Cherlin's conjecture on finite primitive binary permutation groups

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In memory of Jan Saxl.

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Chapter 1

Introduction

In this monograph, we are concerned with the problem of classifying the finite primitive binary permutation groups. Let G be a permutation group on the set Ω . Given a positive integer n, given $I := (\omega_1, \omega_2, \ldots, \omega_n)$ in the Cartesian product Ω^n and given $g \in G$, we write

$$I^g := (\omega_1^g, \omega_2^g, \dots, \omega_n^g).$$

Moreover, for every $1 \leq i < j \leq n$, we let $I_{ij} := (\omega_i, \omega_j)$ be the 2-subtuple of I corresponding to the i^{th} and to the j^{th} coordinate. Now, the permutation group G on Ω is called *binary* if, for all positive integers n, and for all I and J in Ω^n , there exists $g \in G$ such that $I^g = J$ if and only if for all 2-subtuples, I_{ij} , of I, there exists an element g_{ij} such that $I_{ij}^{g_{ij}} = J_{ij}$. Cherlin has proposed a conjecture listing the finite primitive binary permutation groups [20]. The

Cherlin has proposed a conjecture listing the finite primitive binary permutation groups [20]. The conjecture is as follows, and our task is to complete the proof of this conjecture.

Conjecture 1.1. A finite primitive binary permutation group must be one of the following:

- 1. a symmetric group Sym(n) acting naturally on n elements;
- 2. a cyclic group of prime order acting regularly on itself;
- 3. an affine orthogonal group $V \rtimes O(V)$ with V a vector space over a finite field equipped with a nondegenerate anisotropic quadratic form, acting on itself by translation, with complement the full orthogonal group O(V).

The terminology of Conjecture 1.1 is fully explained in subsequent sections. In particular, we give two equivalent definitions of the adjective "binary" in §1.1, and all three families listed in Conjecture 1.1 are fully discussed in §1.2.

The O'Nan–Scott–Aschbacher theorem describes the structure of finite primitive permutation groups: there are five families of these. Thus, to prove Conjecture 1.1, it is sufficient to prove it for each of these families.

Cherlin himself gave a proof of the conjecture for the family of affine permutation groups, i.e. when G has an abelian socle [21]. Wiscons then studied the remaining cases and showed that Conjecture 1.1 reduces to the following statement concerning almost simple groups [106].

Conjecture 1.2. If G is a finite binary almost simple primitive group on Ω , then $G = \text{Sym}(\Omega)$.

We recall that an *almost simple group* G is a finite group that has a unique minimal normal subgroup S and, moreover, the group S is non-abelian and simple. Note that S is the socle of G.

We now invoke the Classification of Finite Simple Groups which says that a non-abelian simple group is either an alternating group, Alt(n) with $n \ge 5$; a simple group of Lie type; or one of 26 sporadic groups.

In [46], Conjecture 1.2 was proved for groups with socle a simple alternating group; in [34], Conjecture 1.2 was proved for groups with socle a sporadic simple group. In this monograph we deal with the remaining family.

Theorem 1.3. Let G be an almost simple group with socle a finite group of Lie type and assume that G has a primitive and binary action on a set Ω . Then $|\Omega| \in \{5, 6, 8\}$ and $G \cong \text{Sym}(\Omega)$.

The examples in Theorem 1.3 arise via the isomorphisms

1.
$$G \cong SL_2(4).2 \cong PGL_2(5) \cong Sym(5)$$
 and $|\Omega| = 5$;

- 2. $G \cong \operatorname{Sp}_4(2) \cong \operatorname{PSL}_2(9).2 \cong \operatorname{Sym}(6)$ and $|\Omega| = 6$;
- 3. $G \cong SL_4(2).2 \cong Sym(8)$ and $|\Omega| = 8$.

Note that, here, we have not tried to list all isomorphisms between classical groups and the symmetric groups listed in Theorem 1.3. The listed isomorphisms are the ones that crop up in the proof that follows; there are many further isomorphisms with classical groups not listed in the theorem (for example $SO_4^-(2) \cong \Gamma O_3(4) \cong Sym(5)$).

A special case of Theorem 1.3 has already appeared in the literature; in [34], the theorem is proved for the case where G is almost simple with socle a finite group of Lie type of rank 1.

Theorem 1.3 is the final piece in the jigsaw. We can now assert that Cherlin's conjecture is true:¹

Corollary 1.4. Conjecture 1.1 is true.

As will become clear, once the various equivalent definitions of the word "binary" have been introduced, a proof of Conjecture 1.1 is equivalent to a classification of the finite primitive binary relational structures. In particular we have the following (the definition of homogeneous relational structure can be found in Definitions 1.1.1 and 1.1.5):

Corollary 1.5. Let \mathcal{R} be a homogeneous binary relational structure with vertex set Ω , such that $G = \operatorname{Aut}(\mathcal{R})$ acts primitively on Ω . Then the action of G on Ω is one of the actions listed in Conjecture 1.1.

We have not completely described the relational structure \mathcal{R} in our statement of Corollary 1.5 – to do this, we would need to specify the relations in \mathcal{R} . We will not do this here, but we can at least start the task, making use of the fact that all relations of \mathcal{R} must be unions of orbits of G on Ω^2 .

Consider the first family listed in Conjecture 1.1, where $G = \text{Sym}(\Omega)$. In this case G has two orbits on Ω^2 : the set D, of distinct pairs, and the set R, of repeated pairs. Thus the binary relational structures with all relations some union of D and R are:

 $(\Omega), (\Omega, D), (\Omega, R), (\Omega, D, R), (\Omega, R, D) \text{ and } (\Omega, D \cup R).$

One can check directly that every one of these is homogeneous and has automorphism group isomorphic to $Sym(\Omega)$. One needs to repeat this analysis for the other two families; in these cases enumerating orbits and ascertaining which of the resulting structures are homogeneous is much more difficult.

For the remainder of this chapter we have three basic aims: first we seek to give the basic theory of relational complexity for permutation groups including, in particular, the definition of a binary action, and of a binary permutation group. We will also describe some of the key examples.

¹Wiscons informed us of a small gap in his proof of [106, Proposition 4.1]. The next paragraph consists of his comments on this, including a patch. For notation and terminology, we refer to the rest of this chapter.

^{[106,} Proposition 4.1] is devoted to showing that primitive groups of diagonal type are not binary. The gap in the proof stems from an implicit (and accidental) assumption in the first sentence of the proof of [106, Lemma 4.2] that the socle is a product of at least *three* isomorphic nonabelian simple groups. This leaves open the case of two factors, for which it suffices to consider the following setting: let G be a group acting on a nonabelian group T in such a way that the stabilizer of $1 \in T$ satisfies $Inn(T) \leq G_1 \leq Aut(T) \times \langle i \rangle$ for $i: T \to T$ the inversion map. In this context, we show the action of G on T is not binary. To see this, choose noncommuting $a, b \in T$, not both of order 2, and observe that (1, a, b, ab) and (1, a, b, ba) are 2-subtuple complete (witnessed by conjugating by 1, a, or b^{-1}). However, since one of a or b is not of order 2, $G_{1,a,b} \leq Aut(T)$, so $G_{1,a,b}$ fixes $ab \neq ba$. Thus, (1, a, b, ab) and (1, a, b, ba) are not 4-subtuple complete, so such an action is not binary.

Second, we will give some motivation for interest in our result – thus we will survey some related results in the study of relational structures, and in group theory. We will also briefly discuss Cherlin's original motivation for studying binary permutation groups, which arises from model theoretic considerations.

In neither of these first two aspects do we make any claim for originality – instead we seek to draw the key definitions and examples together into one place. Much of the material of this kind that we present below was worked out by Cherlin in his papers [20, 21, 26].

Our third aim in this chapter is to present some of the results and methods concerning binary permutation groups that we consider to be most essential. These will be used in subsequent chapters when we commence our proof of Theorem 1.3.

The remainder of this monograph is occupied with a proof of Theorem 1.3. In Chapter 2 we give a number of general background results concerning groups of Lie type; in Chapter 3 we prove the theorem for the exceptional groups of Lie type; in Chapter 4 we prove the theorem for the classical groups of Lie type.

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1.1 Basics: The definition of relational complexity

The notion of relational complexity can be defined in two different ways. Our job in this section is to present these definitions, and to show that they are equivalent. Throughout this section G is a permutation group on a set Ω of size $t < \infty$. Note that when we write "permutation group" we are assuming that the associated action of G on Ω is faithful – in other words we can think of G as a subgroup of Sym(Ω).

1.1.1 Relational structures

The first approach towards relational complexity is via the concept of a relational structure [21]. Recall that, for a positive integer ℓ , Ω^{ℓ} denotes the set of ℓ -tuples with entries in Ω .

Definition 1.1.1. A relational structure \mathcal{R} is a tuple $(\Omega, R_1, \ldots, R_k)$, where Ω is a set, k is a non-negative integer and, for each $i \in \{1, \ldots, k\}$, there exists an integer $\ell_i \geq 2$ such that $R_i \subseteq \Omega^{\ell_i}$.

The set Ω is called the *vertex set* of the structure, while the sets R_1, \ldots, R_k are referred to as *relations*; in addition, for each *i*, the integer ℓ_i is the *arity* of relation R_i . We say that the relational structure \mathcal{R} is of arity ℓ , where $\ell = \max{\ell_1, \ldots, \ell_k}$.

Example 1.1.2. If a relation, or a relational structure is of arity 2 (resp. 3), then it is commonly called *binary* (resp. *ternary*). Binary relational structures which contain a single relation are nothing more nor less than directed graphs: if $\mathcal{R} = (\Omega, R_1)$ is one such, then the elements of the vertex set Ω are of course the vertices, and each pair in R_1 can be thought of as a directed edge between two elements of Ω . (Note that by "graph" here we implicitly mean a graph with no multiple edges.)

When considering a binary relational structure with more than one relation, it is sometimes helpful to think of it as a directed graph in which there are several different "edge colours" – each relation corresponding to a different "colour".

The notions of isomorphism and automorphism are generalizations of the corresponding definitions for graphs.

Definition 1.1.3. Let $\mathcal{R} = (\Omega, R_1, \ldots, R_k)$ and $\mathcal{S} = (\Lambda, S_1, \ldots, S_k)$ be relational structures. An isomorphism $h : \mathcal{R} \to \mathcal{S}$ is a bijection $h : \Omega \to \Lambda$ such that

$$(\omega_1,\ldots,\omega_{\ell_i})\in R_i\iff (\omega_1^h,\ldots,\omega_{\ell_i}^h)\in S_i.$$

An automorphism g of \mathcal{R} is an element of $\operatorname{Sym}(\Omega)$ that is also an isomorphism $g : \mathcal{R} \to \mathcal{R}$. It is clear that the set of all automorphisms of \mathcal{R} forms a group under composition of bijections; we denote this group by $\operatorname{Aut}(\mathcal{R})$, and note that it is a subgroup of $\operatorname{Sym}(\Omega)$.

Note that we have only defined isomorphisms between relational structures that have the same number of relations; the definition also implies that the (ordered) list of relation-arities must be the same for isomorphic relational structures.²

Our focus will be on those relational structures that exhibit the maximum possible level of symmetry – this requires the notion of *homogeneity*. To state this definition we must first explain what is meant by "an induced substructure" – once again this notion is a direct analogue of the same idea for graphs.

Definition 1.1.4. Let $\mathcal{R} = (\Omega, R_1, \ldots, R_k)$ be a relational structure, with R_i a relation of arity ℓ_i for each $i = 1, \ldots, k$. Let Γ be a subset of Ω . The *induced substructure on* Γ is the relational structure $\mathcal{R}_{\Gamma} = (\Gamma, R'_1, \ldots, R'_k)$ where $R'_i = \Gamma^{\ell_i} \cap R_i$.

So, to clarify what we said above: if $\mathcal{R} = (\Omega, R_1)$ is a binary structure with a single relation (i.e. a directed graph), and Γ is a subset of the vertex set Ω , then \mathcal{R}_{Γ} is precisely the induced subgraph on Γ .

Definition 1.1.5. A relational structure $\mathcal{R} = (\Omega, R_1, \ldots, R_k)$ is called *homogeneous* if, for all $\Gamma, \Gamma' \subset \Omega$ and for all isomorphisms $h : \mathcal{R}_{\Gamma} \to \mathcal{R}_{\Gamma'}$, there exists $g \in \operatorname{Aut}(\mathcal{R})$ such that $g|_{\Gamma} = h$.

The following example will be important shortly.

Example 1.1.6. Given a permutation group G on a set Ω of size t, we define a relational structure $\mathcal{R}_G = (\Omega, R_1, \ldots, R_k)$, where the relations R_1, \ldots, R_k are precisely the orbits of the group G on the sets $\Omega^2, \ldots, \Omega^{t-1}$.

Observe, first, that by definition any element of G maps an element of relation R_i to an element of relation R_i , for all $i \in \{1, \ldots, k\}$; we conclude that $G \leq \operatorname{Aut}(\mathcal{R}_G)$.

On the other hand, suppose that $h \in \operatorname{Aut}(\mathcal{R}_G)$, and let $r = (\omega_1, \ldots, \omega_{t-1})$ be a tuple of distinct elements in Ω lying in relation R_j , for some j. The image of this tuple under h also lies in R_j ; since R_j is an orbit of G, this implies that there exists $g \in G$ such that for all $i \in \{1, \ldots, t-1\}$, $\omega_i^h = \omega_i^g$. It follows that $\omega_t^h = \omega_t^g$, where ω_t is the only element of Ω not represented in the tuple r. We conclude that h = g and so, in particular, $G = \operatorname{Aut}(\mathcal{R}_G)$.

Finally, suppose that Γ and Δ are proper subsets of Ω of size s such that the associated induced relational structures are isomorphic, i.e. there exists an isomorphism $h: (\mathcal{R}_G)_{\Gamma} \to (\mathcal{R}_G)_{\Delta}$. Let $r_{\gamma} = (\gamma_1, \ldots, \gamma_s)$ be a tuple containing all of the distinct elements of Γ , and observe that r_{γ} lies in a relation R_j of \mathcal{R}_G , for some j. Indeed, by construction, r_{γ} lies in the corresponding relation R_j of $(\mathcal{R}_G)_{\Gamma}$, and so $(r_{\gamma})^h$ lies in the corresponding relation R_j of $(\mathcal{R}_G)_{\Delta}$, and hence also lies in the relation R_j of \mathcal{R}_G . In particular, since R_j is an orbit of G, we conclude that there exists $g \in G$ such that for all $i \in \{1, \ldots, s\}, \gamma_i^h = \gamma_i^g$. Since $G = \operatorname{Aut}(\mathcal{R}_G)$, we conclude that \mathcal{R}_G is homogeneous.

²One can imagine a slight weakening of Definition 1.1.3 where one allows an automorphism of \mathcal{R} to map a set of tuples corresponding to an erelation of tuples corresponding to a different relation – for certain relational structures, this would yield a larger automorphism group (which would contain Aut(\mathcal{R}) as defined above, as a normal subgroup). We will not need this extension in what follows.

We are ready to give our first definition of relational complexity. Before stating it, we remind the reader that we are assuming that G is a permutation group on a set Ω , and we recall that if \mathcal{R} is any relational structure with vertex set Ω , then Aut(\mathcal{R}) is also a permutation group on Ω .

Definition 1.1.7. The structural relational complexity of a permutation group G is equal to the smallest integer $s \ge 2$ for which there exists a homogeneous relational structure $\mathcal{R} = (\Omega, R_1, \ldots, R_k)$ of arity s such that $\operatorname{Aut}(\mathcal{R})$ is permutation isomorphic to G.

Note that Example 1.1.6 implies, in particular, that if $|\Omega| \ge 3$, then the structural relational complexity of G is well-defined, and is bounded above by $|\Omega| - 1$ (and is at least 2). In what follows, we will write SRC(G, Ω) for the structural relational complexity of the permutation group G.

One might wonder why we have required that $\operatorname{SRC}(G,\Omega) \geq 2$. The reason is that, in the next section we will define a different statistic $\operatorname{TRC}(G,\Omega)$ using a completely different approach, and we will also require that $\operatorname{TRC}(G,\Omega) \geq 2$. We will then show that $\operatorname{SRC}(G,\Omega) = \operatorname{TRC}(G,\Omega)$ for all permutation groups G on a set Ω . Were we to omit the requirement that $\operatorname{SRC}(G,\Omega) \geq 2$ and $\operatorname{TRC}(G,\Omega) \geq 2$, there would be a number of actions for which $\operatorname{SRC}(G,\Omega) \neq \operatorname{TRC}(G,\Omega)$, for instance the natural action of $\operatorname{Sym}(\Omega)$.

1.1.2 Tuples

In this section we give an alternative approach to the notion of relational complexity based on [26]. We then show that it coincides with the approach of the previous section. As before G is a permutation group on a finite set Ω .

Definition 1.1.8. Let $2 \leq r \leq n$ be positive integers, and let $I = (I_1, \ldots, I_n)$ and $J = (J_1, \ldots, J_n)$ be elements of Ω^n . We say that I and J are r-subtuple complete with respect to G if, for all k_1, k_2, \ldots, k_r integers with $1 \leq k_1, k_2, \ldots, k_r \leq n$, there exists $g \in G$ with $I_{k_i}^g = J_{k_i}$ for $i \in \{1, \ldots, r\}$. In this case we write $I \sim J$.

Note that if $I \simeq J$ and $u \leq r$, then $I \simeq J$.

Definition 1.1.9. The permutation group G has *tuple relational complexity* equal to s if the following two conditions hold:

- 1. If $n \ge s$ is any integer and I, J are elements of Ω^n such that $I \simeq J$, then there exists $g \in G$ such that $I^g = J$.
- 2. $s \ge 2$ is the smallest integer for which (1) holds.

We write $\text{TRC}(G, \Omega)$ for the tuple relational complexity of the permutation group G.

Put another way, the tuple relational complexity of G is the smallest integer $s \ge 2$ such that

$$I \simeq J \Longrightarrow I \simeq J,$$

for any integer $n \geq s$, and any pair of *n*-tuples I and J.

It is not immediately clear, a priori, that $\text{TRC}(G, \Omega)$ exists for every permutation group G on the set Ω . The next lemma deals with this concern.

Lemma 1.1.10. If $SRC(G, \Omega) = s$, then $TRC(G, \Omega)$ exists and is bounded above by s.

Proof. Let $n \ge 2$ be some integer, and let I and J be subsets of Ω^n such that $I \simeq J$. We must prove that there exists $g \in G$ such that $I^g = J$.

Let \mathcal{R} be a homogeneous relational structure of arity s for which $G = \operatorname{Aut}(\mathcal{R})$. Write $\{I\}$ (resp. $\{J\}$) for the underlying set associated with the *n*-tuple I (resp. J); as $s \geq 2$, these sets must be of equal cardinality bounded above by n. Now consider the induced substructures $\mathcal{R}_{\{I\}}$ and $\mathcal{R}_{\{J\}}$ and consider the map $h : \mathcal{R}_{\{I\}} \to \mathcal{R}_{\{J\}}$ for which $h(I_i) = J_i$ for all $i \in \{1, \ldots, n\}$. We claim that h is an isomorphism of relational structures. Let $(I_{i_1}, \ldots, I_{i_u})$ be an element of some relation R_j in $\mathcal{R}_{\{I\}}$. Note that $u \leq s$ and recall that $I \simeq J$ with respect to the action of G. Thus there exists $g \in G$ such that

$$(J_{i_1},\ldots,J_{i_u})=(I_{i_1},\ldots,I_{i_u})^g.$$

Then, since $g \in Aut(\mathcal{R})$, we conclude that $(J_{i_1}, \ldots, J_{i_u})$ is an element of relation R_j in $\mathcal{R}_{\{J\}}$. We conclude that h is an isomorphism as required.

Now, since \mathcal{R} is homogeneous, there exists $g \in G = \operatorname{Aut}(\mathcal{R})$ such that $g_{|\{I\}} = h$; in particular $I^g = J$, as required.

Lemma 1.1.11. $SRC(G, \Omega) \leq TRC(G, \Omega)$.

Proof. Let $r = \text{TRC}(G, \Omega)$. Define $\mathcal{R} = (\Omega, R_1, \ldots, R_k)$, where R_1, \ldots, R_k are the orbits of G on Ω^i for all $i \in \{2, \ldots, r\}$.

Clearly $G \leq \operatorname{Aut}(\mathcal{R})$. Suppose that $\sigma \in \operatorname{Aut}(\mathcal{R})$, and let $I = (\omega_1, \ldots, \omega_t)$ be a *t*-tuple of distinct elements of Ω , where $t = |\Omega|$ (so every entry of Ω occurs as an entry in I). Then $I \simeq I^{\sigma}$, and so there exists $g \in G$ such that $I^g = I^{\sigma}$. This implies that $\sigma = g$, and so $\operatorname{Aut}(\mathcal{R}) \leq G$. We conclude that $G = \operatorname{Aut}(\mathcal{R})$.

We must show that \mathcal{R} is homogeneous. Let Γ and Δ be subsets of Ω of size s such that there exists an isomorphism $\varphi : \mathcal{R}_{\Gamma} \to \mathcal{R}_{\Delta}$. Furthermore, let $I = (\gamma_1, \ldots, \gamma_s)$ be an s-tuple of distinct elements of Γ . Suppose first $s \leq r$. Since \mathcal{R} contains all the orbits of G on Ω^s and since $\mathcal{R}_{\Gamma} \cong \mathcal{R}_{\Delta}$, we deduce that I and $\varphi(I)$ are in the same G-orbit, that is, there exists $g \in G$ such that $I^g = \varphi(I)$. Thus $\varphi = g|_{\Gamma}$, as required. Suppose next s > r. Since all r-subtuples of I occur as relations in \mathcal{R} and since $\mathcal{R}_{\Gamma} \cong \mathcal{R}_{\Delta}$, we conclude that $I \simeq \varphi(I)$. Since $r = \text{TRC}(G, \Omega)$, we deduce $I \simeq \varphi(I)$. As before, this implies that there exists $g \in G = \text{Aut}(\mathcal{R})$ such that $I^g = \varphi(I)$; in other words $\varphi = g|_{\Gamma}$, as required. \Box

Corollary 1.1.12. $SRC(G, \Omega) = TRC(G, \Omega)$.

In light of this corollary, we now drop the distinction between the two types of relational complexity:

Definition 1.1.13. The relational complexity of G is equal to the tuple relational complexity of G (and hence also equal to the structural relational complexity of G), and is denoted $RC(G, \Omega)$.

In particular, a permutation group $G \leq Sym(\Omega)$ is called *binary* if $RC(G, \Omega) = 2$.

Our definition of relational complexity has, to this point, pertained only to permutation groups, i.e. to *faithful* group actions. It is convenient to extend this definition now to any group action:

Definition 1.1.14. Suppose that a group G acts on a set Ω . The *relational complexity* of the action, denoted $\mathrm{RC}(G, \Omega)$, is the relational complexity of the permutation group induced by the action of G on Ω .

Note, finally, that in [26] the word *arity* is used as a synonym for relational complexity.

1.2 Basics: Some key examples

Our focus in this monograph is on actions with small relational complexity, thus the examples we present below are skewed in this direction. In particular, all of the actions listed in Conjecture 1.1 are discussed.

As we shall see, there are times when the structural definition of relational complexity is easiest to work with, and times when we prefer the tuple definition.

Before we outline the primary examples, we need to say a few words about the third family in Conjecture 1.1. This family consists of all groups isomorphic to an affine orthogonal group $V \rtimes O(V)$ with V a vector space over a finite field equipped with a non-degenerate anisotropic quadratic form, acting on itself by translation, with complement the full orthogonal group O(V). It is a straightforward consequence of the classification of non-degenerate quadratic forms that if V admits an anisotropic quadratic form Q (i.e. one for which $Q(\mathbf{v}) \neq 0$ for all $\mathbf{v} \in V \setminus \{\mathbf{0}\}$, then $\dim(V) \leq 2$. We will split this family into two smaller families according to whether $\dim(V)$ is 1 or 2;

- 3a. $\dim(V) = 1$: the associated group G is isomorphic to $\mathbb{F}_q \rtimes C_2$, where C_2 acts as -1 on the finite field \mathbb{F}_q with q elements, and the action is on $\Omega = \mathbb{F}_q$. For G to be primitive we require that q is prime, and we obtain that G is isomorphic to the dihedral group of order 2q, with the action being on the q-gon, as usual.
- 3b. $\dim(V) = 2$ and the associated quadratic form is of minus type: the associated group G is isomorphic to $\mathbb{F}_q^2 \rtimes \mathcal{O}_2^-(q) \cong \mathbb{F}_q^2 \rtimes D_{2(q+1)}$, where $D_{2(q+1)}$ is a dihedral group of order 2(q+1).

First, let us observe that the relational complexity of the natural action of the symmetric group is as small as it can possibly be.

Example 1.2.1. Consider the natural action of G = Sym(t) on the set $\Omega = \{1, \ldots, t\}$. Define

$$R = \{ (i, j) \mid 1 \le i, j \le t \text{ and } i \ne j \}.$$

Then $\mathcal{R} = (\Omega, R)$ is the complete directed graph, \mathcal{R} is homogeneous and $G = \operatorname{Aut}(\mathcal{R})$. We conclude immediately that $\operatorname{RC}(G, \Omega) = 2$.

Note that the first family of permutation groups listed in Conjecture 1.1 is precisely the family of finite symmetric groups in their natural action.

In many group-theoretic respects, the alternating group is very like the symmetric group. The next example shows that relational complexity does not conform to this rule-of-thumb: while, as we have just seen, the natural action of the symmetric group has relational complexity as small as it can possibly be, the natural action of the alternating group has relational complexity as large as it can possibly be.

Example 1.2.2. Consider the natural action of G = Alt(t) on the set $\Omega = \{1, \ldots, t\}$. Consider the tuples

$$I = (1, 2, 3, \dots, t)$$
 and $J = (2, 1, 3, \dots, t)$.

It is straightforward to check that $I_{t-2} J$; it is equally clear that the only permutation h for which $I^h = J$ is $h = (1,2) \notin G$. We conclude that $\operatorname{RC}(G,\Omega) \ge t-1$. Now Example 1.1.6 implies that $\operatorname{RC}(G,\Omega) = t-1$.

The previous two examples are a salutary warning that, in general, relational complexity behaves badly with respect to subgroups. All is not lost however: Lemma 1.6.2 shows that the relational complexity of a group is related to that of some of its subgroups.

Our first aim is to understand the actions listed in Conjecture 1.1. Note that the Families 2 and 3a (using the notation at the start of this section) consist of primitive actions with very small point-stabilizers (size 1 and 2, respectively). In the next couple of examples we consider this situation.

Example 1.2.3. If G acts regularly on Ω , then $\mathrm{RC}(G,\Omega)$ is binary.

Proof: Suppose that $I = (I_1, \ldots, I_n)$ and $J = (J_1, \ldots, J_n)$ satisfy $I \simeq J$. For $i \in \{1, \ldots, n-1\}$, let g_i be an element of G that satisfies $I_i^{g_i} = J_i$ and $I_{i+1}^{g_i} = J_{i+1}$. The regularity of G implies that, for $j \in \{1, \ldots, n\}$, there is a unique element of G satisfying $I_j^g = J_j$. This fact, applied with j = 2, implies that $g_1 = g_2$; then applied with j = 3, implies that $g_2 = g_3$, and so on. Thus $g_1 = \cdots = g_{n-1}$; calling this element g, we see that $I^g = J$ and we conclude that $I \simeq J$, as required. \Box

Recall that the only regular primitive actions are associated with cyclic groups of prime order; we see, then, that the second family of groups in Conjecture 1.1 are precisely the regular primitive groups.

³It may be perhaps better to call such a Q a non-singular form rather than an anisotropic form – a vector \mathbf{v} is generally called singular if $Q(\mathbf{v}) = 0$, and isotropic if $\beta(\mathbf{v}, \mathbf{v}) = 0$ where β is the polar form of Q. If the characteristic of the field is odd, these two definitions coincide, however in characteristic 2 this is not the case. Our definition of an anisotropic form requires that the only singular vector for Q is the zero vector, but note that all vectors are isotropic in the characteristic 2 case. In any case, we will stick to calling such a Q anisotropic as it is consistent with what has come before in the literature.

Example 1.2.4. Suppose that G is transitive and a point-stabilizer H has size 2, and suppose that x is the non-trivial element in H. Let $C = x^G$ be the conjugacy class of x in G. Then

$$\operatorname{RC}(G) = \begin{cases} 2, & \text{if } C \not\subseteq C^2; \\ 3, & \text{otherwise.} \end{cases}$$

Proof: It is an easy exercise to verify that, under these assumptions, $RC(G) \leq 3$. One can use, for instance, Lemma 1.5.1 below.

Since $RC(G) \leq 3$, it is clear that a pair of *n*-tuples will be *n*-subtuple complete if and only if they are 3-subtuple complete. Thus, if there exists an *n*-tuple that is 2-subtuple complete but not *n*-subtuple complete, then there must exist a 3-tuple that is 2-subtuple complete but not 3-subtuple complete.

Suppose that G is not binary, and let $(P,Q) = ((P_1, P_2, P_3), (Q_1, Q_2, Q_3))$ be a pair of 3-tuples that is 2-subtuple complete but not 3-subtuple complete. Then there is, by assumption, an element g of G that maps (P_1, P_2) to (Q_1, Q_2) . Replacing Q by $Q^{g^{-1}}$ and relabelling, we conclude that there exists a pair

$$((P_1, P_2, P_3), (P_1, P_2, P_4))$$

that is 2-subtuple complete but not 3-subtuple complete, in particular $P_3 \neq P_4$. Write H_i for the stabilizer of P_i , and let x_i be the non-trivial element of H_i . Then we must have

$$P_3^{x_1} = P_3^{x_2} = P_4.$$

Since (P,Q) is not 3-subtuple complete, $x_1 \neq x_2$, otherwise $P^{x_1} = Q$. Moreover, since $P_3^{x_1x_2} = P_3$, we conclude that x_1x_2 is the non-trivial element in H_3 . Thus $C \subseteq C^2$, as required.

Suppose now that $C \subseteq C^2$. Let $x_1, x_2, x_3 \in C$ with $x_3 = x_1x_2$. In particular, there exist three points P_1, P_2 and P_3 with $G_{P_1} = \langle x_1 \rangle$, $G_{P_2} = \langle x_2 \rangle$ and $G_{P_3} = \langle x_3 \rangle$. Set $P_4 := P_3^{x_1}$. We claim that $((P_1, P_2, P_3), (P_1, P_2, P_4))$ is a pair of 3-tuples that is 2-subtuple complete. In fact,

$$(P_1, P_2)^{1_G} = (P_1, P_2), (P_1, P_3)^{x_1} = (P_1^{x_1}, P_3^{x_1}) = (P_1, P_4), (P_2, P_3)^{x_2} = (P_2^{x_2}, P_3^{x_2}) = (P_2, P_3^{x_3x_2}) = (P_2, P_3^{x_1}) = (P_2, P_4).$$

If this pair is 3-subtuple complete, then there exists $g \in G$ with $P_1^g = P_1$, $P_2^g = P_2$ and $P_3^g = P_4$. In particular, $g \in \langle x_1 \rangle \cap \langle x_2 \rangle$. If g = 1, then $P_3 = P_4 = P_3^{x_1}$ and hence $x_1 \in \langle x_3 \rangle$. This gives $x_1 = x_3$ and hence $x_2 = 1$ because $x_1x_2 = x_3$. However, this is a contradiction. Thus $g = x_1 = x_2$ and hence $x_3 = x_1x_2 = 1$, again a contradiction. Therefore, $((P_1, P_2, P_3), (P_1, P_2, P_4))$ is a pair of 3-tuples that are 2-subtuple complete but that are not 3-subtuple complete; hence G is not binary. \Box

There is an important special case which occurs when point-stabilizers are of size 2, and G has a regular normal subgroup N. In this case it follows immediately that $C \not\subseteq C^2$ (where C is as in Example 1.2.4), and thus $\operatorname{RC}(G,\Omega) = 2$. Such an action is primitive if and only if N is of prime order, and we now see that Family 3a pertaining to Conjecture 1.1 is precisely this.⁴

Our next example addresses Family 3b in Conjecture 1.1.

$$\hat{C}_i \hat{C}_j = \sum_{v=1}^k a_{ijv} \hat{C}_v,$$

where k is the number of conjugacy classes in G. The non-negative integers a_{ijv} for $1 \le i, j, v \le k$ are the class constants of

⁴The problem of specifying relational complexity when point-stabilizers have size 2 is now reduced to the problem of studying when C, a certain conjugacy class of involutions satisfies $C \subseteq C^2$. This problem is, in general, difficult, however one potential avenue of investigation is via the *class constants* of the finite group G, denoted a_{ijv} . For any conjugacy class C_i in a group G, we define $\hat{C}_i = \sum_{c \in C_i} c_i$ to be the class sum of C_i in the group algebra $\mathbb{C}G$. Now write

Example 1.2.5. This example is Lemma 1.1 of [21]. We identify Ω with a vector space V over a field F, such that V is endowed with a quadratic form Q such that Q is anisotropic, i.e. $Q(v) \neq 0$ for all $v \in V \setminus \{0\}$. We set $G = V \rtimes O(V)$, where O(V) is the isometry group of the form Q, and the semidirect product is the natural one, as is the action of G on $\Omega = V$.

Let us see that this action is binary. Let n be a positive integer, and assume that $\mathbf{u} = (u_0, \ldots, u_n)$ and $\mathbf{u}' = (u'_0, \ldots, u'_n)$ satisfy $\mathbf{u} \simeq \mathbf{u}'$. Let us show that $\mathbf{u}_{n+1}\mathbf{u}'$. We may suppose, without loss of generality that $u_0 = u'_0 = 0$.

Note that $\mathbf{u} \simeq \mathbf{u}'$ implies that $Q(u_i) = Q(u'_i)$ for all $i \in \{1, \ldots, n\}$. What is more, since the isometry group also preserves the polar form β of Q, $\mathbf{u} \simeq \mathbf{u}'$ also implies that

$$\beta(u_i, u_j) = \beta(u'_i, u'_j),$$

for any $1 \leq i, j \leq n$. This, in turn, implies that

$$Q\left(\sum_{j=1}^{n} c_j u_j\right) = Q\left(\sum_{j=1}^{n} c_j u_j'\right),\tag{1.2.1}$$

for any choice of scalars $c_1, \ldots, c_n \in F$.

Let $W = \operatorname{span}(\mathbf{u})$, and let $W' = \operatorname{span}(\mathbf{u}')$ and suppose, without loss of generality, that u_1, \ldots, u_m is a basis for W (for $m = \dim(W)$). We claim that then u'_1, \ldots, u'_m is a basis for W'. To see this, it is enough to show that if u_1, \ldots, u_k are linearly independent, then so too are u'_1, \ldots, u'_k . Suppose that $c_1, \ldots, c_k \in F$ such that $c_1u'_1 + \cdots + c_ku'_k = 0$. Then, clearly,

$$Q(c_1u'_1 + \dots + c_ku'_k) = Q(0) = 0.$$

But, by the observation above, this implies that $Q(c_1u_1 + \cdots + c_ku_k) = 0$, which implies that $c_1u_1 + \cdots + c_ku_k = 0$, which in turn implies that $c_1 = \cdots = c_k = 0$. The claim follows.

Now we can define an isometry $f: W \to W'$ by setting $f(u_i) = u'_i$ for $i \in \{1, \ldots, m\}$, and extending linearly. Then Witt's Lemma implies that there exists $g \in O(V)$ such that $u_i^g = u'_i$ for all $i \in \{1, \ldots, m\}$. Let us now consider $m < i \le n$. Write $u_i = \sum_{j=1}^m c_j u_j$ and now, observe that (1.2.1) yields that

$$Q\left(u'_{i} - \sum_{j=1}^{m} c_{j}u'_{j}\right) = Q\left(u_{i} - \sum_{j=1}^{n} c_{j}u_{j}\right) = Q(0) = 0.$$

Now the fact that Q is anisotropic implies that $u'_i - \sum_{j=1}^m c_j u'_j = 0$, and we conclude that $u'_i = u'_i$, as required.

All of the examples considered so far have been transitive. Let us briefly consider what can happen with intransitive actions.

Example 1.2.6. Suppose that the action of G on Ω is intransitive with orbits $\Delta_1, \ldots, \Delta_v$. It is immediate from the definition that

$$\operatorname{RC}(G,\Omega) \ge \max\{\operatorname{RC}(G,\Delta_1), \operatorname{RC}(G,\Delta_2), \dots, \operatorname{RC}(G,\Delta_v)\}.$$

G. Now a well-known formula asserts that

$$a_{ijv} = \frac{|C_i||C_j|}{|G|} \sum_{\chi \in \operatorname{Irr}_{\mathbb{C}}(G)} \frac{\chi(g_i)\chi(g_j)\chi(g_v^{-1})}{\chi(1)}.$$

We conclude, therefore, that if a point-stabilizer $H = \langle x \rangle$ has size 2, then RC(G) = 2 if and only if

$$\sum_{\chi \in \operatorname{Irr}_{\mathbb{C}}(G)} \frac{\chi(x)^3}{\chi(1)} = 0.$$

On the other hand, let $n \ge 3$ and consider the intransitive action of G = Sym(n) with two orbits, where the action on the first orbit is the natural one of degree n, and the second orbit is of size 2. Clearly the action of G on each orbit is binary; on the other hand, one can check directly that $\text{RC}(G, \Omega) = n = t - 2$.

This example suggests that the problem of calculating the relational complexity of intransitive actions may be rather difficult.

1.2.1 Existing results on relational complexity

Results on relational complexity above and beyond the basic examples discussed above are hard to obtain. Nearly all of the important results are due to Cherlin, and his co-authors, and we briefly mention some of these here. The first result is stated in [20], with a small correction in [21].

Theorem 1.2.7. Let Ω be the set of all k-subsets of a the set $\{1, \ldots, n\}$ with $2k \leq n$. If G = Sym(n), then $\text{RC}(G, \Omega) = 2 + \lfloor \log_2 k \rfloor$. If G = Alt(n), then

$$\operatorname{RC}(G,\Omega) = \begin{cases} n-1, & \text{if } k = 1; \\ \max(n-2,3), & \text{if } k = 2; \\ n-2, & \text{if } k \ge 3 \text{ and } n = 2k+2; \\ n-3, & \text{otherwise.} \end{cases}$$

The actions of the symmetric and alternating groups on partitions, rather than k-sets, are currently being studied by Cherlin and Wiscons [24]. The only general result to date is for Sym(2n) and Alt(2n)acting on Ω , the set of partitions of 2n into n blocks of size 2 (so, for G = Sym(2n), this is the action on cosets of a maximal imprimitive subgroup of form Sym(2) wr Sym(n)). The result they have obtained for $n \geq 2$ is as follows:

$$\operatorname{RC}(\operatorname{Sym}(2n), \Omega) = n;$$
$$\operatorname{RC}(\operatorname{Alt}(2n), \Omega) = \begin{cases} 2, & n = 2; \\ 4, & n \in \{3, 4\}; \\ n, & n > 3 \text{ and } n \equiv 0, 1, 3, 5 \pmod{6}; \\ n - 1, & n > 4 \text{ and } n \equiv 2, 4 \pmod{6}. \end{cases}$$

As we shall see below (Theorem 1.5.2), when considering large relational complexity, an important family of actions involves groups G which are subgroups of $\operatorname{Sym}(m) \operatorname{wr} \operatorname{Sym}(r)$ containing $(\operatorname{Alt}(m))^r$, where the action of $\operatorname{Sym}(m)$ is on k-subsets of $\{1, \ldots, m\}$ and the wreath product has the product action of degree $t = {m \choose k}^r$. The particular situation where $G = \operatorname{Sym}(m) \operatorname{wr} \operatorname{Sym}(r)$ is studied in [26]. We summarise some of the results there, using the notation just established.

Theorem 1.2.8. Let G = Sym(m) wr Sym(r) acting on a set Ω of size $t = {\binom{m}{k}}^r$, as described.

- 1. If m = 2, then k = 1 and $\operatorname{RC}(G, \Omega) = 2 + \lfloor \log_2 r \rfloor$.
- 2. If k = 1, then $RC(G, \Omega) \le m + \lfloor \log_2 r \rfloor$.
- 3. $\operatorname{RC}(G, \Omega) \leq |2 + \log_2 k| |1 + \log_2 r|$ with equality if $m \geq 2k |1 + \log_2 r|$.

The particular situation where k = 1 and G = Sym(m) wr Sym(r) (so we are considering the natural product action of degree m^r) has been taken much further in a series of papers by Saracino [87, 88, 89]. Saracino's results effectively yield an exact value for the relational complexity of this family of actions. We do not write this value here as the precise formulation of the results is slightly involved; instead we refer to [26, §6] and to the papers of Saracino, particularly the first.

1.3 Motivation: On homogeneity

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In his paper [20], Cherlin chooses a quote from Aschbacher as an epigraph. This quote, plus some more, goes as follows:

Define an object X in a category \mathfrak{C} to possess the *Witt property* if, whenever Y and Z are subobjects of X and $\alpha : Y \to Z$ is an isomorphism, then α extends to an automorphism of X. Witt's Lemma says that orthogonal spaces, symplectic spaces, and unitary spaces have the Witt property in the category of spaces with forms and isometries. All objects in the category of sets and functions have the Witt property. But in most categories few objects have the Witt property; those that do are very well behaved indeed. If X is an object with the Witt property and G is its group of automorphisms, then the representation of G on X is usually an excellent tool for studying G. [3, pp. 81, 82]

One should think of "the Witt property" as a generalization of the notion of homogeneity which we have introduced in the specific setting of relational structures. The study of homogeneous objects in different categories has a long and interesting history.⁵

Before discussing this history, let us delve a little deeper into why such objects have received attention: Aschbacher's answer is given above. This approach has its roots in the *Erlangen Programme* of Klein, in which the key features of a particular "geometry" define, and are defined by, the group of automorphisms of said geometry. The idea here is that one studies the geometry in question, one deduces information about the geometry, which one then reinterprets as information about the associated group; one can use this information about the group to deduce further information about the geometry and so on. Thus the process of mathematical inquiry moves back-and-forth between geometrical study and algebraic (group theoretic).

The efficacy of this approach varies considerably - if an object has a very small automorphism group for instance, then group theory may provide very little insight. On the other hand, as Aschbacher suggests, this approach is most spectacularly successful when the object in question is homogeneous. Indeed the two examples which Aschbacher mentions clearly illustrate the success of this approach.

First, we note that the category of sets and functions have the Witt property. If we restrict ourselves to finite objects in this category, then the associated automorphism groups are the finite symmetric groups, Sym(n). Of course, all of the basic group-theoretical information about these groups is most naturally expressed in the language of their natural (homogeneous) action on a set of size n. This includes their conjugacy class structure (via cycle type), and their subgroup structure (via the O'Nan–Scott-Aschbacher Theorem [2, 91]; see also [71]).

Second, in the category of spaces with forms, basic linear algebra asserts that objects associated with a zero form (i.e. naked vector spaces) have the Witt property; Witt's Lemma extends this to cover objects associated with either a non-degenerate quadratic or non-degenerate sesquilinear form. Again, restricting ourselves to finite such objects, we obtain the finite classical groups as the associated automorphism groups. As before, the basic group-theoretical properties of these groups are most naturally expressed in the language of their natural homogeneous action on the associated vector space. This includes their conjugacy class structure (via rational canonical form for $GL_n(q)$, and the variants due to Wall for the other classical groups [101]), and their subgroup structure (via Aschbacher's Theorem [1]).

In light of all this, a natural question when studying some (permutation) group G is whether we can find an object in some category on which G acts homogeneously. Example 1.1.6 gives an easy answer to this: it turns out that there is always such an object in the category of relational structures. The bad news is that the object provided by Example 1.1.6 is little more than an encoding of the complete structure of the permutation group in terms of a relational structure – studying the structure \mathcal{R}_G will hardly be easier than studying the original group and its associated action.

⁵There is some inconsistency in terminology across the literature – homogeneity as we have defined it here is sometimes called "ultra-homogeneity" while homogeneity refers to a strictly weaker property.

The investigation of relational complexity seeks to remedy this disappointing state of affairs: given a group G and an associated action, $\operatorname{RC}(G, \Omega)$ gives us an indication of the efficiency with which we can build a relational structure on which G can act homogeneously. From this point of view, an "efficient" representation of G acting homogeneously on a relational structure is one for which the arity of the structure is as small as possible.

There is an alternative way of viewing efficiency in this context where one is, instead, interested in using relational structures with as few relations as possible (but not necessarily worrying about the arity of the relations used). We will not pursue this point of view here, but we refer to [52] (for the primitive case) and to [22] (for the general case), for results that pertain to this approach.

1.3.1 Existing results on homogeneity

We briefly review some important results on homogeneity for particular finite relational structures.

The classification of homogeneous graphs was partially completed by Sheehan [94], and then completely by Gardiner [42]. Indeed, Gardiner's result applies to a wider class of graphs than those we would call homogeneous. This classification was then extended by Lachlan to homogeneous digraphs [60].

In order to state these results we need some terminology: a *digraph*, Γ , is an ordered pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is a non-empty set, and $E(\Gamma)$ is an irreflexive binary relation on that set. The digraph is *symmetric* (resp. *anti-symmetric*) if, whenever $(x, y) \in E(\Gamma)$, we have (y, x) in (resp. not in) $E(\Gamma)$. So a symmetric digraph is the object commonly called a *graph* in the literature.

If Γ and Δ are two digraphs, then we can construct two new digraphs with vertex set $V(\Gamma) \times V(\Delta)$:

- 1. in the composition of Γ and Δ , $\Gamma[\Delta]$, vertices (u_1, v_1) and (u_2, v_2) are connected if and only if $(u_1, u_2) \in E(\Gamma)$, or $u_1 = u_2$ and $(v_1, v_2) \in E(\Delta)$;
- 2. in the direct product of Γ and Δ , $\Gamma \times \Delta$ vertices (u_1, v_1) and (u_2, v_2) are connected if and only if $(u_1, u_2) \in E(\Gamma)$ and $(v_1, v_2) \in E(\Delta)$.

We write K_n for the complete (symmetric di)graph on n vertices. We also define two infinite families of graphs, both indexed by a parameter $n \in \mathbb{Z}$ with $n \geq 3$:

- 1. Λ_n is the digraph with vertex set $\{0, 1, \dots, n-1\}$ and $(x, y) \in E(\Gamma_n)$ if and only if $x y \equiv 1 \pmod{n}$;
- 2. Δ_n is the symmetric digraph with vertex set $\{0, 1, \ldots, n-1\}$ and $(x, y) \in E(\Delta_n)$ if and only if $x y \equiv \pm 1 \pmod{n}$.

Thus Λ_n is the directed cycle on *n* vertices, and Δ_n is the undirected cycle on *n* vertices. Let S (resp. A) denote the set of homogeneous symmetric (resp. antisymmetric) digraphs. We write $\overline{\Gamma}$ for the complement of Γ . Then Gardiner's result is the following:

Theorem 1.3.1. A digraph Γ is in S if and only if Γ or $\overline{\Gamma}$ is isomorphic to one of

$$\Delta_5, K_3 \times K_3, K_m[\overline{K_n}],$$

where $m, n \in \mathbb{Z}^+$.

Now we will state Lachlan's result in three stages. First we need to define three "sporadic homogeneous digraphs"; this is done in Figure 1.1.

Second we classify the homogeneous antisymmetric digraphs.

Theorem 1.3.2. A digraph Γ is in \mathcal{A} if and only if Γ is isomorphic to one of

 $\Lambda_4, \overline{K_n}, \overline{K_n}[\Lambda_3], \Lambda_3[\overline{K_n}], H_0,$

where $n \in \mathbb{Z}^+$.

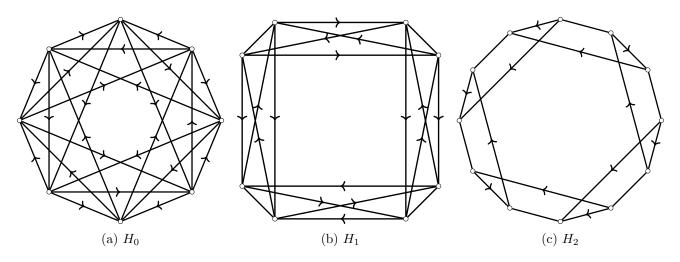


Figure 1.1: Three homogeneous digraphs. The presence of an undirected edge $\{v, w\}$ in the diagrams for H_0 and H_1 indicates that both directed edges between v and w are present. In the diagram for H_2 we have omitted most of the directed edges. To obtain the remaining edges, note first that each vertex in H_2 has a unique *mate*, to which it is connected by an undirected edge (indicated in the diagram). Next, let v and w be vertices, and let w' be the mate of w. Finally, if (v, w) is a directed edge, then (w', v) is a directed edge, then (w, w') is a directed edge. This leads to the insertion of another 36 directed edges.

Finally we can state Lachlan's classification of homogeneous digraphs.

Theorem 1.3.3. A digraph Γ is homogeneous if and only if Γ or $\overline{\Gamma}$ is isomorphic to a digraph with one of the following forms:

 $K_n[A], A[K_n], S, \Lambda_3[S], S[\Lambda_3], H_1, H_2,$

where $n \in \mathbb{Z}^+$, $A \in \mathcal{A}$ and $S \in \mathcal{S}$.

Lachlan's result, expressed in our terms, is *almost* a classification of those homogeneous relational structures $\mathcal{R} = (\Omega, R_1)$ such that R_1 is binary. We write "almost" because Lachlan imposes the condition that R_1 is irreflexive whereas we make no such restriction. Nonetheless, given that in this monograph we are focusing on transitive actions, Lachlan's result is sufficient: any relational structure $\mathcal{R} = (\Omega, R_1)$ for which R_1 is binary and $\operatorname{Aut}(\mathcal{R})$ is transitive on Ω , will either be precisely of the form listed in Theorem 1.3.3, or else will be of the form listed in Theorem 1.3.3 with the addition of a loop at every vertex. We have made no attempt to extend this classification to the situation where $\operatorname{Aut}(\mathcal{R})$ is not transitive on Ω although we note that in this situation, $\operatorname{Aut}(\mathcal{R})$ would have exactly two orbits on Ω – one corresponding to vertices with loops, one corresponding to vertices without.

The groups $\operatorname{Aut}(\Gamma)$ for Γ appearing in Theorem 1.3.3 have not been explicitly listed to our knowledge. We will not calculate this list, but we can at least start the task: It is easy to check that $\operatorname{Aut}(\Lambda_n)$ is the cyclic group of order n, $\operatorname{Aut}(\Delta_n)$ is the dihedral group of order 2n and $\operatorname{Aut}(K_n)$ is the symmetric group of degree n. It is slightly more involved to check the larger sporadic examples; the automorphism group and the action on points (which is necessarily binary) are as follows:

- 1. $\operatorname{Aut}(K_3 \times K_3) = \operatorname{Sym}(3) \operatorname{wr} \operatorname{Sym}(2)$ in the product action on 9 points;
- 2. $\operatorname{Aut}(H_0) \cong \operatorname{SL}_2(3)$ acting on the 8 cosets of a Sylow 3-subgroup;
- 3. Aut(H_1) is the semidihedral group of order 16 it has presentation $\langle x, y | x^8 = y^2, x^y = x^3 \rangle$ in an action of degree 8;
- 4. Aut $(H_2) \cong \text{Alt}(4) \rtimes C_4$ where $C_4 = \langle x \rangle$ acts by conjugation on Alt(4) via $g^x = g^{(1,2,3,4)}$ for all $g \in \text{Alt}(4)$; as an abstract group Aut $(H_2) \cong (\text{Alt}(4) \times 2).2$, and the action is of degree 12.

To complete the enumeration of the automorphism groups of homogeneous digraphs, we would need to study the automorphisms of the various graphs arising from the composition of two others: for instance, we would need to calculate $\operatorname{Aut}(A[K_n])$ and $\operatorname{Aut}(K_n[A])$ for each $A \in \mathcal{A}$. We will not do this.

There are a multitude of results that extend Gardiner, Sheehan and/or Lachlan's results to finite (di)graphs with automorphism groups that satisfy weaker properties than homogeneity. We particularly mention [48] which considers so-called *set-homogeneous* digraphs. In a different direction Cherlin has classified the homogeneous countable digraphs [25] extending work of Lachlan and Woodrow classifying the homogeneous countable graphs [63], and of Lachlan classifying the homogeneous countable tournaments [61].

Analogues for some of the given results exist for relational structures containing a single relation which may not be binary. Lachlan and Tripp have classified the homogeneous 3-graphs [64] and Cameron has done the same for homogeneous k-graphs with $k \ge 6$ [17]; these results are analogues of Gardiner's result for ternary relational structures with a single relation. Devillers has studied a rather similar problem in her work on homogeneous Steiner systems, however the notion of homogeneity considered there is different to ours [37].

1.4 Motivation: On model theory

Cherlin's conjecture arises from model theory considerations rooted in Lachlan's theory of finite homogeneous relational structures (see, for instance, [59, 62]). We give a brief summary of some of the main ideas; the origin of nearly everything we consider here is [20].

Let us consider a family of theorems indexed by parameters k and ℓ , with $k, \ell \in \mathbb{Z}^+$ and $\ell \ge 2$. Theorem (k, ℓ) is a full classification of the homogeneous relational structures with at most ℓ relations, and with arity at most k. So, for instance, the first theorem we are likely to consider is Theorem (2, 1) which (modulo the transitivity assumption we discussed above) is just Theorem 1.3.3, a result of Lachlan himself that classifies finite binary relational structures with one relation; in other words finite simple homogeneous directed graphs.

Lachlan's theory of finite homogeneous relational structures asserted a number of facts about the form of these theorems, and about the relationships between them. With regard to the form of the theorem, Lachlan's theory asserts that each theorem can be written as follows:

"A finite homogeneous relational structure of arity at most k with at most ℓ relations lies in one of a number of infinite families, or else is one of a finite number of sporadic individuals."

The power of this assertion is in the restrictions which Lachlan placed upon the definition of the word "family": a family of finite homogeneous relational structures in Lachlan's sense is an infinite collection of structures that can be constructed from a single infinite relational structure via a set of explicitly described operations.

With regard to the relationships between these theorems, Lachlan's theory gives us information about what the word "sporadic" means in these theorems. Specifically he asserts that any sporadic individual cropping up in Theorem (k, ℓ) , say, will appear later as part of an infinite family in Theorem (k', ℓ') for some $k' \ge k$ and $\ell' \ge \ell$. Thus the "sporadic-ness" of a particular homogeneous relational structure is, in some sense, not genuine – rather, it is an artefact of restricting our investigations to particular values of kand ℓ .

The significance of all of this from a group-theoretic point of view lies in Cherlin's observation that every finite permutation group can be viewed as the automorphism group of a homogeneous relational structure – we demonstrated one way of seeing this in Example 1.1.6. This observation allows us to shift our point of view on the family of theorems studied by Lachlan: we can think of them as being about finite permutation groups.

In this setting the parameters k and ℓ can be seen as providing some kind of stratification on the universe of finite permutations groups, and Lachlan's results concerning "families" and "sporadic-ness" can be seen

as statements about groups as well as structures. Finally, we can rewrite the theorems themselves from a group-theoretic point of view; they take the following form:

"Let G be the automorphism group of a homogeneous relational structure \mathcal{R} on a set Ω of arity at most k with at most ℓ relations. Then, viewed as a permutation group on Ω , G lies in one of a number of infinite families, or else is one of a finite number of sporadic individuals."

With this set-up, any given permutation group G will occur in an infinite number of Theorems (k, ℓ) . Typically, though, we are interested in the *first* such occurrence: we are interested in the pair (k, ℓ) for which k is minimal, and having fixed k as this minimal value, we then seek the minimum possible value of ℓ . The resulting pair (k, ℓ) is a measure of the *complexity* of G from the model-theoretic point of view or, using the point of view espoused in §1.3, gives a measure of the efficiency with which G can be represented as the automorphism group of a homogeneous relational structure.

Of course, plenty remains: we know that these theorems about finite permutation groups exist; we know their form, and we know something about the relationships that exist between them. We would like to know the statements of these theorems, and we would like to prove them!

As described in the previous section, this last task has only been completed for Theorem (2,1) (and, even then, with a small caveat). The main theorem of this monograph completes the task of ascertaining which groups appear as *primitive* permutation groups in any Theorem $(2, \ell)$.

1.5 Motivation: Other important statistics

It turns out that relational complexity is closely connected to a number of other permutation group statistics, some of which have received a great deal of attention in the literature. Our reference for the following definitions is [5].

For $\Lambda = \{\omega_1, \ldots, \omega_k\} \subseteq \Omega$ and for $G \leq \text{Sym}(\Omega)$, we write $G_{(\Lambda)}$ or $G_{\omega_1, \omega_2, \ldots, \omega_k}$ for the point-wise stabilizer. If $G_{(\Lambda)} = \{1\}$, then we say that Λ is a *base*. The size of the smallest possible base is known as the *base size* of G and is denoted b(G).

We say that a base is a *minimal base* if no proper subset of it is a base. We denote the maximum size of a minimal base by B(G).

Given an ordered sequence of elements of Ω , $[\omega_1, \omega_2, \ldots, \omega_k]$, we can study the associated *stabilizer* chain:

$$G \ge G_{\omega_1} \ge G_{\omega_1,\omega_2} \ge G_{\omega_1,\omega_2,\omega_3} \ge \dots \ge G_{\omega_1,\omega_2,\dots,\omega_k}$$

If all the inclusions given above are strict, then the stabilizer chain is called *irredundant*. If, furthermore, the group $G_{\omega_1,\omega_2,\ldots,\omega_k}$ is trivial, then the sequence $[\omega_1,\omega_2,\ldots,\omega_k]$ is called an *irredundant base*. The size of the longest possible irredundant base is denoted I(G).

Finally, let Λ be any subset of Ω . We say that Λ is an *independent set* if its point-wise stabilizer is not equal to the point-wise stabilizer of any proper subset of Λ . We define the *height* of G to be the maximum size of an independent set, and we denote this quantity by H(G).

Note that if G is a transitive permutation group on a set Ω , then H(G) = 1 if and only if G is regular; similarly, H(G) = 2 if and only if the stabilizer of a point is a non-trivial TI-subgroup of G. (Recall that X is said to be a non-trivial TI-subgroup of a group G if X is a proper subgroup of G and $X \cap X^g = 1$, for every $g \in G \setminus N_G(X)$.)

There is a basic connection between the four statistics we have defined so far:

$$b(G) \le B(G) \le H(G) \le I(G) \le b(G) \log t.$$
(1.5.1)

Recall that in this document, if the base is not specified, then "log" always means "log to the base 2"; recall, also, that $t = |\Omega|$. Let us see why (1.5.1) is true:

The first inequality is obvious. For the second, suppose that Λ is a minimal base; then Λ is an independent set. For the third, suppose that $\Lambda := \{\omega_1, \omega_2, \ldots, \omega_k\}$ is an independent set and observe that

$$G > G_{\omega_1} > G_{\omega_1,\omega_2} > G_{\omega_1,\omega_2,\omega_3} > \dots > G_{\omega_1,\omega_2,\dots,\omega_k}$$

is a strictly decreasing sequence of stabilizers. In particular, $[\omega_1, \omega_2, \ldots, \omega_k]$ is irredundant and we may extend this irredundant sequence to an irredundant base. Hence $H(G) \leq I(G)$.

The fourth inequality has been attributed to Blaha [7] who, in turn, describes it as an "observation of Babai" [4]. Suppose that G has a base of size b = b(G). Then, in particular $|G| \le t^b$. On the other hand, any irredundant base and any independent set have size at most $\log |G|$. We conclude that $I(G) \le \log(t^b)$, and the result follows.

We are ready to connect relational complexity to the four statistics we have just defined. The key result is the following.

Lemma 1.5.1. $RC(G) \le H(G) + 1$.

Proof. Let h = H(G) and consider a pair $(I, J) \in \Omega^n$ such that $I_{h+1}J$. We must show that $I \sim J$.

Observe that we can reorder the tuples without affecting their subtuple completeness. Hence, without loss of generality, we can assume that

$$G_{I_1} > G_{I_1,I_2} > \dots > G_{I_1,I_2,\dots,I_{\ell}},$$

for some $\ell \leq h$ and then this chain stabilizers, i.e.

$$G_{I_1,...,I_{\ell}} = G_{I_1,...,I_{\ell+j}},$$

for all $1 \leq j \leq n - \ell$. From the assumption of *h*-subtuple completeness it follows that there exists an element $g \in G$ such that $I_i^g = J_i$ for all $1 \leq i \leq \ell$ and observe that the set of all such elements g forms a coset of G_{I_1,\ldots,I_ℓ} .

The assumption of (h + 1)-subtuple completeness implies, moreover, that for all $1 \le j \le n - \ell$ there exists $g_j \in G$ such that

$$\begin{cases} I_i^{g_j} = J_i, & \text{for} \quad 1 \le i \le \ell, \\ I_{\ell+j}^{g_j} = J_{\ell+j}. \end{cases}$$

The set of all such elements g_j forms a coset of $G_{I_1,\ldots,I_{\ell},I_{\ell+j}}$, which is, again, a coset of $G_{I_1,\ldots,I_{\ell}}$. Since any coset of $G_{I_1,\ldots,I_{\ell}}$ is defined by the image of the points I_1,\ldots,I_{ℓ} under an element of the coset, we conclude that elements of the same coset of $G_{I_1,\ldots,I_{\ell}}$ map $I_{\ell+j}$ to $J_{\ell+j}$ for all $1 \leq j \leq n-\ell$. In particular, $I \simeq J$, as required.

Lemma 1.5.1 has been exploited in [44], where an upper bound on the height of a primitive permutation group is proved, from which the obvious upper bound on relational complexity is deduced. The main result on height is the following:

Theorem 1.5.2. Let G be a finite primitive group of degree t. Then one of the following holds:

- 1. G is a subgroup of $\operatorname{Sym}(m)\operatorname{wr}\operatorname{Sym}(r)$ containing $(\operatorname{Alt}(m))^r$, where the action of $\operatorname{Sym}(m)$ is on k-subsets of $\{1, \ldots, m\}$ and the wreath product has the product action of degree $t = {m \choose k}^r$;
- 2. $H(G) < 9 \log t$.

Note that various members of the family listed at item (1) of Theorem 1.5.2 genuinely violate the bound at item (2): for example, when r = k = 1, we obtain the groups Sym(t) and Alt(t) in their natural action, for which the height is t - 1 and t - 2, respectively. In fact, though, we do not know the exact height of the groups listed at item (1) for all possible values of k, m and r. The proof of Theorem 1.5.2 exploits the rich array of results in the literature giving bounds on b(G) for various families of permutation groups. In particular, use is made of the proof of the Cameron-Kantor conjecture [19] by Liebeck and Shalev [80], and of Cameron's follow-up conjecture giving a value for the associated constant [18] by many authors [11, 13, 14, 15]. These results mean that, in the almost simple case, work is only required for the so-called "standard actions".

Theorem 1.5.2 is an analogue of an existing result for b(G) [70]; now (1.5.1) and Lemma 1.5.1 yield analogues for B(G) and RC(G). With this result for RC(G), and with the proof of Conjecture 1.1, we now have a good handle on those permutation groups G for which RC(G) is either very large, or as small as possible. In the case where RC(G) is large, work remains to be done to ascertain the relational complexity of the groups listed at item (1) of the theorem; the most important results in this direction can be found in [26], and we summarised some of these above in Theorem 1.2.8.

The relationship between the various statistics occurring in (1.5.1), and between these statistics and RC(G) is an intriguing area of investigation, although not one that has hitherto received much attention. Cherlin and Wiscons have started to study some of these questions, and we mention two of their remarks [23]:

- 1. From computational evidence, it appears that $\operatorname{RC}(G)$ and H(G) are "close" (say, $\operatorname{RC}(G) \ge \operatorname{H}(G) 3$). The obvious exceptions to this rule of thumb are the symmetric groups in their natural action; more generally, among primitive groups of degree at most 100, the only groups for which $\operatorname{RC}(G) < \operatorname{H}(G) - 3$ are various members of the family listed at item (1) of Theorem 1.5.2.
- 2. Again, from computational evidence, more often than not, it appears that B(G) and H(G) coincide for primitive groups. Moreover, for all primitive groups of degree at most 100, $H(G) - B(G) \le 3$.

We shy away from making conjectures about the general pattern for larger n but, still, these lines of inquiry seem promising.

1.6 Methods: basic lemmas

Most of the results in this section were first written down in [34, 45, 46]. All of these papers were focused on showing that certain group actions are not binary, hence the lemmas we present here tend to yield lower bounds for relational complexity.

As always G is a group acting on a set Ω . In what follows, we will write $I, J \in \Omega^n$ to mean that $n \geq 2$ is a positive integer and I, J are elements of Ω^n ; we will always assume that $I = (I_1, \ldots, I_n)$ and $J = (J_1, \ldots, J_n)$. We will write $I \approx J$ to mean that the pair (I, J) is k-subtuple complete; we will write $I \approx I$ to mean that the pair (I, J) is k-subtuple complete with respect to the action of G.

1.6.1 Relational complexity and subgroups

Examples 1.2.1 and 1.2.2 serve as a warning that relational complexity can behave badly with respect to arbitrary subgroups of the group G. Nonetheless, something can still be said.

Lemma 1.6.1. Let G be a transitive permutation group on Ω and let M be a point-stabilizer in this action. Let Λ be a non-trivial orbit of M. Then

$$\operatorname{RC}(G, \Omega) \ge \operatorname{RC}(M, \Lambda).$$

Note, in particular, that if G is binary, then the action of M on all non-trivial suborbits must be binary. This will be useful later, particularly when we consider actions in which G is very large and M relatively small (for instance, $G = E_8(2)$, and $M = \text{Aut}(\text{PSU}_3(8))$), in which case it is sometimes possible to use magma to list all of the transitive binary actions of M.

Proof. Write α for an element of Ω stabilized by M. Let $r = \operatorname{RC}(M, \Lambda)$; then there exist $I, J \in \Lambda^n$ such that $I \underset{r-1,n}{\sim} J$ with respect to the action of M on Λ . But now observe that if we define

$$I^* = (\alpha, I_1, \dots, I_n) \text{ and } J^* = (\alpha, J_1, \dots, J_n),$$

then $I^* \underset{r-1,n+1}{\sim} J^*$, and the result follows.

We write (G: M) here, and below, to mean the set of right cosets of M in G.

Lemma 1.6.2. Let M < H < G. Then $RC(G : (G : M)) \ge RC(H, (H : M))$.

Proof. Write r = RC(H : (H : M)), and observe that $\Lambda = (H : M)$ is a subset of $\Omega = (G : M)$. Then there exist $I, J \in \Lambda^n$ such that $I \underset{r-1,n}{\sim} J$ with respect to the action of H.

We must show that $I_{r-1,n} J$ with respect to the action of G. That $I_{r-1} J$ with respect to the action of G is immediate. Suppose that $I_{n} J$ with respect to the action of G. Then there exists $g \in G$ such that $I_{i}^{g} = J_{i}$ for all $i \in \{1, \ldots, n\}$. Since $I_{i}, J_{i} \in (H : M)$ for all $i \in \{1, \ldots, n\}$, we must have $g \in H$. But then $I \underset{n}{\sim} J$ with respect to the action of H, which is a contradiction.

1.6.2 Relational complexity and subsets

For Λ a subset of Ω we write G_{Λ} for the *set-wise* stabilizer of Λ , and $G_{(\Lambda)}$ for the *point-wise* stabilizer of Λ . We write G^{Λ} for the permutation group induced on Λ by G_{Λ} ; note that $G^{\Lambda} \cong G_{\Lambda}/G_{(\Lambda)}$.

In this section we present some results connecting $\mathrm{RC}(G,\Omega)$ with $\mathrm{RC}(G^{\Lambda},\Lambda)$.

Definition 1.6.3. Let $t := |\Omega|$. For $k \in \mathbb{Z}^+$ with $k \ge 2$, we say that the action of G on Ω is *strongly* non-k-ary if there exist $I, J \in \Omega^t$ such that $I_{k,t} J$, and all elements of I (resp. J) are distinct.

Note that this definition requires the existence of $I, J \in \Omega^t$ with $I \underset{k,t}{\sim} J$ and with every element of Ω occurring as an entry of I (and, therefore, also of J). If k = 2, then we tend to write strongly non-binary as a synonym for strongly non-k-ary.

The notion of a strongly non-k-ary set is connected to a classical notion in permutation group theory which was introduced by Wielandt [103].

Definition 1.6.4. Let $G \leq \text{Sym}(\Omega)$ and let $k \in \mathbb{Z}^+$. The k-closure of G is the set

 $G^{(k)} = \{ \sigma \in \operatorname{Sym}(\Omega) \mid \forall I \in \Omega^k, \text{ there exists } g \in G, I^g = I^\sigma \}.$

We say that G is k-closed if $G = G^{(k)}$.

Observe that $G^{(k)}$ is the largest subgroup of $Sym(\Omega)$ that has the same orbits on the set of k-tuples of Ω as G. Now the connection with strongly non-k-ary sets is as follows.

Lemma 1.6.5. The group G is strongly non-k-ary if and only if G is not k-closed.

Proof. Write $\Omega := \{\omega_1, \ldots, \omega_t\}$. If G is not k-closed, then there exists $\sigma \in G^{(k)} \setminus G$. Now, it is easy to verify that $I := (\omega_1, \ldots, \omega_t)$ and $J := I^{\sigma} = (\omega_1^{\sigma}, \ldots, \omega_t^{\sigma})$ are k-subtuple complete (because $\sigma \in G^{(k)}$) and are not t-subtuple complete (because $\sigma \notin G$). Thus $I_{k,t} J$, and we conclude that the action of G on Ω is strongly non-k-ary. The converse is similar.

The most important example, for us, of a permutation group that is not k-closed is as follows.

Example 1.6.6. Let G be a k-transitive permutation group on Ω , for some integer $k \geq 2$. The definition implies that $G^{(k)} = \text{Sym}(\Omega)$.

We immediately conclude that $Alt(\Omega)$ is not (t-2)-closed, and we obtain (again) that $RC(Alt(\Omega), \Omega) \ge t-1$.

Recall that the Classification of Finite Simple Groups implies that examples of k-transitive permutation groups that do not contain Alt(Ω) only exist for $k \leq 5$. What is more, all such groups are classified for $k \geq 2$ (see, for instance [38, §7.7]).

Lemma 1.6.7. Let $\Lambda \subseteq \Omega$. If G^{Λ} is strongly non-k-ary, then $\operatorname{RC}(G, \Omega) > k$.

Proof. Suppose that $|\Lambda| = \ell$, and let I, J be ℓ -tuples of distinct elements of Λ such that $I \underset{k,\ell}{\sim} J$ with respect to the action of G^{Λ} . It is enough to show that $I \underset{k,\ell}{\sim} J$ with respect to the action of G. It is immediate that $I \underset{k}{\sim} J$ with respect to the action of G. On the other hand, if $I \underset{\ell}{\sim} J$, then there exists $g \in G$ such that $I^g = J$. Since I contains all elements of Λ , we conclude that $g \in G_{\Lambda}$ which contradicts the fact that $I \not_{\ell} J$ with respect to the action of G^{Λ} .

1.6.3 Strongly non-binary subsets

Our final few results apply specifically to the study of binary actions. As usual G acts on a set Ω , and we refer to a subset $\Lambda \subseteq \Omega$ as strongly non-binary if G^{Λ} is strongly non-binary.

The next lemma details our first example of such a subset. This example was first described in [46]; its key properties are a consequence of Example 1.6.6 and Lemma 1.6.7.

Lemma 1.6.8. Suppose that there exists a subset $\Lambda \subseteq \Omega$ such that $|\Lambda| \ge 2$ and G^{Λ} is a 2-transitive proper subgroup of Sym(Λ). Then G^{Λ} is strongly non-binary and the action of G on Ω is not binary.

In subsequent chapters, our focus is on proving that certain actions are not binary. Lemma 1.6.8 means that we will be interested in finding subsets which have 2-transitive set-wise stabilizers. The next lemma requires no proof, but we include it as it clarifies when such subsets exist.

Lemma 1.6.9. Let K be some 2-transitive group, and let K_0 be a point-stabilizer in K. Let H be a subgroup of G and suppose that $\varphi : H \to K$ is a surjective homomorphism. Let M be the stabilizer in G of a point $\omega \in \Omega$ and let C be the core of $H \cap M$ in H. If $\text{Ker}(\varphi) = C$ and $\varphi(H \cap M) = K_0$, then H acts 2-transitively on the orbit ω^H .

The next lemma is a useful tool in finding subsets on which a set-stabilizer acts 2-transitively (recall that, when $r \ge 2$, the affine special linear group $ASL_r(q)$ is 2-transitive in its natural action on q^r points).

Lemma 1.6.10. Let G be a finite group acting transitively on a set Ω with point-stabilizer M, and suppose that the following two conditions hold:

- (i) M has a subgroup $A \cong SL_r(q)$, where $r \ge 2$, and
- (ii) G has a subgroup S that is a central quotient of $SL_{r+1}(q)$, such that $A \leq S$ (the natural completely reducible embedding) and $S \not\leq M$.

Then there is a subset Δ of Ω such that $|\Delta| = q^r$ and $G^{\Delta} \ge ASL_r(q)$.

Proof. We have $A \leq S \cap M < S$. Since A is embedded in S via the natural completely reducible embedding, we have $S \cap M \leq P_i(S)$ with $i \in \{1, r\}$, where $P_i(S)$ is a maximal parabolic subgroup of S stabilizing a 1-dimensional or an r-dimensional subspace. Say i = 1 (the case i = r is entirely similar). Then writing matrices with respect to a suitable basis,

$$S \cap M \le P_1(S) = \left\{ \begin{pmatrix} Y & v \\ 0 & \lambda \end{pmatrix} : Y \in \operatorname{GL}_r(q), v \in \mathbb{F}_q^r, \det(Y)\lambda = 1 \right\},$$

where A is the subgroup obtained by setting $\lambda = 1$, $\det(Y) = 1$ and v = 0. Define

$$U = \left\{ egin{pmatrix} I & 0 \ a & 1 \end{pmatrix} : a \in \mathbb{F}_q^r
ight\},$$

and set $\Delta = \{Mu : u \in U\} \subseteq \Omega$ (where we identify Ω with the set (G : M) of right cosets of M in G).

Since $M \cap U = 1$, the cosets $Mu (u \in U)$ are all distinct, and so $|\Delta| = q^r$. Since A normalizes U and $A \leq M$, the subgroup $UA \cong q^r$. $SL_r(q)$ stabilizes Δ , and since $UA \cap M = A$, we have $(UA)^{\Delta} = ASL_r(q) \leq G^{\Delta}$.

It turns out that in the context of almost simple groups, it is convenient to use a variant of Lemma 1.6.8 where we don't just seek proper 2-transitive subgroups of $\text{Sym}(\Omega)$, but also exclude $\text{Alt}(\Omega)$ from our consideration. To that end we include the following definition which first appeared in [46].

Definition 1.6.11. A subset $\Lambda \subseteq \Omega$ is a *G*-beautiful subset if G^{Λ} is a 2-transitive subgroup of $\text{Sym}(\Lambda)$ that is isomorphic to neither $\text{Alt}(\Lambda)$ nor $\text{Sym}(\Lambda)$.

In what follows, if the group G is clear from the context, we will speak of a beautiful subset rather than a G-beautiful subset of Ω . Observe that a beautiful subset of Ω is a strongly non-binary subset. The reason for the stronger definition is explained by the following result.

Lemma 1.6.12. Suppose that G is almost simple with socle S. If Ω contains an S-beautiful subset, then G is not binary.

Proof. Let Λ be an S-beautiful subset and observe that Λ has cardinality at least 5. Then, since S is normal in G, the group $(S_{\Lambda}G_{(\Lambda)})/G_{(\Lambda)}$ is a normal subgroup of $G_{\Lambda}/G_{(\Lambda)}$. This implies that $G_{\Lambda}/G_{(\Lambda)}$ is (isomorphic to) a 2-transitive proper subgroup of Sym(Λ). Then Lemma 1.6.8 implies that G is not binary.

Although in this paper we do not need to deal with C_1 -actions for classical groups since they were dealt with in [46], we include the next lemma because it clearly illustrates the beautiful subsets method. The lemma has the added advantage of giving the reader an idea of how to deal with C_1 -actions in general. (These actions all yield to the method of beautiful subsets provided n and q are not too small.)

Lemma 1.6.13. Let $S = PSL_n(q)$ and for n = 2 assume q > 5. Let M be a maximal parabolic subgroup of S, and let Ω be the set of right cosets of M. Then Ω contains an S-beautiful subset.

Proof. Here M is the stabilizer of a subspace W of V, where V is the natural n-dimensional module for $SL_n(q)$. Since the action of S on the k-dimensional subspaces of V is permutation isomorphic to the action on the (n - k)-subspaces of V, we may assume that $\dim(W) \leq n/2$.

If $\dim(W) = 1$, then the action of S on Ω is 2-transitive. Now Ω itself is an S-beautiful subset, because we are assuming q > 5 when n = 2.

Suppose next that $\dim(W) > 1$. Observe that this implies that $n \ge 4$. Let W' be a subspace of W with $\dim(W') = \dim(W) - 1$ and consider $\Lambda = \{W'' \le V \mid W' \subset W'', \dim(W'') = \dim(W)\}$. Clearly, $S_{\Lambda} = \operatorname{Stab}_{S}(W')$ and the action of S^{Λ} on Λ is permutation isomorphic to the natural 2-transitive action of $\operatorname{GL}(V/W')$ on the 1-dimensional subspaces of V/W'. Since $\dim(V/W') \ge 3$, the action of $\operatorname{GL}(V/W')$ induces neither the alternating nor the symmetric group on the set $P_1(V/W')$ of 1-dimensional subspaces of V/W'; therefore Λ is a beautiful subset.

Our second example of a strongly non-binary subset is taken from [45, Example 2.2]

Example 1.6.14. Let G be a subgroup of $\text{Sym}(\Omega)$, let g_1, g_2, \ldots, g_r be elements of G, and let $\tau, \eta_1, \ldots, \eta_r$ be elements of $\text{Sym}(\Omega)$ with

$$g_1 = \tau \eta_1, \ g_2 = \tau \eta_2, \ \dots, \ g_r = \tau \eta_r.$$

Suppose that, for every $i \in \{1, ..., r\}$, the support of τ is disjoint from the support of η_i ; moreover, suppose that, for each $\omega \in \Omega$, there exists $i \in \{1, ..., r\}$ (which may depend upon ω) with $\omega^{\eta_i} = \omega$. Suppose, in addition, $\tau \notin G$. Now, writing $\Omega = \{\omega_1, ..., \omega_t\}$, observe that

$$((\omega_1, \omega_2, \ldots, \omega_t), (\omega_1^{\tau}, \omega_2^{\tau}, \ldots, \omega_t^{\tau}))$$

is a non-binary witness. Thus the action of G on Ω is strongly non-binary.

Lemma 1.6.15 ([34, Lemma 2.5]). Let G be a transitive permutation group on Ω , let $\alpha \in \Omega$ and let p be a prime with p dividing both $|\Omega|$ and $|G_{\alpha}|$ and with p^2 not dividing $|G_{\alpha}|$. Suppose that G contains an elementary abelian p-subgroup $V = \langle g, h \rangle$ with $g \in G_{\alpha}$, with $\langle h \rangle$ and $\langle gh \rangle$ conjugate to $\langle g \rangle$ via G. Then G is not binary.

In [34, Lemma 2.5], the hypothesis actually requires that h and gh are conjugate to g via G; however the same proof yields the conclusion that G is not binary under the weaker assuption that $\langle h \rangle$ and $\langle gh \rangle$ are conjugate to $\langle g \rangle$ in G, as stated in the lemma. We will need this strengthening in what follows.

Lemma 1.6.16 ([34, Lemma 2.6]). Let G be a permutation group on Ω and suppose that g and h are G-conjugate elements of prime order p, and gh^{-1} is conjugate to g (and so to h). Suppose that $V = \langle g, h \rangle$ is elementary abelian of order p^2 . Suppose, finally, that G does not contain any elements of order p that fix more points of Ω than g. If |Fix(V)| < |Fix(g)|, then G is not binary.

1.7 Methods: Frobenius groups

It turns out that the presence of Frobenius groups can be a powerful tool in proving that certain actions are not binary. We give three lemmas in this direction; the first was proved independently by Wiscons, although a proof has not appeared in the literature.⁶

Lemma 1.7.1. Let G be a Frobenius permutation group on Ω (that is, G acts transitively on Ω , $G_{\omega} \neq 1$ for every $\omega \in \Omega$ and $G_{\omega\omega'} = 1$ for every $\omega, \omega' \in \Omega$ with $\omega \neq \omega'$). If G is binary, then a Frobenius complement has order equal to 2.

Proof. Throughout this proof we write $G = N \rtimes H$ where N is the Frobenius kernel, and H is a Frobenius complement (and point-stabilizer). Suppose that G is binary. Let a and b be distinct non-trivial elements of N. We claim that the binary condition on triples implies that

$$HH^a \cap HH^b = H.$$

To see this, assume H stabilizes $\alpha \in \Omega$. Let $\beta \in \alpha H^a \cap \alpha H^b$. Then the tuples $(\alpha, \alpha^a, \alpha^b) \simeq (\beta, \alpha^a, \alpha^b)$. As G is binary, there exists $g \in G$ mapping the first tuple to the second. Then $g \in H^a \cap H^b$ and since the action is Frobenius, g = 1 and hence $\alpha = \beta$. So $\alpha H^a \cap \alpha H^b = \alpha$, and considering the isomorphic action on cosets of H yields $HH^a \cap HH^b = H$.

We will show that if |H| > 2, then this equality cannot hold. Suppose, then, that $h_1, h_2, h_3, h_4 \in H$ are such that

$$h_1 a^{-1} h_2 a = h_3 b^{-1} h_4 b. (1.7.1)$$

Observe first that if the element represented by the two sides of this equation is equal to an element of H, then $a^{-1}h_2a$ must also be an element of H and so h_2 is equal to 1, as is h_4 , and in addition $h_1 = h_3$.

Thus it suffices to find a solution to (1.7.1) for which $h_1 \neq h_3$. To do this we start by rearranging to obtain that

$$h_3^{-1}h_1a^{-1}h_2ah_4^{-1} = b^{-1}h_4bh_4^{-1}$$

⁶Here is the shorter and more elegant argument due to Wiscon for Lemma 1.7.1

For distinct $a, b, c \in \Omega$, binarity implies that the intersection of the suborbits cG_a and cG_b is equal to $cG_{a,b}$, so as the action is Frobenius, $(cG_a) \cap (cG_b) = \{c\}$. Also, using again that the action is Frobenius, $|cG_a| = |G_a| = |G_b| = |cG_b|$. This shows that $\bigcup_{a\neq c} (cG_a \setminus \{c\})$ is a disjoint union of sets of constant size $|G_a| - 1$. So, letting $N = |\Omega|$, we find that $N - 1 = |\Omega \setminus \{c\}| \ge (N - 1)(|G_a| - 1)$, implying that $|G_a| = 2$.

and observe that the right-hand side lies in N. Thus the left-hand side lies in N. Doing some rearranging we find that the left-hand side can be rewritten as

$$(h_3^{-1}h_1a^{-1}h_1^{-1}h_3)(h_3^{-1}h_1h_2ah_2^{-1}h_1^{-1}h_3)(h_3^{-1}h_1h_2h_4^{-1}).$$

Since the first two bracketed terms lie in N, we conclude that $h_3^{-1}h_1h_2h_4^{-1} = 1$. This allows us to replace h_4 in (1.7.1) to get

$$h_1 a^{-1} h_2 a = h_3 b^{-1} h_3^{-1} h_1 h_2 b,$$

which we rearrange one last time to obtain

$$(a^{-1})(h_2ah_2^{-1})(h_2b^{-1}h_2^{-1}) = h_1^{-1}h_3b^{-1}h_3^{-1}h_1$$
(1.7.2)

where, again, we have put terms that lie in N in brackets.

Now, for $h \in H \setminus \{1\}$, we define

$$\phi_h : N \to N$$
$$n \mapsto n^{-1} h n h^{-1}.$$

We claim that this map is a bijection. We need only show injectivity: suppose that $n_1, n_2 \in N$ with

$$n_1^{-1}hn_1h^{-1} = n_2^{-1}hn_2h^{-1}$$

Then $n_2 n_1^{-1} h n_1 n_2^{-1} = h$ and hence $n_1 n_2^{-1}$ centralizes h. Since we have a Frobenius action, we obtain that $n_1 = n_2$, as required.

Now fix $b \in N$ and $h_2 \in H \setminus \{1\}$ and consider (1.7.2). The first two bracketed terms correspond to $\phi_{h_2}(a)$ and the surjectivity of the function ϕ_{h_2} implies that the left-hand side of (1.7.2) ranges over all values of N as a varies across N. Recall, though, that we require that $a \neq b$: this restriction tells us that the left-hand side equals all but one of the elements of N, as a varies.

On the other hand if H has orbits of size at least 3, we obtain that (1.7.2) has a solution in which $h_1 \neq h_3$. We are done.

Lemma 1.7.2. Let $F \triangleleft G \leq \text{Sym}(\Omega)$ with F having an orbit $\Lambda \subseteq \Omega$ on which it acts as a Frobenius group. (As usual, F^{Λ} is the permutation group induced by the action of F on Λ .) Write $F^{\Lambda} = T \rtimes C$, where T is the Frobenius kernel, and C is a Frobenius complement. If T is cyclic, and C contains an element x of order strictly greater than 2, then G is not binary.

Proof. Let $\alpha \in \Lambda$. Since Λ is a block of imprimitivity for G, the group G_{α} must preserve Λ set-wise. Observe that G_{Λ} normalizes F, because $F \trianglelefteq G$. In particular, $F^{\Lambda} \trianglelefteq G^{\Lambda}$. Since the non-identity elements of T are precisely those elements of F^{Λ} that are fixed-point-free, G^{Λ} also normalizes T. Thus T is a regular normal subgroup of G^{Λ} . As T acts regularly on Λ , from the Frattini argument we obtain $G^{\Lambda} = T \rtimes G^{\Lambda}_{\alpha}$.

We can, therefore, identify T with Λ in such a way that the action of G^{Λ}_{α} on Λ is permutation isomorphic to the conjugation-action of G^{Λ}_{α} on T. To see this, define

$$\theta: T \to \Lambda$$
$$y \mapsto \alpha^y$$

and observe that, for $y \in T$ and $g \in G_{\alpha}$,

$$\theta(y^g) = \alpha^{(y^g)} = \alpha^{g^{-1}yg} = \alpha^{yg} = (\alpha^y)^g = (\theta(y))^g.$$

With this set-up, we write n = |T| and we let y be a generator of T. We will construct (for the action of G) a 2-subtuple complete pair of the form

$$\left((1, y, y^a), (1, y, y^b)\right).$$
 (1.7.3)

We must choose a and b appropriately. Let $x \in C$ having order strictly greater than 2. First, let $k \in \mathbb{Z}^+$ be such that $y^x = y^k$; note that gcd(k, n) = 1, and so k is invertible modulo n. Then we set $a = \frac{1+k}{k} \in \mathbb{Z}_n$ and set b = 1 + k. Now observe that

$$\begin{aligned} (1,y)^{\mathrm{id}} &= (1,y);\\ (1,y^a)^x &= (1,y^{(k+1)/k})^x = (1,y^{k+1}) = (1,y^b);\\ (y,y^a)^{y^{-1}x^2y} &= (y,y^{(k+1)/k})^{y^{-1}x^2y} = (1,y^{1/k})^{x^2t} = (1,y^k)^y = (y,y^{k+1}) = (y,y^b). \end{aligned}$$

We see immediately that the pair (1.7.3) is 2-subtuple complete.

Note on the other hand that, provided $a \neq b$, this pair cannot be 3-subtuple complete: suppose that an element $g \in G$ sends the first triple in (1.7.3) to the second. Then g fixes 1 and, as we saw above, this means that the action of g on Λ is isomorphic to the action of g by conjugation on T. Since $y^g = y$, we conclude that, if $(y^a)^g = y^b$, then we must have a = b modulo n. But now observe that

$$a = b \Longleftrightarrow \frac{1+k}{k} = 1+k \Longleftrightarrow k^2 = 1.$$

Since we chose x to have order strictly greater than 2, we see that $k^2 \neq 1$, and we conclude that (1.7.3) is a pair which is 2-subtuple complete but not 3-subtuple complete. The result follows.

Lemma 1.7.3. Let $F = T \rtimes C \leq G \leq \text{Sym}(\Omega)$ with C acting by conjugation fixed-point-freely on T. Suppose there exists $\alpha \in \Omega$ such that $F_{\alpha} = C$, and let Λ be the orbit of α under F. Define

 $m := \min\{|G_{\gamma_1,\gamma_2}| \mid \gamma_1,\gamma_2 \text{ distinct elements of } \Lambda\}.$

If $\left\lceil \frac{(|C|-1)(|C|-2)}{|\Lambda|-2} \right\rceil \ge m$, then G is not binary. In particular, if $|G:F| \le \left\lceil \frac{(|C|-1)(|C|-2)}{|\Lambda|-2} \right\rceil$, then G is not binary.

Proof. Observe that F acts as a Frobenius group on Λ , where T is the Frobenius kernel, and C is a Frobenius complement. It is useful to observe that the regularity of T on Λ implies that, for every $c \in C$ and for every $\beta \in \Lambda$, there exists a unique $x \in T$ such that $\beta^{xc} = \beta$.

We study triples of the form

$$\left((\alpha,\beta,\gamma), \ (\alpha,\beta,\delta)\right),$$
 (1.7.4)

for $\alpha, \beta, \gamma, \delta \in \Lambda$. We make the following claim:

Claim: for any distinct pair of elements (α, β) , there are at least (|C| - 1)(|C| - 2) choices for (γ, δ) such that the set $\{\alpha, \beta, \gamma, \delta\}$ has size 4, and the pair (1.7.4) is 2-subtuple complete.

Proof of claim: First we consider the set of pairs of distinct non-trivial elements in C, i.e.

$$C^{(2)} := \{ (c_1, c_2) \mid c_1, c_2 \in C \setminus \{1\}, c_1 \neq c_2 \}.$$

Now we construct a function $\phi : C^{(2)} \to \Omega^2$ as follows. For $(c_1, c_2) \in C^{(2)}$, we let t_1 be the unique nontrivial element of T such that $t_1c_1 \in G_\beta$. Now, since $c_1 \neq c_2$, we can define γ to be the unique point in Λ fixed by $t_1c_1c_2^{-1}$. Observe that γ is distinct from both α and β .

Next, we see that

$$\gamma^{t_1c_1c_2^{-1}} = \gamma \Longleftrightarrow \gamma^{t_1c_1} = \gamma^{c_2}.$$

We define $\delta := \gamma^{c_2}$, and we set $\phi(c_1, c_2) = (\gamma, \delta)$. An easy argument shows that δ is distinct from all of α, β and γ . Furthermore we claim that, with these definitions the pair (1.7.4) is 2-subtuple complete. Indeed, observe that

$$(\alpha,\beta)^1 = (\alpha,\beta), \ (\alpha,\gamma)^{c_2} = (\alpha,\delta) \text{ and } (\beta,\gamma)^{t_1c_1} = (\beta,\delta).$$

Thus every element (γ, δ) in the image of ϕ gives rise to a 2-subtuple complete pair as in (1.7.4). Since the domain of ϕ , $C^{(2)}$ has order (|C| - 1)(|C| - 2), the claim will follow if we prove that ϕ is one-to-one.

Suppose, then, that $\phi(c_1, c_2) = (\gamma, \delta) = \phi(c'_1, c'_2)$. Let t_1 (resp. t'_1) be the unique element of T such that t_1c_1 (resp. $t'_1c'_1$) is in G_β . Then $t_1c_1c_2^{-1}$ and $t'_1c'_1(c'_2)^{-1}$ fix γ . What is more $\gamma^{c_2} = \gamma^{c'_2} = \delta$ and so $c'_2c_2^{-1}$ fixes γ . However $c_2, c'_2 \in C = F_\alpha$ and so $c'_2c_2^{-1}$ fixes two points of Λ . We conclude that $c_2 = c'_2$. But now, write $h_1 := t_1c_1$ and $h'_1 := t'_1c'_1$; observe that $h_1, h'_1 \in G_\beta$ and $\gamma^{h_1} = \gamma^{h'_1}$. As before we conclude that $h'_1h_1^{-1}$ fixes β and γ , and so $h_1 = h'_1$. Then $t_1c_1 = t'_1c'_1$ and so $t_1^{-1}t'_1 = c'_1c_1^{-1}$; since $T \cap C = \{1\}$, this gives $c_1 = c'_1$, as required.

The claim and the pigeon-hole principle imply that there exists some $\gamma \in \Lambda \setminus \{\alpha, \beta\}$ for which there are $k := \left\lceil \frac{(|C|-1)(|C|-2)}{|\Lambda|-2} \right\rceil$ choices for δ such that all pairs of the form (1.7.4) are 2-subtuple complete; call these elements $\delta_1, \ldots, \delta_k$. If G is binary, then all of these pairs are 3-subtuple complete and we conclude that the set $\{\gamma, \delta_1, \ldots, \delta_k\}$ is a subset of an orbit of $G_{\alpha,\beta}$. But this is only possible if $k + 1 \leq m$, and the result follows.

1.8 Methods: On computation

We will use magma very frequently in what follows to verify that certain actions are not binary. The methods we use to do this are largely drawn from [34]. We give a brief summary of some of the key methods here. In what follows G acts transitively on the set Ω , and M is the stabilizer of a point.

Test 1: Using the permutation character. Given $\ell \in \mathbb{N} \setminus \{0\}$, we denote by $\Omega^{(\ell)}$ the subset of the Cartesian product Ω^{ℓ} consisting of the ℓ -tuples $(\omega_1, \ldots, \omega_{\ell})$ with $\omega_i \neq \omega_j$, for every two distinct elements $i, j \in \{1, \ldots, \ell\}$. We denote by $r_{\ell}(G)$ the number of orbits of G on $\Omega^{(\ell)}$. The next result is Lemma 2.7 of [34].

Lemma 1.8.1. If G is transitive and binary, then $r_{\ell}(G) \leq r_2(G)^{\ell(\ell-1)/2}$ for each $\ell \in \mathbb{N}$.

Let $\pi : G \to \mathbb{N}$ be the permutation character of G, that is, $\pi(g) = \operatorname{fix}_{\Omega}(g)$ where $\operatorname{fix}_{\Omega}(g)$ is the cardinality of the fixed point set $\operatorname{Fix}_{\Omega}(g) := \{\omega \in \Omega \mid \omega^g = \omega\}$ of g. From the Orbit Counting Lemma, we have

$$r_{\ell}(G) = \frac{1}{|G|} \sum_{g \in G} \operatorname{fix}_{\Omega}(g) (\operatorname{fix}_{\Omega}(g) - 1) \cdots (\operatorname{fix}_{\Omega} - (\ell - 1))$$
$$= \langle \pi(\pi - 1) \cdots (\pi - (\ell - 1)), 1 \rangle_{G},$$

where 1 is the principal character of G and $\langle \cdot, \cdot \rangle_G$ is the natural Hermitian product on the space of \mathbb{C} -class functions of G.

Clearly whenever the permutation character of G is available in magma, we can directly check the inequality in Lemma 1.8.1, and this is often enough to confirm that a particular action is not binary.

Test 2: using Lemma 1.6.5. By connecting the notion of strong-non-binariness to 2-closure, Lemma 1.6.5 yields an immediate computational dividend: there are built-in routines in magma to compute the 2-closure of a permutation group.

Thus if Ω is small enough, say $|\Omega| \leq 10^7$, then we can easily check whether or not the group G is 2-closed. Thus we can ascertain whether or not G is strongly non-binary.

Test 3: a direct analysis. The next test we discuss is feasible once again provided $|\Omega| \leq 10^7$. It simply tests whether or not 2-subtuple-completeness implies 3-subtuple completeness, and the procedure is as follows:

We fix $\alpha \in \Omega$, we compute the orbits of G_{α} on $\Omega \setminus \{\alpha\}$ and we select a set of representatives \mathcal{O} for these orbits. Then, for each $\beta \in \mathcal{O}$, we compute the orbits of $G_{\alpha} \cap G_{\beta}$ on $\Omega \setminus \{\alpha, \beta\}$ and we select a set of representatives \mathcal{O}_{β} . Then, for each $\gamma \in \mathcal{O}_{\beta}$, we compute $\gamma^{G_{\alpha}} \cap \gamma^{G_{\beta}}$. Finally, for each $\gamma' \in \gamma^{G_{\alpha}} \cap \gamma^{G_{\beta}}$, we test whether the two triples (α, β, γ) and (α, β, γ') are *G*-conjugate. If the answer is "no", then *G* is not binary because by construction (α, β, γ) and (α, β, γ') are 2-subtuple complete. In particular, in this circumstance, we can break all the "for loops" and deduce that *G* is not binary.

If the answer is "yes", for every β, γ, γ' , then we cannot deduce that G is binary, but we can keep track of these cases for a deeper analysis. We observe that, if the answer is "yes", for every β, γ, γ' , then

2-subtuple completeness implies 3-subtuple completeness. At this point, we may either use a different method for checking whether the permutation group is genuinely binary or, with a similar method, we can check whether 3-subtuple completeness implies 4-subtuple completeness. This test is very expensive in terms of time, therefore before starting this whole procedure, we do a preliminary check: for 10^6 times, we select β, γ, γ' as above at random and we check this random triple.

Test 4: studying suborbits. Lemma 1.6.1 implies that if G is binary, then the action of M on any suborbit is also binary. This fact is particularly useful for computation in situations where the group G is very large compared to the group M.

In general, our approach is to demonstrate that there must be some suborbit on which the action of M is not binary. For instance, this would follow in the case where $|\Omega| = |G : M|$ is divisible by some integer d, and all non-trivial transitive binary actions of M are also of degree divisible by d.

This last approach sometimes fails for just a few possible actions of M; in this situation, provided the action of G on Ω is primitive, the following lemma is often useful.

Lemma 1.8.2 ([103, Theorem 18.2]). Suppose that G is a finite primitive subgroup of $\text{Sym}(\Omega)$. Let Γ be a non-trivial orbit of M. Then, every simple section of M is isomorphic to a section of the group M^{Γ} which M induces on Γ . In particular, each composition factor of M is isomorphic to a section of M^{Γ} .

This lemma means that when studying possible suborbits of our action we may disregard the actions of M (on a set Γ say) where M has a simple section not isomorphic to a section of the group M^{Γ} . If the resulting set of actions are all not binary, then we can conclude that the action of G on Ω is also not binary. The method is summarised in Lemma 3.1 of [34]:

Lemma 1.8.3. Let G be a primitive group on a set Ω , let α be a point of Ω , let M be the stabilizer of α in G and let d be an integer with $d \geq 2$. Suppose $M \neq 1$ and, for each transitive action of M on a set Λ satisfying:

- 1. $|\Lambda| > 1$, and
- 2. every composition factor of M is isomorphic to some section of M^{Λ} , and
- 3. either $M_{(\Lambda)} = 1$ or, given $\lambda \in \Lambda$, the stabilizer M_{λ} has a normal subgroup N with $N \neq M_{(\Lambda)}$ and $N \cong M_{(\Lambda)}$, and
- 4. M is binary in its action on Λ ,

we have that d divides $|\Lambda|$. Then either d divides $|\Omega| - 1$ or G is not binary.

Test 5: special primes. We have turned Lemmas 1.6.15 and 1.6.16 into a routine in magma. Both of these lemmas are rather convenient from a computational point of view because they do not require us to construct the permutation representation of G on (G : M). For example, the only critical step in the routine for Lemmas 1.6.15 and Lemma 1.6.16 is the construction of the centraliser in G of an element g in M of prime order p. There is a stardard built-in command in magma for constructing centralizers. Most often than not, this command is sufficient for our computations. However, for very large groups, where it is computationally out of reach to use a general command for computing centralizers, we have constructed $C_G(g)$ with *ad hoc* methods exploiting the subgroup structure of the group G under consideration.

Test 6: *M* very small. This method draws on the following lemma.

Lemma 1.8.4 ([46, Lemma 2.5]). Let $\omega_0, \omega_1, \omega_2 \in \Omega$ with $G_{\omega_0} \cap G_{\omega_1} = 1$. Suppose there exists $g \in G_{\omega_0} \cap G_{\omega_2}G_{\omega_1}$ with $g \notin G_{\omega_2}$. Then the two triples $(\omega_0, \omega_1, \omega_2)$ and $(\omega_0, \omega_1, \omega_2^g)$ are 2-subtuple complete but are not 3-subtuple complete. In particular, G is not binary.

This method is particularly useful when M (G_{ω_0} in Lemma 1.8.4) is small compared to G because in this case it is more likely that $G_{\omega_0} \cap G_{\omega_1} = 1$, for some ω_1 . This method also has the benefit that it does not

require us to construct the permutation representation of G on (G : M), and that all the computations are performed locally. Since this method is designed to deal with the case that (G : M) is large compared to M, we do not exhaustively check all triples $\omega_0, \omega_1, \omega_2 \in (G : M)$. In practice, we let $\omega_0 := M$, we generate at random $g_1, g_2 \in G$, we let $G_{\omega_1} := M^{g_1}$ and $G_{\omega_2} := M^{g_2}$ and we check whether Lemma 1.8.4 applies to $\omega_0, \omega_1, \omega_2$. We repeat this routine 10^5 times and if at some point we find a triple satisfying Lemma 1.8.4, then G acting on (G : M) is not binary and we stop the routine. If, after the 10^5 trials, we have not found any triple satisfying Lemma 1.8.4, then we turn to a different method.

Chapter 2

Preliminary results for groups of Lie type

In this chapter we collect a number of results that will be needed when we come to prove Theorem 1.3. All of these results involve the finite groups of Lie type, so let us first establish the notation that we will use in this chapter and those that follow.

Our notation for the classical groups is standard and is consistent with, for instance, [54, Table 2.1.B]. We write, for example, $SO_n^+(q)$ to mean a group of special isometries associated with a +-type quadratic form on an *n*-dimensional vector space over the finite field \mathbb{F}_q having q elements, and we write $PSO_n^+(q)$ for the projective version of the same. We write $SO_n^{\pm}(q)$ or $SO_n^{\varepsilon}(q)$ if we wish to allow the quadratic form to have either + or - type.

We shall also use the general notation $\operatorname{Cl}_n(q)$ to denote a quasisimple classical group with natural module of dimension n over the field \mathbb{F}_q (over \mathbb{F}_{q^2} for unitary groups).

Our Lie notation is also standard: we write $A_n(q)$, $B_n(q)$, $C_n(q)$, and so on, for quasisimple groups of Lie type associated with Dynkin diagrams of type A_n, B_n, C_n, \ldots ; similarly we write ${}^{2}A_n(q)$, ${}^{2}B_2(q)$, and so on, for twisted versions of the same. Note that the Lie notation does not specify the group up to isomorphism in all cases. For instance, $A_n(q)$ can stand for both $SL_{n+1}(q)$ and $PSL_{n+1}(q)$.) We write $A_n^-(q), D_n^-(q)$ and $E_6^-(q)$ as alternative notation for ${}^{2}A_n(q), {}^{2}D_n(q)$ and ${}^{2}E_6(q)$ respectively, and we write $A_n^{\pm}(q), D_n^{\pm}(q), E_6^{\pm}(q)$ or $A_n^{\varepsilon}(q), D_n^{\varepsilon}(q), E_6^{\varepsilon}(q)$ if we wish to consider both the twisted and untwisted version at the same time.

The results collected here are of six kinds:

- 1. Results concerning alternating sections: We consider a simple group of Lie type, G, and we specify for which values of r the alternating group, Alt(r), is a section of G. These results will be used later, in conjunction with Definition 1.6.11, when we study the primitive actions of G one frequently-used method for showing that these actions are not binary will be to show that they exhibit a beautiful subset.
- 2. Stabilizer results: We consider a group G, and we consider all faithful transitive actions of G in which the stabilizer of a point, H, contains a particular element g. We will prove that, for an appropriately chosen G and g, such an action is always not binary. We call these "stabilizer results" because these lemmas will typically be applied in later chapters in contexts where G is a point-stabilizer and we are seeking to use Lemma 1.6.1. These applications motivate the choices of G which we consider in this section.
- 3. Odd degree results: We consider a group M, normally a small group of Lie type, and we use magma to show that all of the transitive actions of odd degree of M are not binary. Although it is not about groups of Lie type, we also include one result Lemma 2.3.2 which does the same thing for the sporadic groups.
- 4. Centralizer results: We will present a number of results giving lower bounds for the size of a centralizer of a non-trivial element in a simple group of Lie type.

- 5. Automorphism results: We present a well-known result classifying the outer automorphisms of prime of rder of finite groups of Lie type.
- 6. Fusion and factorization results: All these results will be used in conjunction with Lemma 1.6.10 to prove the existence of beautiful subsets (Definition 1.6.11).

We will use the stabilizer results in two ways when it comes to the proof of Theorem 1.3. For the proof we study an almost simple group G acting on the cosets of a maximal subgroup M. Now, the first use of our stabilizer results is direct: if M contains the element g, then we immediately know that the action is not binary and we are done.

The second use is slightly less direct. In this case, we wish to apply our stabilizer results to the group M, rather than the group G: so we pick a distinguished element $g \in M$ and appeal to our stabilizer results to assert that if H is any core-free subgroup of M that contains g, then the action of M on (M : H) is not binary. Next we use our centralizer results, to show that, in general $|C_M(g)|$ is smaller than the smallest centralizer in G. We conclude that there exists $x \in C_G(g) \setminus C_M(g)$. Now $M \cap M^x$ is a core-free subgroup of M that contains g. We conclude that the action of M on $(M : M \cap M^x)$ is not binary. Then Lemma 1.6.1 implies that the action of G on (G : M) is not binary.

This second method explains the selection of groups under consideration for our stabilizer results: for instance the group G appearing in Lemma 2.2.1 is studied because such a group is maximal in $E_8(q)$.

The second method also applies to the odd degree results: if we are studying the action of a group G on the cosets of a subgroup M and we know (a) that |G:M| is even, (b) that all odd-degree actions of M are not binary, then Lemma 1.6.1 implies that the action of G on (G:M) is not binary.

2.1 Results on alternating sections

Let G be a simple group of Lie type. We wish to know for which values of r the alternating group, Alt(r), is a section of G.

We first consider classical groups.

Lemma 2.1.1. Let $Cl_n(q)$ be a simple classical group with natural module of dimension n and p is a prime number. If $Cl_n(q)$ has a section isomorphic to the alternating group Alt(r), then

$$n \ge R_p(\operatorname{Alt}(r)),\tag{2.1.1}$$

where $R_p(Alt(r))$ denotes the smallest dimension of a non-trivial projective representation of Alt(r) over a field of characteristic p. In particular, for $r \ge 9$, we have

$$R_p(\operatorname{Alt}(r)) = r - 1 - \delta,$$

where

$$\delta = \begin{cases} 1, & if \ p \mid r, \\ 0, & otherwise. \end{cases}$$

For $5 \le r \le 8$, the values for $R_p(Alt(r))$ are as in Table 2.1.1.

[r	$R_2(\operatorname{Alt}(r))$	$R_3(\operatorname{Alt}(r))$	$R_5(\operatorname{Alt}(r))$	$R_p(\operatorname{Alt}(r)), p \ge 7$
	5	2	2	2	2
	6	3	2	3	3
	7	4	4	3	4
	8	4	7	7	7

Table 2.1.1: Values for $R_p(Alt(r))$, with $5 \le r \le 8$

Proof. The inequality $R_p(Alt(r)) = r - 1 - \delta$ follows from [40, Proposition 4.1]. The values of $R_p(Alt(r))$ are well-known (see [54, Proposition 5.3.7]).

If G is exceptional, then the following lemma gives the result that we need. (Here, $\delta_{x,y}$ is the usual Kronecker delta.)

Lemma 2.1.2. Let G = G(q) be a finite simple group of exceptional Lie type as in the table below, where $q = p^a$, $a \ge 1$ and p is a prime number. If Alt(r) is a section of G, then $r \le N_G$, where N_G is as in the table below.

$$\frac{G}{N_G} \begin{bmatrix} E_8(q) & E_7(q) & E_6^{\epsilon}(q) & F_4(q) & G_2(q), {}^{3}\!D_4(q) & {}^{2}\!F_4(q) \\ \hline N_G & 17 + \delta_{p,3} & 13 + \delta_{p,7} & 11 + \delta_{p,2} + \delta_{p,5} & 10 + \delta_{p,11} & 6 + \delta_{p,5} & 6 \end{bmatrix}$$

Proof. Fix $r \ge 5$, and let $K \triangleleft H \le G$ with $H/K \cong \operatorname{Alt}(r)$, and |H| minimal. Choose a minimal subfield $\mathbb{F}_{q_0} \subseteq \mathbb{F}_q$ such that $H \le G(q_0)$, and a maximal subgroup M of $G(q_0)$ such that $H \le M$.

Consider first $G(q_0) = {}^{2}F_4(q_0)$. The maximal subgroups are given by [83], from which it follows that Alt(r) is a section of one of the groups $Sp_4(q_0)$ or $PSU_3(q_0)$. Hence by Lemma 2.1.1 we have $4 \ge R_2(Alt(r))$, forcing $r \le 8$. As Alt(7) is not a section of $Sp_4(q_0)$ or $PSU_3(q_0)$ (see, for instance, [10]), we in fact have $r \le 6$, as in the conclusion.

The cases where $G(q_0) = G_2(q_0)$ or ${}^{3}D_4(q_0)$ are dealt with similarly, using [29, 56, 57] for the lists of maximal subgroups.

Now consider the remaining cases, where G is of type E_8 , E_7 , E_6^{ϵ} or F_4 . By the minimality of H we have $K \leq \Phi(H)$, where $\Phi(H)$ is the Frattini subgroup of H. So, K is nilpotent.

Suppose $Z(H) \neq 1$, and let $1 \neq x \in Z(H)$. Then $H \leq C_G(x)$, which is contained in a parabolic or a subsystem subgroup, and it follows that Alt(r) is a section of one of the following subgroups of G:

G	$\operatorname{Alt}(r)$ section of one of
$F_4(q)$	$B_4(q), C_3(q)$
$E_6^{\epsilon}(q)$	$A_5^\epsilon(q), D_5^\epsilon(q)$
$E_7(q)$	$A_7^{\pm}(q), D_6(q), E_6^{\pm}(q)$
$E_8(q)$	$D_8(q), A_8^{\pm}(q), E_7(q)$

Working down from $F_4(q)$, the bounds in the conclusion now follow using Lemma 2.1.1. (Note that the possibilities r = 18 in E_8 (p = 2) and r = 14 in E_7 (p = 2) are excluded by the fact that $D_8(2^a)$ (resp. $D_6(2^a)$) does not have a section isomorphic to Alt(18) (resp. Alt(14)) (see [54, (5.3.8)]).

Suppose finally that Z(H) = 1. If Z(K) = 1, then K = 1 (as K is nilpotent), so $H \cong \operatorname{Alt}(r)$, and the conclusion follows from [77, Table 10.1]. So assume $Z(K) \neq 1$. If p divides |Z(K)|, then H is contained in a parabolic subgroup of G by [8], a case already considered above. Hence we may assume that Z(K) has order divisible by a prime s with $s \neq p$. As Z(H) = 1, it must be the case that $H/K \cong \operatorname{Alt}(r)$ acts non-trivially on the elementary abelian group $E = \Omega_1(O_s(Z(K)))$. Say $E \cong (C_s)^{\kappa}$, of rank κ . Then $\kappa \geq R_s(\operatorname{Alt}(r))$. On the other hand, [27] shows that $\kappa \leq R + 1$, where R is the untwisted Lie rank of G. Hence

$$R_s(\operatorname{Alt}(r)) \le R + 1$$

and the bounds for r in the conclusion follow from this. This completes the proof.

2.2 Stabilizers containing certain elements

In this section we prove results that are (more or less) of the following kind: we suppose that x is an element of a group G, and we prove that, if H is any core-free subgroup of G containing x, then the action of G on (G : H) is not binary. In the first subsection we consider groups G of a variety of isomorphism types; in subsequent subsections, G will always be almost simple.

2.2.1 Some groups that are not almost simple

Lemma 2.2.1. Let $S = PGL_2(q) \times Sym(5)$ with q > 5, and suppose that $S \leq G$ with G/S solvable. Let L be the normal subgroup in S that is isomorphic to $PGL_2(q)$, and suppose that $g \in L$ has order q - 1; let M be a subgroup of G that contains g. If the action of G on (G:M) is binary, then M contains L.

Proof. Assume that the action of G on (G:M) is binary. Notice that the element g normalizes, and acts fixed-point-freely by conjugation upon two unipotent subgroups of L of order q; we call these U_1 and U_2 .

Suppose that M does not contain U_1 . Since $g \in M$, we have $M \cap U_1 = \{1\}$. Now, we define $\Lambda = \{Mu \mid u \in U_1\}$. It is easy to see that $U_1 \rtimes \langle g \rangle$ acts 2-transitively on Λ , which is a subset of (G : M) of size q. Since G is binary on (G : M), the group G^{Λ} is isomorphic to the symmetric group of degree q. As q > 5 and G/S is solvable, by Lemma 2.1.1, G has no section isomorphic to Alt(q), which is a contradiction.

Thus M contains U_1 and, by the same reasoning, U_2 . But now $\langle U_1, U_2, g \rangle = L$ and we are done.

Lemma 2.2.2. Let $S = PSL_2(q) \times PSL_2(q)$ with $q \ge 4$ and $q \ne 5$, and suppose that $S = F^*(G)$, where $F^*(G)$ is the generalized Fitting subgroup of G. Let L be a subgroup of S isomorphic to $D_{t(q-1)} \times D_{t(q-1)}$ (where t = (2, q) and where $D_{t(q-1)}$ denotes the dihedral group of order t(q-1)), and let M be a subgroup of G that contains L. If the action of G on (G : M) is binary, then $M \ge S$.

Proof. We write $S = S_1 \times S_2$ and $L = L_1 \times L_2$, where $D_{t(q-1)} \cong L_i < S_i \cong PSL_2(q)$ for $i \in \{1, 2\}$. Assume first that $q \notin \{4, 7, 9, 11\}$.

Suppose, first, that $M \cap S = L$; we must show that the action of G on (G : M) is not binary. Let $H = \langle M, S \rangle = MS$. Lemma 1.6.2 implies that it is sufficient to show that the action of H on (H : M) is not binary. Now observe that

$$H/S = MS/S \cong M/(M \cap S) = M/L.$$

Thus |H:M| = |S:L| and we can identify (H:M) with the set of conjugates of L in S, by using the map

$$(H:M) \to \{L^s \mid s \in S\}, \quad Mg \mapsto L^g.$$

Now define

$$\Gamma = \{ L_1 \times L_2^g \mid g \in S_2 \}.$$

The intersection of the elements of Γ is L_1 and so $H_{\Gamma} \leq N_H(L_1)$. Since the reverse inclusion is also true, we deduce

$$H_{\Gamma} = N_H(L_1). \tag{2.2.1}$$

Observe that the action of H_{Γ} on Γ is isomorphic to the action of an almost simple group with socle $S_2 = \text{PSL}_2(q)$ on the cosets of a subgroup M_2 for which $M_2 \cap S_2 \cong D_{t(q-1)}$. When $q \notin \{4,7,9,11\}$, the action of H_{Γ} on Γ is primitive by [10, Table 8.1] and hence the main theorem of [34] implies that this action is not binary. Thus there is an integer $k \geq 3$ and two k-tuples $I, J \in \Gamma^k$ that are 2-subtuple complete but not k-subtuple complete with respect to the action of H^{Γ} . Using (2.2.1), one can see that any $h \in H$ for which $I^h = J$ must satisfy $h \in H_{\Gamma}$, and so I, J are not k-subtuple complete with respect to the action of H. Thus the action of H on (H : M) is not binary, and so the action of G on (G : M) is not binary, as required.

We conclude that L is a proper subgroup of $M \cap S$. We may assume, without loss of generality, that $M \cap S$ contains S_1 but not S_2 . Then the action of G on (G : M), modulo the kernel, is isomorphic to the action of an almost simple group with socle $S_2 = \text{PSL}_2(q)$ on the cosets of a maximal subgroup M_2 for which $M_2 \cap S_2 \cong D_{t(q-1)}$. Once again the main theorem of [34] implies that this action is not binary.

The only remaining possibility is that $M \ge S$, as required.

Assume now that $q \in \{4, 7, 9, 11\}$. With the help of magma, we have constructed all the groups G with $F^*(G) = S \cong PSL_2(q) \times PSL_2(q)$ and all the subgroups H of G containing $D_{t(q-1)} \times D_{t(q-1)}$. Then we have verified, by witnessing non-binary triples, that the action of G on (G : H) is binary only when $S \leq H$. \Box

2.2.2 Classical groups

Lemma 2.2.3. Let G be almost simple with socle $S = PSL_2(q)$, and let x be the projective image of an element \tilde{x} as given in Table 2.2.1 line 1. Let M < G be core-free with $x \in M \cap S$. Then, provided $q \notin \{4, 5\}$, the action of G on (G : M) is not binary.

Proof. For $q \in \{7, 9, 11, 13, 27\}$, we confirm the result using magma. In particular, for the rest of the proof we suppose q = 8 or q > 13 with $q \neq 27$.

Let d = (2, q - 1). The stated conditions imply that $M \cap S \in \{C_{(q-1)/d}, D_{2(q-1)/d}, B\}$, where B is a Borel subgroup of S. We set $q = p^f$, for a prime p and positive integer f, and we consider the three cases separately.

CASE 1: Suppose that $M \cap S = C_{(q-1)/d}$. In particular, $T := M \cap S$ is a split torus in $\text{PSL}_2(q)$. Since distinct split tori in $\text{PGL}_2(q)$ intersect trivially, we conclude $|G_{\alpha,\beta}| \leq f$. On the other hand, let $B = U \rtimes T$, a Borel subgroup of S, and observe that B acts as a Frobenius group on the set $\Lambda = \{Mu \mid u \in U\} \subset (G : M)$. Clearly $|\Lambda| = q$; if p = 2, then d = 1, Λ is a beautiful subset and we are done. Suppose, then, that q is odd; Lemma 1.7.3 implies that, if G is binary, then

$$\left[\frac{(\frac{q-1}{2} - 1)(\frac{q-1}{2} - 2)}{q}\right] < f$$

This implies $\lceil q/4 - 2 - 15/q \rceil < f$. It is easy to verify that, when q > 13,

$$\lceil q/4 - 2 - 15/(4q) \rceil = \lceil q/4 - 2 \rceil \ge (q - 7)/4$$

and hence, in particular, q - 7 < 4f. However, this inequality is never satisfied when q > 13.

CASE 2: Suppose that $M \cap S = D_{2(q-1)/d}$. The analysis of the previous case still applies: for q even, we obtain a beautiful subset again and are done; for q odd, we proceed as before except that this time $|G_{\alpha,\beta}| \leq 2f$, which implies that q-7 < 8f. However, for q > 13, this inequality is satisfied only when q = 27, but we are excluding this case here.

CASE 3: Suppose that $M \cap S = B$. Let $K = \langle M, S \rangle$. Then (K : M) is a set of size q + 1 that is stabilized by K and on which K acts 2-transitively. By Lemma 2.1.1, any alternating section, Alt(r), of $P\Gamma L_2(q)$ has $r \leq 6$, hence (K : M) is a beautiful subset of (G : M), and we are done.

We shall also need the following variant of Lemma 2.2.3.

Lemma 2.2.4. Let G be almost simple with socle $S = PSL_2(q^2)$, and let x be the projective image of the diagonal matrix diag (a, a^{-1}) , where $a \in \mathbb{F}_{q^2}$ has order (q - 1, 2)(q - 1). Let M < G be core-free with $x \in M \cap S$. Then, provided $q \geq 7$, the action of G on (G : M) is not binary.

Proof. For $\lambda \in \mathbb{F}_{q^2}$, define subgroups U_{λ}^{\pm} of S by

$$U_{\lambda}^+ = \{I + \lambda t E_{12} : t \in \mathbb{F}_q\}, \quad U_{\lambda}^- = \{I + \lambda t E_{21} : t \in \mathbb{F}_q\},$$

where as usual E_{ij} denotes the matrix with ij-entry 1 and 0 elsewhere. Then $T = \langle x \rangle$ normalizes U_{λ}^{\pm} and acts transitively on $U_{\lambda}^{\pm} \setminus \{1\}$. Since $\langle U_{\lambda}^{\pm} : \lambda \in \mathbb{F}_{q^2} \rangle = S$, there exists λ and $\epsilon = \pm$ such that $U = U_{\lambda}^{\epsilon} \not\leq M$. Then UT acts 2-transitively on the set $\Lambda = \{Mu \mid u \in U\} \subset (G : M)$ of size q, and since Alt(q) is not a section of G for $q \geq 7$, it follows that the action of G on (G : M) is not binary for $q \geq 7$.

Lemma 2.2.5. Let G contain a subgroup $S \cong SL_n(q)/Z$, where Z is a central subgroup of $SL_n(q)$, and such that $n \ge 3$. Let $x \in S$ be the projection in S of an element $\tilde{x} \in SL_n(q)$ as given in Table 2.2.1 lines 2 and 3. Let M < G with $x \in M$. Then one of the following holds:

1. G contains a section isomorphic to $\operatorname{Sym}(q^{n-2})$ (if q > 2) or $\operatorname{Sym}(2^{n-1})$ (if q = 2);

Line	S/Z(S)	\widetilde{x}	Conditions
1	$\mathrm{PSL}_2(q)$	$\begin{pmatrix} a \\ & a^{-1} \end{pmatrix}$	a of order $q-1$
2	$\begin{aligned} \operatorname{PSL}_n(q) \\ n \geq 3 \\ q \geq 3 \end{aligned}$	$ \begin{pmatrix} 1 & & \\ & A & \\ & & a^{-1} \end{pmatrix} $	$A \in \operatorname{GL}_{n-2}(q)$ $A \text{ of order } q^{n-2} - 1$ $\det(A) = a \in \mathbb{F}_q$
3	$\begin{aligned} \operatorname{SL}_n(2) \\ n \ge 3 \end{aligned}$	$\begin{pmatrix} 1 \\ & A \end{pmatrix}$	$A \in \operatorname{GL}_{n-1}(2)$ A of order $2^{n-1} - 1$

Table 2.2.1: Auxiliary table for Lemma 2.2.5

- 2. M contains S;
- 3. the action of G on (G:M) is not binary.

Proof. We assume that none of the three possibilities hold, and we reach a contradiction. In particular, the action of G on (G: M) is binary. Since $S \cong SL_n(q)/Z$, there exists a surjective group homomorphism $\pi: SL_n(q) \to S$.

CASE 1: q > 2. We observe first that $\langle \tilde{x} \rangle$ normalizes two distinct elementary abelian subgroups of $SL_n(q)$ of order q^{n-2} , namely those having shape

$$U_{1} = \left\{ \begin{pmatrix} 1 & u_{1} & \cdots & u_{n-2} & 0 \\ 1 & & & \\ & \ddots & & \\ & & & 1 \\ & & & & 1 \end{pmatrix} \mid u_{1}, \dots, u_{n-2} \in \mathbb{F}_{q} \right\}, U_{2} = \left\{ \begin{pmatrix} 1 & & & \\ u_{1} & 1 & & \\ \vdots & \ddots & & \\ u_{n-2} & & 1 & \\ 0 & & & 1 \end{pmatrix} \mid u_{1}, \dots, u_{n-2} \in \mathbb{F}_{q} \right\}.$$

Observe that $\langle \tilde{x} \rangle$ acts by conjugation fixed-point-freely on each of these two groups. Let us suppose that $\pi(U_1) \leq M$. As $\pi(\tilde{x}) = x \in M$, the fixed-point-freeness of the action yields $\pi(U_1) \cap M = \{1\}$.

Now let Λ be the set of cosets of M corresponding to $M\pi(U_1)$, that is, $\Lambda = \{Mh \mid h \in \pi(U_1)\}$. Then Λ is a set of size q^{n-2} on which the group $M_1 = \pi(U_1 \rtimes \langle \tilde{x} \rangle)$ acts 2-transitively. Since we are assuming that G on (G:M) is binary, Λ is not a beautiful subset. Therefore, $G^{\Lambda} \geq \text{Sym}(q^{n-2})$; however this contradicts the fact that we are assuming that G has no section isomorphic to $\text{Sym}(q^{n-2})$. Thus $\pi(U_1) \leq M$.

A similar argument applies to U_2 . Thus $\pi(U_2) \leq M$ and hence $\langle \pi(U_1), \pi(U_2), x \rangle \leq M$. Observe that

$$\left\langle \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \mid u \in \mathbb{F}_q \right\rangle = \mathrm{SL}_2(q).$$

Now, an easy inductive argument on n shows that

$$\langle U_1, U_2 \rangle = \left\{ \begin{pmatrix} Z & 0\\ 0 & 1 \end{pmatrix} \mid Z \in \mathrm{SL}_{n-1}(q) \right\}$$

and hence, from the definition of \tilde{x} , we obtain that $\langle U_1, U_2, \tilde{x} \rangle$ contains all matrices of the form

$$\begin{pmatrix} Z \\ & z^{-1} \end{pmatrix},$$

where $Z \in \operatorname{GL}_{n-1}(q)$ has determinant $z \in \mathbb{F}_q$. But now we define two elementary abelian subgroups of $\operatorname{SL}_n(q)$ of order q^{n-1} , namely those having shape

$$U_{3} = \left\{ \begin{pmatrix} 1 & & u_{1} \\ & \ddots & & \vdots \\ & & \ddots & u_{n-1} \\ & & & 1 \end{pmatrix} \mid u_{1}, \dots, u_{n-1} \in \mathbb{F}_{q} \right\}, U_{4} = \left\{ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ u_{1} & \cdots & u_{n-1} & 1 \end{pmatrix} \mid u_{1}, \dots, u_{n-1} \in \mathbb{F}_{q} \right\}.$$

Repeating the same argument as before, with U_1 and U_2 replaced by U_3 and U_4 , we obtain that M contains $\pi(U_3)$ and $\pi(U_4)$. But then $M \ge \langle \pi(U_1), \pi(U_2), \pi(U_3), \pi(U_4), x \rangle = S$, a contradiction, and we are done.

CASE 2: q = 2. Clearly, in this case, Z = 1 and we may think of π as the identity mapping. We define two elementary abelian subgroups of $SL_n(2)$ of order 2^{n-1} , namely those having shape

$$U_{1} = \left\{ \begin{pmatrix} 1 & & u_{1} \\ & \ddots & & \vdots \\ & & \ddots & u_{n-1} \\ & & & 1 \end{pmatrix} \mid u_{1}, \dots, u_{n-1} \in \mathbb{F}_{q} \right\}, U_{2} = \left\{ \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ u_{1} & \cdots & u_{n-1} & 1 \end{pmatrix} \mid u_{1}, \dots, u_{n-1} \in \mathbb{F}_{q} \right\}.$$

We suppose that $U_1 \not\leq M$. As in the previous proof we use the fact that $\langle x \rangle$ normalizes, and acts fixedpoint-freely on, U_1 . As before, either G contains a section isomorphic to $\operatorname{Sym}(2^{n-1})$ (but this contradicts our hypothesis) or else we obtain a beautiful subset (but this contradicts again our hypothesis). Hence Mcontains U_1 and, by the same argument, U_2 . Since $\langle U_1, U_2 \rangle = \operatorname{SL}_n(2)$, we obtain a contradiction and are done.

Applying Lemma 2.2.5 to the case where G is almost simple, and applying [54, Proposition 5.3.7] to establish when G may contain the relevant alternating section, we obtain the following result.

Lemma 2.2.6. Let G be almost simple with socle $S = PSL_n(q)$, and let x be the projective image of an element \tilde{x} as given in Table 2.2.1. Let M < G be core-free with $x \in M \cap S$. Then, provided $(n,q) \notin \{(2,4), (2,5)\}$ and $(G,M) \notin \{(Sym(8), Alt(7)), (Sym(8), Sym(7))\}$, the action of G on (G : M) is not binary.

Moreover, if $(n,q) \notin \{(2,3), (3,3)\}$, then $|C_S(x)| < q^n$.

Proof. Since M is core-free in G, we have $S \not\leq M$. When n = 2, the proof follows from Lemma 2.2.3. When $n \geq 3$, from Lemma 2.2.5, if the action of G on (G : M) is binary, then G contains a section isomorphic to $\text{Sym}(q^{n-2})$ (if q > 2) or $\text{Sym}(2^{n-1})$ (if q = 2). From this it follows from Lemma 2.1.1 that $(n,q) \in \{(3,3), (3,4), (3,5), (3,2), (4,2)\}$. For these values of (n,q), we have constructed all the permutation representations under consideration and we have checked that none is binary unless (n,q) = (4,2) and (G,M) is one for the cases listed in the statement.

When \tilde{x} is as in Line 1 of Table 2.2.1, $\langle x \rangle$ is a torus in $\text{PSL}_2(q)$ of cardinality (q-1)/2 when q is odd, and q-1 when q is even. Thus $|C_S(x)| \leq q-1 < q$, except when q=3. Similarly, using the fact that, if $A \in \text{GL}_k(q)$ has order $q^k - 1$ (that is, $\langle A \rangle$ is a Singer cycle), then $C_{\text{GL}_k(q)}(A) = \langle A \rangle$, we deduce that $|C_S(x)| = (q^{n-1}-1)(q-1)/(n,q-1) < q^n - 1$ when \tilde{x} is as in Line 2 of Table 2.2.1 and $q \neq 3$, and $|C_S(x)| = 2^{n-1} - 1 < 2^n$ when \tilde{x} is as in Line 3 of Table 2.2.1.

The fact that $|C_S(x)| < q^n$ will be important later on – in Lemma 2.2.5, and in the results that follow, we have tried to pick distinguished elements $x \in S$ for which $C_S(x)$ is relatively small.

For groups with socle $PSL_4(q)$ we shall also need the following special result.

Lemma 2.2.7. Let G be almost simple with socle $S = PSL_4(q)$, and let x be the projective image of a diagonal matrix $\tilde{x} = \text{diag}(1, 1, a, a^{-1})$, where $a \in \mathbb{F}_q$ has order q-1. Let M < G be core-free with $x \in M \cap S$. Then, provided $q \geq 8$, the action of G on (G : M) is not binary.

Proof. The proof is very similar to that of Lemma 2.2.5. Let $T = \langle x \rangle$, and for $i \neq j$ define $U_{ij} = \{I + \alpha E_{ij} : \alpha \in \mathbb{F}_q\}$, where E_{ij} denotes the matrix with ij-entry 1 and 0 elsewhere. Then T acts fixed-point-freely on the groups U_{ij} for $i \in \{1, 2\}, j \in \{3, 4\}$ or vice versa. Since these subgroups U_{ij} generate S, at least one of them is not contained in M. Hence we obtain a subset Δ of size q on which G_{Δ} acts 2-transitively. If Δ is a beautiful subset then (G, (G : M)) is not binary. So suppose Δ is not beautiful. Then Alt(q) is a section of S, and moreover Alt(q-1) is a section of M. By Lemma 2.1.1, Alt(q) is a section of S only if $q \leq 8$; moreover, if q = 8, then Alt(7) can only be a section of a maximal core-free subgroup M of G if M

is a subfield subgroup of type $PSL_4(2)$ (see [10, Tables 8.8, 8.9]) – but such a subgroup does not contain the element x. Hence if $q \ge 8$ we have a contradiction, and the proof is complete.

We now need to prove an analogue of Lemma 2.2.5 for the other classical groups, albeit subject to some conditions (including lower bounds on n). Some of the situations excluded by these conditions are studied in subsequent lemmas. In the statement and proof of the lemma, if S is orthogonal or symplectic, we set $\mathbb{K} = \mathbb{F}_q$; if S is unitary, then we set $\mathbb{K} = \mathbb{F}_{q^2}$. In either case, for a scalar $a \in \mathbb{K}$ we define $\overline{a} := a^q$; for a matrix $A = (a_{ij})_{i,j} \in \operatorname{GL}_d(\mathbb{K})$ we write \overline{A} for the matrix $(\overline{a}_{ij})_{i,j}$.

Lemma 2.2.8. Suppose that one of the following holds:

- 1. G contains a subgroup $S \cong SU_n(q)/Z$ where Z is a central subgroup of $SU_n(q)$ and $n \ge 5$;
- 2. G contains a subgroup $S \cong \operatorname{Sp}_n(q)/Z$ where Z is a central subgroup of $\operatorname{Sp}_n(q)$ and $n \ge 4$;
- 3. G contains a subgroup $S \cong \Omega_n^{\varepsilon}(q)$, q is even and $n \ge 8$;
- 4. G contains a subgroup $S \cong SO_n^{\varepsilon}(q)/Z$ where Z is a central subgroup of $SO_n^{\varepsilon}(q)$, q is odd and $n \ge 7$.

Let k be the Witt index of the associated formed space. If $S \neq SU_n(q)/Z$ with n even, then we define j = k, otherwise j = k - 1. We let $\mathcal{B} = \{e_1, \ldots, e_j, f_1, \ldots, f_j\} \cup Y$ be a hyperbolic basis; thus Y is a set of linearly independent anisotropic vectors if $S \neq SU_{2j+2}(q)/Z$, otherwise $Y = \{e_{j+1}, f_{j+1}\}$. We set $y = |Y| \in \{0, 1, 2\}$.

Let M < G with $x \in M$, where x is the projective image in S of

$$\tilde{x} = \begin{pmatrix} 1 & & & \\ & A & & \\ & & 1 & \\ & & & \overline{A^{-T}} & \\ & & & & J_y \end{pmatrix},$$

written with respect to \mathcal{B} , $A \in \operatorname{GL}_{j-1}(\mathbb{K})$ is of order $|\mathbb{K}|^{j-1} - 1$, and J_y is some y-by-y matrix. If S is not unitary, then we can take J_y to be the identity matrix; if S is unitary, then J_y is a matrix such that $\det(\tilde{x}) = 1$ (thereby ensuring that $x \in S$).

Then one of the following holds:

- 1. G contains a section isomorphic to $\text{Sym}(|\mathbb{K}|^{j-1})$;
- 2. M contains S;
- 3. the action of G on (G:M) is not binary.

In particular if G is almost simple, M is core-free and the action of G on (G: M) is binary, then S is symplectic and one of the following holds:

- 1. (k,q) = (2,2), or
- 2. (k,q) = (2,3) and $M = \langle x \rangle$.

Note that if S is orthogonal and q is even, then [12, Lemmas 2.5.7 and 2.5.9] imply that \tilde{x} lies in $\Omega_n^{\varepsilon}(q)$.

Proof. We suppose throughout that the action of G on (G : M) is binary. Our argument is the same for all families, more or less, but the details are different; we will, therefore, need to do some case work – especially in the third stage of the proof.

STEP 1. We observe first that $\langle x \rangle$ normalizes U_1 and U_2 , two distinct elementary-abelian subgroups of S of order $|\mathbb{K}|^{j-1}$, namely those having shape

$$\begin{pmatrix} 1 & u_1 & \cdots & u_{j-1} & & & \\ & 1 & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & -\overline{u_1} & 1 & \\ & & & & \vdots & \ddots & \\ & & & & -\overline{u_{j-1}} & & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & & & & & \\ u_1 & 1 & & & & \\ \vdots & \ddots & & & & \\ u_{j-1} & & 1 & & \\ & & & & 1 & -\overline{u_1} & \cdots & -\overline{u_{j-1}} \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

respectively. In each case we write only the first 2j rows and columns of each matrix – the remaining rows and columns are completed by setting off-diagonal entries to be 0, and diagonal entries to be 1. The resulting group is the set of all matrices obtained by allowing the parameters u_i to range over K.

Observe that $\langle x \rangle$ acts fixed-point-freely on each of these two groups. Let us suppose that $U_1 \not\leq M$; then the fixed-point-freeness of the action means that $U_1 \cap M = \{1\}$. Now let Λ be the set of cosets $\{Mu \mid u \in U_1\}$ of M corresponding to MU_1 . Then this is a set of size $|\mathbb{K}|^{j-1}$ on which the group $M_1 = U_1 \rtimes \langle x \rangle$ acts 2-transitively. Now Lemma 1.6.8 implies that G contains a section isomorphic to $\operatorname{Sym}(|\mathbb{K}|^{j-1})$ and the result follows. The same argument works with U_2 so we may assume hereafter that M contains $\langle U_1, U_2 \rangle$.

Observe that

$$\left\langle \begin{pmatrix} 1 & u & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\bar{u} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & 1 & 0 & 0 \\ 0 & 0 & 1 & -\bar{u} \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid u \in \mathbb{F}_q \right\rangle = \left\{ \begin{pmatrix} Z & 0 \\ 0 & \bar{Z}^{-T} \end{pmatrix} \mid Z \in \mathrm{SL}_2(q) \right\}.$$

Now, an easy inductive argument on j shows that

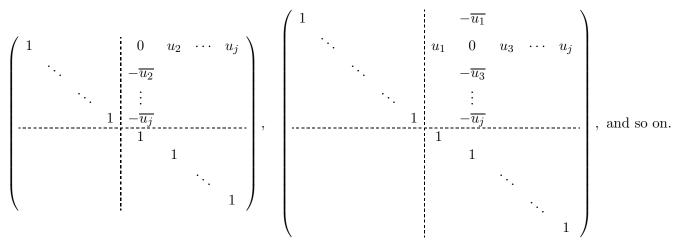
$$\langle U_1, U_2 \rangle = \left\{ \begin{pmatrix} Z & 0\\ 0 & \bar{Z}^{-T} \end{pmatrix} \mid Z \in \mathrm{SL}_j(q) \right\}$$

and hence, from the definition of \tilde{x} , $K = \langle U_1, U_2, x \rangle$ contains all matrices of the form

$$\begin{pmatrix} Z & \\ & \overline{Z}^{-T} \end{pmatrix},$$

where $Z \in \operatorname{GL}_j(\mathbb{K})$.

STEP 2A. Next we define U_3, \ldots, U_{j+2} , j elementary-abelian subgroups of S of order q^{j-1} , namely those having shape



Note, first, that we have placed dotted lines to mark the point where the "e-vectors" change to "f-vectors"; note, second, that we have omitted columns and rows corresponding to basis elements from Y; note, third, that in the case where S is symplectic the given matrices do not lie in S – but this is fixed by removing all minus signs, and proceeding in the same way.

It is easy enough to see that $\langle x \rangle$ normalizes, and acts fixed-point freely on U_3 . Similarly, K contains a conjugate of $\langle x \rangle$ that acts fixed-point-freely on U_4 , and so on. By the same argument as before, we have two possibilities:

- (a) M contains U_3 ;
- (b) there is a set $\Lambda \subset \Omega$ such that $|\Lambda| = |\mathbb{K}|^{j-1}$ and on which S^{Λ} acts 2-transitively; then Lemma 1.6.8 implies that G contains a section isomorphic to $\operatorname{Sym}(|\mathbb{K}|^{j-1})$ and the result follows.

Thus, again, we may assume that M contains U_3 .

Since the same argument works for U_4, \ldots, U_{j+2} , we conclude that the group M must contain the group W_1 , consisting of all matrices of the form

$$\begin{pmatrix} I & Z \\ & I \\ & & I_y \end{pmatrix}, \tag{2.2.2}$$

where Z is a *j*-by-*j* matrix satisfying $Z = -\overline{Z}^T$ and having zero diagonal entries (or, in the case where S is symplectic, Z satisfies $Z = Z^T$ and has zero diagonal entries).

STEP 2B. Now we repeat the argument of Step 2a but this time, all the matrices we use are the transposes of those in Step 2a. We conclude that M must contain the group W_2 , consisting of all matrices of the form

$$\begin{pmatrix} I & & \\ Z & I & \\ & & I_y \end{pmatrix}, \tag{2.2.3}$$

where Z is a *j*-by-*j* matrix satisfying $Z = -\overline{Z}^T$ and having zero diagonal entries (or, in the case where S is symplectic, Z satisfies $Z = Z^T$ and has zero diagonal entries).

STEP 3. We use the fact that M contains the group $\langle K, W_1, W_2 \rangle$ and we split into cases, depending on the particular family of classical groups which we are dealing with.

CASE 3A: S IS UNITARY. In this case an easy argument says that, since M contains the group K, the group M contains all matrices of the form (2.2.2), where Z is a *j*-by-*j* matrix satisfying $Z = -\overline{Z}^T$, i.e. we can drop the requirement that Z has zero diagonal entries. The resulting set of matrices forms an elementary-abelian group U of size q^{j^2} which is the unipotent radical of a parabolic subgroup P_j in $SU_{2j}(q)$.

The same argument works "with transposes" and we obtain that M contains all matrices of the form (2.2.3), where Z is a *j*-by-*j* matrix satisfying $Z = -\overline{Z}^T$. We split into two cases, depending on the parity of n.

Assume, first, that n = 2k + 1 with $k \ge 2$. Then M contains the projective image of $M_0 \cong SU_{2k}(q)$, where M_0 stabilizes the unique non-isotropic basis vector, v, in Y. Without loss of generality, we may suppose that v has norm 1.

Let $\alpha_2, \ldots, \alpha_k \in \mathbb{K}$ and for simplicity set $\alpha_1 := 0$. For each $i \in \{1, \ldots, k\}$, let $\beta_{i,i} \in \mathbb{K}$ with $\beta_{i,i} + \overline{\beta_{i,i}} + \alpha_i \overline{\alpha_i} = 0$. (Observe that the existence of $\beta_{i,i}$ is guaranteed by Hilbert's Theorem 90.) For each $i, j \in \{1, \ldots, k\}$ with $i \neq j$, let $\beta_{i,j} = 0$ when i > j and $\beta_{i,j} = -\overline{\alpha_i}\alpha_j$ when i < j. Now, let $g \in S$ be the element fixing e_1, \ldots, e_k pointwise and which satisfies

$$v \mapsto v + \alpha_2 e_2 + \dots + \alpha_k e_k,$$

 $f_i \mapsto f_i - \overline{\alpha_i} v + \sum_{j=1}^k \beta_{i,j} e_j.$

In particular, the matrix form of g with respect to the basis $(e_1, \ldots, e_k, f_1, \ldots, f_k, v)$ is

$$\begin{pmatrix} I & 0 & 0 \\ B & I & -\overline{d} \\ d^T & 0^T & 1 \end{pmatrix}, \text{ where } d = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix}, \ B = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,k} \\ \beta_{2,1} & \beta_{2,2} & \cdots & \beta_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k,1} & \beta_{k,2} & \cdots & \beta_{k,k} \end{pmatrix}.$$

Let U_0 be the subgroup of S consisting of all of these elements, as $\alpha_2, \ldots, \alpha_k$ run through \mathbb{K} . Let $T_0 = N_{M_0}(U_0)$ and observe that T_0 acts transitively on the non-identity elements of U_0 . We conclude that either M contains U_0 or else $U_0 \rtimes T_0$ acts 2-transitively by right multiplication on the set of right cosets $\{Mu \mid u \in U_0\}$, a set of size \mathbb{K}^{k-1} . In the latter case Lemma 1.6.8 implies that G contains a section isomorphic to $\mathrm{Sym}(|\mathbb{K}|^{k-1})$ and the result follows. In the former case M contains U_0 which in turn implies that $M \geq S$ and the result follows.

Assume, next, that n = 2j + 2. Then M contains the projective image of $M_0 \cong SU_{2k-2}(q)$, where M_0 stabilizes the basis vectors e_k and f_k . In this case we define two subgroups:

- 1. U_1 is the subgroup of S whose elements g fix e_1, \ldots, e_{k-1} and which satisfy $e_k \mapsto e_k + \alpha_1 e_1 + \cdots + \alpha_{k-1} e_{k-1}$ for some $\alpha_1, \ldots, \alpha_{k-1} \in \mathbb{K}$.
- 2. U_2 is the subgroup of S whose elements g fix f_1, \ldots, f_{k-1} and which satisfy $f_k \mapsto f_k + \beta_1 f_1 + \cdots + \beta_{k-1} f_{k-1}$ for some $\beta_1, \ldots, \beta_{k-1} \in \mathbb{K}$.

The same argument as for n odd allows us to conclude that either G contains a section isomorphic to $\text{Sym}(|\mathbb{K}|^{k-1})$ or else M contains U_1 and U_2 and so contains S and the result follows.

From here, we have, by definition, that j = k.

CASE 3B: S IS SYMPLECTIC AND q IS ODD. Again an easy argument asserts that, since M also contains the group K, then M must contain all matrices of the form (2.2.2), where Z is any symmetric matrix. These matrices together form an elementary abelian group U of size $q^{\frac{1}{2}k(k+1)}$, which is the unipotent radical of a parabolic subgroup P_k . Applying the same argument "with transposes" allows us to conclude that $M \geq S$, and the result follows.

CASE 3C: S IS SYMPLECTIC AND q IS EVEN. In this case, the set of matrices of the form (2.2.2), where Z is symmetric with zero diagonal entries, forms an elementary abelian group U of size $q^{\frac{1}{2}k(k-1)}$, which is the unipotent radical of a parabolic subgroup P_k of an orthogonal group $L = \Omega_{2k}^+(q)$ (this is the particular orthogonal group corresponding to the quadratic form for which our basis is hyperbolic).

Now, as in the odd case, we can apply the same argument to the transpose of these matrices to conclude that $M \cap S$ contains the group $L \cong \Omega_{2k}^+(q)$. In particular $M \cap S$ is either L, L.2 or S. The result follows if $M \cap S = S$, so assume $M \cap S$ is either L or L.2. We define an element \tilde{g} whose action on $\langle e_1, f_2 \rangle$ is given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and which fixes all elements of $\mathcal{B} \setminus \{e_1, f_1\}$. Clearly \tilde{g} is an element of $\operatorname{Sp}_{2k}(q)$; we take g to be the projective image of \tilde{g} in S. Observe that g centralizes x but does not normalize L. We can, therefore, repeat all of the preceding argument using subgroups of L^g instead of L. The same case is left: when $M \cap S$ contains both L and L^g . Since $\langle L, L^g \rangle = S$ the result follows.

CASE 3D: S IS ORTHOGONAL AND n = 2k. In this case, $S = \Omega_{2k}^+(q)/Z$ and the groups W_1 and W_2 are both unipotent radicals of parabolic subgroups P_k in S. From this, we conclude that $M \ge S$ and the result follows.

CASE 3E: S IS ORTHOGONAL, AND $n \in \{2k + 1, 2k + 2\}$. In this case, $S = \Omega_{2k+1}(q)$ or $\Omega_{2k+2}^-(q)/Z$ and, arguing à la Case 3D, we see that M contains the projective image of $L \cong \Omega_{2k}^+(q)$, where L fixes all vectors in the non-degenerate subspace $\langle Y \rangle$. Recall that, by construction, the element \tilde{x} fixes all vectors in $\langle Y \rangle$.

Suppose first that q is odd, let $z \in Y$ and suppose that $\varphi(z, z) = \eta$ where φ is the symmetric form associated with the covering group of S. We define an element \tilde{g} whose action on $\langle e_1, z, f_1 \rangle$ is given by the

matrix

$$\begin{pmatrix} 1 & a & -\frac{1}{2}a\eta \\ & 1 & -a\eta \\ & & 1 \end{pmatrix},$$

where a is some non-zero element of \mathbb{F}_q , and \tilde{G} fixes all elements of $\mathcal{B} \setminus \{e_1, f_1, z\}$. Clearly \tilde{g} is an element of $\mathrm{SO}_n^{\varepsilon}(q)$; we take g to be the projective image of \tilde{g} in S. Observe that g centralizes x but does not normalize L. We can, therefore, repeat all of the preceding argument using subgroups of L^g instead of L. The same case is left: when $M \cap S$ contains both L and L^g . Notice that we can repeat this argument for any choice of $z \in Y$ and any choice of $a \in \mathbb{F}_q$. It is straightforward to conclude that the resulting collection of conjugates of L generates S and, hence $M \geq S$ and the result follows.

Suppose next that q is even, in which case n = 2k + 2, $S = \Omega_{2k+2}(q)$ and $Y = \langle x, y \rangle$. Let Q be the quadratic form associated with the covering group of S and consider the restriction of Q to the subspace $W = \langle e_1, f_1, x, y \rangle$. Let \tilde{g} be a linear transformation which fixes all elements of $\mathcal{B} \setminus \{e_1, f_1, x, y\}$ and, on W, restricts to an element of the group $J = \Omega_4^-(q)$ associated with $Q|_W$. By [12, Lemma 2.5.9], \tilde{g} is an element of the covering group of S and we take g to be its projective image in G. Again g centralizes x and, again, we must deal with the case where $M \cap S$ contains L and L^g for all such g. Thus we may assume that M contains $L_1 = \langle L^g \mid g \in J \rangle$. There are two possibilities: either J normalizes L_1 or else we can repeat the same argument with L_1 in place of L and we are able to assume that M contains $L_2 = \langle L^g \mid g \in J \rangle$. Repeating as many times as necessary we are left with the situation where M contains a group L_∞ that contains $L \cong \Omega_{2k}^+(q)$ and is normalized by $J \cong \Omega_4^-(q)$.

Let $X = \langle e_1, f_1, e_2, f_2, x, y \rangle$ and consider the group $H \leq S$ that fixes every vector in X^{\perp} and induces $\Omega_6^-(q)$ on X. Observe that H contains J and so, in particular, J normalizes $H \cap L_{\infty}$. Then $H \cap L_{\infty}$ is a subgroup of $H = \Omega_6^-(q) \cong SU_4(q)$ that contains a group isomorphic to $\Omega_4^+(q) \cong SL_2(q) \times SL_2(q)$ and is normalized in H by a group isomorphic to $\Omega_4^-(q) \cong SL_2(q^2)$. Checking [10, Tables 8.10 and 8.14] we conclude that $H \cap L_{\infty}$ is either H or a subgroup of H isomorphic to $Sp_4(q)$. Suppose $H \cap L_{\infty} \cong Sp_4(q)$. Checking [10, Table 8.10] we see that there is precisely one conjugacy class of subgroups of $H \cong SU_4(q)$ isomorphic to $Sp_4(q)$ hence, regarding H as $\Omega_6^-(q)$ these are the stabilizers of non-singular vectors. Note that $H \cap L_{\infty}$ contains all J-conjugates of $H \cap L$. What is more the non-singular vectors fixed by $H \cap L$ are precisely those in $\langle x, y \rangle$. This, in turn, means that, for all $j \in J$, the non-singular vectors fixed by $(H \cap L)^j$ are precisely those in $\langle x, y \rangle^j$. Thus if v is a non-singular vector fixed by $H \cap L_{\infty}$, then $v^j \in \langle x, y \rangle$ for all $j \in J$. Direct calculation (or using the fact that J is irreducible on W) confirms that no such vector exists. We conclude that $H \cap L_{\infty} = H$.

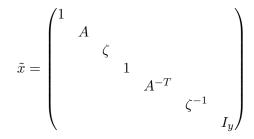
Now observe that, working with respect to the basis \mathcal{B} , M contains all of the fundamental root groups for S, and hence M contains S as required.

Finally, suppose that G is almost simple and M is core-free. Either the action of G on (G : M) is not binary (and we are done) or else G contains a section isomorphic to $\text{Sym}(|\mathbb{K}|^{j-1})$. Lemma 2.1.1 (and [10]) yield the result barring only a few values of k and q. In particular we use magma to verify the result when $S = \text{PSp}_n(q)$ with $(k,q) \in \{(2,2), (2,3), (2,4), (2,5), (2,7), (3,2), (3,3), (4,2)\}$, when $S = \text{PSU}_n(q)$ with (k,q) = (2,2) and when S is an orthogonal group with $(k,q) \in \{(3,2), (3,3), (4,2)\}$.

The following proposition deals with one of the lacunae in the previous: when S is orthogonal, q is odd, and G does not contain $PSO_n^{\varepsilon}(q)$. The statement of the proposition uses the notation established in the statement of the previous; to make matters more straightforward we assume that G is almost simple.

Lemma 2.2.9. Suppose that q is odd, and that $S = P\Omega_n^{\varepsilon}(q) \leq G \leq \operatorname{Aut}(P\Omega_n^{\varepsilon}(q))$ with $n \geq 7$. Let k be the Witt index of the associated formed space and let M < G be core-free with $x \in M \cap S$, where x is the

projective image of



written with respect to \mathcal{B} , $A \in \operatorname{GL}_{k-2}(q)$ is of order $q^{k-2} - 1$, ζ is a non-square in \mathbb{F}_q and I_y is the y-by-y identity matrix. If the action of G on (G:M) is binary, then

$$(k,q) \in \{(3,3), (3,5), (3,7), (3,9), (4,3)\}$$

Proof. We refer, first, to [12, Lemma 2.5.7] to confirm that \tilde{x} is indeed an element of $\Omega_n^{\varepsilon}(q)$. Now the action of \tilde{x} on the subspace $W := \langle e_1, \ldots, e_{k-1}, f_1, \ldots, f_{k-1}, Y \rangle$ is identical to that studied in the previous proposition; the arguments given there allow us to assume that M contains (the projective image of) the group

$$K := \{ g \in \Omega_n^{\varepsilon}(q) \mid e_k^g = e_k, f_k^g = f_k, g \mid_W \in \Omega(W) \}.$$

We should be careful about exceptions however: studying the proof we see that our conclusion is valid only when $Alt(q^{k-2})$ is not a section in S. Now, Lemma 2.1.1 implies that exceptions occur only when $q^{k-2} \leq 2k + 4$; this yields the given list.

Now we study the normalizer in K of four different elementary-abelian subgroups U_1, \ldots, U_4 of S of order q^{k-2} . We choose these groups so that they stabilize the subspaces $E = \langle e_1, \ldots, e_k \rangle$ and $F = \langle f_1, \ldots, f_k \rangle$. We require furthermore that $\langle Y \rangle$ is in the 1-eigenspace of each of the groups, thus to specify the elements of these groups it is enough to specify their action on the subspace E:

$$\begin{aligned} U_1 &:= \left\{ g \mid e_1^g = e_1; \text{ for all } i = 2, \dots, k-1, \text{ there exist } \alpha_i \text{ such that } e_i^g = e_i + \alpha_i e_k \right\}; \\ U_2 &:= \left\{ g \mid e_2^g = e_2; \text{ for all } i = 1, 3, \dots, k-1, \text{ there exist } \alpha_i \text{ such that } e_i^g = e_i + \alpha_i e_k \right\}; \\ U_3 &:= \left\{ g \mid e_1^g = e_1, \dots, e_{k-1}^g = e_{k-1}; \\ \text{ for all } i = 2, \dots, k-1, \text{ there exist } \alpha_i \text{ s.t. } e_k^g = e_k + \alpha_2 e_2 + \dots + \alpha_{k-1} e_{k-1} \right\}; \\ U_4 &:= \left\{ g \mid e_1^g = e_1, \dots, e_{k-1}^g = e_{k-1}; \\ \text{ for all } i = 1, 3, \dots, k-1, \text{ there exist } \alpha_i \text{ s.t. } e_k^g = e_k + \alpha_1 e_1 + \alpha_3 e_3 + \dots + \alpha_{k-1} e_{k-1} \right\}. \end{aligned}$$

It is a simple matter to check that, for each i = 1, ..., 4, $N_K(U_i)$ acts transitively on the non-trivial elements of U_i . Thus, by the same argument as before, we have three possibilities:

- (a) M contains U_i ;
- (b) G admits a beautiful subset of size q^{k-2} ;
- (c) S admits a section isomorphic to $Alt(q^{k-2})$.

The second possibility is ruled out because G is binary on (G : M) and the third is ruled out as before, except for the listed exceptions. Therefore, M contains U_i for each i and hence $\langle K, U_1, \ldots, U_4 \rangle = S$, and the result follows.

Lemmas 2.2.10 and 2.2.11 deal with some small rank cases that were not covered by Lemma 2.2.8.

Lemma 2.2.10. Let G contain a subgroup $S \cong SU_3(q)/Z$, where Z is a central subgroup of $SU_3(q)$ and q > 2. We let $\mathcal{B} := (e_1, f_1, x)$ be a hyperbolic basis for the underlying unitary space. Let

$$\tilde{g} = \begin{pmatrix} t & \\ & t^{-q} & \\ & & 1 \end{pmatrix} \in \mathrm{SU}_3(q),$$

where $t \in \mathbb{F}_q$ is of order q - 1; let

$$\tilde{g}' = \begin{pmatrix} u & & \\ & u^{-q} & \\ & & u^{q-1} \end{pmatrix} \in \mathrm{SU}_3(q),$$

where $u \in \mathbb{F}_q$ is of order $q^2 - 1$. Let g and g' be the projective images of \tilde{g} and \tilde{g}' in S. Let M < G and, if q is odd, then suppose that $g \in M$; if q is even, then suppose that $g' \in M$. Then one of the following holds:

- 1. G contains a section isomorphic to Sym(q);
- 2. M contains S;
- 3. the action of G on (G:M) is not binary.

In particular, if G is almost simple with socle S and M is core-free, then the action of G on (G:M) is not binary.

Proof. Assume first that q is odd and suppose that the action of G on (G:M) is binary. Write $T_0 = \langle g \rangle$ and observe that T_0 normalizes the following groups of order q:

$$U_1 := \left\{ \begin{pmatrix} 1 & -\frac{1}{2}a^2 & a \\ & 1 & \\ & -a & 1 \end{pmatrix} \mid a \in \mathbb{F}_q \right\} \text{ and } U_2 := \left\{ \begin{pmatrix} 1 & & \\ -\frac{1}{2}a^2 & 1 & -a \\ a & & 1 \end{pmatrix} \mid a \in \mathbb{F}_q \right\}$$

What is more, $U_1 \rtimes T_0$ and $U_2 \rtimes T_0$ are both Frobenius subgroups.

Suppose that M does not contain the projective image of U_1 . Then, since M contains T_0 , we conclude that $M \cap U_1 = \{1\}$. Define M_1 to be the projective image in S of $U_1 \rtimes T_0$, and observe that M_1 acts 2-transitively on $(M_1 : M \cap M_1)$. Let $\Lambda := \{Mu \mid u \in U_1\}$; we conclude that the set-wise stabilizer of Λ acts 2-transitive. Lemma 1.6.8 implies that G contains a section isomorphic to Sym(q) and the result follows.

Clearly the same argument applies if M does not contain the projective image of U_2 . Thus we may assume that M contains the projective image of $\langle U_1, U_2 \rangle$. This projective image is a subfield subgroup isomorphic to $SO_3(q)$. Thus $SO_3(q) \leq M$. Now observe that g acts fixed-point-freely by conjugation on the conjugates $U_i^{g'}$ for $i \in \{1, 2\}$. Therefore, we may apply the argument above also to the groups $U_1^{g'}$ and $U_2^{g'}$. We conclude, again, that G contains a section isomorphic to Sym(q), or else that M contains $\langle SO_3(q), U_1^{g'}, U_2^{g'} \rangle = S$ and the result follows.

Assume now that q > 2 is even and $g' \in M$. Let X be the subgroup of S that is isomorphic to $SU_2(q)$ and acts trivially on $\langle x \rangle$, where x is the third basis vector of the basis \mathcal{B} for V. Then $(g')^{q+1}$ acts fixed-point-freely on the unipotent subgroups

$$U_1 := \left\{ \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_q \right\}, U_2 := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_q \right\}$$

of X. Therefore, arguing as usual, we can assume $\langle U_1, U_2 \rangle = X \leq M$. Hence $Y = \langle X, g' \rangle \leq M$.

Now Y is a maximal subgroup of S in the C_1 class (see for instance [10]). Then [46, Prop. 4.2] implies that (S : Y) contains a beautiful subset with respect to the action of S and, checking the proof of [46, Prop. 4.4] we see that there is always a beautiful subset of size at least q. We conclude that either M contains S (and the result follows) or else $M \cap S = Y$ and (G : M) contains a subset Λ of size at least q on which S_{Λ} acts 2-transitively. Then Lemma 1.6.8 implies that G contains a section isomorphic to Sym(q)and the result follows.

Finally, suppose that G is almost simple and M is core-free. Either the action of G on (G:M) is not binary (and we are done) or else G contains a section isomorphic to Sym(q). Lemma 2.1.1 implies that $q \leq 5$; we confirm the result for $q \in \{3, 4, 5\}$ with a magma computation.

Lemma 2.2.11. Let G contain a subgroup $S \cong SU_4(q)/Z$, where Z is a central subgroup of $SU_4(q)$ and q > 2. We let $\mathcal{B} := (e_1, e_2, f_1, f_2)$ be a hyperbolic basis for the underlying unitary space. Let $x \in S$ be the projection in S of

$$\tilde{x} = \begin{pmatrix} a & & & \\ & 1 & & \\ & & a^{-1} & \\ & & & 1 \end{pmatrix}$$

written with respect to \mathcal{B} , where a is an element of \mathbb{F}_q^* of order q-1. Let M < G with $x \in M$. Then one of the following holds:

- 1. G contains a section isomorphic to Sym(q);
- 2. M contains S;
- 3. the action of G on (G:M) is not binary.

In particular, if G is almost simple with socle S and M is core-free, then the action of G on (G:M) is not binary.

Proof. Suppose that the action of G on (G : M) is binary, and write $X = \langle x \rangle$. Let y be any element of one of the following forms:

$$\begin{pmatrix} 1 & \alpha & & \\ & 1 & & \\ & & 1 & \\ & & -\overline{\alpha} & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & & \alpha \\ & 1 & -\overline{\alpha} & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

or the transpose of these forms (in each case $\alpha \in \mathbb{F}_q^*$). Now let $U = \langle y^h \mid h \in X \rangle$. In all four cases, U is a group of order q that is normalized by X. In the usual way, we conclude that either M contains U, or else there is a subset, $\Lambda := \{Mu \mid u \in U\}$, of (G : M), on which G_{Λ} acts 2-transitively. In the latter case Lemma 1.6.8 implies that G contains a section isomorphic to $\operatorname{Sym}(q)$ and the result follows. On the other hand, if the former case holds for all four unipotent subgroups in question, then, since these four subgroups generate S, we conclude that M contains S and the result follows.

Finally, suppose that G is almost simple and M is core-free. Either the action of G on (G:M) is not binary (and we are done) or else G contains a section isomorphic to Sym(q). Lemma 2.1.1 implies that $q \leq 8$. One can check directly that $SU_4(8)$ does not contain a section isomorphic to Alt(8); we confirm the result for $q \in \{3, 4, 5, 7\}$ with a magma computation.

The groups we deal with in Lemmas 2.2.12, 2.2.13 and 2.2.14 have already been considered in previous lemmas; however, here, we choose a different distinguished element and we prove that every faithful transitive action containing this element gives rise to a non-binary action.

Lemma 2.2.12. Let $S = \text{Sp}_4(q)$ where $q = 2^a$ with $a \ge 2$, and suppose that $S \le G \le \text{Aut}(S)$. Let g be the element

$$\begin{pmatrix} a & & & \\ & b & & \\ & & b^{-1} & \\ & & & a^{-1} \end{pmatrix}$$

written with respect to a hyperbolic basis (e_1, e_2, f_2, f_1) , where $a, b \in \mathbb{F}_q$ are of order q - 1. Let M be any core-free subgroup of G that contains g. Then the action of G on (G : M) is not binary.

Proof. We assume that the action of (G : M) is binary, and show a contradiction. Suppose, first, that $q \ge 8$. Let $T = \langle g \rangle$ and consider the groups U_1, \ldots, U_4 , all of order q, which contain elements of shape

$$\begin{pmatrix} 1 & & \\ & 1 & u \\ & & 1 \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & \\ & 1 & \\ & u & 1 \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & u \\ & 1 & \\ & & 1 \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ u & & & 1 \end{pmatrix},$$

respectively. Observe that, for all i = 1, ..., 4, the group T normalizes U_i and acts fixed-point-freely upon it. Thus, using our usual argument, either G contains U_i or G has a section isomorphic to Alt(q). From Lemma 2.1.1, G does not contain a section isomorphic to Alt(7). Thus G contains $\langle U_1, U_2, U_3, U_4 \rangle \cong$ $\operatorname{Sp}_2(q) \times \operatorname{Sp}_2(q)$, where $\langle U_1, U_2, U_3, U_3 \rangle$ is the subgroup of S that stabilizes the subspaces $\langle e_1, f_1 \rangle$ and $\langle e_2, f_2 \rangle$.

Now we repeat the argument with the groups U_5, \ldots, U_8 , all of order q, which contain elements of shape

$$\begin{pmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & u \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & u & \\ & 1 & & u \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ u & 1 & & \\ & & 1 & & \\ & & u & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ u & & 1 & \\ & u & & 1 \end{pmatrix},$$

respectively. As before we find that either there is a beautiful subset, or else G contains each of the groups U_5, \ldots, U_8 . The first possibility is ruled out as before because S does not admit a section isomorphic to Alt(7). Therefore $M \ge \langle \operatorname{Sp}_2(q) \times \operatorname{Sp}_2(q), U_5, \ldots, U_8 \rangle = S$ and hence M contains S, a contradiction.

If q = 4, then the result is confirmed using a magma computation.

Lemma 2.2.13. Let $S = PSU_3(q)$ with q > 2, and suppose that $S \le G \le Aut(S)$. Let T be the projective image of a maximal torus of $SU_3(q)$ of order $(q+1)^2$, and let T.2 be a subgroup of $N_S(T)$. Let M be any core-free subgroup of G that contains T.2. Then the action of G on (G:M) is not binary.

Proof. When $q \in \{3, 4, 5\}$, the veracity of this lemma is verified with the auxiliary help of magma.

Suppose that q > 5. Consulting [10, Table 8.5], we see that there are two possibilities for $M \cap S$: either $M \cap S = T.2$ or $M \cap S = T.$ Sym(3). In the latter case, M is a maximal subgroup of SM. Therefore, the action of SM on (SM : M) is primitive, and we know that the action is not binary, thanks to the argument in [45, Section 6]. Thus, by Lemma 1.6.2, the action of G on (G : M) is also not binary.

It turns out that the argument in [45, Section 6] can be used for the case $M \cap S = T.2$ as well. First, recall that if $M \cap S = T.$ Sym(3), then the action of SM on (SM : M) is permutation equivalent to the natural action of SM on

$$\{\{V_1, V_2, V_3\} \mid \dim_{\mathbb{F}_{q^2}}(V_1) = \dim_{\mathbb{F}_{q^2}}(V_2) = \dim_{\mathbb{F}_{q^2}}(V_3) = 1, V = V_1 \perp V_2 \perp V_3, V_1, V_2, V_3 \text{ non-degenerate}\}.$$

If $M \cap S = T.2$, then the action of SM on (SM : M) is permutation equivalent to the natural action of SM on

$$\Lambda := \{ (V_1, \{V_2, V_3\}) \mid \dim_{\mathbb{F}_{q^2}}(V_1) = \dim_{\mathbb{F}_{q^2}}(V_2) = \dim_{\mathbb{F}_{q^2}}(V_3) = 1, V = V_1 \perp V_2 \perp V_3, V_1, V_2, V_3 \text{ non-degenerate} \}.$$

Now fix $M \cap S = T.2$ and identify (SM : M) with the given set Λ . Let e_1, e_2, e_3 be a basis of V such that the matrix of the Hermitian form with respect to this basis is the identity. Thus $\lambda_0 := (\langle e_1 \rangle, \{\langle e_2 \rangle, \langle e_3 \rangle\}) \in \Lambda$.

Consider $\Lambda_0 := \{(V_1, \{V_2, V_3\}) \in \Lambda \mid V_1 = \langle e_1 \rangle\}$. Clearly, $(SM)_{\Lambda_0} = (SM)_{\langle e_1 \rangle}, (SM)_{\Lambda_0}/Z((SM)_{\Lambda_0})$ is almost simple with socle isomorphic to $PSL_2(q)$ (here we are using q > 3), and the action of $(SM)_{\Lambda_0}$ on Λ_0

G(q)	$E_8(q)$	$E_7(q)$	$E_6(q)$	${}^{2}\!E_{6}(q)$	$F_4(q)$	$G_2(q)$	$^{3}D_{4}(q)$
m	9	8	6	4	4	3	3
r	7	5	4	2	2	2	2
N	q^8	q^7	q^8	q^{18}	q^{10}	q^4	q^{10}
q				q > 3	q > 3	q > 5	q > 5

Table 2.2.2: Values of m such that $SL_m(q)/Z \leq G(q)$

is permutation equivalent to the action of $(SM)_{\langle e_1 \rangle}$ on $\Lambda'_0 := \{\{W_1, W_2\} \mid \dim(W_1) = \dim(W_2), \langle e_1 \rangle^{\perp} = W_1 \perp W_2, W_1, W_2$ non degenerate}. Therefore $(SM)^{\Lambda_0}$ is an almost simple primitive group with socle isomorphic to $PSL_2(q)$ and having degree $|\Lambda_0| = q(q-1)/2$. Applying [45, Theorem 1.3] to $(SM)^{\Lambda_0}$, we obtain that $(SM)^{\Lambda_0}$ is not binary and hence there exist two ℓ -tuples $(\{W_{1,1}, W_{1,2}\}, \ldots, \{W_{\ell,1}, W_{\ell,2}\})$ and $(\{W'_{1,1}, W'_{1,2}\}, \ldots, \{W'_{\ell,1}, W'_{\ell,2}\})$ in Λ^{ℓ}_0 which are 2-subtuple complete for the action of $(SM)_{\Lambda_0}$ but not in the same $(SM)_{\Lambda_0}$ -orbit. By construction the two ℓ -tuples

$$I := ((\langle e_1 \rangle, \{W_{1,1}, W_{1,2}\}), (\langle e_1 \rangle, \{W_{2,1}, W_{2,2}\}), \dots, (\langle e_1 \rangle, \{W_{\ell,1}, W_{\ell,2}\})), J := ((\langle e_1 \rangle, \{W'_{1,1}, W'_{1,2}\}), (\langle e_1 \rangle, \{W'_{2,1}, W'_{2,2}\}), \dots, (\langle e_1 \rangle, \{W'_{\ell,1}, W'_{\ell,2}\}))$$

are in Λ^{ℓ} and are 2-subtuple complete. Moreover, I and J are not in the same SM-orbit. Thus SM is not binary on $\Lambda = (SM : M)$. Now, G is not binary on (G : M) by Lemma 1.6.2.

Lemma 2.2.14. Let $S = PSp_4(q)$ where $q \in \{3,5\}$, and suppose that $S \leq G \leq Aut(S)$. Let T be a torus of S of size $\frac{1}{2}(q-1)^2$, and let M be any core-free subgroup of G that contains T. Then the action of G on (G:M) is not binary.

Proof. In each case we use magma: we consider all almost simple groups G with socle one of these two groups; we then compute all the core-free subgroups M having order divisible by $(q-1)^2/2$; finally we prove, in all cases, that the action of G on the right coset of M is not binary.

To test this, we have divided our algorithm in two cases: when $|M|^3 \leq |G|$, since we could not afford to determine the permutation representation explicitly having too many points available, we have generated, for 10⁶ times, two cosets Mg_1 and Mg_2 of M in G, and we tested whether Lemma 1.8.4 applies with $\omega_0 := M$, $\omega_1 := Mg_1$ and $\omega_2 := Mg_2$ (observe that for this test we do not need to construct the permutation representation of G on the right cosets of M); when $|M|^3 > |G|$, we have constructed the permutation representation of G on the cosets of M and we looked (extensively) for pairs for the form $((\omega_1, \omega_2, \ldots, \omega_\ell), (\omega'_1, \omega'_2, \ldots, \omega'_\ell))$, with $\ell \leq 4$, which are 2-subtuple complete but not in the same orbit. \Box

2.2.3 Exceptional groups

Lemma 2.2.15. Suppose that G is almost simple with socle $G_0 = G(q)$, an exceptional group of Lie type as in Table 2.2.2, and let m be the value given in the table.

- (i) Then G_0 has a subgroup $L \cong SL_m(q)/Z$, where Z is central in $SL_m(q)$.
- (ii) Adopt the assumptions on q in the last line of Table 2.2.2, and let $x \in L$ be the element as in the statement of Lemma 2.2.5, of order $q^{m-2} 1$ (if q > 2) or $2^{m-1} 1$ (if q = 2). If M is any core-free subgroup of G that contains x, then the action of G on (G:M) is not binary.
- (iii) If x is the element in part (ii), then $|C_G(x)| < N$, where N is as in Table 2.2.2.

Proof. (i) The existence of these subgroups L follows easily from inspection of extended Dynkin diagrams (this fact will also be used in the proofs of Propositions 3.3.1 and 3.4.1, where we also provide additional comments).

(ii) Let $L \cong SL_m(q)/Z$ be the subgroup of (i), and let $x \in L$ be the element as in the statement of Lemma 2.2.5. Suppose that M is a core-free subgroup of G containing x, and assume that the action of G on (G:M) is binary. We apply Lemma 2.2.5: by our assumptions on q, the only possibilities are

(a) G has a section isomorphic to $Alt(q^{m-2})$ (if q > 2) or $Alt(2^{m-1})$ (if q = 2), or

(b) M contains L.

The possibility (a) is excluded by Lemma 2.1.2 together with our assumptions on q.

Hence $L \leq M$. Using Theorem 3.1.1, it is straightforward to see that any core-free maximal subgroup H of G(q) containing M is either parabolic, or of maximal rank, or a subgroup $F_4(q)$ or $C_4(q)$ of ${}^2E_6(q)$. (Actually this follows from [75] except for the case where q = 2.) Hence we can argue exactly as in the proofs of Propositions 3.3.1, 3.4.1 and 3.6.1 (case (4) of the proof) that there are subgroups A < S of G with the following properties:

- (1) $A \leq L$ and $A \not\leq H$ for any maximal subgroup H of G(q) containing M, and
- (2) $A \cong \operatorname{SL}_r(q), S \cong \operatorname{SL}_{r+1}(q)/Z$, where r is as in Table 2.2.2.

By Lemma 1.6.10, there is a subset Δ of (G : M) such that $|\Delta| = q^r$ and $G^{\Delta} \ge ASL_r(q)$. Since $Alt(q^r)$ is not a section of G, by Lemma 2.1.2 and our assumptions on q, this contradicts the assumption that the action of G on (G : M) is binary, completing the proof of (ii).

(iii) There is a simple adjoint algebraic group \overline{G} over the algebraic closure $\overline{\mathbb{F}}_q$, and a Frobenius endomorphism F of \overline{G} , such that G(q) is the socle of the fixed point group $(\overline{G}^F)'$. The element x lies in $L = (\overline{L}^F)'$, where \overline{L} is a subsystem subgroup A_{m-1} of \overline{G} . The composition factors of the restriction of the Lie algebra $L(\overline{G})$ to \overline{L} can be found in Tables 8.1 - 8.5 of [76], from which it is easy to work out the action of x on $L(\overline{G})$, and hence obtain the upper bound dim $C_{L(\overline{G})}(x) \leq R$, where R is as follows:

Here is an example of such a computation for the case where $G(q) = {}^{2}E_{6}(q)$: here $\overline{L} = A_{3}$ and

$$L(\bar{G}) \downarrow A_3 = L(A_3)/\lambda_1^4/\lambda_3^4/\lambda_2^4/0^7,$$

where in this notation, λ_1 denotes the irreducible 4-dimensional module $V_{A_3}(\lambda_1)$, and so on. The action of x on the module λ_1 is $(1, a, \lambda, \lambda^q)$, where $\lambda \in \overline{\mathbb{F}}_q$ has order $q^2 - 1$ and $a = \lambda^{-1-q}$. Hence x has fixed point space of dimension 1 on λ_1 and λ_3 , and of dimension 0 on λ_2 (which is $\wedge^2(\lambda_1)$). It follows that

$$\dim C_{L(\bar{G})}(x) \le 3 + 4 + 4 + 7 = 18.$$

In each case there is in fact a subgroup of \bar{G} of dimension R centralizing x: for E_8 and E_7 this is just a maximal torus; for E_6 it is a subgroup T_5A_1 (where T_5 denotes a 1-dimensional torus); for 2E_6 it is a subgroup $T_2A_2A_2$ (since $x \in A_2 < A_3 = \bar{L}$, and this A_2 centralizes A_2A_2 in \bar{G}); similarly in F_4 it is T_2A_2 ; in G_2 it is T_1A_1 and in 3D_4 it is $T_1A_1^3$. Hence these subgroups are the full centralizers of x in \bar{G} (noting that $C_{\bar{G}}(x)$ is connected), and hence taking fixed points under the Frobenius endomorphism F, we see that $|C_{\bar{G}^F}(x)|$ is as follows:

G(q)	$ C_{ar{G}^F}(x) $	G(q)	$ C_{ar{G}^F}(x) $
$E_8(q)$	$(q^7 - 1)(q - 1), q > 2$	$^{2}E_{6}(q)$	$(q^2 - 1) A_2(q^2) $
	$2^8 - 1, q = 2$	$F_4(q)$	$(q^2 - 1) A_2(q) $
$E_7(q)$	$(q^6 - 1)(q - 1), q > 2$	$G_2(q)$	$(q-1) A_1(q) $
	$2^7 - 1, q = 2$	$^{3}D_{4}(q)$	$(q-1) A_1(q^3) $
$E_6(q)$	$(q^4 - 1)(q - 1) A_1(q) , q > 2$		
	$(2^5 - 1) A_1(2) , q = 2$		

Since x is centralized by no graph or field automorphisms, it follows that $|C_G(x)| < N$, where N is as in Table 2.2.2. This completes the proof.

Lemma 2.2.16. Let G contain a subgroup S isomorphic to ${}^{2}F_{4}(q)$ (q > 2), ${}^{2}G_{2}(q)$ (q > 3) or ${}^{2}B_{2}(q)$ (q > 2). Then there exists an element $x \in S$ of order q - 1 such that, if $x \in M < G$, then one of the following holds:

- 1. G contains a section isomorphic to Sym(q);
- 2. M contains S;
- 3. the action of G on (G:M) is not binary.

In particular if G is almost simple, then we can choose x to have the following properties:

- (i) If M is any core-free subgroup of G that contains x, then the action of G on (G:M) is not binary.
- (ii) $|C_G(x)| = (q-1)^2$, q-1 or q-1, according as $S = {}^2F_4(q)$, ${}^2G_2(q)$ or ${}^2B_2(q)$, respectively.

Proof. Suppose first that $S = {}^{2}F_{4}(q)$. Let T be a maximal torus of S of order $(q-1)^{2}$, and choose $x \in T$ of order q-1 such that $C_{Aut(S)}(x) = T$ (such an element exists by [96]). Let M be a subgroup of G containing x and assume that the action of G on (G:M) is binary.

The structure of the root subgroups of S with respect to T can be found in [47, Theorem 2.4.5(d)]. If U_1 is a root subgroup of type A_1^2 with respect to T, then either U_1 is contained in M or else $U_1 \rtimes \langle x \rangle$ acts 2-transitively on $\Lambda = \{Mu \mid u \in U_1\}$, a set of size $q \geq 8$. In the latter case Lemma 1.6.8 implies that G contains a section isomorphic to $\operatorname{Sym}(q)$ and the result follows. Suppose, then that $U_1 \leq M$. The same argument applies to U_1^- , the "opposite" root group of type A_1^2 . We can apply the same argument to a root group U_2 , of type B_2 , although in this case we consider $Z(U_2) \rtimes \langle x \rangle$, and we conclude that $Z(U_2) \leq M$. The same argument applies to U_2^- , the "opposite" root group of type B_2 . Since $\langle U_1^{\pm}, Z(U_2^{\pm}) \rangle = S$, it follows that $M \geq S$ and the result follows.

In the special case where G is almost simple and M is core-free, Lemma 2.1.2 implies that G does not contain a section isomorphic to Alt(q) and the result follows.

Now suppose that $S = {}^{2}G_{2}(q)$. We refer to the main theorem of [102] for basic information about this group. We may choose an element $x \in S$ of order q-1 such that $C_{Aut(S)}(x) = \langle x \rangle$ and x normalizes a Sylow 3-subgroup P of S. If Z = Z(P), then |Z| = q and [102, item (3)] implies that $\langle x \rangle$ acts fixed-point-freely on Z. Let M be a subgroup of G containing x and assume that the action of G on (G : M) is binary.

Suppose that $Z \leq M$. Then $Z \cap M = \{1\}$ and, identifying (G : M) with the cosets of M we can set $\Lambda = \{Mz \mid z \in Z\}$, a subset of (G : M) of size q. Then $Z \rtimes \langle g \rangle$ acts 2-transitively on Λ . Lemma 1.6.8 implies that G contains a section isomorphic to Sym(q) and the result follows.

We may suppose, then, that $Z \leq M$. The same argument applies to the "opposite" Sylow 3-subgroup P^- of S and so we obtain a second subgroup, Z^- , on which $\langle x \rangle$ acts fixed-point-freely, and which is contained in M. Thus $\langle Z, Z^-, x \rangle \leq M$. From the list of maximal subgroups of ${}^2G_2(q)$ in [86], we see that either $\langle Z, Z^-, x \rangle = S$ or $\langle Z, Z^-, x \rangle \leq 2 \times \text{PSL}_2(q)$. The latter is not possible since $\langle x \rangle$ is fixed point free on Z. Hence $\langle Z, Z^-, x \rangle = S$. Therefore, $M \geq S$ and the result follows.

In the special case where G is almost simple and M is core-free, we note that G has no section isomorphic to Alt(5) (because 5 does not divide |S|) and the result follows.

Suppose finally that $S = {}^{2}B_{2}(q)$. We refer to [98]. The first part of the argument for ${}^{2}G_{2}(q)$ applies here word-for-word, except that this time P is a Sylow 2-subgroup of S. As in that previous case, we obtain that M contains $\langle Z_{1}, Z_{2} \rangle$, where Z_{1} and Z_{2} are the centres of two distinct Sylow 2-subgroups of S. From the list of the maximal subgroups of S in [98, Theorem 9] we obtain $\langle Z_{1}, Z_{2} \rangle = S$, completing the proof as before. In the special case where G is almost simple and M is core-free, we note that G has no section isomorphic to Alt(3) (because 3 does not divide |S|) and the result follows.

2.3 Results on odd-degree actions

In this section we present two results, both proved using magma. Our methods are described in full in §1.8.

Lemma 2.3.1. Let M_0 be one of the following groups

$$\begin{split} & \operatorname{PSL}_2(r) \, (r \leq 31), \operatorname{PSL}_3(r) \, (r \in \{2,3,4,5\}), \operatorname{PSL}_4(r) \, (r \in \{2,3,5\}), \\ & \operatorname{PSU}_3(r) \, (r \in \{3,4,5,8\}), \operatorname{PSU}_4(r) \, (r \in \{2,3,4,5,7\}), \operatorname{PSU}_5(r) \, (r \in \{2,3,4,5,7\}), \operatorname{PSU}_6(2), \\ & \operatorname{PSp}_4(r) \, (r \in \{2,3,4,5,7\}), \operatorname{PSp}_6(r) \, (r \in \{2,3\}), \operatorname{PSp}_8(2), \\ & \Omega_7(r) \, (r \in \{3,5,7,9\}), \operatorname{P\Omega}_8^-(r) \, (r \in \{2,3,4\}), \Omega_8^+(2), \Omega_{10}^\pm(2), \, \Omega_{10}^-(3), \\ & ^2B_2(8), ^2B_2(32), G_2(r) \, (r \in \{3,4,5\}), \, ^3D_4(r) \, (r \in \{2,3\}), \, F_4(r) \, (r \in \{2,3\}), \, ^2F_4(2)', \, ^2E_6(2). \end{split}$$

Let M be an almost simple group with socle M_0 and let H be a core-free subgroup of M with |M:H| odd. Then either the action of M on (M:H) is not binary or M, M_0 and H are as in Table 2.3.1.

M_0	M	M:H
Alt(5)	$\operatorname{Sym}(5)$	5
Alt(5)	Alt(5)	15
$PSL_2(8)$	$PSL_2(8)$	63
$PSL_2(8)$	$PSL_{2}(8).3$	189
$PSL_{2}(16)$	$PSL_{2}(16)$	255
$PSL_2(16)$	$PSL_2(16).2$	51

Table 2.3.1: Some odd-degree binary actions

Proof. Suppose first that $M_0 \notin \{F_4(2), F_4(3), {}^3D_4(3), {}^2E_6(2)\}$. We have constructed all the groups M under consideration and all odd index subgroups H of M. The construction of H can be done quite efficiently working recursively: for each group M under consideration, the list of the maximal core-free subgroups X of M is either already available in magma, or it can be constructed. Then, we can simply select the subgroups X with |M : X| odd. In all cases, X is considerably smaller than M and we can directly compute the odd index subgroups of X. Thus, we obtain all odd index subgroups of M.

We then check that the action of M on (M : H) is not binary with a combination of techniques. First, we have checked the permutation character bound, see Lemma 1.8.1, then we have tried to apply Lemma 1.8.4 and finally Lemma 1.6.15. For permutation groups failing this method, the degree of the action was less than 10⁷ and hence we simply searched for non-binary *t*-tuples (with *t* relatively small: except when $M_0 = \text{PSU}_4(2)$, it was sufficient to consider $t \in \{3, 4\}$).

Suppose now that $M_0 = F_4(2)$. In particular, either $M = F_4(2)$ or $M = F_4(2).2$. Let H be a subgroup of M with |M:H| odd and let K be a maximal core-free subgroup of M with $H \leq K$. We have reported in Table 3.2.1 the maximal subgroups of M. We have proved in Proposition 3.2.1 that the action of M on (M:K) is not binary and hence we may suppose that H < K. Using the information on K in Table 3.2.1, we have computed the odd index subgroups of K and we have checked that, except when $M = F_4(2).2$, $K = [2^{22}](\text{Sym}(2)\text{wr Sym}(2))$ and H is a Sylow 2-subgroup of K, the action of K on (K:H) is not binary. In particular, we may suppose that $M = F_4(2).2$, $K = [2^{22}](\text{Sym}(2)\text{wr Sym}(2))$. Now, let T be a maximal subgroup of M with $H \leq T$ and with $T \cong [2^{20}].\text{Alt}(6) \cdot 2^2$ (clearly, this is possible from Table 3.2.1 and from Sylow's theorem). Now, the action of T on (T:H) is not binary and hence so is the action of M on (M:H).

Suppose now that $M_0 = F_4(3)$. In particular, $M = F_4(3) = M_0$. Let H be a subgroup of M with |M : H| odd and let K be a maximal subgroup of M with $H \leq M$. From [73], we see that K is isomorphic to either $2.\Omega_9(3)$ or to $2^2.P\Omega_8^+(3).Sym(3)$. For each of these two groups, we have computed all the subgroups H with |K : H| odd and we have checked that either K is not binary on (K : H), or K = H, or $K = 2^2.P\Omega_8^+(3).Sym(3)$ and |K : H| = 3. In the first case, we deduce that the action of M on (M : H) is not binary and hence we may consider one of the remaining cases. In all cases, 7 is a divisor of both |M : H| and |H| and also 7^2 is the largest power of 7 dividing |M|. Let V be a Sylow 7-subgroup

of M. Now, $N_M(V)$ lies in the maximal rank subgroup ${}^3D_4(3).3$, since the normalizer of V in ${}^3D_4(3)$ is $V.SL_2(3)$, see [57] for example. Therefore, we have $N_M(V) \cong V.(3 \times SL_2(3))$, where the action of $3 \times SL_2(3)$ by conjugation on V has two orbits of cardinality 4 on the subgroups of V having order 7. This shows that we are in the position to apply Lemma 1.6.15 with the prime 7 and we deduce that also the action of M on the remaining cases is not binary.

Suppose now that $M_0 = {}^3D_4(3)$. In this case, the maximal subgroups of M are not available in magma. However, using [73], we see that if H is a core-free maximal subgroup of M and |M : H| is odd, then $H \cap M_0$ is either $G_2(3)$, or $(7 \times SU_3(3)).2$, or $(SL_2(27) \circ SL_2(3)).2$. Now, using the structure of these groups, we may construct them as subgroups of M (for instance, when $H \cap M_0 \cong 7 \times SU_3(3).2$, we may construct H by computing the normalizer of a suitable subgroup of M of order 7). Then we have checked that the action was not binary using the permutation character method. Then, we worked recursively on the subgroups of H, as explained in the first part of this proof.

Suppose now that $M_0 = {}^2E_6(2)$. We use the information in [105]. In this case, the maximal subgroups of M are not available in magma. However, using the information in the work of Wilson [105], we see that if |M:H| is odd, then X is contained in a parabolic subgroup P of M. The information in [105] is also enough to construct the (abstract) group P explicitly using magma. At this point, we have constructed for each parabolic subgroup P of M, all the subgroups H of P with |P:H| odd. We have checked that the action of P on (P:H) is not binary (by witnessing non-binary triples or quadruples), unless $|P:H| \in \{1,3,9,15,45\}$. At this point, the only actions that we need to discuss are the actions of M on (M:H), where $H \leq P$ for some maximal parabolic subgroup P of M and for some subgroup H of P with $|P:H| \in \{1,3,9,15,45\}$.

Using the structure of P, we deduce that 7 divides both |M : H| and |H|. Now, a Sylow 7-subgroup V of M has order $49 = 7^2$ and M has two conjugacy classes of elements of order 7, which are referred to as type 7A and 7B. The group V contains 8 subgroups of order 7, where 4 of these subgroups consist only of 7A elements and 4 of these subgroups consist only of 7B elements. Using this information, it is readily seen that we may use Lemma 1.6.15 to show that M is not binary on (M : H).

Lemma 2.3.2. Let M be an almost simple group with socle M_0 a sporadic simple group. Then every faithful odd degree action of M is not binary.

Proof. We use magma to verify the statement of the lemma. We divide the proof into three cases.

(1) Suppose that M_0 is one of the following groups:

 $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, HS, McL, He, Ru, Suz, Co_1, Co_2, Co_3, M_{11}, M_{12}, M_{12}, M_{23}, M_{24}, J_1, J_2, J_3, HS, McL, He, Ru, Suz, Co_1, Co_2, Co_3, M_{11}, M_{12}, M_{12}, M_{13}, M_{14}, M_{14}, M_{15}, M_{15}$

$$Fi_{22}, Fi_{23}, Fi'_{24}, HN \text{ or } O'N.$$

Let M be an almost simple group with socle M_0 . We use magma to construct all odd index subgroups H of M, using a recursive routine as described at the start of the previous proof. For the groups Co_1 and Fi'_{24} , we used some extra information at the start of the recursion: in these cases, rather than computing all maximal subgroups of M and selecting those of odd index, we explicitly constructed only the maximal subgroups of M having odd index using the information in [28] or in the online atlas of finite group representations. Then, to determine all the other odd index subgroups of M we can simply run the procedure above, for each of the maximal subgroups that we have constructed.

Given (M, H) as above, and using the terminology in §1.8, either Test 3 (direct analysis) or Test 1 (using the permutation character) is enough to complete the magma calculations. When M_0 is one of Fi_{23} , Fi'_{24} , HN, O'N and Co_3 , we used Test 1 as well as Lemma 1.6.15.

(2) Suppose now that M_0 is one of the following groups:

$$J_4, Ly, Th, B.$$

M_0	K	Н	H:K
J_4	$2^{3+12} \cdot (\operatorname{Sym}(5) \times \operatorname{PSL}_3(2))$	$2^{3+12} \cdot (\operatorname{Sym}(4) \times \operatorname{PSL}_3(2))$	5
B	$2^{2+10+20}.(M_{22}: 2 \times \text{Sym}(3))$	$2^{2+10+20}.(M_{22}:2\times 2)$	3
B	$[2^{35}].(Sym(5) \times PSL_3(2))$	$[2^{32}].(Sym(4) \times PSL_3(2))$	5
M	$2^{2+11+22}.(M_{24} \times \text{Sym}(3))$	$2^{2+11+22}.(M_{24} \times 2)$	3
M	$2^{5+10+20}$.(Sym(3) × PSL ₅ (2))	$2^{5+10+20}.(2 \times PSL_5(2))$	3

Table 2.3.2

In this case we proceeded similarly at first, by constructing all core-free odd index subgroups H of M. However in some cases the index |H:M| was too large to prove directly that the action of M on (M:H) is not binary. Thus we took a different approach as follows.

Let M be an almost simple group with socle M_0 in the given list, let H be a core-free subgroup of M with |M:H| odd and let K be a maximal core-free subgroup of M with $H \leq K$. Recall that Conjecture 1.2 has been verified for primitive actions of almost simple groups having socle a sporadic simple group [34]. Therefore, we may suppose that H < K. Now, rather than studying the action of M on (M:H), we study the action of K on (K:H). We use magma to confirm that, except when (M, M_0, K, H) is in Table 2.3.2, the action of K on the right cosets of H is not binary (by witnessing a non-binary triple or a non-binary 4-tuple). Now Lemma 1.6.2 implies that the action of M on (M:H) is not binary.

For the remaining cases in Table 2.3.2, we have computed the permutation character for the action of M on the right cosets of H and we have checked that this action is not binary using the permutation character bound (Test 1 in §1.8).

(3) Finally suppose that M is the Monster group. The maximal subgroups K of M having odd index can be deduced from [104] and are:

$$2^{1+24}.Co_1, \quad 2^{10+16}.O_{10}^+(2), \quad 2^{2+11+22}.(M_{24} \times \text{Sym}(3)),$$

 $2^{5+10+20}.(\text{Sym}(3) \times \text{PSL}_5(2)), \quad 2^{3+6+12+18}.(\text{PSL}_3(2) \times 3.\text{Sym}(6)).$

Let K be one of these groups and let H be a subgroup of K with |M : H| odd. We show that the action of M on the right cosets of H is not binary. If K = H, then this follows from [34]. Suppose then H < K. Observe that, when $K \cong 2^{1+24}.Co_1$, we have $O_2(K) \leq H \leq K$ and $K/O_2(K) \cong Co_1$. Therefore, in this case, the proof follows from the fact that the faithful transitive actions of Co_1 of odd degree are not binary.

For the remaining three groups K, we have constructed all odd index subgroups H of K. Except when (M_0, K, H) is in the last two lines of Table 2.3.2, we have verified that the action of K on the right cosets of H is not binary (by using three techniques: via the permutation character method, or via Lemma 1.8.4, or when the degree of the action is not very large via witnessing a non-binary triple or a non-binary 4-tuple). In particular, the action of M on the right cosets of H is not binary in these cases.

It remains to deal with the cases in Table 2.3.2: here we cannot argue as in the paragraph above, because the information in the character table stored in magma is not enough to construct the permutation character under consideration. When $K = 2^{5+10+20}$.(Sym(3) × $L_5(2)$) and $H = 2^{5+10+20}$.(2 × $L_5(2)$), the action of M on the right cosets of H is not binary by using Lemma 1.6.15 applied with the prime p = 7 (for details see [34, Lemma 5.1 and 5.2]). When $K = 2^{2+11+22}$.($M_{24} \times \text{Sym}(3)$) and $H = 2^{2+11+22}$.($M_{24} \times 2$), the action of M on the right cosets of H is not binary by using Lemma 1.6.15 applied with the prime p = 7 (for details see [34, Lemma 5.1 and 5.2]).

2.4 Results on centralizers

The first result in this subsection is taken from $[41, \S6]$.

Proposition 2.4.1. Let $G = Cl_n(q)$ be a simple classical group, and let $1 \neq g \in H$.

- (i) Then $|C_G(g)| > f(n,q)$, where f(n,q) is as in Table 2.4.1.
- (ii) In particular, for any $G = Cl_n(q)$ we have

$$|C_G(g)| > \frac{q^{\lceil (n-1)/2 \rceil}}{4} \left(\frac{q-1}{2qe(\log_q(2n)+4)}\right)^{1/2}$$

Table 2.4.1: Lower bounds for centralizers in classical groups

Н	f(n,q)
$\mathrm{PSL}_n(q)$	$\frac{q^{n-1}}{e(1+\log_q(n+1))(n,q-1)}$
$\mathrm{PSU}_n(q)$	$\frac{q^{n-1}}{(n,q+1)} \cdot \left(\frac{q-1}{e(q+1)(2+\log_q(n+1))}\right)^{1/2}$
$\mathrm{PSp}_n(q), \mathrm{P}\Omega^{\epsilon}_n(q)$	$\frac{q^{\lceil (n-1)/2\rceil}}{4} \left(\frac{q-1}{2qe(\log_q(2n)+4)}\right)^{1/2}$

The next result is Lemma 5.7 of [84].

Lemma 2.4.2. Let S = G(q) be a be a simple group of Lie type, let d be the untwisted rank of S, and let g be an element of S. Then

$$|C_S(g)| \ge \frac{(q-1)^d}{|\operatorname{Inndiag}(S)|}.$$

2.5 Outer automorphisms of groups of Lie type

Here we record a well-known result which classifies all outer automorphisms of prime order of finite groups of Lie type. In the terminology of [47, Defn. 2.5.13], all such are diagonal, field, graph-field or graph automorphisms. A proof can be found in [68, Prop. 1.1].

Proposition 2.5.1. Let L = L(q) be a simple group of Lie type over \mathbb{F}_q , and let α be an automorphism of L of prime order. If L is classical with natural module V, suppose that α does not lie in PGL(V); and if L is exceptional, suppose that $\alpha \notin Inndiag(L)$. Then one of the following holds:

- (i) α is a field or graph-field automorphism, and $C_L(\alpha)$ is of type $L(q^{1/|\alpha|})$ or ${}^{2}L(q^{1/2})$ (or ${}^{3}D_4(q^{1/3})$ when $L = D_4(q)$);
- (ii) α is a graph automorphism and the possibilities are as in Table 2.5.1. (In the last column of the table, t denotes a long root element.)

L	$ \alpha $	possible types for $C_L(\alpha)$
$\mathrm{PSL}_n^{\epsilon}(q)$	2	$PSO_n(q) (n \text{ odd})$
		$PSO_n^{\pm}(q), PSp_n(q) (n \text{ even}, q \text{ odd})$
		$\operatorname{Sp}_n(q), C_{\operatorname{Sp}_n(q)}(t) (n \text{ even}, q \text{ even})$
$D_4(q), {}^3\!D_4(q)$	3	$G_2(q), A_2^{\epsilon}(q)$ if $(3, q) = 1$
		$G_2(q), C_{G_2(q)}(t)$ if 3 divides q
$E_6^{\epsilon}(q)$	2	$F_4(q), C_4(q) (q \text{ odd})$
		$F_4(q), C_{F_4(q)}(t) (q \text{ even})$

Table 2.5.1

2.6 On fusion and factorization

Before working our way through the families of maximal subgroups given in Theorem 3.1.1 we record a few useful lemmas.

In the next lemma, given a group G and two subgroups X and Y with X < Y < G, we say that Y controls fusion of X in G if, whenever $X^g < Y$ for some $g \in G$, there exists $y \in Y$ such that $X^g = X^y$.

Lemma 2.6.1. Let G be a finite group, and let A < S < H be subgroups of G with the following properties:

- (i) S controls fusion of A in G;
- (ii) H controls fusion of S in G;
- (iii) $S^x \leq H$ for all $x \in N_G(A)$.

Then $N_G(A) = N_H(A) (N_G(S) \cap N_G(A)).$

Proof. Let $x \in N_G(A)$. Then $S^x \leq H$ by (iii), so by (ii) there exists $h \in H$ such that $S^x = S^h$. Hence $x^{-1} \in HN_G(S)$, so

$$N_G(A) \subseteq HN_G(S). \tag{2.6.1}$$

Now let $y \in N_G(S)$. Then $A^y \leq S$, so by (i) there exists $s \in S$ such that $A^y = A^s$. Hence $N_G(S) \subseteq SN_G(A)$, and, by intersecting both sides of this inclusion by $N_G(S)$, it follows that

$$N_G(S) = S (N_G(S) \cap N_G(A)).$$
(2.6.2)

From (2.6.1) and (2.6.2), we deduce

$$N_G(A) \subseteq H\left(N_G(S) \cap N_G(A)\right)$$

and the proof follows by intersecting both sides of this inclusion by $N_G(A)$.

In our application of the above lemma we will also need the following result on factorizations of simple groups, which is a consequence of Theorem A of [69].

Lemma 2.6.2. Let G be an almost simple group with socle $G_0 = \text{PSL}_n(q^a)$, where $a \ge 2$. Suppose G has a factorization G = AB, where A, B are core-free subgroups and A is contained in a subfield subgroup $N_G(\text{PSL}_n(q^b))$, where $\mathbb{F}_{q^b} \subset \mathbb{F}_{q^a}$. Then $(n, q^a) = (2, 4)$, (2, 9), (2, 16) or (3, 4), and the possibilities for A, B are as follows:

G_0	$A \cap G_0$	$B \cap G_0$
$PSL_2(4)$	$\operatorname{Sym}(3), C_3$	D_{10}
$PSL_2(9)$	$\operatorname{Sym}(4), \operatorname{Alt}(4), C_3$	$\operatorname{Alt}(5)$
$PSL_{2}(16)$	$PSL_2(4)$	D_{34}
$PSL_3(4)$	$\mathrm{PSL}_3(2)$	Alt(6)

Proof. Theorem A of [69], together with [72] imply the listed restrictions on the pairs (n, q^a) . What is more we know the maximal subgroups of G_0 which contain $A \cap G_0$ and $B \cap G_0$; these are listed as the first entry in each column in the tables in [72]. We then check directly whether it is possible for $A \cap G_0$ or $B \cap G_0$ to be non-maximal. The proof follows with a case-by-case analysis or with a magma computation.

Chapter 3

Exceptional Groups

In this chapter we prove the following theorem.

Theorem 3.1. Let G be an almost simple primitive permutation group with socle an exceptional group of Lie type. Then G is not binary.

Note that the Suzuki and Ree groups ${}^{2}B_{2}(q)$ and ${}^{2}G_{2}(q)$ have been dealt with in [45], so we do not consider them here. Note, too, that the groups with socle ${}^{2}F_{4}(2)'$ were dealt with in [34], hence these too are excluded from what follows.

Our notation for finite groups of Lie type is in line with standard references such as [47]. Dynkin diagrams are labelled as in [9].

3.1 Maximal subgroups of exceptional groups of Lie type

We shall need a substantial amount of information about maximal subgroups of finite exceptional groups of Lie type, taken from many sources. A summary follows; note that we write Lie(p) to mean the set of simple groups of Lie type that are defined over a field of characteristic p. By the *rank* of a finite group of Lie type G(q), we mean the Lie rank of the corresponding simple algebraic group.

Theorem 3.1.1. ([78, Theorem 8]) Let G be an almost simple group with socle G(q), an exceptional group of Lie type over \mathbb{F}_q , $q = p^a$, and let H be a maximal subgroup of G. Then one of the following holds:

- (I) *H* is a parabolic subgroup;
- (II) H is reductive of maximal rank: the possibilities for H are determined in [74, Tables 5.1, 5.2];

(III) $G(q) = E_7(q), p > 2$ and $H \cap G(q) = (2^2 \times P\Omega_8^+(q).2^2)$. Sym(3) or $H \cap G(q) = {}^{3}D_4(q).3$;

- (IV) $G(q) = E_8(q), p > 5 \text{ and } H \cap G(q) = \operatorname{PGL}_2(q) \times \operatorname{Sym}(5);$
- (V) $H \cap G(q)$ is as in Table 3.1.1 below;
- (VI) H is of the same type as G that is, $H' \cap G(q) = G(q_0)$ or a twisted version, where \mathbb{F}_{q_0} is a subfield of \mathbb{F}_q ;
- (VII) H is an exotic local subgroup, as in Table 3.1.2;
- (VIII) $G(q) = E_8(q), p > 5 \text{ and } H = (Alt(5) \times Alt(6)).2^2;$
- (IX) $F^*(H) = H_0$ is simple, and not in Lie(p): the possibilities for H_0 are given up to isomorphism by [77];

- (X) $F^*(H) = H(q_0)$ is simple and in Lie(p); moreover $rank(H(q_0)) \le \frac{1}{2}rank(G)$, and one of the following holds:
 - (a) $q_0 \le 9;$
 - (b) $H(q_0) = A_2^{\epsilon}(16);$
 - (c) $H(q_0) = A_1(q_0)$, ${}^2B_2(q_0)$ or ${}^2G_2(q_0)$, and $q_0 \le t(G)$ where t(G) is as in Table 3.1.3 (given by [67]).

In cases (I)-(VIII), H is determined up to G(q)-conjugacy.

Note that Table 3.1.1 includes the subgroups $F_4(q) < E_8(q)$ for $q = 3^a$; these were omitted from the list in [78], but discovered later in [30].

Recent work of Craven has eliminated many of the possibilities left in parts (IX) and (X) of the above theorem:

Theorem 3.1.2. ([31, 32, 33]) Let G be as in Theorem 3.1.1, and let H be a maximal subgroup of G.

- (i) Suppose $F^*(H)$ is an alternating group Alt(n). Then $n \in \{6,7\}$. Moreover, if n = 7, then G is of type E_7 or E_8 .
- (ii) Suppose H is as in part (X) of Theorem 3.1.1 (and not in any of the other parts), and $H(q_0) \neq A_1(q_0)$. Then one of the following holds:
 - (a) $G(q) = E_8(q), q = 3^a, and H(q_0) = PSL_3(3) \text{ or } PSU_3(3);$
 - (b) $G(q) = E_8(q), q = 2^a \text{ and } H(q_0) = PSL_3(4), PSU_3(4), PSU_3(8), PSU_4(2) \text{ or } {}^2B_2(8).$
- (iii) Suppose H is as in part (X) of Theorem 3.1.1 (and not in any of the other parts), and $H(q_0) = A_1(q_0)$. Then one of the following holds:
 - (a) $q_0 = q;$
 - (b) $G(q) = E_7(q)$ and $q_0 = 7,8$ or 25;
 - (c) $G(q) = E_8(q)$.

Table 3.1.1: Possibilities for H in (V) of Theorem 3.1.1

G(q)	possibilities for $F^*(H \cap G(q))$
$G_2(q)$	$A_1(q) (p \ge 7)$
$^{3}D_{4}(q)$	$G_2(q)', \ A_2^{\pm}(q)$
$F_4(q)$	$A_1(q) (p \ge 13), \ G_2(q) (p = 7), \ A_1(q)G_2(q) (p \ge 3, q \ge 5)$
$E_6^{\epsilon}(q)$	$A_{2}^{\pm}(q) \text{ (only for } \epsilon = + \text{ and } p \ge 5), \ G_{2}(q)' \ (p \ne 7, (q, \epsilon) \ne (2, -)),$
	$C_4(q) \ (p \ge 3), \ F_4(q), \ A_2^{\epsilon}(q)G_2(q)'$
$E_7(q)$	$A_1(q)$ (2 classes, $p \ge 17, 19$), $A_2^{\epsilon}(q)$ ($p \ge 5$), $A_1(q)A_1(q)$ ($p \ge 5$),
	$A_1(q)G_2(q) \ (p \ge 3, q \ge 5), \ A_1(q)F_4(q) \ (q \ge 4), \ G_2(q)'C_3(q)$
$E_8(q)$	$A_1(q)$ (3 classes, $p \ge 23, 29, 31$), $B_2(q)$ ($p \ge 5$), $F_4(q)$ ($p = 3$), $A_1(q)A_2^{\epsilon}(q)$ ($p \ge 5$),
	$G_2(q)'F_4(q), \ A_1(q)G_2(q)G_2(q) \ (p \ge 3, q \ge 5),$
	$A_1(q)G_2(q^2) (p \ge 3, q \ge 5)$

Note that Table 3.1.1 contains a small refinement of the corresponding table in [78, Theorem 8] for $G(q) = E_6^{\epsilon}(q)$ and the $A_2^{\pm}(q)$ and $A_2^{\epsilon}(q)G_2(q)'$ entries. This refinement is justified in Remark 5.2 of [16]. Note also that in Table 3.1.1, we write $G_2(q)'$ rather than $G_2(q)$ whenever q = 2 is allowed; this is because $G_2(2)$ is not itself simple, but its derived subgroup is. There is another fact, concerning $A_2^{-}(2)$, we need to clarify in Table 3.1.1: there are two embeddings involving $A_2^{-}(q)$ with q = 2, namely $A_2^{-}(q)$ in ${}^{3}D_4(q)$,

Table 3.1.2: Exotic local subgroups in (VII) of Theorem 3.1.1

and $A_2^-(q)G_2(q)'$ in $E_6^-(q)$. Since $A_2^-(2)$ is not simple and not nilpotent for q = 2, the listed groups are not equal to $F^*(H \cap G(q))$ in these cases; instead one should replace $A_2^-(2)$ by 3^2 .

We shall divide the proof of Theorem 3.1 according to the various parts of Theorem 3.1.1. Note for future reference that by Proposition 2.5.1, the maximal subgroups in the theorem that centralize field, graph-field or graph automorphisms of G(q) are as follows:

- (i) subfield or twisted subgroups as in part (VI);
- (ii) the following subgroups in part (V):

$$C_4(q), F_4(q) < G(q) = E_6^{\epsilon}(q), G_2(q), A_2^{\epsilon}(q) < G(q) = {}^{3}D_4(q).$$

3.2 Small exceptional groups of Lie type

In this section, we deal with some small exceptional groups of Lie type; this will allow us to avoid some degeneracies in later arguments.

Proposition 3.2.1. Theorem 3.1 holds when the socle of G is one of the following exceptional groups of Lie type:

$${}^{2}B_{2}(q), {}^{2}G_{2}(q),$$

 ${}^{2}F_{4}(2)', {}^{3}D_{4}(2), F_{4}(2),$
 $G_{2}(3), G_{2}(4), G_{2}(5).$

Proof. The groups with socle ${}^{2}B_{2}(q)$, ${}^{2}G_{2}(q)$ were dealt with in [45]; groups with socle ${}^{2}F_{4}(2)'$ were dealt with in [34]. The other possibilities have been handled using computational methods, and we describe these in turn.

Socle ${}^{3}D_{4}(2)$. Let G be an almost simple group with socle ${}^{3}D_{4}(2)$. We have computed all the core-free maximal subgroups M of G and we have checked that the action of G on (G:M) is not binary. Except when $M = 3^{2}: 2 \operatorname{Alt}(4)$ or M = 13: 4 and $G = {}^{3}D_{4}(2)$, or $M = 3^{2}: 2 \operatorname{Alt}(4) \times 3$ or M = 13: 12 and $G = {}^{3}D_{4}(2): 3$, we have used the permutation character method, a.k.a. Lemma 1.8.1. In the remaining cases, where the permutation character method does not work, we have used Lemma 1.8.4.

Socle $F_4(2)$. Note that $|F_4(2)| = 2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$ and $G = F_4(2)$ or $F_4(2).2$. In Table 3.2.1 we list the maximal subgroups of G and their indices in G, as given in [85]. Let M be a core-free maximal subgroup of G.

Observe that G has a unique conjugacy class of elements of order 5. Moreover, $F_4(2).2$ has a unique conjugacy class of elements of order 7. In $F_4(2)$ this conjugacy class of 7-elements splits into two distinct

Line	Max. subgroups $F_4(2)$	Index	Max. subgroups of $F_4(2).2$	Index
1	$(2^{1+8}_+ \times 2^6) : \operatorname{Sp}_6(2)$	$3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17$	$[2^{20}]$: Alt(6) · 2 ²	$3^4 \cdot 5 \cdot 7^2 \cdot 13 \cdot 17$
2	$Sp_{8}(2)$	$2^8 \cdot 3 \cdot 7 \cdot 13$	$\operatorname{Sp}_4(4):4$	$2^{15} \cdot 3^4 \cdot 7^2 \cdot 13$
3	$[2^{20}]: (Sym(3) \times PSL_3(2))$	$3^4 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	$(\mathrm{Sym}(6)\mathrm{wr}2).2$	$2^{15} \cdot 3^2 \cdot 7^2 \cdot 13 \cdot 17$
4	$O_8^+(2) : Sym(3)$	$2^{11}\cdot 7\cdot 13\cdot 17$	$[2^{22}](Sym(3) \operatorname{wr} 2)$	$3^4 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$
5	$^{3}D_{4}(2):3$	$2^{12}\cdot 3\cdot 5^2\cdot 17$	$7^2:(3 imes 2\mathrm{Sym}(4))$	$2^{21}\cdot 3^4\cdot 5^2\cdot 13\cdot 17$
6	${}^{2}\!F_{4}(2)$	$2^{12} \cdot 3^3 \cdot 7^2 \cdot 17$	${}^{2}F_{4}(2) \times 2$	$2^{12} \cdot 3^3 \cdot 7^2 \cdot 17$
7	$\mathrm{PSL}_4(3).2$	$2^{16}\cdot 5\cdot 7^2\cdot 17$	$[PSL_4(3).2].2$	$2^{16}\cdot 5\cdot 7^2\cdot 17$
8	$(PSL_3(2) \times L_3(2)) : 2$	$2^{17} \cdot 3^4 \cdot 5^2 \cdot 13 \cdot 17$	$[(PSL_3(2) \times PSL_3(2)): 2].2$	$2^{17} \cdot 3^4 \cdot 5^2 \cdot 13 \cdot 17$
9	$3.(3^2: Q_8 \times 3^2: Q_8)$. Sym(3)	$2^{17} \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$	$[3.(3^2:Q_8 \times 3^2:Q_8).$ Sym(3)].2	$2^{17} \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$

Table 3.2.1: Maximal subgroups of $F_4(2)$ and $F_4(2).2$

 $F_4(2)$ -conjugacy classes; furthermore, $F_4(2)$ has two conjugacy classes of cyclic subgroups of order 7. This information can be deduced from [85].

Let $p \in \{5,7\}$. From the information in the previous paragraph and from Lemma 1.6.15, we deduce that, if $p \mid |M|$ and $p \mid |G : M|$, then the action of G on the cosets of M is not binary. In particular, it remains to consider the case that, for each $p \in \{5,7\}$, p^2 divides |M| or p^2 divides |G : M|.

When $M \in \{{}^{3}D_{4}(2) : 3, {}^{2}F_{4}(2), (PSL_{3}(2) \times PSL_{3}(2)) : 2\}$ and $G = F_{4}(2)$, or when $M \in \{Sp_{4}(4) : 4, (Sym(6) wr 2).2, {}^{2}F_{4}(2) \times 2, [(PSL_{3}(2) \times PSL_{3}(2)) : 2].2\}$ and $G = F_{4}(2).2$, we have verified that the hypothesis of Lemma 1.8.3 with d = 2 holds true (by computing all proper subgroups X of M with |M : X| odd). Thus, we deduce that either G is not binary in its action on (G : M) or 2 divides |G : M| - 1. However, the second possibility yields a contradiction (in each case under consideration |G : M| - 1 is odd). Therefore, G is not binary on (G : M). (Observe that for this computation we only need M as an abstract group and we do not require the embedding of M in G.)

When $M = 3.(3^2 : Q_8 \times 3^2 : Q_8)$. Sym(3) and $G = F_4(2)$, or when $M = [3.(3^2 : Q_8 \times 3^2 : Q_8)$. Sym(3)].2 and $G = F_4(2).2$, since there is not enough information in Table 3.2.1 to determine the isomorphism class of M, we have used magma to construct M inside G. For this we used the fact that M is the normalizer of a cyclic group of order 3 generated by an element in the conjugacy class 3C. This was possible because generators of G and an element in the class 3C are available in the online atlas webpage. Then, we have argued as in the previous paragraph applying Lemma 1.8.3. The group M contains a unique subgroup X(up to conjugacy), such that

- |M:X| is odd,
- the permutation group M_X induced by M on (M:X) is binary and
- every section of M is isomorphic to some section of M_X .

This subgroup X has index 3 in M and $M_X \cong \text{Sym}(3)$. As M is maximal in G, we obtain that G in its action on (G:M) is primitive. If G acting on (G:M) has a suborbit of cardinality 3, then it follows from [97] that |M| divides 48, which is clearly a contradiction. Therefore, G in its action on (G:M) has no suborbits of cardinality 3. Thus, if G is binary in its action on (G:M), then, from the magma computation above, G has no non-trivial suborbits of odd size in its action on (G:M). However, this implies that |G:M| - 1 is even, which is clearly a contradiction.

Using Table 3.2.1, we see that it remains to deal with the action of $G = F_4(2).2$ on the right cosets of $M = [2^{22}]$: (Sym(3) wr 2). First, we work with the restriction of this action to $G' := F_4(2)$. Using the generators of G', we may construct $M \cap G'$ using the fact that it is a local subgroup (first by finding a Sylow 2-subgroup P of G' and then by computing the normalizer of a suitable subgroup of P having index 4). Let K be a Sylow 3-subgroup of $M \cap G'$. We see that K contains four 3-elements in the class 3C, two 3-elements in the class 3A and two more 3-elements in the class 3B. Thus we may write $K = \langle g, h \rangle$, where g and h are 3A and 3B elements (respectively) and gh is a 3C element. Using the formula $|x^G \cap M|/|x^G|$ we may compute the number of fixed points of $x \in M$ without constructing the permutation representation explicitly. We see that g and h both fix 945 points, gh fixes 81 points and K fixes 9 points. Using this information, we see that there exists a K-invariant subset $\Lambda \subseteq (G : M)$ having cardinality 10, say $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_9\}$, such that

$$g^{\Lambda} := (\lambda_0)(\lambda_1, \lambda_2, \lambda_3)(\lambda_4, \lambda_5, \lambda_6)(\lambda_7)(\lambda_8)(\lambda_9),$$

$$h^{\Lambda} := (\lambda_0)(\lambda_1, \lambda_2, \lambda_3)(\lambda_4)(\lambda_5)(\lambda_6)(\lambda_7, \lambda_8, \lambda_9),$$

where g^{Λ} and h^{Λ} are the restrictions of g and h to Λ . It is now easy to verify that the two 10-tuples

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9)$$
 and $(\lambda_0, \lambda_2, \lambda_3, \lambda_1, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9)$

are 2-subtuple complete for the action of G on (G : M). If the action of G on (G : M) is binary, then there exists $a \in G$ mapping the first 10-tuple into the second 10-tuple. As a fixes λ_0 , we get $a \in G_{\lambda_0} = M$. Moreover, a fixes set-wise Λ and $a^{\Lambda} = (\lambda_1, \lambda_2, \lambda_3)$. Therefore, $G_{\Lambda} \cap G_{\lambda_0}$ has a Sylow 3-subgroup of cardinality divisible by 3^3 , but this contradicts the fact that a Sylow 3-subgroup of M has cardinality |K| = 9. (This construction is inspired from Example 2.2 in [45].)

Socle $G_2(q)$ $(q \leq 5)$. These groups (and their automorphism groups) are available in magma. For each possible group G we have computed its maximal subgroups. When q = 3, we have then constructed the permutation representations and checked that the group is not binary by witnessing non-binary triples. When $q \in \{4, 5\}$, we have computed the permutation characters and used Lemma 1.8.1: this test was always successful for proving that the action was not binary except when q = 5 and $M \cong 2^3$.PSL₃(2). In this last case we generated, for 10^6 times, two cosets Mg_1 and Mg_2 of M in G, and we tested whether Lemma 1.8.4 applies with $\omega_0 := M$, $\omega_1 := Mg_1$ and $\omega_2 := Mg_2$. After a few iterations we have found a suitable g_1 and g_2 and hence the action of G on (G:M) is not binary.

In light of Proposition 3.2.1, we assume for the remainder of this section that the socle of G is not one of the groups listed in the proposition.

3.3 Parabolic subgroups

In this section we prove Theorem 3.1 for parabolic actions of exceptional groups of Lie type. We use the notation P_i (resp. P_{ij}) for a parabolic subgroup which corresponds to deleting node *i* (resp. nodes *i*, *j* etc.) from the Dynkin diagram. For twisted groups we shall adopt a similar convention using the untwisted Dynkin diagram: for example for ${}^{3}D_4(q)$ the maximal parabolic subgroups are denoted by P_2 and P_{134} , and so on.

Here is the main result of the section. The cases excluded in the proposition (those in Table 3.3.1) will be dealt with in Lemma 3.3.2.

Proposition 3.3.1. Assume G is almost simple with socle G(q), an exceptional group of Lie type over \mathbb{F}_q , and suppose G(q) is not as in Proposition 3.2.1. Let H be a maximal parabolic subgroup of G, and $\Omega = (G:H)$. Suppose further that (G(q), H) is not as in Table 3.3.1. Then (G, Ω) is not binary.

G(q)	Н
${}^{3}D_{4}(q), q \in \{3, 4, 5\}$	P_2
$E_{6}(2)$	P_2
${}^{2}\!E_{6}(2)$	P_2, P_4, P_{16}

Table 3.3.1: Exceptions in Prop. 3.3.1

Proof. Let $H = P_i$ (or P_{ij} in cases where G contains a graph automorphism of G(q)). Inspection of extended Dynkin diagrams shows that there exist subgroups $A \cong \operatorname{SL}_r(q)$ in H, and $S \cong \operatorname{SL}_{r+1}(q)/Z$ in G (where Z is central), such that $A \leq S$ and $S \not\leq H$, where r is as in Table 3.3.2. Hence Lemma 1.6.10 produces a subset Δ of Ω for which G^{Δ} contains the 2-transitive group $\operatorname{ASL}_r(q)$ of degree q^r . If G^{Δ} does not contain $\operatorname{Alt}(\Delta)$, this implies that G is not binary by Lemma 1.6.12, as required. So assume that $G^{\Delta} \geq \operatorname{Alt}(\Delta)$. Then $q^r \leq N_G$, where N_G is as defined in Lemma 2.1.2. Also $\operatorname{Alt}(q^r - 1)$ must be a section of H. This implies that (G, H, q) is either as in Table 3.3.1, or is one of the following:

$$\begin{array}{c|c} G(q) & G_2(q) & {}^3D_4(q) & {}^2F_4(q) \\ \hline H & P_1, \text{ Borel } & P_2 & \text{any parabolic} \end{array}$$

Consider $G_2(q)$. Here H contains $T_1 = \{h_{\alpha_1}(c) : c \in \mathbb{F}_q\} \cong C_{q-1}$, and this acts fixed-point-freely on the root group $U = U_{-\alpha_0}$, where α_0 is the longest root. Observe that $T_1U \cap H = T_1$. Hence, if we set $\Delta = \{Hu : u \in U\} \subseteq \Omega$, then $|\Delta| = q$ and $G^{\Delta} \ge (T_1U)^{\Delta} = \operatorname{AGL}_1(q)$. Hence, if $q > N_G = 6 + \delta_{p,5}$ (which is the case, as $q \ne 3, 4, 5$ by hypothesis), then as above, G is not binary. A similar proof applies to the case $G(q) = {}^{3}D_4(q)$: here q = 3, 4, 5 are not excluded in the hypothesis, so these cases are included in Table 3.3.1.

Finally, consider $G(q) = {}^{2}F_{4}(q)$, and note that q > 2 here, by hypothesis. In this case, the maximal parabolics are $P_{i} = Q_{i}L_{i}$ for i = 1, 2, where Q_{i} is the unipotent radical and

$$L_1 = \operatorname{GL}_2(q), \ L_2 = {}^2B_2(q) \times (q-1).$$

Let $H = P_i$ and $\Omega = (G : H)$, and suppose (G, Ω) is binary. Let $S \cong \mathbb{F}_q$ be the root subgroup corresponding to the highest root, and S^- its negative. For i = 1, 2 there is a torus $T_1 < L_i$ of order q - 1 acting fixedpoint-freely on both S and S^- . Since $S^- \not\leq P_i$, the Frobenius group $F = S^-T_1$ satisfies $F \cap P_i = T_1$, and so we obtain in the usual way a subset Δ of Ω with $G^{\Delta} \geq \operatorname{AGL}_1(q)$, forcing $q \leq 8$ by Lemma 2.1.2. If q = 8 and $G^{\Delta} \geq \operatorname{Alt}(8)$, then $H = P_i$ must contain a section isomorphic to $\operatorname{Alt}(7)$, which is not the case. This final contradiction completes the proof.

$G(q) = E_8(q)$	$H = P_i, i =$	1	2	3	4	5	6	7	8
	r =	$\overline{7}$	8		5	5	5	6	7
$G(q) = E_7(q)$	$H = P_i, i =$	1	2	3	4	5	6	7	
	r =	6	7	5	4	4	5	6	
$G(q) = E_6(q)$	$H = P_i, i =$	1	2	3	4	16	35		
	r =	5	2	4	3	4	3		
$G(q) = {}^{2}E_{6}(q)$	$H = P_i, i =$	16	2	35	4				
	r =	3	2	3	2				
$G(q) = F_4(q)$	$H = P_i, i =$	1	2	3	4	14	23		
	r =	2	2	3	3	2	2		
$G(q) = G_2(q)$	$H = P_i, i =$	2							
	r =	2							
$G(q) = {}^{3}D_4(q)$	$H = P_i, i =$	134							
	r =	2							

Table 3.3.2: Values of r in proof of Prop. 3.3.1

The remaining cases are resolved by magma computations:

Lemma 3.3.2. Let G be as in Proposition 3.3.1, and let H be a maximal parabolic subgroup of G as listed in Table 3.3.1. Let $\Omega = (G : H)$. Then (G, Ω) is not binary.

Proof. Suppose first that $G(q) = {}^{3}D_{4}(q)$; we refer to [43] for a description of the parabolic subgroup H here (note that, although [43] assumes that q is odd, [49] confirms that the same description applies for q even). Let T be a maximal torus contained in $H \cap G(q)$ that is isomorphic to $C_{q-1} \times C_{q^{3}-1}$. We assume that $H \cap G(q)$ contains the Borel subgroup generated by all positive root subgroups. Let α (resp. β) be the short (resp. long) fundamental root, and let U be the short root group $X_{-2\alpha-\beta}$; then $|U| = q^{3}$ and U is not contained in H. What is more, [43, Table 2.3] confirms that T acts transitively on the non-identity elements of U. Define $\Gamma = \{Hu : u \in U\}$, a subset of Ω of order q^{3} . Then $U \rtimes T$ stabilizes Γ and acts 2-transitively on Γ . Since $q \geq 3$, G(q) does not contain a section isomorphic to $Alt(q^{3})$ by Lemma 2.1.1. Therefore, Γ is a beautiful subset and Lemma 1.6.12 yields the conclusion.

In the case where $G = E_6(2)$ or $E_6(2).2$ and $H = P_2$, we compute the index |G:H| and we select the complex irreducible characters of G having degree at most |G:H|. Then we find all non-negative integer linear combinations of these irreducible characters having degree |G:H|. These combinations are our putative permutation characters. Then, for each of these characters, we use Lemma 1.8.1 to prove that the action under consideration is not binary.

Finally, for $G(q) := {}^{2}E_{6}(2)$, let H be one of the parabolic subgroups in Table 3.3.1. Then we see that 5 divides both |G:H| and |H|, but 5² does not divide |H|. Moreover, G contains a unique conjugacy class of elements of order 5 (see [28]). Therefore Lemma 1.6.15 implies that the action of G on (G:H) is not binary.

3.4 Maximal rank subgroups

In this section we prove Theorem 3.1 in the case where the point stabilizer H is a subgroup of maximal rank that is not the normalizer of a maximal torus in G. Such maximal subgroups are listed in Table 5.1 of [74]. They will be listed in Tables 3.4.2 - 3.4.8 below, where for notational convenience we list each possibility for H as a "type", which is a subgroup (usually equal to $H^{(\infty)}$) of small index in H.

Here is the main result of this section. The cases excluded in the proposition (those in Table 3.4.1 and also the case of socle ${}^{2}F_{4}(q)$) will be handled later in Lemmas 3.4.2, 3.4.3 and 3.4.4.

Proposition 3.4.1. Assume G is almost simple with socle G(q), an exceptional group of Lie type over \mathbb{F}_q . Suppose G(q) is not as in Proposition 3.2.1, and suppose also that $G(q) \neq {}^2F_4(q)$. Let H be a maximal subgroup of maximal rank in G, as in [74, Table 5.1], and let $\Omega = (G : H)$. Then either (G, Ω) is not binary, or (G(q), H) is as in Table 3.4.1.

Table 3.4.1: Exceptions in Prop. 3.4.1

G(q)	type of H
$E_7(q)$	$A_1(q^7)$
$E_6^{\epsilon}(q)$	$A_2^\epsilon(q^3)$
$E_{8}(2)$	$A_2^-(2)^4, A_2^-(2^4)$
$^{2}E_{6}(2)$	$A_2^{-}(2)^3, D_4(2)T_2$

Proof. We adopt the same method as in the previous section, using Lemma 1.6.10. In Tables 3.4.2 - 3.4.8 we have listed the possibilities for H, together with a subgroup $A \cong \operatorname{SL}_r(q^u)$ of H (where u = 1 or 2), such that A is contained in a subgroup $S \cong \operatorname{SL}_{r+1}(q^u)/Z$ of G that does not lie in H. We shall justify these assertions below.

Given the assertions on the tables, the argument proceeds as in the proof of Proposition 3.3.1: Lemma 1.6.10 produces a subset Δ of Ω of size q^{ru} , for which G^{Δ} contains $ASL_r(q^u)$. If $G^{\Delta} \ge Alt(\Delta)$, then $q^{ru} \le N_G$

type of H	$D_8(q)$	$A_1(q)E_7(q)$	$A_8(q)$	$A_2(q)E_6(q)$	$A_4(q)^2$	$D_4(q)^2$	$A_2(q)^4$
A	$\mathrm{SL}_8(q)$	$\operatorname{SL}_7(q)$	$\operatorname{SL}_7(q)$	$\operatorname{SL}_6(q)$	$\operatorname{SL}_5(q)$	$\mathrm{SL}_4(q)$	$SL_3(q)$
type of H	$A_1(q)^8(q>2)$	$A_8^-(q)$	$A_2^-(q)E_6^-(q)$		$A_{4}^{-}(q^{2})$	$D_4(q^2)$	$^{3}D_{4}(q)^{2}$
A	$\operatorname{SL}_2(q)$	$SL_3(q^2)$	$\mathrm{SL}_3(q^2)$	$\operatorname{SL}_2(q^2)$	$\operatorname{SL}_2(q^2)$	$\operatorname{SL}_2(q^2)$	$SL_3(q)$
type of H		$A_{2}^{-}(q)^{4}$	$A_{2}^{-}(q^{2})^{2}$	$A_{2}^{-}(q^{4})$			
A	$SL_3(q^2)$	$\operatorname{SL}_2(q)$	$\operatorname{SL}_2(q^2)$	$\operatorname{SL}_2(q)$			

Table 3.4.2: Subgroups H and A for $E_8(q)$

Table 3.4.3: Subgroups H and A for $E_7(q)$

type of H	$A_1(q)D_6(q)$	$A_7(q)$	$A_2(q)A_5(q)$	$A_1(q)^3 D_4(q)$	$A_1(q)^7(q>2)$	$E_6(q)T_1$
A	$\operatorname{SL}_6(q)$	$\mathrm{SL}_5(q)$	$\mathrm{SL}_5(q)$	$\operatorname{SL}_4(q)$	$\operatorname{SL}_2(q)$	$\operatorname{SL}_6(q)$
type of H	$A_{7}^{-}(q)$	$A_2^-(q)A_5^-(q)$	$A_1(q^3) {}^3\!D_4(q)$	$E_{6}^{-}(q)T_{1}$		
A	$\operatorname{SL}_2(q^2)$	$\mathrm{SL}_3(q^2)$	$\mathrm{SL}_3(q)$	$\operatorname{SL}_3(q^2)$	_	

Table 3.4.4: Subgroups H and A for $E_6(q)$

t	type of H	$A_1(q)A_5(q)$	$A_2(q)^3$	$A_2(q^2)A_2^-(q)$	$D_4(q)T_2$	$^{3}D_{4}(q)T_{2}$	$D_5(q)T_1$
	A	$\mathrm{SL}_4(q)$	$SL_3(q)$	$\operatorname{SL}_2(q^2)$	$\mathrm{SL}_4(q)$	$\mathrm{SL}_3(q)$	$\mathrm{SL}_5(q)$

Table 3.4.5: Subgroups H and A for ${}^2\!E_6(q)$

type of H	$A_1(q)A_5^-(q)$	$A_{2}^{-}(q)^{3}$	$A_2(q^2)A_2(q)$	$D_4(q)T_2$	$D_5^-(q)T_1$
A	$\operatorname{SL}_2(q^2)$	$\operatorname{SL}_2(q)$	$\mathrm{SL}_3(q)$	$SL_3(q)$	$SL_2(q^2)$

Table 3.4.6: Subgroups H and A for $F_4(q)$

type of H	$A_1(q)C_3(q)$	$B_4(q)$	$D_4(q)$	${}^{3}D_{4}(q)$	$A_2(q)^2$	$A_2^-(q)^2$
A	$\mathrm{SL}_2(q)$	$\operatorname{SL}_2(q)$	$\operatorname{SL}_2(q)$	$SL_3(q)$	$\mathrm{SL}_3(q)$	$\operatorname{SL}_2(q)$
type of H	$B_2(q)^2$	$B_2(q^2)$				
A	$\mathrm{SL}_2(q)$	$\operatorname{SL}_2(q)$				

Table 3.4.7: Subgroups H and A for $G_2(q)$

type of H	$A_1(q)^2$	$A_2(q)$	$A_2^-(q)$
A	$\mathrm{SL}_2(q)$	$\mathrm{SL}_2(q)$	$\overline{\mathrm{SL}}_2(q)$

Table 3.4.8: Subgroups H and A for ${}^{3}D_{4}(q)$

type of H	$A_1(q)A_1(q^3)$	$A_2(q)$	$A_2^-(q)$
A	$\operatorname{SL}_2(q)$	$\mathrm{SL}_2(q)$	$\operatorname{SL}_2(q)$

(as defined in Lemma 2.1.2), and also $Alt(q^{ru} - 1)$ must be a section of H. By Lemmas 2.1.1 and 2.1.2,

this eliminates all possibilities except for the list in Table 3.4.1, together with the following cases:

$$\begin{array}{c|c}
G & F_4(q), q = 3 \\
\hline
H & B_4(q), D_4(q)
\end{array}$$
(3.4.1)

We shall handle the cases in (3.4.1) after first justifying the assertions in Tables 3.4.2 - 3.4.8. For the cases in the tables where the maximal rank subgroup H is just an untwisted subsystem subgroup over \mathbb{F}_q , the existence of the subgroups A < S is clear from inspection of the extended Dynkin diagram of G. (The cases $(A, H) = (\mathrm{SL}_2(q), D_4(q))$ for $F_4(q)$, and also $(A, H) = (\mathrm{SL}_2(q), A_2(q))$ for $G_2(q)$ and ${}^{3}D_4(q)$ require some small additional observations: in the first case, $D_4(q)$ contains a subgroup $A = \mathrm{SL}_2(q)$ corresponding to a short root in the F_4 -system, and this lies in a short root $S = \mathrm{SL}_3(q)$ which is not contained in $D_4(q)$; and in the second case, there exists $x \in C_G(A) \setminus H$, and we can take $S = H^x$.)

Now consider cases in Tables 3.4.2 - 3.4.8 where H involves a twisted group, or a group over a proper extension field of \mathbb{F}_q .

Consider first Table 3.4.2, where $G(q) = E_8(q)$. In the cases where H is of type ${}^{3}D_4(q)^2$ or $A_2^-(q)^4$, we choose A to be a subsystem subgroup $SL_3(q)$ or $SL_2(q)$ of one of the factors. Now suppose H is of type $A_8^-(q)$. Then H has a Levi subgroup $S = SL_4(q^2)$, and we let A be a natural subgroup $SL_3(q^2)$ of this. We use Lemma 2.6.1 to show that there is a conjugate S^x such that $A < S^x \leq H$. First observe that the fusion control hypotheses of the lemma for A < S < H clearly hold. Now $N_G(A)$ contains a subgroup $A_2(q)A_2^-(q)$ (a subgroup $A_2(q^2)A_2(q)A_2^-(q)$ can be seen inside a subsystem subgroup of type E_6A_2), whereas $N_H(A)$ normalizes a subgroup $A_2^-(q)T_2A$ of H, where T_2 is a torus of order $q^2 - 1$. The factor $A_2(q)$ of $N_G(A)$ does not have a factorization with one of the factors being $N(T_2)$ (see [69]); hence $N_G(A) \neq N_H(A) (N_G(S) \cap N_G(A))$ and the required conjugate of S exists by Lemma 2.6.1.

Next suppose H is of type $A_2^-(q)E_6^-(q)$. Here we choose A to be a subgroup $SL_3(q^2)$ of a Levi subgroup $SU_6(q)$ of the $E_6^-(q)$ factor; this is contained in a subgroup $S = SL_4(q^2)$ as defined in the previous paragraph, and $S \not\leq H$. A similar argument applies to produce a suitable subgroup $A = SL_2(q^2)$ when H has type $A_4^-(q)^2$, and also a subgroup $A = SL_3(q^2)$ when H has type $^3D_4(q^2)$. In the case where H is of type $A_4^-(q^2)$, we choose A to be a subgroup $SL_2(q^2)$ corresponding to a natural subgroup $SU_2(q^2)$ of the unitary group; this arises from a subsystem A_1A_1 of the ambient algebraic group, and is conjugate to the subgroup $SL_2(q^2)$ of the previous case. The same subgroup $A = SL_2(q^2)$ pertains when H is of type $A_2^-(q^2)^2$ or $D_4(q^2)$. In the latter case, we also need to apply Lemma 2.6.1 to produce a subgroup $S = SL_3(q^2)$ such that $A < S \not\leq H$: here $N_G(A)$ contains $D_6^-(q)$, which does not factorize as $N_H(A)$ ($N_G(S) \cap N_G(A)$).

Finally, for H of type $A_2^-(q^4)$, let A be a natural subgroup $SL_2(q)$ of $A_2^-(q^4)$ (acting as $2 \oplus 1$ on the associated 3-dimensional unitary module). Then A is a diagonal subgroup of a subsystem subgroup of type $A_1(q)^4$ which lies in a subsystem $A_2(q)^4$, and hence A lies in a diagonal $A_2(q)$ in the latter. This completes the justification for Table 3.4.2.

For $G(q) = E_7(q)$, $E_6^{\epsilon}(q)$, $F_4(q)$ or $G_2(q)$ the justification for the existence of the subgroups A < S uses the same arguments as above. Extra argument using Lemma 2.6.1 is needed just for the cases

$$(G(q), H) = (E_7(q), A_7^-(q)), (E_6(q), A_2(q^2)A_2^-(q)) \text{ and } ({}^2E_6(q), A_1(q)A_5^-(q));$$

observe that $C_G(A)$ contains ${}^{2}D_4(q)A_1(q)$, ${}^{2}A_3(q)$ or $A_3(q)$ in the respective cases, from which it can be seen that $N_G(A)$ does not factorize as $N_H(A)$ ($N_G(S) \cap N_G(A)$), so that Lemma 2.6.1 applies.

We have now justified all the assertions in Tables 3.4.2 - 3.4.8.

It remains to handle the cases in (3.4.1). Let $G = F_4(3)$, and let H be a maximal rank subgroup $B_4(3)$ or $D_4(3)$. Sym(3). First consider the case where $H = D_4(3)$. Sym(3). Let $S = SL_4(3) < H$ be generated by root subgroups, and $A = SL_3(3) < S$. Then A < S < H, and H controls fusion of S in G (as all subgroups $SL_4(3)$ generated by root groups in H are H-conjugate). We claim that $N_G(A)$ does not factorize as $N_H(A)$ ($N_G(S) \cap N_G(A)$). To see this, observe that $N_G(A)/A$ contains $\tilde{A}_2(3)$ (generated by short root groups); while $|N_H(A)/A|_3 = 3$ and $|(N_G(S) \cap N_G(A))/A|_3 = |N_{A_1(3)}S(A)/A|_3 = 3$, proving the claim. It then follows from Lemma 2.6.1 that there is a conjugate S^g such that $A < S^g \not\leq H$, so as usual there is a subset Δ with $G^{\Delta} \geq ASL_3(3)$, showing that (G, (G : H)) is not binary.

Now consider the case $H = B_4(3)$. Again take A < S < H with $A = SL_3(3)$, $S = SL_4(3)$ generated by root subgroups. This time H does not control fusion of S in G, as there are two classes of subgroups isomorphic to $SL_4(3)$ in $B_4(3)$ with representatives S_1 , S_2 of types $SL_4(3)$ and $\Omega_6^+(3)$, respectively. We again aim to find a conjugate S^g such that $A < S^g \not\leq H$, but we need to do this a little differently. Define

$$\Lambda = \{ R < G : A \le R, R \text{ conjugate to } S \text{ in } G \},\$$

$$\Phi = \{ R \in \Lambda : R < H \}.$$

We shall show that $|\Lambda| > |\Phi|$, which will achieve our aim, completing the proof that the action of G on (G:H) is not binary.

First observe that $N_G(A)$ acts transitively on Λ , since

$$R \in \Lambda \quad \Rightarrow R = S^{g} (g \in G)$$

$$\Rightarrow A, A^{g^{-1}} < S$$

$$\Rightarrow A^{g^{-1}} = A^{s} (s \in S)$$

$$\Rightarrow R = S^{sg}, sg \in N_{G}(A).$$

Hence $|\Lambda| = |N_G(A) : N_G(A) \cap N_G(S)| = |\tilde{A}_2(3).2 : T_1A_1(3).2|$, which is divisible by 3².13.

In similar fashion, we see that $N_H(A)$ has two orbits Φ_1 , Φ_2 on Φ , with orbit representatives S_1 and S_2 . The orbit sizes are $|\Phi_i| = |N_H(A) : N_H(A) \cap N_H(S_i)|$ for i = 1, 2. Hence $|\Phi_1| = |T_1A_1(3).2 : T_2.2|$ divides 24, while $|\Phi_2| = 1$. Therefore $|\Lambda| > |\Phi|$, as required.

The next three results deal with the cases not covered by Proposition 3.4.1 (those in Table 3.4.1 and also the ${}^{2}F_{4}(q)$ case).

Lemma 3.4.2. Let G be as in Proposition 3.4.1, and suppose that (G(q), H) is as in line 1 or 2 of Table 3.4.1. If $\Omega = (G : H)$, then (G, Ω) is not binary.

Proof. In these cases $G(q) = E_7(q)$ or $E_6^{\epsilon}(q)$ ($\epsilon = \pm$), and H is of type $A_1(q^7)$ or $A_2^{\epsilon}(q^3)$, respectively. In the first case, $H \cap G(q) = A_1(q^7).7$, and we choose a subfield subgroup $X = A_1(q) \times 7 = X_0 \times 7$ of H. The factor $X_0 = A_1(q)$ is contained diagonally in a subsystem subgroup $A_1(q)^7$ of $E_7(q)$, which has normalizer acting as $L_3(2)$ on the 7 factors (see [74, Table 5.1]). Hence $N_G(X)$ is not contained in H, and so there is a suborbit here on which H acts as (H : Y), where $X \leq Y \leq N_H(X_0)$. If q > 2, then Y is maximal in $\langle H \cap G(q), Y \rangle$, and we know, by [45] that the associated action on cosets is not binary. Appealing to Lemma 1.6.2 if necessary, we conclude that the action of H on (H : Y) is not binary, and now Lemma 1.6.1 implies that the action of G on (G : H) is not binary. Suppose now q = 2. Let x be an element of order 7 in H. From [6, Table 2], we see that there exists $g \in C_G(x) \setminus H$ and hence $H \cap H^g$ is a proper subgroup of H containing x. We have calculated with magma the faithful transitive actions of Hhaving point stabiliser of order divisible by 7. We find that all such actions are not binary. Therefore, the action of H on $(H : H \cap H^g)$ is not binary, and hence by Lemma 1.6.1 so is the action of G on $(G : \Omega)$.

For the $E_6^{\epsilon}(q)$ cases, we argue similarly. Choose a subfield subgroup $X = A_2^{\epsilon}(q) \times 3$ of H. The $A_2^{\epsilon}(q)$ factor is contained diagonally in a subsystem $A_2^{\epsilon}(q)^3$, which has normalizer acting as Sym(3) on the factors. Hence again $N_G(X) \not\leq H$ and so there is a suborbit here on which H acts as (H : Y) where $X \leq Y \leq N_H(A_2^{\epsilon}(q))$. It will be sufficient to show that this action is not binary.

If $\epsilon = +$, then we take a subgroup $S = \operatorname{SL}_2(q)$ of Y, and it is easy to verify that S normalizes and acts transitively on an elementary abelian subgroup $E = E_{q^2}$ of H that is not contained in Y. Defining $\Delta = \{Ye : e \in E\}$, we obtain, in the usual way, that either Δ is a beautiful subset in the action of Hon (H : Y) (and we are done), or else $H^{\Delta} \ge \operatorname{Alt}(\Delta)$. But by Lemma 2.1.2, this is not possible unless q = 2. When q = 2, for G with $E_6(2) \le G \le \operatorname{Aut}(E_6(2))$ and for $H := N_G(A_2(8))$, we have computed the subgroups K of H with |H : K| odd. For each such pair (H, K), we have checked that, if the action of H on (H : K) is binary, then K contains $A_2(8)$. Hence there is no binary action of H of odd degree with $A_2(8)$ acting non-trivially. From this we deduce that, if the action of G on (G : H) is binary, then each non-trivial subdegree of G must be even, which implies that |G : H| is odd, a contradiction. If $\epsilon = -$, then we argue similarly with a cyclic subgroup K of order q - 1 in Y. There is a subgroup $E = E_q$ that K normalizes and upon which it acts fixed-point-freely. The same line of argument rules out all cases with q > 8. Also, using Lemma 2.1.1, we can rule out q = 8 (since Alt(7) is not a section of SU₃(8)) and q = 7 (since Alt(6) is not a section of SU₃(7)). To deal with q < 7, we use magma as follows.

Let M be a maximal subgroup of G with socle ${}^{2}A_{2}(q^{3})$. We consider the permutation action of G on the right cosets Ω of M in G. Observe that q divides $|\Omega| = |G : M|$ and hence G has a suborbit of cardinality relatively prime to q. Using magma, we have verified that all faithful transitive actions of M on a set of cardinality relatively prime to q are non binary. Therefore, the action of M on each non-trivial suborbit is non binary. From this, it follows that G is not binary on Ω .

Lemma 3.4.3. Let G be as in Proposition 3.4.1, and suppose that (G(q), H) is as in line 3 or 4 of Table 3.4.1. If $\Omega = (G : H)$, then (G, Ω) is not binary.

Proof. Adopt the hypothesis of the lemma, and assume that (G, Ω) is binary.

Suppose, first, that $G(q) = {}^{2}E_{6}(2)$ and that H is of type $A_{2}^{-}(2)^{3}$. An inspection of extended Dynkin diagrams confirms that H contains a subgroup K of type $A_{2}^{-}(2)^{2}$ that, in turn, is embedded in the natural way inside a subgroup $L = \text{PSU}_{6}(2)$ in G(q). Now K has a "diagonal" subgroup $Q \cong Q_{8}$ which normalizes, and acts fixed-point-freely upon an elementary abelian subgroup, E_{9} , of L. Now we study the conjugates of E_{9} under $N_{L}(Q)$ and observe that Q normalizes each of these conjugates. There are two possibilities: first, one of these conjugate subgroups, say E, does not lie in H. In this case, $E \cap H = \{1\}$ and setting $\Delta = H^{E}$ we see that the set-wise-stabilizer of Δ acts 2-transitively on Δ . Hence, as (G, Ω) is binary, we have $G^{\Delta} \geq \text{Alt}(\Delta)$. However $|\Delta| = 9$, and H does not have a section isomorphic to Alt(8), so this is impossible. This possibility is, therefore, excluded, and we conclude that all of the conjugate subgroups lie in H. But now direct calculation, using for instance GAP, confirms that $\langle E_{9}^{g} | g \in N_{L}(Q) \rangle = L$, which is a contradiction. Thus this possibility is also excluded.

Consider, next, the situation where $G(q) = E_8(2)$ and H is of type $A_2^-(2)^4$. In this case a version of the previous argument yields a contradiction, this time using a subgroup K of type $A_2^-(2)^3$ embedded in a subgroup isomorphic to $L = PSU_9(2)$ in G(q).

Suppose, next, that $G(q) = E_8(2)$ and H has type $A_2^-(2^4)$. In this case, $G = E_8(2)$ and $H \cong \operatorname{Aut}(\operatorname{PSU}_3(16)) = \operatorname{PSU}_3(16).8$ (see [74]). We have computed all the core-free subgroups K of H with |H:K| odd and shown that, for each of these subgroups, the action of H on (H:K) is not binary, by witnessing a non-binary triple. Hence, as (G, Ω) is binary (by assumption), |G:H| must be odd, which is clearly a contradiction.

Suppose, finally, that $G(q) = {}^{2}E_{6}(2)$ and that H is of type $D_{4}(2)T_{2}$. Consulting [105], we see that $H \cap G(q)$ has shape $(3 \times \Omega_{8}^{+}(2) : 3) : 2$, extending to $(3^{2} : 2 \times \Omega_{8}^{+}(2)) : \text{Sym}(3)$ in G(q). Sym(3) = Aut(G(q)). We have calculated the transitive actions of all groups of the relevant shapes on sets of odd cardinality using magma. We find that the only binary actions for such groups occur when the set is of size 1, 3 or 9, in which case the kernel of the action contains $\Omega_{8}^{+}(2)$. As (G, Ω) is binary, we conclude, therefore, that all non-trivial subdegrees must be even. This contradicts the fact that |G : H| is even.

Lemma 3.4.4. Let G be as in Proposition 3.4.1, and suppose that $G(q) = {}^{2}F_{4}(q) \ (q > 2)$. If $\Omega = (G : H)$, then (G, Ω) is not binary.

Proof. Here the possibilities for the maximal rank subgroup $H(q) := H \cap G(q)$ are:

 $SU_3(q).2$, $PGU_3(q).2$, $(Sz(q) \times Sz(q)).2$, $Sp_4(q).2$.

We write $G = G(q)\langle \phi \rangle$, where ϕ is a field automorphism of odd order f (possibly f = 1).

In the first two cases, let T < H(q) be a maximal torus of order $(q+1)^2$. Then $N_{H(q)}(T) = T.(\text{Sym}(3) \times 2)$, while $N_{G(q)}(T) = T.\text{GL}_2(3)$. Hence $N_{H(q)}(T.2^2) = T.2^2$ and $N_{G(q)}(T.2^2) = T.D_8$. It follows that there exists $x \in G \setminus H$ such that $H \cap H^x$ contains $T.2^2$ but does not contain H. We can chose ϕ to normalize all

of the above subgroups, so that we also have $H \cap H^x \ge T.(2^2 \times f)$. Now Lemma 2.2.13 implies that the action of H on $(H : H \cap H^x)$ is not binary, and the result follows from Lemma 1.6.1.

Next consider the possibility that $H(q) = (Sz(q) \times Sz(q)).2$. Here, as we shall see, when we study suborbits we find a primitive group via a product action hence we could appeal to [106]; nonetheless we give a direct argument. First, let T < H(q) be the direct product of two maximal tori of Sz(q) of order q-1. Then $N_{H(q)}(T) = T.[8]$, while $N_{G(T)}(T) \ge T.[16]$ (one can see this, for instance, by using the fact that T is also a subgroup of $\text{Sp}_4(q).2$). Again we obtain an $x \in G \setminus H$ such that $H \cap H^x$ contains $T.2^2$ but does not contain H. Choosing ϕ appropriately we have $H \cap H^x \ge T.([8] \times f)$. Then it must be the case that $H \cap H^x = T.([8] \times f)$. In particular, we can write $H \cap H^x = (M \times M).(2 \times f)$, where M is a maximal subgroup of Sz(q) of order 2(q-1).

Identify (Sz(q): M) with the set of conjugates of M in Sz(q) and identify $(H: H \cap H^x)$ with the set

$$\Gamma := \{ (M_1, M_2) \mid M_1, M_2 \in (Sz(q) : M) \}.$$

Now we fix M and define

$$\Gamma_0 := \{ (M, M_1) \mid M_1 \in (Sz(q) : M) \}.$$

Clearly $H_{\Gamma_0} \cong (M \times S).f$ and H^{Γ_0} is almost simple and isomorphic to Sz(q).f, with the action on Γ being isomorphic to the action of Sz(q).f on (Sz(q):M). We know that this action is not binary by [45, Theorem 1.3]; thus there exist k-tuples $I = (M_1, \ldots, M_k)$ and $J = (M'_1, \ldots, M'_k)$ such that (I, J) is 2-subtuple complete but not k-subtuple complete with respect to the action of Sz(q).f. Now the same is true for the pair of elements of Γ^k ,

 $(((M, M_1), (M, M_2), \dots, (M, M_k)), ((M, M'_1), (M, M'_2), \dots, (M, M'_k)))),$

with respect to the action of H. Now, the result follows from Lemma 1.6.1.

Consider, finally, the possibility that $H(q) = \operatorname{Sp}_4(q).2$. In this case we use the fact that H(q) contains an element g of order q-1 that is centralized in G(q) by a subgroup isomorphic to ${}^2B_2(q)$. In [96, Table IV] a parametrization of such elements g is given: they are conjugate to the element t_1 in the table. Working in the F_4 root system, it can be seen that there is a conjugate g of t_1 that can be written in $H' = \operatorname{Sp}_4(q)$ as a diagonal matrix with all its eigenvalues of order q-1. In particular, there is an element $x \in G \setminus H$ that centralizes g and so we conclude that there is a suborbit of G on which the action of H is isomorphic to the action of H on (H:M), where M is a subgroup of H containing g. Since, by assumption, $q \geq 8$, Lemma 2.2.12 implies that this action is not binary, and the result follows by Lemma 1.6.1.

3.5 Maximal torus normalizers

In this section we prove Theorem 3.1 in the case where the point stabilizer H is the normalizer of a maximal torus. Such maximal subgroups are listed in Table 5.2 of [74]. The main result of the section follows. The cases excluded in the proposition (those in Table 3.5.1) will be dealt with in Lemma 3.5.3.

Proposition 3.5.1. Assume G is almost simple with socle G(q), an exceptional group of Lie type over \mathbb{F}_q , and suppose G(q) is not as in Proposition 3.2.1. Let H be a maximal subgroup of G that is the normalizer of a maximal torus T, as in [74, Table 5.2], and let $\Omega = (G : H)$. Then either (G, Ω) is not binary, or $(G(q), |T \cap G(q)|)$ is as in Table 3.5.1.

For the proof we need the following lemma. In the statement, by a semisimple group we mean a perfect group that is a central product of quasisimple groups.

Lemma 3.5.2. Let G be an almost simple group with socle of Lie type, and let $H = N_G(T)$ be a maximal subgroup of G that is the normalizer of a maximal torus T. Write $\Omega = (G : H)$.

(A) Suppose there exist subgroups A, D of G with the following properties:

Table 3.5.1: Exceptions in Prop. 3.5.1

G(q)	$ T \cap G(q) $
$E_{7}(2)$	3^{7}
${}^{2}\!E_{6}(2)$	3^5

- (i) A is quasisimple, D is either semisimple or a torus containing Z(A), [A, D] = 1 and $C_G(A) = DZ(A)$;
- (ii) $T \leq N_G(A)$ and $T \cap AD = T_1T_0$, where $T_1 = T \cap A$, $T_0 = T \cap D$;

(iii) $C_G(T_0)' = A$.

Define

$$\Delta = \{T^g : g \in N_D(T_0)A\} \subseteq \Omega.$$

Then G^{Δ} has socle A/Z(A) acting on $(A: N_A(T_1))$.

(B) Suppose that in addition to (i)-(iii) above, the following hold:

- (iv) $C_G(T_1T_0) = T;$
- (v) for any distinct $T', T'' \in \Delta$ we have $T' \cap T'' = \bigcap_{a \in A} T^a$;
- (vi) for any $g \in N_G(A)$, there exists $a \in A$ such that $T_1^g = T_1^a$;
- (vii) the action of G^{Δ} on Δ is not binary.

Then the action of G on $\Omega = (G : H)$ is not binary.

Proof. (A) Write $K = G_{(\Delta)}$, the point-wise stabilizer of Δ . We claim first that K normalizes A. To see this, observe first that K normalizes $X := \bigcap_{a \in A} T^a$. Since X is A-invariant and A is quasisimple, $X \cap A = Z(A)$. Also, $X \leq T$ and so X normalizes A by (ii). Hence $[X, A] \leq X \cap A \leq Z(A)$. As A is perfect, this implies that [X, A] = 1, and hence $X \leq DZ(A)$ by (i). It follows that $X = T_0Z(A)$. By (iii) we have $C_G(X)' = A$, and hence K normalizes A, as claimed.

Next, we claim that

$$C_G(K)' = A.$$
 (3.5.1)

Clearly $T_0 \leq K$, so $C_G(K)' \leq C_G(T_0)' = A$, by (iv). For the reverse containment, let $x \in K$. Then $T^{ax} = T^a$ for all $a \in A$. Now x normalizes A, hence normalizes $T^a \cap A = T_1^a$ for all $a \in A$. In other words, x induces an automorphism of A that lies in the kernel, L say, of the action on the set of A-conjugates of T_1 . As A is quasisimple, either $L \leq Z(A)$ or $A \leq L$. If $L \leq Z(A)$, then x commutes with A and hence (3.5.1) holds. So assume the latter. Since the action in question is on A-conjugates of T_1 and $A \leq L$, we get $T_1 \leq A$. As A is quasisimple, this means that $T_1 \leq Z(A)$. Therefore, A centralizes T, a maximal torus of G. But then A must be in the centre of G which is a contradiction to the fact that A is quasisimple. Summing up, in all cases x commutes with A and hence (3.5.1) holds.

Now G_{Δ} normalizes $K = G_{(\Delta)}$, hence normalizes A, by (3.5.1). Therefore by (i), G_{Δ} also normalizes DA. Let $g \in G_{\Delta}$. Then $T^g \in \Delta$, so by definition of Δ , intersecting with DA gives $(T_0T_1)^g = T_0T_1^a$ for some $a \in A$, and since $g \in N_G(DA)$ this implies that $T_0^g = T_0$. Hence $G_{\Delta} \leq N_G(T_0)$ and $G_{\Delta} \cap DA = N_D(T_0)A$. As $K \geq N_D(T_0)Z(A)$, it follows that $G^{\Delta} = G_{\Delta}/K$ has socle A/Z(A) acting on the conjugates of T_1 , as required (note that $N_A(T_1) \leq N_A(T)$ since $N_A(T_1)$ normalizes T_1T_0 , hence normalizes $C_G(T_1T_0) = T$).

(B) By condition (vii), there is a non-binary witness (δ, λ) for G^{Δ} , where $\delta = (\delta_1, \delta_2, ...)$, $\lambda = (\lambda_1, \lambda_2, ...)$. Suppose there exists $g \in G$ sending $\delta \to \lambda$. Then g sends $\delta_1 \cap \delta_2 \to \lambda_1 \cap \lambda_2$, and so by condition (v), g normalizes the group $X = T_0 Z(A)$. Hence as above, g normalizes A, hence also D and T_0 . Now for $x \in N_D(T_0)A$ we have $T^x \cap DA = T_0T_1^a$ for some $a \in A$, and hence using (vi),

$$T^{xg} \cap DA = (T^x \cap DA)^g = T_0 T_1^{ag} = T_0 T_1^{a'},$$

for some $a' \in A$. Hence by (iv) we see that $T^{xg} \in \Delta$. This shows that $g \in G_{\Delta}$, contradicting the fact that (δ, λ) is a non-binary witness for G^{Δ} . Hence δ and λ are in different *G*-orbits, showing that (G, Ω) is not binary.

Remark The proof shows that condition (v) could be replaced by

(v) there exists a non-binary witness (δ, λ) for G^{Δ} such that

$$\bigcap_{i=1}^k \delta_i = \bigcap_{i=1}^k \lambda_i = \bigcap_{a \in A} T^a$$

G(q)	T	A	D	Maximality	Comment
~->				condition	
$E_8(q)$	$(q-1)^8$	$A_1(q)$	$E_7(q)$	$q \ge 5$	
	$(q+1)^8$	$A_1(q)$	$E_7(q)$		q > 3
	$(q+1)^8$	$A_4^-(q)$	$A_4^-(q)$		$q \leq 3$
	$(q^2 + \epsilon q + 1)^4$	$A_2^{\epsilon}(q)$	$E_6^{\epsilon}(q)$	$(q,\epsilon) \neq (2,-)$	
	$(q^2 + 1)^4$	$A_1(q^2)$	$D_6^-(q)$		$AD < D_8(q)$
	$(q^4 + \epsilon q^3 + q^2 + \epsilon q + 1)^2$	$A_4^{\epsilon}(q)$	$A_4^{\epsilon}(q)$		
$E_7(q)$	$(q-1)^{7}_{-}$	$A_1(q)$	$D_6(q)$	$q \ge 5$	
	$(q+1)^{7}_{-}$	$A_1(q)$	$D_6(q)$		q > 3
	$(q+1)^7$	$A_2^-(q)$	$A_5^-(q)$		$q \leq 3$
$E_6^{\epsilon}(q)$	$(q-1)^6 (\epsilon = +)$	$A_1(q)$	$A_5(q)$	$q \ge 5$	
	$(q+1)^6 (\epsilon = -)$	$A_1(q)$	$A_{5}^{-}(q)$		q > 3
	$(q+1)^6 (\epsilon = -)$	$A_2^-(q)$	$A_{2}^{-}(q)^{2}$		$q \leq 3$
	$(q^2 + \epsilon q + 1)^3$	$A_2^{\epsilon}(q)$	$A_2^{\epsilon}(q)^2$	$(q,\epsilon) \neq (2,-)$	
$F_4(q),$	$(q-\epsilon)^4$	$A_1(q)$	$C_3(q)$	$q \ge 4$	
q even	$(q^2 + \epsilon q + 1)^2$	$A_2^{\epsilon}(q)$	$A_2^{\epsilon}(q)$	$(q,\epsilon) \neq (2,-)$	
	$\frac{(q^2+1)^2}{(q-\epsilon)^2}$	$A_1(q^2)$	$B_2(q)$		$AD < B_4(q)$
$G_2(q),$	$(q-\epsilon)^2$	$A_1(q)$	$A_1(q)$	$q \ge 9$	
$q = 3^a$					
${}^{2}F_{4}(q)',$	$(q+1)^2$	$A_1(q)$	$A_1(q)$	$q \ge 8$	
	$(q + \epsilon \sqrt{2q} + 1)^2$	${}^{2}\!B_{2}(q)$	${}^{2}\!B_{2}(q)$	$(q,\epsilon) \neq (2,-)$	
$^{3}D_{4}(q)$	$(q^2 + \epsilon q + 1)^2$	$A_2^{\epsilon}(q)$	$q^2 + \epsilon q + 1$		

Table 3.5.2: Possibilities for T, A, D

Table 3.5.3: Remaining possibilities for T

G	$\mathcal{L}(q)$	T	$N_{G(q)}(T)/T$
E_{i}	$_8(q)$	$q^8 + \epsilon q^7 - \epsilon q^5 - q^4 - \epsilon q^3 + \epsilon q + 1$	Z_{30}
$F_4(q), q$	$=2^{a}>2$	$q^4 - q^2 + 1$	Z_{12}
$G_2(q), q$	$= 3^a > 3$	$q^2 + \epsilon q + 1$	Z_6
$^{2}F_{c}$	$_4(q)'$	$q^2 + \epsilon \sqrt{2q^3} + q + \epsilon \sqrt{2q} + 1$	Z_{12}
^{3}D	$P_4(q)$	$q^4 - q^2 + 1$	Z_4

Proof of Proposition 3.5.1. Let G be almost simple with socle G(q) an exceptional group of Lie type, and let $H = N_G(T)$ be a maximal subgroup of G normalizing a maximal torus T, as in [74, Table 5.2]. We aim

to apply Lemma 3.5.2. Tables 3.5.2 and 3.5.3 together list all possibilities for T, and the first table also lists a pair of subgroups A, D that, as we shall see, satisfy the hypotheses of Lemma 3.5.2 (there are no such subgroups for the cases in Table 3.5.3). Note that in the tables, the values for |T| are those for the relevant maximal torus in the inner-diagonal group InnDiag(G(q)) rather than in the simple group G(q)itself.

Suppose T, A, D are as in Table 3.5.2. The cases where A is not quasisimple are those listed in Table 3.5.1, and so are excluded from further consideration here. Thus A is quasisimple, and we must check that A, D satisfy the first three hypotheses of Lemma 3.5.2. In all cases except the two with entries in the "Comment" column of the table, AD is a subsystem subgroup with maximal normalizer in G(q) as in [74, Table 5.1], so condition (i) holds. Moreover, $N_G(AD)$ contains a maximal torus $T = T_1T_0$ of order as in column 2 of the table, giving (ii). Finally, we can check that condition (iii) holds by computing the action of T_0 on the Lie algebra $L(\bar{G})$ (where \bar{G} is the ambient algebraic group) and seeing that the zero-weight space has dimension equal to that of A. Hence, by Lemma 3.5.2, there is a subset Δ of $\Omega = (G : H)$ such that G^{Δ} has socle A/Z(A) acting on $(A : N_A(T_1))$, where $T_1 = T \cap A$.

Suppose A is of type A_1 . We check that the further conditions (iv) - (vii) of Lemma 3.5.2 hold. Condition (vii) holds, since the group G^{Δ} is not binary, by [45]; and to verify (iv), we compute the action of T_1T_0 on $L(\bar{G})$ again to see that $C_G(T_1T_0)$ is a maximal torus, which must be T. For (v), let $T', T'' \in \Delta$. Then $T' \cap DA = T_0T_1^{a'}$ and $T'' \cap DA = T_0T_1^{a''}$, for some $a', a'' \in A$. Since $A = \mathrm{SL}_2(q)$, we have $T_1^{a'} \cap T_1^{a''} = Z(A)$, and so $T' \cap T'' \cap DA = T_0Z(A)$. As in the proof of Lemma 3.5.2(A), it follows that $T' \cap T'' = T_0Z(A)$, and this is equal to $\cap_{a \in A} T^a$, giving (v). Finally, (vi) is a standard property of tori in $\mathrm{SL}_2(q)$. Hence conditions (iv) - (vii) in Lemma 3.5.2 hold, and so the lemma shows that G is not binary.

If A is not of type A_1 , then we have three families of examples and three sporadic examples. Let us consider the families first: we find that

$$(A, |T_1|) = (A_2^{\epsilon}(q), q^2 + \epsilon q + 1), \ (A_4^{\epsilon}(q), q^4 + \epsilon q^3 + q^2 + \epsilon q + 1) \ \text{or} \ (^2B_2(q), q + \epsilon \sqrt{2q} + 1).$$

Lemma 1.7.2 implies that G^{Δ} is not binary and we again check that Lemma 3.5.2 applies to show that G is not binary.

Finally we must deal with the remaining sporadic examples: here

$$(A, |T_1|) = (A_4^-(q), (q+1)^4) (q=2, 3) \text{ or } (A_2^-(q), (q+1)^2) (q=3).$$

A magma calculation verifies that, in each case, the action of an almost simple group X with socle A on $\Delta = (X : N_X(T_1))$ is not binary and, what is more, there exists a non-binary witness $(\delta, \lambda) = ((\delta_1, \delta_2, \delta_3, \delta_4), (\lambda_1, \lambda_2, \lambda_3, \lambda_4))$ of length 4 for X^{Δ} such that

$$\bigcap_{i=1}^{4} \delta_i = \bigcap_{i=1}^{4} \lambda_i = \bigcap_{a \in A} T^a.$$

Note that the computation here is straightforward: we have constructed the permutation representations under consideration and then we have checked 4-tuples until we found one satisfying the required property. Thus condition (vii), and also condition (v') of the Remark following Lemma 3.5.2 hold. Conditions (iv) is verified as before, and (vi) is straightforward, as T_1 is the unique maximal torus of its order up to conjugacy in A. Hence Lemma 3.5.2 gives the conclusion in these cases also.

Suppose finally that T is as in Table 3.5.3. In these cases T is cyclic. One can check that for each prime t dividing |T|, T contains a Sylow t-subgroup of G. Now [35, 36, 39, 95] imply that, for every $g \in T \setminus \{1\}$, $C_G(g) = T$, and we conclude that $N = N_{G(q)}(T)$ is a Frobenius group, with T the Frobenius kernel. Let C be a Frobenius complement; observe that C is cyclic, and let c be a generator of C. Now Lemma 2.4.2 implies that $C_G(c) > C$ and so we can choose an element $x \in C_G(c) \setminus N_G(T)$. Then the action of N on $(N : N \cap N^x)$ is a Frobenius action and, since |C| > 2 in every case, and, since $N \cap N^x = N \cap H \cap H^x$, Lemma 1.7.2 implies that the action of H on $(H : H \cap H^x)$ is not binary; hence G is not binary by Lemma 1.6.1.

The remaining cases are resolved by calculations with magma:

Lemma 3.5.3. Let G be as in Proposition 3.5.1, and suppose that $H = N_G(T)$ where (G(q), |T|) is as listed in Table 3.5.1. If $\Omega = (G : H)$, then (G, Ω) is not binary.

Proof. For the group $G(q) := {}^{2}E_{6}(2)$, observe that $N_{G(q)}(T) \cong 3^{5}.SO_{5}(3)$ and also that G(q) has a unique conjugacy class of elements of order 5 (see [28]). Hence Lemma 1.6.15, applied with the prime p := 5, gives the conclusion.

When $G(q) := E_7(2)$, we gain use Lemma 1.6.15 with the prime p := 7. Using information from [81], we see that there exists a unique conjugacy class of elements of order 7. Furthermore, from [74], we have $H \cong 3^7.2.\mathrm{Sp}_6(2)$. Since the Sylow 7-subgroup of G(q) is elementary-abelian of order 7³ and a Sylow 7-subgroup of H is of order 7, Lemma 1.6.15 implies that G(q) is not binary.

3.6 Maximal subgroups in (V) of Theorem 3.1.1

The main result of this section is the following proposition. The cases excluded in the proposition (those in Table 3.6.1) will be dealt with in Lemma 3.6.2.

Proposition 3.6.1. Assume G is almost simple with socle G(q), an exceptional group of Lie type over \mathbb{F}_q , and suppose G(q) is not as in Proposition 3.2.1. Let H be a maximal subgroup of G as in part (V) of Theorem 3.1.1. Let $\Omega = (G : H)$. Then either (G, Ω) is not binary, or (G, H) is as in Table 3.6.1.

Table 3.6.1:	Exceptions	in Prop.	3.6.1
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G(q)	$H \cap G(q)$
$E_{6}(2)$	$G_2(2)$
$^{2}E_{6}^{-}(2)$	$F_{4}(2)$
$E_{7}(2)$	$G_2(q)C_3(2)$

Proof. Here H is one of the subgroups given in Table 3.1.1.

If *H* has socle $A_1(q)$, then q > 5 and we consider an element $x \in H$ of order $\frac{q-1}{(2,q-1)}$, as given in Table 2.2.1. As $C_G(x)$ contains a maximal torus of G(q), there exists $g \in C_G(x) \setminus H$. Now Lemmas 2.2.5 and 2.2.6 imply that the action of *H* on $(H : H \cap H^g)$ is not binary. The result then follows by Lemma 1.6.1.

If *H* has socle $B_2(q)$ then $q \ge 5$ and we proceed in the same way, using an element of order q-1 together with Lemma 2.2.8 in place of Lemma 2.2.5. A similar argument applies in the case where $F^*(H \cap G(q)) = C_4(q) \cong PSp_8(q)$.

Now consider the cases listed in Table 3.6.2. In each of these cases $F^*(H)$ has a factor that is generated by long root subgroups of G, from which it follows easily that there are subgroups $A \cong \operatorname{SL}_r(q)$ of H, and $S \cong \operatorname{SL}_{r+1}(q)/Z$ of G, satisfying the hypotheses of Lemma 1.6.10, where r is as indicated in Table 3.6.2. Thus Lemmas 1.6.10 provides a subset Δ of Ω of size q^r , and this is a beautiful subset unless $\operatorname{Alt}(q^r)$ and $\operatorname{Alt}(q^r - 1)$ are sections of G(q) and H, respectively. Hence Lemmas 2.1.1 and 2.1.2 show that if (G, Ω) is binary, the only possibility for (G(q), H) is $(E_7(2), G_2(2)C_3(2))$, as in Table 3.6.1.

Similarly, the subgroup $A_1(q)G_2(q^2)$ of $E_8(q)$ in Table 3.1.1 is a twisted version of the subgroup $A_1G_2G_2$ in the algebraic group E_8 ; hence this contains a subgroup $A \cong SL_3(q^2)$ which lies in a subgroup $S \cong SL_4(q^2)$ of G, and again Lemmas 1.6.10 and 2.1.2 give the conclusion.

It remains to deal with the following subgroups from Table 3.1.1:

- (1) $E_8(q)$: $F^*(H) = F_4(q) (p = 3)$ or $A_1(q)A_2^{\epsilon}(q) (p \ge 5)$
- (2) $E_7(q)$: $F^*(H) = A_2^{\epsilon}(q) \ (p \ge 5), \ A_1(q)A_1(q) \ (p \ge 5), \ \text{or} \ A_1(q)G_2(q) \ (p \ge 3, q \ge 5)$

G(q)	Н	A
$E_8(q)$	$G_2(q)F_4(q)$	$SL_4(q)$
	$A_1(q)G_2(q)G_2(q)$	$SL_3(q)$
$E_7(q)$	$G_2(q)C_3(q)$	$SL_3(q)$
	$A_1(q)F_4(q)$	$SL_4(q)$
$E_6(q)$	$F_4(q)$	$SL_4(q)$
	$A_2(q)G_2(q)$	$SL_3(q)$
$E_{6}^{-}(q)$	$A_2^-(q)G_2(q)$	$SL_3(q)$
$F_4(q)$	$A_1(q)G_2(q)$	$SL_3(q)$

Table 3.6.2: Subgroups in Table 3.1.1 with a long root factor

- (3) $E_6^{\epsilon}(q)$: $F^*(H) = A_2^{\pm}(q) (\epsilon = +, p \ge 5)$ or $G_2(q) (p \ne 7)$
- (4) $E_6^-(q)$: $F^*(H) = F_4(q)$
- (5) $F_4(q)$: $F^*(H) = G_2(q) (p = 7)$
- (6) ${}^{3}D_{4}(q)$: $F^{*}(H) = G_{2}(q)$ or $A_{2}^{\epsilon}(q)$.

Case (1) Here $G(q) = E_8(q)$. First consider $F^*(H) = F_4(q)$ with $q = 3^a$. If q > 3, let $x \in F^*(H)$ be the semisimple element defined in Lemma 2.2.15. Then $C_G(x)$ contains a maximal torus of G(q), hence there exists $g \in C_G(x) \setminus H$, and so $x \in H \cap H^g$, a core-free subgroup of H. By Lemma 2.2.15, the action of H on $(H : H \cap H^g)$ is not binary, giving the conclusion. If q = 3, we use the result of Lemma 2.3.1 for the group $H = F_4(3)$: since |G : H| is even, there exists a non-trivial orbit of H on Ω of odd size, and the action of H on this orbit is not binary by Lemma 2.3.1, giving the conclusion.

Now consider the other possibility $F^*(H) = A_1(q)A_2^{\epsilon}(q)$ $(p \ge 5)$. Let R be the $A_2^{\epsilon}(q)$ factor of $F^*(H)$. From the construction of the corresponding maximal subgroup A_1A_2 in the algebraic group E_8 given in [93, p.46], we see that R lies in a Levi subgroup $L = A_7(q)$ of G, with embedding given by the adjoint representation. Let A be a natural subgroup $SL_2(q)$ of R (i.e. acting as $1 \oplus 0$ on the natural 3-dimensional module, where we denote by a non-negative integer r the irreducible $\mathbb{F}_q A$ -module of highest weight r). The restriction of the natural 8-dimensional L-module to A is $2 \oplus 1^2 \oplus 0$, so in particular $C_L(A)$ contains a subgroup $SL_2(q)$ and also A lies in a subgroup $S = A_2(q)$ of L. Hence there exists $x \in C_L(A) \setminus H$ such that $A < S^x \not\leq H$. Now an application of Lemma 1.6.10 yields a subset Δ of Ω such that $G^{\Delta} \ge ASL_2(q)$, giving the conclusion in the usual way using Lemma 2.1.2.

Case (2) Here $G(q) = E_7(q)$. First consider $F^*(H) = A_2^{\epsilon}(q)$. Again let A be a natural $SL_2(q)$ in $F^*(H)$. Since maximal subgroups $A_2^{\epsilon}(q)$ exist for both $\epsilon = +$ and $\epsilon = -$, and each of these arises from a fixed maximal A_2 in the algebraic group E_7 , it follows that there is a subgroup $S \cong A_2(q)$ of G containing A (it could be that $S = F^*(H)$). From [93, p.83], it follows that A is contained in a Levi subgroup L of G of type $A_1A_4T_2$. The torus T_2 centralizes A, and so there exists $x \in C_G(A) \setminus H$. Then $A < S^x \not\leq H$, and the conclusion follows as in Case (1) above.

Next consider $F^*(H) = A_1(q)G_2(q)$. Let A be an $SL_3(q)$ subgroup of the $G_2(q)$ factor. From [93, 3.12] we see that the $G_2(q)$ factor lies in a Levi subgroup $L = A_6(q)$ of G, acting irreducibly on the natural 7-dimensional L-module V_7 . Then $V_7 \downarrow A = 10 \oplus 01 \oplus 00$. Now L lies in a subsystem subgroup $M = A_7(q)$ of G, and so we see that A is contained in a subgroup $S \cong A_3(q)$ of M acting on the natural M-module as $100 \oplus 001$. Hence $A < S \leq H$, and now Lemmas 1.6.10 and 2.1.2 give the conclusion.

Finally consider $F^*(H) = A_1(q)A_1(q) = A_1^{(1)}A_1^{(2)} \cong PSL_2(q)^2$ $(p \ge 5)$. We assume that the action is binary, and for i = 1, 2 let $T^{(i)}$ be a torus in $A_1^{(i)}$ of order $\frac{q-1}{2}$. Write $T = T_1^{(1)}T_1^{(2)}$ and observe that $N_{F^*(H)}(T)$ contains $D_{q-1} \times D_{q-1}$. From [93, p.37], we see that (re-labelling the $A_1^{(i)}$ if necessary), we have $C_G(T^{(1)}) \ge T^{(1)}A_2(q)A_4(q)$, and $A_1^{(2)}$ is embedded in this via the irreducibles of highest weight 2 and 4. Hence $T_1^{(2)}$ is diagonal in $A_2(q)A_4(q)$, and so $C_{G(q)}(T)$ contains a maximal torus of order $(q-1)^7/2$. Hence a Sylow 2-subgroup of $C_{G(q)}(T)$ is strictly larger than the Sylow 2-subgroup of T. This means, in particular, that the group $(D_{q-1} \times D_{q-1})/T$, which is a Klein 4-group, is a proper subgroup of a Sylow 2-subgroup of $N_{G(q)}(T)/T$. The group $H \cap G(q)$ is $(PSL_2(q) \times PSL_2(q))$.2, of index 2 in $PGL_2(q) \times PGL_2(q)$. Now, observe that $|C_{G(q)}(T)|_2$ is strictly larger than $|C_{H\cap G(q)}(T)|_2$. We conclude that there exists x in $G(q) \setminus H$ such that $H^x \cap H$ contains $D_{q-1} \times D_{q-1}$. We therefore obtain a suborbit of G on which the action of the stabiliser H is isomorphic to the action of H on (H: M) where M is a subgroup of H containing $D_{q-1} \times D_{q-1}$. But now, when q > 5, Lemma 1.6.1 and Lemma 2.2.2 imply that if (G, Ω) is binary then M must contain $F^*(H)$; this would mean that $x \in H$, a contradiction, as required. Suppose now q = 5. We have $F^*(H) \cong Alt(5) \times Alt(5)$. Write $F^*(H) = A \times B$, with $A \cong Alt(5) \cong B$. Pick F < H with F elementary abelian of order 5². Clearly, there exists x in $N_G(F) \setminus H$, so $F \leq H^x \cap H$. Suppose one of the factors, say A, of $F^*(H)$ is contained in $H^x \cap H$. Then A and $A^{x^{-1}}$ are contained in H. Hence $A^{x^{-1}}$ is equal to A, to B or to a diagonal subgroup of $A \times B$. Unipotent elements of order 5 in B or a diagonal subgroup are in different classes to those in A (see [66, Table 34]). Hence $A^{x^{-1}} = A$ and so $x \in N_G(A) = H$, a contradiction. We conclude that neither factor A or B is contained in $H^x \cap H$. Now, the proof follows with a magma computation: we have verified that, for every group H with $F^*(H) = Alt(5) \times Alt(5)$ and for every subgroup X of H, the action of H on (H:X) is binary only when X contains the whole of $F^*(H)$ or one of the two simple factors Alt(5) of $F^*(H)$.

Case (3) Let $G(q) = E_6^{\epsilon}(q)$. First consider $F^*(H) = A_2^{\pm}(q)$. Here $\epsilon = +$. Let A be a natural $SL_2(q)$ in $F^*(H)$. Since maximal subgroups $A_2^+(q)$ and $A_2^-(q)$ both exist (actually just for $q \equiv \epsilon \mod 4$), and each of these arises from a fixed maximal A_2 in the algebraic group E_6 , it follows that there is a subgroup $S \cong A_2(q)$ of G containing A (it could be that $S = F^*(H)$). From [93, 5.5] we know that $L(E_6) \downarrow A_2 = 11 \oplus 41 \oplus 14$. Hence we can work out $L(E_6) \downarrow A$, and in particular compute that, if t denotes the central involution of A, then dim $C_{L(E_6)}(t) = 38$, whence $C_{E_6}(t) = A_1A_5$. Also using $L(E_6) \downarrow A$, the only possible embedding of A in A_1A_5 is via the representations $1, 2^2$. Hence $C_{A_5}(A) = A_1$, and so $C_G(A) \not\leq H$. If we pick $x \in C_G(A) \setminus H$, then $A < S^x \not\leq H$, and the conclusion follows in the usual way.

Now consider $F^*(H) = G_2(q)$ $(p \neq 7)$. First suppose that $p \neq 2$. Then $G_2(q)$ has an involution t with centralizer $A\tilde{A}$, where A and \tilde{A} are long and short $SL_2(q)$ subgroups (respectively) in $G_2(q)$. Arguing as above using $L(E_6) \downarrow G_2$ (given in [79, Table 10.1]), we see that $A\tilde{A} < C_{E_6}(t) = A_1A_5$, with embedding given by $0 \otimes 1, 1 \otimes 2$. Hence $C_{A_5}(A) = A_2$, and so $N_G(A)$ contains a subgroup $A_2^{\epsilon}(q)A$.

Now $G_2(q)$ has a subgroup $S \cong SL_3(q)$ containing A. The composition factors of S on $L(E_6)$ are given in [99, Table 5 and Lemma 5.5], from which we can deduce that the only subsystem subgroup containing S is A_2^3 (this is $C_{E_6}(Z(S))$ unless p = 3). It follows that $C_G(S) = Z(S)$, and so $N_{G(q)}(S) = S.2 < H$. In particular, it follows that $N_G(A) \neq N_H(A)$ ($N_G(S) \cap N_G(A)$). Therefore Lemma 2.6.1 implies that there exists $x \in N_G(A)$ such that $S^x \leq H$. Hence $A < S^x \leq H$, giving the conclusion in the usual way.

It remains to consider the case where p = 2. Again let $SL_3(q) \cong S < G_2(q)$, let A be a natural subgroup $SL_2(q)$ of S, and let $\tilde{A} = C_{G_2(q)}(A) \cong SL_2(q)$. Now [99, 5.5] shows that S lies in a subsystem subgroup A_2^3 of the algebraic group E_6 , and so $A < A_1^3 < A_5$. Therefore $C_{A_5}(A)$ contains an A_1 subgroup, and so $C_G(A)$ contains $A_1(q)^2$. Hence we see as in the previous paragraph that $N_G(A) \neq N_H(A)$ ($N_G(S) \cap N_G(A)$), and now the argument goes through as before, the only difference being that this time Lemma 2.1.2 does not give a contradiction when q = 2, leaving that possibility in Table 3.6.1.

Case (4) Let $G(q) = E_6^-(q)$ and $F^*(H) = F_4(q)$. There are long root subsystem subgroups A < S < H with $A \cong SL_3(q)$, $S \cong SL_4(q)/Z$. Their centralizers can be read off using [75, Sec. 4], and we have $C_G(A) = A_2(q^2)$, $C_G(S) = A_1(q^2)T_1$ and $C_H(A) = A_1(q)T_1$. Hence $N_G(A) \neq N_H(A)$ ($N_G(S) \cap N_G(A)$), and so Lemma 2.6.1 yields an element $x \in G$ such that $A < S^x \not\leq H$. Now Lemmas 1.6.10 and 2.1.2 give a contradiction, except when q = 2, leaving that possibility in Table 3.6.1.

Case (5) Let $G(q) = F_4(q)$ and $F^*(H) = G_2(q)$ with p = 7. We argue as for $G_2(q)$ in case (3) above. For an involution $t \in G_2(q)$ we have $C_{G_2(q)}(t) = A\tilde{A}$ where A and \tilde{A} are long and short $SL_2(q)$ subgroups, and also $A < S < G_2(q)$ with $S \cong SL_3(q)$. Then $A\tilde{A} < C_{F_4}(t) = A_1C_3$ with embedding $0 \otimes 1, 1 \otimes 2$, and so $C_{C_3}(A) = A_1$. Again $N_G(S) < H$, and hence $N_G(A) \neq N_H(A) (N_G(S) \cap N_G(A))$. Now Lemma 2.6.1 yields an element $x \in G$ such that $A < S^x \not\leq H$ and we proceed as before.

Case (6) Here $G(q) = {}^{3}D_{4}(q)$ and $F^{*}(H) = G_{2}(q)$ or $A_{2}^{\epsilon}(q)$. Consider the first case. Let $A < S < G_{2}(q)$ with $A \cong SL_{2}(q)$, $S \cong SL_{3}(q)$ generated by long root subgroups of H (and of G). Then $C_{G}(A) = A_{1}(q^{3})$, $C_{H}(A) = A_{1}(q)$ and $N_{G(q)}(S) = S.(q^{2} + q + 1).2$. There is no factorization of a group with socle $A_{1}(q^{3})$ with one of the factors being $N(A_{1}(q))$ (see Lemma 2.6.2), and so Lemma 2.6.1 applies to give an element $x \in G$ such that $A < S^{x} \not\leq H$. Now Lemmas 1.6.10 and 2.1.2 give a contradiction (except when q = 2, in which case $G(q) = {}^{3}D_{4}(2)$, excluded by hypothesis).

Now let $F^*(H) = A_2^{\epsilon}(q)$. From [57], we see that $H \cap G(q) = \operatorname{PGL}_3^{\epsilon}(q)$ with $q \equiv \epsilon \mod 3$ and q > 2. First assume that $\epsilon = +$, and let A be a natural $\operatorname{SL}_2(q)$ subgroup of H. Then A centralizes an element g of order q - 1 in H, and from the list of centralizers in G (see for example [57, p.184]), we see that $C_G(g)'$ must be $\operatorname{SL}_2(q^3)$, or possibly $\operatorname{SL}_3(q)$ when q = 4. Excluding the latter possibility, it follows that $C_G(A)$ contains the centralizer of $\operatorname{SL}_2(q^3)$, which is a root subgroup $\operatorname{SL}_2(q)$. Hence in any case (including the extra q = 4 possibility), there is a group $S \cong A_2(q)$ such that $A < S \leq H$, giving the result in the usual way.

This leaves the case where $\epsilon = -$, so that $H \cap G(q) = \operatorname{PGU}_3(q)$ with $q \equiv -1 \mod 3$. We refer to Lemma 2.2.10, and let g be the element of $H \cap G(q)$ defined in that lemma. Observe that g is semisimple in G(q), hence, using the list of maximal tori of G(q) given in [53], we can conclude that there exists $x \in G(q) \setminus H$ such that $x \in C_G(g)$. Now consider the action of H on the cosets of $H \cap H^x$, a subgroup containing the element g and not containing $\operatorname{PSU}_3(q)$. If $q \leq 5$, we use magma to show that this action is not binary, giving the conclusion. And if $q \geq 7$, Lemma 2.1.1 shows that H has no section $\operatorname{Sym}(q)$, and so Lemma 2.2.10 shows that the action $(H, (H : H \cap H^x))$ is not binary, again giving the conclusion.

The remaining cases are resolved with the aid of magma:

Lemma 3.6.2. Let G be as in Proposition 3.6.1, and suppose that (G(q), H) is listed in Table 3.6.1. Then (G, Ω) is not binary.

Proof. Suppose that $G(q) = E_6(2)$ and $H \cap G(q) = G_2(2)$. Referring to [58], we see that H is maximal in G only when $G = G(q) = E_6(2)$, thus we assume this from here on. Now, using magma, we have computed all the binary transitive actions of $G_2(2)$, and we have found that these have degree 1, 2, 4032, 6048 and 12096. Now Lemma 1.6.1 implies that, if the action of G on (G : H) is binary, then the action of H on each of its suborbits must be binary – thus all suborbits must have size one of the five listed numbers. There is precisely one suborbit of size 1 (by maximality), the other suborbits are of even size, hence $|E_6(2) : G_2(2)|$ is odd, a contradiction.

Next assume that $G(q) = {}^{2}E_{6}(2)$ and $H \cap G(q) = F_{4}(2)$. Here H is either $F_{4}(2)$ or $F_{4}(2) \times 2$. Now $F_{4}(2)$ has a maximal subgroup isomorphic to $D_{4}(2)$. Sym(3); let X be the subgroup $D_{4}(2)$ of this. Then X is centralized by an element g of order 3 in $G(q) \setminus H$ (see [28]). Hence $X \triangleleft H \cap H^{g}$. At this point we can argue as in the proof of Proposition 3.2.1 (the $F_{4}(2)$ case); indeed, Lemma 1.6.15 applied with p = 7 shows that $(H, (H : H \cap H^{g}))$ is not binary. The conclusion follows.

Finally, assume that $G = E_7(2)$ and $H = G_2(2)C_3(2)$. Choose $x \in G \setminus H$ normalizing a Sylow 2-subgroup of H, so that $|H : H \cap H^x|$ is odd and greater than 1. A magma computation show that all transitive actions of H of odd degree greater than 1 are not binary, so the conclusion follows.

3.7 Maximal subgroups in (VI) of Theorem 3.1.1

In this section we prove

Proposition 3.7.1. Assume G is almost simple with socle G(q), an exceptional group of Lie type over \mathbb{F}_q , and suppose G(q) is not as in Proposition 3.2.1. Let H be a maximal subgroup of G as in part (VI) of Theorem 3.1.1. Let $\Omega = (G : H)$. Then either (G, Ω) is not binary, or (G, H) is as in Table 3.7.1.

Table 3.7.1: Exceptions in Prop. 3.7.1

G(q)	$H \cap G(q)$
$G_2(2^e), e \text{ prime}$	$G_2(2)$
$F_4(2^e), e \text{ prime}$	$F_{4}(2)$
$E_6^{\epsilon}(2^e), e \text{ prime}$	$E_6^{\epsilon}(2)$

Proof. Here H is of the same type as G – that is, one of the following holds:

- (i) $H \cap G(q) = G(q_0)$, where $\mathbb{F}_{q_0} \subset \mathbb{F}_q$;
- (ii) $H \cap G(q) < G(q)$ is a twisted subgroup, namely one of

$${}^{2}E_{6}(q^{1/2}) < E_{6}(q),$$

 ${}^{2}F_{4}(q) < F_{4}(q),$
 ${}^{2}G_{2}(q) < G_{2}(q).$

Consider first case (i). Here for each possible G(q), we define subgroups $A < S < G(q_0)$ with $A \cong$ $\operatorname{SL}_r(q_0)$ and $S \cong \operatorname{SL}_{r+1}(q_0)$, both subsystem subgroups of $G(q_0)$, as in Table 3.7.2. In each case $C_G(A)'$ and $C_G(S)'$ are as indicated in Table 3.7.2 and $C_H(A)$ is of the same type as $C_G(A)$ over the subfield \mathbb{F}_{q_0} . It then follows from Lemma 2.6.2 that $N_G(A) \neq N_H(A)$ ($N_G(S) \cap N_G(A)$), and so Lemma 2.6.1 yields an element $x \in G$ such that $A < S^x \not\leq H$. Now Lemmas 1.6.10 and 2.1.2 give a contradiction, except in the cases with $q_0 = 2$ in Table 3.7.1.

Table 3.7.2

G(q)	r	$C_G(A)$	$C_H(A)$	$C_G(S)$
$G_2(q)$	2	$A_1(q)$	$A_1(q_0)$	$(3, q_0 - 1)$
$F_4(q)$	3	$A_2(q)$	$A_2(q_0)$	$A_1(q)$
$E_6(q)$	3	$A_2(q)^2$	$A_2(q_0)^2$	$A_{1}(q)^{2}$
${}^{2}\!E_{6}(q)$	3	$A_2(q^2)$	$A_2(q_0^2)$	$A_1(q^2)$
$E_7(q)$	4	$A_3(q)A_1(q)$	$A_3(q_0)A_1(q_0)$	$A_2(q)T_1$
$E_8(q)$	5	$A_4(q)$	$A_4(q_0)$	$A_2(q)A_1(q)$

Now consider case (ii). In the first case, $F^*(H) = {}^2E_6(q^{1/2}) < E_6(q)$, and as above we pick A < S with $A \cong \mathrm{SL}_4(q^{1/2})$ and $S \cong \mathrm{SL}_5(q^{1/2})$. Then $S \not\leq H$ as H has no subgroup of type $A_4(q^{1/2})$, and the conclusion follows as usual.

Next let $F^*(H) = {}^2F_4(q) < F_4(q)$ with $q = 2^{2a+1}$, and note that q > 2 by hypothesis. Regard $F^*(H)$ as the centralizer in $F_4(q)$ of a graph automorphism τ . Then H has a subgroup $A \cong \mathrm{SL}_2(q)$ arising as the fixed point group of τ on a subsystem subgroup $A_1(q)\tilde{A}_1(q)$ in $F_4(q)$, and this lies in a subgroup $S = A_2(q)$ of $F_4(q)$ that is a diagonal subgroup of a subsystem $A_2(q)\tilde{A}_2(q)$. As H has no subgroup $A_2(q)$, we have $A < S \leq H$, giving the conclusion.

Now consider the case where $H \cap G(q) = {}^{2}G_{2}(q) < G_{2}(q)$, and note that q > 3 by hypothesis. Choose $x \in H \cap G(q)$ of order q - 1, as in Lemma 2.2.16. Since $C_{G(q)}(x)$ is a torus of order $(q - 1)^{2}$, there exists $g \in C_{G}(x) \setminus H$. Then $x \in H \cap H^{g}$, and the action of H on $(H : H \cap H^{g})$ is not binary by Lemma 2.2.16, giving the conclusion.

The treatment of groups of type (VI) is completed with the following result.

Lemma 3.7.2. Let G be as in Proposition 3.7.1, and suppose that (G(q), H) is listed in Table 3.7.1. Then (G, Ω) is not binary.

Proof. Consider the action in Line 1 of the table. Here $G = G_2(2^e)\langle \phi \rangle$ where ϕ is a field automorphism of order 1 or e, and $H = G_2(2) \times \langle \phi \rangle$. Choose $x \in G \setminus H$ normalizing a Sylow 2-subgroup P of $H \cap G_2(2)$, and let $X = H \cap H^x \cap G_2(2)$, so that $P \leq X$. The subgroups of $H \cap G(q) = G_2(2)$ containing P are the Borel subgroup P itself, and two maximal parabolics of shape $[2^5]$. Sym(3). These are all self-normalizing in $G_2(2)$, so it follows that $H \cap H^x = X \times \langle \sigma \rangle$, where $\sigma = \phi$ or 1. Note that σ is in the kernel of the action of H on $(H : H \cap H^x)$. Thus the latter action is either $(G_2(2), (G_2(2) : X))$ or $(G_2(2) \times e, (G_2(2) \times e : X))$. Using magma we check that the action $(G_2(2), (G_2(2) : X))$ is not binary for each of the three possibilities for X. Hence also $(G_2(2) \times e, (G_2(2) \times e : X))$ is not binary, by Lemma 1.6.2. It follows that the action of H on $(H : H \cap H^x)$ is not binary, giving the conclusion.

Now consider Line 3 of Table 3.7.1. First suppose $H \cap G(q) = {}^{2}E_{6}(2)$, with $G(q) = {}^{2}E_{6}(2^{e})$. Let $D_{0} = {}^{2}D_{5}(2)$ be a subsystem subgroup of $H \cap G(q)$. Then $D < {}^{2}D_{5}(q) < G(q)$, a subgroup centralized by a torus of order $\frac{q+1}{3}$. Choosing $g \in C_{G(q)}(D_{0}) \setminus H$, we have $H \cap H^{g} \triangleright D$. As in the proof of Proposition 3.4.1, there is a subgroup $A = \operatorname{SL}_{2}(4)$ of D and a subgroup $S = \operatorname{PSL}_{3}(4)$ of H such that $A < S \not\leq H \cap H^{g}$. Hence it follows in the usual way using Lemmas 1.6.10 and 2.1.2 that the action of H on $(H : H \cap H^{g})$ is not binary, giving the conclusion in this case. A similar argument handles the case where $H = E_{6}(2)$: here we take $D = A_{5}(2)$ and again choose $g \in C_{G(q)}(D_{0}) \setminus H$, so that $H \cap H^{g} \triangleright D$. There is a subgroup $A = \operatorname{SL}_{4}(2)$ of D and a subgroup $S = \operatorname{SL}_{5}(2)$ of H such that $A < S \not\leq H$, and the conclusion again follows.

Finally, consider $H \cap G(q) = F_4(2)$ with $G(q) = F_4(2^e)$. Choose a subgroup $D = A_2(2) \times 7$ lying in a subsystem subgroup $A_2(2) \times \tilde{A}_2(2)$ of H, where the factor $\tilde{A}_2(2)$ is generated by short root elements. There is an element $x \in N_G(D) \setminus H$, and so $D \leq H \cap H^x < H$. From [85], it follows that $H \cap H^x$ is contained in a subsystem subgroup $(A_2(2) \times \tilde{A}_2(2)).2$ of H. The factor $A_2(2)$ of D lies in a subgroup $A_3(2)$ of H that is not contained in $H \cap H^x$, and so it follows, using Lemma 1.6.10 in the usual way, that the action of H on the suborbit $(H : H \cap H^x)$ is not binary. This completes the proof.

3.8 The remaining families in Theorem 3.1.1

We proceed family by family.

3.8.1 Type (III)

Lemma 3.8.1. Assume G is almost simple with socle G(q), an exceptional group of Lie type over \mathbb{F}_q , and let H be a maximal subgroup of G as in part (III) of Theorem 3.1.1. If $\Omega = (G : H)$, then (G, Ω) is not binary.

Proof. Here $G(q) = E_7(q)$, p > 2 and $H \cap G(q) = (2^2 \times D_4(q).2^2)$. Sym(3) or ${}^{3}D_4(q).3$. Let $D := D_4(q)$ or ${}^{3}D_4(q)$ in H, and let A be a subsystem subgroup $SL_3(q)$ of D. Here D arises from a subgroup D_4 of the algebraic group $E_7(\bar{\mathbb{F}}_q)$ that lies in a subsystem A_7 (see the discussion after [78, Theorems 1,7]), and we see that A lies in a subgroup $A_7(q)$ of G(q), acting on the natural 8-dimensional module as $10 \oplus 01 \oplus 00^2$. Then A lies in a subgroup $A_3(q)$ of this $A_7(q)$ that does not lie in D. At this point we can apply Lemma 1.6.10 to see that there is a subset Δ of Ω such that $G^{\Delta} \geq ASL_3(q)$. This shows that G is not binary in the usual way using Lemma 2.1.2.

3.8.2 Type (IV)

Lemma 3.8.2. Assume G is almost simple with socle G(q), an exceptional group of Lie type over \mathbb{F}_q , and let H be a maximal subgroup of G as in part (IV) of Theorem 3.1.1. If $\Omega = (G : H)$, then (G, Ω) is not binary.

Proof. In this case $G(q) = E_8(q)$ with p > 5 and $H \cap G(q) = \text{PGL}_2(q) \times \text{Sym}(5)$. Let L be the factor $\text{PGL}_2(q)$ and let g be an element of order q - 1 in L. A consideration of the centralizers of semisimple elements in $E_8(q)$ implies that there exists $x \in C_{G(q)}(g) \setminus H$. Note that $x \notin N_G(L)$ because the maximality

of $H \cap G(q)$ requires that $N_{G(q)}(L) = H \cap G(q)$. Then $H \cap H^x$ contains the element g but does not contain the subgroup L, and now Lemma 2.2.1 implies that the action of H on $(H : H \cap H^x)$ is not binary. The result follows by Lemma 1.6.1.

3.8.3 Type (VII)

Lemma 3.8.3. Assume G is almost simple with socle G(q), an exceptional group of Lie type over \mathbb{F}_q , and let H be a maximal subgroup of G as in part (VII) of Theorem 3.1.1. If $\Omega = (G : H)$, then (G, Ω) is not binary.

Proof. Here H is an exotic local subgroup as listed in Table 3.1.2. When $H \cap G(q) = 2^3$.SL₃(2), let r = 3; and when $H \cap G(q) \in \{3^3$.SL₃(3), 3^{3+3} .SL₃(3), 5^3 .SL₃(5), 2^{5+10} .SL₅(2)}, let r = 2. We have verified with magma that every non-trivial transitive action of H of degree coprime to r is not binary. In particular, if the action of G on (G : H) is binary, then every non-trivial suborbit of G has cardinality divisible by r and hence r divides |G : H| - 1. However in all cases r divides |G : H|, and hence we reach a contradiction. \Box

3.8.4 Type (VIII)

Lemma 3.8.4. Assume G is almost simple with socle G(q), an exceptional group of Lie type over \mathbb{F}_q , and let H be a maximal subgroup of G as in part (VIII) of Theorem 3.1.1. If $\Omega = (G : H)$, then (G, Ω) is not binary.

Proof. Here $H = (Alt(5) \times Alt(6)).2^2 < E_8(q)$, where the Klein 4-group acts faithfully on $F^*(H) = Alt(5) \times Alt(6)$. There are several non-isomorphic groups having this shape, but a magma calculation confirms that if H is any such group, and M is a subgroup of H of odd index, then either the action of H on cosets of M is not binary or M contains the simple factor Alt(6) of H. Now let x be any member of $G \setminus H$ that normalizes a Sylow 2-subgroup of H. If the action of H on $(H : H \cap H^x)$ is binary, then by the previous sentence, $H \cap H^x$ contains Alt(6), and hence $x \in N_G(Alt(6)) = H$, a contradiction.

3.8.5 Type (IX)

Lemma 3.8.5. Assume G is almost simple with socle G(q), an exceptional group of Lie type over \mathbb{F}_q , and let H be a maximal subgroup of G as in part (IX) of Theorem 3.1.1. If $\Omega = (G : H)$, then (G, Ω) is not binary.

Proof. Here $F^*(H)$ is a simple group not in Lie(p), as listed in Tables 10.1–10.4 of [77]. Using also Theorem 3.1.2(i), we see that the possibilities for $F^*(H)$ are:

- (1) Alt(6), Alt(7);
- (2) $M_{11}, M_{12}, M_{22}, J_1, J_2, J_3, Ru, Fi_{22}, HS, Th;$
- (3) $PSL_2(r)$ for $r \le 61$;
- (4) $PSL_3(3)$, $PSL_3(4)$, $PSL_3(5)$, $PSL_4(3)$, $PSL_4(5)$, $PSU_3(3)$, $PSU_3(8)$, $PSU_4(2)$, $PSU_4(3)$, $PSp_4(5)$, $Sp_6(2)$, $\Omega_7(3)$, $\Omega_8^+(2)$, $G_2(3)$, ${}^{3}D_4(2)$, ${}^{2}F_4(2)'$, ${}^{2}B_2(8)$, ${}^{2}B_2(32)$.

Suppose first that $F^*(H)$ is not Alt(6), Alt(7) or $PSL_2(r)$, so that H is as in (2) or (4). Observe that |G:H| is even (see [73]), so there must be a non-trivial odd subdegree. However Lemmas 2.3.1 and 2.3.2 imply that if M is any core-free subgroup of H of odd index, then the action of H on cosets of M is not binary. Now Lemma 1.6.1 implies that (G, Ω) is not binary.

Suppose next that $F^*(H) \cong Alt(7)$. Then $G(q) = E_7(q)$ or $E_8(q)$ by Theorem 3.1.2(i), and hence |G| is divisible by 7². Therefore there is an element $g \in G \setminus H$ such that $H \cap H^g$ has order divisible by 7. However a magma computation shows that all faithful transitive actions of H of degree coprime to 7 are not binary, completing the proof in this case.

Suppose next that $F^*(H) \cong PSL_2(r)$ for some $r \leq 61$. If r = 4 or 5 then $F^*(H) \cong Alt(5)$, contrary to Theorem 3.1.2(i). Hence $r \geq 7$. Let g be an element of H of order $\frac{r-1}{(2,r-1)}$. Note that g has order at most 31. We claim that $C_G(g) \not\leq H$: for if $C_G(g) \leq H$, then $C_G(g) = C_H(g)$ is a cyclic maximal torus of G(q) of order either $\frac{r-1}{(2,r-1)}$ or r-1 (the latter only if H contains $PGL_2(r)$). The orders of cyclic maximal tori are given in [53, Sec. 2]. Recalling that G(q) is not as in Proposition 3.2.1, we see that the only possibility is that $G(q) = {}^2E_6(2)$ and g has order 13, 19 or 21. However, $PSL_2(r)$ is not a subgroup of ${}^2E_6(2)$ for r = 27, 39 or 43, as shown in [105, Sec. 12]. Thus $C_G(g) \not\leq H$, as claimed. Hence there exists $x \in C_G(g) \setminus H$, and so $H \cap H^x$ is a core-free subgroup of H containing g. Now Lemma 2.2.3 implies that $(H, (H : H \cap H^x))$ is not binary and the conclusion follows from Lemma 1.6.1.

$3.8.6 \quad \text{Type (X)}$

Lemma 3.8.6. Assume G is almost simple with socle G(q), an exceptional group of Lie type over \mathbb{F}_q , and let H be a maximal subgroup of G as in part (X) of Theorem 3.1.1. If $\Omega = (G : H)$, then (G, Ω) is not binary.

Proof. Here $F^*(H)$ is a simple group in Lie(p). By Theorem 3.1.2(ii),(iii), the possibilities for $F^*(H)$ are

- (1) $PSL_2(q_0), q_0 \le t(G)$ and as in Theorem 3.1.2(iii);
- (2) $PSL_3(3)$, $PSU_3(3)$ (with $G(q) = E_8(q)$, $q = 3^a$);
- (3) $PSL_3(4)$, $PSU_3(4)$, $PSU_3(8)$, $PSU_4(2)$, ${}^2B_2(8)$ (with $G(q) = E_8(q)$, $q = 2^a$).

Suppose first that $F^*(H)$ is as in (2) or (3). Using similar magma computations to those described in the proof of Lemma 3.8.5, we verify that if M is any core-free subgroup of H of index coprime to p, then the action of H on (H : M) is not binary. Since |G : H| is divisible by p, there exists $x \in G \setminus H$ normalizing a Sylow p-subgroup of H. Hence $H \cap H^x$ is a core-free subgroup of H of index coprime to p, and so $(H, (H : H \cap H^x))$ is not binary, completing the proof in cases (2) and (3).

Suppose finally that $F^*(H)$ is isomorphic to $PSL_2(q_0)$ as in (1), and note that $q_0 \neq 4, 5$ by Theorem 3.1.2(i). Let g be an element of H of order $\frac{q_0-1}{(2,q_0-1)}$. As in the last paragraph of the proof of Lemma 3.8.5, it is enough to show that $C_G(g) \not\leq H$. So assume that $C_G(g) \leq H$, in which case $C_G(g) = C_H(g)$ is a cyclic maximal torus of G of order $\frac{q_0-1}{(2,q_0-1)}$ or $q_0 - 1$. Also, by Theorem 3.1.2(ii), if $G(q) \neq E_8(q)$, then either $q_0 = q$ or $G(q) = E_7(q)$ and $q_0 = 7, 8$ or 25. The orders of cyclic maximal tori of G(q) are given in [53, Sec. 2], and there are none of order $q_0 - 1$ or $(q_0 - 1)/2$ with q_0 as in the previous sentence. Hence we may assume that $G(q) = E_8(q)$. Here the only possible cyclic maximal tori of order $q_0 - 1$ or $(q_0 - 1)/2$ (and also with $q_0 \leq t(G)$) have q = 2 and $q_0 = 2^7$ or 2^8 . However, $PSL_2(2^8) \not\leq E_8(2)$, as $E_8(2)$ has no torus of order $2^8 + 1$. And if $F^*(H) = PSL_2(2^7)$, then the element $g \in H$ of order $2^7 - 1$ lies in a subgroup $E_7(2)$ of G, and is centralized by an element of order 3 in $G \setminus H$, so $C_G(g) \not\leq H$ and the conclusion follows.

This completes the proof of Theorem 3.1.

Chapter 4

Classical Groups

In this chapter we prove Theorem 1.3 for classical groups:

Theorem 4.1. Let G be an almost simple group with socle a classical group, and assume that G has a primitive and binary action on a set Ω . Then $|\Omega| \in \{5, 6, 8\}$ and $G \cong \text{Sym}(\Omega)$.

The examples with $|\Omega| \in \{5, 6, 8\}$ arise via the isomorphisms listed after the statement of Theorem 1.3.

The case where G has socle isomorphic to $PSL_2(q)$ or $PSU_3(q)$ has been dealt with in [45], so Theorem 4.1 is already proved in this case.

4.1 Background on classical groups

Let us set up the group-theoretic notation that we need to prove Theorem 4.1. We assume throughout that our group G is almost simple with socle a finite simple classical group. We write M for the stabilizer in G of a point in the action on Ω . Since the action is primitive, M is a maximal subgroup of G, and so we can use the classification of the maximal subgroups of the almost simple finite classical groups due to Aschbacher [1]. This classification divides the maximal subgroups into nine families, labelled C_1 - C_8 and S. We shall give rough descriptions of these families at the beginning of each section of this chapter; full details can be found in [54, Chapter 4], to which we will often refer. The case where M is in family C_1 has been handled in [46]. In this chapter we deal with the families C_2 - C_8 and S, in Sections 4.2- 4.9. Some almost simple groups with socle $P\Omega_8^+(q)$ or $Sp_4(2^a)$ have extra families of maximal subgroups, and these are handled in the last Section 4.10.

In what follows we shall take S to be a certain quasisimple classical group for which S/Z(S) is isomorphic to the socle of G: namely, S will be one of $SL_n(q)$, $Sp_n(q)$, $SU_n(q)$, $\Omega_n(q)$ (with nq odd) or $\Omega_n^{\varepsilon}(q)$ (with n even and $\varepsilon \in \{+, -\}$). As in [54], we denote these cases by L, S, U and O. Sometimes, for uniformity of notation, we shall allow ourselves to write $\Omega_n^{\varepsilon}(q)$ also in the case where n is odd – in which case it just denotes $\Omega_n(q)$. Note that we can think of S as acting on the set Ω – although we emphasise that this action is not necessarily primitive, and not necessarily faithful. We shall always write \overline{S} for the simple group S/Z(S).

The group S is a subgroup of the group of isometries of some fixed bilinear, quadratic or sesquilinear form φ . We will write V for the associated vector space of dimension n over the field K where $\mathbb{K} = \mathbb{F}_{q^u}$ with u = 2 in case U, and u = 1 otherwise. The form φ is either non-degenerate or the zero form (in the case $S = \mathrm{SL}_n(q)$).

When φ is non-degenerate, we will make use of a hyperbolic basis \mathcal{B} of V of form

$$\{e_1,\ldots,e_k,f_1,\ldots,f_k\}\cup\mathcal{A},$$

where k is the Witt index of φ , $\langle e_i, f_i \rangle$ are hyperbolic lines for $i = 1, \ldots, k$ and either \mathcal{A} is empty, or S is orthogonal and \mathcal{A} has size at most 2 and spans an anisotropic subspace of V.

4.1.1 Basic assumptions

We make use of isomorphisms between classical groups of small dimension, as well as known results on Cherlin's conjecture to make the following assumptions.

- 1. If $S = SL_n(q)$, then $n \ge 3$ (using the main result of [45]).
- 2. If $S = SU_n(q)$, then $n \ge 4$ (using the main result of [45]).

3. If $S = \text{Sp}_n(q)$, then $n \ge 4$.

- 4. If $S = \Omega_n(q)$ with n odd, then q is odd and $n \ge 7$.
- 5. If $S = \Omega_n^{\varepsilon}(q)$ with *n* even and with $\varepsilon \in \{+, -\}$, then $n \ge 8$.

Notice that, under these assumptions, S is quasisimple, unless $S = \text{Sp}_4(2)$, in which case $S \cong \text{Sym}(6)$.

In addition, by [46], we can assume that M does not lie in Aschbacher's family C_1 . We also use magma to exclude some small cases:

Lemma 4.1.1. Let G be an almost simple primitive group with socle one of the following groups

- 1. $PSL_3(q)$ with $q \le 25$, $PSL_4(q)$ with $2 < q \le 9$ or $q \in \{16, 25\}$, $PSL_5(q)$ with $q \le 7$, $PSL_6(q)$ with $q \le 4$, $PSL_7(3)$, $PSL_8(q)$ with $q \le 3$;
- 2. $\text{PSU}_4(q)$ with $q \leq 7$, $\text{PSU}_5(q)$ with $q \leq 5$, $\text{PSU}_6(q)$ with $q \leq 3$, $\text{PSU}_7(q)$ with $q \leq 3$, $\text{PSU}_8(2)$;
- 3. $PSp_4(q)$ with $q \in \{4, 5, 8, 16\}$, $PSp_6(q)$ with $q \le 5$, $PSp_8(q)$ with $q \le 3$;
- 4. $P\Omega_7(3)$, $P\Omega_8^-(2)$, $P\Omega_8^+(2)$, $P\Omega_8^+(3)$, $P\Omega_8^+(4)$, $P\Omega_9(5)$, $P\Omega_{10}^-(2)$, $P\Omega_{12}^+(2)$.

Then the action of G is not binary.

Proof. The magma computations here are all rather similar. We give an indication of what we have done in the unitary case only.

We have computed all the possible almost simple groups G and all of their (faithful) primitive actions on a set Ω . We have tested that each of these actions is not binary. Indeed, except when $S = SU_4(2)$ and $|\Omega| = 27$, or $S = SU_4(3)$ and $|\Omega| = 112$, or $S = SU_4(4)$ and $|\Omega| = 325$, we can witness that G is non binary by applying Lemmas 1.6.15, 1.6.16, 1.8.1, 1.8.4, or by finding a suitable non-binary triple. When $S = SU_4(3)$ and $|\Omega| = 112$, or $S = SU_4(4)$ and $|\Omega| = 325$, we can witness that G is not binary by finding a suitable non-binary 4-tuple. The case $S = SU_4(2)$ and $|\Omega| = 27$ requires a little more care because triples and 4-tuples are not enough to witness that G is not binary. We have proved that this group is non-binary using longer tuples (of length 7).

Finally, from here on, except for the final two sections (§4.9 and §4.10), we will assume that

- if $S = \text{Sp}_4(2^a)$, then $G \leq \Gamma \text{Sp}_4(2^a)$ (and so does not contain a graph automorphism); and
- if $S = \Omega_8^+(q)$, then $G \leq \Pr O_8^+(q)$ (and so does not contain a triality automorphism).

These assumptions ensure that if V denotes the natural n-dimensional module for S, then $G \leq P\Gamma L(V)$, except for the case where S = SL(V), in which case $G \leq P\Gamma L(V).2$ (where the .2 denotes a graph automorphism).

Note also, for future reference, that by Proposition 2.5.1, the maximal subgroups of G that centralize field, graph-field or graph automorphisms are in families C_5 (subfield subgroups) and C_8 (classical subgroups).

4.2 Family C_2

In this case M is the projective image of the stabilizer of a direct sum decomposition D of V into t subspaces W_1, \ldots, W_t , each of dimension m, as described in [54, §4.2]. In particular, n = mt. The possibilities are summarized in Table 4.2.1.

case	type	conditions
L	$\operatorname{GL}_m(q)\operatorname{wr}\operatorname{Sym}(t)$	
U	$\operatorname{GU}_m(q)\operatorname{wr}\operatorname{Sym}(t)$	W_i non-degenerate
S	$\operatorname{Sp}_m(q)\operatorname{wr}\operatorname{Sym}(t)$	W_i non-degenerate
0	$\mathrm{O}_m^\delta(q) \operatorname{wr} \operatorname{Sym}(t)$	W_i non-degenerate
U, S, O^+	$\operatorname{GL}_{n/2}(q^u).2$	W_i totally singular,
	,	q odd in case S

Table 4.2.1: Maximal subgroups in family C_2

The main result of this section is the following. The result will be proved in a series of lemmas.

Proposition 4.2.1. Suppose that G is an almost simple group with socle $\overline{S} = Cl_n(q)$, and assume that

- (i) $n \geq 3, 4, 4, 7$ in cases L, U, S, O respectively, and
- (ii) $\operatorname{Cl}_n(q)$ is not one of the groups listed in Lemma 4.1.1.

Let M be a maximal subgroup of G in the family C_2 . Then the action of G on (G:M) is not binary.

4.2.1 Case $S = SL_n(q)$

Assume that $S = SL_n(q)$ with $n \ge 3$, and the socle of G is not as in Lemma 4.1.1(1). Assume also that $\Omega = (G:M)$, where M is in the family C_2 (as in the first row of Table 4.2.1).

Lemma 4.2.2. In this case, Ω contains a beautiful subset.

Proof. There is a basis $\mathcal{B} = \{v_1, \ldots, v_{mt}\}$ of V such that M stabilizes the deomposition $V = W_1 \oplus \cdots \oplus W_t$, where

$$W_i = \langle v_{m(i-1)+1}, v_{m(i-1)+2}, \dots, v_{mi} \rangle$$

First, assume that $q \ge 5$. We let U be the subgroup whose elements fix all elements of \mathcal{B} except v_1 and satisfy

$$v_1 \mapsto v_1 + k_1 v_{m+1},$$

for some $k_1 \in \mathbb{F}_q$, and we define $\Lambda = D^U$. For $k \in \mathbb{F}_q$, we define

$$W_1(k) = \langle v_1 + kv_{m+1}, v_2, \dots, v_m \rangle,$$

and observe that $\Lambda = \{D(k) \mid k \in \mathbb{F}_q\}$, where

$$D(k) = W_1(k) \oplus W_2 \oplus \cdots \oplus W_t$$

Note, in particular, that D(0) = D.

Let T be the maximal split torus whose elements are diagonal when written with respect to \mathcal{B} . Then $U \rtimes T$ is 2-transitive on $\Lambda = D^U$.

Now suppose that $g \in S_{\Lambda}$ and suppose that g maps W_i to $W_1(k)$ for some i > 1 and some $k \in \mathbb{F}_q$. This implies that there exists $v \in W_i$ such that $v^g = v_1 + kv_{m+1}$. But $v_1 + kv_{m+1}$ lies in $W_1(k)$ and not in $W_1(\ell)$ for all $\ell \neq k$ and, similarly, $v_1 + kv_{m+1}$ does not lie in W_j for all j > 1. Thus g does not preserve A, a contradiction. We conclude that g preserves $\{W_1(k) \mid k \in \mathbb{F}_q\}$ set-wise, and preserves $\{W_2, \ldots, W_t\}$ set-wise.

Our aim now is to show that Λ is an S-beautiful subset; to do this, we will show that $S^{\Lambda} \cong U \rtimes T$. For this, we suppose that $g \in S_{\Lambda}$ fixes both D(0) and D(1) and we will show that $g \in S_{(\Lambda)}$. Observe first that, since g fixes D(0), it follows that g fixes W_1 and so

$$v_1^g \in \langle v_1, v_2, \dots, v_m \rangle$$

Similarly, since g fixes D(1), we conclude that g fixes $W_1(1)$ and so

$$(v_1 + v_{m+1})^g \in \langle v_1 + v_{m+1}, v_2, \dots, v_m \rangle.$$

Hence $v_{m+1}^g \in \langle v_1, v_2, \dots, v_m, v_{m+1} \rangle$. On the other hand g preserves $\{W_2, \dots, W_t\}$ set-wise, and so

$$v_{m+1}^g \in \langle v_{m+1}, \dots, v_{mt} \rangle$$

which implies that $v_{m+1}^g \in \langle v_{m+1} \rangle$. In other words, for some $\ell_1 \in \mathbb{F}_q$, we have

$$v_{m+1}^g = \ell_1 v_{m+1}. \tag{4.2.1}$$

Now, since g fixes W_1 , we conclude that there exist ℓ_2, c_2, \ldots, c_m such that

$$v_1^g = \ell_2 v_1 + \sum_{i=2}^m c_i v_i. \tag{4.2.2}$$

Finally, since g fixes $W_1(1)$, there exist ℓ_3, d_2, \ldots, d_m such that

$$(v_1 + v_{m+1})^g = \ell_3(v_1 + v_{m+1}) + \sum_{i=2}^m d_i v_i.$$
(4.2.3)

From, (4.2.1), (4.2.2) and (4.2.3), we conclude that $c_i = d_i$ for all i = 2, ..., m and that $\ell_1 = \ell_2 = \ell_3 = \ell$. We finally obtain that, for each $k \in \mathbb{F}_q$,

$$(v_1 + kv_{m+1})^g = (v_1 + v_{m+1})^g + (k-1)v_{m+1}^g$$

= $\ell(v_1 + v_{m+1}) + \sum_{i=2}^m c_i v_i + (k-1)\ell v_{m+1}$
= $\ell(v_1 + kv_{m+1}) + \sum_{i=2}^m c_i v_i \in W_1(k).$

Thus g fixes $W_1(k)$ for each $k \in \mathbb{F}_q$, and so g fixes D(k) for each $k \in \mathbb{F}_q$. We conclude that $g \in S_{(\Lambda)}$, as required.

Next, assume that $q \in \{3, 4\}$; then [54, Tables 3.5.A and 3.5.H] allows us to assume that $m \ge 2$. We let U be the subgroup whose elements fix all elements of \mathcal{B} except v_1 and satisfy

$$v_1 \mapsto v_1 + k_1 v_{m+1} + k_2 v_{m+2},$$

for some $k_1, k_2 \in \mathbb{F}_q$, and we define $\Lambda = D^U$. For $k_1, k_2 \in \mathbb{F}_q$, we define

$$W_1(k_1, k_2) = \langle v_1 + k_1 v_{m+1} + k_2 v_{m+2}, v_2, \dots, v_m \rangle,$$

and observe that $\Lambda = \{D(k_1, k_2) \mid k_1, k_2 \in \mathbb{F}_q\}$, where

$$D(k_1,k_2) = W_1(k_1,k_2) \oplus W_2 \oplus \cdots \oplus W_t.$$

Note, in particular, that D(0,0) = D. Let X be the stabilizer of the subspaces

$$\langle v_1 \rangle, \ldots, \langle v_m \rangle, \langle v_{m+1}, v_{m+2} \rangle, \langle v_{m+3} \rangle, \ldots, \langle v_{mt} \rangle.$$

Then $U \rtimes X$ is 2-transitive on $\Lambda = D^U$. Our aim now is to show that Λ is a beautiful subset.

Take $g \in S_{\Lambda}$ and suppose that Λ is not beautiful. An analogous argument to the previous case allows us to conclude that g preserves $\{W_1(k_1, k_2) \mid k_1, k_2 \in \mathbb{F}_q\}$ set-wise, and preserves $\{W_2, \ldots, W_t\}$ set-wise. This implies, moreover, that g preserves the subspaces

$$Y_1 := \operatorname{span}_{\mathbb{F}_q} \{ W_1(k_1, k_2) \mid k_1, k_2 \in \mathbb{F}_q \} \text{ and } Y_0 := \bigcap_{k_1, k_2 \in \mathbb{F}_q} W_1(k_1, k_2).$$

Thus there is a homomorphism $\theta: S_{\Lambda} \to \operatorname{GL}(Y_1/Y_0) \cong \operatorname{GL}_3(q)$. Since $\operatorname{GL}_3(q)$ does not contain a subgroup with a composition factor isomorphic to $\operatorname{Alt}(s)$ for $s \geq 8$, we conclude that the action of ker (θ) on Λ must induce $\operatorname{Alt}(\Lambda)$ or $\operatorname{Sym}(\Lambda)$. However ker (θ) is not transitive on Λ so we have a contradiction.

Next, assume that q = 2; then [54, Tables 3.5.A and 3.5.H] allows us to assume that $m \ge 3$. We let U be the subgroup whose elements fix all elements of \mathcal{B} except v_1 and satisfy

$$v_1 \mapsto v_1 + k_1 v_{m+1} + k_2 v_{m+2} + k_3 v_{m+3},$$

for some $k_1, k_2, k_3 \in \mathbb{F}_q$, and we define $\Lambda = D^U$. For $k_1, k_2, k_3 \in \mathbb{F}_q$, we define

$$W_1(k_1, k_2, k_3) = \langle v_1 + k_1 v_{m+1} + k_2 v_{m+2} + k_3 v_{m+3}, v_2, \dots, v_m \rangle,$$

and observe that $\Lambda = \{D(k_1, k_2, k_3) \mid k \in \mathbb{F}_q\}$, where

$$D(k_1, k_2, k_3) = W_1(k_1, k_2, k_3) \oplus W_2 \oplus \cdots \oplus W_t.$$

Note, in particular, that D(0,0,0) = D. Let X be the stabilizer of the subspaces

$$\langle v_1 \rangle, \ldots, \langle v_m \rangle, \langle v_{m+1}, v_{m+2}, v_{m+3} \rangle, \langle v_{m+4} \rangle, \ldots, \langle v_{mt} \rangle$$

Then $U \rtimes X$ is 2-transitive on $\Lambda = D^U$. Suppose that $g \in S_{\Lambda}$ fixes both D(0,0,0) and D(1,0,0). Observe first that, since g fixes D(0,0,0), it follows that g fixes W_1 and so

$$v_1^g \in \langle v_1, v_2, \dots, v_m \rangle$$

Similarly, since g fixes D(1,0,0), it also fixes $W_1(1,0,0)$ and so

$$(v_1 + v_{m+1})^g \in \langle v_1 + v_{m+1}, v_2, \dots, v_m \rangle.$$

We conclude that $v_{m+1}^g \in \langle v_1, v_2, \dots, v_m, v_{m+1} \rangle$.

On the other hand g preserves $\{W_2, \ldots, W_t\}$ set-wise, and so

$$v_{m+1}^g \in \langle v_{m+1}, \dots, v_{mt} \rangle$$

which implies that $v_{m+1}^g \in \langle v_{m+1} \rangle$. Since we are working over \mathbb{F}_2 , we conclude that $v_{m+1}^g = v_{m+1}$. Now, we can repeat this same argument assuming that g also fixes D(0,1,0) and D(0,0,1) and we see that $v_{m+2}^g = v_{m+2}$ and $v_{m+3}^g = v_{m+3}$. But in this case g clearly fixes $W_1(k_1, k_2, k_3)$ for all $k_1, k_2, k_3 \in \mathbb{F}_2$, and so g fixes $D(k_1, k_2, k_3)$ for all $k_1, k_2, k_3 \in \mathbb{F}_2$. Thus if $g \in S_{\Lambda}$ and fixes the four points D(0,0,0), D(1,0,0), D(0,1,0) and D(0,0,1) of Λ , then g fixes all of Λ . Since $|\Lambda| = 8$, we conclude that S^{Λ} does not contain Alt(Λ), and so Λ is a beautiful subset.

4.2.2 The totally singular case

In this section we deal with the case when S preserves a non-degenerate form on V, t = 2 and W_1 and W_2 are both totally singular. This occurs when S is unitary, symplectic with q odd, or of type O⁺ (as in the last row of Table 4.2.1). So assume that S is one of these types, and also that the socle of G is not as in Lemma 4.1.1.

Lemma 4.2.3. In this case Ω contains a beautiful subset.

Proof. We can assume that $W_1 = \langle e_1, \ldots, e_m \rangle$ and $W_2 = \langle f_1, \ldots, f_m \rangle$, where $\mathcal{B} = \{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ is a hyperbolic basis for V (and $m \geq 2$). Note that, with respect to the basis \mathcal{B} , M contains the group of matrices

$$\left\{ \begin{pmatrix} A \\ & A^{-T} \end{pmatrix} \mid A \in \operatorname{GL}_m(\mathbb{F}_q) \right\},$$
(4.2.4)

except in the O⁺ case with q odd, when we need to add the requirement that det(A) is a square in \mathbb{F}_q for such matrices. Notice that in the unitary case we have written A^{-T} rather than $A^{-T\sigma}$ (where σ is the involutory automorphism of the field \mathbb{F}_{q^2}) since we are only considering matrices with entries from the field \mathbb{F}_q .

First, assume that $q \ge 5$. We let U be the subgroup whose elements fix all elements of \mathcal{B} except e_1 and e_2 and satisfy

$$e_1 \mapsto e_1 + kf_2$$
 and $e_2 \mapsto e_2 \pm kf_1$,

for some $k \in \mathbb{F}_q$. (The choice of sign for the image of e_2 will depend on the type of form preserved by S.) We define $\Lambda = D^U$ and, for $k \in \mathbb{F}_q$, we define

$$W_1(k) = \langle e_1 + kf_2, e_2 \pm kf_1, e_3, \dots, e_m \rangle.$$

Observe that $\Lambda = \{D(k) \mid k \in \mathbb{F}_q\}$, where

$$D(k) = W_1(k) \oplus W_2.$$

Note, in particular, that D(0) = D.

Let T be the maximal split torus whose elements are diagonal when written with respect to \mathcal{B} . Then $U \rtimes T$ is 2-transitive on $\Lambda = D^U$. Our aim now is to show that Λ is an S-beautiful subset; to do this, we will show that $S^{\Lambda} \cong U \rtimes T$. For this, we suppose that $g \in S_{\Lambda}$ fixes both D(0) and D(1) and we will show that $g \in S_{(\Lambda)}$.

Observe that, since g fixes D(0) and D(1), g must fix W_2 and hence must fix W_1 and $W_1(1)$. The fact that g fixes W_1 implies that

$$e_1^g \in \langle e_1, e_2, \dots, e_m \rangle.$$

Similarly, the fact that g fixes $W_1(1)$ implies that

$$(e_1 + f_2)^g \in \langle e_1 + f_2, e_2 \pm f_1, e_3, \dots, e_m \rangle.$$

We conclude that $f_2^g \in \langle e_1, e_2, \ldots, e_m, f_1, f_2 \rangle$.

On the other hand g also fixes W_2 , and so $f_2^g \in W_2$ and we conclude that $f_2^g \in \langle f_1, f_2 \rangle$; in other words, there exist $\ell_1, \ell'_1 \in \mathbb{F}_q$ such that

$$f_2^g = \ell_1' f_1 \pm \ell_1 f_2. \tag{4.2.5}$$

Now, since g fixes W_1 , we conclude that there exist $\ell_2, \ell'_2, c_3, \ldots, c_m$ such that

$$e_1^g = \ell_2 e_1 + \ell'_2 e_2 + \sum_{i=3}^m c_i e_i.$$
(4.2.6)

Finally, since g fixes $W_1(1)$, we conclude that there exist $\ell_3, \ell'_3, d_3, \ldots, d_m$ such that

$$(e_1 + f_2)^g = \ell_3(e_1 + f_2) + \ell'_3(e_2 \pm f_1) + \sum_{i=3}^m d_i e_i.$$
(4.2.7)

From (4.2.5), (4.2.6) and (4.2.7), we conclude that $c_i = d_i$ for all i = 3, ..., m, $\ell_1 = \ell_2 = \ell_3$ and $\ell'_1 = \ell'_2 = \ell'_3$. Set $\ell := \ell_1$ and $\ell' := \ell'_1$.

We finally obtain that, for each $k \in \mathbb{F}_q$,

(e

$$\begin{aligned} &(1+kf_2)^g = (e_1 + f_2)^g + (k-1)f_2^g \\ &= \ell(e_1 + f_2) + \ell'(e_2 \pm f_1) + \sum_{i=3}^m c_i e_i + (k-1)(\pm \ell' f_1 + \ell f_2) \\ &= \ell(e_1 + kf_2) + \ell'(e_2 \pm kf_1) + \sum_{i=3}^m c_i e_i \in W_1(k). \end{aligned}$$

Thus g fixes $W_1(k)$ for each $k \in \mathbb{F}_q$, and so g fixes D(k) for each $k \in \mathbb{F}_q$. We conclude that $g \in S_{(\Lambda)}$, as required.

Next assume that $q \in \{3, 4\}$. Since $SU_4(3), SU_4(4)$ and $Sp_4(3)$ were dealt with at the start (recall that in the symplectic case, q is odd), we require $n \ge 6$. Our argument is similar to before. We let U be the subgroup whose elements fix all elements of \mathcal{B} except e_1 , e_2 and e_3 and satisfy

$$e_1 \mapsto e_1 + k_2 f_2 + k_3 f_3$$

for some $k_2, k_3 \in \mathbb{F}_q$, and we define $\Lambda = D^U$. For $k_2, k_3 \in \mathbb{F}_q$, we define

$$W_1(k_2, k_3) = \langle e_1 + k_2 f_2 + k_3 f_3, e_2, \dots, e_m \rangle,$$

and observe that $\Lambda = \{D(k_2, k_3) \mid k_2, k_3 \in \mathbb{F}_q\}$ where

$$D(k_2, k_3) = W_1(k_2, k_3) \oplus W_2.$$

Note, in particular, that D(0,0) = D. Let X be the stabilizer of the subspaces

$$\langle e_1 \rangle, \langle e_2, e_3 \rangle, \langle e_4 \rangle, \dots, \langle e_m \rangle, \langle f_1 \rangle, \langle f_2, f_3 \rangle, \langle f_4 \rangle, \dots, \langle f_m \rangle.$$

Then $U \rtimes X$ is 2-transitive on $\Lambda = D^U$ (making use of Lemma 1.6.9 and the fact that M contains the matrices in (4.2.4)). Our aim now is to show that Λ is a beautiful subset of size q^2 .

Take $g \in S_{\Lambda}$ and suppose that Λ is not beautiful. An analogous argument to the previous case allows us to conclude that g preserves $\{W_1(k_2, k_3) \mid k_2, k_3 \in \mathbb{F}_q\}$ set-wise. This implies that g preserves the subspaces

$$Y_1 := \operatorname{span}_{\mathbb{K}} \{ W_1(k_2, k_3) \mid k_2, k_3 \in \mathbb{F}_q \} \text{ and } Y_0 := \bigcap_{k_2, k_3 \in \mathbb{F}_q} W_1(k_2, k_3)$$

Thus there is a homomorphism $\theta: S_{\Lambda} \to \operatorname{GL}(Y_1/Y_0) \cong \operatorname{GL}_3(\mathbb{K})$. Since $\operatorname{GL}_3(\mathbb{K})$ does not contain a subgroup with a composition factor isomorphic to $\operatorname{Alt}(s)$ for s > 7, we conclude that the action of ker (θ) on Λ must induce $\operatorname{Alt}(\Lambda)$ or $\operatorname{Sym}(\Lambda)$. However ker (θ) is not transitive on Λ so we have a contradiction.

Finally, assume that q = 2. In this case, we require that $n \ge 8$. This requirement excludes the groups $SU_4(2)$ and $SU_6(2)$ which were dealt with at the start. In addition Lemma 4.1.1 allows us to exclude the case when $S = \Omega_8^+(2)$ (notice that, in the proof, we prove that there is a beautiful subset). If $n \ge 10$, then our method here is similar to the previous case, but we start with a subgroup U whose elements fix all elements of \mathcal{B} except e_1 , e_2 , e_3 , e_4 and e_5 and satisfy

$$e_1 \mapsto e_1 + k_2 f_2 + k_3 f_3 + k_4 f_4 + k_5 f_5,$$

for some $k_2, k_3, k_4, k_5 \in \mathbb{F}_q$. We obtain a beautiful subset of cardinality 16; we leave the details to the reader. For the case when $S = SU_8(2)$, we use a subgroup U whose elements fix all elements of \mathcal{B} except e_1, e_2 and e_3 and satisfy

$$e_1 \mapsto e_1 + k_2 f_2 + k_3 f_3,$$

for some $k_2, k_3 \in \mathbb{F}_4$. Again we obtain a beautiful subset of cardinality 16.

4.2.3 A general reduction

In light of the previous subsections, we can now assume that we are in the case when S preserves a nondegenerate form on V and W_1, \ldots, W_t are all non-degenerate subspaces of V of dimension m. In the end we will need to split into separate cases, depending on the type of the form, but before we do that we give three general lemmas that significantly reduce the subsequent case work.

Lemma 4.2.4. If $q \ge 5$, then either Ω contains a beautiful subset or else one of the following holds:

1. m = 1;

2.
$$m = 2, S = \Omega_n^{\varepsilon}(q)$$
 for some $\varepsilon \in \{+, -\}$, and W_1, \ldots, W_t are all of type O_2^- .

Proof. Suppose that neither of the listed outcomes occurs – we must show that Ω contains a beautiful subset. Choose a hyperbolic basis for each of W_1, \ldots, W_t and let \mathcal{B} be the union of these bases.

Since we have excluded the two listed outcomes, we can let (e_1, f_1) (resp. (e_2, f_2)) be hyperbolic pairs whose elements are in $\mathcal{B} \cap W_1$ (resp. $\mathcal{B} \cap W_2$). Let U be the subgroup whose elements fix all elements of \mathcal{B} except e_1 and f_2 and satisfy

$$e_1 \mapsto e_1 + ke_2, \quad f_2 \mapsto f_2 \pm kf_1,$$

for some $k \in \mathbb{F}_q$, and we let $\Lambda = M^U$. (The choice of sign for the image of f_2 will depend on the type of form preserved by S.) For $k \in \mathbb{F}_q$, we define

$$W_1(k) = \langle e_1 + ke_2, f_1, x_1 \dots, x_{m-2} \rangle,$$

$$W_2(k) = \langle e_2, f_2 \pm kf_1, y_1, \dots, y_{m-2} \rangle,$$

where $\mathcal{B} \cap W_1 = \{e_1, f_1, x_1, ..., x_{m-2}\}$ and $\mathcal{B} \cap W_2 = \{e_2, f_2, y_1, ..., y_{m-2}\}$. Observe that $\Lambda = \{D(k) \mid k \in \mathbb{F}_q\}$, where

$$D(k) = W_1(k) \oplus W_2(k) \oplus W_3 \oplus \cdots \oplus W_t.$$

Note, in particular, that D(0) = D. Now we follow the argument of Lemma 4.2.3 with some slight adjustments. As before, we are able to conclude that, if $g \in S_{\Lambda}$, then g preserves the set $\{W_3, \ldots, W_t\}$ as well as the set

$$\{W_1(k) \mid k \in \mathbb{F}_q\} \cup \{W_2(k) \mid k \in \mathbb{F}_q\}.$$

Now suppose that $g \in S_{\Lambda}$ and g fixes both D(0) and D(1). We study the image of the spaces $W_1, W_2, W_1(1)$ and $W_2(1)$.

Suppose that $W_1^g = W_1$ and $W_1(1)^g = W_2(1)$. This implies that $f_1^g \in W_1 \cap W_2(1) = \{0\}$, a contradiction. Similarly, we cannot have $W_1^g = W_2$ and $W_1(1)^g = W_1(1)$.

Suppose next that $W_1^g = W_1$ and $W_1(1)^g = W_1(1)$. Then there exist $a, b, c_1, \ldots, c_{m-2}$ such that

$$e_1^g = ae_1 + bf_1 + \sum_{i=1}^{m-2} c_i x_i$$

Similarly there exist $a', b', c'_1, \ldots, c'_{m-2}$ such that

$$(e_1 + e_2)^g = a'(e_1 + e_2) + b'f_1 + \sum_{i=1}^{m-2} c'_i x_i.$$

We obtain that

$$e_2^g = (a'-a)e_1 + a'e_2 + (b'-b)f_1 + \sum_{i=1}^{m-2} (c'_i - c_i)x_i.$$

But, since $e_2 \in W_2 \cap W_2(1)$ and since g fixes D(0) and D(1), we deduce $e_2^g \in W_2 \cap W_2(1) = \operatorname{span}_{\mathbb{K}} \{e_2, y_1, \ldots, y_{m-2}\}$. Now, we conclude $e_2^g \in \operatorname{span}_{\mathbb{K}} \{e_2\}$. This implies, in particular, that $a' = a, b' = b, c'_1 = c_1, \ldots, c'_{m-2} = c_{m-2}$ and, in particular $e_2^g = ae_2$.

But now observe that

$$(e_1 + ke_2)^g = (e_1 + e_2)^g + (k - 1)e_2^g$$

= $a(e_1 + e_2) + bf_1 + \sum_{i=1}^{m-2} c_i x_i + (k - 1)ae_2$
= $a(e_1 + ke_2) + bf_1 + \sum_{i=1}^{m-2} c_i x_i \in W_1(k).$

This shows that $W_1(k)^g = W_1(k)$, for every $k \in \mathbb{F}_q$. We conclude that $D(k)^g = D(k)$ for all $k \in \mathbb{F}_q$.

So let us consider the remaining case, when $W_1^g = W_2$ and $W_1(1)^g = W_2(1)$. Then there exist $a, b, c_1, \ldots, c_{m-2}$ such that

$$e_1^g = ae_2 + bf_2 + \sum_{i=1}^{m-2} c_i y_i.$$

Similarly there exist $a', b', c'_1, \ldots, c'_{m-2}$ such that

$$(e_1 + e_2)^g = a'e_2 + b'(f_2 \pm f_1) + \sum_{i=1}^{m-2} c'_i y_i$$

We obtain that

$$e_2^g = (a'-a)e_2 + b'(f_2 \pm f_1) - bf_2 + \sum_{i=1}^{m-2} (c'_i - c_i)y_i.$$

But, since $e_2 \in W_2 \cap W_2(1)$ and since g fixes D(0) and D(1), we deduce $e_2^g \in W_1 \cap W_1(1) = \operatorname{span}_{\mathbb{K}} \{f_1, x_1, \ldots, x_{m-2}\}$, and we conclude that $e_2^g \in \operatorname{span}_{\mathbb{K}} \{f_1\}$. This implies, in particular, that $a' = a, b' = b, c'_1 = c_1, \ldots, c'_{m-2} = c_{m-2}$ and, in particular $e_2^g = \pm bf_1$.

But now observe that

$$(e_1 + ke_2)^g = (e_1 + e_2)^g + (k - 1)e_2^g$$

= $ae_2 + b(f_2 \pm f_1) + \sum_{i=1}^{m-2} c_i y_i + (k - 1)(\pm b)f_1$
= $ae_2 + b(f_2 \pm kf_1) + \sum_{i=1}^{m-2} c_i y_i \in W_2(k).$

This shows that $W_1(k)^g = W_2(k)$, for every $k \in \mathbb{F}_q$. We conclude that $D(k)^g = D(k)$ for all $k \in \mathbb{F}_q$.

In all cases, then, we conclude that, if $g \in S_{\Lambda}$ and g fixes the two points D(0) and D(1) of Λ , then g fixes all elements of Λ . But this implies that S^{Λ} does not contain $Alt(\Lambda)$ and we are done.

The next case deals with the first outcome of the preceding lemma, but also applies when q = 4.

Lemma 4.2.5. If m = 1 and $q \ge 4$, then either the action is not binary or else S is orthogonal and q = 5.

Proof. Our method is based on the treatment of this case for $PSU_3(q)$ in [45]. Note that m = 1 cannot occur if S is symplectic – thus we may assume that S is either unitary or orthogonal. In the orthogonal case, m = 1 implies that q is an odd prime number by [54, §4.2].

The action of G on Ω is permutation equivalent to the natural action of G on

$$\left\{ \{X_1, X_2, \dots, X_n\} \middle| \begin{array}{c} \dim_{\mathbb{K}}(X_1) = \dim_{\mathbb{K}}(X_2) = \dots = \dim_{\mathbb{K}}(X_n) = 1; \\ V = X_1 \perp X_2 \perp \dots \perp X_n; X_1, X_2, \dots, X_n \text{ non-degenerate} \end{array} \right\}.$$

Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for V; thus $\omega_0 := \{\langle v_1 \rangle, \langle v_2 \rangle, \ldots, \langle v_n \rangle\} \in \Omega$. If S is unitary, then define $n_0 = 3$; if S is orthogonal, then define $n_0 = 4$.

Now consider

$$\Lambda := \{ \{ X_1, X_2, \dots, X_n \} \in \Omega \mid X_i = \langle v_i \rangle \text{ for } i = n_0, \dots, n \}$$

Then G_{Λ} is equal to the stabilizer of $\{\langle v_{n_0} \rangle, \ldots, \langle v_n \rangle\}$. In the unitary case G^{Λ} is almost simple with socle $\mathrm{PSU}_2(q)$; in the orthogonal case, G^{Λ} is almost simple with socle $\Omega_3(q)$ (here we are using $q \geq 4$). In both cases the socle is isomorphic to $\mathrm{PSL}_2(q)$, and the action of G_{Λ} on Λ is permutation equivalent to the action of $G_{\{\langle v_{n_0} \rangle, \ldots, \langle v_n \rangle\}}$ on

$$\Lambda' := \{ \{X_1, \cdots, X_{n_0-1}\} \mid \dim(X_1) = \cdots = \dim(X_{n_0-1}), \\ \langle v_{n_0}, \dots, v_n \rangle^{\perp} = X_1 \perp \cdots \perp X_{n_0-1}, X_1, \dots, X_{n_0-1} \text{ non-degenerate} \}.$$

Suppose first that S is unitary. Then, this action of G^{Λ} has degree $|\Lambda| = q(q-1)/2$. By consulting the table of the maximal subgroups of almost simple groups with socle $PSL_2(q)$ in [10], we see that provided $q \notin \{7,9\}$ this action is primitive and hence by applying [45, Theorem 1.1] to G^{Λ} , we obtain that G^{Λ} is not binary. Moreover, when $q \in \{7,9\}$, it can be easily checked with magma that the action of G^{Λ} is not binary.

Suppose now that S is orthogonal. (Recall that q is a prime number.) In particular, $G^{\Lambda} \cong PSL_2(q)$ or $G^{\Lambda} \cong PGL_2(q)$. Let us denote by X the socle of G^{Λ} and by Y the stabilizer in G^{Λ} of an element of Λ . By consulting the table of the maximal subgroups of almost simple groups with socle $PSL_2(q)$ in [10, Table 8.7], we have

$$X \cap Y \cong \begin{cases} \text{Sym}(4), & \text{when } q \equiv \pm 1 \pmod{8} \\ \text{Alt}(4), & \text{when } q \equiv \pm 3, 5, \pm 11, \pm 13, \pm 19 \pmod{40}. \end{cases}$$

From the same table we infer that $X \cap Y$ is a maximal subgroup of X unless

$$G^{\Lambda} = X$$
 and $q \equiv \pm 11, \pm 19 \pmod{40}$.

Therefore, except for the cases where $q = p \equiv \pm 11, \pm 19 \pmod{40}$ and $G^{\Lambda} = X \cong \text{PSL}_2(q)$, by applying [45, Theorem 1.1] to G^{Λ} we obtain that G^{Λ} is not binary for $q \neq 5$. (Observe that q = 5 is the exception listed in the statement of the lemma.) We claim that this is the case also when $q = p \equiv \pm 11, \pm 19 \pmod{40}$ and $G^{\Lambda} = X \cong \text{PSL}_2(q)$. To see this, observe that from [10, Table 8.7], there exists a subgroup Z of $X = G^{\Lambda}$ with $Y < Z, Z \cong \text{Alt}(5)$ and with Z maximal in $X = G^{\Lambda}$. Now, the action of Z on (Z : Y) is not binary because it is permutation isomorphic to the non-binary degree 5 action of Alt(5). Hence G^{Λ} is not binary by Lemma 1.6.2, as claimed.

Summing up, for the rest of the proof we may suppose that G^{Λ} is not binary. In particular there exist two ℓ -tuples

$$(\{W_{1,1},\ldots,W_{1,n_0-1}\},\ldots,\{W_{\ell,1},\ldots,W_{\ell,n_0-1}\})$$

and

$$(\{W'_{1,1},\ldots,W'_{1,n_0-1}\},\ldots,\{W'_{\ell,1},\ldots,W'_{\ell,n_0-1}\})$$

in Λ^{ℓ} which are 2-subtuple complete for the action of G_{Λ} but not in the same G_{Λ} -orbit. By construction the two ℓ -tuples

$$I := (\{W_{1,1}, \dots, W_{1,n_0-1}, \langle v_{n_0} \rangle, \dots, \langle v_n \rangle\}, \{W_{2,1}, \dots, W_{2,n_0-1}, \langle v_{n_0} \rangle, \dots, \langle v_n \rangle\}, \dots, \\ \{W_{\ell,1}, \dots, W_{\ell,n_0-1}, \langle v_{n_0} \rangle, \dots, \langle v_n \rangle\}, \\ J := (\{W'_{1,1}, \dots, W'_{1,n_0-1}, \langle v_{n_0} \rangle, \dots, \langle v_n \rangle\}, \{W'_{2,1}, \dots, W'_{2,n_0-1}, \langle v_{n_0} \rangle, \dots, \langle v_n \rangle\}, \dots, \\ \{W'_{\ell,1}, \dots, W'_{\ell,n_0-1}, \langle v_{n_0} \rangle, \dots, \langle v_n \rangle\})$$

are in Ω^{ℓ} and are 2-subtuple complete. Furthermore it is easy to see that I and J are not in the same G-orbit. Thus G is not binary.

In the next lemma we write $X_n(q)$ to represent one of the three families of classical groups associated with non-degenerate forms. So, for instance, to get the result in the unitary case, the reader should read "PSU" wherever X occurs.

Lemma 4.2.6. Let m be a fixed positive integer, and let n_0 be a multiple of m such that $X_{n_0}(q)$ is almost simple. If the primitive C_2 -action of $X_{n_0}(q)$ on m-decompositions of fixed type is not binary, then the same is true of $X_n(q)$, for all n that are multiples of m and that exceed n_0 .

Note that the caveat "of fixed type" is included to account for the orthogonal case with m even, where we have decompositions of type O^+ and O^- .

Proof. We assume that $n > n_0$ and we proceed similarly to the previous lemma. First we identify Ω with the set

$$\left\{ \{X_1, X_2, \dots, X_t\} \middle| \begin{array}{c} \dim_{\mathbb{K}}(X_1) = \dim_{\mathbb{K}}(X_2) = \dots = \dim_{\mathbb{K}}(X_t) = m; \\ V = X_1 \perp X_2 \perp \dots \perp X_t; X_1, X_2, \dots, X_t \text{ non-degenerate} \end{array} \right\}.$$

Now, fix an element $\{W_1, \ldots, W_t\}$ of Ω and consider

$$\Lambda := \left\{ \{X_1, X_2, \dots, X_t\} \in \Omega \ \middle| \ X_i = W_i \text{ for } i = \frac{n_0}{m} + 1, \dots, t \right\}.$$

Clearly, G_{Λ} is equal to the stabilizer of $\{W_{n_0/m+1}, \ldots, W_t\}$ and G^{Λ} is almost simple with socle isomorphic to $X_{n_0}(q)$, and the action of G_{Λ} on Λ is permutation equivalent to the action of $G_{\{W_{n_0/m+1},\ldots,W_t\}}$ on

$$\Lambda' := \left\{ \{ W_1, \dots, W_{n_0/m} \} \left| \begin{array}{c} \dim(W_1) = \dots = \dim(W_{n_0/m}); W_1, \dots, W_{n_0/m} \text{ non-degenerate}; \\ \langle W_{n_0/m+1}, \dots, W_t \rangle^{\perp} = W_1 \perp \dots \perp W_{n_0/m} \end{array} \right\} \right\}$$

Therefore G^{Λ} is an almost simple primitive group with socle isomorphic to $X_{n_0}(q)$ in a C_2 -action on *m*-decompositions of given type. By assumption, G^{Λ} is not binary and hence there exist two ℓ -tuples $(\{W_{1,1},\ldots,W_{1,n_0/m}\},\ldots,\{W_{\ell,1},\ldots,W_{\ell,n_0/m}\})$ and $(\{W'_{1,1},\ldots,W'_{1,n_0/m}\},\ldots,\{W'_{\ell,1},\ldots,W'_{\ell,n_0/m}\})$ in Λ^{ℓ} which are 2-subtuple complete for the action of G_{Λ} but not in the same G_{Λ} -orbit. By construction the two ℓ -tuples

$$I := (\{W_{1,1}, \dots, W_{1,n_0/m}, W_{n_0/m+1}, \dots, W_t\}, \{W_{2,1}, \dots, W_{2,n_0/m}, W_{n_0/m+1}, \dots, W_t\}, \dots, \\ \{W_{\ell,1}, \dots, W_{\ell,n_0/m}, W_{n_0/m+1}, \dots, W_t\}), \\ J := (\{W'_{1,1}, \dots, W'_{1,n_0/m}, W_{n_0/m+1}, \dots, W_t\}, \{W'_{2,1}, \dots, W'_{2,n_0/m}, W_{n_0/m+1}, \dots, W_t\}, \dots, \\ \{W'_{\ell,1}, \dots, W'_{\ell,n_0/m}, W_{n_0/m+1}, \dots, W_t\})$$

are in Ω^{ℓ} and are 2-subtuple complete. As before, I and J are not in the same G-orbit. Thus G is not binary.

Group	Details of action
$\mathrm{SU}_n(q)$	m = 1
$SU_4(3), SU_4(4)$	m=2
$\mathrm{SU}_n(2)$	m = 3

Table 4.2.2: $C_2 - S = SU_n(q)$ and the W_i are non-degenerate.

4.2.4 Case where $S = SU_n(q)$ and the W_i are non-degenerate

Assume that $S = SU_n(q)$, the W_i are non-degenerate, and the socle of G is not as in Lemma 4.1.1. Here $V = V_n(\mathbb{K})$ where $\mathbb{K} = \mathbb{F}_{q^2}$, and we denote by σ the involutory field automorphism of \mathbb{K} .

Lemma 4.2.7. In this case either Ω contains a beautiful subset or else S is listed in Table 4.2.2.

Proof. Lemma 4.2.4 implies that when $q \ge 5$, either Ω contains a beautiful subset or else we obtain the first line of Table 4.2.2. Now assume that $q \le 4$ and $m \ge 2$.

If $q \in \{3, 4\}$, then we repeat the same set-up as Lemma 4.2.4, except that this time U is the subgroup whose elements fix all elements of \mathcal{B} except e_1 and f_2 and satisfy

$$e_1 \mapsto e_1 + ke_2,$$

 $f_2 \mapsto f_2 - k^{\sigma} f_1,$

for some $k \in \mathbb{K} = \mathbb{F}_{q^2}$, and we let $\Lambda = M^U$. Notice that Λ is of size q^2 rather than q as in Lemma 4.2.4. Now the same argument as before allows us to conclude that Λ is a beautiful subset of order q^2 , provided that n > 4. (When n = 4, we cannot conclude that G_{Λ} acts 2-transitively on Λ ; notice that the groups $SU_4(3)$ and $SU_4(4)$ are listed as exceptions in Table 4.2.2.) Now the argument of Lemma 4.2.4 implies that Λ is a beautiful subset, as required.

If q = 2 and $m \ge 2$, then [54, Table 3.5.H] implies that $m \ge 3$; if m = 3, the action is listed in Table 4.2.2, hence we assume that $m \ge 4$. We consider hyperbolic pairs from \mathcal{B} as before; this time assume that $e_1, f_1, e_2, f_2 \in W_1$ and $e_3, f_3, e_4, f_4 \in W_2$. Let $x_1, \ldots, x_{m-4}, y_1, \ldots, y_{m-4} \in \mathcal{B}$ be such that

$$W_1 = \operatorname{span}_{\mathbb{K}} \{ e_1, e_2, f_1, f_2, x_1, \dots, x_{m-4} \}$$
 and $W_2 = \operatorname{span}_{\mathbb{K}} \{ e_3, e_4, f_3, f_4, y_1, \dots, y_{m-4} \}.$

We let U be the subgroup whose elements fix all elements of \mathcal{B} except e_1, f_3 and f_4 and satisfy

$$e_1 \mapsto e_1 + k_3 e_3 + k_4 e_4$$

$$f_3 \mapsto f_3 - k_3^{\sigma} f_1,$$

$$f_4 \mapsto f_4 - k_4^{\sigma} f_1,$$

for some $k_3, k_4 \in \mathbb{K}$, and we define $\Lambda = D^U$. For $k_1, k_2 \in \mathbb{K}$, we define

$$W_1(k_1, k_2) = \langle e_1 + k_1 e_3 + k_2 e_4, e_2, f_1, f_2, x_1, \dots, x_{m-4} \rangle \text{ and} W_2(k_1, k_2) = \langle e_3, e_4, f_3 - k_1^{\sigma} f_1, f_4 - k_2^{\sigma} f_1, y_1, \dots, y_{m-4} \rangle.$$

Observe that $\Lambda = \{ D(k_1, k_2) \mid k_1, k_2 \in \mathbb{K} \}$, where

$$D(k_1,k_2) = W_1(k_1,k_2) \oplus W_2(k_1,k_2) \oplus W_3 \oplus \cdots \oplus W_t$$

Note, in particular, that D(0,0) = D. Let X be the stabilizer of the subspaces

$$\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3, e_4 \rangle, \langle f_1 \rangle, \langle f_2 \rangle, \langle f_3, f_4 \rangle, \langle x_1, \dots, x_{m-4} \rangle, \langle y_1, \dots, y_{m-4} \rangle, W_3, \dots, W_t.$$

Group	Details of action
$\operatorname{Sp}_n(2), \operatorname{Sp}_n(3), \operatorname{Sp}_n(4)$	m = 2

Table 4.2.3: $C_2 - S = \text{Sp}_n(q)$ and the W_i are non-degenerate.

Then $U \rtimes X$ is 2-transitive on $\Lambda = D^U$, a set of size 16. Our aim now is to show that Λ is a beautiful subset. Let $g \in S_{\Lambda}$. As in Lemma 4.2.4 we can see that g preserves the set

$$\{W_1(k_1, k_2) \mid k_1, k_2 \in \mathbb{K}\} \cup \{W_2(k_1, k_2) \mid k_1, k_2 \in \mathbb{K}\}.$$

Now suppose that there exist $k_1, k_2, k'_1, k'_2 \in \mathbb{K}$ such that $W_1(k_1, k_2)^g = W_2(k'_1, k'_2)$; then, by considering the vectors e_2^g, f_1^g, f_2^g , it is clear that for all $k_1, k_2 \in \mathbb{K}$, there exist $k'_1, k'_2 \in \mathbb{K}$ such that $W_1(k_1, k_2)^g = W_2(k'_1, k'_2)$. We conclude that S_Λ has a subgroup H of index at most 2 such that, if $g \in H$, then for all $k_1, k_2 \in \mathbb{K}$ there exist $k'_1, k'_2 \in \mathbb{K}$ (which may depend upon g, k_1, k_2) such that $W_1(k_1, k_2)^g = W_1(k'_1, k'_2)$.

This implies that H preserves the subspaces

$$Y_1 := \operatorname{span}_{\mathbb{K}} \{ W_1(k_1, k_2) \mid k_1, k_2 \in \mathbb{K} \} \text{ and } Y_0 := \bigcap_{k_1, k_2 \in \mathbb{K}} W_1(k_1, k_2)$$

Thus there is a homomorphism $\theta : H \to \operatorname{GL}(Y_1/Y_0) \cong \operatorname{GL}_3(\mathbb{K})$. Since $\operatorname{GL}_3(\mathbb{K})$ does not contain a subgroup with a composition factor isomorphic to $\operatorname{Alt}(s)$ for s > 7, we conclude that either Λ is beautiful or the action of ker(θ) on Λ must induce $\operatorname{Alt}(\Lambda)$ or $\operatorname{Sym}(\Lambda)$. However ker(θ) is not transitive on Λ and the result follows.

We need to deal with the cases listed in Table 4.2.2. Lemma 4.1.1 deals with the second line of the table. Now Lemma 4.2.5 means that, to deal with the first line of Table 4.2.2, we may assume that $q \in \{2, 3\}$. Thus the next lemma deals with what remains.

Lemma 4.2.8. Suppose that (q,m) is one of (2,3), (2,1) or (3,1). Then the action is not binary.

Proof. By Lemma 4.1.1 we have $n \ge 7$. Now Lemma 4.2.6 implies that the result holds for $n \ge 7$.

4.2.5 Case where $S = Sp_n(q)$ and the W_i are non-degenerate

Assume that $S = \text{Sp}_n(q)$ with $n \ge 4$, the W_i are non-degenerate, and the socle of G is not as in Lemma 4.1.1.

Lemma 4.2.9. In this case either Ω contains a beautiful subset or else S is listed in Table 4.2.3.

Proof. Lemma 4.2.4 implies that, when $q \ge 5$, Ω contains a beautiful subset. Now assume that $q \le 4$. Choose a hyperbolic basis for each of W_1, \ldots, W_t and let \mathcal{B} be the union of these bases. Write $m = 2\ell$ and order the hyperbolic basis so that $e_1, f_1, \ldots, e_\ell, f_\ell \in W_1$; $e_{\ell+1}, f_{\ell+1}, \ldots, e_{2\ell}, f_{2\ell} \in W_2$ and so on.

We exclude the case m = 2, since this is listed in Table 4.2.3 and we assume that $m \ge 4$. Now let U be the subgroup whose elements fix all elements of \mathcal{B} except $e_1, e_{\ell+1}, e_{\ell+2}, f_{\ell+1}$ and $f_{\ell+2}$ and satisfy

$$e_{1} \mapsto e_{1} + k_{1}e_{\ell+1} + k_{2}e_{\ell+2} + k_{3}f_{\ell+1} + k_{4}f_{\ell+2},$$

$$e_{\ell+1} \mapsto e_{\ell+1} + k_{3}f_{1},$$

$$e_{\ell+2} \mapsto e_{\ell+2} + k_{4}f_{1},$$

$$f_{\ell+1} \mapsto f_{\ell+1} - k_{1}f_{1},$$

$$f_{\ell+2} \mapsto f_{\ell+2} - k_{2}f_{1},$$

for some $k_1, k_2, k_3, k_4 \in \mathbb{F}_q$, and we define $\Lambda = D^U$. For $k_1, k_2, k_3, k_4 \in \mathbb{F}_q$, we define

$$W_1(k_1, k_2, k_3, k_4) = \langle e_1 + k_1 e_{\ell+1} + k_2 e_{\ell+2} + k_3 f_{\ell+1} + k_4 f_{\ell+2}, e_2, \dots, e_\ell, f_1, \dots, f_\ell \rangle \text{ and } W_2(k_1, k_2, k_3, k_4) = \langle e_{\ell+1} + k_3 f_1, e_{\ell+2} + k_4 f_1, e_{\ell+3}, \dots, e_{2\ell}, f_{\ell+1} - k_1 f_1, f_{\ell+2} - k_2 f_1, f_{\ell+3}, \dots, f_{2\ell} \rangle$$

Group	Details of action
$\Omega_n(p) \\ \Omega_n(3)$	m = 1 $m = 3$

Table 4.2.4: $C_2 - S = \Omega_n(q)$ with nq odd and the W_i are non-degenerate.

and observe that $\Lambda = \{ D(k_1, k_2, k_3, k_4) \mid k_1, k_2, k_3, k_4 \in \mathbb{F}_q \},$ where

$$D(k_1, k_2, k_3, k_4) = W_1(k_1, k_2, k_3, k_4) \oplus W_2(k_1, k_2, k_3, k_4) \oplus W_3 \oplus \cdots \oplus W_t$$

Note, in particular, that D(0,0,0,0) = D. Let X be the stabilizer of the subspaces

$$\langle e_{\ell+1}, e_{\ell+2}, f_{\ell+1}, f_{\ell+2} \rangle, \langle e_1 \rangle, \dots, \langle e_\ell \rangle, \langle e_{\ell+3} \rangle, \dots, \langle e_{\ell t} \rangle, \langle f_1 \rangle, \dots, \langle f_\ell \rangle, \langle f_{\ell+3} \rangle, \dots, \langle f_{\ell t} \rangle.$$

Then $U \rtimes X$ is 2-transitive on $\Lambda = D^U$ (making use of Lemma 1.6.9 and the fact that $\operatorname{Sp}_4(q)$ acts transitively on the set of non-zero vectors in the natural \mathbb{F}_q -module), a set of size q^4 . Our aim now is to show that Λ is a beautiful subset. Let $g \in S_{\Lambda}$. As before we can see that g preserves the set

$$\{W_1(k_1, k_2, k_3, k_4) \mid k_1, k_2, k_3, k_4 \in \mathbb{F}_q\} \cup \{W_1(k_1, k_2, k_3, k_4) \mid k_1, k_2, k_3, k_4 \in \mathbb{F}_q\}.$$

Now suppose that there exist $k_1, k_2, k_3, k_4, k'_1, k'_2, k'_3, k'_4 \in \mathbb{F}_q$ such that $W_1(k_1, k_2, k_3, k_4)^g = W_2(k'_1, k'_2, k'_3, k'_4)$; then, by considering the vectors e_2^g, f_1^g, f_2^g , it is clear that, for all $k_1, k_2, k_3, k_4 \in \mathbb{F}_q$, there exist $k'_1, k'_2, k'_3, k'_4 \in \mathbb{F}_q$ such that $W_1(k_1, k_2, k_3, k_4)^g = W_2(k'_1, k'_2, k'_3, k'_4)$. We conclude that S_Λ has a subgroup H of index at most 2 such that, if $h \in H$, then for all $k_1, k_2, k_3, k_4 \in \mathbb{F}_q$ there exist $k'_1, k'_2, k'_3, k'_4 \in \mathbb{F}_q$ with $W_1(k_1, k_2, k_3, k_4)^g = W_1(k'_1, k'_2, k'_3, k'_4)$.

We conclude that H preserves the subspaces

$$Y_1 := \operatorname{span}_{\mathbb{K}} \{ W_1(k_1, k_2, k_3, k_4) \mid k_1, k_2, k_3, k_4 \in \mathbb{F}_q \} \text{ and } Y_0 := \bigcap_{k_1, k_2, k_3, k_4 \in \mathbb{F}_q} W_1(k_1, k_2, k_3, k_4)$$

Thus there is a homomorphism $\theta: S_{\Lambda} \to \operatorname{GL}(Y_1/Y_0) \cong \operatorname{GL}_5(\mathbb{F}_q)$. By Lemma 2.1.1, $\operatorname{GL}_5(\mathbb{F}_q)$ does not have a section isomorphic to $\operatorname{Alt}(s)$ for s > 8, so we conclude that either Λ is a beautiful set, or the action of $\operatorname{ker}(\theta)$ on Λ must induce $\operatorname{Alt}(\Lambda)$ or $\operatorname{Sym}(\Lambda)$. However $\operatorname{ker}(\theta)$ is not transitive on Λ , and we conclude that Λ is beautiful as required.

We must show that the actions listed in Table 4.2.3 are not binary; the next lemma does the job.

Lemma 4.2.10. Suppose that (q,m) is one of (2,2), (3,2) or (4,2). Then the action is not binary.

Proof. Lemma 4.1.1 gives the result for n = 4. Now Lemma 4.2.6 implies that the result holds for n > 4.

4.2.6 Case where $S = \Omega_n(q)$ for nq odd, and the W_i are non-degenerate

Lemma 4.2.11. In this case either Ω contains a beautiful subset or else S is listed in Table 4.2.4.

Proof. Lemma 4.2.4 implies that when $q \ge 5$, either Ω contains a beautiful subset or else m = 1; in the latter case, [54, Table 3.5.D] implies that q = p, a prime, and we obtain the first line of Table 4.2.4. If q = 3 and m = 3, then we obtain the second line of Table 4.2.4.

Assume, then, that q = 3 and $m \ge 5$. The assumption on m means that each W_i contains at least two hyperbolic pairs. Now the argument of Lemma 4.2.7 for q = 2 carries over here and we obtain a beautiful subset of size 9.

We must show that the actions listed in Table 4.2.4 are not binary. Lemma 4.2.5 deals with the first line, provided q > 5; the next lemma deals with what remains.

Group	Details of action
$ \begin{array}{l} \Omega_n^+(p) \\ \Omega_n^+(q) \\ \Omega_n^+(4) \\ \Omega_n^+(3) \\ \Omega_n^+(2) \end{array} $	$p \geq 3, m = 1$ $W_i \text{ of type } O_2^-$ $W_i \text{ of type } O_2^\pm \text{ or } O_4^-$ $W_i \text{ of type } O_2^\pm, O_3 \text{ or } O_4^-$ $W_i \text{ of type } O_2^\pm, O_4^\pm \text{ or } O_6^-$

Table 4.2.5: $C_2 - S = \Omega_n^+(q)$ – and the W_i are non-degenerate.

Group	Details of action
$\begin{array}{l} \Omega_n^-(p) \\ \Omega_n^-(q) \\ \Omega_n^-(4) \\ \Omega_n^-(3) \\ \Omega_n^-(2) \end{array}$	$p \ge 3, m = 1$ $W_i \text{ of type } O_2^-$ $W_i \text{ of type } O_2^- \text{ or } O_4^-$ $W_i \text{ of type } O_2^-, O_3 \text{ or } O_4^-$ $W_i \text{ of type } O_2^-, O_4^- \text{ or } O_6^-$

Table 4.2.6: $C_2 - S = \Omega_n^-(q)$ – and the W_i are non-degenerate.

Lemma 4.2.12. Suppose that (q,m) is one of (3,1), (5,1), (3,3). Then the action is not binary.

Proof. We begin by checking the truth of this statement for $S \in \{\Omega_5(3), \Omega_5(5), \Omega_9(3)\}$. For the first two cases it follows from Lemma 4.1.1. And when $S = \Omega_9(3)$, we use the permutation character method. Let M be a maximal subgroup of S in the Aschbacher class C_2 , let 1_S be the principal character of S and let π_M be the permutation character for the action of S on the right cosets of M. We have verified that in all cases

$$\langle \pi(\pi - 1_S)(\pi - 2 \cdot 1_S), 1_S \rangle > (|\operatorname{Out}(S)| \langle \pi(\pi - 1_S), 1_S \rangle)^3$$

In particular, all actions under consideration are not binary in view of Lemma 1.8.1.

Now Lemma 4.2.6 implies that the result holds for all $n \ge 7$, as required.

4.2.7 Case where $S = \Omega_n^{\pm}(q)$ and the W_i are non-degenerate

Lemma 4.2.13. In this case either Ω contains a beautiful subset or else S is listed in Table 4.2.5 or Table 4.2.6.

Proof. If $q \ge 5$, then Lemma 4.2.4 yields the first two lines of each table. We also use the fact, from [54, Tables 3.5.E and 3.5.F], that if m = 1, then $q = p \ge 3$, where p is prime. Assume, then, that $q \le 4$. Recall that if q is even, then m is even. We consider the case where $q \in \{3, 4\}$ first. We require that W_1 contains at least two orthogonal hyperbolic lines. All cases that do not satisfy this requirement are listed in the tables.

Now we let e_1, f_1 be a hyperbolic pair in W_1 , and $e_{\ell+1}, f_{\ell+1}, e_{\ell+2}, f_{\ell+2}$ two hyperbolic pairs in W_2 . We let U be the subgroup whose elements fix all elements of \mathcal{B} except $e_1, f_{\ell+1}$ and $f_{\ell+2}$ and satisfy

$$e_{1} \mapsto e_{1} + k_{1}e_{\ell+1} + k_{2}e_{\ell+2}, \\ f_{\ell+1} \mapsto f_{\ell+1} - k_{1}f_{1}, \\ f_{\ell+2} \mapsto f_{\ell+2} - k_{2}f_{1},$$

for some $k_1, k_2 \in \mathbb{F}_q$, and we define $\Lambda = D^U$. We now proceed using the argument for q = 2 in Lemma 4.2.7 to conclude that we have a beautiful subset of size q^2 . (Note that S^{Λ} contains $ASL_2(q)$, hence the 2-transitivity of S^{Λ} is immediate.)

If q = 2, then the argument is similar, but we require that W_1 contains at least three orthogonal hyperbolic lines. All cases that do not satisfy this requirement are listed in the tables. We let U be the subgroup whose elements fix all elements of \mathcal{B} except $e_1, f_{\ell+1}, f_{\ell+2}$ and $f_{\ell+3}$ and satisfy

$$\begin{aligned} e_{1} &\mapsto e_{1} + k_{1}e_{\ell+1} + k_{2}e_{\ell+2} + k_{3}e_{\ell+3}, \\ f_{\ell+1} &\mapsto f_{\ell+1} - k_{1}f_{1}, \\ f_{\ell+2} &\mapsto f_{\ell+2} - k_{2}f_{1}, \\ f_{\ell+3} &\mapsto f_{\ell+3} - k_{3}f_{1}, \end{aligned}$$

for some $k_1, k_2, k_3 \in \mathbb{F}_q$, and we define $\Lambda = D^U$. As before we get a homomorphism $\theta: S_\Lambda \mapsto \operatorname{GL}(Y_1/Y_0) \cong \operatorname{GL}_4(2)$. Moreover, Λ corresponds to a set of 8 vectors in Y_1/Y_0 , namely the set of vectors $e_1 + k_1e_{\ell+1} + k_2e_{\ell+2} + k_3e_{\ell+3} + Y_0$. Since the stabilizer in $\operatorname{GL}_4(2)$ of this set does not induce $\operatorname{Alt}(8)$, it follows as before that Λ is a beautiful subset of size 8, completing the proof.

We must show that the actions listed in Tables 4.2.5 and 4.2.6 are not binary. Lemma 4.2.5 deals with the first line of each table, provided q > 5. The next lemma deals with the second line of each table.

Lemma 4.2.14. If the W_i are of type O_2^- , then the action is not binary.

Proof. The proof is similar to that of Lemma 4.2.5.

Note that the action of G on Ω is permutation equivalent to the natural action of G on

$$\left\{ \{X_1, X_2, \dots, X_{n/2}\} \middle| \begin{array}{c} X_1, \dots, X_{n/2} \text{ of type } \mathcal{O}_2^-; \\ V = X_1 \perp X_2 \perp \dots \perp X_{n/2}; X_1, X_2, \dots, X_{n/2} \text{ non-degenerate} \end{array} \right\}.$$

Now consider

$$\Lambda := \{ \{ X_1, X_2, \dots, X_{n/2} \} \in \Omega \mid X_i = W_i \text{ for } i \in \{4, \dots, n/2\} \}.$$

Then G_{Λ} is equal to the stabilizer of $\{W_4, \ldots, W_{n/2}\}$ and G^{Λ} is almost simple with socle $P\Omega_6^-(q)$; therefore, the socle of G^{Λ} is isomorphic to $PSU_4(q)$ and the action is isomorphic to a C_2 -action of an almost simple group with socle $PSU_4(q)$ on non-degenerate 1-spaces of $\mathbb{F}_{q^2}^4$. We saw in §4.2.4 that this action is not binary, thus there exist two ℓ -tuples ($\{W_{1,1}, W_{1,2}, W_{1,3}\}, \ldots, \{W_{\ell,1}, W_{\ell,2}, W_{\ell,3}\}$) and ($\{W'_{1,1}, W'_{1,2}, W'_{1,3}\}, \ldots$, $\{W'_{\ell,1}, W'_{\ell,2}, W'_{\ell,3}\}$) in Λ^{ℓ} which are 2-subtuple complete for the action of G_{Λ} but not in the same G_{Λ} -orbit. By construction the two ℓ -tuples

$$I := (\{W_{1,1}, W_{1,2}, W_{1,3}, W_4, \dots, W_{n/2}\}, \{W_{2,1}, W_{2,3}, W_{2,3}, W_4, \dots, W_{n/2}\}, \dots, \\ \{W_{\ell,1}, W_{\ell,2}, W_{\ell,3}, W_4, \dots, W_{n/2}\}), \\ J := (\{W'_{1,1}, W'_{1,2}, W'_{1,3}, W_4, \dots, W_{n/2}\}, \{W'_{2,1}, W'_{2,2}, W'_{2,3}, W_4, \dots, W_{n/2}\}, \dots, \\ \{W'_{\ell,1}, W'_{\ell,2}, W'_{\ell,3}, W_4, \dots, W_{n/2}\})$$

are in Ω^{ℓ} and are 2-subtuple complete. Moreover, I and J are not in the same G-orbit. Thus G is not binary.

The next lemma deals with the remaining lines of Tables 4.2.5 and 4.2.6.

Lemma 4.2.15. Suppose that (q, m) is in

 $\{(3,1), (5,1), (2,2), (3,2), (4,2), (3,3), (2,4), (3,4), (4,4), (2,6)\}.$

Then the action is not binary.

Proof. If m = 1, then we use the fact that we have already studied all C_2 actions for n odd. In particular Lemma 4.2.12 attends to the case where $(m, n, q) \in \{(1, 7, 3), (1, 7, 5)\}$. Now Lemma 4.2.6 yields the result for $(q, m) \in \{(3, 1), (5, 1)\}$ and $n \ge 8$.

If m = 3, a similar argument works, using the fact that Lemma 4.2.12 attends to the case where (m, n, q) = (3, 9, 3).

If m = 2 or 4, then we must deal with the cases where $q \in \{2, 3, 4\}$ (note that when m = 2, Lemma 4.2.14 allows us to assume that the W_i are of type O_2^+). When n = 8, Lemma 4.1.1 gives the result for q = 2. We use magma to verify that, when n = 8 and $q \in \{3, 4\}$, then the corresponding C_2 actions are not binary. Lemma 4.2.6 then implies the result for n > 8.

If m = 6, then we must deal with the case q = 2 and the situation where the W_i are of type O_6^- . We consider first what happens when n = 12: note that, in this case, $S = \Omega_{12}^+(2)$, since n/m is even. Now we use magma, with the permutation character method (using Lemma 1.8.1), to confirm that, in the case $S = \Omega_{12}^+(2)$, the action is not binary. Now Lemma 4.2.6 implies the result for n > 12.

4.3 Family C_3

In this section, the subgroup M is a "field extension subgroup". Such subgroups are described in [54, Section 4.3], and are listed in Table 4.3.1. In every case we start with a field extension $\mathbb{K}_{\#}$ of \mathbb{K} of prime degree. We will usually denote this degree by the letter "r", although in a few subfamilies, the degree is always equal to 2. We set m = n/r.

case	type	conditions
L	$\operatorname{GL}_m(q^r)$	
U	$\mathrm{GU}_m(q^r)$	$r \operatorname{odd}$
\mathbf{S}	$\operatorname{Sp}_m(q^r)$	
O^{ϵ}	$\mathrm{O}_m^\epsilon(q^r)$	$m \ge 3$
$\mathbf{S},\mathbf{O}^\epsilon$	$\mathrm{GU}_{n/2}(q)$	r=2,q odd in case S

Table 4.3.1: Maximal subgroups in family C_3

In the case $S = \mathrm{SL}_n(q)$, the group M is embedded in G by considering the group $\Gamma \mathrm{L}_m(\mathbb{K}_{\#})$ acting on an *m*-dimensional vector space $V_{\#}$ over $\mathbb{K}_{\#}$ and then considering those $\mathbb{K}_{\#}$ -semilinear transformations of $V_{\#}$ that induce \mathbb{K} -linear transformations on V, where V is simply $V_{\#}$ viewed as a \mathbb{K} -vector space. For the other cases, one must also consider $\mathbb{K}_{\#}$ -forms defined on $V_{\#}$; full details are given in [54].

It is convenient to give a geometrical interpretation for the set of right cosets of M (which is the set on which we are acting). To do this, we take $\mathbb{K}_{\#}$ to be a field extension of our field \mathbb{K} and we wish to define a $\mathbb{K}_{\#}$ -structure on V.

We start by considering a K-linear isomorphism $\phi: V_{\#} \to V$. Let Σ be the set of all such isomorphisms, and we observe that two groups act naturally on Σ :

- 1. $\operatorname{GL}_m(\mathbb{K}_{\#})$ acts on Σ via $\phi^g(\mathbf{v}) = \phi(\mathbf{v}^{g^{-1}});$
- 2. $\operatorname{GL}_n(\mathbb{K})$ acts on Σ via $\phi^h(\mathbf{v}) = (\phi(\mathbf{v}))^h$.

Clearly these two actions commute. Thus we define a $\mathbb{K}_{\#}$ -structure on V to be an orbit of the group $\operatorname{GL}_m(\mathbb{K}_{\#})$ on Σ , and (using commutativity of the actions) we observe that $\operatorname{GL}_n(\mathbb{K})$ acts on the set of all $\mathbb{K}_{\#}$ -structures on V. What is more, the stabilizer of such an action is a field extension subgroup M, hence we have the geometrical interpretation that we require.

Note that we can replace the word "linear" with the word "semilinear" in the previous paragraph to extend this geometrical interpretation to subgroups of $\Gamma L_n(\mathbb{K})$.

The main result of this section is the following. The result will be proved in a series of lemmas.

- (i) $n \geq 3, 4, 4, 7$ in cases L, U, S, O respectively, and
- (ii) $\operatorname{Cl}_n(q)$ is not one of the groups listed in Lemma 4.1.1.

Let M be a maximal subgroup of G in the family C_3 . Then the action of G on (G:M) is not binary.

4.3.1 Case $S = SL_n(q)$

Lemma 4.3.2. Suppose that G is almost simple with socle equal to $PSL_n(q)$. Let M be a C_3 -maximal subgroup such that $F^*(M)$ contains M_1 , a quasisimple cover of $PSL_m(q^r)$, where n = mr, r is prime and $m \ge 3$. Then the action of G on (G:M) is not binary.

Proof. We define

$$\tilde{x} = \begin{pmatrix} 1 & & \\ & A & \\ & & a^{-1} \end{pmatrix} \in \mathrm{SL}_m(q^r),$$

where A is an element of $\operatorname{GL}_{m-2}(q^r)$ of order $(q^r)^{m-2} - 1$ and $a = \det(A)$. We let x be the element in $F^*(M)$ which is the projective image of \tilde{x} in M_1 . Observe that \tilde{x} has a 1-eigenspace over \mathbb{F}_{q^r} of dimension 1, and so has a 1-eigenspace over \mathbb{F}_q of dimension r; we conclude that $C_M(x)$ is a proper subgroup of $C_G(x)$. Thus there is a suborbit, Δ , on which the action of M is isomorphic to the action of M on (M : H), where H is a subgroup of M that does not contain M_1 and does contain x.

We now refer to Lemma 2.2.5. This shows that either the action of M on Δ is not binary, or M has a section Alt $(q^{r(m-2)})$. In the former case, the conclusion of the lemma holds. In the latter case, Lemma 2.1.1 implies that the only possibility is m = 3, q = 2, r = 2. But then $S = SL_6(2)$, a case covered by Lemma 4.1.1.

The remaining lemmas deal with the case when $m \leq 2$.

Lemma 4.3.3. Suppose that G is an almost simple group with socle $PSL_n(q)$, where n = 2r for r a prime. Suppose that M is a C_3 -maximal subgroup such that $M \triangleright PSL_2(q^r)$. Then the action of G on (G : M) is not binary.

Proof. Let $\mathbb{K}_{\#}$ be the field \mathbb{F}_{q^r} , and let $\{v_1, v_2\}$ be a $\mathbb{K}_{\#}$ -basis for V. Let λ be an element of $\mathbb{K}_{\#}$ such that $\{\lambda, \lambda^q, \ldots, \lambda^{q^{r-1}}\}$ is a basis for $\mathbb{K}_{\#}$ over $\mathbb{K} = \mathbb{F}_q$ and observe that

$$\mathcal{B} = \{v_1\lambda, v_1\lambda^q, \dots, v_1\lambda^{q^{r-1}}, v_2\lambda, v_2\lambda^q, \dots, v_2\lambda^{q^{r-1}}\}$$

is an \mathbb{F}_q -basis for V. We take $M \cap \mathrm{PGL}(V)$ to be the subgroup that preserves the (semilinear) $\mathbb{K}_{\#}$ -vector space structure of V.

Suppose first that r > 2 and q > 2. Then $M \cap PSL(V)$ has the structure $(C \times PSL_2(q^r)).r$ with C cyclic (see [54, 4.3.6]). Let M_0 be a subgroup of $M \cap PSL_n(q)$ isomorphic to $PSL_2(q^r).r$. Then M_0 has a subgroup $H = H_0 \times \langle \sigma \rangle$, where $H_0 \cong PSL_2(q)$ and σ is a field automorphism of order r. Moreover H is maximal in M_0 (see [10, Table 8.1]).

Consider the direct sum \mathbb{F}_q -decomposition

$$V = \langle v_1 \lambda, v_2 \lambda \rangle \oplus \langle v_1 \lambda^q, v_2 \lambda^q \rangle \oplus \dots \oplus \langle v_1 \lambda^{q^{r-1}}, v_2 \lambda^{q^{r-1}} \rangle.$$
(4.3.1)

Observe that H_0 stabilizes each subspace in the decomposition, while the field automorphism $\sigma : x \mapsto x^q$ induces an *r*-cycle on the *r* subspaces in the decomposition. Thus *H* stabilizes the decomposition. Since r > 2, there is an element *g* of order r - 1 stabilizing the decomposition that is not in *M*, centralizes H_0 and normalizes $\langle \sigma \rangle$. Then $H \leq M \cap M^g$, and so $M \cap M^g$ is a maximal subgroup of $M_1 = (M \cap M^g)M'_0$. Hence the action of M_1 on $(M_1 : M \cap M^g)$ is isomorphic to the action of an almost simple group with socle $PSL_2(q^r)$ on a maximal \mathcal{C}_5 -subgroup containing $PSL_2(q)$. By [45], this action is not binary, and now Lemma 1.6.2 implies that the action of M on $(M : M \cap M^g)$ is not binary. Then Lemma 1.6.1 implies that the action of G on (G : M) is not binary.

Suppose, next, that r = 2. Again M has a subgroup $M_0 \cong PSL_2(q^2)$ preserving the decomposition (4.3.1), and M_0 has a subgroup $H_0 \cong PGL_2(q)$. Define U to be the set of elements in S that fix all elements of \mathcal{B} except $v_1\lambda$ and which satisfy

$$v_1 \lambda \mapsto v_1 + \alpha v_2 \lambda \quad (\alpha \in \mathbb{F}_q).$$

Then U is a group of order q with $U \cap M = 1$, and U is normalized by a torus $T < H_0$ of order q - 1, acting fixed-point-freely. In the usual way we obtain a subset Δ of Ω of size q for which G^{Δ} is 2-transitive. Now, by Lemma 2.1.1, M does not have a section isomorphic to Alt(t) for $t \geq 7$ and, by Lemma 1.6.12 the conclusion of the lemma follows for q > 5. If q = 2, then $S = SL_4(2) \cong Alt(8)$ and the result follows from [46]. And if $q \in \{3, 4, 5\}$, then the result follows from Lemma 4.1.1.

Finally assume that q = 2 and r > 2. In this case, writing matrices with respect to \mathcal{B} , M contains an element

$$g = \begin{pmatrix} I_r & \\ & A \end{pmatrix},$$

where $A \in \operatorname{GL}_r(2)$ is an element of order $2^r - 1$, and we let $T = \langle g \rangle$. Now let U be the subgroup of S consisting of elements u that fix all elements of \mathcal{B} except $v_1 \lambda$ and for which

$$v_1\lambda^u - v_1\lambda \in \operatorname{span}_{\mathbb{F}_q}\{v_2\lambda, v_2\lambda^q, \dots, v_2\lambda^{q^{r-1}}\}$$

Then U is a group of order q^r and $U \rtimes T$ is a Frobenius group. Then the set $\Lambda = M^U$ is a set of size q^r on which G^{Λ} acts 2-transitively. By Lemma 2.1.1, M does not contain a section isomorphic to Alt(t) for t > 6. Thus, we conclude that Λ is a beautiful set, and Lemma 1.6.12 yields the result.

Lemma 4.3.4. Suppose that G is almost simple with socle equal to $PSL_n(q)$ and n is an odd prime. Let M be the normalizer of a Singer cycle in G. Then the action of G on cosets of M is not binary.

Proof. We can write the group $F = M \cap \text{PSL}_n(q)$ as a semidirect product $T \rtimes C$ where T is cyclic of order $\frac{q^n-1}{(q-1)(q-1,n)}$ and C is cyclic of order n, and acts fixed-point-freely on T. Choosing an appropriate basis we may take C to be generated by a permutation matrix c corresponding to an n-cycle, and one sees immediately that $C_{\text{PSL}_n(q)}(c) > \langle c \rangle$. Let $x \in C_{\text{PSL}_n(q)}(c) \setminus \langle c \rangle$ and observe that the group F acts as a Frobenius group on the set $(F : F \cap F^x)$.

Since n > 2, Lemma 1.7.2 implies that the action of M on $(M : M \cap M^x)$ is not binary. Now Lemma 1.6.1 yields the result.

4.3.2 Case $S = SU_n(q)$

Lemma 4.3.5. Suppose that G is almost simple with socle equal to $PSU_n(q)$. Let M be a C_3 -subgroup. Then the action of G on (G:M) is not binary.

Proof. Note that $F^*(M)$ contains a normal subgroup M_1 which is a quasisimple cover of $PSU_m(q^r)$ and, by [54, Table 3.5.B], $r \ge 3$.

First suppose that $m \ge 5$. Here we refer to Lemma 2.2.8 and we write elements of M_1 with respect to the basis \mathcal{B} of $V_m(q^{2r})$ in that lemma. Define $j = \lfloor \frac{m-1}{2} \rfloor$ and y = (m, 2), so that m = 2j + y. Then let

$$\tilde{x} = \begin{pmatrix} 1 & & & & \\ & A & & & \\ & & 1 & & \\ & & & \overline{A^{-T}} & \\ & & & & J_y \end{pmatrix},$$

where A is an element of $\operatorname{GL}_{j-1}(q^{2r})$ of order $(q^{2r})^{j-1} - 1$ and J_y is a y-by-y matrix chosen so that \tilde{x} is an element of $\operatorname{SU}_m(q^r)$. Observe that 1 is not an eigenvalue for J_y . Now we let x be the element of M_1 that is an image of \tilde{x} .

Observe that \tilde{x} has a 1-eigenspace over \mathbb{F}_{q^r} of dimension 2, and so has a 1-eigenspace over \mathbb{F}_q of dimensions 2r. From this we conclude that $C_M(x)$ is a proper subgroup of $C_G(x)$. Let $g \in C_G(x) \setminus C_M(x)$ and set $H := M \cap M^g$. Then H contains x but does not contain M_1 , and there is a suborbit, Δ , on which the action of M is isomorphic to the action of M on (M : H).

We now refer to Lemma 2.2.8. This shows that (M : H) has a subset Δ of size $q^{2r(j-1)}$ such that M^{Δ} is 2-transitive. Since M does not have a section isomorphic to $\operatorname{Alt}(q^{2r(j-1)})$ by Lemma 2.1.1, it follows that Δ is a beautiful subset, and the conclusion holds by Lemma 1.6.1.

If m = 4, then we use the same argument with Lemma 2.2.11 in place of Lemma 2.2.8. Now suppose that m = 3. Choose a hyperbolic basis, $\mathcal{B}_0 = \{e_1, x, f_1\}$, for a 3-dimensional Hermitian space V over $\mathbb{K}_{\#} = \mathbb{F}_{q^{2r}}$ associated with a form φ . We will use the fact that the isometry group of φ contains an element

$$g := \begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix}$$

where a is an element of \mathbb{F}_{q^r} of order $q^r - 1$. Now we can take M to contain a projective image of the special isometry group of φ , and let S be the special isometry group of the form $\operatorname{Tr}_{\mathbb{K}_{\#}/\mathbb{K}}(\varphi)$ on V, considered as an \mathbb{F}_q -space.

Set $E = \operatorname{span}_{\mathbb{K}_{\#}} \{e_1\}, F = \operatorname{span}_{\mathbb{K}_{\#}} \{f_1\}$ and $X = \operatorname{span}_{\mathbb{K}_{\#}} \{x\}$ and observe that $\langle E, F \rangle$ is a non-degenerate 2r-dimensional \mathbb{F}_q -subspace of V, while X is a non-degenerate r-dimensional \mathbb{F}_q -subspace of V. Choose a hyperbolic \mathbb{F}_q -basis for $\langle E, F \rangle, \mathcal{B}_1 = \{e_1, \ldots, e_r, f_1, \ldots, f_r\}$, where $e_1, \ldots, e_r \in E$ and $f_1, \ldots, f_r \in F$. Let \mathcal{B}_2 be a hyperbolic basis \mathbb{F}_q -basis for X and assume that \mathcal{B}_2 contains elements e_{r+1}, f_{r+1} such that (e_{r+1}, f_{r+1}) is a hyperbolic pair. Observe that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a hyperbolic \mathbb{F}_q -basis for V.

Define a group U in S whose elements fix all elements in \mathcal{B} except e_1, \ldots, e_r and f_{r+1} , and which satisfy

$$e_i \mapsto e_i + c_i e_{r+1}$$
 for $1 \le i \le r$,
 $f_{r+1} \mapsto f_{r+1} - c_1 f_1 - c_2 f_2 - \dots - c_r f_r$ $(c_1, \dots, c_r \in \mathbb{F}_q)$.

Now U is of order q^r and $\langle g \rangle$ normalizes and acts fixed-point-freely on U. What is more, U is not in M (since it contains non-trivial elements with a 1-eigenspace of dimension at least 2r + 1 over \mathbb{F}_q). Thus, in the usual way, we obtain Δ , a set of size q^r whose stabilizer is 2-transitive. By Lemma 4.1.1, M does not have a section isomorphic to $\operatorname{Alt}(q^r)$ (recall that $r \geq 3$ here), so Δ is a beautiful set and Lemma 1.6.12 yields the result.

Suppose now that m = 2 and q > 2. Here we proceed in a similar fashion to Lemma 4.3.3. In this case we start with $\mathcal{B}_1 = \{e, f\}$, a hyperbolic $\mathbb{K}_{\#}$ -basis for V with respect to a unitary form $\varphi_{\#}$. Let λ be an element of $\mathbb{K}_{\#}$ such that $\mathcal{B}_2 = \{\lambda, \lambda^q, \ldots, \lambda^{q^{r-1}}\}$ is a basis for $\mathbb{K}_{\#}$ over \mathbb{K} . Then taking tensor products of elements of \mathcal{B}_1 and \mathcal{B}_2 we obtain a \mathbb{K} -basis for V. We write V_1 for the \mathbb{K} -span of \mathcal{B}_1 and V_2 for the \mathbb{K} -span of \mathcal{B}_2 .

Now we take S to preserve the form $\varphi = \operatorname{Tr}_{\mathbb{K}_{\#}/\mathbb{K}}(\varphi_{\#})$, and M to preserve $\varphi_{\#}$, so that M is a subgroup of G. What is more we observe that

$$\begin{split} \varphi(e \otimes \lambda^{q^{i}}, e \otimes \lambda^{q^{j}}) &= 0, \\ \varphi(f \otimes \lambda^{q^{i}}, f \otimes \lambda^{q^{j}}) &= 0, \\ \varphi(e \otimes \lambda^{q^{i}}, f \otimes \lambda^{q^{j}}) &= \operatorname{Tr}_{\mathbb{K}_{\#}/\mathbb{K}}(\lambda^{q^{i}+q^{r+j}}) \end{split}$$

In particular this means that φ can be written as a product $\varphi = \varphi_1 \varphi_2$, where $\varphi(u_1 \otimes u_2, v_1 \otimes v_2) = \varphi_1(u_1, v_1)\varphi_2(u_2, v_2)$; here $u_1, v_1 \in V_1, u_2, v_2 \in V_2$, $\varphi_1 = \varphi|_{V_1}$ and φ_2 is defined by setting $\varphi_2(\lambda^{q^i}, \lambda^{q^j}) = \operatorname{Tr}_{\mathbb{K}_{\#}/\mathbb{K}}(\lambda^{q^i+q^{r+j}})$ and extending linearly on V_2 .

With this set-up we see that the group M contains a subfield subgroup isomorphic to the projective image of a group $H = \mathrm{SU}_2(q) \times r$ which preserves this tensor product structure (observe that the Frobenius automorphism on V_2 preserves the form φ_2). Then H lies in a group of the form $K = \mathrm{SU}_2(q) \times \mathrm{SU}_r(q)$, and it is clear that H is not self-normalizing in K.

Now we proceed as before: we obtain a suborbit of M whose action is isomorphic to the action of an almost simple group with socle $PSL_2(q^r)$ on a maximal \mathcal{C}_5 -subgroup containing $PSL_2(q)$; by [45], this action is not binary, and now Lemma 1.6.2 implies that the action of M on $(M : M \cap M^g)$ is not binary. Then Lemma 1.6.1 implies that the action of G on (G : M) is not binary, as required.

Suppose next that m = 2 and q = 2. Then $S = SU_{2r}(2)$. If r = 3, then the result follows from Lemma 4.1.1. Assume, then, that r > 3 and notice that S is simple. For convenience we shall work with the group $X = GU_{2r}(2) = Z \times S$, where the centre Z has order 3, and replace M by ZM; the centre Z will act trivially on all the sets we consider in the rest of the proof.

We have $N_X(M \cap X) = \operatorname{GU}_2(q_0).r$, where $q_0 = 2^r$, and this contains a maximal torus T of order $(q_0 + 1)^2$. Then

$$N_X(T) \ge T.(r \times r)$$
 and $T.(r \times r) \not\leq M$,

while $N_M(T) \ge T.r.$ Hence there exists $g \in N_X(T) \setminus M$ such that $T.r \le M \cap M^g$. Note also that $N := N_M(T) \cap X = T.2r.$ Since $N_X(N) \le M$, it follows that $N \le M \cap M^g$, and hence $M \cap M^g \cap X = T.r.$

Let H be the subgroup $\operatorname{GU}_2(q_0)$ of M, so that $H \cap M \cap M^g = T$. We consider the action of H on (H:T); the kernel of this action is the centre Z_0 of H, of order $q_0 + 1$. Let V be a 2-space over $\mathbb{F}_{q_0^2}$ with unitary form (,) preserved by H, and let v_1, v_2 be an orthonormal basis of V. Replacing T by a conjugate if necessary, we may take T to be the stabilizer in H of $\langle v_1 \rangle$ (hence also of $\langle v_2 \rangle$). So the action of H on (H:T) is equivalent to the action on the set $\Lambda = \{\langle v \rangle : (v, v) = 1\}$.

Write $\alpha \to \bar{\alpha}$ for the involutory automorphism of $\mathbb{F}_{q_0^2}$. The orbits of the point-stabilizer $H_{\langle v_1 \rangle}$ on Λ are the singletons $\langle v_1 \rangle$, $\langle v_2 \rangle$ and the sets Λ_{λ} , for $\lambda \bar{\lambda} \notin \{0, 1\}$, where $\lambda \in \mathbb{F}_{q_0^2}$ and

$$\Lambda_{\lambda} = \{ \langle w \rangle \in \Lambda : (w, w) = 1, (v_1, w) = \lambda \}.$$

Note that $\Delta_{\lambda} = \Delta_{\alpha\lambda}$ if and only if $\alpha \bar{\alpha} = 1$, so there are precisely $q_0 - 2$ suborbits Λ_{λ} , all of size $q_0 + 1$. In particular, $H_{ab} = Z_0$, for any distinct $a, b \in \Lambda$ such that $b \neq a^{\perp}$.

We claim that there exist scalars $\lambda_1, \lambda_2, \beta \in \mathbb{F}_{q_0^2}$ with the following properties:

- (i) $\lambda_1 \bar{\lambda}_1 \neq \lambda_2 \bar{\lambda}_2$ and $\lambda_i \bar{\lambda}_i \neq 0, 1$ for i = 1, 2,
- (ii) $\beta \bar{\beta} = 1 + \lambda_1 \bar{\lambda}_1$,
- (iii) $\lambda_1 \bar{\lambda}_2 \bar{\beta} = 1.$

To see this, first choose λ_1 and β such that $\lambda_1 \overline{\lambda}_1 \neq 0, 1$ and $\beta \overline{\beta} = 1 + \lambda_1 \overline{\lambda}_1$. Define $\lambda_2 = \beta^{-1} \overline{\lambda}_1^{-1}$. Then setting $y = \lambda_1 \overline{\lambda}_1$, we have

$$\lambda_2 \bar{\lambda}_2 = (\beta \bar{\beta})^{-1} (\lambda_1 \bar{\lambda}_1)^{-1} = \frac{y^{-1}}{1+y} = \frac{1}{y+y^2}$$

If this is equal to y, then $y^3 + y^2 + 1 = 0$; but there is no such $y \in \mathbb{F}_{q_0} = \mathbb{F}_{2^r}$, as $r \ge 5$ by assumption. Similarly, $\lambda_2 \overline{\lambda}_2$ is not equal to 1 or 0. Thus (i)-(iii) hold.

Now choose $\gamma \in \mathbb{F}_{q_0^2} \setminus \mathbb{F}_{q_0}$ such that $\gamma \bar{\gamma} = 1 + \lambda_2 \bar{\lambda}_2$, and define the following four points $a, b, c, d \in \Lambda$:

$$a = \langle v_1 \rangle, b = \langle \lambda_1 v_1 + \beta v_2 \rangle, c = \langle \lambda_2 v_1 + \gamma v_2 \rangle, d = \langle \lambda_2 v_1 + \bar{\gamma} v_2 \rangle.$$

We shall show that the triples (a, b, c) and (a, b, d) are 2-subtuble complete, but not 3-subtuple complete under the action of H.

Since $c, d \in \Lambda_{\lambda_2}$, we have $(a, c) \sim (a, d)$. Also $(b, c) \sim (b, d)$ if and only if

$$(\lambda_1 v_1 + \beta v_2, \lambda_2 v_1 + \gamma v_2) = \nu(\lambda_1 v_1 + \beta v_2, \lambda_2 v_1 + \bar{\gamma} v_2),$$

for some $\nu \in \mathbb{F}_{q_0^2}$ satisfying $\nu \bar{\nu} = 1$. This is equivalent to the equation

$$(\lambda_1\bar{\lambda}_2 + \beta\bar{\gamma})(\bar{\lambda}_1\lambda_2 + \bar{\beta}\gamma) = (\lambda_1\bar{\lambda}_2 + \beta\gamma)(\bar{\lambda}_1\lambda_2 + \bar{\beta}\bar{\gamma}),$$

which boils down to $\lambda_1 \bar{\lambda}_2 \bar{b}(\gamma + \bar{\gamma}) = \bar{\lambda}_1 \lambda_2 \beta(\gamma + \bar{\gamma})$. This holds, since $\lambda_1 \bar{\lambda}_2 \bar{\beta} = 1$ by (iii).

Hence (a, b, c) and (a, b, d) are 2-subtuple complete. They are clearly not 3-subtuple complete under the action of H, since $H_{ab} = Z_0$ which is the kernel of the action on Λ .

Thus the action of H on Λ is non-binary. The same is true when we add field automorphisms to get the group M = H.r or H.(2r) acting on Λ : for any non-trivial field automorphism does not fix any of the suborbits Λ_{λ} , and hence $M_{ab} = H_{ab} = Z_0$, so (a, b, c) and (a, b, d) are not in the same orbit under the action of M.

We have now established that the action of M on $(M : M \cap M^g)$ is not binary. Hence the result follows by Lemma 1.6.1.

Suppose finally that m = 1. Here we use the method of Lemma 4.3.4: first we write the group $F = M \cap \text{PSU}_n(q)$ as a semidirect product $T \rtimes C$, where T is cyclic of order $\frac{q^{n+1}}{(q+1)(q+1,n)}$ and C is cyclic of order n, and acts fixed-point-freely on T. Proposition 2.4.1 implies that $C_{\text{PSU}_n(q)}(c) > \langle c \rangle$ unless (n,q) = (5,2), but this case can be excluded since M is not maximal, see [10, Table 8.20]. Let $x \in C_{\text{PSU}_n(q)}(c) \setminus \langle c \rangle$ and observe that the group F acts as a Frobenius group on the set $(F : F \cap F^x)$. Since n > 2, Lemma 1.7.2 implies that the action of M on $(M : M \cap M^x)$ is not binary. Now Lemma 1.6.1 yields the result.

4.3.3 Case $S = Sp_n(q)$

Lemma 4.3.6. Suppose that G is almost simple with socle equal to $PSp_n(q)$ with $n \ge 4$ and $(n,q) \ne (4,2)$, and let M be a C_3 -subgroup. Then the action of G on (G:M) is not binary.

Proof. There are two cases to consider here, namely $M \triangleright PSp_m(q^r)$ with mr = n, and $M \triangleright SU_{n/2}(q)/Z$ (q odd).

Consider the first case, where M is almost simple with socle $PSp_m(q^r)$ and m is even, r is prime and n = mr. If $m \ge 4$, then we let x be the element in Lemma 2.2.8 (applied to the group M rather than G). Since x has a 1-eigenspace of dimension 2 over \mathbb{F}_{q^r} , it is easy to see that $C_G(x) \setminus C_M(x)$ is non-empty, and so we take an element g in $C_G(x) \setminus C_M(x)$ and appeal to Lemma 2.2.8 to see that the action of M on $(M: M \cap M^g)$ is not binary. Then Lemma 1.6.1 yields the result.

Now suppose that m = 2 and r > 2 (we will deal with the case where m = r = 2 in the last paragraph of the proof). Here $M \cap S/Z(S)$ is the projective image of $M_0 \cong \operatorname{Sp}_2(q^r).r$, and we let $H_0 \cong \operatorname{SL}_2(q) \times \langle \sigma \rangle$ be a subfield subgroup of M_0 , where σ is a field automorphism of order r; this is maximal in M_0 for q > 2, see for instance [10, Table 8.1]. We claim that there exists $g \in S \setminus M_0$ normalizing H_0 . Once we have shown this, then in a similar manner to the previous paragraph, we obtain a suborbit for which the action is isomorphic to the action of an almost simple group with socle $\operatorname{PSL}_2(q^r)$ on a maximal subfield subgroup; then [45] yields the result for q > 2.

To see the existence of the element g we note that the subfield subgroup H_0 preserves a tensor product structure on V and so lies in a maximal subgroup $\operatorname{Sp}_2(q) \circ I_r(q)$, where $I_r(q)$ is the isometry group of a symmetric bilinear form having matrix I_r , the identity. We can choose g in $I_r(q)$ of order r-1 normalizing the subgroup $\langle \sigma \rangle$ of order r in $I_r(q)$. The claim follows.

We also need to deal with the case where m = 2, r > 2 and q = 2. In this case an element g as above exists, but this time the group $M \cap M^g$ is not necessarily maximal in M. However, in this case $M \cap M^g$ contains $\text{Sp}_2(2)$ and has order not divisible by 4, so the conditions for applying Lemma 1.6.15 to the action $(M, (M : M \cap M^g))$ are met with the prime 2, and the result follows.

Now consider the second case, where $M \triangleright SU_{n/2}(q)/Z$ with q odd. For $n \ge 6$, we proceed as in Lemma 4.3.5, taking x to be the element given in Lemma 2.2.8 (for $n/2 \ge 5$), in Lemma 2.2.11 (for n/2 = 4) and in Lemma 2.2.10 (for n/2 = 3). Lemma 1.6.1 implies that, choosing (as we may) $g \in C_G(x) \setminus M$, the action of M on $(M : M \cap M^g)$ is not binary and thus the listed lemmas imply that M must contain a section isomorphic to Sym(s) where

$$s = \begin{cases} q, \text{ if } n/2 = 3 \text{ or } 4; \\ q^{2\lfloor (n-6)/4 \rfloor}, \text{ if } n/2 \ge 5. \end{cases}$$

By Lemma 2.1.1, this implies that one of the following holds:

- (i) q = 7, n/2 = 3 or 4,
- (ii) $q \le 5, n/2 = 3$ or 4.

Using [10, Table 8.5], we see that Alt(7) is not a section of $PSU_3(7)$. Hence the socle of G is $PSp_6(q)$ $(q \le 5)$ or $PSp_8(q)$ $(q \le 7)$. All of these groups apart from $PSp_8(q)$ $(5 \le q \le 7)$ are covered by Lemma 4.1.1; and the C_3 -actions of the remaining groups were shown to be not binary by a computation using the permutation character method of Lemma 1.8.1.

We are left with the situation when $S = \text{Sp}_4(q)$ and r = 2. Here we need to consider both the case where M is almost simple with socle $\text{PSp}_2(q^2)$ (the situation we have postponed above) and the case where q is odd and M contains a subgroup isomorphic to $\text{GU}_2(q).2$. Lemma 4.1.1 and [46] imply that we can assume that $q \ge 7$.

For the case where M has socle $PSp_2(q^2)$, we choose a hyperbolic basis, $\mathcal{B}_0 = \{e_1, f_1\}$, for a 2dimensional space V over $\mathbb{K}_{\#} = \mathbb{F}_{q^2}$ associated with an alternating form φ . We will use the fact that the isometry group of φ contains an element

$$g := \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix},$$

where a is an element of \mathbb{F}_{q^2} of order (q, 2)(q - 1). Now we can take M to contain a projective image of the special isometry group of φ , and let S be the special isometry group of the form $\operatorname{Tr}_{\mathbb{K}_{\#}/\mathbb{K}}(\varphi)$ on V, considered as an \mathbb{F}_q -space.

Set $E = \operatorname{span}_{\mathbb{K}_{\#}} \{e_1\}, F = \operatorname{span}_{\mathbb{K}_{\#}} \{f_1\}$. Choose a hyperbolic \mathbb{F}_q -basis for $V, \mathcal{B} = \{e_1, e_2, f_2, f_1\}$, where $e_1, e_2 \in E$ and $f_1, f_2 \in F$.

Define a group U in S whose elements can be written with respect to \mathcal{B} as

$$\begin{pmatrix}1&&\\&1&b\\&&1\\&&&1\end{pmatrix}$$

for some $b \in \mathbb{F}_q$. Now U is of size q and $\langle g \rangle$ normalizes, and acts transitively on the set of non-trivial elements of, U. What is more, U is not in M (since it contains non-trivial elements with a 1-eigenspace of dimension 3 over \mathbb{F}_q). We therefore obtain Δ , a set of size q whose stabilizer is 2-transitive. Note that M does not contain a section isomorphic to $\operatorname{Alt}(q-1)$ for $q \geq 7$, thus Δ is a beautiful set and Lemma 1.6.12 yields the result.

The other case is handled very similarly: here q is odd and M contains $\operatorname{GU}_2(q)$. As before we start with a hyperbolic basis $\mathcal{B}_0 = \{e_1, f_1\}$, but this time for a 2-dimensional space V over $\mathbb{K}_{\#} = \mathbb{F}_{q^2}$ associated with a unitary σ -form φ . We use the same element g, and we let S be the special isometry group of the form $\operatorname{Tr}_{\mathbb{K}_{\#}/\mathbb{K}}(\zeta\varphi)$ on V, considered as an \mathbb{F}_q -space; here ζ is an element of $\mathbb{K}_{\#}$ that satisfies $\zeta^{\sigma} = -\zeta$. Now the proof proceeds as before.

4.3.4 Case $S = \Omega_n^{\varepsilon}(q)$

Lemma 4.3.7. Suppose that G is almost simple with socle equal to $P\Omega_n^{\varepsilon}(q)$ $(n \ge 7)$, and let M be a maximal C_3 -subgroup. Then the action of G on (G:M) is not binary.

Proof. First assume that n is odd, so q is odd. In this case M is almost simple with socle $\Omega_m(q^r)$ where n = mr. For $m \ge 7$, let $x \in M$ be as in Lemma 2.2.9; then there exists $g \in C_G(x) \setminus C_M(x)$, and the lemma shows that the action of M on $(M : M \cap M^g)$ is not binary, giving the conclusion. If m = 5 we use the isomorphism $\Omega_5(q^r) \cong PSp_4(q^r)$: the element $x \in PSp_4(q^r)$ defined in Lemma 2.2.8 acts as diag $(1, a, a, a^{-1}, a^{-1})$ in $\Omega_5(q^r)$, so again there exists $g \in C_G(x) \setminus C_M(x)$, and the action $(M, (M : M \cap M^g))$ is not binary by the lemma. Finally, if m = 3, the element $x \in PSL_2(q^r)$ defined in Lemma 2.2.3 acts as diag $(1, a^2, a^{-2}) \in \Omega_3(q^r)$, and so again there exists g for which $(M, (M : M \cap M^g))$ is not binary.

Next assume that n is even and $S = \Omega_n^{\varepsilon}(q)$, where $\varepsilon \in \{+, -\}$. We refer to [54, Tables 3.5.E and 3.5.F] and split into three cases:

- (1) $m = n/r \ge 4$ is even and M is of type $O_m^{\varepsilon}(q^r)$;
- (2) qm = qn/2 is odd, r = 2 and M is of type $O_{n/2}(q^2)$;
- (3) $m = n/2 \ge 4$, r = 2 and M is of type $SU_m(q)$.

Case (1) Assume that $m = n/r \ge 4$ is even and M is of type $O_m^{\varepsilon}(q^r)$. Observe that provided $(m, \varepsilon) \ne (4, +), M$ is almost simple with socle $P\Omega_m^{\varepsilon}(q^r)$.

Proceeding as before, using Lemma 2.2.9, the conclusion follows directly for $m \geq 8$, except for M of type $\Omega_8^-(9)$ in $S = \Omega_{16}^-(3)$. Therefore, we look closer at this embedding. Choose Sylow 13-subgroups Q of M and P of S such that Q < P. Then |Q| = 13, $P \cong 13^2$. Observe that a Sylow 13-subgroup of the C_1 -subgroup $\Omega_8^-(3) \times \Omega_8^+(3)$ has order 13^2 ; therefore, we may assume that $P \leq \Omega_8^-(3) \times \Omega_8^+(3)$ and $P = P_- \times P_+$, where P_- is a Sylow 13-subgroup of $\Omega_8^-(3)$ and P_+ is a Sylow 13-subgroup of $\Omega_8^+(3)$. Thus $13 = |P_-| = |P_+|$. Write $P_- = \langle g_- \rangle$ and $P_+ = \langle g_+ \rangle$. Now, as $Q \leq P$, we may write $Q = \langle g_-^i g_+^j \rangle$, for some positive integers i and j. Replacing the original generators of P_- and P_+ if necessary, we may suppose that i = j = 1: observe that neither i nor j cannot be 0, because a Sylow 13-subgroup of $\Omega_8^-(9)$ cannot fix an 8-dimensional subspace of the underlying vector space for $\Omega_{16}^-(3)$. It can be easily verified (for instance with [28] or with magma) that g_+^4 and g_+ are in the same $\Omega_8^+(3)$ -conjgacy class. Therefore, $h := g_-g_+^4$ and $\langle gh \rangle = \langle (gh)^2 \rangle = \langle g_-^4 g_+^{10} \rangle$. Again, it is easy to verify that g_- and g_-^4 are in the same $\Omega_8^-(3)$ -conjugacy class and g_+ and $\langle gh \rangle$ are conjugate in $\Omega_8^-(3)$ -conjugacy class. Therefore, $\langle h \rangle$ and $\langle gh \rangle$ are conjugate in $\Omega_8^-(3) \times \Omega_8^+(3)$ -conjugacy class. Therefore, $h := g_-g_+^5$ and $\langle gh_-^2 = \langle g_-^4 g_+^{10} \rangle$. Again, it is easy to verify that g_- and g_-^4 are in the same $\Omega_8^-(3)$ -conjugacy class. Therefore, $\langle h \rangle$ and $\langle gh \rangle$ are conjugate in $\Omega_8^-(3) \times \Omega_8^+(3)$ -conjugacy class. Therefore, $\langle h \rangle$ and $\langle gh \rangle$ are conjugate in $\Omega_8^-(3) \times \Omega_8^+(3)$ -conjugacy class. Therefore, $\langle h \rangle$ and $\langle gh \rangle$ are conjugate in $\Omega_8^-(3) \times \Omega_8^+(3)$ -conjugacy class. Therefore, $\langle h \rangle$ and $\langle gh \rangle$ are conjugate in $\Omega_8^-(3) \times \Omega_8^+(3)$ -conjugacy class. Therefore, $\langle h \rangle$ and $\langle gh \rangle$ are conjugate. Hence (G, Ω) is not binary in this

Now consider m = 6. Here M has socle $P\Omega_6^{\varepsilon}(q^r) \cong PSL_4^{\varepsilon}(q^r)$, and we use the usual argument using the elements given in Lemma 2.2.7 (for $\varepsilon = +$) and Lemma 2.2.11 for $\varepsilon = -$. These give the conclusion unless $\varepsilon = +$ and $q^r = 4$. In the latter case, $S = \Omega_{12}^+(2)$, which is covered by Lemma 4.1.1.

If m = 4 and M has socle $P\Omega_4^-(q^r) \cong PSL_2(q^{2r})$, then we use Lemma 2.2.4. The element x given in that lemma acts in $\Omega_4^-(q^r)$ as diag $(b, b^{-1}, -1, -1)$ for some b, so has a larger centralizer in G than in M, and so the result follows as usual, unless $q^r = 4$, in which case $S = \Omega_8^-(2)$, which is covered by Lemma 4.1.1.

Suppose now that m = 4 and M has socle $P\Omega_4^+(q^r) \cong PSL_2(q^r) \times PSL_2(q^r)$. Thus $S \cong \Omega_{4r}^+(q)$. Note that, if r = 2, then [10] implies that M is conjugate by a triality automorphism to a maximal subgroup of the C_2 -class, hence we know already that the action in this case is not binary. Assume, from here on, that $r \ge 3$. First assume that q is odd. Here $M \cap S = P\Omega_4^+(q^r).r \cong (PSL_2(q^r) \times PSL_2(q^r)).r$ (see [54, 4.3.14]). Write $M_0 = M \cap S$, and let H be a maximal subgroup $\Omega_3(q^r).r$ of M_0 . Then $H < \Omega_{3r}(q) < S$, and $C_S(H)$ contains a subgroup $\Omega_r(q)$. Picking $g \in C_S(H) \setminus M$, we have $M_0^g \cap M_0 = H$, a maximal subgroup of M_0 . The action of M_0 on $(M_0 : H)$ is a primitive permutation action of diagonal type, and is not binary by [106, Proposition 4.1]. Hence the action of S on $(S: M_0)$ is also not binary. To prove the same assertion for (G: M), let $G_1 = \langle M^g \cap M, S \rangle$, and $M_1 = M \cap G_1$. Then $M^g \cap M$ is maximal in M_1 , and the action of M_1 on $(M_1: M^g \cap M)$ is not binary, again by [106, Proposition 4.1]. Hence $(M, (M: M^g \cap M))$ is also not binary, by Lemma 1.6.2, and so (G, (G: M)) is not binary by Lemma 1.6.1.

Next consider the case where q is even. Again let $H = \Omega_3(q^r) \cong \text{PSL}_2(q^r)$ be a diagonal subgroup of $M_0 = \text{soc}(M) \cong \text{PSL}_2(q^r) \times \text{PSL}_2(q^r)$, and let T be a cyclic torus of order $q^r - 1$ in H. Then T lies in a subgroup $\Omega_2^+(q^r)$ of H, so is centralized by a subgroup $\Omega_{2r}^+(q)$ of S. Pick $g \in C_S(T) \setminus M$, so that $T < M^g \neq M$. As T centralizes a 2r-subspace of $V = V_{4r}(q)$, it must act on the natural module $V_4(q^r)$ for M^g with eigenvalues $(\lambda, \lambda^{-1}, 1, 1)$ for some $\lambda \in GF(q^r)$, and so T lies in $q^r - 1$ nonsingular point-stabilizers $\Omega_3(q^r)$ in M^g . These generate M^g , so not all of them can lie in M. Hence there exists a subgroup $H_1 = \Omega_3(q^r)$ of M^g such that $T < H_1 \not\leq M$. Hence there is a Frobenius group $UT < H_1$ of order $q^r(q^r - 1)$ with $U \not\leq M$, and so in the usual way we obtain a subset Δ of $\Omega = (G : M)$ such that G^{Δ} contains the 2-transitive group $\text{AGL}_1(q^r)$. Then Δ is a beautiful subset, unless possibly $\text{Alt}(q^r - 1)$ is a section of M. Lemma 2.1.1 implies that $q^r - 1 \leq 6$ and, since r > 2, we obtain a contradiction as required.

Case (2) Next assume that r = 2 and qm = qn/2 is odd, and that M has socle $\Omega_{n/2}(q^2)$. If $m \ge 7$, then we proceed as before using Lemma 2.2.9: the result follows except for the embedding $\Omega_7(9)$ in $P\Omega_{14}^{\varepsilon}(3)$. For this embedding we observe that |G:M| is even and M is almost simple. Let P be a Sylow 2-subgroup of M, let Q be a Sylow 2-subgroup of G that contains P, and let x be an element in $G \setminus M$ that normalizes P. Then $|M:M \cap M^x|$ is odd and $M \cap M^x$ is core-free. Now Lemma 2.3.1 implies that the action of Mon $(M:M \cap M^x)$ is not binary, and Lemma 1.6.1 yields the result. If m = 5, then we use the fact that $\Omega_5(q^2) \cong \operatorname{Sp}_4(q^2)$ and the result follows using the same method replacing Lemma 2.2.9 with Lemma 2.2.8.

Case (3) Finally assume that r = 2 and $m = n/2 \ge 4$, and that M contains a normal subgroup that is a quotient of $SU_m(q)$. Note that when n = 8, [54] implies that $\varepsilon = +$ while [10] implies that these C_3 -subgroups of $P\Omega_8^+(q)$ are conjugate, via a triality automorphism, to C_1 -subgroups, hence are already dealt with in [46]. Assume, then, that $n \ge 10$.

We proceed as before: let $x \in M$ be the element given in Lemma 2.2.8. As this has a non-trivial 1-eigenspace, there exists $g \in C_G(x) \setminus C_M(x)$, and so in the action of M on $(M : M \cap M^g)$, the stabilizer is a subgroup of M containing x but not containing a homomorphic image of $SU_m(q)$. If the action of G on (G : M) is binary, then Lemma 1.6.1 implies that the action of M on $(M : M \cap M^g)$ is binary and Lemma 2.2.8 implies that M contains a section isomorphic to Sym(s) where $s = q^{2(\lfloor (m-3)/2 \rfloor)}$. Then Lemma 2.1.1 implies that (m,q) = (5,2) or (6,2). In the former case $S = \Omega_{10}^{-}(2)$, while in the latter case $S = \Omega_{12}^{+}(2)$; both are covered by Lemma 4.1.1. Thus in all cases the action of M on $(M : M \cap M^g)$ is non-binary, and hence the same is true of the action of G on (G : M), completing the proof.

4.4 Family C_4

In this section, the subgroup M preserves a tensor product. We start with two K-vector spaces, W_1 and W_2 , of dimension n_1 and n_2 , respectively, and satisfying $n = n_1 n_2$. Roughly speaking, we identify V with the tensor product $W_1 \otimes W_2$, and M is the subgroup of G that preserves this identification. The list of subgroups is given in Table 4.4.1; the details of their precise structure and embeddings can be found in [54, §4.4].

As in the C_3 -case we give a geometrical interpretation to the set of cosets of M (which is the set on which we are acting); this will involve defining a *tensor product structure on* V. Let us start with the case where $S = SL_n(q)$.

We begin with a K-linear isomorphism $\phi: W_1 \otimes W_2 \to V$. Let Σ be the set of all such isomorphisms, and we observe that two groups act naturally on Σ :

- 1. $\operatorname{GL}(W_1) \circ \operatorname{GL}(W_2)$ acts on Σ via $\phi^g(\mathbf{w_1} \otimes \mathbf{w_2}) = \phi(\mathbf{w_1}^{g^{-1}} \otimes \mathbf{w_2}^{g^{-1}})$ (and extended linearly);
- 2. $\operatorname{GL}_n(\mathbb{K})$ acts on Σ via $\phi^h(\mathbf{w_1} \otimes \mathbf{w_2}) = (\phi(\mathbf{w_1} \otimes \mathbf{w_2}))^h$ (and extended linearly).

case	type	conditions
L^{ϵ}	$\operatorname{GL}_{n_1}^{\epsilon}(q) \otimes \operatorname{GL}_{n_2}^{\epsilon}(q)$	$n_1 < n_2$
\mathbf{S}	$\operatorname{Sp}_{n_1}(q)\otimes \operatorname{O}_{n_2}^\epsilon(q)$	$n_2 \geq 3, q \text{ odd}$
O^+	$\operatorname{Sp}_{n_1}(q)\otimes \operatorname{Sp}_{n_2}(q)$	$n_1 < n_2$
0	$\mathrm{O}_{n_1}^{\epsilon_1}(q)\otimes\mathrm{O}_{n_2}^{\epsilon_2}(q)$	$n_i \geq 3, q \text{ odd}$

Table 4.4.1: Maximal subgroups in family C_4

As in the C_3 -case, these two actions commute. Thus we define a *tensor product structure on* V to be an orbit of the group $\operatorname{GL}(W_1) \circ \operatorname{GL}(W_2)$ on Σ , and (using commutativity of the actions) we observe that $\operatorname{GL}_n(\mathbb{K})$ acts on the set of all tensor product structures on V. What is more the stabilizer of this action is the subgroup M, hence we have the geometrical interpretation that we require.

Again, just as before, we can replace the word "linear" with the word "semilinear" in the previous paragraph to extend this geometrical interpretation to subgroups of $\Gamma L_n(\mathbb{K})$.

For the remaining classical groups, we need to clarify what is meant by a tensor product structure on a vector space equipped with a form. So let us assume that our two vector spaces, W_1 and W_2 are equipped with forms \langle , \rangle_1 and \langle , \rangle_2 , respectively. Now we define

$$\langle \cdot, \cdot \rangle : (W_1 \otimes W_2) \times (W_1 \otimes W_2) \to \mathbb{K}, \\ \left(\sum_i v_1^i \otimes v_2^i, \sum_j w_1^j \otimes w_2^j \right) \mapsto \sum_{i,j} \langle v_1^i, w_1^j \rangle_1 \langle v_2^i, w_2^j \rangle_2,$$

where $v_1^i, w_1^j \in W_1$ and $v_2^i, w_2^j \in W_2$ for all *i* and *j*. One can check that \langle , \rangle is a well-defined form on $W_1 \otimes W_2$. Now our map $\phi : W_1 \otimes W_2 \to V$ carries this form onto the vector space *V*, and we obtain a map to a formed space. Following the same approach as above, we see that there are actions of $\operatorname{Isom}(\langle , \rangle_1) \circ \operatorname{Isom}(\langle , \rangle_2)$ and of $\operatorname{Isom}(\langle , \rangle)$ acting on the set of all such maps; this yields a definition of a tensor product structure on a formed space, and provides an embedding of $\operatorname{Isom}(\langle , \rangle_1) \circ \operatorname{Isom}(\langle , \rangle_2)$ in the group $\operatorname{Isom}(\langle , \rangle)$, as the stabilizer of such a tensor product structure. Moreover, in the case where the characteristic p = 2 and both \langle , \rangle_1 and \langle , \rangle_2 are symplectic, the group $\operatorname{Isom}(\langle , \rangle_1) \circ \operatorname{Isom}(\langle , \rangle_2)$ also preserves a quadratic form on $W_1 \otimes W_2$ with associated bilinear form \langle , \rangle , yielding an embedding into $O^+(V)$. Thus we obtain all the embeddings listed in Table 4.4.1.

In the formed space case, it is useful to observe that if we start with hyperbolic bases e_1, \ldots, f_1, \ldots for W_1 and u_1, \ldots, v_1, \ldots for W_2 , then, by taking pure tensors, we obtain a hyperbolic basis for $W_1 \otimes W_2$; the hyperbolic pairs are

$$(e_i \otimes u_j, f_i \otimes v_j)$$
 and $(e_i \otimes v_j, f_i \otimes u_j)$.

Similarly, if W_1 contains a vector x such that $\langle x, x \rangle_1 = 1$, then $(x \otimes u_i, x \otimes v_i)$ is a hyperbolic pair in $W_1 \otimes W_2$; and also if $W_{2,0}$ is a non-degenerate subspace of W_2 , then $x \otimes W_{2,0}$ is a non-degenerate subspace of the tensor product.

The main result of this section is the following. The result will be proved in a series of lemmas.

Proposition 4.4.1. Suppose that G is an almost simple group with socle $\overline{S} = Cl_n(q)$, and assume that

- (i) $n \geq 3, 4, 4, 7$ in cases L, U, S, O respectively, and
- (ii) $\operatorname{Cl}_n(q)$ is not one of the groups listed in Lemma 4.1.1.

Let M be a maximal subgroup of G in the family \mathcal{C}_4 . Then the action of G on (G:M) is not binary.

Group	Details of action
(-)	$q \in \{3, 4, 5\}, n_1 = 2, n_2 = 3: M \triangleright \text{PSL}_2(q) \times \text{PSL}_3(q).$ $n_1 = 3, n_2 = 4: M \triangleright \text{PSL}_3(2) \times \text{PSL}_4(2).$

Table 4.4.2: $C_4 - SL_n(q)$ – Cases where a beautiful subset was not found.

4.4.1 Case $S = SL_n(q)$

Lemma 4.4.2. In this case either Ω contains a beautiful subset or else S is listed in Table 4.4.2.

Proof. In this case M contains a normal subgroup isomorphic to $SL_{n_1}(q) \circ SL_{n_2}(q)$ where $n = n_1 n_2$ and we may assume that $2 \le n_1 < n_2$.

Let $\mathcal{B}_1 = \{u_1, \ldots, u_{n_1}\}$ be a basis for W_1 , $\mathcal{B}_2 = \{w_1, \ldots, w_{n_2}\}$ a basis for W_2 . Now $\mathcal{B} = \{u_i \otimes w_j : all i, j\}$ is a basis for $W_1 \otimes W_2$, which is mapped to V via a map ϕ contained in a tensor product structure \mathcal{P} , which is stabilized by M.

Assume that $q \ge 7$. Let T_1 (resp. T_2) be the maximal split torus in $GL(W_1)$ (resp. $GL(W_2)$) that is diagonal with respect to \mathcal{B}_1 (resp. \mathcal{B}_2). Let T be the intersection of the tensor product of T_1 and T_2 with the group S.

Now we let U be the subgroup whose elements fix all elements of \mathcal{B} except $u_1 \otimes w_1$, where we require

$$u_1 \otimes w_1 \mapsto u_1 \otimes w_1 + \alpha u_1 \otimes w_2,$$

for some $\alpha \in \mathbb{F}_q$. Note that $U \not\leq M$ (consider, for instance, the 1-eigenspace of non-trivial elements of U), and that n_2 is necessarily greater than or equal to 3. Hence the group $H = U \rtimes T$ acts 2-transitively on the set $\Lambda = \mathcal{P}^U := \{\mathcal{P}u : u \in U\}$, and $|\Lambda| = q$.

Let $G_1 \cong \operatorname{GL}_{n_1-1}(q)$ be the subgroup of $\operatorname{GL}(W_1)$ fixing u_1 and $\langle u_2, \ldots, u_{n_1} \rangle$; and let $G_2 \cong \operatorname{GL}_{n_2-2}(q)$ be the subgroup of $\operatorname{GL}(W_2)$ fixing w_1, w_2 and $\langle w_3, \ldots, w_{n_2} \rangle$. Then $M_{(\Lambda)}$, the pointwise stabilizer of Λ in M, contains $(G_1 \otimes G_2) \cap S$ (since this subgroup commutes with U). It follows that any (non-abelian) simple section of $M^{\Lambda} = M_{\Lambda}/M_{(\Lambda)}$ is isomorphic to a section of $\operatorname{GL}_2(q)$. By Lemma 2.1.1, for $q \geq 7$ this precludes the possibility that $M^{\Lambda} \geq \operatorname{Alt}(q-1)$, and we obtain that Λ is a beautiful subset; now Lemma 1.6.12 allows us to conclude that there are no such binary actions.

For $q \in \{3, 4, 5\}$ we exclude the case $(n_1, n_2) = (2, 3)$ (since this appears on the table), and so we assume that $n_2 \ge 4$. Now we proceed as before, this time taking U to be the subgroup whose elements fix all elements of \mathcal{B} except $u_1 \otimes w_1$ and

$$u_1 \otimes w_1 \mapsto u_1 \otimes w_1 + \alpha u_1 \otimes w_2 + \beta u_1 \otimes w_3,$$

for some $\alpha, \beta \in \mathbb{F}_q$. Now we take T_2 to be a maximal torus of $GL(W_2)$ that preserves the decomposition

$$\langle w_1 \rangle \oplus \langle w_2, w_3 \rangle \oplus \langle w_4 \rangle \oplus \cdots$$

and induces a Singer cycle on the subspace $\langle w_2, w_3 \rangle$. Defining T as before, we obtain a beautiful set of size q^2 unless Alt $(q^2 - 1)$ is isomorphic to a section of GL₃(q). By Lemma 2.1.1, Alt $(q^2 - 1)$ is not isomorphic to a section of GL₃(q) and hence the result follows.

For q = 2, [54, Table 3.5.A] allows us to assume that $n_1 > 2$. Now we exclude the case $(n_1, n_2) = (3, 4)$ (this is in Table 4.4.2), and we conclude that $n_2 \ge 5$. The argument proceeds as before, taking U to be the subgroup whose elements fix all elements of \mathcal{B} except $u_1 \otimes w_1$ and

$$u_1 \otimes w_1 \mapsto u_1 \otimes w_1 + \alpha u_1 \otimes w_2 + \beta u_1 \otimes w_3 + \gamma u_1 \otimes w_4 + \delta u_2 \otimes w_5$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$. We obtain a beautiful set of order 16 unless Alt(15) is a isomorphic to a section of GL₄(2). Again, by Lemma 2.1.1, Alt(15) is not isomorphic to a section of GL₄(2) and hence the result follows.

Group	Details of action
$\begin{array}{c} \mathrm{SU}_6(q),\mathrm{SU}_8(q)\\ \mathrm{SU}_{12}(q),\\ \mathrm{SU}_n(2) \end{array}$	$q \in \{3, 4, 5\}, n_1 = 2: M \triangleright \mathrm{PSU}_2(q) \times \mathrm{PSU}_{n_2}(q).$ $q \in \{3, 4, 5\}, n_1 = 3: M \triangleright \mathrm{PSU}_3(q) \times \mathrm{PSU}_4(q).$ $3 \le n_1 < n_2 \le 5: M \triangleright \mathrm{PSU}_{n_1}(2) \times \mathrm{PSU}_{n_2}(2).$

Table 4.4.3: $C_4 - SU_n(q)$ – Cases where a beautiful subset was not found.

4.4.2 Case $S = SU_n(q)$

Lemma 4.4.3. In this case either Ω contains a beautiful subset or else S is listed in Table 4.4.3.

Proof. In this case M contains a normal subgroup isomorphic to $SU_{n_1}(q) \circ SU_{n_2}(q)$ where $n = n_1 n_2$ and we may assume that $2 \le n_1 < n_2$.

Assume first that $q \ge 7$. Our method here exploits the existence of a Frobenius group inside $SU_3(q)$, as follows: let $W_{2,0} = \langle u_1, x, v_1 \rangle$ be a non-degenerate 3-dimensional subspace of W_2 and observe that we have two subgroups of $SU_3(q)$ consisting of elements of the form

$$U = \left\{ \begin{pmatrix} 1 & b & c \\ & 1 & -b^q \\ & & 1 \end{pmatrix} \mid b, c \in \mathbb{K} \text{ with } b^{q+1} + c + c^q = 0 \right\};$$
(4.4.1)

$$T = \left\{ \begin{pmatrix} r & \\ & r^{q-1} & \\ & & r^{-q} \end{pmatrix} \mid r \in \mathbb{K}^{\times} \right\}.$$
(4.4.2)

Then $U \rtimes T$ is a Borel subgroup of $SU_3(q)$.

Now, first, assume that q is odd. Take U_0 to be the subgroup of U obtained by requiring that $b \in \mathbb{F}_q$ and that $c = -\frac{1}{2}b^2$; take $y \in W_1$ such that $\langle y, y \rangle_1 = 1$. We now define an isomorphic group in S: let U_1 consist of those elements for which there exists $b \in \mathbb{F}_q$ such that

$$y \otimes u_1 \mapsto y \otimes u_1 + by \otimes x - \frac{1}{2}b^2y \otimes v_1,$$

$$y \otimes x \mapsto y \otimes x - by \otimes v_1,$$

$$y \otimes v_1 \mapsto y \otimes v_1,$$

and all elements of $\langle y \otimes u_1, y \otimes x, y \otimes v_1 \rangle^{\perp}$ are fixed. Then U_1 is a subgroup of order q that is not contained in M.

Similarly, think of T as a subgroup of $SU(W_2)$ by requiring that it fixes all elements in $W_{2,0}^{\perp}$, and take T_0 to be the subgroup of T obtained by requiring that $r \in \mathbb{F}_q$; let T_1 be the subgroup in S obtained by tensoring elements of T_0 with $1 \in SU(W_1)$. Then T_1 is a group of order q-1 that normalizes U_1 and acts transitively on the set of non-trivial elements in U_1 .

In the case when q is even, we do similarly; this time U_0 is the subgroup of U obtained by setting b = 0and letting c range through \mathbb{F}_q , while $T_0 = T$. Again, T_0 acts transitively upon the non-trivial elements of U_0 ; the same is therefore true of U_1 .

In both cases in the usual way, we set Λ to be \mathcal{P}^{U_1} , where \mathcal{P} is the tensor product structure stabilized by M, and we see that S^{Λ} acts 2-transitively upon Λ , with Λ a set of size q.

We wish to show that this set is beautiful. As before, we see that $M_{(\Lambda)}$ contains $S \cap (\mathrm{GU}_{n_1-1}(q) \circ \mathrm{GU}_{n_2-3}(q))$, where the first factor fixes y and the second fixes u_1, x, v_1 . Hence we see that any non-abelian simple section of M^{Λ} is isomorphic to a section of $\mathrm{GU}_3(q)$. Since $q \geq 7$, by Lemma 2.1.1 this precludes the possibility that $M^{\Lambda} \geq \mathrm{Alt}(q-1)$, and hence Λ is a beautiful subset; now Lemma 1.6.12 allows us to conclude that there are no such binary actions.

For $q \in \{3, 4, 5\}$ we assume that $n_2 \ge 5$ (the first two lines of Table 4.4.3 cover $n_2 \le 4$). We proceed as for $q \ge 7$ but we use the existence of a Frobenius group in $SU_5(q)$ this time. We let $W_{2,0} := \langle u_1, u_2, x, v_2, v_1 \rangle$ be a non-degenerate 5-subspace of W_2 , and consider the group:

$$U \rtimes T = \left\langle \begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & \\ & & & 1 & -a^q \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} r & & & \\ & 1 & & \\ & & r^{q-1} & & \\ & & & 1 & \\ & & & & r^{-q} \end{pmatrix} \mid a, r \in \mathbb{K}, r \neq 0 \right\rangle$$

Now we define U_1 , the subgroup for which there exists $a \in \mathbb{K}$ such that

$$y \otimes u_1 \mapsto y \otimes u_1 + ay \otimes u_2,$$

$$y \otimes v_2 \mapsto y \otimes v_2 - a^q y \otimes v_1,$$

and which fixes $y \otimes u_1$, $y \otimes v_1$, and the orthogonal complement of $\langle y \otimes u_1, y \otimes u_2, y \otimes v_1, y \otimes v_2 \rangle$. Note that this group is not contained in M, so we define $\Lambda = \mathcal{P}^{U_1}$ as before, this time a set of size q^2 .

We take T_1 to be the group obtained by tensoring elements of T_0 with $1 \in SU(W_1)$. Then T_1 is a group of order $q^2 - 1$ that normalizes U_1 and acts transitively on $U_1 \setminus \{1\}$, and as usual we conclude that S^{Λ} acts 2-transitively on Λ .

Arguing as above we see that that any simple section of M^{Λ} must appear as a section of $\operatorname{GU}_5(q)$ and so Λ is beautiful provided $\operatorname{Alt}(q^2 - 1)$ is not a section of $\operatorname{GU}_5(q)$ – this is true for $q \geq 3$ by Lemma 2.1.1.

Finally for q = 2, we do as in the previous case, but we use the existence of a 2-transitive group in $SU_6(q)$ this time. We require that $n_2 \ge 6$ and we let $W_{2,0} = \langle u_1, u_2, u_3, v_1, v_2, v_3 \rangle$ be a non-degenerate 6-subspace of W_2 . Now consider the group

$$U \rtimes L = \left\langle \begin{pmatrix} 1 & a & b & & \\ & 1 & & & \\ & & 1 & & \\ & & & -a^q & 1 \\ & & & -b^q & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & A & & \\ & & 1 & \\ & & & \bar{A}^{-T} \end{pmatrix} \mid a, b \in \mathbb{K}, A \in \mathrm{SL}_2(\mathbb{K}) \right\rangle,$$

where we write \overline{A} to denote the matrix obtained from A by raising each entry to the q-th power. Proceeding as before we obtain a beautiful set provided $\operatorname{Alt}(2^4 - 1)$ does not appear as a section of $\operatorname{GU}_6(2)$ – it does not, so we are done.

4.4.3 Case where S is symplectic or orthogonal

In all of the remaining cases the formed spaces W_1 and W_2 are either symplectic or orthogonal. These are the embeddings in the last three lines of Table 4.4.1. Our strategy is similar to the one already used, namely:

- 1. We identify a subspace $W_{2,0}$ in W_2 , and we identify a group $U \rtimes T$ in $\text{Isom}(W_{2,0})$ for which T acts transitively on the non-identity elements of U.
- 2. If W_1 is orthogonal, we choose a non-degenerate 1-space $X = \langle x \rangle \subseteq W_1$, and identify a subgroup U_1 of Isom(V) whose action on $X \otimes W_{2,0}$ is isomorphic to the action of U on $W_{2,0}$, and which fixes the vectors in $(X \otimes W_{2,0})^{\perp}$. In particular, since dim $(W_1) > 1$, this means that U_1 is not a subgroup of M. If W_1 is symplectic, we do soemthing similar, working with a non-degenerate 2-space $X \subseteq W_1$.
- 3. We define T_1 to be $1 \otimes T$, and observe that T_1 normalizes U_1 , and lies in M. This then allows us to define $\Lambda = \mathcal{P}^{U_1}$, where \mathcal{P} is the tensor product structure stabilized by M, and we observe that S^{Λ} is 2-transitive.

Group	Details of action
$\operatorname{Sp}_n(q)$	$q \in \{3, 5, 7\}, n_1 \in \{2, 4\}, n_2 \in \{3, 4\}: M \triangleright PSp_{n_1}(q) \times P\Omega_{n_2}^{\varepsilon}(q).$

Table 4.4.4: $C_4 - \text{Sp}_n(q)$ – Cases where a beautiful subset was not found.

Group	Details of action
$\Omega_{15}(q)$	$q \in \{5,7\}, M \triangleright \Omega_3(q) \times \Omega_5(q)$

Table 4.4.5: $C_4 - \Omega_n(q)$ – Cases where a beautiful subset was not found.

4. We then identify M_{Λ} and use this to define a monomorphism from M^{Λ} into a small rank classical group, H. The proof is complete, by Lemma 1.6.12, provided M does not contain a section isomorphic to $Alt(|\Lambda| - 1)$.

We start by considering the possibilities for $W_{2,0}$ for q not too small. First, suppose that W_2 is symplectic and contains a subspace $W_{2,0} := \langle u_1, u_2, v_2, v_1 \rangle$, where $\langle u_i, v_i \rangle$ are mutually perpendicular hyperbolic pairs. Then with respect to this basis we define

$$U \rtimes T := \left\langle \begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & -a \\ & & & 1 \end{pmatrix}, \begin{pmatrix} r & & & \\ & 1 & & \\ & & 1 & \\ & & & r^{-1} \end{pmatrix} \mid a, r \in \mathbb{F}_q, r \neq 0 \right\rangle.$$

Observe that $U \rtimes T < \operatorname{Sp}(W_2)$, where we take this subgroup to fix $W_{2,0}^{\perp}$ pointwise.

Suppose next that W_2 is orthogonal, in which case q is odd. In some cases we need $U \rtimes T$ to lie in $\Omega(W_2)$, and in such cases we will assume that W_2 has a non-degenerate subspace $W_{2,0}$ with standard basis u_1, u_2, x, v_2, v_1 . With respect to this basis we define

$$U \rtimes T := \left\langle \begin{pmatrix} 1 & a & -\frac{1}{2}a^2 \\ 1 & & \\ & 1 & -a \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} r & & \\ s & & \\ & 1 & \\ & & s^{-1} \\ & & & r^{-1} \end{pmatrix} \mid a, r, s \in \mathbb{F}_q, r, s \neq 0, rs \text{ a square in } \mathbb{F}_q \right\rangle.$$

Note that $U \rtimes T < \Omega(W_2)$ by [12, Lemmas 2.5.7 and 2.5.9].

If we only need $U \rtimes T$ to lie in the special isometry group $SO(W_2)$, then we take $W_{2,0} = \langle u_1, x, v_1 \rangle$, and with respect to this basis we define

$$U \rtimes T := \left\langle \begin{pmatrix} 1 & a & -\frac{1}{2}a^2 \\ & 1 & -a \\ & & 1 \end{pmatrix}, \begin{pmatrix} r & & \\ & 1 & \\ & & r^{-1} \end{pmatrix} \mid a, r \in \mathbb{F}_q, r \neq 0 \right\rangle.$$

Observe that $U \rtimes T < SO(W_2)$.

Lemma 4.4.4. In this case, if $q \ge 8$, then Ω contains a beautiful subset.

Proof. Suppose that (relabelling W_1 and W_2 if necessary) W_2 satisfies one of the following possibilities:

- 1. W_2 is symplectic of dimension at least 4;
- 2. W_2 is orthogonal of dimension at least 5;

Group	Details of action
$\Omega_{24}^{+}(2), \Omega_{32}^{+}(2), \Omega_{48}^{+}(2) \\ \Omega_{16}^{+}(q) \\ \Omega_{12}^{+}(q) \\ \Omega_{18}^{+}(q)$	$4 \le n_1 < n_2 \le 8: \ M \triangleright \operatorname{PSp}_{n_1}(2) \times \operatorname{PSp}_{n_2}(2).$ $q \in \{3, 5, 7\}, \ n_1 = n_2 = 4: \ M \triangleright \operatorname{P\Omega}_4^-(q) \times \operatorname{P\Omega}_4^+(q).$ $q \in \{5, 7\}, \ M \triangleright \operatorname{P\Omega}_3(q) \times \operatorname{P\Omega}_4^+(q)$ $q \in \{5, 7\}, \ M \triangleright \operatorname{\Omega}_3(q) \times \operatorname{P\Omega}_6^+(q)$

Table 4.4.6: $C_4 - \Omega_n^+(q)$ – Cases where a beautiful subset was not found.

Group	Details of action
$\begin{array}{c} \Omega^{12}(q) \\ \Omega^{18}(q) \end{array}$	$\begin{array}{l} q \in \{5,7\}, \ M \triangleright \Omega_3(q) \times \mathrm{P}\Omega_4^-(q) \\ q \in \{5,7\}, \ M \triangleright \Omega_3(q) \circ \mathrm{P}\Omega_6^-(q) \end{array}$

Table 4.4.7: $C_4 - \Omega_n^-(q)$ – Cases where a beautiful subset was not found.

3. W_2 is orthogonal of dimension 3 or 4, and there exists $X \leq \text{Isom}(W_1)$ such that $X \otimes \text{SO}(W_2)$ embeds in S.

In each of these cases we take $W_{2,0}$ to be the space described above, and $U \rtimes T$ as defined above.

Suppose, first, that W_1 is not symplectic. Then [54, §4.4] confirms that q is odd, and we take x to be a non-isotropic element of W_1 . Then we proceed as detailed above: so U_1 is a subgroup of S whose action on $x \otimes W_{2,0}$ is isomorphic to the action of U on $W_{2,0}$, and T_1 is the group $1 \otimes T$. Setting $\Lambda = \mathcal{P}^{U_1}$, we observe that Λ is a set of size q such that S^{Λ} is 2-transitive.

Now let Y_2 be the subgroup of $\text{Isom}(W_2)$ that fixes point-wise the elements of $W_{2,0}$, and let Y_1 be the subgroup of $\text{Isom}(W_1)$ that fixes the element x. Then $M_{(\Lambda)}$ contains $S \cap (Y_1 \otimes Y_2)$; hence any non-abelian simple section of M^{Λ} is isomorphic to a section of $\text{Isom}(W_{2,0})$. Now $W_{2,0}$ is either symplectic of dimension 4, or orthogonal of dimension 3 or 5. By Lemma 2.1.1, for $q \geq 9$, $\text{Isom}(W_{2,0})$ does not have a section Alt(q-1). Thus Λ is a beautiful subset, and the action is not binary by Lemma 1.6.12.

This argument yields the result except when one of the following holds:

- (a) both W_1 and W_2 are symplectic;
- (b) both W_1 and W_2 are orthogonal, and cannot be labeled so that W_2 satisfies the restrictions stated at the start;
- (c) labelling appropriately, W_1 is symplectic, and W_2 is orthogonal and does not satisfy the restrictions stated at the start.

We see that situation (b) occurs only if $n_i = \dim(W_i) \leq 4$ for i = 1, 2; however [54, 4.4.13] implies that $1 \otimes SO(W_2) < S$ for $n_2 = 3$ or 4, so in fact case 3 above pertains and we are done. Situation (c) is similarly ruled out, except when W_1 is symplectic of dimension 2. Suppose, then, that we are in this case: W_1 is symplectic of dimension 2, and W_2 is orthogonal. Again, q is odd here, S is symplectic, and [54, Lemma 4.4.11] implies that $1 \otimes O(W_2)$ embeds in S. We write $W_1 = \langle e_1, f_1 \rangle$, and we take $W_{2,0}$ to be the 3-dimensional subspace of W_2 described before the statement of the lemma. Then $W_1 \otimes W_{2,0}$ is a 6-dimensional non-degenerate symplectic space with a hyperbolic basis given as follows (we omit the tensor sign for clarity, and we list hyperbolic pairs together, starting with the first two):

$$\{e_1u_1, f_1v_1, e_1v_1, f_1u_1, e_1x, f_1x\}.$$

Now if we consider the group $T_1 = 1 \otimes T$ with respect to this basis, we see that it is diagonal with entries

$$[r, r^{-1}, r^{-1}, r, 1, 1].$$

On the other hand, we can take U_0 to be the set of elements given with respect to this basis by

and we take U_1 to be the subgroup of S which acts like U_0 on $W_1 \otimes W_{0,2}$, and fixes the elements in its orthogonal complement. Then U_1 is not a subgroup of M, and is normalized by T_1 , and so we obtain a set $\Lambda = \mathcal{P}^{U_1}$ of size q for which S^{Λ} is 2-transitive. Arguing as before, we find that a simple section of M^{Λ} is isomorphic to a section of either $\mathrm{Isom}(W_{2,0})$ or $\mathrm{Isom}(W_1)$. We obtain, therefore, a beautiful subset, provided $\mathrm{Alt}(q-1)$ is not a section of $\mathrm{O}_3(q)$ or $\mathrm{Sp}_2(q)$. This is true for $q \geq 7$, and we are done.

Finally, we suppose that situation (a) holds. Here both W_1 and W_2 are symplectic, $S = \Omega_n^+(q)$ and, since $n \ge 8$, we can assume without loss of generality that $\dim(W_2) \ge 4$. In this case, we set $W_{2,0} = \langle u_1, u_2, v_2, v_1 \rangle$ as detailed above, and we consider the basis of $\langle e_1, f_1 \rangle \otimes W_{2,0}$ given by

$$\{-f_1u_1, e_1v_1, -f_1u_2, e_1v_2, f_1v_1, e_1u_1, f_1v_2, e_1u_2\}$$

Again $T_1 = 1 \otimes T$ is given by the diagonal matrix with entries

$$[r, r^{-1}, 1, 1, r^{-1}, r, 1, 1].$$

On the other hand, we can take U_0 to be the set of elements given with respect to the subspace, Y, spanned by the first four of these elements

$$\left\{ \begin{pmatrix} 1 & a \\ & 1 & \\ & & 1 \\ & & 1 \\ & -a & 1 \end{pmatrix} \mid a \in \mathbb{F}_q \right\},$$

and we take U_1 to be the subgroup of S which acts like U_0 on Y, and fixes the elements in its orthogonal complement. Then $U_1 < SO_n^+(q)$.

Now U_1 is not a subgroup of M, is normalized by T_1 , and so we obtain a set $\Lambda = \mathcal{P}^{U_1}$ of size q for which S^{Λ} is 2-transitive. Arguing as before, we find that a simple section of M^{Λ} is isomorphic to a section of either Isom $(W_{2,0})$ or Isom $(\langle e_1, f_1 \rangle)$. We obtain, therefore, a beautiful subset, provided $\operatorname{Alt}(q-1)$ is not a section of $\operatorname{Sp}_4(q)$. This is true for $q \geq 8$, and we are done.

Lemma 4.4.5. In this case, if $q \leq 7$, then Ω contains a beautiful subset or else S is listed in Tables 4.4.4, 4.4.5, 4.4.6 or 4.4.7.

Proof. Let us suppose first that W_1 and W_2 are symplectic, and so $S = \Omega_n^+(q)$; then [54, Table 4.4.A] implies that we can assume that $n_2 > n_1$.

Suppose, first, that $n_2 = 4$; then $n_1 = 2$ and n = 8. Now [10, Table 8.50] confirms that no C_4 -maximal subgroups exist when q is even. What is more, when q is odd, all C_4 -subgroups are conjugate, via a triality automorphism, to certain maximal C_1 -subgroups; then [46, Proposition 4.6] asserts that Ω contains a beautiful subset.

Suppose from here on that $n_2 \ge 6$. If q > 2, then the procedure is virtually identical to that in the previous lemma, but this time we build a beautiful set of size q^2 . To do this we start with a 6-dimensional

subspace of W_2 : define $W_{2,0} = \langle u_1, u_2, u_3, v_1, v_2, v_3 \rangle$. Then with respect to this basis we define

$$U \rtimes T := \left\langle \begin{pmatrix} 1 & a_1 & a_2 & & \\ & 1 & & \\ & & 1 & & \\ & & & -a_1 & 1 \\ & & & -a_2 & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & A & & \\ & & 1 & \\ & & & A^{-T} \end{pmatrix} \mid a_1, a_2 \in \mathbb{F}_q, A \in \mathrm{GL}_2(q) \right\rangle.$$
(4.4.3)

Observe that $U \rtimes T < \operatorname{Sp}(W_2)$. We set $T_1 = 1 \otimes T$, and we set U_1 to be the set of elements given by the same matrices as U above, but with respect to the basis

$$\{e_1u_1, e_1u_2, e_1u_3, f_1v_1, f_1v_2, f_1v_3\},\$$

and fixing the elements in the orthogonal complement. As before we obtain a beautiful subset of size q^2 , provided $Alt(q^2 - 1)$ is not a section of $Sp_6(q)$. This is true for $q \ge 3$.

Now assume that q = 2, in which case [54, Table 3.5.E] implies that $n_1 > 2$. We proceed as in the previous paragraph, but this time, we assume that $n_2 \ge 10$, and we take T to be a group isomorphic to $GL_4(q)$. We obtain a beautiful subset of size $q^4 = 16$, provided $Alt(q^4 - 1) = Alt(15)$ is not a section of $Sp_{10}(2)$; it is not (by Lemma 2.1.1), so the result follows. The exceptions occur when $4 \le n_1 < n_2 \le 8$, and are listed in Table 4.4.6. This completes the case where both W_1 and W_2 are symplectic.

Suppose now that W_1 is orthogonal. Then q is odd, so $q \in \{3, 5, 7\}$.

First, assume that W_2 is symplectic of dimension at least 6. As W_1 is orthogonal, it contains a nonisotropic vector x. We define $U \rtimes T$ exactly as for (4.4.3). We let $T_1 := 1 \otimes T$, and we set U_1 to be the set of elements given by the same matrices as U above, but with respect to the basis

$$\{xu_1, xu_2, xu_3, xv_1, xv_2, xv_3\}$$

and fixing the elements in the orthogonal complement. As before we obtain a beautiful subset of size q^2 , provided $Alt(q^2 - 1)$ is not a section of $Sp_6(q)$ (which is true, by Lemma 2.1.1, since $q \ge 3$).

Now assume that W_2 is symplectic of dimension 2 or 4, and also that dim $W_1 \ge 5$. To keep notation consistent, relabel W_2 as W_1 and vice versa. Then W_2 is orthogonal of dimension at least 5, and we define $W_{2,0} = \langle u_1, u_2, x, v_1, v_2 \rangle$. Then with respect to this basis we define

$$U \rtimes T := \left\langle \begin{pmatrix} 1 & a_1 & -\frac{1}{2}a_1^2 \\ 1 & a_2 & & -\frac{1}{2}a_2^2 \\ 1 & -a_1 & -a_2 \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} A & \\ & 1 & \\ & & A^{-T} \end{pmatrix} \mid a_1, a_2 \in \mathbb{F}_q, A \in \mathrm{GL}_2(q) \right\rangle.$$
(4.4.4)

As usual we set $T_1 = 1 \otimes T$. To define U_1 we let e_1, f_1 be a hyperbolic pair in W_1 , and we consider the space

$$W'_{2,0} = \langle e_1 u_1, e_1 u_2, e_1 x, f_1 x, f_1 v_1, f_1 v_2 \rangle,$$

which we observe is a non-degenerate symplectic 6-space. We define U_1 to act as $1 \otimes U$ on $W'_{2,0}$, and to fix $W'^{\perp}_{2,0}$. Now, proceeding as above we obtain a beautiful subset of size q^2 , provided $\operatorname{Alt}(q^2 - 1)$ is not a section of $O_5(q)$ (which is true, by Lemma 2.1.1, since $q \geq 3$).

The previous two paragraphs cover all cases where $S = \text{Sp}_n(q)$, since Table 4.4.4 contains the remaining cases with $n_1, n_2 \leq 4$.

Finally, suppose that W_1 and W_2 are both orthogonal. Recall that $q \in \{3, 5, 7\}$.

Assume n_1 and n_2 are both even. Then [54, 4.4.2, 4.4.13] implies that $S = \Omega_n^+(q)$, and that $1 \otimes SO(W_2)$ and $SO(W_1) \otimes 1$ both embed into S. We suppose now that $n_2 \ge n_1$ with $n_2 \ge 6$. Then we define $W_{2,0}$, U and T via (4.4.4). We set $T_1 = 1 \otimes T$, and we set U_1 to be the set of elements given by the same matrices as U above, but with respect to the basis

$$\{yu_1, yu_2, yx, yv_1, yv_2\}$$

(where y is an anisotropic element of W_1), and fixing the elements in the orthogonal complement. As before we obtain a beautiful subset of size q^2 , provided $\operatorname{Alt}(q^2 - 1)$ is not a section of $\operatorname{SO}_5(q)$. This is true, by Lemma 2.1.1, for $q \ge 3$. This leaves the case where $n_1 = n_2 = 4$, which is in Table 4.4.6.

Notice that the same argument works if n_1 is even and n_2 is odd with $n_2 \ge 5$ (again using the fact, given in [54, 4.4.13], that $1 \otimes SO(W_2)$ embeds into S). Again we obtain a beautiful subset since $q \ge 3$.

We are left with the possibility that both n_1 and n_2 are odd (in which case we may assume that $3 \leq n_1 < n_2$), or (relabelling if necessary), that $n_1 = 3$ and n_2 is even. In this case we assume now that $n_2 \geq 7$. Now the argument of the previous paragraph works except that we cannot assume that $1 \otimes SO(W_2) \leq S$; this means that we must adjust the definition of $W_{2,0}$ to ensure that $T \leq \Omega(W_2)$. To do this we define $W_{2,0} = \langle u_1, u_2, u_3, x, v_1, v_2, v_3 \rangle$; we take U to be identical to that given in (4.4.4), except that we prescribe that the 7 × 7-matrices of U fix u_3 and v_3 ; then we define

$$T = \left\{ \begin{pmatrix} A & & & \\ & s & & \\ & & 1 & \\ & & & A^{-T} & \\ & & & s^{-1} \end{pmatrix} \middle| s \in \mathbb{F}_q^*, A \in \mathrm{GL}_2(q) \text{ and } s. \det(A) \text{ is a square} \right\}.$$

Now the argument proceeds as before.

The cases not yet covered have $n_1 = 3$, $n_2 = 4, 5, 6$, and are listed in Tables 4.4.6 and 4.4.7. Note that [54, Tables 3.5.D, 3.5.E and 3.5.F] imply that if $n_1 = 3$, then we can exclude q = 3.

4.4.4 The remaining cases

The remaining cases are dealt with by the following result.

Lemma 4.4.6. If the action is listed in Tables 4.4.2, 4.4.3, 4.4.4, 4.4.5, 4.4.6 or 4.4.7, then it is not binary.

Proof. The socle of M is a direct product, as given by the tables. Our method for most cases is as follows: suppose that the action of G on (G : M) is binary. In every case we can see that |G : M| is even. Thus, given a Sylow 2-subgroup P of M, there exists an element x of order a power of 2 in $G \setminus M$ that normalizes P. Then $|M : M \cap M^x|$ is odd and $M \cap M^x$ is core-free in M. Now a magma computation shows that every faithful transitive action of odd degree of a group M, with socle as given in one of the tables, is not binary. Hence $(M, (M : M \cap M^x))$ is not binary, and the conclusion follows by Lemma 1.6.1.

In a couple of cases where the magma computation required too much time we have, instead, found a suitable group H < M with the property that $N_G(H)$ is not contained in M. This guarantees that there is a suborbit of M for which the stabilizer contains H. Now we use magma to show that the action of M on such a suborbit is not binary, and the result follows, again, by Lemma 1.6.1.

4.5 Family C_5

In this case M is a "subfield subgroup": let \mathbb{F}_{q_0} be a subfield of \mathbb{K} (with $|\mathbb{K}| = q_0^r$ for some prime r), and let \mathcal{B} be a basis of V. Then $V_0 = \operatorname{span}_{\mathbb{F}_0}(\mathcal{B})$ is an n-dimensional \mathbb{F}_{q_0} -vector space. The group $\operatorname{GL}_n(q)$ acts naturally on the set of all such vector spaces and M can be taken to be a subgroup of the stabilizer of V_0 in this action.

When G is not $SL_n(q)$, the group S is a set of isometries for some non-degenerate form φ on V. Now we require that M is also a subset of the set of isometries of the form φ_0 on V_0 which is the restriction of φ (or a scalar multiple of φ); full details are given in [54, §4.5]. We list the embeddings in Table 4.5.1. Note that the subfield subgroups are centralized by outer automorphisms of S (see Proposition 2.5.1), so M may not be almost simple. We will need to take account of this possibility in the proofs below.

case	type	conditions
L	$\operatorname{GL}_n(q^{1/r})$	
S	$\operatorname{Sp}_n(q^{1/r})$	
O^{ϵ}	$\mathcal{O}_n^{\delta}(q^{1/r})$	$\epsilon = \delta^r$
U	$\operatorname{GU}_n(q^{1/r})$	$r \operatorname{odd}$
U	$\mathrm{O}_n^\epsilon(q)$	r = 2, q odd
U	$\operatorname{Sp}_n(q)$	r = 2, n even

Table 4.5.1: Maximal subgroups in family C_5

The main result of this section is the following. The result will be proved in a series of lemmas.

Proposition 4.5.1. Suppose that G is an almost simple group with socle $\overline{S} = Cl_n(q)$, and assume that

- (i) $n \geq 3, 4, 4, 7$ in cases L, U, S, O respectively, and
- (ii) $\operatorname{Cl}_n(q)$ is not one of the groups listed in Lemma 4.1.1.

Let M be a maximal subgroup of G in the family C_5 . Then the action of G on (G:M) is not binary.

4.5.1 Case $S = SL_n(q)$

Lemma 4.5.2. In this case either Ω contains a beautiful subset or else S is listed in Table 4.5.2.

Proof. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V, and we assume that M stabilizes the \mathbb{F}_{q_0} -span of \mathcal{B} . We use Lemma 1.6.10 for which we need to exhibit two subgroups, as follows.

We set $A \cong \mathrm{SL}_{n-2}(q_0)$ to be the subgroup of M that fixes v_{n-1} and v_n ; we let $B_0 \cong \mathrm{SL}_{n-1}(q_0)$ be the subgroup of M which fixes v_n . Then let $g \in C_G(A)$ such that v_{n-1}^g is not in the \mathbb{F}_{q_0} -span of \mathcal{B} ; we set $B = B_0^g$ and note that $B \not\leq M$. Now Lemma 1.6.10 implies that there is a subset Δ of Ω such that $|\Delta| = q_0^{n-2}$ and G_{Δ} acts 2-transitively on Δ .

If Δ is a beautiful subset, then Lemma 1.6.12 yields the result; if Δ is not a beautiful subset, then $\operatorname{Alt}(q_0^{n-2})$ must be a section of $\operatorname{SL}_n(q)$. By Lemma 2.1.1, this is impossible unless $(n, q_0) \in \{(4, 2), (5, 2)\}$ or $n = 3, q_0 \leq 7$.

Consider the remaining situations, and set $A \cong SL_{n-1}(q_0)$ to be the subgroup of M that fixes v_n . Let g be the diagonal matrix with entries

 $(\lambda,\ldots,\lambda,\lambda^{-n+1}),$

where λ is an element of $\mathbb{F}_q \setminus \mathbb{F}_{q_0}$ such that $\lambda^n \notin \mathbb{F}_{q_0}$; this is possible unless (n,q) = (3,4). Setting $B = M^g$, and applying Lemma 1.6.10 yields a set Δ as above, except that this time $|\Delta| = q_0^{n-1}$. Again we obtain a beautiful subset unless $\operatorname{Alt}(q_0^{n-1})$ is a section of $\operatorname{SL}_n(q)$; we conclude that $n \leq 4$ and $q_0 = 2$.

Lemma 4.5.3. If S is listed in Table 4.5.2, then the action is not binary.

Group S	Details of action
$\frac{\operatorname{SL}_3(2^r)}{\operatorname{SL}_4(2^r)}$	r prime, $M \triangleright SL_3(2)$. r prime, $M \triangleright SL_4(2)$.

Table 4.5.2: $C_5 - SL_n(q)$ – Cases where a beautiful subset was not found.

Group S	Details of action
$\mathrm{SU}_n(q_0^r)$	$n \in \{4, 5\}, q_0 \in \{2, 3, 4, 5, 7\}, r \text{ odd prime}, M \triangleright \text{PSU}_n(q_0)$
$SU_8(3)$	$M \triangleright \mathrm{P}\Omega_8^{\pm}(3), \mathrm{PSp}_8(3)$
$SU_8(2)$	$M \triangleright \mathrm{PSp}_8(2)$
$SU_7(3)$	$M \triangleright \mathrm{P}\Omega_7(3)$
$SU_6(2)$	$M \triangleright \mathrm{PSp}_6(2)$
$SU_6(q)$	$q \in \{3, 5, 7\}, \ M \triangleright \mathrm{P}\Omega_{6}^{-}(q)$
$\mathrm{SU}_5(q)$	$q \in \{3, 5, 7\}, M \triangleright P\Omega_5(q)$
$SU_4(q)$	$q \in \{2, 3, 4, 5, 7\}, M \triangleright \mathrm{PSp}_4(q)$
$\mathrm{SU}_4(q)$	$q \in \{3, 5, 7\}, \ M \triangleright \mathrm{P}\Omega_4^{\pm}(q)$

Table 4.5.3: $C_5 - SU_n(q)$ – Cases where a beautiful subset was not found.

Proof. Here $n \in \{3,4\}$, $S = SL_n(2^r)$ with r a prime, and M contains a normal subgroup $SL_n(2)$. If $r \in \{2,3\}$, then Lemma 4.1.1 yields the result.

Assume from here on that $r \ge 5$. We have $M = M_0 \times \langle \phi \rangle$, where $M_0 \cong SL_n(2).a$ with $a \in \{1, 2\}$, and ϕ is either 1 or a field automorphism of S of order r.

Let Q be a Sylow 2-subgroup of M_0 . As |G:M| is even, there exists $g \in N_G(Q) \setminus M$. Then $M_0 \cap M_0^g$ contains Q, hence is a parabolic subgroup P of M_0 , and $N_{M_0}(P) = P$. It follows that $M \cap M^g = P \times \langle \sigma \rangle$, where $\sigma = 1$ or ϕ . In particular, σ is in the kernel of the action of M on $(M: M \cap M^g)$. Hence this action is isomorphic to either $(M_0, (M_0: P))$ or $(M_0 \times \langle \phi \rangle, (M_0 \times \langle \phi \rangle : P))$. Lemma 2.3.1 shows that the first action is not binary, and it follows using Lemma 1.6.2 that the second action is also not binary.

4.5.2 Case $S = SU_n(q)$

Note that we are assuming that $n \ge 4$, since the case where $S = SU_3(q)$ is covered in [45]. Note, though, that an inspection of the proof [45] shows up a missing case when $q_0 = 2$. Let us deal with that case now.

Lemma 4.5.4. Suppose that $S = SU_3(2^r)$ with r an odd prime, and that M is a subfield subgroup of G containing $PSU_3(2)$. Then the action of G on (G : M) is not binary.

Proof. We use magma, first, to confirm the result when r = 3. For the rest of the proof we suppose $r \ge 5$. We have $M = M_0 \times \langle \phi \rangle$, where $M_0 \in \{ PSU_3(2), PSU_3(2), 2, PGU_3(2), PGU_3(2), 2 \}$ and ϕ is either 1 or a field automorphism of S of order r. Another magma computation confirms that all non-trivial odd-degree core-free actions of M_0 are not binary.

The proof is now similar to that of the previous lemma. Let $Q \in Syl_2(M_0)$ and $g \in N_G(Q) \setminus M$. Then $Q \leq M_0 \cap M_0^g \leq Q\langle h \rangle$, where h has order 1 or 3. If $M \cap M^g \leq M_0$, then $M \cap M^g$ contains an element $h^i \phi$ for some i, and hence also contains ϕ (as ϕ has order r > 3). Thus $M \cap M^g = (M_0 \cap M_0^g) \times \langle \sigma \rangle$, where $\sigma = 1$ or ϕ . Now we complete the proof as in Lemma 4.5.3.

Lemma 4.5.5. In this case either Ω contains a beautiful subset or else S is listed in Table 4.5.3.

Proof. Our proof splits into two cases, depending on whether r is odd or even. Suppose, first, that r is odd. In this case we let $\mathcal{B} = \{e_1, \ldots, e_m, x, f_1, \ldots, f_m\}$ be a hyperbolic basis for V (for n even we do not need the element x); then φ_0 , the restriction of φ to the $\mathbb{F}_{q_0^2}$ -span of \mathcal{B} , is unitary.

First assume that $m \ge 3$, in which case we will use Lemma 1.6.10. We start by defining $A \cong SL_{m-1}(q_0^2)$ to be the set of elements stabilizing the $\mathbb{F}_{q_0^2}$ -subspaces

$$\langle e_1, \ldots, e_{m-1} \rangle, \langle e_m \rangle, \langle f_1, \ldots, f_{m-1} \rangle, \langle f_m \rangle \text{ (and } \langle x \rangle \text{ if } n \text{ is odd}),$$

and acting on each as an element of determinant 1.

Now let g be the diagonal element with respect to \mathcal{B} whose diagonal entries are 1 except in the entries corresponding to e_m and f_m , in which case the entries are μ and μ^{-1} , respectively, where $\mu \in \mathbb{F}_q^* \setminus \mathbb{F}_{q_0}^*$. Let $B_0 \cong \mathrm{SL}_m(q_0^2)$ be the subgroup of M stabilizing the subspaces

$$\langle e_1, \ldots, e_{m-1}, e_m \rangle$$
 and $\langle f_1, \ldots, f_{m-1}, f_m \rangle$ (and $\langle x \rangle$ if *n* is odd),

and acting on each as an element of determinant 1. Let $B = B_0^g$, and observe that A < B and $B \leq M$; thus Lemma 1.6.10 implies that we have a subset $\Lambda \subseteq \Omega$ of cardinality $q_0^{2(m-1)}$ such that S^{Λ} is a 2-transitive group. This yields a beautiful subset unless $\operatorname{Alt}(q_0^{2(m-1)})$ is a section of $\operatorname{SU}_n(q)$; since we are assuming that $m \geq 3$, Lemma 2.1.1 eliminates the latter possibility, and we are done.

We are left with the possibility that m = 2, in which case $n \in \{4, 5\}$. We define two subgroups, T and U, as follows. First, if n = 5, then both subgroups fix the vector x. Then, in both cases, we fix an element $\zeta \in \mathbb{F}_q^* \setminus \mathbb{F}_{q_0}^*$ and describe the action of the two groups on the space $\langle e_1, e_2, f_1, f_2 \rangle$ (writing elements with respect to the ordered basis $\{e_1, e_2, f_1, f_2\}$):

$$T = \left\{ \begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} & \\ & & & 1 \end{pmatrix} \mid a \in \mathbb{F}_{q_0}^* \right\}, \quad U = \left\{ \begin{pmatrix} 1 & \zeta x & & \\ & 1 & & \\ & & 1 & \\ & & -\zeta x & 1 \end{pmatrix} \mid x \in \mathbb{F}_{q_0} \right\}.$$
(4.5.1)

As usual, we can check that $T \leq M$, $U \not\leq M$, and T normalizes U and acts transitively on the set of non-identity elements of U. Then $\Delta = M^U$ is a subset of Ω on which G_{Δ} acts 2-transitively, and we have a beautiful subset unless $\operatorname{Alt}(q_0)$ is a section of $\operatorname{SU}_n(q)$, which by Lemma 2.1.1 can only occur if $q_0 \leq 7$, as listed in the first line of Table 4.5.3.

Suppose, next, that r = 2. In this case φ_0 is either symmetric (and q is odd) or alternating (and q can be either even or odd). These are the embeddings in the last two lines of Table 4.5.1. In the case where φ_0 is symmetric and not of type O⁻, we take \mathcal{B} , as before, to be a hyperbolic basis.

For the other two cases, we adjust \mathcal{B} slightly in order to see more clearly the embeddings (namely, $SO_n^-(q) < SU_n(q)$ and $Sp_n(q) < SU_n(q)$). In the symplectic case we take $\mathcal{B} = \{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ such that

$$\varphi(e_i, e_j) = \varphi(f_i, f_j) = 0 \text{ and } \varphi(e_i, f_j) = \delta_{ij}\zeta,$$

where $\zeta \in \mathbb{F}_{q^2}$ satisfies $\zeta^q = -\zeta$. It is easy to see that the restriction φ_0 of φ to the \mathbb{F}_q -span of \mathcal{B} is symplectic; what is more the matrix for φ_0 written in block form with respect to \mathcal{B} is

$$\zeta \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

a scalar multiple of the "usual" alternating matrix; hence $\text{Isom}(\varphi_0)$ is a symplectic group $\text{Sp}_n(q)$.

In the O⁻ case we take $\mathcal{B} = \{e_1, \ldots, e_m, f_1, \ldots, f_m, x, y\}$ to be a hyperbolic basis for φ_0 over \mathbb{F}_q , and we simply define φ to be the Hermitian form obtained by extending φ_0 to include scalars over \mathbb{F}_{q^2} .

In all cases, m is the Witt index of φ_0 , and we now proceed as in the first part of the proof. First assume that $m \geq 3$ and define $A \cong SL_{m-1}(q)$ to be the set of elements in M stabilizing the \mathbb{F}_q -subspaces

$$\langle e_1, \ldots, e_{m-1} \rangle$$
, $\langle e_m \rangle$, $\langle f_1, \ldots, f_{m-1} \rangle$, $\langle f_m \rangle$ (and $\langle x \rangle$ and $\langle y \rangle$ if needed).

and acting on each as an element of determinant 1.

Now we define g according to two cases:

1. If φ_0 is orthogonal with n odd, or of type O⁻ with n even, then let g send

$$e_m \mapsto \mu e_m, \ f_m \mapsto \mu^{-q} f_m, \ x \mapsto \mu^{q-1} x,$$

and fix all other elements of \mathcal{B} , where μ is a primitive element of \mathbb{F}_{q^2} .

2. If φ_0 is symplectic or of type O⁺, then we let $\lambda, \mu \in \mathbb{F}_{q^2}$ with λ primitive, and let g act as

$$\lambda I \text{ on } \langle e_1, \dots, e_{m-1} \rangle,$$

$$\mu \text{ on } \langle e_m \rangle,$$

$$\lambda^{-q} I \text{ on } \langle f_1, \dots, f_{m-1} \rangle,$$

$$\mu^{-q} \text{ on } \langle f_m \rangle.$$

We require that $\lambda^{-(q-1)(m-1)} = \mu^{q-1}$ to ensure that $\det(g) = 1$, and we require that $\lambda \mu^{-1} \notin \mathbb{F}_q$ (this condition ensures that $B = B_0^g \not\leq M$, see next paragraph). This can be done provided q+1 does not divide m – we defer this remaining case for the moment.

Let $B_0 \cong \operatorname{SL}_m(q)$ be the subgroup of M stabilizing the subspaces

$$\langle e_1, \ldots, e_{m-1}, e_m \rangle$$
 and $\langle f_1, \ldots, f_{m-1}, f_m \rangle$ (and $\langle x \rangle$ and $\langle y \rangle$ if needed),

and acting on each as an element of determinant 1. Let $B = B_0^g$, and observe that A < B and $B \not\leq M$; thus Lemma 1.6.10 implies that we have a subset $\Lambda \subseteq \Omega$ of order q^{m-1} such that S_{Λ} is 2-transitive. This yields a beautiful subset unless $\operatorname{Alt}(q^{m-1})$ is a section of $\operatorname{SU}_n(q)$. By Lemma 2.1.1, the latter is only possible if (n,q) is one of (6,2), (7,3), (8,2), (8,3) (recall that we are assuming $m \geq 3$ here), and M is as in Table 4.5.3.

Now let us deal with the deferred case: we suppose that φ_0 is symplectic or of type O⁺ and q + 1 divides $m = \frac{n}{2}$. Then we repeat the argument with m redefined to equal $\frac{n-2}{2}$. Note, though, that for the argument to work we must have $(n-2)/2 = m \ge 3$, that is, $n \ge 8$. We set g to act as

$$I \text{ on } \langle e_1, \dots, e_{m-1} \rangle,$$

$$\mu \text{ on } \langle e_m \rangle,$$

$$\mu^{-1} \text{ on } \langle e_{m+1} \rangle,$$

$$I \text{ on } \langle f_1, \dots, f_{m-1} \rangle,$$

$$\mu^{-q} \text{ on } \langle f_m \rangle,$$

$$\mu^q \text{ on } \langle e_{m+1} \rangle.$$

We obtain the same outcome: a beautiful subset of size q^{m-1} unless $Alt(q^{m-1})$ is a section of $SU_n(q)$. The latter is only possible when (n, q) = (8, 3), which situation is listed in the second line of Table 4.5.3.

If we are in the deferred case with n < 8, then n = 6. As q + 1 divides m + 1 = 3, we have q = 2 and $S = SU_6(2)$, and \overline{S} is listed in Lemma 4.1.1. This concludes the analysis of the deferred case.

Next, we consider the possibility that m = 2 (defined, as it was originally, to be the Witt index of φ_0) in which case $n \in \{4, 5, 6\}$. In this case we proceed as at the start of this proof – defining two subgroups U and T as in (4.5.1) – so that we obtain a beautiful subset unless Alt(q) is a section of $SU_n(q)$. Using Lemma 2.1.1, we conclude that $q \leq 7$ in the latter case, giving the cases listed in Table 4.5.3.

Finally, if m = 1, then n = 4 and M is of type O⁻. We shall work with the quasisimple group $S = SU_4(q)$ with centre Z of order d = (4, q + 1). In this group, the corresponding maximal subgroup, which we shall also denote as M, has structure $SO_4^-(q).d$ (see [10, Table 8.10]). Let $X = SO_3(q) < M$, and let $T = \{(\lambda, \lambda^{-1}, 1) : \lambda \in \mathbb{F}_q^*\}$ be a maximal torus of order q - 1 in X (matrices relative to a standard basis for the O₃-space). Thus

$$T < X < M < S.$$
 (4.5.2)

We claim that there is an S-conjugate Y of X such that $T < Y \leq M$. Given the claim, we can complete the proof as follows. Since $Y \cong SO_3(q) \cong PGL_2(q)$, there are subgroups U_+, U_- of order q in Y such that T acts by conjugation fixed-point-freely on both of them. These cannot both be contained in M, as $Y \leq M$. Hence, say, $U_+ \cap M = 1$. Then in the usual way, $\Delta = M^{U_+}$ is a set of q points on which TU_+ acts 2-transitively. For q > 7, Alt(q) is not a section of G, and so Δ is a beautiful subset of Ω , giving the conclusion; when $q \leq 7$, these case are listed in the last line of Table 4.5.3.

So it remains to prove the claim. The claim would follow by applying Lemma 2.6.1 to the sequence (4.5.2) if we knew that M controls fusion of X in S, but this may not be the case: there are two conjugacy classes of subgroups $SO_3(q)$ in $SO_4^-(q)$, with representatives X_1, X_2 , say; then X_1 and X_2 are S-conjugate, but may or may not be M-conjugate (this depends on certain congruences of q which we do not need to state here). Therefore, our argument is different. Define

$$\Lambda = \{ Y < S : T < Y, Y \text{ conjugate to } X \text{ in } S \},\\ \Phi = \{ Y \in \Lambda : Y < M \}.$$

We shall compute the sizes of Λ and Φ , showing that $|\Lambda| > |\Phi|$, hence proving the claim.

First observe that $N_S(T)$ acts on Λ . The action of $N_S(T)$ on Λ is transitive; indeed,

$$Y \in \Lambda \quad \Rightarrow Y = X^s \ (s \in S)$$

$$\Rightarrow T, T^{s^{-1}} < X$$

$$\Rightarrow T^{s^{-1}} = T^x \text{ for some } x \in X$$

$$\Rightarrow Y = X^{xs} \text{ with } xs \in N_S(T).$$

Hence $|\Lambda| = |N_S(T) : N_S(T) \cap N_S(X)|$. Since $N_S(T)$ has a subgroup of order $|\operatorname{GU}_2(q).(q-1)/|Z|$, while the order of $N_S(T) \cap N_S(X)$ divides $2(q^2 - 1)/|Z|$, it follows that $|\Lambda|$ is divisible by $\frac{1}{2}q(q^2 - 1)$.

In the same way we see that $N_M(T)$ has at most 2 orbits on Φ . The orbit Φ_1 of X_1 has size $|N_M(T) : N_M(T) \cap N_M(X_1)|$. Since $|N_M(T)|$ divides $4(q^2 - 1)$ and $|N_M(T) \cap N_M(X_1)|$ is divisible by 2(q - 1), it follows that $|\Phi_1|$ divides 2(q + 1). If there is a second orbit Φ_2 , its size also divides 2(q + 1). Hence $|\Phi|$ divides 4(q + 1).

As q > 7, it is clear from the previous two paragraphs that $|\Lambda| > |\Phi|$. This yields the claim and completes the proof.

Lemma 4.5.6. If S is listed in Table 4.5.3, then the action is not binary.

Proof. Suppose, first, that r = 2 – this covers all but the first line of Table 4.5.3. Now Lemma 4.1.1 deals with all the possible groups S except for $SU_8(3)$, $SU_6(5)$, $SU_6(7)$ and $SU_5(7)$. We handled these cases with magma computations using the permutation character method.

Suppose, next, that $r \geq 3$, so we are in the first line of the table. Here $M = M_0 \times \langle \phi \rangle$, where M_0 has socle $PSU_4(q_0)$ or $PSU_5(q_0)$ with $q_0 \leq 7$, and ϕ is either 1 or a field automorphism of S of order r. We adopt the strategy of the proof of Lemma 4.5.3. Let p be the characteristic of \mathbb{F}_q , let $Q \in Syl_p(M_0)$, and choose $g \in N_G(Q) \setminus M$. Then $Q \leq M_0 \cap M_0^g$, and so (by the well-known "Tits lemma", or by computation) there is a parabolic subgroup P of M_0 such that $UL' \leq M_0 \cap M_0^g \leq P$, where U is the unipotent radical and L a Levi factor.

Write $M_1 = M_0 \cap M_0^g$, a core-free subgroup of M_0 . A magma computation shows that any transitive action of M_0 of p'-degree is not binary. Hence, if $M \cap M^g = M_1 \times \langle \sigma \rangle$ with $\sigma = 1$ or ϕ , then we obtain the conclusion as in the proof of Lemma 4.5.3. Otherwise, $M \cap M^g = M_1 \langle h \phi \rangle$, where $h \in N_{M_0}(M_1)$. Analysing this normalizer, we see that we can take h to be diagonal of order dividing $q_0^2 - 1$. Since $\phi \notin M \cap M^g$, the order of h must be divisible by r, and hence as $q_0 \leq 7$, we must have r = 3 or 5. Hence $M = M_0 \times r, r = 3$ or 5, and $|M : M \cap M^g|$ is coprime to p. Now a further magma computation shows that any such action $(M, (M : M \cap M^g))$ (with $\operatorname{soc}(M_0) \notin M \cap M^g$) is not binary.

4.5.3 Case $S = Sp_n(q)$

Lemma 4.5.7. In this case either Ω contains a beautiful subset or else S is listed in Table 4.5.4. In all cases the action of G on Ω is not binary.

Group S	Details of action
$\operatorname{Sp}_4(2^r)$	r prime, $M \triangleright \operatorname{Sp}_4(2)$.

Table 4.5.4: $C_5 - \text{Sp}_n(q)$ – Cases where a beautiful subset was not found.

Proof. Let $\mathcal{B} = \{e_1, \ldots, e_k, f_k, \ldots, f_1\}$ be a hyperbolic basis for V with $k = \frac{n}{2}$. Let M be the group stabilizing the \mathbb{F}_{q_0} -span of \mathcal{B} .

First fix an element $\zeta \in \mathbb{F}_q \setminus \mathbb{F}_{q_0}$. We define two subgroups, writing elements with respect to \mathcal{B} :

$$T = \left\{ \begin{pmatrix} 1 & & \\ & A & \\ & & 1 \end{pmatrix} \mid A \in \operatorname{Sp}_{n-2}(q_0) \right\}, \quad U = \left\{ \begin{pmatrix} 1 & \zeta x & & \\ & I_{n-2} & \zeta x'^T \\ & & 1 \end{pmatrix} \mid x \in \mathbb{F}_{q_0}^{n-2} \right\},$$

where x' = xJ, J being the matrix of the form relative to the basis \mathcal{B} , omitting e_1, f_1 . As usual, we can check that $T \leq M, U \leq M$, and T normalizes U and acts transitively on $U \setminus 1$. Then $\Delta = M^U$ is a subset of Ω of size q_0^{n-2} on which G_{Δ} acts 2-transitively, and we have a beautiful subset unless $\operatorname{Alt}(q_0^{n-2})$ is a section of $\operatorname{Sp}_n(q)$. By Lemma 2.1.1, the latter is only possible if n = 4 and $q_0 = 2$, the case listed in Table 4.5.4. Finally, the case in the table is dealt with exactly as in Lemma 4.5.3.

4.5.4 Case S is orthogonal

In this section we deal with all of the orthogonal families in one go. Recall from Section 4.1.1 that $n \ge 7$, and also if $S = \Omega_8^+(q)$, then we are assuming that $G \le P\Gamma\Omega_8^+(q)$.

Group S	Details of action
$\Omega_7(3^r)$	$q_0 = 3, M \triangleright \Omega_7(3)$
$\Omega_8^-(q_0^r)$	$q_0 \in \{2,3\}, r \text{ odd}, M \triangleright \mathrm{P}\Omega_8^-(q_0)$
$\Omega_{8}^{+}(q_{0}^{2})$	$q_0 \in \{2,3\}, \ M \triangleright \mathrm{P}\Omega_8^-(q_0)$
$\Omega_8^+(2^r)$	$q_0 = 2, \ M \triangleright \Omega_8^+(2)$
$\Omega_{10}^{-}(2^{r})$	$q_0 = 2, r \text{ odd}, M \triangleright \Omega_{10}^-(2)$
$\Omega_{10}^{+}(4)$	$q_0 = 2, \ M \triangleright \Omega^{10}(2)$

Table 4.5.5: $C_5 - \Omega_n^{\varepsilon}(q)$ – Cases where a beautiful subset was not found.

Lemma 4.5.8. In this case either Ω contains a beautiful subset or else S is listed in Table 4.5.5.

Proof. Let W be an *n*-dimensional orthogonal space over \mathbb{F}_{q_0} , with associated quadratic form Q_W , and let \mathcal{B} be a hyperbolic basis for W. Define $V = W \otimes_{\mathbb{F}_{q_0}} \mathbb{F}_q$, with Q_W extended to a quadratic form, Q_V , on V. This yields an embedding of $\mathrm{Isom}(Q_W) \leq \mathrm{Isom}(Q_V)$. The embeddings listed in row 3 of Table 4.5.1 follow immediately. Note that, in the case where $\Omega_n^-(q_0)$ is embedded in $\Omega_n^+(q_0^2)$, \mathcal{B} is not a hyperbolic basis for V.

We write $\mathcal{B} = \{e_1, \ldots, e_k, f_1, \ldots, f_k, x, y\}$ (omitting x if n is odd, and omitting x and y if n is even and $\varepsilon = +$). We write \mathcal{A} for the \mathbb{F}_q -span of the anisotropic vectors in \mathcal{B} ; so dim $(\mathcal{A}) \in \{0, 1, 2\}$.

We define two subgroups:

$$A = \{g \in M \mid g \text{ stabilizes } \langle e_k \rangle, \langle f_k \rangle, \langle e_1, \dots, e_{k-1} \rangle \text{ and } \langle f_1, \dots, f_{k-1} \rangle; v^g = v \,\forall v \in \mathcal{A}\}; \\B_0 = \{g \in M \mid g \text{ stabilizes } \langle e_1, \dots, e_k \rangle \text{ and } \langle f_1, \dots, f_k \rangle; v^g = v \,\forall v \in \mathcal{A}\}.$$

Observe that $A \triangleright \operatorname{SL}_{k-1}(q_0)$ and $B_0 \triangleright \operatorname{SL}_k(q_0)$. Now define $g \in G$ to send $e_k \mapsto \lambda e_k$, $f_k \mapsto \lambda^{-1} f_k$ and to fix the other elements of \mathcal{B} , where $\lambda \in \mathbb{F}_q \setminus \mathbb{F}_{q_0}$. Set $B = B_0^g$ and observe that B contains A but is not

contained in M. Then Lemma 1.6.10 implies that there is a subset Δ of Ω such that $|\Delta| = q_0^{k-1}$ and G_{Δ} acts 2-transitively on Δ . Then Δ is a beautiful subset (and we are done) or else $\operatorname{Alt}(q_0^{k-1})$ is a section of $\Omega_n^{\varepsilon}(q)$. In the latter case, Lemma 2.1.1 implies that S, M are as in Table 4.5.5.

Lemma 4.5.9. If S is listed in Table 4.5.5, then the action is not binary.

Proof. Suppose, first, that r = 2. In this case, $S \in \{\Omega_8^+(4), \Omega_8^+(9), \Omega_{10}^+(4)\}$ and we confirm the result using magma.

Now suppose that $r \geq 3$. Then $M = M_0 \times \langle \phi \rangle$, where M_0 has socle $P\Omega_n^{\epsilon}(q_0)$ with $n \leq 10$ and $q_0 \leq 3$, and ϕ is either 1 or a field automorphism S of order r. We use the same argument as for Lemma 4.5.6. First, a magma computation shows that any transitive action of M_0 of p'-degree is not binary. Then the argument shows that there exists $g \in G$ such that $(M, (M : M \cap M^g))$ is not binary, unless possibly rdivides $q_0^2 - 1$. As $q_0 \leq 3$, this forces r = 3, and now a further magma computation shows that any transitive p'-action of $M_0 \times 3$ is not binary, completing the proof.

4.6 Family C_6

The members in the Aschbacher class C_6 arise as local subgroups; more specifically they are normalizers of certain absolutely irreducible *r*-groups *R* of symplectic-type, where *r* is a prime number with $r \neq p$ and *p* is the characteristic of the defining field for the classical group. For *r* odd, the *r*-group *R* is extraspecial of exponent *r*, denoted by its order r^{1+2a} ; and for r = 2, either *R* is an extraspecial group 2^{1+2a}_{\pm} , or is a central product $4 \circ 2^{1+2a}$. These *r*-groups have absolutely irreducible embeddings in various classical groups of dimension r^a , and the normalizers of *R* in these classical groups comprise the C_6 subgroups; more precisely, if *G* is an almost simple classical group and \overline{R} is the projective image of *R* in *G*, then $M = N_G(\overline{R})$ is in the C_6 class. Full details are given in [54, §4.6], and we give a list of the embeddings in Table 4.6.1.

case	normalizer	conditions
L^{ϵ}	$r^{1+2a}.\mathrm{Sp}_{2a}(r) < \mathrm{GL}_{r^a}^{\epsilon}(q)$	$r \text{ odd}, q \equiv \epsilon \mod r$
L^{ϵ}	$4 \circ 2^{1+2a} \cdot \operatorname{Sp}_{2a}(2) < \operatorname{GL}_{2^a}^{\epsilon}(q)$	$q = p \equiv \epsilon \mod 4$
\mathbf{S}	$2^{1+2a}_{-}.O^{-}_{2a}(2) < \mathrm{GSp}_{2^a}(q)$	q = p
O^+	$2^{1+2a}_{+}.O^{+}_{2a}(2) < O^{+}_{2a}(q)$	q = p

Table 4.6.1: Maximal subgroups in family C_6

In Line 1 of Table 4.6.1 there is a further condition on q: namely, let e be the smallest positive integer such that $p^e \equiv 1 \mod r$. If e is odd, then $\epsilon = +$ and $q = p^e$; and if e is even, then $\epsilon = -$ and $q = p^{e/2}$.

The main result of this section is the following. The result will be proved in a series of lemmas.

Proposition 4.6.1. Suppose that G is an almost simple group with socle $\overline{S} = Cl_n(q)$, and assume that

- (i) $n \geq 3, 4, 4, 7$ in cases L, U, S, O respectively, and
- (ii) $\operatorname{Cl}_n(q)$ is not one of the groups listed in Lemma 4.1.1.

Let M be a maximal subgroup of G in the family C_6 . Then the action of G on (G:M) is not binary.

Our first lemma deals with the situation when r is odd, in which case $S = \operatorname{SL}_{r^a}^{\varepsilon}(q)$ and q is as given above, so that $\mathbb{K} = \mathbb{F}_{p^e}$. To prove the lemma we recall the set-up described in [54, §4.6] and establish some notation.

We let $R := \langle x_1, \ldots, x_a, y_1, \ldots, y_a, z \rangle$ be the extraspecial *r*-group with center $Z(R) = \langle z \rangle$ and where, for every $i, j \in \{1, \ldots, a\}$,

$$x_i^r = y_j^r = [x_i, x_j] = [y_i, y_j] = 1$$

and

$$[y_i, x_j] = \begin{cases} z & \text{when } j = i, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, $\overline{R} := R/Z(R)$ is an elementary abelian r-group and one can see that \overline{R} embeds naturally in $C_{\text{Aut}(R)}(Z(R))$. We use an additive notation for the elements of \overline{R} and a multiplicative notation for the elements of R and observe that the commutator function

$$\mathfrak{B}: \bar{R} \times \bar{R} \longrightarrow Z(R)$$
$$(gZ(R), h(Z(R)) \longmapsto [g, h]$$

defines a non-degenerate symplectic form on \overline{R} , which endows \overline{R} with the structure of a symplectic space over the field \mathbb{F}_r . Using the basis $(\overline{x}_1, \ldots, \overline{x}_a, \overline{y}_1, \ldots, \overline{y}_a)$ of \overline{R} , the symplectic form \mathfrak{B} on \overline{R} is represented by the skew-symmetric matrix in block form

$$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Observe that, under the natural projection $R \to \overline{R}$, the abelian subgroups of R correspond to the totally isotropic subspaces of \overline{R} . We let $X := \langle x_1, \ldots, x_a \rangle$ and $Y := \langle y_1, \ldots, y_a \rangle$. Observe that X and Y are elementary abelian subgroups of R of cardinality r^a and \overline{X} and \overline{Y} are maximal totally isotropic subspaces of \overline{R} .

From the structure of R, it is clear that each element of R can be written uniquely in the form

$$x_1^{\varepsilon_1}\cdots x_a^{\varepsilon_a}y_1^{\eta_1}\cdots y_a^{\eta_a}z^{\nu},$$

where $\varepsilon_1, \ldots, \varepsilon_a, \eta_1, \ldots, \eta_a, \nu$ can be taken in \mathbb{F}_r . Given $v = \sum_{i=1}^a \varepsilon_i \bar{x}_i + \sum_{i=1}^a \eta_i \bar{y}_i$, an element in \bar{R} , we write $\underline{v} = x_1^{\varepsilon_1} \cdots x_a^{\varepsilon_a} y_1^{\eta_1} \cdots y_a^{\eta_a}$, a corresponding element in R.

Given a matrix $A \in \operatorname{GL}_{2a}(r)$ that preserves the symplectic form \mathfrak{B} , we find that the function

$$\begin{array}{c} \theta_A : R \longrightarrow R \\ \underline{x_i} \longmapsto \underline{Ax_i} \\ \underline{y_i} \longmapsto \underline{Ay_i} \end{array}$$

defined on the generators of R extends to an automorphism of R that centralizes Z(R). In this way we obtain an embedding \overline{R} .Sp_{2a}(r) in $C_{Aut(R)}(Z(R))$ and now [54, Table 4.6.A] asserts that in fact $C_{Aut(R)}(Z(R)) = \overline{R}$.Sp_{2a}(r) $\cong r^{2a}$.Sp_{2a}(r).

Now [54, p.151] describes an absolutely irreducible representation of R over \mathbb{K} of dimension r^a that induces an embedding of $C_{\operatorname{Aut}(R)}(Z(R))$ into $\operatorname{PGL}_{r^a}(\mathbb{K})$; this embedding yields the \mathcal{C}_6 subgroups for r odd.

Lemma 4.6.2. Let r be an odd prime, let G be almost simple with socle $PSL_{r^a}^{\epsilon}(q)$, and let $M = N_G(\bar{R})$, where $R = r^{1+2a}$, as in line 1 of Table 4.6.1. Then the action of G on (G:M) is not binary.

Proof. We adopt the above notation and, since $M = N_G(\bar{R})$, we may identify the set $\Omega = (G : M)$ with the set of conjugates $\{\bar{R}^g : g \in G\}$. Recall that X is an elementary abelian group of order r^a , so $X \cong \mathbb{F}_r^a$. Moreover, since $\mathbb{F}_r^a \cong \mathbb{F}_r^a$ as \mathbb{F}_r -vector spaces, GL(X) contains an automorphism acting as scalar multiplication by a field element of order $r^a - 1$. Let B be the matrix of this automorphism of X with respect to the basis $\{x_1, \ldots, x_a\}$. Then the matrix

$$A = \begin{pmatrix} B & 0\\ 0 & B^{-T} \end{pmatrix}$$

preserves the bilinear form \mathfrak{B} . Thus θ_A is an automorphism of R and hence θ_A determines an element of $N_{\text{PGL}(V)}(\bar{R})$. In fact, from [54, Proposition 4.6.5], $\theta_A \in M$ except when a = 1, r = 3 and p = q. We leave the case a = 1, r = 3 and p = q aside for the time being; indeed let us assume, for now, that n > 5.

Let $C = \langle \theta_A \rangle \in M$ and let $T = \overline{X} \rtimes C \leq M$. By construction T is a Frobenius group with Frobenius kernel \overline{X} of cardinality r^a and cyclic Frobenius complement C of cardinality $r^a - 1$. We claim that

$$\exists g \in N_G(C) \text{ with } M \cap T^g = C. \tag{4.6.1}$$

We argue by contradiction and we suppose that $M \cap T^g \neq C$, for every $g \in N_G(C)$. Let $g \in N_G(C)$. Since $T^g = \bar{X}^g \rtimes C$, $M \cap T^g \ge C$ and C acts transitively by conjugation on the non-identity elements of \bar{X}^g , we deduce $M \cap T^g = T^g$, that is, $T^g \le M$. Suppose that $\bar{X}^g \nleq \bar{R}$. Then T^g is a Frobenius group and is isomorphic to a subgroup of $\operatorname{Sp}_{2a}(r)$. Since r is odd, $r^a - 1$ is even and hence $A^{(r^a-1)/2}$ is the -I matrix. In particular, $A^{(r^a-1)/2}$ centralizes \bar{X}^g . However, this is a contradiction because (since $T^g = \bar{X}^g \rtimes C$ is a Frobenius group) the action of $A^{(r^a-1)/2}$ by conjugation on \bar{X}^g is fixed-point-free. This contradiction yields $\bar{X}^g \le \bar{R}$. Thus \bar{X}^g is a totally isotropic subspace of \bar{R} normalized by C. The only such totally isotropic subspaces are \bar{X} and \bar{Y} , and hence $\bar{X}^g = \bar{X}$ or $\bar{X}^g = \bar{Y}$. Now, consider $T' := \bar{Y} \rtimes C$. As T and T' are conjugate in M, we obtain $T'^g \cap M \ne C$ because $T^g \cap M \ne C$. Therefore, repeating the argument in this paragraph with the group T replaced by T', we deduce that g normalizes $\bar{X}\bar{Y} = \bar{R}$. Thus $N_G(C) \le N_G(\bar{R}) = M$. Now we apply Proposition 2.4.1 and Lemma 2.4.2 to establish the existence of an element $g \in G$ normalizing C but not lying in M. Therefore our claim (4.6.1) is now proved.

Let $g \in N_G(C)$ with $M \cap T^g = C$ and let $\Lambda := \{\overline{R}^t \mid t \in T^g\}$. Then Λ is a set of cardinality $|T^g: T^g \cap M| = r^a$ and T^g induces on Λ a permutation group isomorphic to a Frobenius group of order $r^a(r^a-1)$. If Λ is a beautiful subset, the conclusion follows by Lemma 1.6.12. Otherwise, $\operatorname{Alt}(r^a-1)$ must be isomorphic to a section of M, hence to a section of $\operatorname{Sp}_{2a}(r)$. Since n > 5, Lemma 2.1.1 rules out the latter possibility and we are done.

Consider next the case a = 1, r = 5. Here the embedding is $5^{1+2}.\operatorname{Sp}_2(5) < S = \operatorname{SL}_5^{\epsilon}(q)$, where q is minimal such that $q \equiv \epsilon \mod 5$. If p = 2 then $S = \operatorname{SU}_5(4)$, which is covered by Lemma 4.1.1. So now assume p > 2. It is well-known that the extension $R.\operatorname{Sp}_2(5)$ splits. Let $S_0 \cong \operatorname{Sp}_2(5)$ be a complement, and let $t \in S_0$ be the central involution. If V is the natural 5-dimensional module for S, then $V_5 \downarrow S_0 = V_3 \oplus V_2$, where $V_3 = C_V(t), V_2 = C_V(-t)$, of dimensions 3,2 respectively. Hence there exists a diagonal element of S of the form $\hat{g} = (\lambda I_3, \mu I_2)$ such that $\hat{g} \in C_S(S_0) \setminus N_S(R)$. Denoting by g the projective image of \hat{g} , we then have $\overline{RS}_0 \cap (\overline{RS}_0)^g = S_0$. Since $\overline{RS}_0 = 5^2.\operatorname{Sp}_2(5)$ is a Frobenius group, this give a 2-transitive subset of size 25 in the usual way, and the conclusion follows.

Finally, the case where a = 1, r = 3 is dealt with in similar fashion. Here the embedding is $3^{1+2}.Q_8 < SL_3^{\epsilon}(p)$ with $p \equiv \epsilon \mod 3$ and $p \neq 2$. As $V \downarrow Q_8 = V_2 \oplus V_1$, there exists $g \in C_G(Q_8) \setminus M$, and hence as above we obtain a subset Λ of size 9 with G^{Λ} 2-transitive. This completes the proof.

Lemma 4.6.3. Let r = 2, let G be almost simple with socle $PSL_{2a}^{\epsilon}(p)$ $(a \ge 2)$, $PSp_{2a}(p)$ $(a \ge 2)$ or $P\Omega_{2a}^{+}(p)$ $(a \ge 3)$, and let $M = N_G(\bar{R})$, where $R = 4 \circ 2^{1+2a}$ or 2^{1+2a}_{\pm} , as in Lines 2, 3, 4 of Table 4.6.1. Then the action of G on (G:M) is not binary.

Proof. Since $M = N_G(\bar{R})$, we may identify the set $\Omega = (G : M)$ with the set of conjugates $\{\bar{R}^g : g \in G\}$. Referring to [54, §4.6], we have

$$R = \langle z \rangle \circ \langle x_1, y_1 \rangle \circ \cdots \circ \langle x_a, y_a \rangle,$$

where

z has order 4 in types L, U, and has order 2 in types S, O^+ ,

 $\langle x_i, y_i \rangle \cong D_8 \text{ for } i \ge 3,$ $\langle x_1, y_1 \rangle \cong \langle x_2, y_2 \rangle \cong Q_8 \text{ in types } L, U, O^+,$ $\langle x_1, y_1 \rangle \cong Q_8 \text{ and } \langle x_2, y_2 \rangle \cong D_8 \text{ in type } S.$

The natural 2^a -dimensional module V has a tensor product decomposition $V = W_1 \otimes \cdots \otimes W_a$ under the action of R, where each W_i is an irreducible 2-dimensional module for $\langle z, x_i, y_i \rangle$.

case	$M\cap ar{S}$
L^{ϵ}	2^4 . Alt(6), if $n = 4, p \equiv \epsilon 5 \mod 8$
	2^{2a} .Sp _{2a} (2), otherwise
S	$2^{2a}.O_{2a}^{-}(2), \text{ if } p \equiv \pm 1 \mod 8$
	$2^{2a} \cdot \Omega_{2a}^{-}(2)$, if $p \equiv \pm 3 \mod 8$
O^+	$2^{2a}.O_{2a}^+(2), \text{ if } p \equiv \pm 1 \mod 8$
	$2^{2a} \cdot \Omega_{2a}^+(2)$, if $p \equiv \pm 3 \mod 8$

From [54, 4.6.6, 4.6.8, 4.6.9], writing $\overline{S} = \operatorname{soc}(G)$, the precise structure of $M \cap \overline{S}$ is as follows:

We now divide the proof in two parts (A) and (B), depending on whether $p \ge 7$ or p < 7.

(A) Assume first that $p \ge 7$. Define $W = W_2 \otimes \cdots \otimes W_a$, and note that $SL(W_1) = Sp(W_1)$. The subgroup of *G* preserving the tensor decomposition $V = W_1 \otimes W$ is the normalizer of the image of $SL(W_1) \otimes Cl(W)$, where Cl(W) is $SL^{\epsilon}(W)$, $\Omega^+(W)$ or Sp(W) for case L^{ϵ} , *S* or O^+ repectively.

Again we use the bar notation for the natural homomorphism to the projective version of our classical group. As before, M preserves on \overline{R} a non-degenerate symplectic form \mathfrak{B} in types L and U defined as above. In types O^+ and S, the group M not only preserves \mathfrak{B} but also a particular quadratic form $\mathfrak{q}: \overline{R} \to \mathbb{F}_r$ that polarizes to \mathfrak{B} . Rather than defining \mathfrak{q} explicitly we remark only that, for $i = 1, \ldots, a$, the 2-spaces corresponding to $\langle x_i, y_i \rangle \cong Q_8$ (resp. $\langle x_i, y_i \rangle \cong D_8$) are of type $O_2^-(2)$ (resp. $O_2^+(2)$). Then \overline{R}_1 is a non-degenerate 2-space in \overline{R} , and is of type $O_2^-(2)$ in cases O^+ and S.

Define $\bar{R}_0 = \prod_{i=2}^a \bar{R}_i = \bar{R}_1^{\perp}$, and $W = W_2 \otimes \cdots \otimes W_a$. By [54, 4.4.3], $N_G(\bar{R}_0)$ preserves the tensor decomposition $V = W_1 \otimes W$ and contains the image of $SL(W_1) \otimes 1_W$. Define a subset Δ of $\Omega = \{\bar{R}^g : g \in G\}$ by

$$\Delta = \{ \bar{R}^g : g \in N_G(\bar{R}_0) \}$$

Let X be the image of the group induced on W_1 by $N_G(\bar{R}_0)$. Then $X \cong PSL_2(p)$ or $PGL_2(p)$, and

$$\Delta = \{ \bar{R}_0 \times \bar{R}_1^x : x \in X \}$$

Also, from the structure of $M \cap \overline{S}$, we see that

$$N_X(\bar{R}_1) \cong 2^2.\mathrm{Sp}_2(2) \cong 2^2.\mathrm{O}_2^-(2) \cong \mathrm{Sym}(4).$$

Since the intersection of all the subgroups in Δ is \overline{R}_0 , we have $G_{\Delta} = N_G(\overline{R}_0)$. Hence the action of G_{Δ} on Δ is isomorphic to the action of X on the cosets of Sym(4).

Recall that we are assuming $p \ge 7$. Hence Sym(4) is a maximal subgroup of either X or X', and [45] (together with Lemma 1.6.2) shows that (X, (X : Sym(4))) is not binary. Thus there is an integer $k \ge 3$, and k-tuples $I = (I_1, \ldots, I_k), J = (J_1, \ldots, J_k) \in \Delta^k$ such that $I \simeq J$ and $I \not\sim J$ with respect to the action of G_{Δ} . Since $I \simeq J$ we can assume that $I_1 = J_1$ and $I_2 = J_2$; we also assume that there are no repeated entries in I (and hence there are none in J either).

We need to show that $I \not\sim J$ with respect to the action of G. Suppose $I^g = J$ for some $g \in G$. Observe that for each j we have $I_j = \overline{R}_0 \times \overline{R}_1^{x_j}$ for some $x_j \in G_\Delta$. We claim that

$$\bigcap_{j=1}^k \bar{R}_1^{x_j} = 1$$

Proof of claim: Suppose otherwise. Since \bar{R}_1 is a Klein 4-group we must have $\bigcap_{j=1}^k \bar{R}_1^{x_j} = \langle g_I \rangle$ where g_I is an involution. Now for each j we have $J_j = \bar{R}_0 \times \bar{R}_1^{y_j}$ for some $y_j \in G_\Delta$. Since $I \simeq J$ with respect to the action of G, we conclude that $\bigcap_{j=1}^k \bar{R}_1^{y_j} = \langle g_J \rangle$ for some involution g_J . Observe that, for distinct i and j, we have

$$\bar{R}_1^{x_i} \cap \bar{R}_1^{x_j} = \langle g_I \rangle$$
 and $\bar{R}_1^{y_i} \cap \bar{R}_1^{y_j} = \langle g_J \rangle$.

Since $I_1 = J_1$ and $I_2 = J_2$ we conclude that $g_I = g_J$. Consider what this means for the action of Xon (X : Sym(4)): we can think of this action as being the conjugation action of X on a class of Klein 4-subgroups. The tuples I and J correspond to k-tuples, I_X and J_X , whose entries are Klein 4-subgroups of X all of which contain an involution g_X . What is more $I_X \simeq J_X$ and $I_X \not\sim J_X$ with respect to the action of X. Now if i and j are distinct in $\{1, \ldots, k\}$ and

$$I_i^h = J_i$$
 and $I_j^h = J_j$ for some $h \in X$,

then $h \in C_X(g_X)$. The group $C_X(g_X)$ is a maximal dihedral subgroup of X and we define $Y = C_X(g_X)/\langle g_X \rangle$ which is also a dihedral group. The tuples I_X and J_X correspond to k-tuples, I_Y and J_Y , whose entries are involutions in Y. Since $I_X \cong J_X$ with respect to X, we have $I_Y \cong J_Y$ with respect to Y. But this action is binary (see the discussion of Family 3a at the start of §1.2) and so $I_Y \cong J_Y$ with respect to Y. But this implies that $I_X \cong J_X$ with respect to X which is a contradiction. Hence the claim is proved.

It now follows that

$$\bigcap_{j=1}^{k} I_j = \bar{R}_0$$

and similarly $\bigcap_{j=1}^{k} J_j = \bar{R}_0$. Therefore $g \in N_G(\bar{R}_0) = G_\Delta$, which is a contradiction. This completes the proof under the assumption that $p \ge 7$.

(B) Now assume that p < 7, so that p = 3 or 5. First note that the cases where a = 2 (in which case $L = PSL_4^{\epsilon}(p)$ or $PSp_4(p)$) are covered by Lemma 4.1.1. So we may assume that $a \ge 3$.

Define $R_a = R_1 \times R_3$ and $R_b = \prod_{i \neq 1,3} R_i$, and let $W_a = W_1 \otimes W_3$, $W_b = \bigotimes_{i \neq 1,3} W_i$. Then in case S or O^+ , the subgroup of G preserving the tensor decomposition $W_a \otimes W_b$ is the normalizer of the image of $\operatorname{Sp}(W_a) \otimes \operatorname{Cl}(W_b)$, where $\operatorname{Cl}(W_b)$ is orthogonal or symplectic, respectively. The normalizer $N_G(\bar{R}_b)$ preserves this tensor decomposition.

Define a subset Δ of Ω by $\Delta = \{\bar{R}^x : x \in N_G(\bar{R}_b)\}$. Let X be the image of the group induced on W_a by $N_G(\bar{R}_b)$. Then X has socle $\mathrm{PSL}_4^e(p)$ or $\mathrm{PSp}_4(p)$, and $\Delta = \{\bar{R}_b \times \bar{R}_a^x : x \in X\}$. Also, from the structure of $M \cap L$, we see that

$$N_X(\bar{R}_a) \cong \begin{cases} 2^4.\mathrm{Sp}_4(2), \text{ case } L^{\epsilon} \\ 2^4.\mathrm{O}_4^-(2), \text{ cases } S, O^+. \end{cases}$$

As above, the action of G_{Δ} on Δ is isomorphic to the action of X on the cosets of $N_X(R_a)$. Using magma, we check in all possible cases that this action is not binary, and that there exist k-tuples $I = (I_1, \ldots, I_k)$, $J = (J_1, \ldots, J_k) \in \Delta^k$ such that $I \cong J$ and $I \not\sim J$ with respect to the action of G_{Δ} , and also such that $\bigcap_{j=1}^k I_j = \bigcap_{j=1}^k J_j = \bar{R}_b$. Now we see exactly as in the argument at the end of part (A) that $I \not\sim J$ with respect to the action of G. Hence G is not binary, and the proof is complete.

4.7 Family C_7

In this case M is the stabilizer of a tensor decomposition of V, in much the same way as was detailed at the start of §4.4. In this case, though, M stabilizes a tensor product of two or more subspaces of the same dimension: we write $V = W_1 \otimes \cdots \otimes W_t$, and $m := \dim(W_1) = \cdots = \dim(W_t)$. Observe that $\dim(V) = n = m^t$. If $G = \operatorname{PGL}_n(q)$, the stabilizer M has the structure $\operatorname{PGL}_m(q) \operatorname{wr} \operatorname{Sym}(t)$, where $\operatorname{Sym}(t)$ permutes the tensor factors.

In the case where S is not $SL_n(q)$, i.e. S preserves a non-degenerate form φ , the spaces $W_1 \ldots, W_t$ are mutually similar spaces equipped with non-degenerate forms $\varphi_1, \ldots, \varphi_t$, and

$$\varphi = \begin{cases} Q(\varphi_1 \otimes \cdots \otimes \varphi_t), & \text{if } q \text{ is even and } \varphi_1, \dots, \varphi_t \text{ are non-degenerate alternating;} \\ \varphi_1 \otimes \cdots \otimes \varphi_t, & \text{otherwise.} \end{cases}$$

The definition of the quadratic form $Q(\varphi_1 \otimes \cdots \otimes \varphi_t)$ is given on [54, p.127]: it is the unique non-degenerate quadratic form Q such that

- 1. $Q(w_1 \otimes \cdots \otimes w_t) = 0$ for all $w_i \in W_i$, and
- 2. the polarization of Q is equal to $\varphi_1 \otimes \cdots \otimes \varphi_t$.

Again the stabilizer M has a wreath product structure. It is convenient to set φ to be the zero map when $S = SL_n(q)$. We have given a list of all the C_7 embeddings in Table 4.7.1, taken from [54, §4.7], where the precise structures of the C_7 subgroups can be found.

case	type	conditions
L^{ϵ}	$\operatorname{PGL}_m^{\epsilon}(q) \operatorname{wr} \operatorname{Sym}(t)$	$m \ge 3$
\mathbf{S}	$\mathrm{PSp}_m(q) \operatorname{wr} \operatorname{Sym}(t)$	qt odd
O^+	$\mathrm{PO}_m^{\pm}(q) \operatorname{wr} \operatorname{Sym}(t)$	$q \operatorname{odd}$
O^+	$\mathrm{PSp}_m(q) \operatorname{wr} \operatorname{Sym}(t)$	qt even
0	$\mathrm{PO}_m(q) \operatorname{wr} \operatorname{Sym}(t)$	qm odd

Table 4.7.1: Maximal subgroups in family C_7

Note that [54, p. 156] details a further restriction on the subgroup M, namely that the relevant subgroup $\operatorname{Cl}_m(q)$ must be quasisimple. For instance, in the O⁺ case that is listed on Line 3 of the table, we require that $\Omega_m^{\pm}(q)$ is quasisimple; thus, for this case, $m \ge 6$ or $(m, \varepsilon) = (4, -)$. In general we have that the socle of M is $(\operatorname{Cl}_m(q))^t$.

The main result of this section is the following. The result will be proved in a series of lemmas.

Proposition 4.7.1. Suppose that G is an almost simple group with socle $\overline{S} = Cl_n(q)$, and assume that

- (i) $n \geq 3, 4, 4, 7$ in cases L, U, S, O respectively, and
- (ii) $\operatorname{Cl}_n(q)$ is not one of the groups listed in Lemma 4.1.1.

Let M be a maximal subgroup of G in the family C_7 . Then the action of G on (G:M) is not binary.

The following lemma will be used in various special cases.

Lemma 4.7.2. Let t_0 be an integer, at least 2; in the case where $\varphi_1, \ldots, \varphi_t$ are non-degenerate alternating bilinear forms, we require that q is odd and that $m^{t_0} \ge 8$.

Let V_0 be the m^{t_0} -dimensional formed space that is a tensor product of t_0 formed spaces all similar to (W_1, φ_1) . Let $\bar{S}_0 = X_{m^{t_0}}(q)$, where $X \in \{\text{PSL}, \text{PSU}, \text{PSp}, \text{P}\Omega^{\varepsilon}\}$, be the simple group associated with V_0 . Consider all pairs (G_0, M_0) where G_0 is an almost simple group with socle \bar{S}_0 and M_0 is the subgroup of G_0 from class C_7 associated with the tensor product decomposition. Let $\Omega_0 = (G_0 : M_0)$.

- 1. Suppose that $t_0 = 2$ and that, for all such pairs (G_0, M_0) , the action of G_0 on Ω_0 is not binary. Then the action of G on (G:M) is not binary.
- 2. Suppose that $t_0 > 2$ and that, for all such pairs (G_0, M_0) we can find an integer $k \ge 3$ and tuples $(I_1, \ldots, I_k), (J_1, \ldots, J_k) \in \Omega_0^k$ such that
 - (a) $(I_1, \ldots, I_k) \simeq (J_1, \ldots, J_k);$
 - (b) $(I_1, \ldots, I_k) \not\sim (J_1, \ldots, J_k);$
 - (c) there is no group isomorphic to $X_m(q)$ that is a normal subgroup of each of the socles of $(G_0)_{I_1}, \dots, (G_0)_{I_k}$.

Then the action of G on (G:M) is not binary.

Note, first, that the family in which \overline{S}_0 lies (i.e. the particular choice of X from {PSL, PSU, PSp, $P\Omega^{\varepsilon}$ }) is determined by the type of W_1 and the value of t_0 (see Table 4.7.1). Then [54, Tables 3.5.H and 3.5.I] (and, when $n \in \{8, 9\}$, [10]) imply that, since M is maximal in G, we know that M_0 will be a maximal C_7 -subgroup of G_0 unless $\varphi_1, \ldots, \varphi_t$ are non-degenerate alternating and q is even – but this case is explicitly ruled out by our hypotheses in the statement of this lemma.

Note, second, that in most cases the groups \bar{S} and \bar{S}_0 will lie in the same family, i.e. if $\bar{S}_0 = X_{m^{t_0}}(q)$, then $\bar{S} = X_{m^t}(q)$. The exception to this occurs when $\varphi_1, \ldots, \varphi_t$ are non-degenerate alternating bilinear forms and t and t_0 have different parity (see Table 4.7.1).

Proof. We noted above that the socle of M is L^t where $L = \operatorname{Cl}_m(q)$, a non-abelian simple group. Write Γ for the set of semilinear similarities of φ , so $S = F^*(\Gamma)$, and let ι be the inverse transpose map. Write \underline{G} (resp. \underline{M}) for the preimage of G (resp. M) in Γ (or in $\Gamma : \langle \iota \rangle$ if $S = \operatorname{SL}_n(q)$). We write \mathcal{D} for the decomposition preserved by \underline{M} :

$$(V,\varphi) = (W_1,\varphi_1) \otimes \cdots \otimes (W_t,\varphi_t).$$

We define $U = W_1 \otimes \cdots \otimes W_{t_0}$ and $\varphi_U = \varphi_1 \otimes \cdots \otimes \varphi_{t_0}$. Let G_U be the stabilizer in <u>G</u> of the decomposition

$$\mathcal{D}_U: (V, \varphi) = (U, \varphi_U) \otimes (W_{t_0+1}, \varphi_{t_0+1}) \otimes \cdots \otimes (W_t, \varphi_t).$$

Now we consider the action of G_U , the projective image of $\underline{G_U}$ in G, on (G : M). In particular we can consider the action on the set of cosets $M.G_U$; the action on this set is isomorphic to the action of G_U on $(G_U : M_U)$ where $M_U = M \cap G_U$.

Clearly the kernel of this action contains the image in G_U of

$$\underline{G_U} \cap (\{1\} \otimes \Delta_{t_0+1} \cdots \otimes \Delta_t) J_U$$

where $J_U \cong \text{Sym}(t - t_0)$. The quotient of G_U by the kernel of this action is an almost simple group G_0 with socle $X_{m^{t_0}}(q)$ and the stabilizer in G_0 of a point is a subgroup M_0 of G_0 from class \mathcal{C}_7 associated with a decomposition of the associated m^{t_0} -dimensional formed space into a tensor product of t_0 formed spaces all similar to (W_1, φ_1) .

By assumption we know that this action is not binary. Let I, J be elements of $(G_U : M_U)^k$ for some integer $k \ge 3$ such that $I \simeq J$ and $I \not\sim J$ with respect to the action of G_U . Identify the entries of I and J with the corresponding elements of (G : M). We can think of the entries of I and J as conjugates of M in G; now, if $t_0 > 2$, then assume that I has the property listed at 2(c). It is sufficient to prove that $I \not\sim J$ with respect to the action of G.

It is at this point that we use the fact that the socle of M is L^t where $L = \operatorname{Cl}_m(q)$. Define

$$K = \underbrace{\{1\} \times \cdots \times \{1\}}_{t_0} \times L^{t-t_0}.$$

Observe that K is a normal subgroup of the socle of M. By construction, K is a normal subgroup of each of the socles of I_1, \ldots, I_k and J_1, \ldots, J_k . Suppose that $I^g = J$. Then $\langle K, K^g \rangle$ is a normal subgroup of the socle of J_i for $i = 1, \ldots, k$ and we see that $\langle K, K^g \rangle$ is isomorphic to L^{t-s} for some $0 \le s \le t_0$. We can relabel so that

$$\langle K, K^g \rangle = \underbrace{\{1\} \times \cdots \times \{1\}}_{s} \times L^{t-s}$$

If $s = t_0$, then $K = K^g$. But $N_G(K) = G_U$ and so $g \in G_U$ which is a contradiction. Thus $s < t_0$. If s = 0, then J_1, \ldots, J_k have the same socle and so $J_1 = J_2 = \cdots = J_k$. Since $I \simeq J$ we obtain that $I_1 = I_2 = \cdots = I_k$ and so $I \simeq J$ with respect to the action of G_U , which is a contradiction. Thus $0 < s < t_0$.

Now we refer to [54, Lemma 4.4.3] from which we deduce that $C_S(\langle K, K^g \rangle)$ must be a subgroup of $GL(W_1 \otimes \cdots \otimes W_s) \times 1^{t-s}$ and, of course, must preserve the form φ . If s = 1, then this means that

Group	Details of action
$\frac{\mathrm{SL}_{3^t}(2)}{\mathrm{SL}_{4^t}(2)}$	$m = 3: M \triangleright \mathrm{PSL}_3(2)^t$ $m = 4: M \triangleright \mathrm{PSL}_4(2)^t$

Table 4.7.2: $C_7 - SL_n(q)$ – Cases where a beautiful subset was not found.

there is a unique conjugate of M whose socle contains $\langle K, K^g \rangle$ and, again, $J_1 = J_2 = \cdots = J_k$ which is a contradiction as before. This proves the result when $t_0 = 2$.

If $t_0 > 2$, then the property listed at 2(c) implies that the only conjugate of K that is normal in each of the socles of I_1, \ldots, I_k is K itself. Hence, since $J = I^g$ and since K is normal in each of the socles of J_1, \ldots, J_k , we conclude that the only conjugate of K that is normal in each of the socles of J_1, \ldots, J_k is K itself. But this means that $K^g = K$ and, again, the fact that $N_G(K) = G_U$ implies that $g \in G_U$, a contradiction.

4.7.1 Case $S = SL_n(q)$

In this case [54, Table 3.5.A] allows us to assume that $m \ge 3$.

Lemma 4.7.3. In this case either Ω contains a beautiful subset or else the action is listed in Table 4.7.2.

Proof. We write $W_1 = \cdots = W_t$, and let $\mathcal{B}_1 = \{e_1, \ldots, e_m\}$ be a basis for W_1 . Then

$$\mathcal{B} = \{e_{i_1} \otimes \cdots \otimes e_{i_t}\} \mid 1 \le i_1, \dots, i_t \le m\}$$

is a basis for V, and we take M to be the stabilizer the associated tensor decomposition, so that $M \cap \overline{S} = (\operatorname{PGL}_m(q) \operatorname{wr} \operatorname{Sym}(t)) \cap \overline{S}$.

First assume that $q \ge 7$, and let T_1 be a split maximal torus in $SL_m(q)$ that is diagonal with respect to \mathcal{B}_1 ; then $T = T_1 \otimes 1 \otimes \cdots \otimes 1$ is a subgroup of (the preimage of) M. Define U to be the set of elements in S for which there exists $\alpha \in \mathbb{F}_q$ such that

$$e_1 \otimes \cdots \otimes e_1 \mapsto e_1 \otimes \cdots \otimes e_1 + \alpha e_2 \otimes e_1 \otimes \cdots \otimes e_1,$$

and which fixes all elements $e_{i_1} \otimes \cdots \otimes e_{i_t} \in \mathcal{B}$ for which $i_j > 1$ for some j. Observe that U is not a subgroup of M, that T normalizes U and that T acts transitively on the non-identity elements of U. We define $\Delta = M^U$, a subset of Ω of size q and observe that $U \rtimes T$ acts 2-transitively on Δ . On the other hand,

$$M_{(\Delta)} \ge C_M(U) \ge \left[\operatorname{GL}_{m-2}(q) \circ (\underbrace{\operatorname{GL}_{m-1}(q) \circ \cdots \circ \operatorname{GL}_{m-1}(q)}_{t-1}) \cdot \operatorname{Sym}(t-1)\right] \cap \bar{S}.$$
(4.7.1)

Assuming that Δ is not beautiful, G^{Δ} induces at least $\operatorname{Alt}(q)$ on Δ , hence the point stabilizer M^{Δ} induces at least $\operatorname{Alt}(q-1)$ on Δ . However, $M_{(\Delta)}$ contains $C_M(U)$, which contains the group on the right hand side of 4.7.1. It follows that any simple section of M^{Δ} is a section of $\operatorname{GL}_2(q)$. Since $q \geq 7$, it follows from Lemma 2.1.1 that $\operatorname{Alt}(q-1)$ is not a section of M^{Δ} . This implies that Δ is a beautiful subset and Lemma 1.6.12 yields the result.

Next assume that $q \in \{3, 4, 5\}$, and let T_2 be a maximal torus in $GL_m(q)$ that preserves the decomposition

$$\langle e_1 \rangle \oplus \langle e_2, e_3 \rangle \oplus \langle e_4 \rangle \oplus \cdots \oplus \langle e_m \rangle,$$

and that acts on $\langle e_2, e_3 \rangle$ as a Singer cycle; let T_1 be as above. Then $T_2 \otimes T_1 \otimes 1 \otimes \cdots \otimes 1$ is a subgroup of M. Define U to be the set of elements in S, for which there exists $\alpha, \beta \in \mathbb{F}_q$ such that

$$e_1 \otimes \cdots \otimes e_1 \mapsto e_1 \otimes \cdots \otimes e_1 + \alpha e_2 \otimes e_1 \otimes \cdots \otimes e_1 + \beta e_3 \otimes e_1 \otimes \cdots \otimes e_1,$$

Group	Details of action
	$q \in \{3, 4, 5\}, m = 3: M \triangleright \mathrm{PSU}_3(q)^t$ $m \in \{4, 5\}: M \triangleright \mathrm{PSU}_m(2)^t$

Table 4.7.3: $C_7 - SU_n(q)$ – Cases where a beautiful subset was not found.

and which fixes all other elements of \mathcal{B} . Observe that U is not a subgroup of M, that T normalizes U and that T acts transitively on the non-identity elements of U. We define $\Delta = M^U$, a subset of Ω of size q^2 and observe that $U \rtimes T$ acts 2-transitively on Δ . Arguing as above, we see that any non-abelian simple section of M^{Δ} is isomorphic to a section of $GL_3(q)$; hence, since $q \geq 3$, $Alt(q^2-1)$ is not a section of M^{Δ} . This implies that Δ is a beautiful subset and Lemma 1.6.12 yields the result.

When q = 2, we assume that $m \ge 5$ and we proceed similarly: we construct a beautiful subset of size $q^4 = 16$, using the same method but this time we choose a maximal torus T_4 in $GL_m(q)$ preserving the decomposition

$$\langle e_1 \rangle \oplus \langle e_2, e_3, e_4, e_5 \rangle \oplus \langle e_6 \rangle \oplus \cdots \oplus \langle e_m \rangle,$$

and acting on $\langle e_2, e_3, e_4, e_5 \rangle$ as a Singer cycle. At the final stage, we use the fact that Alt $(q^4 - 1) = Alt(15)$ is not a section of $GL_5(2)$ to conclude that the set we have constructed is indeed beautiful.

Lemma 4.7.4. If the action is listed in Table 4.7.2, then the action is not binary.

Proof. We begin with the case when t = 2 for which we use magma. Let S be either $SL_{16}(2)$ or $SL_{9}(2)$ and let M be a maximal subgroup of G in the Aschbacher class \mathcal{C}_7 . With magma, we have first computed a Sylow 2-subgroup of M, say Q. Then, we have computed $P = N_G(N_G(Q))$ and we have found an element $g \in P$, with the property that

- $|M: M \cap M^g| = 294$ when $G = SL_9(2)$,
- $|M: M \cap M^g| = 588$ when $G = Aut(SL_9(2))$,
- $|M: M \cap M^g| = 11025$ when $G \in {SL_{16}(2), Aut(SL_{16}(2))}.$

In particular, in the faithful primitive action of S on the right cosets Ω of M, a point stabilizer has a suborbit Δ with the property that the action of M on Δ is permutation isomorphic to the action of M on the right cosets of $M \cap M^{g}$. We have constructed the permutation representation of M under consideration (that is, on the right cosets of $M \cap M^g$) and we have verified that in this action $(M: M \cap M^g)$ contains a beautiful subset of cardinality 7 when $S = SL_9(2)$ and cardinality 5 when $S = SL_{16}(2)$. This immediately yields that the action of M on Δ is not binary and hence the action of S on Ω is also not binary.

If t > 2, then we use the result for t = 2 combined with Lemma 4.7.2.

Case $S = SU_n(q)$ 4.7.2

In this case [54, Table 3.5.B] allows us to assume that $m \ge 3$, and that $(q, m) \ne (2, 3)$.

Lemma 4.7.5. In this case either Ω contains a beautiful subset or else the action is listed in Table 4.7.3.

Proof. Our method here will be very reminiscent of that used in Lemma 4.4.3. We start by writing $W = W_1 = \cdots = W_t$, and letting $\mathcal{B}_1 = \{u_1, \ldots, v_1, \ldots, x\}$ be a hyperbolic basis for W_1 (omitting x if m is even). Taking pure tensors we obtain a hyperbolic basis, \mathcal{B} , for V, and we let M be the stabilizer of the associated tensor decomposition. Then $M \cap \overline{S} = \mathrm{PGU}_m(q) \mathrm{wr} \mathrm{Sym}(t)$.

If $q \ge 7$, then we consider subgroups U and T of $GU(\langle u_1, x, v_1 \rangle)$ defined as per (4.4.1) and (4.4.2). (In what follows, for $i \in \mathbb{Z}^+$, we write x^i to mean $\underbrace{x \otimes \cdots \otimes x}_{i}$.)

As in Lemma 4.4.3 we now split into two cases. If q is odd, then we take U_0 to be the subgroup of U obtained by requiring that $b \in \mathbb{F}_q$ and that $c = \frac{1}{2}b^2$; we define an isomorphic group in S: U_1 consists of those elements for which there exists $b \in \mathbb{F}_q$ such that

$$u_1 \otimes x^{t-1} \mapsto u_1 \otimes x^{t-1} + bx^t - \frac{1}{2}b^2v_1 \otimes x^{t-1},$$
$$x^t \mapsto x^t - bv_1 \otimes x^{t-1},$$
$$v_1 \otimes x^{t-1} \mapsto v_1 \otimes x^{t-1},$$

and all elements of $\langle u_1 \otimes x^{t-1}, x^t, v_1 \otimes x^{t-1} \rangle^{\perp}$ are fixed. Then U_1 is a subgroup of order q that is not contained in M. Now we take T_0 to be the subgroup of T obtained by requiring that $r \in \mathbb{F}_q$ and let $T_1 = T \circ 1 \circ \cdots \circ 1$, a group of order q-1 that normalizes U_1 and acts transitively on the set of non-trivial elements in U_1 .

If q is even, the set-up is slightly different but follows the procedure in Lemma 4.4.3 as above. In both cases, identifying Ω with conjugates of M we set $\Lambda = M^{U_1} \subset \Omega$, and see that S^{Λ} acts 2-transitively upon Λ , a set of size q. The usual argument shows that that any non-abelian simple section in M^{Λ} is isomorphic to a section of $\operatorname{GU}_3(q)$. By Lemma 2.1.1, for $q \geq 7$, we conclude that $\operatorname{Alt}(q-1)$ is not a section of M^{Λ} and so Λ is a beautiful subset, and Lemma 1.6.12 implies that the action is not binary.

For $q \in \{3, 4, 5\}$ we diverge from the argument given in Lemma 4.4.3, and we assume that $m \ge 4$ (the first line of Table 4.7.3 covers m = 3). We proceed as for $q \ge 7$ but we use the existence of a Frobenius group in $\operatorname{GU}_4(q)$ this time. We let $W_0 := \langle u_1, u_2, v_2, v_1 \rangle$ be a non-degenerate 4-subspace of W, and consider the group:

$$U \rtimes T = \left\langle \begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & -a^{q} \\ & & & 1 \end{pmatrix}, \begin{pmatrix} r & & & \\ & 1 & & \\ & & 1 & \\ & & & r^{-q} \end{pmatrix} \mid a, r \in \mathbb{K}, r \neq 0 \right\rangle.$$

Now we define T_0 to be the subgroup of $\mathrm{GU}(W)$ which stabilizes W_0 , whose action on W_0 is equal to the action of T, and which fixes W_0^{\perp} point-wise. Then we define

$$T_1 = (T_0 \otimes T_0 \otimes 1 \otimes \cdots \otimes 1) \cap S.$$

On the other hand we let x be an element of W_1 for which (x, x) = 1, and we define

$$V_0 := \langle u_1 \otimes u_2 \otimes x^{t-2}, u_2 \otimes u_2 \otimes x^{t-2}, v_2 \otimes v_2 \otimes x^{t-2}, v_1 \otimes v_2 \otimes x^{t-2} \rangle.$$

Observe that V_0 is a non-degenerate 4-subspace of V; indeed there exists an isomorphism between W_0 and V_0 which maps the listed ordered basis for W_0 to that of V_0 . We can, therefore, define U_1 to be the subgroup of SU(W) which stabilizes V_0 , whose action on V_0 is equal to the action of U, and which fixes V_0^{\perp} point-wise.

Observe that U_1 is of order q^2 , is contained in S but not in M, and is normalized by T_1 . Observe, moreover, that T_1 acts transitively on the set of non-identity elements of U_1 . Defining $\Lambda = M^{U_1} \subseteq \Omega$, we therefore conclude that S_{Λ} acts 2-transitively on Λ . The usual argument shows that any simple section in M^{Λ} is necessarily isomorphic to a section of $GU_4(q)$. However Lemma 2.1.1 implies that for $q \geq 3$, $Alt(q^2 - 1)$ is not a section of M^{Λ} , and so Λ is beautiful and we are done as before.

Finally, for q = 2, we assume that $m \ge 6$ (the cases where $m \le 5$ are listed in Table 4.7.3). We proceed as in the previous paragraph, using the 2-transitive group constructed in Lemma 4.4.3 for the q = 2 case. As there, the fact that Alt(15) is not a section of $GU_6(2)$ allows us to construct a beautiful subset. \Box

Lemma 4.7.6. If the action is listed in Table 4.7.3, then the action is not binary.

Group	Details of action
$\operatorname{Sp}_{2^t}(5)$	$m = 2, t \text{ odd: } M \triangleright PSp_2(5)^t$

Table 4.7.4: $C_7 - \text{Sp}_n(q)$ – Cases where a beautiful subset was not found.

Proof. Our method is entirely analogous to that used in Lemma 4.7.4. We begin with the case when t = 2.

Suppose, first, that $M \triangleright \mathrm{PSU}_3(q)^2$ and $S = \mathrm{SU}_9(q)$ with $q \in \{3, 4, 5\}$. Let $\{e, f, x\}$ and $\{v, w, y\}$ be hyperbolic bases for a Hermitian space of dimension 3 (where (e, f) and (v, w) are hyperbolic pairs and xand y are anisotropic); taking tensor products we obtain a hyperbolic basis \mathcal{B} for a 9-dimensional Hermitian space, and we obtain our embedding of M in S. We choose an order for \mathcal{B} as follows:

$$e\otimes v, e\otimes w, e\otimes y, x\otimes v, x\otimes w, f\otimes y, f\otimes v, f\otimes w, x\otimes y.$$

Define T to be the subgroup of M, whose elements when written with respect to \mathcal{B} consist of all diagonal matrices

diag
$$[a^q, a^{-1}, 1, a, a^{-q}, 1, a^q, a^{-1}, a^{1-q}],$$

with $a \in \mathbb{F}_{q^2}^*$. Observe that T normalizes and acts fixed-point-freely upon the group U, whose elements fix all elements of \mathcal{B} except $e \otimes y$ and $x \otimes w$, and for which there exists $a \in \mathbb{F}_{q^2}$ such that

$$e \otimes y \mapsto e \otimes y + ax \otimes v,$$

$$x \otimes w \mapsto x \otimes w - a^q f \otimes y.$$

Since U is not in M, we obtain, in the usual way, a set Λ of size q^2 , on which S_{Λ} acts 2-transitively. Now observe that an alternating section, Alt(t) of M satisfies $t \leq 7$, and so we conclude that M^{Λ} does not have a section $Alt(q^2 - 1)$. We conclude that the set Λ is a beautiful subset and Lemma 1.6.12 yields the result.

Next suppose that $M \triangleright \mathrm{PSU}_m(2)^2$ and $S = \mathrm{SU}_{m^2}(2)$ with $m \in \{4, 5\}$. In both cases we take a pair of hyperbolic bases $\{e_1, e_2, f_1, f_2\}$ and $\{v_1, v_2, w_1, w_2\}$ (adding in an anisotropic element when m = 5), and we take tensor products to obtain a hyperbolic basis, \mathcal{B} , for an m^2 -dimensional Hermitian space. Now M has a subgroup isomorphic to $A = \mathrm{SL}_2(4)$ that preserves the subspaces $\langle e_1 \otimes v_1, e_1 \otimes v_2 \rangle$ and $\langle f_1 \otimes w_1, f_1 \otimes w_2 \rangle$ and fixes all other elements of \mathcal{B} .

What is more A lies inside a subgroup $X \cong SL_3(4) \leq S$ that preserves the subspace $\langle e_1 \otimes v_1, e_1 \otimes v_2, f_1 \otimes v_1 \rangle$, and note that $X \not\leq M$. Then Lemma 1.6.10 implies that there is a subset Δ of Ω of size 16 on which S^{Δ} acts 2-transitively. Since $SU_5(2)$ does not contain a section isomorphic to Alt(16), we obtain that Δ is a beautiful subset and, as before, Lemma 1.6.12 yields the result.

Now for t > 2 we use Lemma 4.7.2 and the fact that the result is proved for t = 2.

4.7.3 Case $S = Sp_n(q)$

In this case [54, Table 3.5.C] implies that qt is odd, that m is even, that $t \ge 3$, and that $(m, q) \ne (2, 3)$.

Lemma 4.7.7. In this case either Ω contains a beautiful subset or else the action is listed in Table 4.7.4.

Proof. We start by writing $W = W_1 = \cdots = W_t$, and letting $\mathcal{B}_1 = \{u_1, \ldots, u_{m/2}, v_{m/2}, \ldots, v_1\}$ be a hyperbolic basis for W_1 . Taking pure tensors we obtain a hyperbolic basis, \mathcal{B} , for V, and we let M be the subgroup of G that stabilizes the associated tensor decomposition. Then $M \cap \overline{S} = (\mathrm{PGSp}_m(q) \mathrm{wr} \mathrm{Sym}(t)) \cap \overline{S}$.

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First suppose that $m \ge 4$. We define two subgroups of $\text{Sp}_m(q)$:

$$U := \left\{ \begin{pmatrix} 1 & a_1 & \cdots & a_{m-2} \\ & 1 & & & a_{m-2} \\ & & \ddots & & \vdots \\ & & & 1 & -a_1 \\ & & & & 1 \end{pmatrix} \middle| a_1, \dots a_{m-2} \in \mathbb{F}_q \right\},$$
$$T := \left\{ \begin{pmatrix} 1 & & \\ & A & \\ & & 1 \end{pmatrix} \middle| A \in \operatorname{Sp}_{m-2}(q) \right\}.$$

Our construction is inspired by the observation that T normalizes U and acts transitively on the set of non-trivial elements of U. We define $T_1 = T \circ 1 \circ \cdots \circ 1 < S$ and we define the group U_1 to be the set of elements for which there exist a_1, \ldots, a_{m-2} such that

$$u_{1}^{t} \mapsto u_{1}^{t} + a_{1}u_{2} \otimes u_{1}^{t-1} + \dots + a_{(m-2)/2}u_{m/2} \otimes u_{1}^{t-1} + a_{m/2}v_{m/2} \otimes u_{1}^{t-1} + \dots + a_{m-2}v_{2} \otimes u_{1}^{t-1},$$

$$v_{i} \otimes v_{1}^{t-1} \mapsto v_{i} \otimes v_{1}^{t-1} - a_{i-1}v_{1}^{t},$$

$$u_{i} \otimes v_{1}^{t-1} \mapsto u_{i} \otimes v_{1}^{t-1} + a_{m-i}v_{1}^{t},$$

for $i = 2, \ldots, \frac{m}{2}$, and all other elements of \mathcal{B} are fixed. Observe that U_1 is of order q^{m-2} , is contained in S but not in M, and is normalized by T_1 . Furthermore T_1 acts transitively on the set of non-identity elements of U_1 . Defining $\Lambda = M^{U_1} \subseteq \Omega$, we conclude that S_{Λ} acts 2-transitively on the elements of Λ . The usual argument shows that any simple section of M^{Λ} is necessarily isomorphic to a section of $\operatorname{Sp}_{m-2}(q)$. We conclude that either Λ is beautiful or else $\operatorname{Sp}_{m-2}(q)$ contains a section isomorphic to $\operatorname{Alt}(q^{m-2}-1)$, which is impossible by Lemma 2.1.1 (recall that q is odd).

We are left with the situation where m = 2, in which case we use the fact that a Borel subgroup of $GSp_2(q) = GL_2(q)$ has a 2-transitive action on q points. We use the basis $\mathcal{B}_1 = \{u_1, v_1\}$, and consider the group:

$$B = U \rtimes T = \left\langle \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}, \begin{pmatrix} r & \\ & s \end{pmatrix} \mid a, r, s \in \mathbb{F}_q, r \neq 0 \neq s \right\rangle.$$
(4.7.2)

Then we define

 $T_1 = (T \circ \cdots \circ T) \cap S.$

Next we define the group U_1 in S to be the set of elements for which there exists $b \in \mathbb{F}_q$ such that

$$u_1^t \mapsto u_1^t + bv_1 \otimes u_1^{t-1},$$

$$u_1 \otimes v_1^{t-1} \mapsto u_1 \otimes v_1^{t-1} + bv_1^t,$$

and all elements of $\langle u_1^t, u_1 \otimes v_1^{t-1} \rangle^{\perp}$ are fixed. Observe that U_1 is of order q, is contained in S but not in M, and is normalized by T_1 . We can use [54, Proposition 4.7.4] to check that T_1 acts transitively on the set of non-identity elements of U_1 . Defining $\Lambda = M^{U_1} \subseteq \Omega$, we conclude that S_{Λ} acts 2-transitively on the elements of Λ . As usual, either Λ is beautiful or else $\operatorname{Sp}_2(q)$ contains a section isomorphic to $\operatorname{Alt}(q-1)$. This yields the result for $q \geq 7$. We are left with the case listed in Table 4.7.4 (recall that (m, q) = (2, 3) is excluded).

Lemma 4.7.8. If the action is listed in Table 4.7.4, then the action is not binary.

Proof. If t = 3, then $S = \text{Sp}_8(5)$ and we use magma to verify the result. If t > 3, then we use the result for t = 3 combined with Lemma 4.7.2. Our application of Lemma 4.7.2 requires that we check the property listed at 2(c): suppose that G_0 has socle $\overline{S}_0 \cong \text{Sp}_8(5)$, that $k, I_1, \ldots, I_k, J_1, \ldots, J_k$ are as given in the lemma and that they satisfy the properties listed at 2(a) and 2(b) – our magma calculations confirm that

Group	Details of action
$\Omega_{3^t}(5)$	$m = 3: M \triangleright \Omega_3(5)^t$

Table 4.7.5: $C_7 - \Omega_n(q)$ – Cases where a beautiful subset was not found.

such cosets do exist. Suppose that the property listed at 2(c) is not satisfied, in which case there exists a group $K \cong PSp_2(5)$ that is a normal subgroup of the socles of $(G_0)_{I_1}, \ldots, (G_0)_{I_k}$. Then [54, Lemma 4.4.3] implies that $C_{\overline{S}_0}(K)$ is isomorphic to a subgroup of $O_4^+(5)$, which has socle isomorphic to $PSp_2(5) \times PSp_2(5)$. Since the socles of $(G_0)_{I_1}, \ldots, (G_0)_{I_k}$ are isomorphic to $PSp_2(5) \times PSp_2(5) \times PSp_2(5)$, we conclude that the socles of $(G_0)_{I_1}, \ldots, (G_0)_{I_k}$ are all equal and hence $I_1 = \cdots = I_k$. Then the property listed at 2(a) implies that $J_1 = \cdots = J_k$ and now the property listed at 2(b) yields a contradiction. We conclude, therefore, that the property listed at 2(c) is satisfied.

4.7.4 Case $S = \Omega_n(q)$, *n* odd

In this case note that m and q are odd, and [54, Table 3.5.D] implies that $(m, q) \neq (3, 3)$.

Lemma 4.7.9. In this case either Ω contains a beautiful subset or else the action is listed in Table 4.7.5.

Proof. We start by writing $W = W_1 = \cdots = W_t$, and letting $\mathcal{B}_1 = \{u_1, \ldots, v_1, \ldots, x\}$ be a hyperbolic basis for W_1 . Taking pure tensors we obtain a hyperbolic basis, \mathcal{B} , for V, and we let M be the subgroup of G that stabilizes the associated tensor decomposition. Then $M \cap \overline{S} = (\Omega_m(q) \operatorname{wr} \operatorname{Sym}(t)) \cap \overline{S}$.

First suppose that $q \ge 7$; here our method is very similar to that used in Lemma 4.7.5. We define analogues of the groups defined at (4.4.1) and (4.4.2): we consider subgroups of SO($\langle u_1, x, v_1 \rangle$) consisting of elements of the form

$$U = \left\{ \begin{pmatrix} 1 & b & -\frac{b^2}{2} \\ 1 & -b \\ & & 1 \end{pmatrix} \mid b \in \mathbb{F}_q \right\};$$
(4.7.3)

$$T = \left\{ \begin{pmatrix} r & & \\ & 1 & \\ & & r^{-1} \end{pmatrix} \mid r \in \mathbb{F}_q \text{ with } r \neq 0 \right\}.$$
(4.7.4)

Then $U \rtimes T$ is a Borel subgroup of $SO_3(q)$, and is a Frobenius group. We let $T_1 = (T \circ \cdots \circ T) \cap S$, a subgroup of M; we let U_1 be the group consisting of elements for which there exists $b \in \mathbb{F}_q$ such that

$$u_1 \otimes x^{t-1} \mapsto u_1 \otimes x^{t-1} + bx^t - \frac{1}{2}b^2v_1 \otimes x^{t-1},$$
$$x^t \mapsto x^t - bv_1 \otimes x^{t-1},$$
$$v_1 \otimes x^{t-1} \mapsto v_1 \otimes x^{t-1},$$

and all elements of $\langle u_1 \otimes x^{t-1}, x^t, v_1 \otimes x^{t-1} \rangle^{\perp}$ are fixed. Then U_1 is a subgroup of order q that is not contained in M. Now T_1 normalizes U_1 and acts transitively on the set of non-trivial elements in U_1 .

In the same way as before we obtain a beautiful subset, provided Alt(q-1) is not a section of $SO_3(q)$; this is true for $q \ge 7$ by Lemma 2.1.1.

Suppose that $q \in \{3, 5\}$ and $m \geq 5$. We define

$$T = \left\{ g \circ \underbrace{1 \circ \cdots \circ 1}_{t-1} \mid \begin{array}{c} g \in \Omega_m(q), \ x^g = x, \\ g \text{ stabilizes both } \langle u_1, \dots, u_{(m-1)/2} \rangle \end{array} \text{ and } \langle v_1, \dots, v_{(m-1)/2} \rangle \right\}.$$

Group	Details of action
$\Omega_{2^t}^+(5)$	U_1 symplectic: $M \triangleright \mathrm{PSp}_2(5)^t$
$\Omega_{8^t}^{+}(3)$	U_1 orthogonal: $M \triangleright \mathrm{P}\Omega_8^-(3)^t$
$\Omega^+_{4^t}(q)$	$q \in \{3,5\}, U_1 \text{ orthogonal: } M \triangleright P\Omega_4^-(q)^t$

Table 4.7.6: $C_7 - \Omega_n^+(q)$ – Cases where a beautiful subset was not found.

Now define U to be the set of elements g such that, for $i = 1, \ldots, \frac{m-1}{2}$, there exist $a_i \in \mathbb{F}_q$ such that

$$x \otimes u_1^{t-1} \mapsto x \otimes u_1^{t-1} + a_1 u_1^t + a_2 u_2 \otimes u_1^{t-1} + \dots + a_{(m-1)/2} u_{(m-1)/2} \otimes u_1^{t-1} \\ v_i \otimes v_1^{t-1} \mapsto v_i \otimes v_1^{t-1} - a_i x \otimes v_1^{t-1},$$

and all other members of \mathcal{B} are fixed. In exactly the same way as before, we see that T normalizes U, that T acts transitively on the set of non-trivial elements of U, that T is in M, and that U is not contained in M. Then, identifying Ω with conjugates of M, and setting $\Delta = M^U$, we conclude that Δ is a set of size $q^{(m-1)/2}$ whose set-wise stabilizer acts 2-transitively.

As usual, either Δ is a beautiful subset and we are done, or M^{Δ} has a section $Alt(q^{(m-1)/2} - 1)$, in which case $SO_m(q)$ also has such a section. This is not the case by Lemma 2.1.1.

The remaining case m = 3, q = 5 is in Table 4.7.5 (recall that (m, q) = (3, 3) is excluded).

Lemma 4.7.10. If the action is listed in Table 4.7.5, then the action is not binary.

Proof. When t = 2, we have $S = \Omega_9(5)$ and we use magma to verify the result, see Lemma 4.1.1. The remainder of the proof, for t > 2, proceeds using the result for t = 2 along with Lemma 4.7.2.

4.7.5 Case $S = \Omega_n^+(q)$

In this case there are two subfamilies, as listed in Table 4.7.1.

Lemma 4.7.11. In this case either Ω contains a beautiful subset or else the action is listed in Table 4.7.6.

Proof. Note that [10, Table 8.50] allows us to exclude n = 8 in all cases; in particular this means $n \ge 16$. We split into two cases.

First consider line 4 of table 4.7.1. In this case W is equipped with an alternating form, $M \cap \overline{S} = (\operatorname{PGSp}_m(q) \operatorname{wr} \operatorname{Sym}(t)) \cap \overline{S}$, and both m and qt are even. Furthermore in the case where q is even, [54, Tables 3.5.E and 3.5.I] (and the explanation on p.69) imply that $m \geq 6$.

Our method is virtually identical to that of Lemma 4.7.7. For m > 2 we proceed as before, except that we make a sign adjustment for the elements of U_1 .

We obtain the same conclusion as in Lemma 4.7.7 – the existence of a beautiful subset of size q^{m-2} – and we are done.

For m = 2 our proof is, again, the same as that of Lemma 4.7.7. Note that, by [54, Table 3.5.E], q is odd, $q \ge 5$; in addition we may assume that $t \ge 4$. We obtain a beautiful set except when q = 5, and this case is listed in Table 4.7.6.

Now consider line 3 of Table 4.7.1. Here q is odd, W is equipped with a symmetric form of type $\varepsilon \in \{+, -\}$ and $M \cap \overline{S} = (\operatorname{PO}_m^{\epsilon}(q) \operatorname{wr} \operatorname{Sym}(t)) \cap \overline{S}$. Furthermore [54, Table 3.5.E] implies that $m \geq 5 + \varepsilon 1$. This time $\mathcal{B}_1 = \{u_1, \ldots, u_{m/2-1}, v_1, \ldots, v_{m/2-1}, x, y\}$ is a hyperbolic basis for W if $\varepsilon = -$, while $\mathcal{B}_1 = \{u_1, \ldots, u_{m/2}, v_1, \ldots, v_{m/2}\}$ is a hyperbolic basis for W if $\varepsilon = +$. Taking pure tensors we obtain a basis, \mathcal{B} , for V, and we let M be the subgroup of G that stabilizes the associated tensor decomposition. Write k for the Witt index of W.

Assume first that $k \geq 3$. Consider the following group:

$$T = \left\{ g \circ \underbrace{1 \circ \cdots \circ 1}_{t-1} \mid \begin{array}{c} g \in \Omega_m^{\varepsilon}(q), \ u_k^g = u_k, \ v_k^g = v_k \\ g \text{ stabilizes both } \langle u_1, \dots, u_{k-1} \rangle \text{ and } \langle v_1, \dots, v_{k-1} \rangle \end{array} \right\}.$$

Now define U to be the set of elements g such that, for i = 1, ..., k - 1, there exist $a_i \in \mathbb{F}_q$ such that

$$u_k \otimes u_1^{t-1} \mapsto u_k \otimes u_1^{t-1} + a_1 u_1^t + a_2 u_2 \otimes u_1^{t-1} + \dots + a_{k-1} u_{k-1} \otimes u_1^{t-1}$$
$$v_i \otimes v_1^{t-1} \mapsto v_i \otimes v_1^{t-1} - a_i v_k \otimes v_1^{t-1},$$

and all other members of \mathcal{B} are fixed. One can check directly that T normalizes U, that T acts transitively on the set of non-trivial elements of U, that T is in M, and that U is not in M. Then, identifying Ω with conjugates of M, and setting $\Lambda = M^U$, we conclude that Λ is a set of size q^{k-1} whose set-wise stabilizer acts 2-transitively.

Either Δ is a beautiful subset and we are done, or Alt $(q^{k-1}-1)$ is a section of SO^{ϵ}_m(q). By Lemma 2.1.1, since $k \geq 3$ and q is odd, the latter can only hold if q = 3 and $(m, \epsilon) = (8, -)$, a case listed in Table 4.7.6.

We are left with the possibility that $k \leq 2$, in which case $\varepsilon = -$ and $m \in \{4, 6\}$. Suppose, first, that m = 6. We define

$$T = \left\{ g \circ \underbrace{1 \circ \cdots \circ 1}_{t-1} \mid \begin{array}{c} g \in \Omega_m(q), \ x^g = x, y^g = y \\ g \text{ stabilizes both } \langle u_1, u_2 \rangle \text{ and } \langle v_1, v_2 \rangle \end{array} \right\}.$$

Note that we take x to satisfy $\varphi(x, x) = 1$. Now define U to be the set of elements g for which there exist $a_1, a_2 \in \mathbb{F}_q$ such that

$$x \otimes u_1^{t-1} \mapsto x \otimes u_1^{t-1} + a_1 u_1^t + a_2 u_2 \otimes u_1^{t-1}, v_1^t \mapsto v_1^t - a_1 x \otimes v_1^{t-1}, v_2 \otimes v_1^{t-1} \mapsto v_2 \otimes v_1^{t-1} - a_2 x \otimes v_1^{t-1}$$

and all other members of \mathcal{B} are fixed. As before we obtain a set Δ of size q^2 on which M^{Δ} acts 2-transitively. Either Δ is a beautiful subset and we are done, or Alt $(q^2 - 1)$ is a section of M^{Δ} , in which case SO₆⁻(q) also has such a section. This is not the case by Lemma 2.1.1.

Finally, suppose that m = 4 and take

$$U_0 \rtimes T_0 \cong [q] \rtimes C_{q-1} < \Omega_4^-(q) \leq \operatorname{Isom}(W).$$

Define $T = T_0 \circ \underbrace{1 \circ \cdots \circ 1}_{t-1}$. To define U, we first let $W_0 = W \otimes x^{t-1}$, and we define U to be the subgroup of

 $\Omega_m^-(q)$ which fixes, point-wise, every element of W_0^{\perp} , and whose action on W_0 is isomorphic to the action of U_0 on W. We can check that T and U have the same properties as before. Thus, following the same argument we are done unless $O_4^-(q)$ contains a section isomorphic to Alt(q-1). Now Lemma 2.1.1 shows this can only happen if $q \in \{3, 5\}$, as in Table 4.7.6.

Lemma 4.7.12. If the action is listed in Table 4.7.6, then the action is not binary.

Proof. We work through Table 4.7.6 line-by-line.

First consider Line 1. We apply Lemma 4.7.2 with $t_0 = 3$. In this case $\bar{S}_0 \cong \text{Sp}_8(5)$ and we confirm, using magma, that the actions of almost simple groups with socle \bar{S}_0 on maximal C_7 -subgroups of type $\text{Sp}_2(5) \text{ wr Sym}(3)$ are not binary. This yields the result for $t \ge 4$, as required. (Note that to confirm the property listed at 2(c) in Lemma 4.7.2 we argue as per Lemma 4.7.8.)

Next consider Line 2. If t = 2, then we let $\{u_1, u_2, u_3, v_1, v_2, v_3, x, y\}$ be a hyperbolic basis for W. Define

$$T_1 := \left\{ \begin{pmatrix} A & & \\ & A^{-T} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \mid A \in \mathrm{SL}_3(3) \right\},\$$

a subgroup of $\Omega_8^-(3)$, and let $T = T_1 \circ 1$, a subgroup of M. Now consider the subspace

$$X := \langle x \otimes u_1, u_1 \otimes u_2, u_2 \otimes u_1, u_3 \otimes u_1, x \otimes v_1, v_1 \otimes v_1, v_2 \otimes v_1, v_3 \otimes v_1 \rangle,$$

and observe that X is a non-degenerate subspace of V of type O_8^+ . We define U to be the set of elements in S for which there exist $a, b, c \in \mathbb{F}_q$ such that

 $\begin{aligned} x \otimes u_1 &\mapsto x \otimes u_1 + au_1 \otimes u_1 + bu_2 \otimes u_1 + cu_3 \otimes u_1, \\ v_1 \otimes v_1 &\mapsto v_1 \otimes v_1 - ax \otimes v_1, \\ v_2 \otimes v_1 &\mapsto v_2 \otimes v_1 - bx \otimes v_1, \\ v_3 \otimes v_1 &\mapsto v_3 \otimes v_1 - cx \otimes v_1, \end{aligned}$

and all elements of X^{\perp} are fixed. We see that U is a subgroup of S that is not contained in M, that T normalizes U and that T acts transitively on the set of non-identity elements of U. We obtain, in the usual way, a set Λ of size |U| = 27 on which G^{Λ} acts 2-transitively. Since Alt(26) is not a section of M, we obtain a beautiful subset and we conclude that the action is not binary by Lemma 1.6.12. For t > 2, we use the result for t = 2 along with Lemma 4.7.2.

Finally consider Line 3 and suppose, first, that t = 2, $S = \Omega_{16}^+(q)$ and $M \triangleright M_0 := P\Omega_4^-(q)^2$ with $q \in \{3,5\}$. We confirm the result with magma, in the following way. For all groups M, we calculate all actions of M on the cosets of a subgroup $H \leq M$ where (M : H) is odd. We find that the only binary actions occur when H = M.

Now, observe that |M : H| is even, thus a Sylow 2-subgroup, P, of M is normalized by a 2-group Q that strictly contains P. Let $x \in Q \setminus P$ and consider $H = M \cap M^x$. Our magma calculation implies that the action of M on $(M : M \cap M^x)$ is not binary, and so Lemma 1.6.1 implies that the action of G on (G : M) is not binary.

Again the proof for t > 2 is completed using the result for t = 2 and Lemma 4.7.2.

4.8 Family C_8

In this case M is the normalizer of a classical subgroup of G having the same natural module V. The possiblilities are listed in Table 4.8.1, taken from [54, §4.8]. Note that in case L, the classical subgroup M is centralized by a graph or graph-field automorphism of S (see Proposition 2.5.1), so M may not be almost simple.

case	type	conditions
L	$\operatorname{Sp}_n(q)$	$n \geq 4, n$ even
L	$\mathrm{SU}_n(q^{1/2})$	$n \geq 3, q$ square
L	$\Omega_n^\epsilon(q)$	$n \geq 3, q \text{ odd}$
S	$\Omega_n^\epsilon(q)$	$n \geq 4, q$ even

Table 4.8.1: Maximal subgroups in family C_8

The main result of this section is the following. The result will be proved in a series of lemmas.

Proposition 4.8.1. Suppose that G is an almost simple group with socle $\overline{S} = Cl_n(q)$, and assume that

- (i) $n \geq 3, 4, 4, 7$ in cases L, U, S, O respectively, and
- (ii) $\operatorname{Cl}_n(q)$ is not one of the groups listed in Lemma 4.1.1.

Let M be a maximal subgroup of G in the family C_8 . Then the action of G on (G:M) is not binary.

4.8.1 Case $S = SL_n(q)$

Lemma 4.8.2. Suppose that G is almost simple with socle equal to $PSL_n(q)$. If M is a maximal C_8 -subgroup, then the action of G on (G:M) is not binary.

Proof. By Lemma 4.1.1, we may assume that q > 25 when n = 3, that q > 9 when n = 4, that q > 7 when n = 5, that q > 4 when n = 6 and that q > 3 when n = 8. In what follows we suppose, for a contradiction, that the action of G on (G : M) is binary.

Suppose first that $n \ge 5$ when M is unitary, and $n \ge 7$ when M is orthogonal. We refer to Lemma 2.2.8; let x be the element listed there and observe that $C_S(x)$ is strictly greater than $C_M(x)$. We conclude that there is a suborbit, Δ , on which the action of M is isomorphic to the action of M on (M : H), where $H = M \cap M^g$ (for some $g \in C_S(x) \setminus C_M(x)$) is a subgroup of M containing the element x (and not containing $M \cap \overline{S}$). Lemma 1.6.1 implies that the action of M on (M : H) is binary, and now Lemma 2.2.8 implies that M must contain a section isomorphic to Sym(t) where t is as follows:

- 1. if M is unitary and n is even, then $t = q^{n-4}$; Lemma 2.1.1 implies a contradiction.
- 2. if M is unitary and n is odd, then $t = q^{n-3}$; Lemma 2.1.1 implies a contradiction.
- 3. if M is symplectic or orthogonal of type O⁺ with n even, then $t = q^{(n-2)/2}$; given the excluded cases for small n and q, Lemma 2.1.1 implies a contradiction.
- 4. if M is orthogonal and n is odd, then $t = q^{(n-3)/2}$; Lemma 2.1.1 implies a contradiction.
- 5. if M is orthogonal of type O⁻ with n even, then $t = q^{(n-4)/2}$; given the excluded cases for small n and q, Lemma 2.1.1 implies a contradiction.

Next assume that M is unitary and $n \in \{3, 4\}$. Here we adopt the same argument using Lemmas 2.2.10 and 2.2.11 in place of Lemma 2.2.8. Again Lemmas 2.1.1 and 4.1.1 yield a contradiction except when (n, q) = (4, 49). This final case was dealt with using magma and the permutation character method (a.k.a. Lemma 1.8.1).

It remains to consider the case where M is orthogonal (so q is odd) and $3 \le n \le 6$. First assume $n \in \{5, 6\}$. We think of V as a formed space with form, φ , preserved by M. Let $W = \langle e_1, e_2, f_1, f_2 \rangle$ be a non-degenerate subspace of V of type O_4^+ , and consider the group

$$T := \left\{ \begin{pmatrix} A \\ & A^{-t} \end{pmatrix} \mid A \in \mathrm{SL}_2(q) \right\},\$$

inside M; here we specify the action of the elements of T on W (with respect to the given basis) and we require that elements of T fix all elements of W^{\perp} . Then T is isomorphic to $SL_2(q)$.

Now let $\{x\}$ or $\{x, y\}$ be a basis for W^{\perp} and consider the group, U < S, consisting of elements for which there exist $a_1, a_2 \in \mathbb{F}_q$ such that

$$x \mapsto x + a_1 e_1 + a_2 e_2,$$

and all vectors in W are fixed, as is y if n = 6. Then U is a group of order q^2 , U does not lie in M, U is normalized by T, and T acts transitively on the set of non-identity elements of U. Thus, in the usual way, we obtain a set $\Lambda \subset \Omega$ of size q^2 such that G^{Λ} is 2-transitive. This set is a beautiful set unless $\operatorname{Alt}(q^2)$ is a section of $\operatorname{SL}_n(q)$; however, this is not the case by Lemma 2.1.1. Hence Lemma 1.6.12 implies the result. Finally, for $n \in \{3, 4\}$ we argue similarly. Let $W = \langle e_1, f_1 \rangle$ be a non-degenerate subspace of V of type O_2^+ , and consider the group

$$T:=\left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^* \right\},$$

inside M; as before we specify the action of the elements of T on W (with respect to the given basis) and we require that elements of T fix all elements of W^{\perp} . Then T is cyclic of order q-1. Again let $\{x\}$ or $\{x, y\}$ be a basis for W^{\perp} . Consider the group, U < S, consisting of elements for which there exists $a \in \mathbb{F}_q$ such that

$$x \mapsto x + ae_1,$$

and all vectors in W are fixed, as is y if n = 4. As before we obtain a beautiful set of size q unless Alt(q) is a section of $SL_n(q)$, which is not the case as q > 9.

4.8.2 Case $S = Sp_n(q)$

This case is line 4 of Table 4.8.1.

Lemma 4.8.3. Suppose that G is almost simple with socle $PSp_n(q)$, where q is even, $n \ge 4$ and $(n,q) \ne (4,2)$. Let $M = N_G(O_n^{\epsilon}(q))$ be a maximal C_8 -subgroup. Then the action of G on (G:M) is not binary.

Proof. First observe that for q = 2, the action of G on (G : M) is 2-transitive, hence is clearly not binary. So assume from now on that q > 2.

Suppose now that $\epsilon = +$, and let $H = O_n^+(2)$ be a subfield subgroup of M. There is a subfield subgroup $K = \operatorname{Sp}_n(2)$ of G containing H, and $K \cap M = H$. If we let $\Lambda = \{Mk : k \in K\}$, then the action of K on Λ is isomorphic to the action of $\operatorname{Sp}_n(2)$ on the cosets of $O_n^+(2)$, which is 2-transitive of degree $d := 2^{n/2-1}(2^{n/2}+1)$. As $\operatorname{Alt}(d)$ is not a section of $\operatorname{Sp}_n(q)$ by Lemma 2.1.1, it follows that Λ is a beautiful subset, giving the conclusion in this case.

Suppose finally that $\epsilon = -$. The argument of the previous paragraph does not work, as $O_n^-(2^a)$ does not possess a subfield subgroup $O_n^{\pm}(2)$ if a is even, so we use a different argument.

For $n \ge 8$, let $x \in M$ be the element defined in Lemma 2.2.8. This has larger centralizer in G than in M, so we can choose $g \in C_G(x) \setminus M$. Then $x \in M \cap M^g$, so Lemmas 2.2.8 and 2.1.1 imply that the action of M on $(M : M \cap M^g)$ is not binary.

Next suppose that n = 6, so $M \triangleright \Omega_6^-(q) \cong \text{PSU}_4(q)$. This time we use the element $x = \text{diag}(1, 1, a, a^{-1})$ of $\text{PSU}_4(q)$, defined in Lemma 2.2.11, where $a \in \mathbb{F}_q$ has order q - 1. This acts as $\text{diag}(1, 1, a, a, a^{-1}, a^{-1})$ in $\Omega_6^-(q)$, so there exists $g \in C_G(x) \setminus M$. Now, provided q > 8, we finish the proof as above, using Lemmas 2.2.11 and 2.1.1. If q = 8, then we use the same argument with Lemma 2.2.11 and the fact that Alt(8) is not a section of $\text{PSU}_4(8)$; if q = 4, then the result follows from Lemma 4.1.1.

Finally, suppose that n = 4, so $M \triangleright \Omega_4^-(q) \cong PSL_2(q^2)$. This time we use the element x defined in Lemma 2.2.4 in exactly the same way as in the previous paragraph to obtain the conclusion.

4.9 Family S

Let us first define the family S of subgroups of classical groups. Let G be an almost simple group with socle $\operatorname{Cl}_n(q)$, a classical simple group with associated natural module V of dimension n over \mathbb{F}_{q^u} , where u = 2 if $\operatorname{Cl}_n(q)$ is unitary and u = 1 otherwise. We say that a subgroup M of G is in the family S if the following hold:

- (a) M is almost simple, with socle M_0 ,
- (b) the action of the preimage of M_0 on V is absolutely irreducible, and cannot be realised over a proper subfield of \mathbb{F}_{q^u} ,
- (c) M_0 is not contained in a member of the family \mathcal{C}_8 of subgroups of G.

In this section we prove the following result. We shall adopt the assumptions on the dimension n made at the beginning of Section 4.1.1.

Proposition 4.9.1. Suppose that G is an almost simple group with socle $\overline{S} = Cl_n(q)$, and assume that

- (i) $n \geq 3, 4, 4, 7$ in cases L, U, S, O respectively, and
- (ii) $\operatorname{Cl}_n(q)$ is not one of the groups listed in Lemma 4.1.1.

Let M be a maximal subgroup of G in the family S. Then the action of G on (G:M) is not binary.

Note that, in all sections up to this point we have assumed (as stipulated in Section 4.1.1) that if $\bar{S} = P\Omega_8^+(q)$, then G does not contain a triality automorphism. In the current section we shall drop this assumption. To clarify what we mean by assuming that "M is in the family S" in this special case: we are allowing the possibility that $\bar{S} = P\Omega_8^+(q)$, that G contains a triality automorphism, that M is almost simple with socle M_0 , and that M_0 satisfies the defining conditions (a,b,c) given above for the family S.

We have a number of strategies, which we outline first.

4.9.1 Strategies

Strategy 1: Subgroups containing centralizers

This strategy is based on the following definition, the value of which is demonstrated in the ensuing proposition. It will be used for the case where the socle M_0 of M is an alternating group.

Definition 4.9.2. Let *L* be a simple group and *r* a positive integer. We say that *L* satisfies Property(r) if there exists an element $x_r \in L$ of order *r* such that the following hold for any almost simple group *M* with socle *L*:

- (1) $Z(C_M(x_r)) = \langle x_r \rangle;$
- (2) for any core-free subgroup H of M such that $C_L(x_r) \leq H$, the action of M on (M:H) is not binary;
- (3) If $\langle x_r \rangle \leq N \triangleleft C_M(x_r)$ with $C_M(x_r)/N$ solvable, then N contains $C_L(x_r)$.

Lemma 4.9.3. Let G be an almost simple group with socle a classical group $G_0 = \text{Cl}_n(q)$, and suppose M is a maximal subgroup of G in the family S, with socle L. Assume that L satisfies Property(r) for each $r \in \{3, 5, 7, 11, 13\}$. Then the action of G on (G : M) is not binary.

Proof. Let G, G_0, L and M be as in the statement, and let $q = p^a$ with p prime. Consider Property(r), satisfied by L for $r \in \{3, 5, 7, 11, 13\}$. If $r \neq p$, then x_r is a semisimple element of G, and the structure of $C := C_G(x_r)$ is described in [47, Thm. 4.2.2]: there is a normal subgroup C^0 of C that has a non-trivial central torus T_r containing x_r , and such that C/C^0 is solvable (C^0 is called the connected centralizer in [47, 4.2.2]).

Suppose that $T_r \neq \langle x_r \rangle$, and let $g \in T_r \setminus \langle x_r \rangle$ and $H = M \cap M^g$. Then g centralizes $C^0 \cap M$, a normal subgroup of $C_M(x_r)$ with solvable quotient, and hence H contains $C_L(x_r)$ by condition (3) in the definition of Property(r). If also H contains L, then $g \in N_G(L) = M$, and so $g \in Z(C_M(x_r))$, which is a contradiction as $Z(C_M(x_r)) = \langle x_r \rangle$ by assumption (1) in the definition of Property(r). As M is almost simple with socle L, H is core-free in M, and so Property(r) implies that (M, (M : H)) is not binary, whence also (G, (G : M)) is not binary, as required. Hence we may assume from now on that

if
$$p \neq r$$
, then $T_r = \langle x_r \rangle$. (4.9.1)

CASE G_0 SYMPLECTIC OR ORTHOGONAL. Assume that $G_0 = PSp_n(q)$, or $P\Omega_n^{\pm}(q)$ with n even, or $P\Omega_n(q)$ with n odd. If $p \neq 3$, then the torus T_3 has order divisible by $\frac{q-\epsilon}{d}$, where $\epsilon = \pm 1$, $q \equiv \epsilon \mod 3$ and d is 1,2 or 4 (it can only be 4 in the orthogonal case). By (4.9.1) we have $|T_3| = 3$. Hence we see that

$$q = 2, 4, 5, 7, 11, 13$$
 or 3^a .

If q = 7, 13 or 3^a with a > 2, then the torus T_5 has order greater than 5, so these possibilities are excluded by (4.9.1).

Now consider the torus T_7 . For q = 4, 5, 9 or 11, this has order divisible by $\frac{q^3-\delta}{e}$, where $q^3 \equiv \delta = \pm 1 \mod 7$ and $e \in \{1, 2, 4\}$, and so $|T_7| > 7$, contradicting (4.9.1).

We are left with the cases q = 2 or 3. For these we consider T_{11} , which has order divisible by $2^5 + 1$ or $\frac{3^5-1}{2}$, respectively, again contrary to (4.9.1). This completes the proof for the case of symplectic and orthogonal groups.

CASE G_0 LINEAR. Now assume that $G_0 = \text{PSL}_n(q)$. Suppose $p \neq 3$ and let $q \equiv \epsilon \mod 3$ with $\epsilon = \pm 1$. Consider Property(3). A preimage of the element x_3 in $\text{SL}_n(q)$ acts on $\bar{V} = V_n(q) \otimes \bar{\mathbb{F}}_q$ with at most three eigenspaces. Hence the central torus T_3 (of order 3 by (4.9.1)) in $C_G(x_3)$ has order either $q - \epsilon$ or $\frac{q-\epsilon}{(n,q-1)}$, and so one of the following holds:

(i)
$$q = 2, 4$$
 or 5,

(ii)
$$\epsilon = 1$$
 and $\frac{q-1}{(n,q-1)} = 3$,

(iii)
$$q = 3^a$$
.

Now consider Property(5), assuming $p \neq 5$. As above, T_5 has order q-1 or $\frac{q-1}{(n,q-1)}$ (if $q \equiv 1 \mod 5$), order $\frac{q+1}{c}$ with $c \in \{1,2\}$ (if $q \equiv -1 \mod 5$), and order $\frac{q^4-1}{(q-1)c}$ (if $q \equiv \pm 2 \mod 5$). Since $|T_5| = 5$, it follows that one of the following holds:

(iv) q = 4, 5 or 9,

(v)
$$q = 5^{2k}$$
 and $\frac{q-1}{(n,q-1)} = 3$,

(vi)
$$q = 3^{4k}$$
 and $\frac{q-1}{(n,q-1)} = 5$.

Now Property(7) rules out all possibilities except for q = 4, since for all the other cases we must have $|T_7| > 7$. Finally, Property(11) excludes q = 4, since in this case T_{11} must have order divisible by $\frac{4^5-1}{3}$.

CASE G_0 UNITARY. To complete the proof of the theorem, assume that $G_0 = \text{PSU}_n(q)$. This is very similar to the linear case. If $p \neq 3$ then consideration of Property(3) shows that either $q \in \{2, 4, 7\}$ or $q \equiv -1 \mod 3$ and $\frac{q+1}{(n,q+1)} = 3$. Then Property(5) implies that one of the following holds:

(i) q = 2, 4 or 11,

(ii)
$$q = 5^k$$
 and $\frac{q+1}{(n,q+1)} = 3$,

(iii)
$$q = 3^{2k}$$
 and $\frac{q+1}{(n,q+1)} = 5$

Now Property(7) excludes all possibilities except for q = 2, and that is ruled out by Property(13).

Strategy 2: Odd degree actions

Our second strategy has been used already at various stages; however it is convenient to write down an explicit statement. Note that the proof of the next proposition appeals to results of [73] and [51] which detail, amongst other things, all primitive actions of odd-degree for all of the almost simple groups. Note that both sources omit one family of actions for the groups with socle ${}^{2}G_{2}(q)$ (here the stabilizer contains a group isomorphic to $(2^{2} \times D_{\frac{1}{2}(q+1)}): 3$), however this omission does not affect the proof given below.

Lemma 4.9.4. Let G be an almost simple group with socle a classical group $G_0 = \operatorname{Cl}_n(q)$, not one of the groups listed in Lemma 4.1.1. Suppose M is a maximal subgroup of G in the family S, with socle M_0 . Then one of the following occurs:

- 1. the action of G on (G:M) is not binary;
- 2. there is a suborbit on which M has a transitive faithful action of odd-degree that is binary;
- 3. $(G_0, M \cap G_0) = (P\Omega_7(p), Sp_6(2))$ or $(P\Omega_8^+(p), \Omega_8^+(2))$, where p is an odd prime.

Proof. Suppose that the third listed possibility does not occur. Then [73] (or, equivalently, [51]) implies that |G:M| is even. Thus there exists a non-trivial odd subdegree for the action of G on (G:M). Hence there exists $g \in G \setminus M$ such that $|M:M \cap M^g|$ is odd; moreover, by the maximality of M in G, $M_0 \not\leq M \cap M^g$, so the action of M on $(M:M \cap M^g)$ is faithful.

Now suppose, in addition, that the second listed possibility does not occur, so that the action of M on $(M : M \cap M^g)$ is not binary. Then Lemma 1.6.1 implies that the action of G on (G : M) is not binary, and so the first listed possibility occurs, as required.

Strategy 3: Using distinguished elements

The strategy here is used primarily for the situation where M_0 is a group of Lie type. It has already been used multiple times for other families, and was briefly discussed at the start of Chapter 2. We briefly summarise:

- 1. We pick a distinguished element $g \in M$ and show that, if H is any core-free subgroup of M that contains g, then the action of M on (M : H) is not binary. This was done in §2.2.
- 2. We give an upper bound for $|C_M(g)|$ and we use results of §2.4 to show that, in general, $|C_M(g)|$ is smaller than the smallest centralizer in G. We conclude that there exists $x \in C_G(g) \setminus C_M(g)$.
- 3. Now $M \cap M^x$ is a core-free subgroup of M that contains g. We conclude that the action of M on $(M: M \cap M^x)$ is not binary. Then Lemma 1.6.1 implies that the action of G on M is not binary.

We shall also need the well-known lower bounds for dimensions of cross-characteristic representations of groups of Lie type, taken from [65], with improvements as given in [100]:

Proposition 4.9.5. Let S be a simple group of Lie type over \mathbb{F}_r , not isomorphic to one of the following groups:

PSL₂(r) (r
$$\leq 9$$
), PSL₄[±](r) (r = 2, 3), $\Omega_8^+(2)$, $\Omega_7(3)$,
G₂(r) (r ≤ 4), ²E₆(2), F₄(2), ²F₄(2)', ²B₂(8).

If V is a non-trivial irreducible module for a quasisimple cover of S over a field of characteristic coprime to r, then dim $V \ge R(S)$, where R(S) is as given in Table 4.9.1.

S	$\operatorname{PSL}_d(r) \ (d \ge 3)$	$\mathrm{PSU}_d(r)$	$\mathrm{PSp}_{2k}(r)(r \text{ odd})$	$\mathrm{PSp}_{2k}(r)\left(r \text{ even}\right)$	$\mathrm{P}\Omega^{\epsilon}_{2k+y}(r) (y \le 1)$
R(S)	$\frac{r^d - r}{r - 1} - 1$	$\frac{r^d - 1}{r + 1}$	$\frac{1}{2}(r^k - 1)$	$\frac{(r^k-1)(r^k-r)}{2(r+1)}$	$\frac{(r^k-1)(r^{k-1}-1)}{r^2-1}$
S	$E_8(r)$	$E_7(r)$	$E_6^\epsilon(r)$	$F_4(r)$	$^2\!F_4(r)$
R(S)	$r^{29} - r^{27}$	$r^{17} - r^{15}$	$r^{11} - r^9$	$r^{8} - r^{6}$	$r^{5} - r^{4}$
S	$G_2(r)$	${}^{3}\!D_{4}(r)$	${}^{2}\!G_{2}(r)$	$^{2}B_{2}(r)$	
R(S)	$r^3 - r$	$r^{5} - r^{3}$	$r^2 - r$	$(r-1)\sqrt{r/2}$	

Table 4.9.1: Lower bounds for cross-characteristic representations

4.9.2 The case where M_0 is alternating

In this case, we use a combination of Strategies 1 and 2.

Lemma 4.9.6. Let G be an almost simple group with socle a classical group $G_0 = \operatorname{Cl}_n(q)$, and suppose M is a maximal subgroup of G in the family S, with socle $M_0 \cong \operatorname{Alt}(d)$ for some $d \ge 27$. Then the action of G on (G:M) is not binary.

Proof. We use Strategy 1: Lemma 4.9.3 yields the result provided we can verify Property(r) for r = 3, 5, 7, 11 and 13.

In every case, we take x_r to be the *r*-cycle (1, 2, ..., r). Then $C_M(x_r) \cong (\langle x_r \rangle \times \text{Sym}(d-r)) \cap M$ and $C_{M_0}(x_r) \cong \langle x_r \rangle \times \text{Alt}(d-r)$. Parts (1) and (3) in the definition of Property (*r*) follow immediately. Hence to prove the result we must prove part (2): if *H* is a core-free subgroup of *M* containing $C_{M_0}(x_r)$, then the action of *M* on (*M* : *H*) is not binary.

We claim that the group H satisfies

$$\langle x_r \rangle \times \operatorname{Alt}(d-r) \leq H \leq (\operatorname{Sym}(r) \times \operatorname{Sym}(d-r)) \cap M.$$

To see this observe that the first inclusion is true by definition; the second will follow if we can show that H is intransitive in the natural action on d points. Suppose, instead, that H is transitive. If H is imprimitive, then H is isomorphic to a subgroup of Sym(e) wr Sym(f) where ef = d. Then, since $\max\{e, f\} \leq \frac{d}{2}$, any alternating section of H is of form Alt(s) with $s \leq \frac{d}{2}$. But, since $d \geq 27$, $r \leq 13$ and H contains Alt(d-r), we have a contradiction and we conclude that H is primitive. But H contains a 3-cycle hence, by a classical theorem of Jordan, H contains Alt(d), a contradiction. Thus the claim follows.

Suppose, first, that $M = \operatorname{Alt}(d)$ and let $K = (\operatorname{Sym}(r) \times \operatorname{Sym}(d-r)) \cap M$. We have just seen that K contains H. Now Lemma 1.6.2 implies that if the action of K on (K : H) is not binary, then the result follows. The kernel of the action of K on (K : H) contains a subgroup isomorphic to $\operatorname{Alt}(d-r)$ and we see that the action of K on (K : H) is isomorphic to the action of $\operatorname{Sym}(r)$ on some subgroup H_1 that is the projection of H to $\operatorname{Sym}(r)$. Using magma we confirm that, for $r \in \{5, 7, 11, 13\}$, all such actions are not binary, provided H_1 is core-free. Thus we are left with the case where $H_1 = \operatorname{Alt}(r)$ or $\operatorname{Sym}(r)$ and we have

$$H = \operatorname{Alt}(r) \times \operatorname{Alt}(d-r)$$
 or $(\operatorname{Alt}(r) \times \operatorname{Alt}(d-r)).2$.

We repeat this analysis with M = Sym(d) and $K = \text{Sym}(r) \times \text{Sym}(d-r)$. In this case the kernel of the action of K on (K : H) is isomorphic to either Alt(d-r) or Sym(d-r) and we see that the action of K on (K : H) is isomorphic to the action of either Sym(r) or $\text{Sym}(r) \times C_2$ on some subgroup H_1 . This time magma confirms that in all but one case these actions are not binary, provided H_1 does not contain Alt(r).

Let us deal first with the one exceptional case in which H_1 does not contain Alt(r): here r = 5 and the action of K on (K : H) is isomorphic to the action of Sym $(5) \times C_2 = \langle (1, 2, 3, 4, 5), (4, 5), (6, 7) \rangle$ on $\langle (1, 2, 3, 4, 5), (2, 5)(3, 4)(6, 7) \rangle$. In particular, we can take H to contain

$$H_0 = \langle (1, 2, 3, 4, 5) \rangle \times \operatorname{Alt}(\{6, 7, 8, \dots, d\})$$

as an index 2 subgroup and we have $H = \langle H_0, (2,5)(3,4)(6,7) \rangle$. We will show directly that the action of M on (M:H) is not binary. We define

$$I_1 = J_1 = H;$$

$$I_2 = J_2 = H(2, 3, 4, 5, 6, 7);$$

$$I_3 = H(1, 3, 4, 5, 6, 7);$$

$$J_3 = H(1, 6, 7, 5, 4, 3).$$

In addition we set

$$g_{12} = (1), g_{13} = (1, 5, 4, 3, 2) \text{ and } g_{23} = (1, 5, 3, 6, 4).$$

Direct calculation confirms that for $i, j \in \{1, 2, 3\}$, $I_i^{g_{ij}} = J_i$ and $I_j^{g_{ij}} = J_j$; in other words, $(I_1, I_2, I_3) \simeq (J_1, J_2, J_3)$. Now suppose that there exists $g \in \text{Sym}(d)$ such that $I_i^g = J_i$ for $i \in \{1, 2, 3\}$. We note that the stabilizer in Sym(d) of an element in (M : H) contains a unique normal cyclic subgroup generated by a 5-cycle. For I_1 we can take this 5-cycle to be (1, 2, 3, 4, 5), for I_2 we can take this 5-cycle to be (1, 3, 4, 5, 6). Since $I_1 = J_1$ and $I_2 = J_2$ we conclude that g must normalize the two groups generated by these 5-cycles. Direct calculation confirms that g is, therefore, a subgroup of $\text{Sym}(\{7, 8, 9, \ldots, d\})$. But now we require that $I_3^g = J_3$; the stabilizer of I_3 (resp. J_3) contains a normal cyclic subgroup generated by (2, 4, 5, 6, 3) (resp. (1, 3, 4, 6, 2)) and g must conjugate the first subgroup to the second. But g clearly commutes with these subgroups and we have a contradiction. Thus $(I_1, I_2, I_3) \not\prec (J_1, J_2, J_3)$ and we are done.

We are left with the situation where

$$\operatorname{Alt}(r) \times \operatorname{Alt}(d-r) \le H \le \operatorname{Sym}(r) \times \operatorname{Sym}(d-r).$$

Observe that, for fixed d and r, there are five such groups. We now divide the proof in two parts, depending on whether H contains $C_M(x_r)$ or not.

SUPPOSE THAT H CONTAINS $C_M(x_r)$. We define a function from (M : H) to the power set of the conjugacy class x_r^M :

$$\psi: (M:H) \longrightarrow \mathcal{P}(x_r^M)$$
$$Hk \mapsto \omega_k := \{k_0^{-1} x_r k_0 \mid k_0 \in Hk\}.$$

Notice that $\omega_{x_r} = x_r^H$. We claim that the image, $\psi(M:H)$ is a partition of x_r^M . It is clear that

$$\bigcup_{X \in \psi(M:H)} X = x_r^M,$$

thus suppose that $\omega_{k_1} \cap \omega_{k_2} \neq \emptyset$. This implies that $(k'_1)^{-1}x_r(k'_1) = (k'_2)^{-1}x_r(k'_2)$ for some $k'_1 \in Hk_1, k'_2 \in Hk_2$. But this implies that $(k'_2)(k'_1)^{-1} \in C_M(x_r) < H$ and so $Hk'_1 = Hk'_2$ which means that $Hk_1 = Hk_2$ and so $\omega_{k_1} = \omega_{k_2}$, as required.

Now we define an action of M on $\psi(M:H)$ via

$$\omega_{k_1}^k = \{k^{-1}xk \mid x \in \omega_{k_1}\}.$$

This action is well-defined and is isomorphic to the action of M on (M : H).

Notice that x_r^M is the set of all r-cycles in Sym(d). We showed above that

$$\operatorname{Alt}(r) \times \operatorname{Alt}(d-r) \le H \le (\operatorname{Sym}(r) \times \operatorname{Sym}(d-r)) \cap M.$$

This implies that the partition $\psi(M:H)$ of x_r^M is a refinement of the partition where two r-cycles are in the same part if and only if they have the same underlying r-set.

Our method will vary slightly depending on precise properties of this partition. To divide our method into cases we define H_1 to be the projection of H onto $\text{Sym}(\{1,\ldots,r\})$ and we recall that H_1 is either Alt(r) or Sym(r).

CASE 1: x_r IS CONJUGATE TO x_r^{-1} IN H_1 . In this case we define

$$I_{1} = J_{1} = \left[(1, 2, 3, \dots, r) \right],$$

$$I_{2} = J_{2} = \left[\left(1, 2, \dots, \frac{r-1}{2}, r+1, r+2, \dots, \frac{3r+1}{2} \right) \right],$$

$$I_{3} = \left[\left(1, 2, \dots, \frac{r-1}{2}, \frac{3r+3}{2}, \frac{3r+5}{2} \dots, 2r+1 \right) \right],$$

$$J_{3} = \left[\left(r, r-1, \dots, \frac{r+3}{2}, \frac{3r+3}{2}, r+2, r+3, \dots, \frac{3r+1}{2} \right) \right]$$

where we use " $\begin{bmatrix} - \end{bmatrix}$ " to denote the part of $\psi(M:H)$ containing the listed cycle.

It is easy to see that $I \not\prec_3 J$: the cycles representing I_1, I_2, I_3 all move the points $1, 2, \ldots, \frac{1}{2}(r-1)$, whereas the cycles representing J_1, J_2, J_3 have no moved points in common.

To see that $I \simeq J$ we must define g_{13} and g_{23} such that $I_i^{g_{ij}} = J_i$ and $I_j^{g_{ij}} = J_j$. To this end, we set:

$$g_{13} = \left(1, r\right) \left(2, r-1\right) \cdots \left(\frac{r-1}{2}, \frac{r+3}{2}\right) \left(2r+1, \frac{3r+1}{2}\right) \left(2r, \frac{3r-1}{2}\right) \cdots \left(\frac{3r+5}{2}, r+2\right);$$

$$g_{23} = \left(1, \frac{3r+1}{2}\right) \left(2, \frac{3r-1}{2}\right) \cdots \left(\frac{r-1}{2}, r+2\right) \left(2r+1, r\right) \left(2r, r-1\right) \cdots \left(\frac{3r+5}{2}, \frac{r+3}{2}\right).$$

It is easy to check that these even permutations do the job; more specifically, we can see that the representative r-cycle listed above in the definition of I_i is mapped to either the representative r-cycle listed for J_i , or to its inverse.

CASE 2: x_r IS NOT CONJUGATE TO x_r^{-1} IN H_1 . Since $H_1 = \operatorname{Alt}(r)$ or $\operatorname{Sym}(r)$, we conclude that $H_1 = \operatorname{Alt}(r)$ with $r \equiv 3 \pmod{4}$. In particular $r \in \{3, 7, 11\}$ and $H_1 = \operatorname{Alt}(r)$.

Suppose, first, that r = 3. In this case H is the centralizer of a 3-cycle in M and the set $\psi(M : H)$ can be identified with set of 3-cycles in Alt(d). We define

$$I_1 = J_1 = g_{13} = (1, 2, 3),$$

$$I_2 = J_2 = g_{23} = (1, 2, 4),$$

$$I_3 = (2, 3, 4),$$

$$J_3 = (3, 1, 4).$$

Finally we define $g_{12} = 1$, and now one can check directly that, for all i, j such that $1 \le i < j \le 3$, we have $I_i^{g_{ij}} = J_i$ and $I_j^{g_{ij}} = J_j$. In particular $I \simeq J$.

We wish to show that $I \not\prec_3 J$. Suppose that $g \in M$ such that $I^g = J$. Clearly g must stabilize the set $\Delta = \{1, 2, 3, 4\}$. But now, since g must fix both I_1 and I_2 , we obtain that $g|_{\Delta} = 1$. This contradicts the fact that $I_3^g = J_3$ and the result follows.

Suppose, next, that $r \ge 7$. In this case we exhibit the presence of a beautiful subset and the result follows thanks to Lemma 1.6.12. We consider the set

$$\Lambda = \left\{ \begin{array}{c} \left[(1, 2, 4, 8, 9, 11, 15 \dots, r+8) \right], \left[(2, 3, 5, 9, 10, 12, 15, \dots, r+8) \right], \left[(3, 4, 6, 10, 11, 13, 15, \dots, r+8) \right], \\ \left[(4, 5, 7, 11, 12, 14, 15, \dots, r+8) \right], \left[(5, 6, 1, 12, 13, 8, 15, \dots, r+8) \right], \left[(6, 7, 2, 13, 14, 9, 15, \dots, r+8) \right], \\ \left[(7, 1, 3, 14, 8, 10, 15, \dots, r+8) \right] \end{array} \right\}$$

Note that the parts of the partition of x_r^M correspond to the conjugacy classes of *r*-cycles for the alternating group of the underlying *r*-set. In particular, for instance, the *r*-cycle $(1^{\tau}, 2^{\tau}, 4^{\tau}, 8^{\tau}, 9^{\tau}, 11^{\tau}, 15, \ldots, r+8)$ is in $[(1, 2, 4, 8, 9, 11, 15, \ldots, r+8)]$ (where τ is some permutation of $\{1, 2, 4, 8, 9, 11\}$) if and only if τ is even.

We have chosen seven r-tuples $(\mu_1, \ldots, \mu_6, 15, \ldots, r+8)$ that satisfy two properties:

(a) the seven 3-tuples given by (μ_1, μ_2, μ_3) form the lines of a Fano plane;

(b)
$$\mu_{i+3} = \mu_i + 7$$
 for $i = 1, 2, 3$.

It is clear that a group preserving Λ must stabilize the set $\{15, \ldots, r+8\}$; in addition we claim that if $g \in M_{\Lambda}$, then $\mu_{i+3}^g = \mu_i^g \pm 7$ for i = 1, 2, 3. To see this, let $g \in M_{\Lambda}$ and observe that, for $i = 1, \ldots, 3$, the number $\mu_i + 7$ is the only one that occurs in every listed tuple where μ_i occurs. Thus $(\mu_i + 7)^g$ must occur in every listed tuple where $\mu_i^g = 1$ as required.

These two properties allow us to conclude that M^{Λ} is isomorphic to a subgroup of $GL_3(2)$ and so, in particular, does not contain Alt(7). We wish to show that, in fact, $M^{\Lambda} = GL_3(2)$ and the result will then follow.

Write Λ_1 for the set of seven 3-tuples obtained by projecting the listed tuple in each element of Λ onto its first three entries; similarly Λ_2 is the set of seven 3-tuples obtained by projecting the listed tuple in each element of Λ onto entries 4,5,6. Both Λ_1 and Λ_2 correspond to Fano planes. Let θ_1 be a permutation of $\{1, \ldots, 7\}$ corresponding to an automorphism of the Λ_1 -Fano plane and let θ_2 be a permutation of $\{8, \ldots, 14\}$ corresponding to the automorphism of the Λ_2 -Fano plane obtained by increasing each entry in the cycle notation of θ_1 by 7. Now the permutation $\theta_1\theta_2$ is an element of Alt(14).

Consider the image of a listed tuple λ under $\theta_1\theta_2$. The projection of this image onto its first three entries yields a 3-tuple which is a permutation of the 3-tuple given by the first three entries of the listed permutation in an element of Λ . Likewise the projection of this image onto entries 4, 5, 6 yields a 3-tuple which is a permutation of the 3-tuple given by entries 4,5 and 6 of the same listed permutation. The two resulting permutations are of the same type and so, since $\theta_1\theta_2$ fixes the points $15, \ldots, r+8$, we conclude that $\lambda^{\theta_1\theta_2}$ is of the form $(\mu_1^{\tau}, \mu_2^{\tau}, \ldots, \mu_6^{\tau}, 15, \ldots, r+8)$ where $(\mu_1, \ldots, \mu_6, 15, \ldots, r+8)$ is one of the listed permutations and $\tau = \theta_1\theta_2$ is even. In particular $\lambda^{\theta_1\theta_2}$ lies in an element $[\gamma]$ of Λ , where γ is one of the listed tuples. Now both $[\lambda]$ and $[\gamma]$ are conjugacy classes in conjugates, H_{λ} and H_{γ} , of H. Then $H_{\lambda}^{\theta_1\theta_2} = H_{\gamma}$ and, since $\lambda^{\theta_1\theta_2} \in [\gamma]$ we conclude that $[\lambda]^{\theta_1\theta_2} = [\gamma]$. We conclude that $\theta_1\theta_2$ is in M_{Λ} and the result follows.

SUPPOSE THAT H CONTAINS $C_{M_0}(x_r)$ BUT NOT $C_M(x_r)$. In this case M = Sym(d) and H is one of the following groups:

$$\operatorname{Alt}(r) \times \operatorname{Alt}(d-r), \ \operatorname{Sym}(r) \times \operatorname{Alt}(d-r) \ \operatorname{or} \ (\operatorname{Alt}(r) \times \operatorname{Alt}(d-r)).2.$$

Observe, first, that if H < Alt(d), then the action of Alt(d) on cosets of H is considered above. Since we know that this action is not binary, the result follows by Lemma 1.6.2.

Thus we assume that $H \not\leq \operatorname{Alt}(d)$, in which case $H = \operatorname{Sym}(r) \times \operatorname{Alt}(d-r)$. But now the analysis of Case 2 for $r \in \{7, 11\}$ works, with the rôles of r and d-r interchanged. (Note that in Case 2 our only use of the fact that $r \in \{7, 11\}$ was when we needed $r \geq 6$ in order to make our definition of Λ work; in the current situation we just observe that $d-r \geq 6$ in all cases.)

For the alternating groups of degree less than 27, we shall use a magma computation together with the following result.

Lemma 4.9.7. Let G be a group with socle $G_0 = \operatorname{Cl}_n(q)$ $(q = p^a)$, a classical group, not one of the groups listed in Lemma 4.1.1. Suppose $M = N_G(M_0)$ is a maximal subgroup of G in the family S, where $M_0 = \operatorname{Alt}(r)$ with r odd, $r \leq 25$.

- (i) If r is prime, then for any $x \in M_0$ of order r, we have $C_G(x) \neq \langle x \rangle$.
- (ii) If r = 9,15 or 21, then $|G|_3 > |M|_3$; if r = 25, then $|G|_5 > |M|_5$.

Proof. (i) First assume $r \ge 11$. Then $n \ge r - \delta_{p,r}$ (see Lemma 2.1.1). The orders of centralizers of elements of prime order in classical groups are given in Tables B3 - B12 of [12], and it is straightforward to read off from these tables that for $n \ge r - \delta_{p,r}$, no such centralizer in $G_0 = \operatorname{Cl}_n(q)$ can have order equal to $r \in \{11, 13, 17, 19, 23\}$.

Now suppose r = 7. The modular character tables of Alt(7) and its covering groups are given in [50]. We have $n \ge 3$. If $n \ge 9$, then [12] gives a contradiction as above. And if n = 7 or 8, then the characteristic $p \ge 5$, and again we can use [12] to rule this out. Hence $n \le 6$.

If n = 3, then the only characteristic in which there is an irreducible modular representation is 5, yielding a maximal subgroup Alt(7) < PSU₃(5) - but this possibility is excluded by Lemma 4.1.1.

If n = 4, then $p \ge 5$ yields a contradiction using [12] as above; and $p \le 3$ is again excluded by Lemma 4.1.1.

If n = 5, then the only possible characteristic is p = 7 with Alt(7) < $\Omega_5(7)$; but then clearly $C_G(x) \neq \langle x \rangle$.

Finally consider n = 6. Here p = 2 is excluded by Lemma 4.1.1. If p = 3, then there are two possible embeddings, Alt(7) < $\Omega_6^-(3)$ or PSp₆(9). The first is out by Lemma 4.1.1, and the second is excluded using [12]. Finally, [12] rules out all possibilities with $p \ge 5$.

It remains to consider r = 5. Here $n \leq 6$. If n = 2 then $G_0 = \text{PSL}_2(q)$ and assuming $p \neq 5$, $C_{G_0}(x)$ has order $q \pm 1/(2, q - 1)$. Hence q = 9 or 11, excluded by Lemma 4.1.1. When n = 3, the embedding is $\text{Alt}(5) < \Omega_3(q) \cong \text{PSL}_2(q)$, already handled. When n = 4, the embeddings are $\text{Alt}(5) < \Omega_4^-(p) \cong \text{PSL}_2(p^2)$ and $\text{Alt}(5) < \text{PSp}_4(p)$; in the latter case $N_G(S)$ is non-maximal. If n = 5 then $\text{Alt}(5) < \Omega_5(q) \cong \text{PSp}_4(q)$. Finally, if n = 6, then $\text{Alt}(5) < \text{PSp}_6(p)$, and [12] gives a contradiction.

(ii) Suppose r = 9. Then $n \ge 8 - \delta_{p,3}$. If p = 3, the conclusion is clear, so assume $p \ne 3$. If n = 8 then $G_0 = P\Omega_8^+(q)$, which has order divisible by 3^5 , greater than $|\operatorname{Sym}(9)|_3 = 3^4$. And if $n \ge 9$ then $|G_0|$ is divisible by $\frac{1}{d} \prod_{i=1}^{4} (q^{2i} - 1)$ (where d = (2, q - 1)), hence is also divisible by 3^5 .

For r = 15 we have $n \ge 14 - \delta_{p,3} - \delta_{p,5}$ and so $|G_0|$ is divisible by $\frac{1}{d} \prod_{1}^6 (q^{2i} - 1)$, hence by $3^8 > |\operatorname{Sym}(15)|_3$. A similar argument works for r = 21 or 25; the only extra point to note is that if r = 25 and n = 24 then $G_0 = P\Omega_{24}^+(q)$ (rather than $P\Omega_{24}^-(q)$), and this has order divisible by $5^7 > |\operatorname{Sym}(25)|_5$. This completes the proof.

We can now complete the proof of Proposition 4.9.1 for the case of alternating groups:

Lemma 4.9.8. Let G be an almost simple group with socle a classical group $G_0 = \operatorname{Cl}_n(q)$, under the hypotheses of Propsition 4.9.1, and suppose M is a maximal subgroup of G in the family S, with socle $M_0 \cong \operatorname{Alt}(d)$ for some $5 \leq d \leq 26$. Then the action of G on (G:M) is not binary.

Proof. Recall our assumptions on n in the hypothesis: namely, $n \ge 3, 4, 4, 7$ in cases L, U, S, O respectively. Next we check using magma the following facts, where $6 \le d \le 26$:

- (a) every non-trivial binary action of Alt(d) has even degree;
- (b) for d even, every non-trivial binary action of Sym(d) has even degree;
- (c) every non-trivial binary action of M_{10} , PGL₂(9) and P Γ L₂(9) has even degree;
- (d) every non-trivial binary action of Alt(5) and Sym(5) has degree divisible by 5;
- (e) for d odd, every non-trivial binary action of Sym(d) (with core-free point stabilizer) has degree divisible by a prime s, as in the following table:

d	7	9	11	13	15	$\frac{17}{17}$	19	21	23	25
s	7	3	11	13	3	17	19	3	23	5

Given these facts, we can complete the proof as follows. Assume for a contradiction that the action of G on (G:M) is binary. We know by Lemma 4.9.4 that in this action, there is a non-trivial suborbit of odd degree on which the action of M is binary. Hence by Fact (a), M cannot be Alt(d) for d > 5. Thus, either d = 5, d = 6 or M is Sym(d). But now Fact (b) rules out the possibility that M is Sym(d) of even degree, and Fact (c) rules out all the possibilities when d = 6. Thus, in any case, d is odd and M = Sym(d) except, possibly, when d = 5 and M = Alt(5).

Suppose now that d is a prime (so is 5,7,11,13,17,19 or 23). Let $x \in M$ have order d. Then by Lemma 4.9.7(i), there exists $g \in C_G(x) \setminus M$. Thus there is a suborbit $(M : M \cap M^g)$ of size coprime to d. Now the action of M on this suborbit is not binary, by Facts (d) and (e). Hence G is not binary on (G:M) by Lemma 1.6.1, a contradiction.

The remaining cases d = 9, 15, 21 or 25 succumb to a similar argument. For these cases, we let P be a Sylow 3-subgroup of M (a Sylow 5-subgroup in the last case), and observe that by Lemma 4.9.7(ii), there exists $g \in N_G(P) \setminus M$. Hence the suborbit $(M : M \cap M^g)$ has size coprime to 3 (or 5), and the action of M on this is not binary, by Fact (e), giving a contradiction as before.

This completes the proof of Proposition 4.9.1 for the case where the socle M_0 is an alternating group.

4.9.3 The case where M_0 is sporadic

In this case we use Strategy 2 and some earlier computations with magma.

Lemma 4.9.9. Let G be an almost simple group with socle a classical group $G_0 = \operatorname{Cl}_n(q)$, and suppose M is a maximal subgroup of G in the family S, with socle M_0 a sporadic simple group. Then the action of G on (G:M) is not binary.

Proof. The proof follows immediately from Lemmas 2.3.2 and 4.9.4.

4.9.4 The case where M_0 is of Lie type

In this section we prove Proposition 4.9.1 for the case where M_0 is of Lie type. We will use the strategy outlined in §4.9.1; in particular we will make use of Propositions 2.4.1 and 4.9.5.

To start we use magma to rule out a number of small possibilities for M.

Lemma 4.9.10. Let M_0 be one of the simple groups listed in Lemma 2.3.1, and let G be an almost simple group with socle a classical group $G_0 = \operatorname{Cl}_n(q)$, not one of the groups listed in Lemma 4.1.1. Suppose M is a maximal subgroup of G in the family S, with socle M_0 . Then the action of G on (G:M) is not binary.

Proof. Lemmas 2.3.1 and 4.9.4 imply the result unless $(G_0, M_0) = (P\Omega_7(p), Sp_6(2))$ or $(P\Omega_8^+(p), \Omega_8^+(2))$, where p is an odd prime.

If $G_0 = P\Omega_7(p)$ and $M_0 = \operatorname{Sp}_6(2)$, then we let $g \in M_0$ be the element of order 3 defined in Lemma 2.2.8. In that lemma it is proved that if H is any subgroup of M that contains g, then the action of M on (M : H) is not binary. Suppose that there exists $x \in C_G(g) \setminus M$. Then the action of M on $(M : M \cap M^x)$ is not binary, and Lemma 1.6.1 yields the result. It remains to show, therefore, that $C_G(g)$ is strictly larger that $C_M(g)$. Direct calculation implies that $|C_M(g)| = 108$ and now Lemma 2.4.2 implies the result for q > 7. For $q \leq 7$, the result is confirmed with magma or by direct calculation.

If $G_0 = P\Omega_8^+(p)$ and $M_0 = \Omega_8^+(2)$, then we let g be the element of order 7 defined in Lemma 2.2.8. We proceed as before but must confirm that there exists $x \in C_G(g) \setminus M$. Using [10] we see that $G_0 \cap M = M_0$, and using [28] we see that $C_{M_0}(g) = \langle g \rangle$. Thus it is sufficient to prove that $C_{G_0}(g) \neq \langle g \rangle$. When p = 7, this is immediate; when $p \neq 7$, one can confirm this using, for instance, [12].

Let us next deal with some troublesome groups that are just a little too big to be easily handled with magma.

Lemma 4.9.11. Let M_0 be one of the following groups

$${}^{3}D_{4}(4), {}^{3}D_{4}(5), {}^{2}E_{6}(3).$$

Let G be an almost simple group with socle a classical group $G_0 = \operatorname{Cl}_n(q) (q = p^a)$, and suppose M is a maximal subgroup of G in the family S, with socle M_0 . Then the action of G on (G:M) is not binary.

Proof. (1) Suppose, first that $M_0 \cong {}^{3}D_4(5)$.

Claim: There exists an element, g, of order 24 in M_0 such that if M is almost simple with socle M_0 and H is any core-free subgroup of M containing g, then the action of M on (M : H) is not binary.

Proof of claim: Assume (M, (M : H)) is binary for some such H. We use the fact that if Alt(t) is a section of ${}^{3}D_{4}(5)$, then $t \leq 7$ by Lemma 2.1.2. We also use the existence of the following subgroup chain:

$$SL_3(5) < G_2(5) < {}^3D_4(5).$$

The group $SL_3(5)$ contains a maximal parabolic subgroup with unipotent radical E, an elementary abelian group of order 25. In addition $SL_3(5)$ contains an element g of order 24 that normalizes, and acts fixedpoint-freely upon, E. Since $E \rtimes \langle g \rangle$ is a Frobenius group, we conclude that either H contains E or else

the set of cosets H.E forms a beautiful subset of size 25, and Lemma 1.6.12 yields a contradiction. Thus H contains E.

We can now repeat the same argument with the "opposite" unipotent radical, E_1 . The same element g acts fixed-point-freely on E_1 and we can now also assume that H contains E_1 . Since $\langle E, E_1 \rangle = \text{SL}_3(5)$, we conclude that H contains $\text{SL}_3(5)$.

The group $SL_3(5)$ is a subgroup of $K := SL_3(5) : 2$, a maximal subgroup of $G_2(5)$ of maximal rank. We revisit our argument for the action of $G_2(q)$ on cosets of K – see Table 3.4.8 and the proof Propositions 3.4.1. As in that proof, we conclude either that we have a beautiful subset of order 25 (a contradiction) or else H contains $G_2(5)$. Thus the latter holds.

The group $L := G_2(5)$ is a maximal subgroup of ${}^{3}D_4(5)$ in family (V) of Theorem 3.1.1. Now we revisit our argument for the action of ${}^{3}D_4(q)$ on cosets of L – see Case (6) of the proof of Proposition 3.6.1. Once again we obtain a beautiful subset of order 25. We conclude, as required, that H contains M_0 . The claim is proved.

We now show that the claim implies the conclusion of the lemma. Suppose that there exists $x \in C_{G_0}(g) \setminus C_M(g)$. Then the claim implies that the action of M on $(M : M \cap M^x)$ is not binary and Lemma 1.6.1 yields the result.

Thus to complete the proof for this case we must check that the element x exists. Note that g is a regular semisimple element of ${}^{3}D_{4}(5)$ (which we can see by computing the action of G on the 8-dimensional module for ${}^{3}D_{4}(5)$). Hence $C_{M_{0}}(g)$ is a maximal torus of M_{0} , the sizes of which are listed in [57]. We conclude that $|C_{M_{0}}(g)| \leq 756$, and so $|C_{M}(g)| \leq 2268$. On the other hand, if $p \neq 5$ we have $n \geq 5^{5} - 5^{3}$ by Lemma 4.9.5, and if p = 5, then either $n = 8, q = 5^{3}$ or $n \geq 24$ by [54, 5.4.8]; hence $|C_{G_{0}}(g)| > 2268$ by Lemma 2.4.2.

(2) Suppose next that $M_0 \cong {}^{3}D_4(4)$. The proof here is very similar to the previous case.

Claim: There exists an element, $g \in M_0$ of order 15, such that if M is almost simple with socle M_0 and H is any core-free subgroup of M containing g, then the action of M on (M : H) is not binary.

Proof of claim: We use the fact that ${}^{3}D_{4}(4)$ contains a maximal group isomorphic to $J \cong PGL_{3}(4)$ (see [57]). The group J contains an element g of order 15 that normalizes and acts fixed-point-freely upon an elementary abelian group E of order 16. Assume that H is a subgroup of M containing g, for which the action of M on (M : H) is binary. We will show that H contains M_{0} .

Arguing exactly as in the previous case, we see that either there is a beautiful subset of size 16 or H contains the group PGL₃(4); hence we conclude the latter. Now we revisit our argument for the action of ${}^{3}D_{4}(q)$ on cosets of J – see Case (6) of the proof of Proposition 3.6.1. Once again we obtain a beautiful subset of order 16. We conclude, as required, that H contains M_{0} . The claim is proved.

We now show that the claim implies the conclusion of the lemma. This proceeds as before, relying on the existence of $x \in C_{G_0}(g) \setminus C_M(g)$. As before, g is regular semisimple and, using [57], we conclude that $|C_M(g)| \leq 945$. Now, as before, we find that $|C_{G_0}(g)| > 945$ and we are done.

(3) Suppose now that $M_0 \cong {}^2E_6(3)$. From [74, Table 5.1], we see that M_0 has a maximal subgroup $SU_3(27).3$. Write L for the simple subgroup $SU_3(27)$ of this. Let $x \in L$ be an element of order 26 which, written with respect to a hyperbolic basis $\{e_1, f_1, x\}$ of the corresponding unitary 3-space, is diagonal with entries $(t, t^{-1}, 1)$. We proceed in a series of steps.

Claim 1: If X is an almost simple group with socle SU₃(27), and Y < X is a core-free subgroup such that |X:Y| is not divisible by 3^2 , then the action of X on (X:Y) is not binary.

Proof of Claim 1: This is a magma computation.

Claim 2: Let $x \in L$ be the element defined above, and suppose $x \in H < M$ with H a core-free subgroup of M. Then the action of M on (M : H) is not binary.

Proof of Claim 2: Assume that the action of M on (M : H) is binary. We shall repeatedly use Lemma 2.1.2 which asserts that M does not contain a section isomorphic to Alt(d) for any $d \ge 12$.

Let $L_1 \cong SO_3(27) \cong PGL_2(27)$ be a subfield subgroup of L containing x. Then there are two Sylow 3-subgroups U_1, U_2 of L_1 such that $\langle x \rangle$ normalizes and acts transitively on the set of non-trivial elements of each of them. This implies (using Lemma 1.6.9) that either U_1 is in H, or else HU_1 is a subset of (M : H)on which the set-wise stabilizer acts 2-transitively. But in the latter case we obtain a beautiful subset, a contradiction to Lemma 1.6.12. Thus H contains U_1 and, similarly, U_2 . Thus $H \ge \langle U_1, U_2, x \rangle = L_1$.

Now let g be a diagonal element of $L = SU_3(27)$ of order $27^2 - 1$ and such that $g^{28} = x$. Then $\langle x \rangle$ acts transitively on the set of non-trivial elements of U_1^g and U_2^g . We deduce, as in the previous paragraph, that H contains U_1^g and U_2^g , and we conclude that $H \ge L = SU_3(27)$. From the information on the maximal subgroups of M given by Theorem 3.1.1, it follows that $L \le H \le N_M(L) \le L.6$.

Write $\Omega = (M : H)$. Now $N_M(L)$ acts transitively on $\operatorname{fix}_{\Omega}(L)$, so the number of fixed points $f = |\operatorname{fix}_{\Omega}(L)|$ divides 6. Also $|\Omega| = f + \sum_i |\Delta_i|$, where the Δ_i are the faithful *H*-orbits on Ω . Since $|\Omega|$ is divisible by 3^2 , it follows that there is an *H*-orbit Δ_i of size not divisible by 3^2 . Hence the action of *H* on Δ_i is not binary, by Claim 1. However the action of *M* on (M : H) is binary by assumption, so this is a contradiction to Lemma 1.6.1. This establishes Claim 2.

We now show that Claim 2 implies the conclusion of the lemma. Just as before, we need to show that there exists $x \in C_G(x) \setminus M$.

We start by computing the order of $C_M(x)$. The subgroup $L = SU_3(27)$ arises as the fixed point group of a Frobenius endomorphism of the algebraic group E_6 acting on a subsystem A_2^3 (see [74]). By [76, Prop. 2.1], the Lie algebra $L(E_6)$ restricts to A_2^3 as the sum of $L(A_2^3)$ together with the tensor product $V_1 \otimes V_2 \otimes V_2$ and its dual, where each V_i is a natural 3-dimensional module for one of the A_2 factors. The element xacts on this tensor product as $(t, t^{-1}, 1) \otimes (t^3, t^{-3}, 1) \otimes (t^9, t^{-9}, 1)$, so has fixed point space 0. Hence the fixed point space of x on $L(E_6)$ has dimension 6, and it follows that x is regular semisimple in M_0 , with centralizer $C_{M_0}(x)$ of order $27^2 - 1$.

On the other hand, for the classical group $G_0 = \operatorname{Cl}_n(q)$, $q = p^e$, we have $n \ge 27$ if p = 3 by [82], and $n \ge 3^{11} - 3^9$ if $p \ne 3$ by Proposition 4.9.5. Hence $|C_G(x)|$ is far greater than $|C_M(x)|$ by Proposition 2.4.1. This final contradiction completes the proof.

In light of the preceding two results, to prove Proposition 4.9.1 when M_0 of Lie type, we may exclude the following list of possibilities for M_0 :

$$\begin{aligned} &\text{PSL}_{2}(r) \ (r \leq 31), \ \text{PSL}_{3}(r) \ (r \leq 5), \ \text{PSL}_{4}(2), \\ &\text{PSU}_{3}(r) \ (r \leq 5), \ \text{PSU}_{4}(r) \ (r \leq 5), \ \text{PSU}_{5}(2), \ \text{PSU}_{6}(2), \\ &\text{PSp}_{4}(r) \ (r \leq 7), \ \text{PSp}_{6}(r) \ (r \leq 3), \ \text{PSp}_{8}(2), \\ &\Omega_{7}(r) \ (r \leq 9), \ \Omega_{8}^{-}(r) \ (r \leq 9), \ \Omega_{10}^{\pm}(2), \ \Omega_{10}^{-}(3), \\ &G_{2}(r) \ (r \leq 5), \ ^{3}D_{4}(r) \ (r \leq 5), \ F_{4}(r) \ (r \leq 3), \ ^{2}E_{6}(r) \ (r \leq 3), \ ^{2}F_{4}(2)'. \end{aligned}$$

For convenience, we restate Proposition 4.9.1 for this case:

Lemma 4.9.12. Let G be an almost simple group with socle a classical group $G_0 = \operatorname{Cl}_n(q)$, and suppose M is a maximal subgroup of G in the family S, with socle M_0 a group of Lie type not in the list (4.9.2). Then the action of G on (G:M) is not binary.

Note that the list (4.9.2) includes all the exceptions in the conclusions of Lemmas 2.2.6, 2.2.8, 2.2.9, 2.2.10, 2.2.11 and 2.2.15.

Now let $x \in M_0$ to be the element defined in these propositions, as detailed in Table 4.9.2.

We shall need upper bounds for the order of the centralizer of x in M, given in the next result.

Lemma 4.9.13. Let M be almost simple, with socle M_0 of Lie type over \mathbb{F}_r , and let $x \in M_0$ be as defined in Table 4.9.2.

- (i) For M_0 classical, we have $|C_M(x)| < N$, where N is as in Table 4.9.3.
- (ii) For M_0 exceptional, upper bounds for $|C_M(x)|$ are given in Lemmas 2.2.15 and 2.2.16.

M_0	x as in Lemma
$\mathrm{PSL}_d(r)$	2.2.6
$\mathrm{PSU}_d(r)$	$2.2.10 \ (d=3)$
	$2.2.11 \ (d=4)$
	$2.2.8 \ (d \ge 5)$
$PSp_d(r)$	2.2.8
$P\Omega_d^{\epsilon}(r)$	$2.2.8 \ (r \ \text{even})$
	$2.2.9 \ (r \ \text{odd})$
exceptional	2.2.15, 2.2.16

Table 4.9.2: Definition of the element $x \in M_0$

Proof. The argument for (i) is very similar for all types of classical groups. For $M_0 = \text{PSL}_d(r)$ with r > 2, the element x has preimage in $\text{SL}_d(r)$ of the form $\text{diag}(1, A, a^{-1})$ and the centralizer in $\text{GL}_d(r)$ of this element has order $(r-1)^2 |\text{GL}_1(r^{d-2})|$. Moreover x is not centralized by any non-trivial field automorphism, and can only be centralized by a graph automorphism if d = 3. It follows that for $d \neq 3$, $|C_M(x)| \leq C_{\text{PGL}_d(r)}(x) \leq (r-1)(r^{d-2}-1)$, and for d = 3 there is an extra factor of 2.

For $M_0 = \operatorname{PSU}_d(r)$ with $d = 2j + \delta$ (where $\delta = 1$ or 2), the centralizer in $\operatorname{GU}_d(r)$ of a preimage of x is contained in $\operatorname{GU}_2(r) \times \operatorname{GU}_\delta(r) \times \operatorname{GL}_1(r^{2(j-1)})$, and there are no outer automorphisms centralizing x unless d = 3 or 4 (in which case there is a graph automorphism); this leads to the bound in Table 4.9.3. Similarly for $M_0 = \operatorname{PSp}_{2k}(r)$, the centralizer is $\operatorname{Sp}_2(r) \times \operatorname{GL}_1(r^{k-1})$.

Next consider $M_0 = P\Omega_{2k}^-(r)$. If r is odd, then x has preimage diag $(I_4, \zeta, \zeta^{-1}, A, A^{-T})$, and the centralizer of this in $O_{2k}^-(r)$ is $O_4^-(r) \times \operatorname{GL}_1(r) \times \operatorname{GL}_1(r^{k-3})$, leading to the required bound. Similar considerations give the result for $P\Omega_{2k}^+(r)$ and $P\Omega_{2k+1}(r)$.

M_0	N
$\mathrm{PSL}_d(r)$	$r^{d-1}(1+\delta_{d,3})$
$\operatorname{PSU}_d(r) \ (d \ge 3)$	$2r^{d+3}, d \ge 6$ even
	$2r^{d+1}, d \ge 5 \text{ odd}$
	$2r^5, d = 4$
	$2r^2, d = 3$
$\mathrm{PSp}_{2k}(r) (k \ge 2)$	r^{k+2}
$P\Omega_{2k}^+(r) \ (k \ge 4)$	$2r^k$
$P\Omega_{2k}^{\overline{-}}(r) \ (k \ge 4)$	$2r^{k+4}$
$P\Omega_{2k+1}(r) \ (k \ge 3)$	$2r^{k+2}$

Table 4.9.3: Upper bounds for $|C_M(x)|$

For the proof of Lemma 4.9.12, we now adopt the following assumptions:

- (1) G is an almost simple group with socle $G_0 = \operatorname{Cl}_n(q)$ $(q = p^a)$, a classical group.
- (2) M is a maximal subgroup of G in the family S with socle M_0 , a group of Lie type over \mathbb{F}_r ; moreover, M_0 is not one of the groups in the list (4.9.2).
- (3) The action of G on (G:M) is binary.

We aim for a contradiction. This will prove Lemma 4.9.12.

Lemma 4.9.14. Adopt the above assumptions (1)–(3), and let $x \in M_0$ be as defined in Table 4.9.2. Then $C_G(x) = C_M(x)$.

Proof. Suppose there exists $g \in C_G(x) \setminus M$. Then $x \in M \cap M^g$. If $M_0 \leq M \cap M^g$, then $g \in N_G(M_0) = M$ which is not the case; hence $M \cap M^g$ is a core-free subgroup of M containing x. It now follows from the results listed in the last column of Table 4.9.2 that the action of M on $(M : M \cap M^g)$ is not binary. But then (G, (G : M)) is also not binary by Lemma 1.6.1, a contradiction.

Recall that the classical group $G_0 = \operatorname{Cl}_n(q)$ is defined over the field \mathbb{F}_q of characteristic p, while the subgroup M_0 is a group of Lie type over \mathbb{F}_r . At this point we divide the analysis into two cases: the cross-characteristic case (where $p \nmid r$) and the defining characteristic case (where $p \mid r$).

Lemma 4.9.15. Under the assumptions (1)–(3), the cross-characteristic case $p \nmid r$ does not occur.

Proof. Suppose $p \nmid r$. Then the following hold:

- (a) $n \ge R(M_0)$, as given in Table 4.9.1.
- (b) By Lemma 2.4.1, we have

$$|C_G(x)| > \frac{q^{\lceil (n-1)/2 \rceil}}{4} \left(\frac{q-1}{2qe(\log_q(2n)+4)}\right)^{1/2}.$$

(c) By Lemma 4.9.13 we also have $|C_M(x)| \leq N$, where N is as defined in Tables 4.9.3 for M_0 classical, and in Table 2.2.2 and Lemma 2.2.16 for M_0 exceptional.

By Lemma 4.9.14, it follows that N is greater than the right hand side of the inequality in (b). However, when combined with the inequality $n \ge R(M_0)$, it is routine to check that this leads to a contradiction.

It remains to handle the defining characteristic case, where $p \mid r$. Recall that $G_0 = \operatorname{Cl}_n(q) \ (q = p^a)$, and M_0 , the socle of the maximal subgroup M, is a group of Lie type over \mathbb{F}_r . Let V be the natural *n*-dimensional module for G_0 . According to [92, Cor. 6] together with [90], there are two possibilities:

- (A) $\mathbb{F}_r \supset \mathbb{F}_q$: in this case $r = q^k$ with $k \ge 2$, and the embedding $M_0 < G_0$ is as in [90, Table 1B], and takes the form $\operatorname{Cl}_d(q^k) < \operatorname{Cl}_{d^k}(q)$;
- (B) $\mathbb{F}_r \subseteq \mathbb{F}_q$: in this case the representation of M_0 on V corresponds to a restricted representation of the overlying simple algebraic group over $\overline{\mathbb{F}}_p$.

First we deal with Case (B).

Lemma 4.9.16. Under the assumptions (1)–(3), the defining characteristic case (B) above does not occur.

Proof. Assume we are in case (B), so that $M_0 = M_0(r) < G_0 = \operatorname{Cl}_n(q)$ with $\mathbb{F}_r \subseteq \mathbb{F}_q$. We shall use the lower bounds for the dimensions of restricted representations of simple algebraic groups given by [82]. For an integer d, define $\epsilon_{p,d}$ to be 1 if $p \mid d$, and 0 otherwise.

(α) Assume first that $M_0 = \text{PSL}_d^{\epsilon}(r)$. If the restriction of V to M_0 is self-dual, then G_0 is symplectic or orthogonal; otherwise, $G_0 = \text{PSL}_n^{\epsilon}(q)$. Hence using [82], we see that one of the following holds:

(i) $G_0 = PSp_n(q)$ or $P\Omega_n(q)$:

$$d = 2, n \ge 4, \text{ or}$$

$$d \ge 3, n \ge d^2 - 1 - \epsilon_{p,d};$$

(ii) $G_0 = \operatorname{PSL}_n^{\epsilon}(q)$:

 $d = 3, n \ge 6, \text{ or}$ $d = 4, n \ge 10, \text{ or}$ $d = 5, n = 10 \text{ or } n \ge 15, \text{ or}$ $d \ge 6, n \ge \frac{1}{2}d(d-1).$ Consider the element $x \in M_0$ defined in Table 4.9.2. By Lemma 4.9.13, we have $|C_M(x)| < N$, where N is as in Table 4.9.3; and by Proposition 2.4.1 we have $|C_G(g)| > f(n,q)$, where f(n,q) is as in Table 2.4.1. Hence Lemma 4.9.14 gives

$$N > f(n,q).$$

Combined with the lower bounds on n in (i) and (ii) above, this gives a contradiction except for the following cases:

(1) d = 3, p = 3, n = 7: here $M_0 = \text{PSL}_3^{\epsilon}(q) < G_0 = \Omega_7(q), q = 3^a$,

(2)
$$d = 5, n = 10, \epsilon = -$$
: here $M_0 = \text{PSU}_5(q) < G_0 = \text{PSU}_{10}(q)$.

In case (1) the element $x \in M_0$ has preimage diag $(1, t, t^{-1})$ in $\mathrm{SL}_3^{\epsilon}(q)$, where $t \in \mathbb{F}_q$ has order q-1. Moreover, the natural 7-dimensional module V for G_0 is a constituent of the adjoint module for M_0 , and hence the action of x on V is diag $(1, t, t, t^{-1}, t^{-1}, t^2, t^{-2})$. Clearly then, $C_M(x) \neq C_G(x)$, contradicting Lemma 4.9.14.

In case (2) above, x has preimage diag $(1, 1, 1, t, t^{-1})$ in $SU_5(q)$, and V is the exterior square of the 5dimensional natural module for M_0 . Hence x acts on V as diag $(1^4, t, t, t, t^{-1}, t^{-1}, t^{-1})$, and again $C_M(x) \neq C_G(x)$. This completes the proof for the case where $M_0 = PSL_d^{\epsilon}(r)$.

(β) Next assume that $M_0 = PSp_{2k}(r)$ with $k \ge 2$. In this case [82] gives

$$k = 2, n = 10 \text{ or } n \ge 12, \text{ or}$$

 $k = 3, n = 8 (p = 2) \text{ or } n = 14 - \delta_{p,3} \text{ or } n \ge 21, \text{ or}$
 $k \ge 4, n = 2^k (p = 2) \text{ or } n \ge k(2k - 1) - 1 - \epsilon_{p,k}.$

Again we have $|C_M(x)| < N$ with N is as in Table 4.9.3, and also $|C_G(g)| > f(n,q)$ with f(n,q) as in Proposition 2.4.1(ii). Hence Lemma 4.9.14 gives N > f(n,q), and combined with the above lower bounds for n, this yields a contradiction apart from the following cases:

- (1) k = 2, n = 10,
- (2) k = 3, n = 8 (p = 2) or $n = 14 \delta_{p,3}$,
- (3) k = 4, n = 16 (p = 2).

In case (1), the element $x \in M_0$ has preimage diag $(1, 1, t, t^{-1})$ in $\text{Sp}_4(q)$, where $t \in \mathbb{F}_q$ has order q - 1; also $p \neq 2$ and V is the symmetric square of the natural 4-dimensional module for M_0 . Hence x acts on V as diag $(1^4, t, t, t, t^{-1}, t^{-1}, t^{-1})$, and $C_M(x) \neq C_G(x)$, contradicting Lemma 4.9.14.

In case (2), x has preimage diag(1, 1, A, A^{-T}) in $\operatorname{Sp}_6(q)$, where $A \in \operatorname{GL}_2(q)$ has order $q^2 - 1$. If $n = 14 - \delta_{p,3}$, then V is a constituent of the exterior square of the natural 6-dimensional module, and so the action of x has diagonal blocks diag($A, A, A^{-T}, A^{-T}, \ldots$). But this implies that $C_G(x)$ contains a subgroup $\operatorname{SL}_2(q^2)$, so again $C_M(x) \neq C_G(x)$. And if n = 8 with p = 2, then V is a spin module for $M_0 = \operatorname{Sp}_6(q)$. Observe that x lies in a subgroup $\operatorname{Sp}_2(q) \times \operatorname{Sp}_4(q)$, and on a spin module this acts as $\operatorname{Sp}_2(q) \otimes \operatorname{Sp}_4(q)$. Hence x acts as $I_2 \otimes \operatorname{diag}(A, A^{-T})$, and so as before, $C_G(x)$ contains a subgroup $\operatorname{SL}_2(q^2)$.

A similar argument applies in case (3), where V is a spin module for $M_0 = \operatorname{Sp}_8(q)$. Here $x = \operatorname{diag}(1, 1, A, A^{-T}) \in M_0$, where $A \in \operatorname{GL}_3(q)$ has order $q^3 - 1$. This lies in a subgroup $\operatorname{Sp}_2(q) \times \operatorname{Sp}_6(q)$, hence as above, acts on a spin module as $(I_4, A, A, A^{-T}, A^{-T})$. Then $C_G(x)$ contains a subgroup $\operatorname{SL}_2(q^3)$, so again $C_M(x) \neq C_G(x)$.

(γ) Now consider the case where M_0 is an orthogonal group $P\Omega_d^{\epsilon}(r)$ with $d \geq 7$. In this case the dimension bounds are:

$$d = 7, n = 8 \text{ or } n \ge 21, \text{ or} d = 8, n = 8 \text{ or } n \ge 26, \text{ or} d \ge 9, n = 2^{\lfloor \frac{d-1}{2} \rfloor} \text{ or } n \ge \frac{1}{2}d(d-1) - 2$$

As above, the inequality N > f(n,q) now gives a contradiction apart from the following cases:

- (1) d = 7, n = 8,
- (2) d = 8, n = 8,
- (3) d = 9 or 10, n = 16.

Consider (1). Here $M_0 = \Omega_7(q) < G_0 = P\Omega_8^+(q)$, and the element $x = (1^3, \zeta, \zeta^{-1}, a, a^{-1}) \in M_0$ with $a, \zeta \in \mathbb{F}_q^{\times}$ of order q-1 and $\zeta \neq a^{\pm 1}$. Write $x = (I_3, X) \in \Omega_3(q) \times \Omega_4^+(q) < M_0$, where $X = (\zeta^{-1}, a, a^{-1})$. In $SL_2(q) \otimes SL_2(q) \cong \Omega_4^+(q)$, X takes the form $(\alpha, \alpha^{-1}) \otimes (\beta, \beta^{-1})$, where $\alpha\beta = a, \alpha\beta^{-1} = \zeta$. Hence in the spin representation on V, x acts as $(I_2 \otimes (\alpha, \alpha^{-1}), I_2 \otimes (\beta, \beta^{-1}))$. It follows that $C_G(x)$ contains a subgroup $(SL_2(q))^2$, so $C_M(x) \neq C_G(x)$, giving the usual contradiction.

Now consider (2). In this case $M_0 = P\Omega_8^-(q^{1/2}) < G_0 = P\Omega_8^+(q)$, where M_0 is the image of a subfield subgroup under a triality automorphism of G_0 . We can write the element x as (I_2, X) with $X \in \Omega_6^+(q^{1/2})$ given by Lemmas 2.2.8 (for p = 2) and 2.2.9 (for p odd). Arguing in similar fashion to the previous paragraph, we see that $C_G(x)$ contains a subgroup $SL_2(q^2)$ (p = 2) or $(SL_2(q))^2$ (p odd). Hence again $C_M(x) \neq C_G(x)$.

Finally, consider case (3). For d = 9 we have $M_0 = \Omega_9(q) < G_0 = P\Omega_{16}^+(q)$ with q odd, and V is a spin module for M_0 . We have $x = (1^3, \zeta, \zeta^{-1}, A, A^{-T}) \in M_0$, where $\zeta \in \mathbb{F}_q^{\times}$ has order q - 1 and $A \in GL_2(q)$ has order $q^2 - 1$. Then $x \in \Omega_3(q) \times \Omega_6^+(q) < M_0$, and this subgroup acts on the spin module V as $(V_2 \otimes V_4) \oplus (V_2 \otimes V_4^*)$, where the action of x on V_4 is computed via the isomorphism $\Omega_6^+(q) \cong SL_4(q)/\langle -I \rangle$. It follows that $C_G(x)$ contains $(SL_2(q^2))^2$, hence $C_M(x) \neq C_G(x)$. A very similar computation applies in the case where d = 10.

(δ) To complete the proof of the lemma, it remains to handle the case where M_0 is an exceptional group of Lie type over \mathbb{F}_r with $\mathbb{F}_r \subseteq \mathbb{F}_q$. From the bounds for the dimensions of restricted representations of groups of Lie type given in [82], it follows that either $n = R_0$, or $n \ge R$, where R_0, R are as in the following table:

We have $|C_M(x)| < N$ with N as given in Lemmas 2.2.15, 2.2.16, and also $|C_G(g)| > f(n,q)$ with f(n,q) as in Proposition 2.4.1(ii). Hence as usual, Lemma 4.9.14 gives N > f(n,q), and combined with the above bounds for n, this yields a contradiction apart from the cases where $n = R_0$ and

$$M_0 = E_6^{\epsilon}(q), \ F_4(q), \ G_2(q), \ ^2B_2(q) \text{ or } \ ^3D_4(q^{1/3})$$

(note that r = q in all but the last case, by the maximality of M).

If $M_0 = E_6^{\epsilon}(q)$, then n = 27, the module V has highest weight λ_1 in the usual notation, and is not self-dual, so that $G_0 = \text{PSL}_{27}^{\epsilon}(q)$. Now the inequality N > f(n,q), with f(n,q) as in Table 2.4.1, gives a contradiction.

Next consider $M_0 = F_4(q)$ with $n = 26 - \delta_{p,3}$. By the exclusions in the list (4.9.2), we have $q \ge 4$. The element $x \in M_0$ is as defined in Lemma 2.2.15: it lies in a subsystem subgroup $A_3 \cong SL_4(q)$ and takes the form diag(1, a, A) where A is a 2×2 matrix of order $q^2 - 1$ and determinant a^{-1} . The restriction $V \downarrow A_3$ is given in [76, Table 8.7]: in terms of highest weight modules, the composition factors are $V(\lambda_1)^2/V(\lambda_3)^2/V(\lambda_2)/0^{4-\delta_{p,3}}$. Here $W := V(\lambda_1)$ is the natural 4-dimensional A_3 -module, $V(\lambda_3) = W^*$, $V(\lambda_2) = \wedge^2 W$ and 0 is the trivial module. Hence we compute that dim $C_V(x) = 8 - \delta_{p,3}$, and so $C_G(x)$ has a subgroup $\Omega_7(q)$. However $C_M(x)$ has no such subgroup by Lemma 2.2.15, a contradiction.

Now let $M_0 = G_2(q)$ with $n = 7 - \delta_{p,2}$ (and $q \ge 7$ by the exclusions in (4.9.2)). Here G_0 is $\operatorname{Sp}_6(q)$ if q is even, and $\Omega_7(q)$ if q is odd. We have $x = \operatorname{diag}(1, a, a^{-1})$ in a subsystem subgroup $\operatorname{SL}_3(q)$, where $a \in \mathbb{F}_q^{\times}$ has order q-1. Hence x acts on V as $\operatorname{diag}(a, a, a^{-1}, a^{-1}, 1^{3-\delta_{p,2}})$, and it follows that $C_G(x)$ has a subgroup $\operatorname{Sp}_2(q) \times \operatorname{SL}_2(q)$ or $\Omega_3(q) \times \operatorname{SL}_2(q)$, whereas $C_M(x)$ has no such subgroup.

If $M_0 = {}^2B_2(q)$ with n = 4, then $G_0 = \operatorname{Sp}_4(q)$ and we have $|C_M(x)| = q - 1$ by Lemma 2.2.16, whereas $|C_G(x)| = (q-1)^2$. Finally, if $M_0 = {}^3D_4(q^{1/3})$ with n = 8, then $G_0 = \operatorname{P\Omega}_8^+(q)$ and x acts on V as diag $(a, a, a^{-1}, a^{-1}, 1^4)$; hence $C_G(x)$ contains $\Omega_4^+(q)$, so once again $C_M(x) \neq C_G(x)$. This completes the proof.

Table 4.9.4: Embeddings $M_0 = \operatorname{Cl}_d(q^k) < G_0 = \operatorname{Cl}_{d^k}(q)$ (k prime)

M_0	G_0	conditions
$\mathrm{PSL}_d^\epsilon(q^k)$	$\mathrm{PSL}_{d^k}^{\epsilon}(q)$	$d \ge 3, (k,\epsilon) \ne (2,-)$
$\mathrm{PSL}_d(q^2)$	$\mathrm{PSU}_{d^2}^{-}(q)$	$d \ge 3$
$\mathrm{PSp}_d(q^k)$	$\mathrm{PSp}_{d^k}(q)$	$kq \operatorname{odd}$
$\mathrm{PSp}_d(q^k)$	$\mathrm{P}\Omega^+_{d^k}(q)$	k odd, q even
$\mathrm{PSp}_d(q^2)$	$\mathrm{P}\Omega^{\widetilde{\epsilon}}_{d^2}(q)$	$d \ge 4, \ \epsilon = (-)^{d/2}$
$\mathrm{P}\Omega_d^{\pm}(q^k)$	$\mathrm{P}\Omega^{\tilde{\epsilon}}_{d^k}(q)$	$d \geq 6$ even, q odd
$P\Omega_d(q^k)$	$\mathrm{P}\Omega_{d^k}(q)$	$dq \text{ odd}, d \geq 3$

Lemma 4.9.17. Under the assumptions (1)–(3), the defining characteristic case (A) above does not occur.

Proof. Assume we are in case (A), so that $M_0 = \operatorname{Cl}_d(q^k) < G_0 = \operatorname{Cl}_{d^k}(q)$ with $k \ge 2$. Specifically, the embeddings $M_0 < G_0$ are as given by [90, Table 1B], and are as in Table 4.9.4, with k prime. With one exception, the natural module for G_0 is of the form $V = W \otimes W^{(q)} \otimes \cdots \otimes W^{(q^{k-1})}$, where W is the natural d-dimensional module for M_0 ; the exception is for the embedding $\operatorname{PSL}_d(q^2) \le \operatorname{PSU}_{d^2}(q)$ in the second row of the table, where $V = W \otimes W^{*(q)}$.

The argument is very similar for all entries in the table: we have $x \in M_0$, a semisimple element with centralizer as described in the proof of Lemma 4.9.13. Then $C_G(x)$ contains a maximal torus of G, whereas we argue that $C_M(x)$ cannot contain such a torus: in most cases this is obvious, as G has much larger rank than M, but nevertheless we shall give a sketch for each case below.

Consider the first row of the table, $\mathrm{PSL}_d^{\epsilon}(q^k) \leq \mathrm{PSL}_{d^k}^{\epsilon}(q)$ with $d \geq 3$, $(k, \epsilon) \neq (2, -)$. For $\epsilon = +$ the element $x \in M_0$ has preimage of the form diag(1, a, A) where $A \in \mathrm{GL}_{d-2}(q^k)$ has order $q^{k(d-2)} - 1$. This acts on V as $(1, a, A) \otimes (1, a, A)^{(q)} \otimes \cdots \otimes (1, a, A)^{(q^{k-1})}$, and hence we see that $C_G(x)$ has order divisible by $(q^{k(d-2)} - 1)^2$ if $d \geq 4$, and by $(q^k - 1)^3/(q - 1)$ if d = 3. Hence $C_G(x) \neq C_M(x)$. Now consider $\epsilon = -$. Here the semisimple element $x \in M_0$ has at least two eigenvalues 1 if $d \geq 4$, and is diag $(1, a, a^{-1})$ if d = 3, where a generates $\mathbb{F}_{q^k}^{\times}$. From the action of x on V, we see that $C_G(x)$ contains $\mathrm{SU}_{2^k}(q)$ if $d \geq 4$, and contains $(\mathrm{GL}_1(q^{2k}))^2$ if d = 3. Hence again $C_G(x) \neq C_M(x)$.

The argument for the second row of Table 4.9.4 is entirely similar: here $C_G(x)$ has order divisible by $(q^{2(d-2)}-1)^2$ if $d \ge 4$, and by $(q^2-1)^3$ if d = 3.

Next consider $M_0 = \operatorname{PSp}_d(q^k)$, with embedding as in rows 3-5 of Table 4.9.4. Suppose first that d = 2, so that k is odd and G_0 is $\operatorname{PSp}_{2^k}(q)$ or $\operatorname{P\Omega}_{2^k}^+(q)$, according as q is odd or even, respectively. Also $q^k > 31$, by the exclusions of (4.9.2). The element $x = \operatorname{diag}(a, a^{-1}) \in M_0$ has centralizer in M of order dividing $q^k - 1$. If k = 3, then q > 3 and the action of x on V has eigenvalues $\mu^{\pm 1}, \mu^{\pm q}, \mu^{\pm q^2}, \lambda, \lambda^{-1}$, where $\mu = a^{q^2+q-1}$, $\lambda = a^{q^2+q+1}$; hence $C_G(x)$ has order divisible by $(q^3 - 1)(q - 1)/(2, q - 1)$, and so $C_G(x) \neq C_M(x)$. And if $k \geq 5$, then we reach the same contradiction as $|C_G(x)|$ is divisible by $(q^k - 1)^{(2^{k-1}-1)/k}$.

This deals with d = 2, so suppose now that $M_0 = \operatorname{PSp}_d(q^k)$ with $d \ge 4$. Here $x = (I_2, A, A^{-T})$, where $A \in \operatorname{GL}_{\frac{d}{2}-1}(q^k)$ has order $q^{k(\frac{d}{2}-1)} - 1$. If $k \ge 3$, then the fixed point space of x on V has dimension 2^k and $C_G(x)$ contains $\operatorname{Cl}_{2^k}(q)$. Hence k = 2 and x acts on V as $(I_2, A, A^{-T}) \otimes (I_2, A, A^{-T})^{(q)}$. This has centralizer in G containing $\Omega_4^{\epsilon}(q) \times \operatorname{SL}_2(q^{k(\frac{d}{2}-1)})$, so once again $C_G(x) \neq C_M(x)$.

Finally, consider $M_0 = P\Omega_d^{\epsilon}(q^k)$, as in the last two rows of Table 4.9.4.

Because of exceptional isomorphisms of low-dimensional orthogonal groups, we need to consider separately the cases d = 3, 5 and 6. If d = 3 then $x \in M_0 = \Omega_3(q^k) \cong \text{PSL}_2(q^k)$ has the form diag $(1, a, a^{-1})$, where $a \in \mathbb{F}_{q^k}$ has order $(q^k - 1)/2$, and we argue in the usual way that $|C_G(x)|$ is divisible by $(q^k - 1)^2/2$, so $C_G(x) \neq C_M(x)$.

If d = 5 then $x \in M_0 = \Omega_5(q^k) \cong PSp_4(q^k)$; in $PSp_4(q^k)$, x takes the form (I_2, a, a^{-1}) , so in $\Omega_5(q^k)$, we have $x = (1, aI_2, a^{-1}I_2)$. Now we argue that $C_G(x)$ contains $(SL_2(q^k))^2$ if $k \ge 3$, and contains $SL_2(q^2) \times SU_2(q)$ if k = 2. In both cases, $C_G(x) \neq C_M(x)$.

Next, if d = 6 then $x \in M_0 = P\Omega_6^{\epsilon}(q^k) \cong PSL_4^{\epsilon}(q^k)$. For $\epsilon = +, x$ takes the form (1, a, A) in $PSL_4(q)$, hence $x = (a, a^{-1}A, A^{-T}) \in P\Omega_6^+(q^k)$; and for e = -, x is $(I_2, a, a^{-1} \in PSU_4(q)$, hence $x = (I_2, aI_2, a^{-1}I_2) \in P\Omega_6^-(q^k)$. Now argue in the usual way that $C_G(x) \neq C_M(x)$.

Finally, if $d \ge 7$ then $x = (I_{2+y}, \zeta, \zeta^{-1}, A, A^{-T}) \in M_0 = P\Omega_d^{\epsilon}(q^k)$ (where $y \in \{0, 1, 2\}$), and so $C_G(x)$ contains $\Omega_{(2+y)^k}(q) \times SL_{2^{k-1}}(q^{k(d-4-y)/2})$, and once again we have the contradiction $C_G(x) \neq C_M(x)$. This completes the proof.

This completes the proof of Lemma 4.9.12, and hence also the proof of Proposition 4.9.1.

4.10 Exceptional automorphisms

As explained in Section 4.1.1, in our proof of Theorem 4.1, we have so far been assuming that our almost simple group G contains no graph automorphisms when G has socle $\text{Sp}_4(2^a)$, and no triality automorphisms when G has socle $\text{P}\Omega_8^+(q)$ (except when M is in the Aschbacher class \mathcal{S}). In this final section, we complete the proof of Theorem 4.1 by handling these cases.

Thus we assume in this section that G is an almost simple group such that one of the following holds:

- 1. the socle of G is isomorphic to $\text{Sp}_4(2^a)$ with a > 1, and G contains a graph automorphism;
- 2. the socle of G is isomorphic to $P\Omega_8^+(q)$ and G contains a triality automorphism.

Note that we omit the case $\text{Sp}_4(2)$, as the theorem is already proved for groups with alternating socle in [46].

We slightly adjust terminology for this final section: we use S to denote the socle of G.

Lemma 4.10.1. Let $S = \text{Sp}_4(q)$ where $q = 2^a$ with a > 1, and suppose that $S \le G \le \text{Aut}(S)$. Let M be a core-free maximal subgroup of G. Then the action of G on (G : M) is not binary.

Proof. For $q \in \{4, 8, 16\}$, we refer to Lemma 4.1.1. Assume that $q \ge 32$. We refer to [10, Table 8.14] for the maximal subgroups of G. One checks that with three exceptions, all of them contain an element g as defined in Lemma 2.2.12; hence these can be excluded. The exceptions are

$$M \cap S = \text{Sp}_4(q_0), \ (q^2 + 1) : 4 \text{ or } (q + 1)^2 : D_8.$$

If $M \cap S = \text{Sp}_4(q_0)$, then $q = q_0^r$, where r is prime, and the argument of Lemma 4.5.7 gives the conclusion.

In the remaining two cases, M is a torus normalizer, and we use arguments similar to those in §3.5. Suppose that $M \cap S = (q^2 + 1) : 4$. Then $N = M \cap S$ is a Frobenius group with $T = q^2 + 1$, the Frobenius kernel. Let $g \in M \cap S$ be of order 4; again we check that there exists $c \in C_G(g) \setminus N_G(T)$. Then the action of N on $(N : N \cap N^x)$ is a Frobenius action and, since $N \cap N^x = N \cap M \cap M^x$, Lemma 1.7.2 implies that the action of M on $(M : M \cap M^x)$ is not binary; hence the action of G on (G : M) is not binary by Lemma 1.6.1.

Suppose finally that $M \cap S = (q+1)^2 : D_8$. We apply Lemma 3.5.2 with $A \cong D \cong SL_2(q)$, and $T_0 \cong T_1 \cong q+1$. The listed conditions are all easy to verify; in particular, item (vii) of the lemma is verified using [34], which asserts that the action of a group with socle $SL_2(q)$ on the set of cosets of the normalizer of a non-split torus is not binary.

Lemma 4.10.2. Let $S = P\Omega_8^+(q)$, suppose that $S \leq G \leq \operatorname{Aut}(S)$, and suppose that G contains an element in the coset of a triality automorphism of S. If M is a maximal core-free subgroup of G, then the action of G on $\Omega = (G : M)$ is not binary.

Proof. This is covered by Lemma 4.1.1 when $q \leq 4$, so assume that $q \geq 5$.

We refer to [55] for a list of the maximal subgroups of G. Following [55] we set $d = \gcd(2, q - 1)$ and, when giving the isomorphism type of a subgroup, we prefix a circumflex symbol to indicate that we are giving the structure of the group in $\Omega_8^+(q)$, rather than its projective image in $P\Omega_8^+(q)$.

Suppose, first, that $M \cap S$ is a maximal subgroup of S in the C_1 family. Then [46, Proposition 4.6] implies that Ω contains a beautiful subset, and the result follows immediately. Suppose, next, that M is a novelty maximal subgroup of G such that $M \cap S$ is a proper subgroup of a maximal C_1 subgroup of S. There are four possibilities for M, and we list them in Table 4.10.1 together with an integer r. The integer r indicates the presence of a subgroup $A = \operatorname{SL}_r(q)$ in M, together with a subgroup of S that is isomorphic to a central quotient of $\operatorname{SL}_{r+1}(q)$ satisfying the conditions of Lemma 1.6.10. The lemma then implies that there is a subset Δ of Ω of size q^r on which G^{Δ} acts 2-transitively. Now Lemma 2.1.1 implies that $\operatorname{P}\Omega_8^+(q)$ does not contain a section isomorphic to $\operatorname{Alt}(q^r)$, and the result follows.

$M \cap S$	
$\left[\left[q^{11} : \left[\frac{q-1}{d} \right] : \frac{1}{d} \operatorname{GL}_2(q) . d^2 \right] \right]$	2
$\ddot{G}_2(q)$	3
$\left(\frac{q-1}{d} \times \frac{1}{d} \operatorname{GL}_3(q)\right).[2d]$	
$\left(\int \frac{q+1}{d} \times \frac{1}{d} \mathrm{GU}_3(q) \right) . [2d]$	2

Table 4.10.1: Novelty C_1 -subgroups in $P\Omega_8^+(q)$

Next suppose that $M \cap S$ is a maximal subgroup of S in the C_2 family, stabilizing a decomposition of $V = V_8(q)$ as a direct sum of *m*-spaces. Lemma 4.2.13 implies that either

- there is a beautiful subset (and we are done), or
- the parameter m = 1 (and [55] implies that we can exclude this case, since such groups are not maximal given our assumption that G contains a triality automorphism), or
- $M \cap S$ is of type $O_2^-(q) \operatorname{wr} \operatorname{Sym}(4)$.

In this last case M is the normalizer of a torus, with $M \cap S \cong \left(\frac{q+1}{d}\right)^4 d^3 \cdot 2^3$. Sym(4). Now, as in the previous lemma, we appeal to Lemma 3.5.2 with $A = A_1(q)$ and $D = A_1(q)^3$, and we conclude that the action of G on (G:M) is not binary.

Suppose now that M is a novelty maximal subgroup of G such that $M \cap S$ is a proper subgroup of a maximal subgroup of S in the C_2 class. Then [55] implies that there are two possibilities: $M \cap S \cong$ $[2^9]$.PSL₃(2) or $(D_{2(q^2+1)/d})^2$. $[2^2]$. For the first we use magma to check that any transitive action of M of degree k with $k \not\equiv 0 \pmod{4}$ and with M/K non-solvable, where K is the kernel of the action, is not binary. Now if $q \equiv 1,7 \pmod{8}$, then |G:M| is even and hence M must have a non-trivial suborbit of odd degree; so we are done in this case (note that we can ignore actions where M/K is solvable by Lemma 1.8.2). If $q \equiv 3,5 \pmod{8}$, then $|G:M| \equiv 3 \pmod{4}$. Therefore $|G:M| - 1 \equiv 2 \pmod{4}$ and hence M cannot have all suborbits of cardinality a multiple of 4. Again, we are done. For the second group $(D_{2(q^2+1)/d})^2 \cdot [2^2]$, we use Lemma 3.5.2 with $A = A_1(q^2)$ and $D = A_1(q^2)$ and we conclude that the action of G on (G:M) is not binary.

Assume next that $M \cap S$ is a maximal subgroup of S in the C_5 class. Then Lemma 4.5.8 implies that either there is a beautiful subset (and we are done), or $F^*(M \cap S) = \Omega_8^-(q_0)$ with $q_0 \in \{2,3\}$ and $S = \Omega_8^+(q_0^2)$ (but this case does not occur when G contains a triality automorphism), or $F^*(M \cap S) = \Omega_8^+(2)$ and $q = 2^r$ with r odd. For this last case M equals either L or $L \times r$ where L is almost simple with socle $\Omega_8^+(2)$; now we obtain the result arguing in exactly the same way as in Lemma 4.5.9.

The final case to consider is that in which $M \cap S$ is a subgroup in the family S of subgroups of S. However this case has already been dealt with in Proposition 4.9.1, thanks to our relaxation of the triality assumption at the beginning of Section 4.9.

This completes our consideration of the exceptional automorphisms. The proof of Theorem 4.1 is now complete.

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