aut}(R)}(Z(R))$ into $\mathrm{PGL}_{r^{a}}(\mathbb{K})$; this embedding yields the $\mathcal{C}_{6}$ subgroups for $r$ odd.

Lemma 4.6.2. Let $r$ be an odd prime, let $G$ be almost simple with socle $\operatorname{PSL}_{r^{a}}^{\epsilon}(q)$, and let $M=N_{G}(\bar{R})$, where $R=r^{1+2 a}$, as in line 1 of Table 4.6.1. Then the action of $G$ on $(G: M)$ is not binary.
Proof. We adopt the above notation and, since $M=N_{G}(\bar{R})$, we may identify the set $\Omega=(G: M)$ with the set of conjugates $\left\{\bar{R}^{g}: g \in G\right\}$. Recall that $X$ is an elementary abelian group of order $r^{a}$, so $X \cong \mathbb{F}_{r}^{a}$. Moreover, since $\mathbb{F}_{r^{a}} \cong \mathbb{F}_{r}^{a}$ as $\mathbb{F}_{r}$-vector spaces, $\mathrm{GL}(X)$ contains an automorphism acting as scalar multiplication by a field element of order $r^{a}-1$. Let $B$ be the matrix of this automorphism of $X$ with respect to the basis $\left\{x_{1}, \ldots, x_{a}\right\}$. Then the matrix

$$
A=\left(\begin{array}{cc}
B & 0 \\
0 & B^{-T}
\end{array}\right)
$$

preserves the bilinear form $\mathfrak{B}$. Thus $\theta_{A}$ is an automorphism of $R$ and hence $\theta_{A}$ determines an element of $N_{\mathrm{PGL}(V)}(\bar{R})$. In fact, from [54, Proposition 4.6.5], $\theta_{A} \in M$ except when $a=1, r=3$ and $p=q$. We leave the case $a=1, r=3$ and $p=q$ aside for the time being; indeed let us assume, for now, that $n>5$.

Let $C=\left\langle\theta_{A}\right\rangle \in M$ and let $T=\bar{X} \rtimes C \leq M$. By construction $T$ is a Frobenius group with Frobenius kernel $\bar{X}$ of cardinality $r^{a}$ and cyclic Frobenius complement $C$ of cardinality $r^{a}-1$. We claim that

$$
\begin{equation*}
\exists g \in N_{G}(C) \text { with } M \cap T^{g}=C . \tag{4.6.1}
\end{equation*}
$$

We argue by contradiction and we suppose that $M \cap T^{g} \neq C$, for every $g \in N_{G}(C)$. Let $g \in N_{G}(C)$. Since $T^{g}=\bar{X}^{g} \rtimes C, M \cap T^{g} \geq C$ and $C$ acts transitively by conjugation on the non-identity elements of $\bar{X}^{g}$, we deduce $M \cap T^{g}=T^{g}$, that is, $T^{g} \leq M$. Suppose that $\bar{X}^{g} \not \leq \bar{R}$. Then $T^{g}$ is a Frobenius group and is isomorphic to a subgroup of $\mathrm{Sp}_{2 a}(r)$. Since $r$ is odd, $r^{a}-1$ is even and hence $A^{\left(r^{a}-1\right) / 2}$ is the $-\bar{I}$ matrix. In particular, $A^{\left(r^{a}-1\right) / 2}$ centralizes $\bar{X}^{g}$. However, this is a contradiction because (since $T^{g}=\bar{X}^{g} \rtimes C$ is a Frobenius group) the action of $A^{\left(r^{a}-1\right) / 2}$ by conjugation on $\bar{X}^{g}$ is fixed-point-free. This contradiction yields $\bar{X}^{g} \leq \bar{R}$. Thus $\bar{X}^{g}$ is a totally isotropic subspace of $\bar{R}$ normalized by $C$. The only such totally isotropic subspaces are $\bar{X}$ and $\bar{Y}$, and hence $\bar{X}^{g}=\bar{X}$ or $\bar{X}^{g}=\bar{Y}$. Now, consider $T^{\prime}:=\bar{Y} \rtimes C$. As $T$ and $T^{\prime}$ are conjugate in $M$, we obtain $T^{\prime g} \cap M \neq C$ because $T^{g} \cap M \neq C$. Therefore, repeating the argument in this paragraph with the group $T$ replaced by $T^{\prime}$, we deduce that $g$ normalizes $\bar{X} \bar{Y}=\bar{R}$. Thus $N_{G}(C) \leq N_{G}(\bar{R})=M$. Now we apply Proposition 2.4.1 and Lemma 2.4.2 to establish the existence of an element $g \in G$ normalizing $C$ but not lying in $M$. Therefore our claim (4.6.1) is now proved.

Let $g \in N_{G}(C)$ with $M \cap T^{g}=C$ and let $\Lambda:=\left\{\bar{R}^{t} \mid t \in T^{g}\right\}$. Then $\Lambda$ is a set of cardinality $\left|T^{g}: T^{g} \cap M\right|=r^{a}$ and $T^{g}$ induces on $\Lambda$ a permutation group isomorphic to a Frobenius group of order $r^{a}\left(r^{a}-1\right)$. If $\Lambda$ is a beautiful subset, the conclusion follows by Lemma 1.6.12. Otherwise, $\operatorname{Alt}\left(r^{a}-1\right)$ must be isomorphic to a section of $M$, hence to a section of $\operatorname{Sp}_{2 a}(r)$. Since $n>5$, Lemma 2.1.1 rules out the latter possibility and we are done.

Consider next the case $a=1, r=5$. Here the embedding is $5^{1+2} \cdot \operatorname{Sp}_{2}(5)<S=\operatorname{SL}_{5}^{\epsilon}(q)$, where $q$ is minimal such that $q \equiv \epsilon \bmod 5$. If $p=2$ then $S=\mathrm{SU}_{5}(4)$, which is covered by Lemma 4.1.1. So now assume $p>2$. It is well-known that the extension $R . \mathrm{Sp}_{2}(5)$ splits. Let $S_{0} \cong \mathrm{Sp}_{2}(5)$ be a complement, and let $t \in S_{0}$ be the central involution. If $V$ is the natural 5 -dimensional module for $S$, then $V_{5} \downarrow S_{0}=V_{3} \oplus V_{2}$, where $V_{3}=C_{V}(t), V_{2}=C_{V}(-t)$, of dimensions 3,2 respectively. Hence there exists a diagonal element of $S$ of the form $\hat{g}=\left(\lambda I_{3}, \mu I_{2}\right)$ such that $\hat{g} \in C_{S}\left(S_{0}\right) \backslash N_{S}(R)$. Denoting by $g$ the projective image of $\hat{g}$, we then have $\bar{R} S_{0} \cap\left(\bar{R} S_{0}\right)^{g}=S_{0}$. Since $\bar{R} S_{0}=5^{2} . \mathrm{Sp}_{2}(5)$ is a Frobenius group, this give a 2 -transitive subset of size 25 in the usual way, and the conclusion follows.

Finally, the case where $a=1, r=3$ is dealt with in similar fashion. Here the embedding is $3^{1+2} . Q_{8}<$ $\mathrm{SL}_{3}^{\epsilon}(p)$ with $p \equiv \epsilon \bmod 3$ and $p \neq 2$. As $V \downarrow Q_{8}=V_{2} \oplus V_{1}$, there exists $g \in C_{G}\left(Q_{8}\right) \backslash M$, and hence as above we obtain a subset $\Lambda$ of size 9 with $G^{\Lambda} 2$-transitive. This completes the proof.

Lemma 4.6.3. Let $r=2$, let $G$ be almost simple with socle $\operatorname{PSL}_{2^{a}}^{\epsilon}(p)(a \geq 2), \operatorname{PSp}_{2^{a}}(p)(a \geq 2)$ or $\mathrm{P} \Omega_{2^{a}}^{+}(p)(a \geq 3)$, and let $M=N_{G}(\bar{R})$, where $R=4 \circ 2^{1+2 a}$ or $2_{ \pm}^{1+2 a}$, as in Lines $2,3,4$ of Table 4.6.1. Then the action of $G$ on $(G: M)$ is not binary.

Proof. Since $M=N_{G}(\bar{R})$, we may identify the set $\Omega=(G: M)$ with the set of conjugates $\left\{\bar{R}^{g}: g \in G\right\}$. Referring to [54, §4.6], we have

$$
R=\langle z\rangle \circ\left\langle x_{1}, y_{1}\right\rangle \circ \cdots \circ\left\langle x_{a}, y_{a}\right\rangle,
$$

where
$z$ has order 4 in types $L, U$, and has order 2 in types $S, O^{+}$,

$$
\begin{aligned}
& \left\langle x_{i}, y_{i}\right\rangle \cong D_{8} \text { for } i \geq 3, \\
& \left\langle x_{1}, y_{1}\right\rangle \cong\left\langle x_{2}, y_{2}\right\rangle \cong Q_{8} \text { in types } L, U, O^{+}, \\
& \left\langle x_{1}, y_{1}\right\rangle \cong Q_{8} \text { and }\left\langle x_{2}, y_{2}\right\rangle \cong D_{8} \text { in type } S .
\end{aligned}
$$

The natural $2^{a}$-dimensional module $V$ has a tensor product decomposition $V=W_{1} \otimes \cdots \otimes W_{a}$ under the action of $R$, where each $W_{i}$ is an irreducible 2-dimensional module for $\left\langle z, x_{i}, y_{i}\right\rangle$.

From [54, 4.6.6, 4.6.8, 4.6.9], writing $\bar{S}=\operatorname{soc}(G)$, the precise structure of $M \cap \bar{S}$ is as follows:

| case | $M \cap S$ |
| :--- | :---: |
| $L^{\epsilon}$ | $2^{4} \cdot \operatorname{Alt}(6)$, if $n=4, p \equiv \epsilon 5 \bmod 8$ |
|  | $2^{2 a} \cdot \mathrm{Sp}_{2 a}(2)$, otherwise |
| $S$ | $2^{2 a} \cdot \mathrm{O}_{2 a}^{-}(2)$, if $p \equiv \pm 1 \bmod 8$ |
|  | $2^{2 a} \cdot \Omega_{2 a}^{-}(2)$, if $p \equiv \pm 3 \bmod 8$ |
| $O^{+}$ | $2^{2 a} \cdot \mathrm{O}_{2 a}^{+}(2)$, if $p \equiv \pm 1 \bmod 8$ |
|  | $2^{2 a} \cdot \Omega_{2 a}^{+}(2)$, if $p \equiv \pm 3 \bmod 8$ |

We now divide the proof in two parts (A) and (B), depending on whether $p \geq 7$ or $p<7$.
(A) Assume first that $p \geq 7$. Define $W=W_{2} \otimes \cdots \otimes W_{a}$, and note that $\operatorname{SL}\left(W_{1}\right)=\operatorname{Sp}\left(W_{1}\right)$. The subgroup of $G$ preserving the tensor decomposition $V=W_{1} \otimes W$ is the normalizer of the image of $\mathrm{SL}\left(W_{1}\right) \otimes \mathrm{Cl}(W)$, where $\mathrm{Cl}(W)$ is $\mathrm{SL}^{\epsilon}(W), \Omega^{+}(W)$ or $\mathrm{Sp}(W)$ for case $L^{\epsilon}, S$ or $O^{+}$repectively.

Again we use the bar notation for the natural homomorphism to the projective version of our classical group. As before, $M$ preserves on $\bar{R}$ a non-degenerate symplectic form $\mathfrak{B}$ in types $L$ and $U$ defined as above. In types $O^{+}$and $S$, the group $M$ not only preserves $\mathfrak{B}$ but also a particular quadratic form $\mathfrak{q}: \bar{R} \rightarrow \mathbb{F}_{r}$ that polarizes to $\mathfrak{B}$. Rather than defining $\mathfrak{q}$ explicitly we remark only that, for $i=1, \ldots, a$, the 2 -spaces corresponding to $\left\langle x_{i}, y_{i}\right\rangle \cong Q_{8}$ (resp. $\left\langle x_{i}, y_{i}\right\rangle \cong D_{8}$ ) are of type $\mathrm{O}_{2}^{-}(2)$ (resp. $\mathrm{O}_{2}^{+}(2)$ ). Then $\bar{R}_{1}$ is a non-degenerate 2 -space in $\bar{R}$, and is of type $\mathrm{O}_{2}^{-}(2)$ in cases $O^{+}$and $S$.

Define $\bar{R}_{0}=\prod_{i=2}^{a} \bar{R}_{i}=\bar{R}_{1}^{\perp}$, and $W=W_{2} \otimes \cdots \otimes W_{a}$. By [54, 4.4.3], $N_{G}\left(\bar{R}_{0}\right)$ preserves the tensor decomposition $V=W_{1} \otimes W$ and contains the image of $\mathrm{SL}\left(W_{1}\right) \otimes 1_{W}$. Define a subset $\Delta$ of $\Omega=\left\{\bar{R}^{g}: g \in G\right\}$ by

$$
\Delta=\left\{\bar{R}^{g}: g \in N_{G}\left(\bar{R}_{0}\right)\right\}
$$

Let $X$ be the image of the group induced on $W_{1}$ by $N_{G}\left(\bar{R}_{0}\right)$. Then $X \cong \operatorname{PSL}_{2}(p)$ or $\mathrm{PGL}_{2}(p)$, and

$$
\Delta=\left\{\bar{R}_{0} \times \bar{R}_{1}^{x}: x \in X\right\}
$$

Also, from the structure of $M \cap \bar{S}$, we see that

$$
N_{X}\left(\bar{R}_{1}\right) \cong 2^{2} \cdot \operatorname{Sp}_{2}(2) \cong 2^{2} \cdot \mathrm{O}_{2}^{-}(2) \cong \operatorname{Sym}(4) .
$$

Since the intersection of all the subgroups in $\Delta$ is $\bar{R}_{0}$, we have $G_{\Delta}=N_{G}\left(\bar{R}_{0}\right)$. Hence the action of $G_{\Delta}$ on $\Delta$ is isomorphic to the action of $X$ on the cosets of $\operatorname{Sym}(4)$.

Recall that we are assuming $p \geq 7$. Hence $\operatorname{Sym}(4)$ is a maximal subgroup of either $X$ or $X^{\prime}$, and [45] (together with Lemma 1.6.2) shows that $(X,(X: \operatorname{Sym}(4))$ is not binary. Thus there is an integer $k \geq 3$, and $k$-tuples $I=\left(I_{1}, \ldots, I_{k}\right), J=\left(J_{1}, \ldots, J_{k}\right) \in \Delta^{k}$ such that $I \widetilde{2} J$ and $I \not{\underset{k}{k}} J$ with respect to the action of $G_{\Delta}$. Since $I \widetilde{2} J$ we can assume that $I_{1}=J_{1}$ and $I_{2}=J_{2}$; we also assume that there are no repeated entries in $I$ (and hence there are none in $J$ either).

We need to show that $I \not{\widehat{k}} J$ with respect to the action of $G$. Suppose $I^{g}=J$ for some $g \in G$. Observe that for each $j$ we have $I_{j}=\bar{R}_{0} \times \bar{R}_{1}^{x_{j}}$ for some $x_{j} \in G_{\Delta}$. We claim that

$$
\bigcap_{j=1}^{k} \bar{R}_{1}^{x_{j}}=1
$$

Proof of claim: Suppose otherwise. Since $\bar{R}_{1}$ is a Klein 4-group we must have $\bigcap_{j=1}^{k} \bar{R}_{1}^{x_{j}}=\left\langle g_{I}\right\rangle$ where $g_{I}$ is an involution. Now for each $j$ we have $J_{j}=\bar{R}_{0} \times \bar{R}_{1}^{y_{j}}$ for some $y_{j} \in G_{\Delta}$. Since $I \widetilde{k} J$ with respect to the action of $G$, we conclude that $\bigcap_{j=1}^{k} \bar{R}_{1}^{y_{j}}=\left\langle g_{J}\right\rangle$ for some involution $g_{J}$. Observe that, for distinct $i$ and $j$, we have

$$
\bar{R}_{1}^{x_{i}} \cap \bar{R}_{1}^{x_{j}}=\left\langle g_{I}\right\rangle \text { and } \bar{R}_{1}^{y_{i}} \cap \bar{R}_{1}^{y_{j}}=\left\langle g_{J}\right\rangle .
$$

Since $I_{1}=J_{1}$ and $I_{2}=J_{2}$ we conclude that $g_{I}=g_{J}$. Consider what this means for the action of $X$ on $(X: \operatorname{Sym}(4))$ : we can think of this action as being the conjugation action of $X$ on a class of Klein 4 -subgroups. The tuples $I$ and $J$ correspond to $k$-tuples, $I_{X}$ and $J_{X}$, whose entries are Klein 4 -subgroups of $X$ all of which contain an involution $g_{X}$. What is more $I_{X} \widetilde{2} J_{X}$ and $I_{X} \not \chi_{\hat{k}} J_{X}$ with respect to the action of $X$. Now if $i$ and $j$ are distinct in $\{1, \ldots, k\}$ and

$$
I_{i}^{h}=J_{i} \text { and } I_{j}^{h}=J_{j} \text { for some } h \in X,
$$

then $h \in C_{X}\left(g_{X}\right)$. The group $C_{X}\left(g_{X}\right)$ is a maximal dihedral subgroup of $X$ and we define $Y=$ $C_{X}\left(g_{X}\right) /\left\langle g_{X}\right\rangle$ which is also a dihedral group. The tuples $I_{X}$ and $J_{X}$ correspond to $k$-tuples, $I_{Y}$ and $J_{Y}$, whose entries are involutions in $Y$. Since $I_{X} \widetilde{2} J_{X}$ with respect to $X$, we have $I_{Y} \widetilde{2} J_{Y}$ with respect to $Y$. But this action is binary (see the discussion of Family 3a at the start of §1.2) and so $I_{Y} \widetilde{k} J_{Y}$ with respect to $Y$. But this implies that $I_{X} \widetilde{k} J_{X}$ with respect to $X$ which is a contradiction. Hence the claim is proved.

It now follows that

$$
\bigcap_{j=1}^{k} I_{j}=\bar{R}_{0}
$$

and similarly $\bigcap_{j=1}^{k} J_{j}=\bar{R}_{0}$. Therefore $g \in N_{G}\left(\bar{R}_{0}\right)=G_{\Delta}$, which is a contradiction. This completes the proof under the assumption that $p \geq 7$.
(B) Now assume that $p<7$, so that $p=3$ or 5 . First note that the cases where $a=2$ (in which case $L=\operatorname{PSL}_{4}^{\epsilon}(p)$ or $\left.\operatorname{PSp}_{4}(p)\right)$ are covered by Lemma 4.1.1. So we may assume that $a \geq 3$.

Define $R_{a}=R_{1} \times R_{3}$ and $R_{b}=\prod_{i \neq 1,3} R_{i}$, and let $W_{a}=W_{1} \otimes W_{3}, W_{b}=\bigotimes_{i \neq 1,3} W_{i}$. Then in case $S$ or $O^{+}$, the subgroup of $G$ preserving the tensor decomposition $W_{a} \otimes W_{b}$ is the normalizer of the image of $\mathrm{Sp}\left(W_{a}\right) \otimes \mathrm{Cl}\left(W_{b}\right)$, where $\mathrm{Cl}\left(W_{b}\right)$ is orthogonal or symplectic, respectively. The normalizer $N_{G}\left(\bar{R}_{b}\right)$ preserves this tensor decomposition.

Define a subset $\Delta$ of $\Omega$ by $\Delta=\left\{\bar{R}^{x}: x \in N_{G}\left(\bar{R}_{b}\right)\right\}$. Let $X$ be the image of the group induced on $W_{a}$ by $N_{G}\left(\bar{R}_{b}\right)$. Then $X$ has socle $\operatorname{PSL}_{4}^{\epsilon}(p)$ or $\operatorname{PSp}_{4}(p)$, and $\Delta=\left\{\bar{R}_{b} \times \bar{R}_{a}^{x}: x \in X\right\}$. Also, from the structure of $M \cap L$, we see that

$$
N_{X}\left(\bar{R}_{a}\right) \cong\left\{\begin{array}{l}
2^{4} \cdot \mathrm{Sp}_{4}(2), \text { case } L^{\epsilon} \\
2^{4} \cdot \mathrm{O}_{4}^{-}(2), \text { cases } S, O^{+}
\end{array}\right.
$$

As above, the action of $G_{\Delta}$ on $\Delta$ is isomorphic to the action of $X$ on the cosets of $N_{X}\left(\bar{R}_{a}\right)$. Using magma, we check in all possible cases that this action is not binary, and that there exist $k$-tuples $I=\left(I_{1}, \ldots, I_{k}\right)$, $J=\left(J_{1}, \ldots, J_{k}\right) \in \Delta^{k}$ such that $I \widetilde{2} J$ and $I{\underset{\sim}{k}} J$ with respect to the action of $G_{\Delta}$, and also such that $\bigcap_{j=1}^{k} I_{j}=\bigcap_{j=1}^{k} J_{j}=\bar{R}_{b}$. Now we see exactly as in the argument at the end of part (A) that $I \not \widehat{x}_{k} J$ with respect to the action of $G$. Hence $G$ is not binary, and the proof is complete.

### 4.7 Family $\mathcal{C}_{7}$

In this case $M$ is the stabilizer of a tensor decomposition of $V$, in much the same way as was detailed at the start of 84.4 . In this case, though, $M$ stabilizes a tensor product of two or more subspaces of the same dimension: we write $V=W_{1} \otimes \cdots \otimes W_{t}$, and $m:=\operatorname{dim}\left(W_{1}\right)=\cdots=\operatorname{dim}\left(W_{t}\right)$. Observe that $\operatorname{dim}(V)=n=m^{t}$. If $G=\operatorname{PGL}_{n}(q)$, the stabilizer $M$ has the structure $\mathrm{PGL}_{m}(q)$ wr $\operatorname{Sym}(t)$, where $\operatorname{Sym}(t)$ permutes the tensor factors.

In the case where $S$ is not $\mathrm{SL}_{n}(q)$, i.e. $S$ preserves a non-degenerate form $\varphi$, the spaces $W_{1} \ldots, W_{t}$ are mutually similar spaces equipped with non-degenerate forms $\varphi_{1}, \ldots, \varphi_{t}$, and

$$
\varphi= \begin{cases}Q\left(\varphi_{1} \otimes \cdots \otimes \varphi_{t}\right), & \text { if } q \text { is even and } \varphi_{1}, \ldots, \varphi_{t} \text { are non-degenerate alternating; } \\ \varphi_{1} \otimes \cdots \otimes \varphi_{t}, & \text { otherwise. }\end{cases}
$$

The definition of the quadratic form $Q\left(\varphi_{1} \otimes \cdots \otimes \varphi_{t}\right)$ is given on [54, p.127]: it is the unique non-degenerate quadratic form $Q$ such that

1. $Q\left(w_{1} \otimes \cdots \otimes w_{t}\right)=0$ for all $w_{i} \in W_{i}$, and
2. the polarization of $Q$ is equal to $\varphi_{1} \otimes \cdots \otimes \varphi_{t}$.

Again the stabilizer $M$ has a wreath product structure. It is convenient to set $\varphi$ to be the zero map when $S=\mathrm{SL}_{n}(q)$. We have given a list of all the $\mathcal{C}_{7}$ embeddings in Table 4.7.1, taken from [54, §4.7], where the precise structures of the $\mathcal{C}_{7}$ subgroups can be found.

| case | type | conditions |
| :---: | :---: | :---: |
| $\mathrm{L}^{\epsilon}$ | $\mathrm{PGL}_{m}^{\epsilon}(q) \mathrm{wr} \operatorname{Sym}(t)$ | $m \geq 3$ |
| S | $\mathrm{PSp}_{m}(q) \operatorname{wrSym}(t)$ | $q t$ odd |
| $\mathrm{O}^{+}$ | $\mathrm{PO}_{m}^{ \pm}(q) w r \operatorname{Sym}(t)$ | $q$ odd |
| $\mathrm{O}^{+}$ | $\mathrm{PSp}_{m}(q) \operatorname{wrSym}(t)$ | $q t$ even |
| O | $\mathrm{PO}_{m}(q)$ wr $\operatorname{Sym}(t)$ | $q m$ odd |

Table 4.7.1: Maximal subgroups in family $\mathcal{C}_{7}$
Note that [54, p. 156] details a further restriction on the subgroup $M$, namely that the relevant subgroup $\mathrm{Cl}_{m}(q)$ must be quasisimple. For instance, in the $\mathrm{O}^{+}$case that is listed on Line 3 of the table, we require that $\Omega_{m}^{ \pm}(q)$ is quasisimple; thus, for this case, $m \geq 6$ or $(m, \varepsilon)=(4,-)$. In general we have that the socle of $M$ is $\left(\mathrm{Cl}_{m}(q)\right)^{t}$.

The main result of this section is the following. The result will be proved in a series of lemmas.
Proposition 4.7.1. Suppose that $G$ is an almost simple group with socle $\bar{S}=\mathrm{Cl}_{n}(q)$, and assume that
(i) $n \geq 3,4,4,7$ in cases $L, U, S, O$ respectively, and
(ii) $\mathrm{Cl}_{n}(q)$ is not one of the groups listed in Lemma 4.1.1.

Let $M$ be a maximal subgroup of $G$ in the family $\mathcal{C}_{7}$. Then the action of $G$ on $(G: M)$ is not binary.
The following lemma will be used in various special cases.
Lemma 4.7.2. Let $t_{0}$ be an integer, at least 2; in the case where $\varphi_{1}, \ldots, \varphi_{t}$ are non-degenerate alternating bilinear forms, we require that $q$ is odd and that $m^{t_{0}} \geq 8$.

Let $V_{0}$ be the $m^{t_{0}}$-dimensional formed space that is a tensor product of $t_{0}$ formed spaces all similar to $\left(W_{1}, \varphi_{1}\right)$. Let $\bar{S}_{0}=X_{m^{t_{0}}}(q)$, where $X \in\left\{\mathrm{PSL}, \mathrm{PSU}, \mathrm{PSp}, \mathrm{P} \Omega^{\varepsilon}\right\}$, be the simple group associated with $V_{0}$. Consider all pairs $\left(G_{0}, M_{0}\right)$ where $G_{0}$ is an almost simple group with socle $\bar{S}_{0}$ and $M_{0}$ is the subgroup of $G_{0}$ from class $\mathcal{C}_{7}$ associated with the tensor product decomposition. Let $\Omega_{0}=\left(G_{0}: M_{0}\right)$.

1. Suppose that $t_{0}=2$ and that, for all such pairs $\left(G_{0}, M_{0}\right)$, the action of $G_{0}$ on $\Omega_{0}$ is not binary. Then the action of $G$ on $(G: M)$ is not binary.
2. Suppose that $t_{0}>2$ and that, for all such pairs $\left(G_{0}, M_{0}\right)$ we can find an integer $k \geq 3$ and tuples $\left(I_{1}, \ldots, I_{k}\right),\left(J_{1}, \ldots, J_{k}\right) \in \Omega_{0}^{k}$ such that
(a) $\left(I_{1}, \ldots, I_{k}\right) \widetilde{2}\left(J_{1}, \ldots, J_{k}\right)$;
(b) $\left(I_{1}, \ldots, I_{k}\right) \not{\widehat{k}}\left(J_{1}, \ldots, J_{k}\right)$;
(c) there is no group isomorphic to $X_{m}(q)$ that is a normal subgroup of each of the socles of $\left(G_{0}\right)_{I_{1}}, \cdots,\left(G_{0}\right)_{I_{k}}$.

Then the action of $G$ on $(G: M)$ is not binary.

Note, first, that the family in which $\bar{S}_{0}$ lies (i.e. the particular choice of $X$ from $\left\{\mathrm{PSL}, \mathrm{PSU}, \mathrm{PSp}, \mathrm{P} \Omega^{\varepsilon}\right\}$ ) is determined by the type of $W_{1}$ and the value of $t_{0}$ (see Table 4.7.1). Then [54, Tables 3.5.H and 3.5.I] (and, when $n \in\{8,9\}$, [10]) imply that, since $M$ is maximal in $G$, we know that $M_{0}$ will be a maximal $\mathcal{C}_{7}$-subgroup of $G_{0}$ unless $\varphi_{1}, \ldots, \varphi_{t}$ are non-degenerate alternating and $q$ is even - but this case is explicitly ruled out by our hypotheses in the statement of this lemma.

Note, second, that in most cases the groups $\bar{S}$ and $\bar{S}_{0}$ will lie in the same family, i.e. if $\bar{S}_{0}=X_{m^{t_{0}}}(q)$, then $\bar{S}=X_{m^{t}}(q)$. The exception to this occurs when $\varphi_{1}, \ldots, \varphi_{t}$ are non-degenerate alternating bilinear forms and $t$ and $t_{0}$ have different parity (see Table 4.7.1).

Proof. We noted above that the socle of $M$ is $L^{t}$ where $L=\mathrm{Cl}_{m}(q)$, a non-abelian simple group. Write $\Gamma$ for the set of semilinear similarities of $\varphi$, so $S=F^{*}(\Gamma)$, and let $\iota$ be the inverse transpose map. Write $\underline{G}($ resp. $\underline{M})$ for the preimage of $G\left(\right.$ resp. $M$ ) in $\Gamma$ (or in $\Gamma:\langle\iota\rangle$ if $S=\mathrm{SL}_{n}(q)$ ). We write $\mathcal{D}$ for the decomposition preserved by $\underline{M}$ :

$$
(V, \varphi)=\left(W_{1}, \varphi_{1}\right) \otimes \cdots \otimes\left(W_{t}, \varphi_{t}\right) .
$$

We define $U=W_{1} \otimes \cdots \otimes W_{t_{0}}$ and $\varphi_{U}=\varphi_{1} \otimes \cdots \otimes \varphi_{t_{0}}$. Let $\underline{G_{U}}$ be the stabilizer in $\underline{G}$ of the decomposition

$$
\mathcal{D}_{U}:(V, \varphi)=\left(U, \varphi_{U}\right) \otimes\left(W_{t_{0}+1}, \varphi_{t_{0}+1}\right) \otimes \cdots \otimes\left(W_{t}, \varphi_{t}\right)
$$

Now we consider the action of $G_{U}$, the projective image of $G_{U}$ in $G$, on $(G: M)$. In particular we can consider the action on the set of cosets M. $G_{U}$; the action on this set is isomorphic to the action of $G_{U}$ on $\left(G_{U}: M_{U}\right)$ where $M_{U}=M \cap G_{U}$.

Clearly the kernel of this action contains the image in $G_{U}$ of

$$
\underline{G_{U}} \cap\left(\{1\} \otimes \Delta_{t_{0}+1} \cdots \otimes \Delta_{t}\right) J_{U}
$$

where $J_{U} \cong \operatorname{Sym}\left(t-t_{0}\right)$. The quotient of $G_{U}$ by the kernel of this action is an almost simple group $G_{0}$ with socle $X_{m^{t_{0}}}(q)$ and the stabilizer in $G_{0}$ of a point is a subgroup $M_{0}$ of $G_{0}$ from class $\mathcal{C}_{7}$ associated with a decomposition of the associated $m^{t_{0}}$-dimensional formed space into a tensor product of $t_{0}$ formed spaces all similar to $\left(W_{1}, \varphi_{1}\right)$.

By assumption we know that this action is not binary. Let $I, J$ be elements of $\left(G_{U}: M_{U}\right)^{k}$ for some integer $k \geq 3$ such that $I \widetilde{2} J$ and $I{\underset{\sim}{k}} J$ with respect to the action of $G_{U}$. Identify the entries of $I$ and $J$ with the corresponding elements of $(G: M)$. We can think of the entries of $I$ and $J$ as conjugates of $M$ in $G$; now, if $t_{0}>2$, then assume that $I$ has the property listed at $2(\mathrm{c})$. It is sufficient to prove that $I \underset{\widehat{k}}{ } J$ with respect to the action of $G$.

It is at this point that we use the fact that the socle of $M$ is $L^{t}$ where $L=\mathrm{Cl}_{m}(q)$. Define

$$
K=\underbrace{\{1\} \times \cdots \times\{1\}}_{t_{0}} \times L^{t-t_{0}} .
$$

Observe that $K$ is a normal subgroup of the socle of $M$. By construction, $K$ is a normal subgroup of each of the socles of $I_{1}, \ldots, I_{k}$ and $J_{1}, \ldots, J_{k}$. Suppose that $I^{g}=J$. Then $\left\langle K, K^{g}\right\rangle$ is a normal subgroup of the socle of $J_{i}$ for $i=1, \ldots, k$ and we see that $\left\langle K, K^{g}\right\rangle$ is isomorphic to $L^{t-s}$ for some $0 \leq s \leq t_{0}$. We can relabel so that

$$
\left\langle K, K^{g}\right\rangle=\underbrace{\{1\} \times \cdots \times\{1\}}_{s} \times L^{t-s}
$$

If $s=t_{0}$, then $K=K^{g}$. But $N_{G}(K)=G_{U}$ and so $g \in G_{U}$ which is a contradiction. Thus $s<t_{0}$. If $s=0$, then $J_{1}, \ldots, J_{k}$ have the same socle and so $J_{1}=J_{2}=\cdots=J_{k}$. Since $I \widetilde{2} J$ we obtain that $I_{1}=I_{2}=\cdots=I_{k}$ and so $I \widetilde{k} J$ with respect to the action of $G_{U}$, which is a contradiction. Thus $0<s<t_{0}$.

Now we refer to [54, Lemma 4.4.3] from which we deduce that $C_{S}\left(\left\langle K, K^{g}\right\rangle\right)$ must be a subgroup of $\mathrm{GL}\left(W_{1} \otimes \cdots \otimes W_{s}\right) \times 1^{t-s}$ and, of course, must preserve the form $\varphi$. If $s=1$, then this means that

| Group | Details of action |
| :--- | :--- |
| $\mathrm{SL}_{3^{t}}(2)$ | $m=3: M \triangleright \mathrm{PSL}_{3}(2)^{t}$ |
| $\mathrm{SL}_{4^{t}}(2)$ | $m=4: M \triangleright \mathrm{PSL}_{4}(2)^{t}$ |

Table 4.7.2: $\mathcal{C}_{7}-\mathrm{SL}_{n}(q)$ - Cases where a beautiful subset was not found.
there is a unique conjugate of $M$ whose socle contains $\left\langle K, K^{g}\right\rangle$ and, again, $J_{1}=J_{2}=\cdots=J_{k}$ which is a contradiction as before. This proves the result when $t_{0}=2$.

If $t_{0}>2$, then the property listed at 2(c) implies that the only conjugate of $K$ that is normal in each of the socles of $I_{1}, \ldots, I_{k}$ is $K$ itself. Hence, since $J=I^{g}$ and since $K$ is normal in each of the socles of $J_{1}, \ldots, J_{k}$, we conclude that the only conjugate of $K$ that is normal in each of the socles of $J_{1}, \ldots, J_{k}$ is $K$ itself. But this means that $K^{g}=K$ and, again, the fact that $N_{G}(K)=G_{U}$ implies that $g \in G_{U}$, a contradiction.

### 4.7.1 Case $S=\mathrm{SL}_{n}(q)$

In this case [54, Table 3.5.A] allows us to assume that $m \geq 3$.
Lemma 4.7.3. In this case either $\Omega$ contains a beautiful subset or else the action is listed in Table 4.7.2.,
Proof. We write $W_{1}=\cdots=W_{t}$, and let $\mathcal{B}_{1}=\left\{e_{1}, \ldots, e_{m}\right\}$ be a basis for $W_{1}$. Then

$$
\left.\mathcal{B}=\left\{e_{i_{1}} \otimes \cdots \otimes e_{i_{t}}\right\} \mid 1 \leq i_{1}, \ldots, i_{t} \leq m\right\}
$$

is a basis for $V$, and we take $M$ to be the stabilizer the associated tensor decomposition, so that $M \cap \bar{S}=$ $\left(\operatorname{PGL}_{m}(q)\right.$ wr $\left.\operatorname{Sym}(t)\right) \cap \bar{S}$.

First assume that $q \geq 7$, and let $T_{1}$ be a split maximal torus in $\mathrm{SL}_{m}(q)$ that is diagonal with respect to $\mathcal{B}_{1}$; then $T=T_{1} \otimes 1 \otimes \cdots \otimes 1$ is a subgroup of (the preimage of) $M$. Define $U$ to be the set of elements in $S$ for which there exists $\alpha \in \mathbb{F}_{q}$ such that

$$
e_{1} \otimes \cdots \otimes e_{1} \mapsto e_{1} \otimes \cdots \otimes e_{1}+\alpha e_{2} \otimes e_{1} \otimes \cdots \otimes e_{1}
$$

and which fixes all elements $e_{i_{1}} \otimes \cdots \otimes e_{i_{t}} \in \mathcal{B}$ for which $i_{j}>1$ for some $j$. Observe that $U$ is not a subgroup of $M$, that $T$ normalizes $U$ and that $T$ acts transitively on the non-identity elements of $U$. We define $\Delta=M^{U}$, a subset of $\Omega$ of size $q$ and observe that $U \rtimes T$ acts 2 -transitively on $\Delta$. On the other hand,

$$
\begin{equation*}
M_{(\Delta)} \geq C_{M}(U) \geq[\mathrm{GL}_{m-2}(q) \circ(\underbrace{\mathrm{GL}_{m-1}(q) \circ \cdots \circ \mathrm{GL}_{m-1}(q)}_{t-1}) \cdot \operatorname{Sym}(t-1)] \cap \bar{S} \tag{4.7.1}
\end{equation*}
$$

Assuming that $\Delta$ is not beautiful, $G^{\Delta}$ induces at least $\operatorname{Alt}(q)$ on $\Delta$, hence the point stabilizer $M^{\Delta}$ induces at least $\operatorname{Alt}(q-1)$ on $\Delta$. However, $M_{(\Delta)}$ contains $C_{M}(U)$, which contains the group on the right hand side of 4.7.1. It follows that any simple section of $M^{\Delta}$ is a section of $\mathrm{GL}_{2}(q)$. Since $q \geq 7$, it follows from Lemma 2.1.1 that $\operatorname{Alt}(q-1)$ is not a section of $M^{\Delta}$. This implies that $\Delta$ is a beautiful subset and Lemma 1.6.12 yields the result.

Next assume that $q \in\{3,4,5\}$, and let $T_{2}$ be a maximal torus in $\mathrm{GL}_{m}(q)$ that preserves the decomposition

$$
\left\langle e_{1}\right\rangle \oplus\left\langle e_{2}, e_{3}\right\rangle \oplus\left\langle e_{4}\right\rangle \oplus \cdots \oplus\left\langle e_{m}\right\rangle
$$

and that acts on $\left\langle e_{2}, e_{3}\right\rangle$ as a Singer cycle; let $T_{1}$ be as above. Then $T_{2} \otimes T_{1} \otimes 1 \otimes \cdots \otimes 1$ is a subgroup of $M$. Define $U$ to be the set of elements in $S$, for which there exists $\alpha, \beta \in \mathbb{F}_{q}$ such that

$$
e_{1} \otimes \cdots \otimes e_{1} \mapsto e_{1} \otimes \cdots \otimes e_{1}+\alpha e_{2} \otimes e_{1} \otimes \cdots \otimes e_{1}+\beta e_{3} \otimes e_{1} \otimes \cdots \otimes e_{1}
$$

| Group | Details of action |
| :---: | :--- |
| $\mathrm{SU}_{3^{t}}(q)$ | $q \in\{3,4,5\}, m=3: M \triangleright \operatorname{PSU}_{3}(q)^{t}$ |
| $\mathrm{SU}_{m^{t}}(2)$ | $m \in\{4,5\}: M \triangleright \operatorname{PSU}_{m}(2)^{t}$ |

Table 4.7.3: $\mathcal{C}_{7}-\mathrm{SU}_{n}(q)-$ Cases where a beautiful subset was not found.
and which fixes all other elements of $\mathcal{B}$. Observe that $U$ is not a subgroup of $M$, that $T$ normalizes $U$ and that $T$ acts transitively on the non-identity elements of $U$. We define $\Delta=M^{U}$, a subset of $\Omega$ of size $q^{2}$ and observe that $U \rtimes T$ acts 2-transitively on $\Delta$. Arguing as above, we see that any non-abelian simple section of $M^{\Delta}$ is isomorphic to a section of $\mathrm{GL}_{3}(q)$; hence, since $q \geq 3, \operatorname{Alt}\left(q^{2}-1\right)$ is not a section of $M^{\Delta}$. This implies that $\Delta$ is a beautiful subset and Lemma 1.6.12 yields the result.

When $q=2$, we assume that $m \geq 5$ and we proceed similarly: we construct a beautiful subset of size $q^{4}=16$, using the same method but this time we choose a maximal torus $T_{4}$ in $\mathrm{GL}_{m}(q)$ preserving the decomposition

$$
\left\langle e_{1}\right\rangle \oplus\left\langle e_{2}, e_{3}, e_{4}, e_{5}\right\rangle \oplus\left\langle e_{6}\right\rangle \oplus \cdots \oplus\left\langle e_{m}\right\rangle,
$$

and acting on $\left\langle e_{2}, e_{3}, e_{4}, e_{5}\right\rangle$ as a Singer cycle. At the final stage, we use the fact that $\operatorname{Alt}\left(q^{4}-1\right)=\operatorname{Alt}(15)$ is not a section of $\mathrm{GL}_{5}(2)$ to conclude that the set we have constructed is indeed beautiful.

Lemma 4.7.4. If the action is listed in Table 4.7.2, then the action is not binary.
Proof. We begin with the case when $t=2$ for which we use magma. Let $S$ be either $\mathrm{SL}_{16}(2)$ or $\mathrm{SL}_{9}(2)$ and let $M$ be a maximal subgroup of $G$ in the Aschbacher class $\mathcal{C}_{7}$. With magma, we have first computed a Sylow 2-subgroup of $M$, say $Q$. Then, we have computed $P=N_{G}\left(N_{G}(Q)\right)$ and we have found an element $g \in P$, with the property that

- $\left|M: M \cap M^{g}\right|=294$ when $G=\operatorname{SL}_{9}(2)$,
- $\left|M: M \cap M^{g}\right|=588$ when $G=\operatorname{Aut}\left(\operatorname{SL}_{9}(2)\right)$,
- $\left|M: M \cap M^{g}\right|=11025$ when $G \in\left\{\mathrm{SL}_{16}(2), \operatorname{Aut}\left(\mathrm{SL}_{16}(2)\right)\right\}$.

In particular, in the faithful primitive action of $S$ on the right $\operatorname{cosets} \Omega$ of $M$, a point stabilizer has a suborbit $\Delta$ with the property that the action of $M$ on $\Delta$ is permutation isomorphic to the action of $M$ on the right cosets of $M \cap M^{g}$. We have constructed the permutation representation of $M$ under consideration (that is, on the right cosets of $M \cap M^{g}$ ) and we have verified that in this action ( $M: M \cap M^{g}$ ) contains a beautiful subset of cardinality 7 when $S=\mathrm{SL}_{9}(2)$ and cardinality 5 when $S=\mathrm{SL}_{16}(2)$. This immediately yields that the action of $M$ on $\Delta$ is not binary and hence the action of $S$ on $\Omega$ is also not binary.

If $t>2$, then we use the result for $t=2$ combined with Lemma 4.7.2.

### 4.7.2 Case $S=\mathrm{SU}_{n}(q)$

In this case [54, Table 3.5.B] allows us to assume that $m \geq 3$, and that $(q, m) \neq(2,3)$.
Lemma 4.7.5. In this case either $\Omega$ contains a beautiful subset or else the action is listed in Table 4.7.3,
Proof. Our method here will be very reminiscent of that used in Lemma 4.4.3, We start by writing $W=W_{1}=\cdots=W_{t}$, and letting $\mathcal{B}_{1}=\left\{u_{1}, \ldots, v_{1}, \ldots, x\right\}$ be a hyperbolic basis for $W_{1}$ (omitting $x$ if $m$ is even). Taking pure tensors we obtain a hyperbolic basis, $\mathcal{B}$, for $V$, and we let $M$ be the stabilizer of the associated tensor decomposition. Then $M \cap \bar{S}=\operatorname{PGU}_{m}(q) \operatorname{wr~Sym}(t)$.

If $q \geq 7$, then we consider subgroups $U$ and $T$ of $\mathrm{GU}\left(\left\langle u_{1}, x, v_{1}\right\rangle\right)$ defined as per (4.4.1) and (4.4.2). (In what follows, for $i \in \mathbb{Z}^{+}$, we write $x^{i}$ to mean $\underbrace{x \otimes \cdots \otimes x}_{i}$.)

As in Lemma 4.4.3 we now split into two cases. If $q$ is odd, then we take $U_{0}$ to be the subgroup of $U$ obtained by requiring that $b \in \mathbb{F}_{q}$ and that $c=\frac{1}{2} b^{2}$; we define an isomorphic group in $S: U_{1}$ consists of those elements for which there exists $b \in \mathbb{F}_{q}$ such that

$$
\begin{aligned}
u_{1} \otimes x^{t-1} & \mapsto u_{1} \otimes x^{t-1}+b x^{t}-\frac{1}{2} b^{2} v_{1} \otimes x^{t-1}, \\
x^{t} & \mapsto x^{t}-b v_{1} \otimes x^{t-1}, \\
v_{1} \otimes x^{t-1} & \mapsto v_{1} \otimes x^{t-1},
\end{aligned}
$$

and all elements of $\left\langle u_{1} \otimes x^{t-1}, x^{t}, v_{1} \otimes x^{t-1}\right\rangle^{\perp}$ are fixed. Then $U_{1}$ is a subgroup of order $q$ that is not contained in $M$. Now we take $T_{0}$ to be the subgroup of $T$ obtained by requiring that $r \in \mathbb{F}_{q}$ and let $T_{1}=T \circ 1 \circ \cdots \circ 1$, a group of order $q-1$ that normalizes $U_{1}$ and acts transitively on the set of non-trivial elements in $U_{1}$.

If $q$ is even, the set-up is slightly different but follows the procedure in Lemma 4.4.3 as above. In both cases, identifying $\Omega$ with conjugates of $M$ we set $\Lambda=M^{U_{1}} \subset \Omega$, and see that $S^{\Lambda}$ acts 2-transitively upon $\Lambda$, a set of size $q$. The usual argument shows that that any non-abelian simple section in $M^{\Lambda}$ is isomorphic to a section of $\mathrm{GU}_{3}(q)$. By Lemma 2.1.1, for $q \geq 7$, we conclude that $\operatorname{Alt}(q-1)$ is not a secion of $M^{\Lambda}$ and so $\Lambda$ is a beautiful subset, and Lemma 1.6 .12 implies that the action is not binary.

For $q \in\{3,4,5\}$ we diverge from the argument given in Lemma 4.4.3, and we assume that $m \geq 4$ (the first line of Table 4.7 .3 covers $m=3$ ). We proceed as for $q \geq 7$ but we use the existence of a Frobenius group in $\mathrm{GU}_{4}(q)$ this time. We let $W_{0}:=\left\langle u_{1}, u_{2}, v_{2}, v_{1}\right\rangle$ be a non-degenerate 4 -subspace of $W$, and consider the group:

$$
U \rtimes T=\left\langle\left(\begin{array}{cccc}
1 & a & & \\
& 1 & & \\
& & 1 & -a^{q} \\
& & & 1
\end{array}\right), \left.\left(\begin{array}{cccc}
r & & & \\
& 1 & & \\
& & 1 & \\
& & & r^{-q}
\end{array}\right) \right\rvert\, a, r \in \mathbb{K}, r \neq 0\right\rangle .
$$

Now we define $T_{0}$ to be the subgroup of $\mathrm{GU}(W)$ which stabilizes $W_{0}$, whose action on $W_{0}$ is equal to the action of $T$, and which fixes $W_{0}^{\perp}$ point-wise. Then we define

$$
T_{1}=\left(T_{0} \otimes T_{0} \otimes 1 \otimes \cdots \otimes 1\right) \cap S .
$$

On the other hand we let $x$ be an element of $W_{1}$ for which $(x, x)=1$, and we define

$$
V_{0}:=\left\langle u_{1} \otimes u_{2} \otimes x^{t-2}, u_{2} \otimes u_{2} \otimes x^{t-2}, v_{2} \otimes v_{2} \otimes x^{t-2}, v_{1} \otimes v_{2} \otimes x^{t-2}\right\rangle .
$$

Observe that $V_{0}$ is a non-degenerate 4 -subspace of $V$; indeed there exists an isomorphism between $W_{0}$ and $V_{0}$ which maps the listed ordered basis for $W_{0}$ to that of $V_{0}$. We can, therefore, define $U_{1}$ to be the subgroup of $\mathrm{SU}(W)$ which stabilizes $V_{0}$, whose action on $V_{0}$ is equal to the action of $U$, and which fixes $V_{0}^{\perp}$ point-wise.

Observe that $U_{1}$ is of order $q^{2}$, is contained in $S$ but not in $M$, and is normalized by $T_{1}$. Observe, moreover, that $T_{1}$ acts transitively on the set of non-identity elements of $U_{1}$. Defining $\Lambda=M^{U_{1}} \subseteq \Omega$, we therefore conclude that $S_{\Lambda}$ acts 2 -transitively on $\Lambda$. The usual argument shows that any simple section in $M^{\Lambda}$ is necessarily isomorphic to a section of $\mathrm{GU}_{4}(q)$. However Lemma 2.1.1 implies that for $q \geq 3$, $\operatorname{Alt}\left(q^{2}-1\right)$ is not a section of $M^{\Lambda}$, and so $\Lambda$ is beautiful and we are done as before.

Finally, for $q=2$, we assume that $m \geq 6$ (the cases where $m \leq 5$ are listed in Table 4.7.3). We proceed as in the previous paragraph, using the 2-transitive group constructed in Lemma 4.4.3 for the $q=2$ case. As there, the fact that $\operatorname{Alt}(15)$ is not a section of $\mathrm{GU}_{6}(2)$ allows us to construct a beautiful subset.

Lemma 4.7.6. If the action is listed in Table 4.7.3, then the action is not binary.

| Group | Details of action |
| :---: | :--- |
| $\mathrm{Sp}_{2^{t}}(5)$ | $m=2, t$ odd: $M \triangleright \mathrm{PSp}_{2}(5)^{t}$ |

Table 4.7.4: $\mathcal{C}_{7}-\operatorname{Sp}_{n}(q)$ - Cases where a beautiful subset was not found.

Proof. Our method is entirely analogous to that used in Lemma 4.7.4. We begin with the case when $t=2$.
Suppose, first, that $M \triangleright \operatorname{PSU}_{3}(q)^{2}$ and $S=\operatorname{SU}_{9}(q)$ with $q \in\{3,4,5\}$. Let $\{e, f, x\}$ and $\{v, w, y\}$ be hyperbolic bases for a Hermitian space of dimension 3 (where $(e, f)$ and $(v, w)$ are hyperbolic pairs and $x$ and $y$ are anisotropic); taking tensor products we obtain a hyperbolic basis $\mathcal{B}$ for a 9 -dimensional Hermitian space, and we obtain our embedding of $M$ in $S$. We choose an order for $\mathcal{B}$ as follows:

$$
e \otimes v, e \otimes w, e \otimes y, x \otimes v, x \otimes w, f \otimes y, f \otimes v, f \otimes w, x \otimes y
$$

Define $T$ to be the subgroup of $M$, whose elements when written with respect to $\mathcal{B}$ consist of all diagonal matrices

$$
\operatorname{diag}\left[a^{q}, a^{-1}, 1, a, a^{-q}, 1, a^{q}, a^{-1}, a^{1-q}\right]
$$

with $a \in \mathbb{F}_{q^{2}}^{*}$. Observe that $T$ normalizes and acts fixed-point-freely upon the group $U$, whose elements fix all elements of $\mathcal{B}$ except $e \otimes y$ and $x \otimes w$, and for which there exists $a \in \mathbb{F}_{q^{2}}$ such that

$$
\begin{gathered}
e \otimes y \mapsto e \otimes y+a x \otimes v, \\
x \otimes w \mapsto x \otimes w-a^{q} f \otimes y .
\end{gathered}
$$

Since $U$ is not in $M$, we obtain, in the usual way, a set $\Lambda$ of size $q^{2}$, on which $S_{\Lambda}$ acts 2-transitively. Now observe that an alternating section, $\operatorname{Alt}(t)$ of $M$ satisfies $t \leq 7$, and so we conclude that $M^{\Lambda}$ does not have a section $\operatorname{Alt}\left(q^{2}-1\right)$. We conclude that the set $\Lambda$ is a beautiful subset and Lemma 1.6.12 yields the result.

Next suppose that $M \triangleright \operatorname{PSU}_{m}(2)^{2}$ and $S=\operatorname{SU}_{m^{2}}(2)$ with $m \in\{4,5\}$. In both cases we take a pair of hyperbolic bases $\left\{e_{1}, e_{2}, f_{1}, f_{2}\right\}$ and $\left\{v_{1}, v_{2}, w_{1}, w_{2}\right\}$ (adding in an anisotropic element when $m=5$ ), and we take tensor products to obtain a hyperbolic basis, $\mathcal{B}$, for an $m^{2}$-dimensional Hermitian space. Now $M$ has a subgroup isomorphic to $A=\mathrm{SL}_{2}(4)$ that preserves the subspaces $\left\langle e_{1} \otimes v_{1}, e_{1} \otimes v_{2}\right\rangle$ and $\left\langle f_{1} \otimes w_{1}, f_{1} \otimes w_{2}\right\rangle$ and fixes all other elements of $\mathcal{B}$.

What is more $A$ lies inside a subgroup $X \cong \mathrm{SL}_{3}(4) \leq S$ that preserves the subspace $\left\langle e_{1} \otimes v_{1}, e_{1} \otimes\right.$ $\left.v_{2}, f_{1} \otimes v_{1}\right\rangle$, and note that $X \not \leq M$. Then Lemma 1.6 .10 implies that there is a subset $\Delta$ of $\Omega$ of size 16 on which $S^{\Delta}$ acts 2-transitively. Since $\mathrm{SU}_{5}(2)$ does not contain a section isomorphic to Alt(16), we obtain that $\Delta$ is a beautiful subset and, as before, Lemma 1.6 .12 yields the result.

Now for $t>2$ we use Lemma 4.7.2 and the fact that the result is proved for $t=2$.

### 4.7.3 Case $S=\operatorname{Sp}_{n}(q)$

In this case [54, Table 3.5.C] implies that $q t$ is odd, that $m$ is even, that $t \geq 3$, and that $(m, q) \neq(2,3)$.
Lemma 4.7.7. In this case either $\Omega$ contains a beautiful subset or else the action is listed in Table 4.7.4.
Proof. We start by writing $W=W_{1}=\cdots=W_{t}$, and letting $\mathcal{B}_{1}=\left\{u_{1}, \ldots u_{m / 2}, v_{m / 2}, \ldots v_{1}\right\}$ be a hyperbolic basis for $W_{1}$. Taking pure tensors we obtain a hyperbolic basis, $\mathcal{B}$, for $V$, and we let $M$ be the subgroup of $G$ that stabilizes the associated tensor decomposition. Then $M \cap \bar{S}=\left(\operatorname{PGSp}_{m}(q)\right.$ wr $\left.\operatorname{Sym}(t)\right) \cap \bar{S}$.

First suppose that $m \geq 4$. We define two subgroups of $\operatorname{Sp}_{m}(q)$ :

$$
\begin{aligned}
& U:=\left\{\left.\left(\begin{array}{ccccc}
1 & a_{1} & \cdots & a_{m-2} & \\
& 1 & & & a_{m-2} \\
& & \ddots & & \vdots \\
& & & 1 & -a_{1} \\
& & & 1
\end{array}\right) \right\rvert\, a_{1}, \ldots a_{m-2} \in \mathbb{F}_{q}\right\}, \\
& T:=\left\{\left.\left(\begin{array}{lll}
1 & & \\
& A & \\
& & 1
\end{array}\right) \right\rvert\, A \in \operatorname{Sp}_{m-2}(q)\right\} .
\end{aligned}
$$

Our construction is inspired by the observation that $T$ normalizes $U$ and acts transitively on the set of non-trivial elements of $U$. We define $T_{1}=T \circ 1 \circ \cdots \circ 1<S$ and we define the group $U_{1}$ to be the set of elements for which there exist $a_{1}, \ldots, a_{m-2}$ such that

$$
\begin{aligned}
u_{1}^{t} \mapsto u_{1}^{t}+a_{1} u_{2} \otimes u_{1}^{t-1}+\cdots+a_{(m-2) / 2} u_{m / 2} \otimes u_{1}^{t-1}+a_{m / 2} v_{m / 2} \otimes u_{1}^{t-1}+\cdots+a_{m-2} v_{2} \otimes u_{1}^{t-1}, \\
v_{i} \otimes v_{1}^{t-1} \mapsto v_{i} \otimes v_{1}^{t-1}-a_{i-1} v_{1}^{t}, \\
u_{i} \otimes v_{1}^{t-1} \mapsto u_{i} \otimes v_{1}^{t-1}+a_{m-i} v_{1}^{t},
\end{aligned}
$$

for $i=2, \ldots, \frac{m}{2}$, and all other elements of $\mathcal{B}$ are fixed. Observe that $U_{1}$ is of order $q^{m-2}$, is contained in $S$ but not in $M$, and is normalized by $T_{1}$. Furthermore $T_{1}$ acts transitively on the set of non-identity elements of $U_{1}$. Defining $\Lambda=M^{U_{1}} \subseteq \Omega$, we conclude that $S_{\Lambda}$ acts 2 -transitively on the elements of $\Lambda$. The usual argument shows that any simple section of $M^{\Lambda}$ is necessarily isomorphic to a section of $\operatorname{Sp}_{m-2}(q)$. We conclude that either $\Lambda$ is beautiful or else $\operatorname{Sp}_{m-2}(q)$ contains a section isomorphic to $\operatorname{Alt}\left(q^{m-2}-1\right)$, which is impossible by Lemma 2.1.1 (recall that $q$ is odd).

We are left with the situation where $m=2$, in which case we use the fact that a Borel subgroup of $\operatorname{GSp}_{2}(q)=\mathrm{GL}_{2}(q)$ has a 2 -transitive action on $q$ points. We use the basis $\mathcal{B}_{1}=\left\{u_{1}, v_{1}\right\}$, and consider the group:

$$
B=U \rtimes T=\left\langle\left(\begin{array}{ll}
1 & a  \tag{4.7.2}\\
& 1
\end{array}\right), \left.\left(\begin{array}{ll}
r & \\
& s
\end{array}\right) \right\rvert\, a, r, s \in \mathbb{F}_{q}, r \neq 0 \neq s\right\rangle .
$$

Then we define

$$
T_{1}=(T \circ \cdots \circ T) \cap S
$$

Next we define the group $U_{1}$ in $S$ to be the set of elements for which there exists $b \in \mathbb{F}_{q}$ such that

$$
\begin{aligned}
u_{1}^{t} & \mapsto u_{1}^{t}+b v_{1} \otimes u_{1}^{t-1}, \\
u_{1} \otimes v_{1}^{t-1} & \mapsto u_{1} \otimes v_{1}^{t-1}+b v_{1}^{t},
\end{aligned}
$$

and all elements of $\left\langle u_{1}^{t}, u_{1} \otimes v_{1}^{t-1}\right\rangle^{\perp}$ are fixed. Observe that $U_{1}$ is of order $q$, is contained in $S$ but not in $M$, and is normalized by $T_{1}$. We can use [54, Proposition 4.7.4] to check that $T_{1}$ acts transitively on the set of non-identity elements of $U_{1}$. Defining $\Lambda=M^{U_{1}} \subseteq \Omega$, we conclude that $S_{\Lambda}$ acts 2-transitively on the elements of $\Lambda$. As usual, either $\Lambda$ is beautiful or else $\operatorname{Sp}_{2}(q)$ contains a section isomorphic to $\operatorname{Alt}(q-1)$. This yields the result for $q \geq 7$. We are left with the case listed in Table 4.7.4 (recall that $(m, q)=(2,3)$ is excluded).

Lemma 4.7.8. If the action is listed in Table 4.7.4, then the action is not binary.
Proof. If $t=3$, then $S=\operatorname{Sp}_{8}(5)$ and we use magma to verify the result. If $t>3$, then we use the result for $t=3$ combined with Lemma 4.7.2, Our application of Lemma 4.7.2 requires that we check the property listed at $2(\mathrm{c})$ : suppose that $G_{0}$ has socle $\bar{S}_{0} \cong \mathrm{Sp}_{8}(5)$, that $k, I_{1}, \ldots, I_{k}, J_{1}, \ldots, J_{k}$ are as given in the lemma and that they satisfy the properties listed at 2(a) and 2(b) - our magma calculations confirm that

| Group | Details of action |
| :--- | :--- |
| $\Omega_{3^{t}}(5)$ | $m=3: M \triangleright \Omega_{3}(5)^{t}$ |

Table 4.7.5: $\mathcal{C}_{7}-\Omega_{n}(q)-$ Cases where a beautiful subset was not found.
such cosets do exist. Suppose that the property listed at 2(c) is not satisfied, in which case there exists a group $K \cong \operatorname{PSp}_{2}(5)$ that is a normal subgroup of the socles of $\left(G_{0}\right)_{I_{1}}, \ldots,\left(G_{0}\right)_{I_{k}}$. Then [54, Lemma 4.4.3] implies that $C_{\bar{S}_{0}}(K)$ is isomorphic to a subgroup of $\mathrm{O}_{4}^{+}(5)$, which has socle isomorphic to $\mathrm{PSp}_{2}(5) \times \mathrm{PSp}_{2}(5)$. Since the socles of $\left(G_{0}\right)_{I_{1}}, \ldots,\left(G_{0}\right)_{I_{k}}$ are isomorphic to $\mathrm{PSp}_{2}(5) \times \mathrm{PSp}_{2}(5) \times \mathrm{PSp}_{2}(5)$, we conclude that the socles of $\left(G_{0}\right)_{I_{1}}, \ldots,\left(G_{0}\right)_{I_{k}}$ are all equal and hence $I_{1}=\cdots=I_{k}$. Then the property listed at $2(\mathrm{a})$ implies that $J_{1}=\cdots=J_{k}$ and now the property listed at 2(b) yields a contradiction. We conclude, therefore, that the property listed at $2(\mathrm{c})$ is satisfied.

### 4.7.4 Case $S=\Omega_{n}(q), n$ odd

In this case note that $m$ and $q$ are odd, and [54, Table 3.5.D] implies that $(m, q) \neq(3,3)$.
Lemma 4.7.9. In this case either $\Omega$ contains a beautiful subset or else the action is listed in Table 4.7.5.
Proof. We start by writing $W=W_{1}=\cdots=W_{t}$, and letting $\mathcal{B}_{1}=\left\{u_{1}, \ldots, v_{1}, \ldots, x\right\}$ be a hyperbolic basis for $W_{1}$. Taking pure tensors we obtain a hyperbolic basis, $\mathcal{B}$, for $V$, and we let $M$ be the subgroup of $G$ that stabilizes the associated tensor decomposition. Then $M \cap \bar{S}=\left(\Omega_{m}(q) \operatorname{wr} \operatorname{Sym}(t)\right) \cap \bar{S}$.

First suppose that $q \geq 7$; here our method is very similar to that used in Lemma 4.7.5. We define analogues of the groups defined at (4.4.1) and (4.4.2): we consider subgroups of $\mathrm{SO}\left(\left\langle u_{1}, x, v_{1}\right\rangle\right)$ consisting of elements of the form

$$
\begin{align*}
& U=\left\{\left.\left(\begin{array}{ccc}
1 & b & -\frac{b^{2}}{2} \\
& 1 & -b \\
& & 1
\end{array}\right) \right\rvert\, b \in \mathbb{F}_{q}\right\} ;  \tag{4.7.3}\\
& T=\left\{\left.\left(\begin{array}{lll}
r & \\
& 1 & \\
& & r^{-1}
\end{array}\right) \right\rvert\, r \in \mathbb{F}_{q} \text { with } r \neq 0\right\} . \tag{4.7.4}
\end{align*}
$$

Then $U \rtimes T$ is a Borel subgroup of $\mathrm{SO}_{3}(q)$, and is a Frobenius group. We let $T_{1}=(T \circ \cdots \circ T) \cap S$, a subgroup of $M$; we let $U_{1}$ be the group consisting of elements for which there exists $b \in \mathbb{F}_{q}$ such that

$$
\begin{aligned}
u_{1} \otimes x^{t-1} & \mapsto u_{1} \otimes x^{t-1}+b x^{t}-\frac{1}{2} b^{2} v_{1} \otimes x^{t-1}, \\
x^{t} & \mapsto x^{t}-b v_{1} \otimes x^{t-1}, \\
v_{1} \otimes x^{t-1} & \mapsto v_{1} \otimes x^{t-1},
\end{aligned}
$$

and all elements of $\left\langle u_{1} \otimes x^{t-1}, x^{t}, v_{1} \otimes x^{t-1}\right\rangle^{\perp}$ are fixed. Then $U_{1}$ is a subgroup of order $q$ that is not contained in $M$. Now $T_{1}$ normalizes $U_{1}$ and acts transitively on the set of non-trivial elements in $U_{1}$.

In the same way as before we obtain a beautiful subset, provided $\operatorname{Alt}(q-1)$ is not a section of $\mathrm{SO}_{3}(q)$; this is true for $q \geq 7$ by Lemma 2.1.1.

Suppose that $q \in\{3,5\}$ and $m \geq 5$. We define

$$
T=\{g \circ \underbrace{1 \circ \cdots \circ 1}_{t-1} \left\lvert\, \begin{array}{l}
g \in \Omega_{m}(q), x^{g}=x, \\
g \text { stabilizes both }\left\langle u_{1}, \ldots, u_{(m-1) / 2}\right\rangle \text { and }\left\langle v_{1}, \ldots, v_{(m-1) / 2}\right\rangle
\end{array}\right.\} .
$$

| Group | Details of action |
| :--- | :--- |
| $\Omega_{2^{+}(5)}^{+}($ | $U_{1}$ symplectic: $M \triangleright \mathrm{PSp}_{2}(5)^{t}$ |
| $\Omega_{8 t^{t}}(3)$ | $U_{1}$ orthogonal: $M \triangleright{\mathrm{P} \Omega_{8}^{-}(3)^{t}}^{\Omega_{4^{t}}^{+}(q)}$ |$q^{t \in\{3,5\}, U_{1} \text { orthogonal: } M \triangleright \mathrm{P} \Omega_{4}^{-}(q)^{t}}$

Table 4.7.6: $\mathcal{C}_{7}-\Omega_{n}^{+}(q)$ - Cases where a beautiful subset was not found.

Now define $U$ to be the set of elements $g$ such that, for $i=1, \ldots, \frac{m-1}{2}$, there exist $a_{i} \in \mathbb{F}_{q}$ such that

$$
\begin{aligned}
& x \otimes u_{1}^{t-1} \mapsto x \otimes u_{1}^{t-1}+a_{1} u_{1}^{t}+a_{2} u_{2} \otimes u_{1}^{t-1}+\cdots+a_{(m-1) / 2} u_{(m-1) / 2} \otimes u_{1}^{t-1} \\
& v_{i} \otimes v_{1}^{t-1} \mapsto v_{i} \otimes v_{1}^{t-1}-a_{i} x \otimes v_{1}^{t-1},
\end{aligned}
$$

and all other members of $\mathcal{B}$ are fixed. In exactly the same way as before, we see that $T$ normalizes $U$, that $T$ acts transitively on the set of non-trivial elements of $U$, that $T$ is in $M$, and that $U$ is not contained in $M$. Then, identifying $\Omega$ with conjugates of $M$, and setting $\Delta=M^{U}$, we conclude that $\Delta$ is a set of size $q^{(m-1) / 2}$ whose set-wise stabilizer acts 2 -transitively.

As usual, either $\Delta$ is a beautiful subset and we are done, or $M^{\Delta}$ has a section $\operatorname{Alt}\left(q^{(m-1) / 2}-1\right)$, in which case $\mathrm{SO}_{m}(q)$ also has such a section. This is not the case by Lemma 2.1.1.

The remaining case $m=3, q=5$ is in Table 4.7.5 (recall that $(m, q)=(3,3)$ is excluded).
Lemma 4.7.10. If the action is listed in Table 4.7.5, then the action is not binary.
Proof. When $t=2$, we have $S=\Omega_{9}(5)$ and we use magma to verify the result, see Lemma 4.1.1. The remainder of the proof, for $t>2$, proceeds using the result for $t=2$ along with Lemma 4.7.2,

### 4.7.5 $\quad$ Case $S=\Omega_{n}^{+}(q)$

In this case there are two subfamilies, as listed in Table 4.7.1.
Lemma 4.7.11. In this case either $\Omega$ contains a beautiful subset or else the action is listed in Table 4.7.6.
Proof. Note that [10, Table 8.50] allows us to exclude $n=8$ in all cases; in particular this means $n \geq 16$. We split into two cases.

First consider line 4 of table 4.7.1. In this case $W$ is equipped with an alternating form, $M \cap \bar{S}=$ $\left(\operatorname{PGSp}_{m}(q) \operatorname{wr} \operatorname{Sym}(t)\right) \cap \bar{S}$, and both $m$ and $q t$ are even. Furthermore in the case where $q$ is even, 54, Tables 3.5.E and 3.5.I] (and the explanation on p.69) imply that $m \geq 6$.

Our method is virtually identical to that of Lemma 4.7.7. For $m>2$ we proceed as before, except that we make a sign adjustment for the elements of $U_{1}$.

We obtain the same conclusion as in Lemma 4.7.7 - the existence of a beautiful subset of size $q^{m-2}$ and we are done.

For $m=2$ our proof is, again, the same as that of Lemma 4.7.7. Note that, by [54, Table 3.5.E], $q$ is odd, $q \geq 5$; in addition we may assume that $t \geq 4$. We obtain a beautiful set except when $q=5$, and this case is listed in Table 4.7.6.

Now consider line 3 of Table 4.7.1. Here $q$ is odd, $W$ is equipped with a symmetric form of type $\varepsilon \in\{+,-\}$ and $M \cap \bar{S}=\left(\mathrm{PO}_{m}^{\epsilon}(q)\right.$ wr $\left.\operatorname{Sym}(t)\right) \cap \bar{S}$. Furthermore [54, Table 3.5.E] implies that $m \geq 5+\varepsilon 1$. This time $\mathcal{B}_{1}=\left\{u_{1}, \ldots, u_{m / 2-1}, v_{1}, \ldots, v_{m / 2-1}, x, y\right\}$ is a hyperbolic basis for $W$ if $\varepsilon=-$, while $\mathcal{B}_{1}=$ $\left\{u_{1}, \ldots, u_{m / 2}, v_{1}, \ldots, v_{m / 2}\right\}$ is a hyperbolic basis for $W$ if $\varepsilon=+$. Taking pure tensors we obtain a basis, $\mathcal{B}$, for $V$, and we let $M$ be the subgroup of $G$ that stabilizes the associated tensor decomposition. Write $k$ for the Witt index of $W$.

Assume first that $k \geq 3$. Consider the following group:

$$
T=\{g \circ \underbrace{1 \circ \cdots \circ 1}_{t-1} \left\lvert\, \begin{array}{l}
g \in \Omega_{m}^{\varepsilon}(q), u_{k}^{g}=u_{k}, v_{k}^{g}=v_{k} \\
g \text { stabilizes both }\left\langle u_{1}, \ldots, u_{k-1}\right\rangle \text { and }\left\langle v_{1}, \ldots, v_{k-1}\right\rangle
\end{array}\right.\} .
$$

Now define $U$ to be the set of elements $g$ such that, for $i=1, \ldots, k-1$, there exist $a_{i} \in \mathbb{F}_{q}$ such that

$$
\begin{aligned}
u_{k} & \otimes u_{1}^{t-1} \mapsto u_{k} \otimes u_{1}^{t-1}+a_{1} u_{1}^{t}+a_{2} u_{2} \otimes u_{1}^{t-1}+\cdots+a_{k-1} u_{k-1} \otimes u_{1}^{t-1} \\
v_{i} & \otimes v_{1}^{t-1} \mapsto v_{i} \otimes v_{1}^{t-1}-a_{i} v_{k} \otimes v_{1}^{t-1},
\end{aligned}
$$

and all other members of $\mathcal{B}$ are fixed. One can check directly that $T$ normalizes $U$, that $T$ acts transitively on the set of non-trivial elements of $U$, that $T$ is in $M$, and that $U$ is not in $M$. Then, identifying $\Omega$ with conjugates of $M$, and setting $\Lambda=M^{U}$, we conclude that $\Lambda$ is a set of size $q^{k-1}$ whose set-wise stabilizer acts 2-transitively.

Either $\Delta$ is a beautiful subset and we are done, or $\operatorname{Alt}\left(q^{k-1}-1\right)$ is a section of $\mathrm{SO}_{m}^{\epsilon}(q)$. By Lemma 2.1.1, since $k \geq 3$ and $q$ is odd, the latter can only hold if $q=3$ and $(m, \epsilon)=(8,-)$, a case listed in Table 4.7.6.

We are left with the possibility that $k \leq 2$, in which case $\varepsilon=-$ and $m \in\{4,6\}$. Suppose, first, that $m=6$. We define

Note that we take $x$ to satisfy $\varphi(x, x)=1$. Now define $U$ to be the set of elements $g$ for which there exist $a_{1}, a_{2} \in \mathbb{F}_{q}$ such that

$$
\begin{aligned}
x \otimes u_{1}^{t-1} & \mapsto x \otimes u_{1}^{t-1}+a_{1} u_{1}^{t}+a_{2} u_{2} \otimes u_{1}^{t-1}, \\
v_{1}^{t} & \mapsto v_{1}^{t}-a_{1} x \otimes v_{1}^{t-1}, \\
v_{2} \otimes v_{1}^{t-1} & \mapsto v_{2} \otimes v_{1}^{t-1}-a_{2} x \otimes v_{1}^{t-1}
\end{aligned}
$$

and all other members of $\mathcal{B}$ are fixed. As before we obtain a set $\Delta$ of size $q^{2}$ on which $M^{\Delta}$ acts 2-transitively. Either $\Delta$ is a beautiful subset and we are done, or $\operatorname{Alt}\left(q^{2}-1\right)$ is a section of $M^{\Delta}$, in which case $\mathrm{SO}_{6}^{-}(q)$ also has such a section. This is not the case by Lemma 2.1.1.

Finally, suppose that $m=4$ and take

$$
U_{0} \rtimes T_{0} \cong[q] \rtimes C_{q-1}<\Omega_{4}^{-}(q) \leq \operatorname{Isom}(W) .
$$

Define $T=T_{0} \circ \underbrace{1 \circ \cdots \circ 1}_{t-1}$. To define $U$, we first let $W_{0}=W \otimes x^{t-1}$, and we define $U$ to be the subgroup of $\Omega_{m}^{-}(q)$ which fixes, point-wise, every element of $W_{0}^{\perp}$, and whose action on $W_{0}$ is isomorphic to the action of $U_{0}$ on $W$. We can check that $T$ and $U$ have the same properties as before. Thus, following the same argument we are done unless $\mathrm{O}_{4}^{-}(q)$ contains a section isomorphic to $\operatorname{Alt}(q-1)$. Now Lemma 2.1.1 shows this can only happen if $q \in\{3,5\}$, as in Table 4.7.6,

Lemma 4.7.12. If the action is listed in Table 4.7.6, then the action is not binary.
Proof. We work through Table 4.7.6 line-by-line.
First consider Line 1 . We apply Lemma 4.7 .2 with $t_{0}=3$. In this case $\bar{S}_{0} \cong \operatorname{Sp}_{8}(5)$ and we confirm, using magma, that the actions of almost simple groups with socle $\bar{S}_{0}$ on maximal $\mathcal{C}_{7}$-subgroups of type $\mathrm{Sp}_{2}(5)$ wr $\operatorname{Sym}(3)$ are not binary. This yields the result for $t \geq 4$, as required. (Note that to confirm the property listed at 2(c) in Lemma 4.7.2 we argue as per Lemma 4.7.8,

Next consider Line 2. If $t=2$, then we let $\left\{u_{1}, u_{2}, u_{3}, v_{1}, v_{2}, v_{3}, x, y\right\}$ be a hyperbolic basis for $W$. Define

$$
T_{1}:=\left\{\left.\left(\begin{array}{cccc}
A & & & \\
& A^{-T} & & \\
& & 1 & \\
& & & 1
\end{array}\right) \right\rvert\, A \in \mathrm{SL}_{3}(3)\right\}
$$

a subgroup of $\Omega_{8}^{-}(3)$, and let $T=T_{1} \circ 1$, a subgroup of $M$. Now consider the subspace

$$
X:=\left\langle x \otimes u_{1}, u_{1} \otimes u_{2}, u_{2} \otimes u_{1}, u_{3} \otimes u_{1}, x \otimes v_{1}, v_{1} \otimes v_{1}, v_{2} \otimes v_{1}, v_{3} \otimes v_{1}\right\rangle
$$

and observe that $X$ is a non-degenerate subspace of $V$ of type $\mathrm{O}_{8}^{+}$. We define $U$ to be the set of elements in $S$ for which there exist $a, b, c \in \mathbb{F}_{q}$ such that

$$
\begin{aligned}
& x \otimes u_{1} \mapsto x \otimes u_{1}+a u_{1} \otimes u_{1}+b u_{2} \otimes u_{1}+c u_{3} \otimes u_{1}, \\
& v_{1} \otimes v_{1} \mapsto v_{1} \otimes v_{1}-a x \otimes v_{1}, \\
& v_{2} \otimes v_{1} \mapsto v_{2} \otimes v_{1}-b x \otimes v_{1}, \\
& v_{3} \otimes v_{1} \mapsto v_{3} \otimes v_{1}-c x \otimes v_{1},
\end{aligned}
$$

and all elements of $X^{\perp}$ are fixed. We see that $U$ is a subgroup of $S$ that is not contained in $M$, that $T$ normalizes $U$ and that $T$ acts transitively on the set of non-identity elements of $U$. We obtain, in the usual way, a set $\Lambda$ of size $|U|=27$ on which $G^{\Lambda}$ acts 2-transitively. Since Alt(26) is not a section of $M$, we obtain a beautiful subset and we conclude that the action is not binary by Lemma 1.6.12, For $t>2$, we use the result for $t=2$ along with Lemma 4.7.2.

Finally consider Line 3 and suppose, first, that $t=2, S=\Omega_{16}^{+}(q)$ and $M \triangleright M_{0}:=\mathrm{P} \Omega_{4}^{-}(q)^{2}$ with $q \in\{3,5\}$. We confirm the result with magma, in the following way. For all groups $M$, we calculate all actions of $M$ on the cosets of a subgroup $H \leq M$ where $(M: H)$ is odd. We find that the only binary actions occur when $H=M$.

Now, observe that $|M: H|$ is even, thus a Sylow 2-subgroup, $P$, of $M$ is normalized by a 2 -group $Q$ that strictly contains $P$. Let $x \in Q \backslash P$ and consider $H=M \cap M^{x}$. Our magma calculation implies that the action of $M$ on $\left(M: M \cap M^{x}\right)$ is not binary, and so Lemma 1.6.1 implies that the action of $G$ on $(G: M)$ is not binary.

Again the proof for $t>2$ is completed using the result for $t=2$ and Lemma 4.7.2,

### 4.8 Family $\mathcal{C}_{8}$

In this case $M$ is the normalizer of a classical subgroup of $G$ having the same natural module $V$. The possiblilities are listed in Table 4.8.1, taken from [54, §4.8]. Note that in case $L$, the classical subgroup $M$ is centralized by a graph or graph-field automorphism of $S$ (see Proposition 2.5.1), so $M$ may not be almost simple.

| case | type | conditions |
| :---: | :---: | :---: |
| L | $\mathrm{Sp}_{n}(q)$ | $n \geq 4, n$ even |
| L | $\mathrm{SU}_{n}\left(q^{1 / 2}\right)$ | $n \geq 3, q$ square |
| L | $\Omega_{n}^{\epsilon}(q)$ | $n \geq 3, q$ odd |
| S | $\Omega_{n}^{\epsilon}(q)$ | $n \geq 4, q$ even |

Table 4.8.1: Maximal subgroups in family $\mathcal{C}_{8}$
The main result of this section is the following. The result will be proved in a series of lemmas.
Proposition 4.8.1. Suppose that $G$ is an almost simple group with socle $\bar{S}=\mathrm{Cl}_{n}(q)$, and assume that
(i) $n \geq 3,4,4,7$ in cases $L, U, S, O$ respectively, and
(ii) $\mathrm{Cl}_{n}(q)$ is not one of the groups listed in Lemma 4.1.1,

Let $M$ be a maximal subgroup of $G$ in the family $\mathcal{C}_{8}$. Then the action of $G$ on $(G: M)$ is not binary.

### 4.8.1 Case $S=\mathrm{SL}_{n}(q)$

Lemma 4.8.2. Suppose that $G$ is almost simple with socle equal to $\mathrm{PSL}_{n}(q)$. If $M$ is a maximal $\mathcal{C}_{8}$ subgroup, then the action of $G$ on $(G: M)$ is not binary.

Proof. By Lemma4.1.1, we may assume that $q>25$ when $n=3$, that $q>9$ when $n=4$, that $q>7$ when $n=5$, that $q>4$ when $n=6$ and that $q>3$ when $n=8$. In what follows we suppose, for a contradiction, that the action of $G$ on $(G: M)$ is binary.

Suppose first that $n \geq 5$ when $M$ is unitary, and $n \geq 7$ when $M$ is orthogonal. We refer to Lemma 2.2.8, let $x$ be the element listed there and observe that $C_{S}(x)$ is strictly greater than $C_{M}(x)$. We conclude that there is a suborbit, $\Delta$, on which the action of $M$ is isomorphic to the action of $M$ on $(M: H)$, where $H=M \cap M^{g}$ (for some $g \in C_{S}(x) \backslash C_{M}(x)$ ) is a subgroup of $M$ containing the element $x$ (and not containing $M \cap \bar{S}$ ). Lemma 1.6 .1 implies that the action of $M$ on $(M: H)$ is binary, and now Lemma 2.2.8 implies that $M$ must contain a section isomorphic to $\operatorname{Sym}(t)$ where $t$ is as follows:

1. if $M$ is unitary and $n$ is even, then $t=q^{n-4}$; Lemma 2.1.1 implies a contradiction.
2. if $M$ is unitary and $n$ is odd, then $t=q^{n-3}$; Lemma 2.1.1 implies a contradiction.
3. if $M$ is symplectic or orthogonal of type $\mathrm{O}^{+}$with $n$ even, then $t=q^{(n-2) / 2}$; given the excluded cases for small $n$ and $q$, Lemma 2.1.1 implies a contradiction.
4. if $M$ is orthogonal and $n$ is odd, then $t=q^{(n-3) / 2}$; Lemma 2.1.1 implies a contradiction.
5. if $M$ is orthogonal of type $\mathrm{O}^{-}$with $n$ even, then $t=q^{(n-4) / 2}$; given the excluded cases for small $n$ and $q$, Lemma 2.1.1 implies a contradiction.

Next assume that $M$ is unitary and $n \in\{3,4\}$. Here we adopt the same argument using Lemmas 2.2.10 and 2.2.11 in place of Lemma 2.2.8. Again Lemmas 2.1.1 and 4.1.1 yield a contradiction except when $(n, q)=(4,49)$. This final case was dealt with using magma and the permutation characther method (a.k.a. Lemma 1.8.1).

It remains to consider the case where $M$ is orthogonal (so $q$ is odd) and $3 \leq n \leq 6$. First assume $n \in\{5,6\}$. We think of $V$ as a formed space with form, $\varphi$, preserved by $M$. Let $W=\left\langle e_{1}, e_{2}, f_{1}, f_{2}\right\rangle$ be a non-degenerate subspace of $V$ of type $\mathrm{O}_{4}^{+}$, and consider the group

$$
T:=\left\{\left.\left(\begin{array}{ll}
A & \\
& A^{-t}
\end{array}\right) \right\rvert\, A \in \mathrm{SL}_{2}(q)\right\},
$$

inside $M$; here we specify the action of the elements of $T$ on $W$ (with respect to the given basis) and we require that elements of $T$ fix all elements of $W^{\perp}$. Then $T$ is isomorphic to $\mathrm{SL}_{2}(q)$.

Now let $\{x\}$ or $\{x, y\}$ be a basis for $W^{\perp}$ and consider the group, $U<S$, consisting of elements for which there exist $a_{1}, a_{2} \in \mathbb{F}_{q}$ such that

$$
x \mapsto x+a_{1} e_{1}+a_{2} e_{2},
$$

and all vectors in $W$ are fixed, as is $y$ if $n=6$. Then $U$ is a group of order $q^{2}, U$ does not lie in $M, U$ is normalized by $T$, and $T$ acts transitively on the set of non-identity elements of $U$. Thus, in the usual way, we obtain a set $\Lambda \subset \Omega$ of size $q^{2}$ such that $G^{\Lambda}$ is 2 -transitive. This set is a beautiful set unless $\operatorname{Alt}\left(q^{2}\right)$ is a section of $\mathrm{SL}_{n}(q)$; however, this is not the case by Lemma 2.1.1. Hence Lemma 1.6.12 implies the result.

Finally, for $n \in\{3,4\}$ we argue similarly. Let $W=\left\langle e_{1}, f_{1}\right\rangle$ be a non-degenerate subspace of $V$ of type $\mathrm{O}_{2}^{+}$, and consider the group

$$
T:=\left\{\left.\left(\begin{array}{ll}
a & \\
& a^{-1}
\end{array}\right) \right\rvert\, a \in \mathbb{F}_{q}^{*}\right\},
$$

inside $M$; as before we specify the action of the elements of $T$ on $W$ (with respect to the given basis) and we require that elements of $T$ fix all elements of $W^{\perp}$. Then $T$ is cyclic of order $q-1$. Again let $\{x\}$ or $\{x, y\}$ be a basis for $W^{\perp}$. Consider the group, $U<S$, consisting of elements for which there exists $a \in \mathbb{F}_{q}$ such that

$$
x \mapsto x+a e_{1},
$$

and all vectors in $W$ are fixed, as is $y$ if $n=4$. As before we obtain a beautiful set of size $q$ unless $\operatorname{Alt}(q)$ is a section of $\mathrm{SL}_{n}(q)$, which is not the case as $q>9$.

### 4.8.2 Case $S=\operatorname{Sp}_{n}(q)$

This case is line 4 of Table 4.8.1.
Lemma 4.8.3. Suppose that $G$ is almost simple with socle $\operatorname{PSp}_{n}(q)$, where $q$ is even, $n \geq 4$ and $(n, q) \neq$ $(4,2)$. Let $M=N_{G}\left(\mathrm{O}_{n}^{\epsilon}(q)\right)$ be a maximal $\mathcal{C}_{8}$-subgroup. Then the action of $G$ on $(G: M)$ is not binary.

Proof. First observe that for $q=2$, the action of $G$ on $(G: M)$ is 2-transitive, hence is clearly not binary. So assume from now on that $q>2$.

Suppose now that $\epsilon=+$, and let $H=\mathrm{O}_{n}^{+}(2)$ be a subfield subgroup of $M$. There is a subfield subgroup $K=\operatorname{Sp}_{n}(2)$ of $G$ containing $H$, and $K \cap M=H$. If we let $\Lambda=\{M k: k \in K\}$, then the action of $K$ on $\Lambda$ is isomorphic to the action of $\mathrm{Sp}_{n}(2)$ on the cosets of $\mathrm{O}_{n}^{+}(2)$, which is 2-transitive of degree $d:=2^{n / 2-1}\left(2^{n / 2}+1\right)$. As $\operatorname{Alt}(d)$ is not a section of $\operatorname{Sp}_{n}(q)$ by Lemma 2.1.1, it follows that $\Lambda$ is a beautiful subset, giving the conclusion in this case.

Suppose finally that $\epsilon=-$. The argument of the previous paragraph does not work, as $\mathrm{O}_{n}^{-}\left(2^{a}\right)$ does not possess a subfield subgroup $\mathrm{O}_{n}^{ \pm}(2)$ if $a$ is even, so we use a different argument.

For $n \geq 8$, let $x \in M$ be the element defined in Lemma 2.2.8. This has larger centralizer in $G$ than in $M$, so we can choose $g \in C_{G}(x) \backslash M$. Then $x \in M \cap M^{g}$, so Lemmas 2.2.8 and 2.1.1 imply that the action of $M$ on ( $M: M \cap M^{g}$ ) is not binary.

Next suppose that $n=6$, so $M \triangleright \Omega_{6}^{-}(q) \cong \operatorname{PSU}_{4}(q)$. This time we use the element $x=\operatorname{diag}\left(1,1, a, a^{-1}\right)$ of $\operatorname{PSU}_{4}(q)$, defined in Lemma 2.2.11, where $a \in \mathbb{F}_{q}$ has order $q-1$. This acts as $\operatorname{diag}\left(1,1, a, a, a^{-1}, a^{-1}\right)$ in $\Omega_{6}^{-}(q)$, so there exists $g \in C_{G}(x) \backslash M$. Now, provided $q>8$, we finish the proof as above, using Lemmas 2.2.11 and 2.1.1. If $q=8$, then we use the same argument with Lemma 2.2.11 and the fact that $\operatorname{Alt}(8)$ is not a section of $\mathrm{PSU}_{4}(8)$; if $q=4$, then the result follows from Lemma 4.1.1.

Finally, suppose that $n=4$, so $M \triangleright \Omega_{4}^{-}(q) \cong \operatorname{PSL}_{2}\left(q^{2}\right)$. This time we use the element $x$ defined in Lemma 2.2.4 in exactly the same way as in the previous paragraph to obtain the conclusion.

### 4.9 Family $\mathcal{S}$

Let us first define the family $\mathcal{S}$ of subgroups of classical groups. Let $G$ be an almost simple group with socle $\mathrm{Cl}_{n}(q)$, a classical simple group with associated natural module $V$ of dimension $n$ over $\mathbb{F}_{q^{u}}$, where $u=2$ if $\mathrm{Cl}_{n}(q)$ is unitary and $u=1$ otherwise. We say that a subgroup $M$ of $G$ is in the family $\mathcal{S}$ if the following hold:
(a) $M$ is almost simple, with socle $M_{0}$,
(b) the action of the preimage of $M_{0}$ on $V$ is absolutely irreducible, and cannot be realised over a proper subfield of $\mathbb{F}_{q^{u}}$,
(c) $M_{0}$ is not contained in a member of the family $\mathcal{C}_{8}$ of subgroups of $G$.

In this section we prove the following result. We shall adopt the assumptions on the dimension $n$ made at the beginning of Section 4.1.1.

Proposition 4.9.1. Suppose that $G$ is an almost simple group with socle $\bar{S}=\mathrm{Cl}_{n}(q)$, and assume that
(i) $n \geq 3,4,4,7$ in cases $L, U, S, O$ respectively, and
(ii) $\mathrm{Cl}_{n}(q)$ is not one of the groups listed in Lemma 4.1.1.

Let $M$ be a maximal subgroup of $G$ in the family $\mathcal{S}$. Then the action of $G$ on $(G: M)$ is not binary.
Note that, in all sections up to this point we have assumed (as stipulated in Section 4.1.1) that if $\bar{S}=\mathrm{P} \Omega_{8}^{+}(q)$, then $G$ does not contain a triality automorphism. In the current section we shall drop this assumption. To clarify what we mean by assuming that " $M$ is in the family $\mathcal{S}$ " in this special case: we are allowing the possibility that $\bar{S}=\mathrm{P} \Omega_{8}^{+}(q)$, that $G$ contains a triality automorphism, that $M$ is almost simple with socle $M_{0}$, and that $M_{0}$ satisfies the defining conditions (a,b,c) given above for the family $\mathcal{S}$.

We have a number of strategies, which we outline first.

### 4.9.1 Strategies

## Strategy 1: Subgroups containing centralizers

This strategy is based on the following definition, the value of which is demonstrated in the ensuing proposition. It will be used for the case where the socle $M_{0}$ of $M$ is an alternating group.

Definition 4.9.2. Let $L$ be a simple group and $r$ a positive integer. We say that $L$ satisfies Property( $r$ ) if there exists an element $x_{r} \in L$ of order $r$ such that the following hold for any almost simple group $M$ with socle $L$ :
(1) $Z\left(C_{M}\left(x_{r}\right)\right)=\left\langle x_{r}\right\rangle$;
(2) for any core-free subgroup $H$ of $M$ such that $C_{L}\left(x_{r}\right) \leq H$, the action of $M$ on ( $M: H$ ) is not binary;
(3) If $\left\langle x_{r}\right\rangle \leq N \triangleleft C_{M}\left(x_{r}\right)$ with $C_{M}\left(x_{r}\right) / N$ solvable, then $N$ contains $C_{L}\left(x_{r}\right)$.

Lemma 4.9.3. Let $G$ be an almost simple group with socle a classical group $G_{0}=\mathrm{Cl}_{n}(q)$, and suppose $M$ is a maximal subgroup of $G$ in the family $\mathcal{S}$, with socle $L$. Assume that $L$ satisfies Property $(r)$ for each $r \in\{3,5,7,11,13\}$. Then the action of $G$ on $(G: M)$ is not binary.
Proof. Let $G, G_{0}, L$ and $M$ be as in the statement, and let $q=p^{a}$ with $p$ prime. Consider Property $(r)$, satisfied by $L$ for $r \in\{3,5,7,11,13\}$. If $r \neq p$, then $x_{r}$ is a semisimple element of $G$, and the structure of $C:=C_{G}\left(x_{r}\right)$ is described in [47, Thm. 4.2.2]: there is a normal subgroup $C^{0}$ of $C$ that has a non-trivial central torus $T_{r}$ containing $x_{r}$, and such that $C / C^{0}$ is solvable ( $C^{0}$ is called the connected centralizer in [47, 4.2.2]).

Suppose that $T_{r} \neq\left\langle x_{r}\right\rangle$, and let $g \in T_{r} \backslash\left\langle x_{r}\right\rangle$ and $H=M \cap M^{g}$. Then $g$ centralizes $C^{0} \cap M$, a normal subgroup of $C_{M}\left(x_{r}\right)$ with solvable quotient, and hence $H$ contains $C_{L}\left(x_{r}\right)$ by condition (3) in the definition of $\operatorname{Property}(r)$. If also $H$ contains $L$, then $g \in N_{G}(L)=M$, and so $g \in Z\left(C_{M}\left(x_{r}\right)\right)$, which is a contradiction as $Z\left(C_{M}\left(x_{r}\right)\right)=\left\langle x_{r}\right\rangle$ by assumption (1) in the definition of $\operatorname{Property}(r)$. As $M$ is almost simple with socle $L, H$ is core-free in $M$, and so $\operatorname{Property}(r)$ implies that $(M,(M: H))$ is not binary, whence also $(G,(G: M))$ is not binary, as required. Hence we may assume from now on that

$$
\begin{equation*}
\text { if } p \neq r \text {, then } T_{r}=\left\langle x_{r}\right\rangle \text {. } \tag{4.9.1}
\end{equation*}
$$

Case $G_{0}$ SYMPlectic or orthogonal. Assume that $G_{0}=\mathrm{PSp}_{n}(q)$, or $\mathrm{P} \Omega_{n}^{ \pm}(q)$ with $n$ even, or $\mathrm{P} \Omega_{n}(q)$ with $n$ odd. If $p \neq 3$, then the torus $T_{3}$ has order divisible by $\frac{q-\epsilon}{d}$, where $\epsilon= \pm 1, q \equiv \epsilon \bmod 3$ and $d$ is 1,2 or 4 (it can only be 4 in the orthogonal case). By (4.9.1) we have $\left|T_{3}\right|=3$. Hence we see that

$$
q=2,4,5,7,11,13 \text { or } 3^{a}
$$

If $q=7,13$ or $3^{a}$ with $a>2$, then the torus $T_{5}$ has order greater than 5 , so these possibilities are excluded by (4.9.1).

Now consider the torus $T_{7}$. For $q=4,5,9$ or 11 , this has order divisible by $\frac{q^{3}-\delta}{e}$, where $q^{3} \equiv \delta=$ $\pm 1 \bmod 7$ and $e \in\{1,2,4\}$, and so $\left|T_{7}\right|>7$, contradicting (4.9.1).

We are left with the cases $q=2$ or 3 . For these we consider $T_{11}$, which has order divisible by $2^{5}+1$ or $\frac{3^{5}-1}{2}$, respectively, again contrary to (4.9.1). This completes the proof for the case of symplectic and orthogonal groups.
Case $G_{0}$ Linear. Now assume that $G_{0}=\operatorname{PSL}_{n}(q)$. Suppose $p \neq 3$ and let $q \equiv \epsilon \bmod 3$ with $\epsilon= \pm 1$. Consider Property (3). A preimage of the element $x_{3}$ in $\mathrm{SL}_{n}(q)$ acts on $\bar{V}=V_{n}(q) \otimes \overline{\mathbb{F}}_{q}$ with at most three eigenspaces. Hence the central torus $T_{3}$ (of order 3 by (4.9.1)) in $C_{G}\left(x_{3}\right)$ has order either $q-\epsilon$ or $\frac{q-\epsilon}{(n, q-1)}$, and so one of the following holds:
(i) $q=2,4$ or 5 ,
(ii) $\epsilon=1$ and $\frac{q-1}{(n, q-1)}=3$,
(iii) $q=3^{a}$.

Now consider Property(5), assuming $p \neq 5$. As above, $T_{5}$ has order $q-1$ or $\frac{q-1}{(n, q-1)}$ (if $q \equiv 1 \bmod 5$ ), order $\frac{q+1}{c}$ with $c \in\{1,2\}($ if $q \equiv-1 \bmod 5)$, and order $\frac{q^{4}-1}{(q-1) c}($ if $q \equiv \pm 2 \bmod 5)$. Since $\left|T_{5}\right|=5$, it follows that one of the following holds:
(iv) $q=4,5$ or 9 ,
(v) $q=5^{2 k}$ and $\frac{q-1}{(n, q-1)}=3$,
(vi) $q=3^{4 k}$ and $\frac{q-1}{(n, q-1)}=5$.

Now Property (7) rules out all possibilities except for $q=4$, since for all the other cases we must have $\left|T_{7}\right|>7$. Finally, Property(11) excludes $q=4$, since in this case $T_{11}$ must have order divisible by $\frac{4^{5}-1}{3}$.
Case $G_{0}$ unitary. To complete the proof of the theorem, assume that $G_{0}=\operatorname{PSU}_{n}(q)$. This is very similar to the linear case. If $p \neq 3$ then consideration of Property(3) shows that either $q \in\{2,4,7\}$ or $q \equiv-1 \bmod 3$ and $\frac{q+1}{(n, q+1)}=3$. Then Property(5) implies that one of the following holds:
(i) $q=2,4$ or 11 ,
(ii) $q=5^{k}$ and $\frac{q+1}{(n, q+1)}=3$,
(iii) $q=3^{2 k}$ and $\frac{q+1}{(n, q+1)}=5$.

Now Property(7) excludes all possibilities except for $q=2$, and that is ruled out by Property(13).

## Strategy 2: Odd degree actions

Our second strategy has been used already at various stages; however it is convenient to write down an explicit statement. Note that the proof of the next proposition appeals to results of [73] and 51] which detail, amongst other things, all primitive actions of odd-degree for all of the almost simple groups. Note that both sources omit one family of actions for the groups with socle ${ }^{2} G_{2}(q)$ (here the stabilizer contains a group isomorphic to $\left(2^{2} \times D_{\frac{1}{2}(q+1)}\right): 3$ ), however this omission does not affect the proof given below.

Lemma 4.9.4. Let $G$ be an almost simple group with socle a classical group $G_{0}=\mathrm{Cl}_{n}(q)$, not one of the groups listed in Lemma 4.1.1. Suppose $M$ is a maximal subgroup of $G$ in the family $\mathcal{S}$, with socle $M_{0}$. Then one of the following occurs:

1. the action of $G$ on $(G: M)$ is not binary;
2. there is a suborbit on which $M$ has a transitive faithful action of odd-degree that is binary;
3. $\left(G_{0}, M \cap G_{0}\right)=\left(\mathrm{P}_{7}(p), \mathrm{Sp}_{6}(2)\right)$ or $\left(\mathrm{P}_{8}^{+}(p), \Omega_{8}^{+}(2)\right)$, where $p$ is an odd prime.

Proof. Suppose that the third listed possibility does not occur. Then [73] (or, equivalently, 51]) implies that $|G: M|$ is even. Thus there exists a non-trivial odd subdegree for the action of $G$ on $(G: M)$. Hence there exists $g \in G \backslash M$ such that $\left|M: M \cap M^{g}\right|$ is odd; moreover, by the maximality of $M$ in $G$, $M_{0} \npreceq M \cap M^{g}$, so the action of $M$ on $\left(M: M \cap M^{g}\right)$ is faithful.

Now suppose, in addition, that the second listed possibility does not occur, so that the action of $M$ on ( $M: M \cap M^{g}$ ) is not binary. Then Lemma 1.6 .1 implies that the action of $G$ on $(G: M)$ is not binary, and so the first listed possibility occurs, as required.

## Strategy 3: Using distinguished elements

The strategy here is used primarily for the situation where $M_{0}$ is a group of Lie type. It has already been used multiple times for other families, and was briefly discussed at the start of Chapter 2. We briefly summarise:

1. We pick a distinguished element $g \in M$ and show that, if $H$ is any core-free subgroup of $M$ that contains $g$, then the action of $M$ on $(M: H)$ is not binary. This was done in 2.2 .
2. We give an upper bound for $\left|C_{M}(g)\right|$ and we use results of $\$ 2.4$ to show that, in general, $\left|C_{M}(g)\right|$ is smaller than the smallest centralizer in $G$. We conclude that there exists $x \in C_{G}(g) \backslash C_{M}(g)$.
3. Now $M \cap M^{x}$ is a core-free subgroup of $M$ that contains $g$. We conclude that the action of $M$ on ( $M: M \cap M^{x}$ ) is not binary. Then Lemma 1.6.1 implies that the action of $G$ on $M$ is not binary.

We shall also need the well-known lower bounds for dimensions of cross-characteristic representations of groups of Lie type, taken from [65], with improvements as given in [100]:

Proposition 4.9.5. Let $S$ be a simple group of Lie type over $\mathbb{F}_{r}$, not isomorphic to one of the following groups:

$$
\begin{aligned}
& \mathrm{PSL}_{2}(r)(r \leq 9), \mathrm{PSL}_{4}^{ \pm}(r)(r=2,3), \Omega_{8}^{+}(2), \Omega_{7}(3), \\
& G_{2}(r)(r \leq 4),{ }^{2} E_{6}(2), F_{4}(2),{ }^{2} F_{4}(2)^{\prime},{ }^{2} B_{2}(8) .
\end{aligned}
$$

If $V$ is a non-trivial irreducible module for a quasisimple cover of $S$ over a field of characteristic coprime to $r$, then $\operatorname{dim} V \geq R(S)$, where $R(S)$ is as given in Table 4.9.1.

Table 4.9.1: Lower bounds for cross-characteristic representations

| $S$ | $\mathrm{PSL}_{d}(r)(d \geq 3)$ | $\mathrm{PSU}_{d}(r)$ | $\mathrm{PSp}_{2 k}(r)(r$ odd $)$ | $\mathrm{PSp}_{2 k}(r)(r$ even $)$ | $\mathrm{P}_{2 k+y}^{\epsilon}(r)(y \leq 1)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $R(S)$ | $\frac{r^{d}-r}{r-1}-1$ | $\frac{r^{d}-1}{r+1}$ | $\frac{1}{2}\left(r^{k}-1\right)$ | $\frac{\left(r^{k}-1\right)\left(r^{k}-r\right)}{2(r+1)}$ | $\frac{\left(r^{k}-1\right)\left(r^{k-1}-1\right)}{r^{2}-1}$ |
| $S$ | $E_{8}(r)$ | $E_{7}(r)$ | $E_{6}^{\epsilon}(r)$ | $F_{4}(r)$ | ${ }^{2} F_{4}(r)$ |
| $R(S)$ | $r^{29}-r^{27}$ | $r^{17}-r^{15}$ | $r^{11}-r^{9}$ | $r^{8}-r^{6}$ | $r^{5}-r^{4}$ |
| $S$ | $G_{2}(r)$ | ${ }^{3} D_{4}(r)$ | ${ }^{2} G_{2}(r)$ | ${ }^{2} B_{2}(r)$ |  |
| $R(S)$ | $r^{3}-r$ | $r^{5}-r^{3}$ | $r^{2}-r$ | $(r-1) \sqrt{r / 2}$ |  |

### 4.9.2 The case where $M_{0}$ is alternating

In this case, we use a combination of Strategies 1 and 2.
Lemma 4.9.6. Let $G$ be an almost simple group with socle a classical group $G_{0}=\mathrm{Cl}_{n}(q)$, and suppose $M$ is a maximal subgroup of $G$ in the family $\mathcal{S}$, with socle $M_{0} \cong \operatorname{Alt}(d)$ for some $d \geq 27$. Then the action of $G$ on $(G: M)$ is not binary.

Proof. We use Strategy 1: Lemma 4.9.3 yields the result provided we can verify Property $(r)$ for $r=$ $3,5,7,11$ and 13.

In every case, we take $x_{r}$ to be the $r$-cycle $(1,2, \ldots, r)$. Then $C_{M}\left(x_{r}\right) \cong\left(\left\langle x_{r}\right\rangle \times \operatorname{Sym}(d-r)\right) \cap M$ and $C_{M_{0}}\left(x_{r}\right) \cong\left\langle x_{r}\right\rangle \times \operatorname{Alt}(d-r)$. Parts (1) and (3) in the definition of Property ( $r$ ) follow immediately. Hence to prove the result we must prove part (2): if $H$ is a core-free subgroup of $M$ containing $C_{M_{0}}\left(x_{r}\right)$, then the action of $M$ on $(M: H)$ is not binary.

We claim that the group $H$ satisfies

$$
\left\langle x_{r}\right\rangle \times \operatorname{Alt}(d-r) \leq H \leq(\operatorname{Sym}(r) \times \operatorname{Sym}(d-r)) \cap M .
$$

To see this observe that the first inclusion is true by definition; the second will follow if we can show that $H$ is intransitive in the natural action on $d$ points. Suppose, instead, that $H$ is transitive. If $H$ is imprimitive, then $H$ is isomorphic to a subgroup of $\operatorname{Sym}(e) \operatorname{wr} \operatorname{Sym}(f)$ where $e f=d$. Then, since $\max \{e, f\} \leq \frac{d}{2}$, any alternating section of $H$ is of form $\operatorname{Alt}(s)$ with $s \leq \frac{d}{2}$. But, since $d \geq 27, r \leq 13$ and $H$ contains $\operatorname{Alt}(d-r)$, we have a contradiction and we conclude that $H$ is primitive. But $H$ contains a 3 -cycle hence, by a classical theorem of Jordan, $H$ contains $\operatorname{Alt}(d)$, a contradiction. Thus the claim follows.

Suppose, first, that $M=\operatorname{Alt}(d)$ and let $K=(\operatorname{Sym}(r) \times \operatorname{Sym}(d-r)) \cap M$. We have just seen that $K$ contains $H$. Now Lemma 1.6 .2 implies that if the action of $K$ on $(K: H)$ is not binary, then the result follows. The kernel of the action of $K$ on $(K: H)$ contains a subgroup isomorphic to $\operatorname{Alt}(d-r)$ and we see that the action of $K$ on $(K: H)$ is isomorphic to the action of $\operatorname{Sym}(r)$ on some subgroup $H_{1}$ that is the projection of $H$ to $\operatorname{Sym}(r)$. Using magma we confirm that, for $r \in\{5,7,11,13\}$, all such actions are not binary, provided $H_{1}$ is core-free. Thus we are left with the case where $H_{1}=\operatorname{Alt}(r)$ or $\operatorname{Sym}(r)$ and we have

$$
H=\operatorname{Alt}(r) \times \operatorname{Alt}(d-r) \text { or }(\operatorname{Alt}(r) \times \operatorname{Alt}(d-r)) \cdot 2 .
$$

We repeat this analysis with $M=\operatorname{Sym}(d)$ and $K=\operatorname{Sym}(r) \times \operatorname{Sym}(d-r)$. In this case the kernel of the action of $K$ on $(K: H)$ is isomorphic to either $\operatorname{Alt}(d-r)$ or $\operatorname{Sym}(d-r)$ and we see that the action of $K$ on $(K: H)$ is isomorphic to the action of either $\operatorname{Sym}(r)$ or $\operatorname{Sym}(r) \times C_{2}$ on some subgroup $H_{1}$. This time magma confirms that in all but one case these actions are not binary, provided $H_{1}$ does not contain $\operatorname{Alt}(r)$.

Let us deal first with the one exceptional case in which $H_{1}$ does not contain $\operatorname{Alt}(r):$ here $r=5$ and the action of $K$ on $(K: H)$ is isomorphic to the action of $\operatorname{Sym}(5) \times C_{2}=\langle(1,2,3,4,5),(4,5),(6,7)\rangle$ on $\langle(1,2,3,4,5),(2,5)(3,4)(6,7)\rangle$. In particular, we can take $H$ to contain

$$
H_{0}=\langle(1,2,3,4,5)\rangle \times \operatorname{Alt}(\{6,7,8, \ldots, d\})
$$

as an index 2 subgroup and we have $H=\left\langle H_{0},(2,5)(3,4)(6,7)\right\rangle$. We will show directly that the action of $M$ on $(M: H)$ is not binary. We define

$$
\begin{aligned}
& I_{1}=J_{1}=H ; \\
& I_{2}=J_{2}=H(2,3,4,5,6,7) ; \\
& I_{3}=H(1,3,4,5,6,7) ; \\
& J_{3}=H(1,6,7,5,4,3) .
\end{aligned}
$$

In addition we set

$$
g_{12}=(1), \quad g_{13}=(1,5,4,3,2) \text { and } g_{23}=(1,5,3,6,4)
$$

Direct calculation confirms that for $i, j \in\{1,2,3\}, I_{i}^{g_{i j}}=J_{i}$ and $I_{j}^{g_{i j}}=J_{j}$; in other words, $\left(I_{1}, I_{2}, I_{3}\right) \widetilde{2}$ $\left(J_{1}, J_{2}, J_{3}\right)$. Now suppose that there exists $g \in \operatorname{Sym}(d)$ such that $I_{i}^{g}=J_{i}$ for $i \in\{1,2,3\}$. We note that the stabilizer in $\operatorname{Sym}(d)$ of an element in $(M: H)$ contains a unique normal cyclic subgroup generated by a 5 -cycle. For $I_{1}$ we can take this 5 -cycle to be $(1,2,3,4,5)$, for $I_{2}$ we can take this 5 -cycle to be $(1,3,4,5,6)$. Since $I_{1}=J_{1}$ and $I_{2}=J_{2}$ we conclude that $g$ must normalize the two groups generated by these 5 -cycles. Direct calculation confirms that $g$ is, therefore, a subgroup of $\operatorname{Sym}(\{7,8,9, \ldots, d\})$. But now we require that $I_{3}^{g}=J_{3}$; the stabilizer of $I_{3}$ (resp. $J_{3}$ ) contains a normal cyclic subgroup generated by $(2,4,5,6,3)$ (resp. ( $1,3,4,6,2)$ ) and $g$ must conjugate the first subgroup to the second. But $g$ clearly commutes with these subgroups and we have a contradiction. Thus $\left(I_{1}, I_{2}, I_{3}\right) \not \chi_{3}\left(J_{1}, J_{2}, J_{3}\right)$ and we are done.

We are left with the situation where

$$
\operatorname{Alt}(r) \times \operatorname{Alt}(d-r) \leq H \leq \operatorname{Sym}(r) \times \operatorname{Sym}(d-r)
$$

Observe that, for fixed $d$ and $r$, there are five such groups. We now divide the proof in two parts, depending on whether $H$ contains $C_{M}\left(x_{r}\right)$ or not.

Suppose that $H$ contains $C_{M}\left(x_{r}\right)$. We define a function from $(M: H)$ to the power set of the conjugacy class $x_{r}^{M}$ :

$$
\begin{aligned}
\psi:(M: H) & \longrightarrow \mathcal{P}\left(x_{r}^{M}\right) \\
H k & \mapsto \omega_{k}:=\left\{k_{0}^{-1} x_{r} k_{0} \mid k_{0} \in H k\right\} .
\end{aligned}
$$

Notice that $\omega_{x_{r}}=x_{r}^{H}$. We claim that the image, $\psi(M: H)$ is a partition of $x_{r}^{M}$. It is clear that

$$
\bigcup_{X \in \psi(M: H)} X=x_{r}^{M}
$$

thus suppose that $\omega_{k_{1}} \cap \omega_{k_{2}} \neq \emptyset$. This implies that $\left(k_{1}^{\prime}\right)^{-1} x_{r}\left(k_{1}^{\prime}\right)=\left(k_{2}^{\prime}\right)^{-1} x_{r}\left(k_{2}^{\prime}\right)$ for some $k_{1}^{\prime} \in H k_{1}, k_{2}^{\prime} \in$ $H k_{2}$. But this implies that $\left(k_{2}^{\prime}\right)\left(k_{1}^{\prime}\right)^{-1} \in C_{M}\left(x_{r}\right)<H$ and so $H k_{1}^{\prime}=H k_{2}^{\prime}$ which means that $H k_{1}=H k_{2}$ and so $\omega_{k_{1}}=\omega_{k_{2}}$, as required.

Now we define an action of $M$ on $\psi(M: H)$ via

$$
\omega_{k_{1}}^{k}=\left\{k^{-1} x k \mid x \in \omega_{k_{1}}\right\} .
$$

This action is well-defined and is isomorphic to the action of $M$ on $(M: H)$.
Notice that $x_{r}^{M}$ is the set of all $r$-cycles in $\operatorname{Sym}(d)$. We showed above that

$$
\operatorname{Alt}(r) \times \operatorname{Alt}(d-r) \leq H \leq(\operatorname{Sym}(r) \times \operatorname{Sym}(d-r)) \cap M
$$

This implies that the partition $\psi(M: H)$ of $x_{r}^{M}$ is a refinement of the partition where two $r$-cycles are in the same part if and only if they have the same underlying $r$-set.

Our method will vary slightly depending on precise properties of this partition. To divide our method into cases we define $H_{1}$ to be the projection of $H$ onto $\operatorname{Sym}(\{1, \ldots, r\})$ and we recall that $H_{1}$ is either $\operatorname{Alt}(r)$ or $\operatorname{Sym}(r)$.

Case 1: $x_{r}$ is conjugate to $x_{r}^{-1}$ in $H_{1}$. In this case we define

$$
\begin{aligned}
& I_{1}=J_{1}=[(1,2,3, \ldots, r)] \\
& I_{2}=J_{2}=\left[\left(1,2, \ldots, \frac{r-1}{2}, r+1, r+2, \ldots, \frac{3 r+1}{2}\right)\right] \\
& I_{3}=\left[\left(1,2, \ldots, \frac{r-1}{2}, \frac{3 r+3}{2}, \frac{3 r+5}{2} \ldots, 2 r+1\right)\right] \\
& J_{3}=\left[\left(r, r-1, \ldots, \frac{r+3}{2}, \frac{3 r+3}{2}, r+2, r+3, \ldots, \frac{3 r+1}{2}\right)\right],
\end{aligned}
$$

where we use " $[-]$ " to denote the part of $\psi(M: H)$ containing the listed cycle.
It is easy to see that $I \not \overbrace{3} J$ : the cycles representing $I_{1}, I_{2}, I_{3}$ all move the points $1,2, \ldots, \frac{1}{2}(r-1)$, whereas the cycles representing $J_{1}, J_{2}, J_{3}$ have no moved points in common.

To see that $I \widetilde{2} J$ we must define $g_{13}$ and $g_{23}$ such that $I_{i}^{g_{i j}}=J_{i}$ and $I_{j}^{g_{i j}}=J_{j}$. To this end, we set:

$$
\begin{aligned}
& g_{13}=(1, r)(2, r-1) \cdots\left(\frac{r-1}{2}, \frac{r+3}{2}\right)\left(2 r+1, \frac{3 r+1}{2}\right)\left(2 r, \frac{3 r-1}{2}\right) \cdots\left(\frac{3 r+5}{2}, r+2\right) ; \\
& g_{23}=\left(1, \frac{3 r+1}{2}\right)\left(2, \frac{3 r-1}{2}\right) \cdots\left(\frac{r-1}{2}, r+2\right)(2 r+1, r)(2 r, r-1) \cdots\left(\frac{3 r+5}{2}, \frac{r+3}{2}\right) .
\end{aligned}
$$

It is easy to check that these even permutations do the job; more specifically, we can see that the representative $r$-cycle listed above in the definition of $I_{i}$ is mapped to either the representative $r$-cycle listed for $J_{i}$, or to its inverse.

Case 2: $x_{r}$ is not conjugate to $x_{r}^{-1}$ in $H_{1}$. Since $H_{1}=\operatorname{Alt}(r)$ or $\operatorname{Sym}(r)$, we conclude that $H_{1}=\operatorname{Alt}(r)$ with $r \equiv 3(\bmod 4)$. In particular $r \in\{3,7,11\}$ and $H_{1}=\operatorname{Alt}(r)$.

Suppose, first, that $r=3$. In this case $H$ is the centralizer of a 3 -cycle in $M$ and the set $\psi(M: H)$ can be identified with set of 3 -cycles in $\operatorname{Alt}(d)$. We define

$$
\begin{aligned}
& I_{1}=J_{1}=g_{13}=(1,2,3), \\
& I_{2}=J_{2}=g_{23}=(1,2,4), \\
& I_{3}=(2,3,4), \\
& J_{3}=(3,1,4) .
\end{aligned}
$$

Finally we define $g_{12}=1$, and now one can check directly that, for all $i, j$ such that $1 \leq i<j \leq 3$, we have $I_{i}^{g_{i j}}=J_{i}$ and $I_{j}^{g_{i j}}=J_{j}$. In particular $I \widetilde{2} J$.

We wish to show that $I \not \chi_{3} J$. Suppose that $g \in M$ such that $I^{g}=J$. Clearly $g$ must stabilize the set $\Delta=\{1,2,3,4\}$. But now, since $g$ must fix both $I_{1}$ and $I_{2}$, we obtain that $\left.g\right|_{\Delta}=1$. This contradicts the fact that $I_{3}^{g}=J_{3}$ and the result follows.

Suppose, next, that $r \geq 7$. In this case we exhibit the presence of a beautiful subset and the result follows thanks to Lemma 1.6.12, We consider the set
$\Lambda=\left\{\begin{array}{l}{[(1,2,4,8,9,11,15 \ldots, r+8)],[(2,3,5,9,10,12,15, \ldots, r+8)],[(3,4,6,10,11,13,15, \ldots, r+8)],} \\ {[(4,5,7,11,12,14,15, \ldots, r+8)],[(5,6,1,12,13,8,15, \ldots, r+8)],[(6,7,2,13,14,9,15, \ldots, r+8)],} \\ {[(7,1,3,14,8,10,15, \ldots, r+8)]}\end{array}\right\}$.
Note that the parts of the partition of $x_{r}^{M}$ correspond to the conjugacy classes of $r$-cycles for the alternating group of the underlying $r$-set. In particular, for instance, the $r$-cycle ( $1^{\tau}, 2^{\tau}, 4^{\tau}, 8^{\tau}, 9^{\tau}, 11^{\tau}, 15, \ldots, r+8$ ) is in $[(1,2,4,8,9,11,15 \ldots, r+8)]$ (where $\tau$ is some permutation of $\{1,2,4,8,9,11\})$ if and only if $\tau$ is even.

We have chosen seven $r$-tuples $\left(\mu_{1}, \ldots, \mu_{6}, 15, \ldots, r+8\right)$ that satisfy two properties:
(a) the seven 3 -tuples given by $\left(\mu_{1}, \mu_{2}, \mu_{3}\right)$ form the lines of a Fano plane;
(b) $\mu_{i+3}=\mu_{i}+7$ for $i=1,2,3$.

It is clear that a group preserving $\Lambda$ must stabilize the set $\{15, \ldots, r+8\}$; in addition we claim that if $g \in M_{\Lambda}$, then $\mu_{i+3}^{g}=\mu_{i}^{g} \pm 7$ for $i=1,2,3$. To see this, let $g \in M_{\Lambda}$ and observe that, for $i=1, \ldots, 3$, the number $\mu_{i}+7$ is the only one that occurs in every listed tuple where $\mu_{i}$ occurs. Thus $\left(\mu_{i}+7\right)^{g}$ must occur in every listed tuple where $\mu_{i}^{g}$ occurs. But this means that $\left(\mu_{i}+7\right)^{g}=\mu_{i}^{g} \pm 7$ as required.

These two properties allow us to conclude that $M^{\Lambda}$ is isomorphic to a subgroup of $\mathrm{GL}_{3}(2)$ and so, in particular, does not contain Alt(7). We wish to show that, in fact, $M^{\Lambda}=\mathrm{GL}_{3}(2)$ and the result will then follow.

Write $\Lambda_{1}$ for the set of seven 3 -tuples obtained by projecting the listed tuple in each element of $\Lambda$ onto its first three entries; similarly $\Lambda_{2}$ is the set of seven 3 -tuples obtained by projecting the listed tuple in each element of $\Lambda$ onto entries $4,5,6$. Both $\Lambda_{1}$ and $\Lambda_{2}$ correspond to Fano planes. Let $\theta_{1}$ be a permutation of $\{1, \ldots, 7\}$ corresponding to an automorphism of the $\Lambda_{1}$-Fano plane and let $\theta_{2}$ be a permutation of $\{8, \ldots, 14\}$ corresponding to the automorphism of the $\Lambda_{2}$-Fano plane obtained by increasing each entry in the cycle notation of $\theta_{1}$ by 7 . Now the permutation $\theta_{1} \theta_{2}$ is an element of $\operatorname{Alt}(14)$.

Consider the image of a listed tuple $\lambda$ under $\theta_{1} \theta_{2}$. The projection of this image onto its first three entries yields a 3 -tuple which is a permutation of the 3 -tuple given by the first three entries of the listed permutation in an element of $\Lambda$. Likewise the projection of this image onto entries 4,5,6 yields a 3 -tuple which is a permutation of the 3 -tuple given by entries 4,5 and 6 of the same listed permutation. The two resulting permutations are of the same type and so, since $\theta_{1} \theta_{2}$ fixes the points $15, \ldots r+8$, we conclude that $\lambda^{\theta_{1} \theta_{2}}$ is of the form $\left(\mu_{1}^{\tau}, \mu_{2}^{\tau}, \ldots, \mu_{6}^{\tau}, 15, \ldots, r+8\right)$ where $\left(\mu_{1}, \ldots, \mu_{6}, 15, \ldots, r+8\right)$ is one of the listed permutations and $\tau=\theta_{1} \theta_{2}$ is even. In particular $\lambda^{\theta_{1} \theta_{2}}$ lies in an element $[\gamma]$ of $\Lambda$, where $\gamma$ is one of the listed tuples. Now both $[\lambda]$ and $[\gamma]$ are conjugacy classes in conjugates, $H_{\lambda}$ and $H_{\gamma}$, of $H$. Then $H_{\lambda}^{\theta_{1} \theta_{2}}=H_{\gamma}$ and, since $\lambda^{\theta_{1} \theta_{2}} \in[\gamma]$ we conclude that $[\lambda]^{\theta_{1} \theta_{2}}=[\gamma]$. We conclude that $\theta_{1} \theta_{2}$ is in $M_{\Lambda}$ and the result follows.

Suppose that $H$ contains $C_{M_{0}}\left(x_{r}\right)$ but not $C_{M}\left(x_{r}\right)$. In this case $M=\operatorname{Sym}(d)$ and $H$ is one of the following groups:

$$
\operatorname{Alt}(r) \times \operatorname{Alt}(d-r), \operatorname{Sym}(r) \times \operatorname{Alt}(d-r) \text { or }(\operatorname{Alt}(r) \times \operatorname{Alt}(d-r)) .2
$$

Observe, first, that if $H<\operatorname{Alt}(d)$, then the action of $\operatorname{Alt}(d)$ on cosets of $H$ is considered above. Since we know that this action is not binary, the result follows by Lemma 1.6.2,

Thus we assume that $H \nless \operatorname{Alt}(d)$, in which case $H=\operatorname{Sym}(r) \times \operatorname{Alt}(d-r)$. But now the analysis of Case 2 for $r \in\{7,11\}$ works, with the rôles of $r$ and $d-r$ interchanged. (Note that in Case 2 our only use of the fact that $r \in\{7,11\}$ was when we needed $r \geq 6$ in order to make our definition of $\Lambda$ work; in the current situation we just observe that $d-r \geq 6$ in all cases.)

For the alternating groups of degree less than 27 , we shall use a magma computation together with the following result.

Lemma 4.9.7. Let $G$ be a group with socle $G_{0}=\mathrm{Cl}_{n}(q)\left(q=p^{a}\right)$, a classical group, not one of the groups listed in Lemma 4.1.1. Suppose $M=N_{G}\left(M_{0}\right)$ is a maximal subgroup of $G$ in the family $\mathcal{S}$, where $M_{0}=\operatorname{Alt}(r)$ with $r$ odd, $r \leq 25$.
(i) If $r$ is prime, then for any $x \in M_{0}$ of order $r$, we have $C_{G}(x) \neq\langle x\rangle$.
(ii) If $r=9,15$ or 21 , then $|G|_{3}>|M|_{3}$; if $r=25$, then $|G|_{5}>|M|_{5}$.

Proof. (i) First assume $r \geq 11$. Then $n \geq r-\delta_{p, r}$ (see Lemma 2.1.1). The orders of centralizers of elements of prime order in classical groups are given in Tables B3 - B12 of [12], and it is straightforward to read off from these tables that for $n \geq r-\delta_{p, r}$, no such centralizer in $G_{0}=\mathrm{Cl}_{n}(q)$ can have order equal to $r \in\{11,13,17,19,23\}$.

Now suppose $r=7$. The modular character tables of Alt(7) and its covering groups are given in 50 . We have $n \geq 3$. If $n \geq 9$, then [12] gives a contradiction as above. And if $n=7$ or 8 , then the characteristic $p \geq 5$, and again we can use [12] to rule this out. Hence $n \leq 6$.

If $n=3$, then the only characteristic in which there is an irreducible modular representation is 5 , yielding a maximal subgroup $\operatorname{Alt}(7)<\mathrm{PSU}_{3}(5)$ - but this possibility is excluded by Lemma 4.1.1.

If $n=4$, then $p \geq 5$ yields a contradiction using [12] as above; and $p \leq 3$ is again excluded by Lemma 4.1.1.

If $n=5$, then the only possible characteristic is $p=7$ with $\operatorname{Alt}(7)<\Omega_{5}(7)$; but then clearly $C_{G}(x) \neq$ $\langle x\rangle$.

Finally consider $n=6$. Here $p=2$ is excluded by Lemma 4.1.1. If $p=3$, then there are two possible embeddings, Alt $(7)<\Omega_{6}^{-}(3)$ or $\operatorname{PSp}_{6}(9)$. The first is out by Lemma 4.1.1, and the second is excluded using [12]. Finally, [12] rules out all possibilities with $p \geq 5$.

It remains to consider $r=5$. Here $n \leq 6$. If $n=2$ then $G_{0}=\mathrm{PSL}_{2}(q)$ and assuming $p \neq 5, C_{G_{0}}(x)$ has order $q \pm 1 /(2, q-1)$. Hence $q=9$ or 11, excluded by Lemma 4.1.1. When $n=3$, the embedding is $\operatorname{Alt}(5)<\Omega_{3}(q) \cong \operatorname{PSL}_{2}(q)$, already handled. When $n=4$, the embeddings are $\operatorname{Alt}(5)<\Omega_{4}^{-}(p) \cong \operatorname{PSL}_{2}\left(p^{2}\right)$ and $\operatorname{Alt}(5)<\operatorname{PSp}_{4}(p)$; in the latter case $N_{G}(S)$ is non-maximal. If $n=5$ then $\operatorname{Alt}(5)<\Omega_{5}(q) \cong \operatorname{PSp}_{4}(q)$. Finally, if $n=6$, then $\operatorname{Alt}(5)<\operatorname{PSp}_{6}(p)$, and [12] gives a contradiction.
(ii) Suppose $r=9$. Then $n \geq 8-\delta_{p, 3}$. If $p=3$, the conclusion is clear, so assume $p \neq 3$. If $n=8$ then $G_{0}=\mathrm{P} \Omega_{8}^{+}(q)$, which has order divisible by $3^{5}$, greater than $|\operatorname{Sym}(9)|_{3}=3^{4}$. And if $n \geq 9$ then $\left|G_{0}\right|$ is divisible by $\frac{1}{d} \prod_{1}^{4}\left(q^{2 i}-1\right)$ (where $d=(2, q-1)$ ), hence is also divisible by $3^{5}$.

For $r=15$ we have $n \geq 14-\delta_{p, 3}-\delta_{p, 5}$ and so $\left|G_{0}\right|$ is divisible by $\frac{1}{d} \prod_{1}^{6}\left(q^{2 i}-1\right)$, hence by $3^{8}>|\operatorname{Sym}(15)|_{3}$. A similar argument works for $r=21$ or 25 ; the only extra point to note is that if $r=25$ and $n=24$ then $G_{0}=\mathrm{P} \Omega_{24}^{+}(q)$ (rather than $\mathrm{P} \Omega_{24}^{-}(q)$ ), and this has order divisible by $5^{7}>|\operatorname{Sym}(25)|_{5}$. This completes the proof.

We can now complete the proof of Proposition 4.9.1 for the case of alternating groups:
Lemma 4.9.8. Let $G$ be an almost simple group with socle a classical group $G_{0}=\mathrm{Cl}_{n}(q)$, under the hypotheses of Propsition 4.9.1, and suppose $M$ is a maximal subgroup of $G$ in the family $\mathcal{S}$, with socle $M_{0} \cong \operatorname{Alt}(d)$ for some $5 \leq d \leq 26$. Then the action of $G$ on $(G: M)$ is not binary.
Proof. Recall our assumptions on $n$ in the hypothesis: namely, $n \geq 3,4,4,7$ in cases $L, U, S, O$ respectively. Next we check using magma the following facts, where $6 \leq d \leq 26$ :
(a) every non-trivial binary action of $\operatorname{Alt}(d)$ has even degree;
(b) for $d$ even, every non-trivial binary action of $\operatorname{Sym}(d)$ has even degree;
(c) every non-trivial binary action of $M_{10}, \mathrm{PGL}_{2}(9)$ and $\mathrm{P}_{2}(9)$ has even degree;
(d) every non-trivial binary action of $\operatorname{Alt}(5)$ and $\operatorname{Sym}(5)$ has degree divisible by 5 ;
(e) for $d$ odd, every non-trivial binary action of $\operatorname{Sym}(d)$ (with core-free point stabilizer) has degree divisible by a prime $s$, as in the following table:

| $d$ | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $s$ | 7 | 3 | 11 | 13 | 3 | 17 | 19 | 3 | 23 | 5 |

Given these facts, we can complete the proof as follows. Assume for a contradiction that the action of $G$ on $(G: M)$ is binary. We know by Lemma 4.9.4 that in this action, there is a non-trivial suborbit of odd degree on which the action of $M$ is binary. Hence by Fact (a), $M$ cannot be $\operatorname{Alt}(d)$ for $d>5$. Thus, either $d=5, d=6$ or $M$ is $\operatorname{Sym}(d)$. But now Fact (b) rules out the possibility that $M$ is $\operatorname{Sym}(d)$ of even degree, and Fact (c) rules out all the possibilities when $d=6$. Thus, in any case, $d$ is odd and $M=\operatorname{Sym}(d)$ except, possibly, when $d=5$ and $M=\operatorname{Alt}(5)$.

Suppose now that $d$ is a prime (so is $5,7,11,13,17,19$ or 23 ). Let $x \in M$ have order $d$. Then by Lemma 4.9.7(i), there exists $g \in C_{G}(x) \backslash M$. Thus there is a suborbit ( $M: M \cap M^{g}$ ) of size coprime to $d$. Now the action of $M$ on this suborbit is not binary, by Facts (d) and (e). Hence $G$ is not binary on ( $G: M$ ) by Lemma 1.6.1, a contradiction.

The remaining cases $d=9,15,21$ or 25 succumb to a similar argument. For these cases, we let $P$ be a Sylow 3-subgroup of $M$ (a Sylow 5 -subgroup in the last case), and observe that by Lemma 4.9.7(ii), there exists $g \in N_{G}(P) \backslash M$. Hence the suborbit ( $M: M \cap M^{g}$ ) has size coprime to 3 (or 5), and the action of $M$ on this is not binary, by Fact (e), giving a contradiction as before.

This completes the proof of Proposition 4.9 .1 for the case where the socle $M_{0}$ is an alternating group.

### 4.9.3 The case where $M_{0}$ is sporadic

In this case we use Strategy 2 and some earlier computations with magma.
Lemma 4.9.9. Let $G$ be an almost simple group with socle a classical group $G_{0}=\mathrm{Cl}_{n}(q)$, and suppose $M$ is a maximal subgroup of $G$ in the family $\mathcal{S}$, with socle $M_{0}$ a sporadic simple group. Then the action of $G$ on $(G: M)$ is not binary.

Proof. The proof follows immediately from Lemmas 2.3 .2 and 4.9.4.

### 4.9.4 The case where $M_{0}$ is of Lie type

In this section we prove Proposition 4.9 .1 for the case where $M_{0}$ is of Lie type. We will use the strategy outlined in $\S 4.9 .1$ in particular we will make use of Propositions 2.4.1 and 4.9.5.

To start we use magma to rule out a number of small possibilities for $M$.
Lemma 4.9.10. Let $M_{0}$ be one of the simple groups listed in Lemma 2.3.1, and let $G$ be an almost simple group with socle a classical group $G_{0}=\mathrm{Cl}_{n}(q)$, not one of the groups listed in Lemma 4.1.1. Suppose $M$ is a maximal subgroup of $G$ in the family $\mathcal{S}$, with socle $M_{0}$. Then the action of $G$ on $(G: M)$ is not binary.

Proof. Lemmas 2.3.1 and 4.9.4 imply the result unless $\left(G_{0}, M_{0}\right)=\left(\mathrm{P}_{7}(p), \mathrm{Sp}_{6}(2)\right)$ or $\left(\mathrm{P} \Omega_{8}^{+}(p), \Omega_{8}^{+}(2)\right)$, where $p$ is an odd prime.

If $G_{0}=\mathrm{P} \Omega_{7}(p)$ and $M_{0}=\mathrm{Sp}_{6}(2)$, then we let $g \in M_{0}$ be the element of order 3 defined in Lemma 2.2.8, In that lemma it is proved that if $H$ is any subgroup of $M$ that contains $g$, then the action of $M$ on $(M: H)$ is not binary. Suppose that there exists $x \in C_{G}(g) \backslash M$. Then the action of $M$ on $\left(M: M \cap M^{x}\right)$ is not binary, and Lemma 1.6 .1 yields the result. It remains to show, therefore, that $C_{G}(g)$ is strictly larger that $C_{M}(g)$. Direct calculation implies that $\left|C_{M}(g)\right|=108$ and now Lemma 2.4.2 implies the result for $q>7$. For $q \leq 7$, the result is confirmed with magma or by direct calculation.

If $G_{0}=\mathrm{P} \Omega_{8}^{+}(p)$ and $M_{0}=\Omega_{8}^{+}(2)$, then we let $g$ be the element of order 7 defined in Lemma 2.2.8. We proceed as before but must confirm that there exists $x \in C_{G}(g) \backslash M$. Using [10] we see that $G_{0} \cap M=M_{0}$, and using [28] we see that $C_{M_{0}}(g)=\langle g\rangle$. Thus it is sufficient to prove that $C_{G_{0}}(g) \neq\langle g\rangle$. When $p=7$, this is immediate; when $p \neq 7$, one can confirm this using, for instance, [12].

Let us next deal with some troublesome groups that are just a little too big to be easily handled with magma.

Lemma 4.9.11. Let $M_{0}$ be one of the following groups

$$
{ }^{3} D_{4}(4),{ }^{3} D_{4}(5),{ }^{2} E_{6}(3) .
$$

Let $G$ be an almost simple group with socle a classical group $G_{0}=\mathrm{Cl}_{n}(q)\left(q=p^{a}\right)$, and suppose $M$ is a maximal subgroup of $G$ in the family $\mathcal{S}$, with socle $M_{0}$. Then the action of $G$ on $(G: M)$ is not binary.

Proof. (1) Suppose, first that $M_{0} \cong{ }^{3} D_{4}(5)$.
Claim: There exists an element, $g$, of order 24 in $M_{0}$ such that if $M$ is almost simple with socle $M_{0}$ and $H$ is any core-free subgroup of $M$ containing $g$, then the action of $M$ on $(M: H)$ is not binary.

Proof of claim: Assume $(M,(M: H))$ is binary for some such $H$. We use the fact that if $\operatorname{Alt}(t)$ is a section of ${ }^{3} D_{4}(5)$, then $t \leq 7$ by Lemma 2.1.2. We also use the existence of the following subgroup chain:

$$
\mathrm{SL}_{3}(5)<G_{2}(5)<{ }^{3} D_{4}(5)
$$

The group $\mathrm{SL}_{3}(5)$ contains a maximal parabolic subgroup with unipotent radical $E$, an elementary abelian group of order 25 . In addition $\mathrm{SL}_{3}(5)$ contains an element $g$ of order 24 that normalizes, and acts fixed-point-freely upon, $E$. Since $E \rtimes\langle g\rangle$ is a Frobenius group, we conclude that either $H$ contains $E$ or else
the set of cosets $H . E$ forms a beautiful subset of size 25, and Lemma 1.6 .12 yields a contradiction. Thus $H$ contains $E$.

We can now repeat the same argument with the "opposite" unipotent radical, $E_{1}$. The same element $g$ acts fixed-point-freely on $E_{1}$ and we can now also assume that $H$ contains $E_{1}$. Since $\left\langle E, E_{1}\right\rangle=\mathrm{SL}_{3}(5)$, we conclude that $H$ contains $\mathrm{SL}_{3}(5)$.

The group $\mathrm{SL}_{3}(5)$ is a subgroup of $K:=\mathrm{SL}_{3}(5): 2$, a maximal subgroup of $G_{2}(5)$ of maximal rank. We revisit our argument for the action of $G_{2}(q)$ on cosets of $K$ - see Table 3.4.8 and the proof Propositions 3.4.1. As in that proof, we conclude either that we have a beautiful subset of order 25 (a contradiction) or else $H$ contains $G_{2}(5)$. Thus the latter holds.

The group $L:=G_{2}(5)$ is a maximal subgroup of ${ }^{3} D_{4}(5)$ in family (V) of Theorem 3.1.1. Now we revisit our argument for the action of ${ }^{3} D_{4}(q)$ on cosets of $L$ - see Case (6) of the proof of Proposition 3.6.1. Once again we obtain a beautiful subset of order 25 . We conclude, as required, that $H$ contains $M_{0}$. The claim is proved.

We now show that the claim implies the conclusion of the lemma. Suppose that there exists $x \in$ $C_{G_{0}}(g) \backslash C_{M}(g)$. Then the claim implies that the action of $M$ on ( $M: M \cap M^{x}$ ) is not binary and Lemma 1.6.1 yields the result.

Thus to complete the proof for this case we must check that the element $x$ exists. Note that $g$ is a regular semisimple element of ${ }^{3} D_{4}(5)$ (which we can see by computing the action of $G$ on the 8 -dimensional module for $\left.{ }^{3} D_{4}(5)\right)$. Hence $C_{M_{0}}(g)$ is a maximal torus of $M_{0}$, the sizes of which are listed in [57]. We conclude that $\left|C_{M_{0}}(g)\right| \leq 756$, and so $\left|C_{M}(g)\right| \leq 2268$. On the other hand, if $p \neq 5$ we have $n \geq 5^{5}-5^{3}$ by Lemma 4.9.5, and if $p=5$, then either $n=8, q=5^{3}$ or $n \geq 24$ by [54, 5.4.8]; hence $\left|C_{G_{0}}(g)\right|>2268$ by Lemma 2.4.2
(2) Suppose next that $M_{0} \cong{ }^{3} D_{4}(4)$. The proof here is very similar to the previous case.

Claim: There exists an element, $g \in M_{0}$ of order 15 , such that if $M$ is almost simple with socle $M_{0}$ and $H$ is any core-free subgroup of $M$ containing $g$, then the action of $M$ on $(M: H)$ is not binary.

Proof of claim: We use the fact that ${ }^{3} D_{4}(4)$ contains a maximal group isomorphic to $J \cong \mathrm{PGL}_{3}(4)$ (see [57]). The group $J$ contains an element $g$ of order 15 that normalizes and acts fixed-point-freely upon an elementary abelian group $E$ of order 16. Assume that $H$ is a subgroup of $M$ containing $g$, for which the action of $M$ on $(M: H)$ is binary. We will show that $H$ contains $M_{0}$.

Arguing exactly as in the previous case, we see that either there is a beautiful subset of size 16 or $H$ contains the group $\mathrm{PGL}_{3}(4)$; hence we conclude the latter. Now we revisit our argument for the action of ${ }^{3} D_{4}(q)$ on cosets of $J$ - see Case (6) of the proof of Proposition 3.6.1. Once again we obtain a beautiful subset of order 16. We conclude, as required, that $H$ contains $M_{0}$. The claim is proved.

We now show that the claim implies the conclusion of the lemma. This proceeds as before, relying on the existence of $x \in C_{G_{0}}(g) \backslash C_{M}(g)$. As before, $g$ is regular semisimple and, using [57], we conclude that $\left|C_{M}(g)\right| \leq 945$. Now, as before, we find that $\left|C_{G_{0}}(g)\right|>945$ and we are done.
(3) Suppose now that $M_{0} \cong{ }^{2} E_{6}(3)$. From [74, Table 5.1], we see that $M_{0}$ has a maximal subgroup $\mathrm{SU}_{3}(27) .3$. Write $L$ for the simple subgroup $\mathrm{SU}_{3}(27)$ of this. Let $x \in L$ be an element of order 26 which, written with respect to a hyperbolic basis $\left\{e_{1}, f_{1}, x\right\}$ of the corresponding unitary 3 -space, is diagonal with entries $\left(t, t^{-1}, 1\right)$. We proceed in a series of steps.

Claim 1: If $X$ is an almost simple group with socle $\mathrm{SU}_{3}(27)$, and $Y<X$ is a core-free subgroup such that $|X: Y|$ is not divisible by $3^{2}$, then the action of $X$ on $(X: Y)$ is not binary.

Proof of Claim 1: This is a magma computation.
Claim 2: Let $x \in L$ be the element defined above, and suppose $x \in H<M$ with $H$ a core-free subgroup of $M$. Then the action of $M$ on $(M: H)$ is not binary.

Proof of Claim 2: Assume that the action of $M$ on $(M: H)$ is binary. We shall repeatedly use Lemma 2.1.2 which asserts that $M$ does not contain a section isomorphic to $\operatorname{Alt}(d)$ for any $d \geq 12$.

Let $L_{1} \cong \mathrm{SO}_{3}(27) \cong \mathrm{PGL}_{2}(27)$ be a subfield subgroup of $L$ containing $x$. Then there are two Sylow 3 -subgroups $U_{1}, U_{2}$ of $L_{1}$ such that $\langle x\rangle$ normalizes and acts transitively on the set of non-trivial elements of each of them. This implies (using Lemma 1.6.9) that either $U_{1}$ is in $H$, or else $H U_{1}$ is a subset of ( $M: H$ ) on which the set-wise stabilizer acts 2 -transitively. But in the latter case we obtain a beautiful subset, a contradiction to Lemma 1.6.12. Thus $H$ contains $U_{1}$ and, similarly, $U_{2}$. Thus $H \geq\left\langle U_{1}, U_{2}, x\right\rangle=L_{1}$.

Now let $g$ be a diagonal element of $L=\mathrm{SU}_{3}(27)$ of order $27^{2}-1$ and such that $g^{28}=x$. Then $\langle x\rangle$ acts transitively on the set of non-trivial elements of $U_{1}^{g}$ and $U_{2}^{g}$. We deduce, as in the previous paragraph, that $H$ contains $U_{1}^{g}$ and $U_{2}^{g}$, and we conclude that $H \geq L=\mathrm{SU}_{3}(27)$. From the information on the maximal subgroups of $M$ given by Theorem 3.1.1, it follows that $L \leq H \leq N_{M}(L) \leq L .6$.

Write $\Omega=(M: H)$. Now $N_{M}(L)$ acts transitively on fix ${ }_{\Omega}(L)$, so the number of fixed points $f=$ $\left|\operatorname{fix}_{\Omega}(L)\right|$ divides 6. Also $|\Omega|=f+\sum_{i}\left|\Delta_{i}\right|$, where the $\Delta_{i}$ are the faithful $H$-orbits on $\Omega$. Since $|\Omega|$ is divisible by $3^{2}$, it follows that there is an $H$-orbit $\Delta_{i}$ of size not divisible by $3^{2}$. Hence the action of $H$ on $\Delta_{i}$ is not binary, by Claim 1. However the action of $M$ on $(M: H)$ is binary by assumption, so this is a contradiction to Lemma 1.6.1. This establishes Claim 2.

We now show that Claim 2 implies the conclusion of the lemma. Just as before, we need to show that there exists $x \in C_{G}(x) \backslash M$.

We start by computing the order of $C_{M}(x)$. The subgroup $L=\mathrm{SU}_{3}(27)$ arises as the fixed point group of a Frobenius endomorphism of the algebraic group $E_{6}$ acting on a subsystem $A_{2}^{3}$ (see [74]). By [76, Prop. 2.1], the Lie algebra $L\left(E_{6}\right)$ restricts to $A_{2}^{3}$ as the sum of $L\left(A_{2}^{3}\right)$ together with the tensor product $V_{1} \otimes V_{2} \otimes V_{2}$ and its dual, where each $V_{i}$ is a natural 3 -dimensional module for one of the $A_{2}$ factors. The element $x$ acts on this tensor product as $\left(t, t^{-1}, 1\right) \otimes\left(t^{3}, t^{-3}, 1\right) \otimes\left(t^{9}, t^{-9}, 1\right)$, so has fixed point space 0 . Hence the fixed point space of $x$ on $L\left(E_{6}\right)$ has dimension 6 , and it follows that $x$ is regular semisimple in $M_{0}$, with centralizer $C_{M_{0}}(x)$ of order $27^{2}-1$.

On the other hand, for the classical group $G_{0}=\mathrm{Cl}_{n}(q), q=p^{e}$, we have $n \geq 27$ if $p=3$ by [82], and $n \geq 3^{11}-3^{9}$ if $p \neq 3$ by Proposition 4.9.5. Hence $\left|C_{G}(x)\right|$ is far greater than $\left|C_{M}(x)\right|$ by Proposition 2.4.1. This final contradiction completes the proof.

In light of the preceding two results, to prove Proposition 4.9.1 when $M_{0}$ of Lie type, we may exclude the following list of possibilities for $M_{0}$ :

$$
\begin{align*}
& \operatorname{PSL}_{2}(r)(r \leq 31), \mathrm{PSL}_{3}(r)(r \leq 5), \operatorname{PSL}_{4}(2), \\
& \mathrm{PSU}_{3}(r)(r \leq 5), \mathrm{PSU}_{4}(r)(r \leq 5), \operatorname{PSU}_{5}(2), \operatorname{PSU}_{6}(2), \\
& \operatorname{PSp}_{4}(r)(r \leq 7), \operatorname{PSp}_{6}(r)(r \leq 3), \operatorname{PSp}_{8}(2),  \tag{4.9.2}\\
& \Omega_{7}(r)(r \leq 9), \Omega_{8}^{-}(r)(r \leq 9), \Omega_{10}^{ \pm}(2), \mathrm{P}_{10}^{-}(3), \\
& G_{2}(r)(r \leq 5),{ }^{3} D_{4}(r)(r \leq 5), F_{4}(r)(r \leq 3),{ }^{2} E_{6}(r)(r \leq 3),{ }^{2} F_{4}(2)^{\prime} .
\end{align*}
$$

For convenience, we restate Proposition 4.9 .1 for this case:
Lemma 4.9.12. Let $G$ be an almost simple group with socle a classical group $G_{0}=\mathrm{Cl}_{n}(q)$, and suppose $M$ is a maximal subgroup of $G$ in the family $\mathcal{S}$, with socle $M_{0}$ a group of Lie type not in the list (4.9.2). Then the action of $G$ on $(G: M)$ is not binary.

Note that the list (4.9.2) includes all the exceptions in the conclusions of Lemmas [2.2.6, 2.2.8, 2.2.9, 2.2.10, 2.2.11 and 2.2.15,

Now let $x \in M_{0}$ to be the element defined in these propositions, as detailed in Table 4.9.2.
We shall need upper bounds for the order of the centralizer of $x$ in $M$, given in the next result.
Lemma 4.9.13. Let $M$ be almost simple, with socle $M_{0}$ of Lie type over $\mathbb{F}_{r}$, and let $x \in M_{0}$ be as defined in Table 4.9.2.
(i) For $M_{0}$ classical, we have $\left|C_{M}(x)\right|<N$, where $N$ is as in Table 4.9.3.
(ii) For $M_{0}$ exceptional, upper bounds for $\left|C_{M}(x)\right|$ are given in Lemmas 2.2.15 and 2.2.16.

Table 4.9.2: Definition of the element $x \in M_{0}$

| $M_{0}$ | $x$ as in Lemma |
| :---: | :---: |
| $\mathrm{PSL}_{d}(r)$ | 2.2.6 |
| $\mathrm{PSU}_{d}(r)$ | 2.2.10 $(d=3)$ |
|  | 2.2.11 $(d=4)$ |
|  | $2.2 .8(d \geq 5)$ |
| $\mathrm{PSp}_{d}(r)$ | 2.2.8 |
| $\mathrm{P} \Omega_{d}^{\epsilon}(r)$ | 2.2.8 ( $r$ even) |
|  | 2.2.9 ( $r$ odd) |
| exceptional | 2.2.15, 2.2.16 |

Proof. The argument for (i) is very similar for all types of classical groups. For $M_{0}=\mathrm{PSL}_{d}(r)$ with $r>2$, the element $x$ has preimage in $\mathrm{SL}_{d}(r)$ of the form $\operatorname{diag}\left(1, A, a^{-1}\right)$ and the centralizer in $\mathrm{GL}_{d}(r)$ of this element has order $(r-1)^{2}\left|\mathrm{GL}_{1}\left(r^{d-2}\right)\right|$. Moreover $x$ is not centralized by any non-trivial field automorphism, and can only be centralized by a graph automorphism if $d=3$. It follows that for $d \neq 3$, $\left|C_{M}(x)\right| \leq C_{\mathrm{PGL}_{d}(r)}(x) \leq(r-1)\left(r^{d-2}-1\right)$, and for $d=3$ there is an extra factor of 2 .

For $M_{0}=\mathrm{PSU}_{d}(r)$ with $d=2 j+\delta$ (where $\delta=1$ or 2 ), the centralizer in $\mathrm{GU}_{d}(r)$ of a preimage of $x$ is contained in $\mathrm{GU}_{2}(r) \times \mathrm{GU}_{\delta}(r) \times \mathrm{GL}_{1}\left(r^{2(j-1)}\right)$, and there are no outer automorphisms centralizing $x$ unless $d=3$ or 4 (in which case there is a graph automorphism); this leads to the bound in Table 4.9.3. Similarly for $M_{0}=\mathrm{PSp}_{2 k}(r)$, the centralizer is $\mathrm{Sp}_{2}(r) \times \mathrm{GL}_{1}\left(r^{k-1}\right)$.

Next consider $M_{0}=\mathrm{P} \Omega_{2 k}^{-}(r)$. If $r$ is odd, then $x$ has preimage $\operatorname{diag}\left(I_{4}, \zeta, \zeta^{-1}, A, A^{-T}\right)$, and the centralizer of this in $\mathrm{O}_{2 k}^{-}(r)$ is $\mathrm{O}_{4}^{-}(r) \times \mathrm{GL}_{1}(r) \times \mathrm{GL}_{1}\left(r^{k-3}\right)$, leading to the required bound. Similar considerations give the result for $\mathrm{P} \Omega_{2 k}^{+}(r)$ and $\mathrm{P} \Omega_{2 k+1}(r)$.

Table 4.9.3: Upper bounds for $\left|C_{M}(x)\right|$

| $M_{0}$ | $N$ |
| :---: | :---: |
| $\mathrm{PSL}_{d}(r)$ | $r^{d-1}\left(1+\delta_{d, 3}\right)$ |
| $\mathrm{PSU}_{d}(r)(d \geq 3)$ | $2 r^{d+3}, d \geq 6$ even |
|  | $2 r^{d+1}, d \geq 5$ odd |
|  | $2 r^{5}, d=4$ |
|  | $2 r^{2}, d=3$ |
| $\mathrm{PSp}_{2 k}(r)(k \geq 2)$ | $r^{k+2}$ |
| $\mathrm{P} \Omega_{2 k}^{+}(r)(k \geq 4)$ | $2 r^{k}$ |
| $\mathrm{P} \Omega_{2 k}^{-}(r)(k \geq 4)$ | $2 r^{k+4}$ |
| $\mathrm{P} \Omega_{2 k+1}(r)(k \geq 3)$ | $2 r^{k+2}$ |

For the proof of Lemma 4.9.12, we now adopt the following assumptions:
(1) $G$ is an almost simple group with socle $G_{0}=\mathrm{Cl}_{n}(q)\left(q=p^{a}\right)$, a classical group.
(2) $M$ is a maximal subgroup of $G$ in the family $\mathcal{S}$ with socle $M_{0}$, a group of Lie type over $\mathbb{F}_{r} ;$ moreover, $M_{0}$ is not one of the groups in the list (4.9.2).
(3) The action of $G$ on $(G: M)$ is binary.

We aim for a contradiction. This will prove Lemma 4.9.12,
Lemma 4.9.14. Adopt the above assumptions (1)-(3), and let $x \in M_{0}$ be as defined in Table 4.9.2, Then $C_{G}(x)=C_{M}(x)$.

Proof. Suppose there exists $g \in C_{G}(x) \backslash M$. Then $x \in M \cap M^{g}$. If $M_{0} \leq M \cap M^{g}$, then $g \in N_{G}\left(M_{0}\right)=M$ which is not the case; hence $M \cap M^{g}$ is a core-free subgroup of $M$ containing $x$. It now follows from the results listed in the last column of Table 4.9 .2 that the action of $M$ on $\left(M: M \cap M^{g}\right)$ is not binary. But then $(G,(G: M))$ is also not binary by Lemma 1.6.1, a contradiction.

Recall that the classical group $G_{0}=\mathrm{Cl}_{n}(q)$ is defined over the field $\mathbb{F}_{q}$ of characteristic $p$, while the subgroup $M_{0}$ is a group of Lie type over $\mathbb{F}_{r}$. At this point we divide the analysis into two cases: the cross-characteristic case (where $p \nmid r$ ) and the defining characteristic case (where $p \mid r$ ).

Lemma 4.9.15. Under the assumptions (1)-(3), the cross-characteristic case $p \nmid r$ does not occur.
Proof. Suppose $p \nmid r$. Then the following hold:
(a) $n \geq R\left(M_{0}\right)$, as given in Table 4.9.1.
(b) By Lemma 2.4.1, we have

$$
\left|C_{G}(x)\right|>\frac{q^{\lceil(n-1) / 2\rceil}}{4}\left(\frac{q-1}{2 q e\left(\log _{q}(2 n)+4\right)}\right)^{1 / 2} .
$$

(c) By Lemma 4.9 .13 we also have $\left|C_{M}(x)\right| \leq N$, where $N$ is as defined in Tables 4.9.3 for $M_{0}$ classical, and in Table 2.2.2 and Lemma 2.2.16 for $M_{0}$ exceptional.

By Lemma 4.9.14, it follows that $N$ is greater than the right hand side of the inequality in (b). However, when combined with the inequality $n \geq R\left(M_{0}\right)$, it is routine to check that this leads to a contradiction.

It remains to handle the defining characteristic case, where $p \mid r$. Recall that $G_{0}=\mathrm{Cl}_{n}(q)\left(q=p^{a}\right)$, and $M_{0}$, the socle of the maximal subgroup $M$, is a group of Lie type over $\mathbb{F}_{r}$. Let $V$ be the natural $n$-dimensional module for $G_{0}$. According to [92, Cor. 6] together with [90], there are two possibilities:
(A) $\mathbb{F}_{r} \supset \mathbb{F}_{q}$ : in this case $r=q^{k}$ with $k \geq 2$, and the embedding $M_{0}<G_{0}$ is as in [90, Table 1B], and takes the form $\mathrm{Cl}_{d}\left(q^{k}\right)<\mathrm{Cl}_{d^{k}}(q)$;
(B) $\mathbb{F}_{r} \subseteq \mathbb{F}_{q}$ : in this case the representation of $M_{0}$ on $V$ corresponds to a restricted representation of the overlying simple algebraic group over $\overline{\mathbb{F}}_{p}$.

First we deal with Case (B).
Lemma 4.9.16. Under the assumptions (1)-(3), the defining characteristic case (B) above does not occur.
Proof. Assume we are in case (B), so that $M_{0}=M_{0}(r)<G_{0}=\mathrm{Cl}_{n}(q)$ with $\mathbb{F}_{r} \subseteq \mathbb{F}_{q}$. We shall use the lower bounds for the dimensions of restricted representations of simple algebraic groups given by 82]. For an integer $d$, define $\epsilon_{p, d}$ to be 1 if $p \mid d$, and 0 otherwise.
( $\boldsymbol{\alpha}$ ) Assume first that $M_{0}=\mathrm{PSL}_{d}^{\epsilon}(r)$. If the restriction of $V$ to $M_{0}$ is self-dual, then $G_{0}$ is symplectic or orthogonal; otherwise, $G_{0}=\mathrm{PSL}_{n}^{\epsilon}(q)$. Hence using [82], we see that one of the following holds:
(i) $G_{0}=\operatorname{PSp}_{n}(q)$ or $\operatorname{P} \Omega_{n}(q)$ :

$$
\begin{aligned}
& d=2, n \geq 4, \text { or } \\
& d \geq 3, n \geq d^{2}-1-\epsilon_{p, d}
\end{aligned}
$$

(ii) $G_{0}=\operatorname{PSL}_{n}^{\epsilon}(q)$ :

$$
\begin{aligned}
& d=3, n \geq 6, \text { or } \\
& d=4, n \geq 10, \text { or } \\
& d=5, n=10 \text { or } n \geq 15, \text { or } \\
& d \geq 6, n \geq \frac{1}{2} d(d-1)
\end{aligned}
$$

Consider the element $x \in M_{0}$ defined in Table 4.9.2, By Lemma 4.9.13, we have $\left|C_{M}(x)\right|<N$, where $N$ is as in Table 4.9.3, and by Proposition 2.4.1 we have $\left|C_{G}(g)\right|>f(n, q)$, where $f(n, q)$ is as in Table 2.4.1. Hence Lemma 4.9.14 gives

$$
N>f(n, q) .
$$

Combined with the lower bounds on $n$ in (i) and (ii) above, this gives a contradiction except for the following cases:
(1) $d=3, p=3, n=7$ : here $M_{0}=\operatorname{PSL}_{3}^{\epsilon}(q)<G_{0}=\Omega_{7}(q), q=3^{a}$,
(2) $d=5, n=10, \epsilon=-$ : here $M_{0}=\operatorname{PSU}_{5}(q)<G_{0}=\operatorname{PSU}_{10}(q)$.

In case (1) the element $x \in M_{0}$ has preimage $\operatorname{diag}\left(1, t, t^{-1}\right)$ in $\mathrm{SL}_{3}^{\epsilon}(q)$, where $t \in \mathbb{F}_{q}$ has order $q-1$. Moreover, the natural 7 -dimensional module $V$ for $G_{0}$ is a constituent of the adjoint module for $M_{0}$, and hence the action of $x$ on $V$ is $\operatorname{diag}\left(1, t, t, t^{-1}, t^{-1}, t^{2}, t^{-2}\right)$. Clearly then, $C_{M}(x) \neq C_{G}(x)$, contradicting Lemma 4.9.14.

In case (2) above, $x$ has preimage $\operatorname{diag}\left(1,1,1, t, t^{-1}\right)$ in $\mathrm{SU}_{5}(q)$, and $V$ is the exterior square of the 5 dimensional natural module for $M_{0}$. Hence $x$ acts on $V$ as $\operatorname{diag}\left(1^{4}, t, t, t, t^{-1}, t^{-1}, t^{-1}\right)$, and again $C_{M}(x) \neq$ $C_{G}(x)$. This completes the proof for the case where $M_{0}=\mathrm{PSL}_{d}^{\epsilon}(r)$.
( $\boldsymbol{\beta}$ ) Next assume that $M_{0}=\mathrm{PSp}_{2 k}(r)$ with $k \geq 2$. In this case [82] gives

$$
\begin{aligned}
& k=2, n=10 \text { or } n \geq 12, \text { or } \\
& k=3, n=8(p=2) \text { or } n=14-\delta_{p, 3} \text { or } n \geq 21, \text { or } \\
& k \geq 4, n=2^{k}(p=2) \text { or } n \geq k(2 k-1)-1-\epsilon_{p, k} .
\end{aligned}
$$

Again we have $\left|C_{M}(x)\right|<N$ with $N$ is as in Table 4.9.3, and also $\left|C_{G}(g)\right|>f(n, q)$ with $f(n, q)$ as in Proposition 2.4.1(ii). Hence Lemma 4.9.14 gives $N>f(n, q)$, and combined with the above lower bounds for $n$, this yields a contradiction apart from the following cases:
(1) $k=2, n=10$,
(2) $k=3, n=8(p=2)$ or $n=14-\delta_{p, 3}$,
(3) $k=4, n=16(p=2)$.

In case (1), the element $x \in M_{0}$ has preimage $\operatorname{diag}\left(1,1, t, t^{-1}\right)$ in $\operatorname{Sp}_{4}(q)$, where $t \in \mathbb{F}_{q}$ has order $q-1$; also $p \neq 2$ and $V$ is the symmetric square of the natural 4-dimensional module for $M_{0}$. Hence $x$ acts on $V$ as $\operatorname{diag}\left(1^{4}, t, t, t, t^{-1}, t^{-1}, t^{-1}\right)$, and $C_{M}(x) \neq C_{G}(x)$, contradicting Lemma 4.9.14.

In case (2), $x$ has preimage $\operatorname{diag}\left(1,1, A, A^{-T}\right)$ in $\mathrm{Sp}_{6}(q)$, where $A \in \mathrm{GL}_{2}(q)$ has order $q^{2}-1$. If $n=14-\delta_{p, 3}$, then $V$ is a constituent of the exterior square of the natural 6 -dimensional module, and so the action of $x$ has diagonal blocks $\operatorname{diag}\left(A, A, A^{-T}, A^{-T}, \ldots\right)$. But this implies that $C_{G}(x)$ contains a subgroup $\mathrm{SL}_{2}\left(q^{2}\right)$, so again $C_{M}(x) \neq C_{G}(x)$. And if $n=8$ with $p=2$, then $V$ is a spin module for $M_{0}=\mathrm{Sp}_{6}(q)$. Observe that $x$ lies in a subgroup $\mathrm{Sp}_{2}(q) \times \mathrm{Sp}_{4}(q)$, and on a spin module this acts as $\mathrm{Sp}_{2}(q) \otimes \mathrm{Sp}_{4}(q)$. Hence $x$ acts as $I_{2} \otimes \operatorname{diag}\left(A, A^{-T}\right)$, and so as before, $C_{G}(x)$ contains a subgroup $\mathrm{SL}_{2}\left(q^{2}\right)$.

A similar argument applies in case (3), where $V$ is a spin module for $M_{0}=\operatorname{Sp}_{8}(q)$. Here $x=$ $\operatorname{diag}\left(1,1, A, A^{-T}\right) \in M_{0}$, where $A \in \mathrm{GL}_{3}(q)$ has order $q^{3}-1$. This lies in a subgroup $\operatorname{Sp}_{2}(q) \times \operatorname{Sp}_{6}(q)$, hence as above, acts on a spin module as $\left(I_{4}, A, A, A^{-T}, A^{-T}\right)$. Then $C_{G}(x)$ contains a subgroup $\mathrm{SL}_{2}\left(q^{3}\right)$, so again $C_{M}(x) \neq C_{G}(x)$.
$(\gamma)$ Now consider the case where $M_{0}$ is an orthogonal group $\mathrm{P} \Omega_{d}^{\epsilon}(r)$ with $d \geq 7$. In this case the dimension bounds are:

$$
\begin{aligned}
& d=7, n=8 \text { or } n \geq 21, \text { or } \\
& d=8, n=8 \text { or } n \geq 26, \text { or } \\
& d \geq 9, n=2^{\left\lfloor\frac{d-1}{2}\right\rfloor} \text { or } n \geq \frac{1}{2} d(d-1)-2 .
\end{aligned}
$$

As above, the inequality $N>f(n, q)$ now gives a contradiction apart from the following cases:
(1) $d=7, n=8$,
(2) $d=8, n=8$,
(3) $d=9$ or $10, n=16$.

Consider (1). Here $M_{0}=\Omega_{7}(q)<G_{0}=\mathrm{P} \Omega_{8}^{+}(q)$, and the element $x=\left(1^{3}, \zeta, \zeta^{-1}, a, a^{-1}\right) \in M_{0}$ with $a, \zeta \in \mathbb{F}_{q}^{\times}$of order $q-1$ and $\zeta \neq a^{ \pm 1}$. Write $x=\left(I_{3}, X\right) \in \Omega_{3}(q) \times \Omega_{4}^{+}(q)<M_{0}$, where $X=\left(\zeta^{-1}, a, a^{-1}\right)$. In $\mathrm{SL}_{2}(q) \otimes \mathrm{SL}_{2}(q) \cong \Omega_{4}^{+}(q), X$ takes the form $\left(\alpha, \alpha^{-1}\right) \otimes\left(\beta, \beta^{-1}\right)$, where $\alpha \beta=a, \alpha \beta^{-1}=\zeta$. Hence in the spin representation on $V, x$ acts as $\left(I_{2} \otimes\left(\alpha, \alpha^{-1}\right), I_{2} \otimes\left(\beta, \beta^{-1}\right)\right)$. It follows that $C_{G}(x)$ contains a subgroup $\left(\mathrm{SL}_{2}(q)\right)^{2}$, so $C_{M}(x) \neq C_{G}(x)$, giving the usual contradiction.

Now consider (2). In this case $M_{0}=\mathrm{P} \Omega_{8}^{-}\left(q^{1 / 2}\right)<G_{0}=\mathrm{P} \Omega_{8}^{+}(q)$, where $M_{0}$ is the image of a subfield subgroup under a triality automorphism of $G_{0}$. We can write the element $x$ as $\left(I_{2}, X\right)$ with $X \in \Omega_{6}^{+}\left(q^{1 / 2}\right)$ given by Lemmas 2.2.8 (for $p=2$ ) and 2.2.9 (for $p$ odd). Arguing in similar fashion to the previous paragraph, we see that $C_{G}(x)$ contains a subgroup $\mathrm{SL}_{2}\left(q^{2}\right)(p=2)$ or $\left(\mathrm{SL}_{2}(q)\right)^{2}$ ( $p$ odd). Hence again $C_{M}(x) \neq C_{G}(x)$.

Finally, consider case (3). For $d=9$ we have $M_{0}=\Omega_{9}(q)<G_{0}=\mathrm{P} \Omega_{16}^{+}(q)$ with $q$ odd, and $V$ is a spin module for $M_{0}$. We have $x=\left(1^{3}, \zeta, \zeta^{-1}, A, A^{-T}\right) \in M_{0}$, where $\zeta \in \mathbb{F}_{q}^{\times}$has order $q-1$ and $A \in \mathrm{GL}_{2}(q)$ has order $q^{2}-1$. Then $x \in \Omega_{3}(q) \times \Omega_{6}^{+}(q)<M_{0}$, and this subgroup acts on the spin module $V$ as $\left(V_{2} \otimes V_{4}\right) \oplus\left(V_{2} \otimes V_{4}^{*}\right)$, where the action of $x$ on $V_{4}$ is computed via the isomorphism $\Omega_{6}^{+}(q) \cong \mathrm{SL}_{4}(q) /\langle-I\rangle$. It follows that $C_{G}(x)$ contains $\left(\mathrm{SL}_{2}\left(q^{2}\right)\right)^{2}$, hence $C_{M}(x) \neq C_{G}(x)$. A very similar computation applies in the case where $d=10$.
( $\boldsymbol{\delta}$ ) To complete the proof of the lemma, it remains to handle the case where $M_{0}$ is an exceptional group of Lie type over $\mathbb{F}_{r}$ with $\mathbb{F}_{r} \subseteq \mathbb{F}_{q}$. From the bounds for the dimensions of restricted representations of groups of Lie type given in [82], it follows that either $n=R_{0}$, or $n \geq R$, where $R_{0}, R$ are as in the following table:

| $M_{0}$ | $E_{8}(r)$ | $E_{7}(r)$ | $E_{6}^{\epsilon}(r)$ | $F_{4}(r)$ | $G_{2}(r)$ | ${ }^{2} F_{4}(r)$ | ${ }^{2} G_{2}(r)$ | ${ }^{2} B_{2}(r)$ | ${ }^{3} D_{4}(r)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $R_{0}$ | 248 | 56 | 27 | $26-\delta_{p, 3}$ | $7-\delta_{p, 2}$ | 26 | 7 | 4 | 8 |
| $R$ | 248 | 132 | 77 | 52 | 14 | 246 | 26 | 16 | 26 |

We have $\left|C_{M}(x)\right|<N$ with $N$ as given in Lemmas 2.2.15, 2.2.16, and also $\left|C_{G}(g)\right|>f(n, q)$ with $f(n, q)$ as in Proposition 2.4.1(ii). Hence as usual, Lemma 4.9.14 gives $N>f(n, q)$, and combined with the above bounds for $n$, this yields a contradiction apart from the cases where $n=R_{0}$ and

$$
M_{0}=E_{6}^{\epsilon}(q), F_{4}(q), G_{2}(q),{ }^{2} B_{2}(q) \text { or }{ }^{3} D_{4}\left(q^{1 / 3}\right)
$$

(note that $r=q$ in all but the last case, by the maximality of $M$ ).
If $M_{0}=E_{6}^{\epsilon}(q)$, then $n=27$, the module $V$ has highest weight $\lambda_{1}$ in the usual notation, and is not self-dual, so that $G_{0}=\operatorname{PSL}_{27}^{\epsilon}(q)$. Now the inequality $N>f(n, q)$, with $f(n, q)$ as in Table 2.4.1, gives a contradiction.

Next consider $M_{0}=F_{4}(q)$ with $n=26-\delta_{p, 3}$. By the exclusions in the list (4.9.2), we have $q \geq 4$. The element $x \in M_{0}$ is as defined in Lemma [2.2.15: it lies in a subsystem subgroup $A_{3} \cong \mathrm{SL}_{4}(q)$ and takes the form $\operatorname{diag}(1, a, A)$ where $A$ is a $2 \times 2$ matrix of order $q^{2}-1$ and determinant $a^{-1}$. The restriction $V \downarrow A_{3}$ is given in [76, Table 8.7]: in terms of highest weight modules, the composition factors are $V\left(\lambda_{1}\right)^{2} / V\left(\lambda_{3}\right)^{2} / V\left(\lambda_{2}\right) / 0^{4-\delta_{p, 3}}$. Here $W:=V\left(\lambda_{1}\right)$ is the natural 4-dimensional $A_{3}$-module, $V\left(\lambda_{3}\right)=W^{*}$, $V\left(\lambda_{2}\right)=\wedge^{2} W$ and 0 is the trivial module. Hence we compute that $\operatorname{dim} C_{V}(x)=8-\delta_{p, 3}$, and so $C_{G}(x)$ has a subgroup $\Omega_{7}(q)$. However $C_{M}(x)$ has no such subgroup by Lemma 2.2.15, a contradiction.

Now let $M_{0}=G_{2}(q)$ with $n=7-\delta_{p, 2}$ (and $q \geq 7$ by the exclusions in (4.9.2)). Here $G_{0}$ is $\operatorname{Sp}_{6}(q)$ if $q$ is even, and $\Omega_{7}(q)$ if $q$ is odd. We have $x=\operatorname{diag}\left(1, a, a^{-1}\right)$ in a subsystem subgroup $\operatorname{SL}_{3}(q)$, where $a \in \mathbb{F}_{q}^{\times}$ has order $q-1$. Hence $x$ acts on $V$ as $\operatorname{diag}\left(a, a, a^{-1}, a^{-1}, 1^{3-\delta_{p, 2}}\right)$, and it follows that $C_{G}(x)$ has a subgroup $\mathrm{Sp}_{2}(q) \times \mathrm{SL}_{2}(q)$ or $\Omega_{3}(q) \times \mathrm{SL}_{2}(q)$, whereas $C_{M}(x)$ has no such subgroup.

If $M_{0}={ }^{2} B_{2}(q)$ with $n=4$, then $G_{0}=\operatorname{Sp}_{4}(q)$ and we have $\left|C_{M}(x)\right|=q-1$ by Lemma 2.2.16, whereas $\left|C_{G}(x)\right|=(q-1)^{2}$. Finally, if $M_{0}={ }^{3} D_{4}\left(q^{1 / 3}\right)$ with $n=8$, then $G_{0}=\mathrm{P} \Omega_{8}^{+}(q)$ and $x$ acts on $V$ as $\operatorname{diag}\left(a, a, a^{-1}, a^{-1}, 1^{4}\right)$; hence $C_{G}(x)$ contains $\Omega_{4}^{+}(q)$, so once again $C_{M}(x) \neq C_{G}(x)$. This completes the proof.

Table 4.9.4: Embeddings $M_{0}=\mathrm{Cl}_{d}\left(q^{k}\right)<G_{0}=\mathrm{Cl}_{d^{k}}(q)$ ( $k$ prime)

| $M_{0}$ | $G_{0}$ | conditions |
| :---: | :---: | :---: |
| $\operatorname{PSL}_{d}^{\epsilon}\left(q^{k}\right)$ | $\operatorname{PSL}_{d^{k}}^{\epsilon}(q)$ | $d \geq 3,(k, \epsilon) \neq(2,-)$ |
| $\operatorname{PSL}_{d}\left(q^{2}\right)$ | $\operatorname{PSU}_{d^{2}}(q)$ | $d \geq 3$ |
| $\operatorname{PSp}_{d}\left(q^{k}\right)$ | $\operatorname{PSp}_{d^{k}}(q)$ | $k q$ odd |
| $\operatorname{PSp}_{d}\left(q^{k}\right)$ | $\operatorname{P\Omega }_{d^{k}}^{+}(q)$ | $k$ odd, $q$ even |
| $\operatorname{PSp}_{d}\left(q^{2}\right)$ | $\operatorname{PS}_{d^{2}}^{\epsilon}(q)$ | $d \geq 4, \epsilon=(-)^{d / 2}$ |
| $\operatorname{P\Omega }_{d}^{ \pm}\left(q^{k}\right)$ | $\mathrm{P}_{d^{k}}(q)$ | $d \geq 6$ even, $q$ odd |
| $\operatorname{P} \Omega_{d}\left(q^{k}\right)$ | $\mathrm{P}_{d^{k}}(q)$ | $d q$ odd, $d \geq 3$ |

Lemma 4.9.17. Under the assumptions (1)-(3), the defining characteristic case (A) above does not occur.
Proof. Assume we are in case (A), so that $M_{0}=\mathrm{Cl}_{d}\left(q^{k}\right)<G_{0}=\mathrm{Cl}_{d^{k}}(q)$ with $k \geq 2$. Specifically, the embeddings $M_{0}<G_{0}$ are as given by [90, Table 1B], and are as in Table 4.9.4, with $k$ prime. With one exception, the natural module for $G_{0}$ is of the form $V=W \otimes W^{(q)} \otimes \cdots \otimes W^{\left(q^{k-1}\right)}$, where $W$ is the natural $d$-dimensional module for $M_{0}$; the exception is for the embedding $\operatorname{PSL}_{d}\left(q^{2}\right) \leq \operatorname{PSU}_{d^{2}}(q)$ in the second row of the table, where $V=W \otimes W^{*(q)}$.

The argument is very similar for all entries in the table: we have $x \in M_{0}$, a semisimple element with centralizer as described in the proof of Lemma4.9.13. Then $C_{G}(x)$ contains a maximal torus of $G$, whereas we argue that $C_{M}(x)$ cannot contain such a torus: in most cases this is obvious, as $G$ has much larger rank than $M$, but nevertheless we shall give a sketch for each case below.

Consider the first row of the table, $\operatorname{PSL}_{d}^{\epsilon}\left(q^{k}\right) \leq \operatorname{PSL}_{d^{k}}^{\epsilon}(q)$ with $d \geq 3,(k, \epsilon) \neq(2,-)$. For $\epsilon=+$ the element $x \in M_{0}$ has preimage of the form $\operatorname{diag}(1, a, A)$ where $A \in \mathrm{GL}_{d-2}\left(q^{k}\right)$ has order $q^{k(d-2)}-1$. This acts on $V$ as $(1, a, A) \otimes(1, a, A)^{(q)} \otimes \cdots \otimes(1, a, A)^{\left(q^{k-1}\right)}$, and hence we see that $C_{G}(x)$ has order divisible by $\left(q^{k(d-2)}-1\right)^{2}$ if $d \geq 4$, and by $\left(q^{k}-1\right)^{3} /(q-1)$ if $d=3$. Hence $C_{G}(x) \neq C_{M}(x)$. Now consider $\epsilon=-$. Here the semisimple element $x \in M_{0}$ has at least two eigenvalues 1 if $d \geq 4$, and is $\operatorname{diag}\left(1, a, a^{-1}\right)$ if $d=3$, where $a$ generates $\mathbb{F}_{q^{k}}^{\times}$. From the action of $x$ on $V$, we see that $C_{G}(x)$ contains $\mathrm{SU}_{2^{k}}(q)$ if $d \geq 4$, and contains $\left(\mathrm{GL}_{1}\left(q^{2 k}\right)\right)^{2}$ if $d=3$. Hence again $C_{G}(x) \neq C_{M}(x)$.

The argument for the second row of Table 4.9 .4 is entirely similar: here $C_{G}(x)$ has order divisible by $\left(q^{2(d-2)}-1\right)^{2}$ if $d \geq 4$, and by $\left(q^{2}-1\right)^{3}$ if $d=3$.

Next consider $M_{0}=\operatorname{PSp}_{d}\left(q^{k}\right)$, with embedding as in rows 3-5 of Table 4.9.4. Suppose first that $d=2$, so that $k$ is odd and $G_{0}$ is $\mathrm{PSp}_{2^{k}}(q)$ or $\mathrm{P} \Omega_{2^{k}}^{+}(q)$, according as $q$ is odd or even, respectively. Also $q^{k}>31$, by the exclusions of (4.9.2). The element $x=\operatorname{diag}\left(a, a^{-1}\right) \in M_{0}$ has centralizer in $M$ of order dividing $q^{k}-1$. If $k=3$, then $q>3$ and the action of $x$ on $V$ has eigenvalues $\mu^{ \pm 1}, \mu^{ \pm q}, \mu^{ \pm q^{2}}, \lambda, \lambda^{-1}$, where $\mu=a^{q^{2}+q-1}$, $\lambda=a^{q^{2}+q+1}$; hence $C_{G}(x)$ has order divisible by $\left(q^{3}-1\right)(q-1) /(2, q-1)$, and so $C_{G}(x) \neq C_{M}(x)$. And if $k \geq 5$, then we reach the same contradiction as $\left|C_{G}(x)\right|$ is divisible by $\left(q^{k}-1\right)^{\left(2^{k-1}-1\right) / k}$.

This deals with $d=2$, so suppose now that $M_{0}=\operatorname{PSp}_{d}\left(q^{k}\right)$ with $d \geq 4$. Here $x=\left(I_{2}, A, A^{-T}\right)$, where $A \in \mathrm{GL}_{\frac{d}{2}-1}\left(q^{k}\right)$ has order $q^{k\left(\frac{d}{2}-1\right)}-1$. If $k \geq 3$, then the fixed point space of $x$ on $V$ has dimension $2^{k}$ and $C_{G}(x)$ contains $\mathrm{Cl}_{2^{k}}(q)$. Hence $k=2$ and $x$ acts on $V$ as $\left(I_{2}, A, A^{-T}\right) \otimes\left(I_{2}, A, A^{-T}\right)^{(q)}$. This has centralizer in $G$ containing $\Omega_{4}^{\epsilon}(q) \times \mathrm{SL}_{2}\left(q^{k\left(\frac{d}{2}-1\right)}\right)$, so once again $C_{G}(x) \neq C_{M}(x)$.

Finally, consider $M_{0}=\mathrm{P} \Omega_{d}^{\epsilon}\left(q^{k}\right)$, as in the last two rows of Table 4.9.4.
Because of exceptional isomorphisms of low-dimensional orthogonal groups, we need to consider separately the cases $d=3,5$ and 6 . If $d=3$ then $x \in M_{0}=\Omega_{3}\left(q^{k}\right) \cong \operatorname{PSL}_{2}\left(q^{k}\right)$ has the form $\operatorname{diag}\left(1, a, a^{-1}\right)$,
where $a \in \mathbb{F}_{q^{k}}$ has order $\left(q^{k}-1\right) / 2$, and we argue in the usual way that $\left|C_{G}(x)\right|$ is divisible by $\left(q^{k}-1\right)^{2} / 2$, so $C_{G}(x) \neq C_{M}(x)$.

If $d=5$ then $x \in M_{0}=\Omega_{5}\left(q^{k}\right) \cong \operatorname{PSp}_{4}\left(q^{k}\right)$; in $\operatorname{PSp}_{4}\left(q^{k}\right), x$ takes the form $\left(I_{2}, a, a^{-1}\right)$, so in $\Omega_{5}\left(q^{k}\right)$, we have $x=\left(1, a I_{2}, a^{-1} I_{2}\right)$. Now we argue that $C_{G}(x)$ contains $\left(\mathrm{SL}_{2}\left(q^{k}\right)\right)^{2}$ if $k \geq 3$, and contains $\mathrm{SL}_{2}\left(q^{2}\right) \times$ $\mathrm{SU}_{2}(q)$ if $k=2$. In both cases, $C_{G}(x) \neq C_{M}(x)$.

Next, if $d=6$ then $x \in M_{0}=\mathrm{P}_{6}^{\epsilon}\left(q^{k}\right) \cong \operatorname{PSL}_{4}^{\epsilon}\left(q^{k}\right)$. For $\epsilon=+, x$ takes the form ( $1, a, A$ ) in $\mathrm{PSL}_{4}(q)$, hence $x=\left(a, a^{-1} A, A^{-T}\right) \in \mathrm{P}_{6}^{+}\left(q^{k}\right)$; and for $e=-, x$ is $\left(I_{2}, a, a^{-1} \in \mathrm{PSU}_{4}(q)\right.$, hence $x=$ $\left(I_{2}, a I_{2}, a^{-1} I_{2}\right) \in \mathrm{P} \Omega_{6}^{-}\left(q^{k}\right)$. Now argue in the usual way that $C_{G}(x) \neq C_{M}(x)$.

Finally, if $d \geq 7$ then $x=\left(I_{2+y}, \zeta, \zeta^{-1}, A, A^{-T}\right) \in M_{0}=\mathrm{P} \Omega_{d}^{\epsilon}\left(q^{k}\right)$ (where $y \in\{0,1,2\}$ ), and so $C_{G}(x)$ contains $\Omega_{(2+y)^{k}}(q) \times \mathrm{SL}_{2^{k-1}}\left(q^{k(d-4-y) / 2}\right)$, and once again we have the contradiction $C_{G}(x) \neq C_{M}(x)$. This completes the proof.

This completes the proof of Lemma 4.9.12, and hence also the proof of Proposition 4.9.1.

### 4.10 Exceptional automorphisms

As explained in Section 4.1.1, in our proof of Theorem 4.1, we have so far been assuming that our almost simple group $G$ contains no graph automorphisms when $G$ has socle $\operatorname{Sp}_{4}\left(2^{a}\right)$, and no triality automorphisms when $G$ has socle $\mathrm{P} \Omega_{8}^{+}(q)$ (except when $M$ is in the Aschbacher class $\mathcal{S}$ ). In this final section, we complete the proof of Theorem 4.1 by handling these cases.

Thus we assume in this section that $G$ is an almost simple group such that one of the following holds:

1. the socle of $G$ is isomorphic to $\operatorname{Sp}_{4}\left(2^{a}\right)$ with $a>1$, and $G$ contains a graph automorphism;
2. the socle of $G$ is isomorphic to $\mathrm{P}_{8}^{+}(q)$ and $G$ contains a triality automorphism.

Note that we omit the case $\mathrm{Sp}_{4}(2)$, as the theorem is already proved for groups with alternating socle in 46 .

We slightly adjust terminology for this final section: we use $S$ to denote the socle of $G$.
Lemma 4.10.1. Let $S=\operatorname{Sp}_{4}(q)$ where $q=2^{a}$ with $a>1$, and suppose that $S \leq G \leq \operatorname{Aut}(S)$. Let $M$ be $a$ core-free maximal subgroup of $G$. Then the action of $G$ on $(G: M)$ is not binary.

Proof. For $q \in\{4,8,16\}$, we refer to Lemma 4.1.1. Assume that $q \geq 32$. We refer to [10, Table 8.14] for the maximal subgroups of $G$. One checks that with three exceptions, all of them contain an element $g$ as defined in Lemma 2.2.12, hence these can be excluded. The exceptions are

$$
M \cap S=\operatorname{Sp}_{4}\left(q_{0}\right), \quad\left(q^{2}+1\right): 4 \text { or }(q+1)^{2}: D_{8}
$$

If $M \cap S=\operatorname{Sp}_{4}\left(q_{0}\right)$, then $q=q_{0}^{r}$, where $r$ is prime, and the argument of Lemma 4.5.7 gives the conclusion.

In the remaining two cases, $M$ is a torus normalizer, and we use arguments similar to those in $\S 3.5$, Suppose that $M \cap S=\left(q^{2}+1\right): 4$. Then $N=M \cap S$ is a Frobenius group with $T=q^{2}+1$, the Frobenius kernel. Let $g \in M \cap S$ be of order 4; again we check that there exists $c \in C_{G}(g) \backslash N_{G}(T)$. Then the action of $N$ on $\left(N: N \cap N^{x}\right)$ is a Frobenius action and, since $N \cap N^{x}=N \cap M \cap M^{x}$, Lemma 1.7.2 implies that the action of $M$ on ( $M: M \cap M^{x}$ ) is not binary; hence the action of $G$ on $(G: M)$ is not binary by Lemma 1.6.1.

Suppose finally that $M \cap S=(q+1)^{2}: D_{8}$. We apply Lemma 3.5 .2 with $A \cong D \cong \mathrm{SL}_{2}(q)$, and $T_{0} \cong T_{1} \cong q+1$. The listed conditions are all easy to verify; in particular, item (vii) of the lemma is verified using [34], which asserts that the action of a group with socle $\mathrm{SL}_{2}(q)$ on the set of cosets of the normalizer of a non-split torus is not binary.

Lemma 4.10.2. Let $S=\mathrm{P} \Omega_{8}^{+}(q)$, suppose that $S \leq G \leq \operatorname{Aut}(S)$, and suppose that $G$ contains an element in the coset of a triality automorphism of $S$. If $M$ is a maximal core-free subgroup of $G$, then the action of $G$ on $\Omega=(G: M)$ is not binary.

Proof. This is covered by Lemma 4.1.1 when $q \leq 4$, so assume that $q \geq 5$.
We refer to [55] for a list of the maximal subgroups of $G$. Following [55] we set $d=\operatorname{gcd}(2, q-1)$ and, when giving the isomorphism type of a subgroup, we prefix a circumflex symbol to indicate that we are giving the structure of the group in $\Omega_{8}^{+}(q)$, rather than its projective image in $\mathrm{P} \Omega_{8}^{+}(q)$.

Suppose, first, that $M \cap S$ is a maximal subgroup of $S$ in the $\mathcal{C}_{1}$ family. Then [46, Proposition 4.6] implies that $\Omega$ contains a beautiful subset, and the result follows immediately. Suppose, next, that $M$ is a novelty maximal subgroup of $G$ such that $M \cap S$ is a proper subgroup of a maximal $\mathcal{C}_{1}$ subgroup of $S$. There are four possibilities for $M$, and we list them in Table 4.10.1 together with an integer $r$. The integer $r$ indicates the presence of a subgroup $A=\mathrm{SL}_{r}(q)$ in $M$, together with a subgroup of $S$ that is isomorphic to a central quotient of $\mathrm{SL}_{r+1}(q)$ satisfying the conditions of Lemma 1.6.10. The lemma then implies that there is a subset $\Delta$ of $\Omega$ of size $q^{r}$ on which $G^{\Delta}$ acts 2-transitively. Now Lemma 2.1.1] implies that $\mathrm{P} \Omega_{8}^{+}(q)$ does not contain a section isomorphic to $\operatorname{Alt}\left(q^{r}\right)$, and the result follows.

| $M \cap S$ | $r$ |
| :---: | :---: |
| $\left\{q^{11}:\left[\frac{q-1}{d}\right]: \frac{1}{d} \mathrm{GL}_{2}(q) \cdot d^{2}\right.$ | 2 |
| $G_{2}(q)$ | 3 |
| $\wedge\left(\frac{q-1}{d} \times \frac{1}{d} \mathrm{GL}_{3}(q)\right) \cdot[2 d]$ | 3 |
| $\wedge\left(\frac{q+1}{d} \times \frac{1}{d} \mathrm{GU}_{3}(q)\right) \cdot[2 d]$ | 2 |

Table 4.10.1: Novelty $\mathcal{C}_{1}$-subgroups in $\mathrm{P}_{8}^{+}(q)$
Next suppose that $M \cap S$ is a maximal subgroup of $S$ in the $\mathcal{C}_{2}$ family, stabilizing a decomposition of $V=V_{8}(q)$ as a direct sum of $m$-spaces. Lemma 4.2.13 implies that either

- there is a beautiful subset (and we are done), or
- the parameter $m=1$ (and [55] implies that we can exclude this case, since such groups are not maximal given our assumption that $G$ contains a triality automorphism), or
- $M \cap S$ is of type $\mathrm{O}_{2}^{-}(q)$ wr $\operatorname{Sym}(4)$.

In this last case $M$ is the normalizer of a torus, with $M \cap S \cong^{\wedge}\left(\frac{q+1}{d}\right)^{4} . d^{3} .2^{3} \cdot \operatorname{Sym}(4)$. Now, as in the previous lemma, we appeal to Lemma 3.5.2 with $A=A_{1}(q)$ and $D=A_{1}(q)^{3}$, and we conclude that the action of $G$ on $(G: M)$ is not binary.

Suppose now that $M$ is a novelty maximal subgroup of $G$ such that $M \cap S$ is a proper subgroup of a maximal subgroup of $S$ in the $\mathcal{C}_{2}$ class. Then [55] implies that there are two possibilities: $M \cap S \cong$ $\left[2^{9}\right] . \mathrm{PSL}_{3}(2)$ or $\left(D_{2\left(q^{2}+1\right) / d}\right)^{2} \cdot\left[2^{2}\right]$. For the first we use magma to check that any transitive action of $M$ of degree $k$ with $k \not \equiv 0(\bmod 4)$ and with $M / K$ non-solvable, where $K$ is the kernel of the action, is not binary. Now if $q \equiv 1,7(\bmod 8)$, then $|G: M|$ is even and hence $M$ must have a non-trivial suborbit of odd degree; so we are done in this case (note that we can ignore actions where $M / K$ is solvable by Lemma 1.8.2). If $q \equiv 3,5(\bmod 8)$, then $|G: M| \equiv 3(\bmod 4)$. Therefore $|G: M|-1 \equiv 2(\bmod 4)$ and hence $M$ cannot have all suborbits of cardinality a multiple of 4 . Again, we are done. For the second group $\left(D_{2\left(q^{2}+1\right) / d}\right)^{2} \cdot\left[2^{2}\right]$, we use Lemma 3.5.2 with $A=A_{1}\left(q^{2}\right)$ and $D=A_{1}\left(q^{2}\right)$ and we conclude that the action of $G$ on $(G: M)$ is not binary.

Assume next that $M \cap S$ is a maximal subgroup of $S$ in the $\mathcal{C}_{5}$ class. Then Lemma 4.5.8 implies that either there is a beautiful subset (and we are done), or $F^{*}(M \cap S)=\Omega_{8}^{-}\left(q_{0}\right)$ with $q_{0} \in\{2,3\}$ and
$S=\Omega_{8}^{+}\left(q_{0}^{2}\right)$ (but this case does not occur when $G$ contains a triality automorphism), or $F^{*}(M \cap S)=\Omega_{8}^{+}(2)$ and $q=2^{r}$ with $r$ odd. For this last case $M$ equals either $L$ or $L \times r$ where $L$ is almost simple with socle $\Omega_{8}^{+}(2)$; now we obtain the result arguing in exactly the same way as in Lemma 4.5.9,

The final case to consider is that in which $M \cap S$ is a subgroup in the family $\mathcal{S}$ of subgroups of $S$. However this case has already been dealt with in Proposition 4.9.1, thanks to our relaxation of the triality assumption at the beginning of Section 4.9.

This completes our consideration of the exceptional automorphisms. The proof of Theorem 4.1 is now complete.

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[^0]:    ${ }^{1}$ Wiscons informed us of a small gap in his proof of [106, Proposition 4.1]. The next paragraph consists of his comments on this, including a patch. For notation and terminology, we refer to the rest of this chapter.
    [106, Proposition 4.1] is devoted to showing that primitive groups of diagonal type are not binary. The gap in the proof stems from an implicit (and accidental) assumption in the first sentence of the proof of [106, Lemma 4.2] that the socle is a product of at least three isomorphic nonabelian simple groups. This leaves open the case of two factors, for which it suffices to consider the following setting: let $G$ be a group acting on a nonabelian group $T$ in such a way that the stabilizer of $1 \in T$ satisfies $\operatorname{Inn}(T) \leq G_{1} \leq \operatorname{Aut}(T) \times\langle i\rangle$ for $i: T \rightarrow T$ the inversion map. In this context, we show the action of $G$ on $T$ is not binary. To see this, choose noncommuting $a, b \in T$, not both of order 2 , and observe that $(1, a, b, a b)$ and ( $1, a, b, b a$ ) are 2 -subtuple complete (witnessed by conjugating by $1, a$, or $b^{-1}$ ). However, since one of $a$ or $b$ is not of order $2, G_{1, a, b} \leq \operatorname{Aut}(T)$, so $G_{1, a, b}$ fixes $a b \neq b a$. Thus, $(1, a, b, a b)$ and ( $1, a, b, b a$ ) are not 4 -subtuple complete, so such an action is not binary.

[^1]:    ${ }^{2}$ One can imagine a slight weakening of Definition 1.1.3 where one allows an automorphism of $\mathcal{R}$ to map a set of tuples corresponding to one relation to the set of tuples corresponding to a different relation - for certain relational structures, this would yield a larger automorphism group (which would contain $\operatorname{Aut}(\mathcal{R})$ as defined above, as a normal subgroup). We will not need this extension in what follows.

[^2]:    ${ }^{3}$ It may be perhaps better to call such a $Q$ a non-singular form rather than an anisotropic form - a vector $\mathbf{v}$ is generally called singular if $Q(\mathbf{v})=0$, and isotropic if $\beta(\mathbf{v}, \mathbf{v})=0$ where $\beta$ is the polar form of $Q$. If the characteristic of the field is odd, these two definitions coincide, however in characteristic 2 this is not the case. Our definition of an anisotropic form requires that the only singular vector for $Q$ is the zero vector, but note that all vectors are isotropic in the characteristic 2 case. In any case, we will stick to calling such a $Q$ anisotropic as it is consistent with what has come before in the literature.

[^3]:    ${ }^{4}$ The problem of specifying relational complexity when point-stabilizers have size 2 is now reduced to the problem of studying when $C$, a certain conjugacy class of involutions satisfies $C \subseteq C^{2}$. This problem is, in general, difficult, however one potential avenue of investigation is via the class constants of the finite group $G$, denoted $a_{i j v}$. For any conjugacy class $C_{i}$ in a group $G$, we define $\hat{C}_{i}=\sum_{c \in C_{i}} c_{i}$ to be the class sum of $C_{i}$ in the group algebra $\mathbb{C} G$. Now write

    $$
    \hat{C}_{i} \hat{C}_{j}=\sum_{v=1}^{k} a_{i j v} \hat{C}_{v}
    $$

    where $k$ is the number of conjugacy classes in $G$. The non-negative integers $a_{i j v}$ for $1 \leq i, j, v \leq k$ are the class constants of

[^4]:    ${ }^{5}$ There is some inconsistency in terminology across the literature - homogeneity as we have defined it here is sometimes called "ultra-homogeneity" while homogeneity refers to a strictly weaker property.

[^5]:    ${ }^{6}$ Here is the shorter and more elegant argument due to Wiscon for Lemma 1.7.1
    For distinct $a, b, c \in \Omega$, binarity implies that the intersection of the suborbits $c G_{a}$ and $c G_{b}$ is equal to $c G_{a, b}$, so as the action is Frobenius, $\left(c G_{a}\right) \cap\left(c G_{b}\right)=\{c\}$. Also, using again that the action is Frobenius, $\left|c G_{a}\right|=\left|G_{a}\right|=\left|G_{b}\right|=\left|c G_{b}\right|$. This shows that $\bigcup_{a \neq c}\left(c G_{a} \backslash\{c\}\right)$ is a disjoint union of sets of constant size $\left|G_{a}\right|-1$. So, letting $N=|\Omega|$, we find that $N-1=|\Omega \backslash\{c\}| \geq(N-1)\left(\left|G_{a}\right|-1\right)$, implying that $\left|G_{a}\right|=2$.

