Non Conservative Products in Fluid Dynamics

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Abstract

Fluid flow in pipes with discontinuous cross section or with kinks is described through balance laws with a non conservative product in the source. At jump discontinuities in the pipes' geometry, the physics of the problem suggests how to single out a solution. On this basis, we present a definition of solution for a general **BV** geometry and prove an existence result, consistent with a limiting procedure from piecewise constant geometries. In the case of a smoothly curved pipe we thus justify the appearance of the curvature in the source term of the linear momentum equation.

These results are obtained as consequences of a general existence result devoted to abstract balance laws with non conservative source terms.

Keywords: Fluid flows in pipes; Non conservative products in balance laws.

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1 Introduction

Conservation laws in one space dimension, i.e., systems of partial differential equations in conservative form of the type

$$\partial_t u + \partial_x f(u) = 0 \qquad t \geq 0,\ x \in \mathbb{R}\,, \tag{1.1}$$

allow to describe, for instance, the movement of a fluid along a rectilinear pipe with constant section. Assume that at a point $\bar{x}$ the pipe's direction or its section changes. Then, equation (1.1) can be used, separately, where $x < \bar{x}$ and where $x > \bar{x}$. At the point $\bar{x}$, on the basis of physical considerations, a further condition is necessary to prescribe the possible defect in the conservation of the various variables. Typically, such a condition is written as

$$\Psi\left(z^+, u(t,\bar{x}+), z^-, u(t,\bar{x}-)\right) = 0 \quad \text{for a.e. } t > 0, \tag{1.2}$$

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where z^+ and z^-, identify the physical parameters that change across $\bar{x}$. Alternatively, (1.2) can be rewritten making the defect in the conservation of the u variable explicit, that is

$$f\left(u(t,\bar{x}+)\right) - f\left(u(t,\bar{x}-)\right) = \Xi\Big(z^+, z^-, u(t,\bar{x}-)\Big) \quad \text{for a.e. } t > 0\,. \tag{1.3}$$

It is then natural to tackle the resulting Riemann Problem, that is, the Cauchy Problem consisting of (1.1)–(1.3) with an initial datum attaining two values, one for $x < 0$ and another one for $x > 0$, as was accomplished, for instance, in [1, § 2] or [7, § 2]. The finite propagation speed, intrinsic to (1.1), allows then to extend the whole construction to any finite number of points $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_k$, essentially solving the Cauchy Problem for the balance law

$$\begin{cases} \partial_t u + \partial_x f(u) = \sum\limits_{i=1}^{k-1} \Xi\left(\zeta_k(\bar{x}_i+), \zeta_k(\bar{x}_i-), u(t,\bar{x}_i-)\right)\, \delta_{\bar{x}_i} \\ u(0,x) = u_o(x), \end{cases} \tag{1.4}$$

where $\delta_{\bar{x}_i}$ denotes the Dirac measure at $\bar{x}_i$ and ζ_k is the piecewise constant function attaining the $k+1$ constant values $z_0, z_1, \ldots, z_k$ on the intervals $]-\infty, \bar{x}_1[$, $]\bar{x}_1, \bar{x}_2[$, …, $]\bar{x}_k, +\infty[$.

This paper provides a detailed description of the rigorous limit $k \to +\infty$ of (1.4), covering the extension of (1.4) to the case of a general $\mathbf{BV}$ function ζ.

In the general setting established below, not limited to fluid dynamics, solutions to (1.4) with initial datum u_o are shown to converge as $k \to +\infty$ to solutions to

$$\begin{cases} \partial_t u + \partial_x f(u) = \sum\limits_{x \in \mathcal{I}} \Xi\left(\zeta(x+), \zeta(x-), u(t,x-)\right) \delta_x + D^+_{v(x)} \Xi\left(\zeta(x), \zeta(x), u(t,x)\right) \|\mu\| \\ u(0,x) = u_o(x)\,. \end{cases} \tag{1.5}$$

The terms in the non conservative source above are defined as follows. Since $\zeta \in \mathbf{BV}(\mathbb{R};\mathbb{R}^p)$, the right and left limits $\zeta(x+)$ and $\zeta(x-)$ are well defined and the distributional derivative $D\zeta$ can be split in a discrete part and a continuous one, which may contain a Cantor part:

$$D\zeta = \sum_{x \in \mathcal{I}} \left(\zeta(x+) - \zeta(x-)\right)\, \delta_x + v\,\|\mu\|\,, \tag{1.6}$$

where the function v is Borel measurable with norm 1, μ is the non atomic part of $D\zeta$ and $\mathcal{I}$ is the set of jump points in ζ. In (1.5) we also used the (one sided) directional derivative

$$D^+_v \Xi(z,z,u) = \lim_{t \to 0+} \frac{\Xi(z + t\,v, z, u) - \Xi(z,z,u)}{t}\,. \tag{1.7}$$

Indeed, one of our motivating examples, namely the case of a curved pipe, leads to a function Ξ that admits directional derivatives but is not differentiable.

On the other hand, note that as soon as Ξ is differentiable with respect to its first argument, the right hand side in (1.5) can be slightly simplified, since

$$D^+_{v(x)} \Xi(a,a,u)\,\|\mu\| = D_1\Xi(a,a,u)\,v(x)\,\|\mu\| = D_1\Xi(a,a,u)\,\mu\,. \tag{1.8}$$

Moreover, in the case $\Xi(z^+, z^-, u) = G(z^+) - G(z^-)$ for a suitable $G \in \mathbf{C}^2(\mathbb{R}^p;\mathbb{R}^n)$, the right hand side above takes a simpler form. Indeed, by [2, Theorem 3.96], (1.5) reduces to the conservative problem

$$\partial_t u + \partial_x f(u) = \partial_x (G \circ \zeta)\,. \tag{1.9}$$

Below, our first task is to provide a definition of solution to (1.5) in its general setting. Indeed, the latter term in the right hand side of (1.5) contains a *non conservative product* between a possibly discontinuous function and a measure. As is well known since the pioneering work [13], such a product intrinsically contains a lack of determinacy. Here, this freedom of choice is used to ensure the convergence of (1.4) to (1.5).

Once the issue of the very meaning of solution is settled, we proceed towards proving the existence of solutions to (1.5). This is achieved sequentially combining wave front tracking [5, § 7.1], a nowadays classical technique that approximates solutions to conservation laws, with the approximation of the equation, in particular of the map ζ. A key role is played by a very careful choice of these approximations. As a byproduct, we characterize the solutions to (1.5) as limits of (suitable subsequences of) solutions to (1.4).

Remark that the above general procedure, when applied to the case of a curved pipe with constant section, amounts to justify the role of the pipe's curvature on the fluid flow inside the pipe. Indeed, if x is the abscissa along the pipe and $\Gamma = \Gamma(x)$ describes the pipe's shape, then the pipe's local direction that enters the equation for fluid flow is $\zeta(x) = \Gamma'(x)$. Problem (1.4) then corresponds to a piecewise linear pipe and (the second component of) (1.3) describes the change in the fluid linear momentum at a kink sited at $\bar{x}$. Assuming that the lack in the conservation of linear momentum depends on the angle in the pipe at $\bar{x}$, i.e., $\Xi(z^+, z^-, u) = K(\|z^+ - z^-\|, u)$ as in [9, 16], automatically implies in the smooth pipe limit, by Theorem 2.2, that the variation in the fluid momentum depends on the pipe's curvature Γ'', see § 3.1 for more details.

The current literature offers a variety of different conditions quantifying the lack in the conservation of linear momentum at a junction where the pipe's section changes, see for instance [3, 4, 6, 7, 8, 10, 11, 15]. As a consequence of Theorem 2.2, we can select those conditions that are consistent with the equations for a pipe with smoothly varying section, both in the isentropic and in the full 3×3 cases, see § 3.2 and § 3.3 below.

While motivated by the above fluid dynamics problems, the present construction also suggests a criterion to select solutions to general balance laws with a non conservative product as a source term, see Definition 3.3. These solutions, whose existence follows from Theorem 2.2, are characterized as limits of solutions to the piecewise constant case (1.4).

The next section is devoted to the main results: the definition of solution and to the existence theorem. Section 3 presents applications to fluid dynamics and to general balance laws with non conservative product in the source. All technical proofs are deferred to Section 4.1.

2 Assumptions and Main Result

Throughout, $|x|$ is the absolute value of the real number x while, as usual, $\|v\|$ is the Euclidean norm of the vector v and $\|\mu\|$ is the total variation of a measure μ. The open ball in $\mathbb{R}^n$ centered at u with radius δ is denoted by $B(u;\delta)$, its closure is $\overline{B(u;\delta)}$. We also use the following standard notation for right/left limits and for differences at a point:

$$F(x-) = \lim_{\xi \to x^-} F(\xi)\,, \quad F(x+) = \lim_{\xi \to x^+} F(\xi) \quad \text{and} \quad \Delta F(x) = F(x+) - F(x-)\,.$$

The problem we tackle is defined by the flow f and by the functions Ξ and ζ. Here we detail the key assumptions.

(f.1) $f \in \mathbf{C}^2(\Omega; \mathbb{R}^n)$, Ω being an open subset of $\mathbb{R}^n$;

(f.2) the system (1.1) is strictly hyperbolic;

(f.3) each characteristic field is either genuinely nonlinear or linearly degenerate.

In the latter assumption we refer to the usual definitions by Lax [17], see also [12, § 7.5].

By **(f.1)** and **(f.2)** we know that, possibly restricting Ω, the eigenvalues $\lambda_1(u), \ldots, \lambda_n(u)$ of $Df(u)$ can be numbered so that, for all $u \in \Omega$,

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u) .$$

We choose $i_o \in \{1, \ldots, n-1\}$ and define the i_o-th non-characteristic set

$$A_{i_o} = \{u \in \Omega \mid \lambda_{i_o}(u) < 0 < \lambda_{i_o+1}(u)\} , \tag{2.1}$$

both the cases of characteristic speeds being either all positive or all negative being simpler.

On the function Ξ in (1.3), used to rewrite the coupling condition induced by Ψ, we require:

(Ξ.1) $\Xi : \mathcal{Z} \times \mathcal{Z} \to \mathbf{C}^1(\Omega; \mathbb{R}^n)$, is a Lipschitz continuous map;

(Ξ.2) $\sup_{z^+, z^- \in \mathcal{Z}} \left\|\Xi(z^+, z^-, \cdot)\right\|_{\mathbf{C}^2(\Omega;\mathbb{R})} < +\infty$;

(Ξ.3) $\Xi(z, z, u) = 0$ for every $z \in \mathcal{Z}$ and $u \in \Omega$;

(Ξ.4) There exists a non decreasing $\sigma\colon [0, \bar{t}[\to \mathbb{R}$ such that for all $(z, v, u) \in \mathcal{Z} \times \overline{B(0;1)} \times \Omega$

$$\left\|\Xi(z + t\, v, z, u) - D_v^+ \Xi(z, z, u)\, t\right\| \leq \sigma(t)\, t$$

and moreover the map $(z, v, u) \to D_v^+ \Xi(z, z, u)$ is Lipschitz continuous.

In the latter condition, recall the definition (1.7) of the Dini derivative. Our requiring this low regularity, i.e. the mere existence of the Dini derivative rather than differentiability, is motivated by the example of a pipe with angles, where Ξ depends on $\left\|z^+ - z^-\right\|$, see § 3.1.

Problem (1.5) requires the introduction of a further function, say $\zeta\colon \mathbb{R} \to \mathbb{R}^p$ describing, for instance, geometrical aspects of the pipeline. We require that $\zeta \in \mathbf{BV}(\mathbb{R}; \mathcal{Z})$. Throughout, the map ζ is assumed to be left continuous and the set of jump discontinuities in ζ is denoted by $\mathcal{I}$, with $\mathcal{I} \subset \mathbb{R}$.

We now precisely state what we mean by *solution* to (1.5).

Definition 2.1. *Let $u_o \in \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \mathbb{R}^n)$. A map $u \in \mathbf{C}^0([0, +\infty[; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \mathbb{R}^n))$ with $u(t) \in \mathbf{BV}(\mathbb{R}; \mathbb{R}^n)$ and left continuous for all $t \in \mathbb{R}_+$, is a* solution *to* (1.5) *if for all test functions $\varphi \in \mathbf{C}^1_c(]0, +\infty[\times \mathbb{R}; \mathbb{R})$,*

$$\begin{aligned} & -\int_0^{+\infty} \int_{\mathbb{R}} \Big(u(t,x)\, \partial_t \varphi(t,x) + f\left(u(t,x)\right)\, \partial_x \varphi(t,x)\Big)\, \mathrm{d}x\, \mathrm{d}t \\ = & \sum_{x \in \mathcal{I}} \int_0^{+\infty} \Xi\left(\zeta(x+), \zeta(x), u(t,x)\right) \varphi(t,x)\, \mathrm{d}t \\ & + \int_0^{+\infty} \int_{\mathbb{R}} D^+_{v(x)} \Xi\left(\zeta(x), \zeta(x), u(t,x)\right) \varphi(t,x)\ \mathrm{d}\|\mu\|\,(x)\ \mathrm{d}t \end{aligned} \tag{2.2}$$

where $\mathcal{I}$ is the set of jump points of ζ and v, μ are as in (1.6)*, and moreover $u(0) = u_o$.*

The main result of this paper is the following.

Theorem 2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be open, f satisfy* **(f.1)**–**(f.3)**, *Ξ satisfy* **(Ξ.1)**–**(Ξ.4)** *and $\zeta \in \mathbf{BV}(\mathbb{R}; \mathcal{Z})$. Fix $\bar u \in A_{i_o}$, $\bar z \in \mathcal{Z}$ and an initial datum u_o in $\mathbf{L^1_{loc}}(\mathbb{R}; A_{i_o})$, with A_{i_o} as defined in* (2.1)*. Then, there exists a positive δ such that if*

$$u_o(\mathbb{R}) \subseteq B(\bar u; \delta), \quad \mathrm{TV}(u_o) < \delta \quad \text{and} \quad \zeta(\mathbb{R}) \subseteq B(\bar z; \delta), \quad \mathrm{TV}(\zeta) < \delta \tag{2.3}$$

the Cauchy Problem for (1.5) *with initial datum u_o admits a solution u_* in the sense of Definition 2.1. Moreover, there exists a sequence of piecewise constant approximations ζ^h of ζ, with $\mathrm{TV}(\zeta^h) < \delta$, such that the corresponding solutions u^h converge to u_* pointwise in time and in $\mathbf{L^1_{loc}}$ in space. In particular, at each discontinuity point y of ζ^h, u^h satisfies the junction condition*

$$f(u^h(t,y+)) - f(u^h(t,y-)) = \Xi(\zeta^h(y+), \zeta^h(y-), u^h(t,y-)).$$

3 Applications

3.1 Isentropic Gas in a Curved Pipe

The well known system of one dimensional isentropic gas dynamics within a pipe with constant section in Eulerian coordinates [12, Formula (7.1.12)] is

$$\left\{\begin{array}{rcl} \partial_t \rho + \partial_x q &=& 0 \\ \partial_t q + \partial_x P(\rho,q) &=& 0 \end{array}\right. \quad \text{where } P(\rho,q) = \frac{q^2}{\rho} + p(\rho), \quad \text{for a.e. } t \geq 0,\ x \in \mathbb{R}. \tag{3.1}$$

Here, x is the abscissa along the pipe, $\rho \in \,]0,+\infty[$ denotes the gas density, $q \in \mathbb{R}$ the momentum density, $p = p(\rho)$ the pressure and $P = P(\rho,q)$ the momentum flux. The pressure law p satisfies

(p) $p \in \mathbf{C}^2(]0,+\infty[,]0,+\infty[)$, $p'(\rho) \geq 0$ and $p''(\rho) \geq 0$ for all $\rho > 0$.

Under this assumption, system (3.1) is strictly hyperbolic, except at the vacuum $\rho = 0$.

We aim to estabilish the existence of solutions to (3.1) in a curved pipeline with constant section lying in a horizontal plane. Parametrize the pipe's support by means of the arc length $\Gamma \colon \mathbb{R} \to \mathbb{R}^2$, so that $\|\Gamma'(x)\| = 1$ for a.e. $x \in \mathbb{R}$. We assume that $\zeta = \Gamma'$ is in $\mathbf{BV}(\mathbb{R};\mathbb{R}^2)$.

As a first step, consider the case of (3.1) at a kink sited at $\bar x$, so that Γ is the glueing of two half lines. Therefore, to solve (3.1), we adopt the usual weak entropy solutions to (3.1) along the straight parts of Γ and match at the kink $\bar x$ a coupling condition of the type

$$\left\{\begin{array}{rcl} q(t,\bar x+) - q(t,\bar x-) &=& 0 \\ P(\rho,q)(t,\bar x+) - P(\rho,q)(t,\bar x-) &=& \Xi_2(\Gamma'(\bar x+), \Gamma'(\bar x-), (\rho,q)(t,\bar x-)) \end{array}\right. \tag{3.2}$$

We set $\Xi_1 \equiv 0$ as it is necessary to comply with mass conservation. Physical considerations suggest that the defect in the conservation of linear momentum is a function, say K, of the norm of the difference in the orientations of the pipes on the sides of the kink:

$$\Xi\left(z^+, z^-, (\rho,q)\right) = \left[\begin{array}{c} 0 \\ K\left(\|z^+ - z^-\|, (\rho,q)\right) \end{array}\right]. \tag{3.3}$$

This holds true in various instances of K considered in the literature. For instance, [16] first introduced the condition

$$K\left(\left\|z^+ - z^-\right\|, (\rho, q)\right) = -\alpha \left\|z^+ - z^-\right\| q \tag{3.4}$$

for a suitable $\alpha > 0$, motivated by

$$\left\|z^+ - z^-\right\| = \sqrt{2(1-\cos\bar\vartheta)} = 2\left|\sin(\bar\vartheta/2)\right|,$$

$\bar\vartheta$ being the angle between the two sides of the kink. It is immediate to see that **(Ξ.1)**–**(Ξ.3)** all hold. Concerning **(Ξ.4)**, we have

$$D_v^+ \Xi\left(z, z, (\rho, q)\right) = \left[\begin{array}{c} 0 \\ -\alpha \left\|v\right\| q \end{array}\right] \qquad \text{with} \qquad \sigma \equiv 0\,.$$

We stress that Ξ_2 is *not* of class $\mathbf{C}^1$.

Theorem 3.1. *Let p satisfy* **(p)** *and $(\bar\rho, \bar q)$ be a subsonic state. Let Γ be piecewise $\mathbf{C}^2(\mathbb{R};\mathbb{R}^2)$, such that $\Gamma' \in \mathbf{BV}(\mathbb{R};\mathbb{R}^2)$ and $\left\|\Gamma'(x)\right\| = 1$ for all $x \in \mathbb{R}$. Let $K \in \mathbf{C}^2([0,r]\times\Omega;\mathbb{R})$ for a positive r, with $K\left(0,(\rho,q)\right) \equiv 0$. Call $\mathcal{I}$ the set of kink points of Γ. Then, there exists a positive δ such that for all initial data (ρ_o, q_o) with*

$$\left\|(\rho_o, q_o) - (\bar\rho, \bar q)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^2)} < \delta\,, \quad \mathrm{TV}\,(\rho_o, q_o) < \delta\,, \quad \mathrm{TV}\,(\Gamma') < \delta$$

the problem

$$\left\{\begin{array}{rcl} \partial_t\rho + \partial_x q &=& 0 \\ \partial_t q + \partial_x P(\rho, q) &=& -\displaystyle\sum_{y\in\mathcal{I}} K\left(\left\|\Gamma'(y+) - \Gamma'(y-)\right\|, (\rho,q)(t, y-)\right)\delta_y \\ && -\left\|\Gamma''(x)\right\| \partial_1 K(0, q) \\ (\rho, q)(0, x) &=& (\rho_o, q_o)(x) \end{array}\right. \tag{3.5}$$

admits a solution (ρ_*, q_*) in the sense of Definition 2.1. Moreover, there exists a sequence of piecewise linear approximations Γ^h of Γ, with $\mathrm{TV}\left((\Gamma^h)'\right) < \delta$, such that the corresponding solutions (ρ^h, q^h) converge to (ρ_*, q_*) pointwise in time and in $\mathbf{L}^1_{\mathbf{loc}}$ in space. In particular, at each discontinuity point $\bar x$ of $(\Gamma^h)'$, (ρ^h, q^h) satisfies condition (3.2).

The proof is deferred to § 4.4.

Remark that the second derivative Γ'' appearing in the right hand side above confirms the relevance of the pipe's curvature. Nevertheless, Theorem 2.2 applies also to less regular functions Γ, but the above simpler formulation then needs to be replaced by the formulation used in Definition 2.1.

3.2 Isentropic Gas in a Pipe with Varying Section

The isentropic flow of a fluid in a pipe with smoothly varying section $a = a(x)$ is described by

$$\left\{\begin{array}{rcl} \partial_t \rho + \partial_x q & = & -\dfrac{a'}{a}\, q \\ \partial_t q + \partial_x P(\rho, q) & = & -\dfrac{a'}{a}\, \dfrac{q^2}{\rho} \end{array}\right. \quad \text{where } P(\rho, q) = \frac{q^2}{\rho} + p(\rho)\,, \quad \text{for a.e. } t \geq 0,\ x \in \mathbb{R}\,, \tag{3.6}$$

see [11, 15, 18]. The case of a piecewise constant, i.e., the section of the pipe changes from a^- to a^+ at a junction sited at $\bar{x}$, is covered in the literature supplementing the p–system (3.1) with a junction condition of the form

$$\left\{\begin{array}{l} a^+ q(t, \bar{x}+) = a^-\, q(t, \bar{x}-) \\ P(\rho, q)(t, \bar{x}+) - P(\rho, q)(t, \bar{x}-) = \Xi_2\left(a^+, a^-, (\rho, q)(t, \bar{x}-)\right)\,. \end{array}\right. \tag{3.7}$$

The former relation in (3.7) ensures the conservation of mass and fits in the framework of Section 2 setting in the first component of (1.3)

$$\Xi_1\left(a^+, a^-, (\rho^-, q^-)\right) = \left(\frac{a^-}{a^+} - 1\right) q^-\,. \tag{3.8}$$

The literature offers a wide range of justifications, often phenomenological, for specific choices of the function Ξ_2 in (3.7), see for instance [8, 11, 15]. Note that, as soon as Ξ_2 is of class $\mathbf{C}^2$ in all variables, with $\Xi_2\left(a, a, (\rho, q)\right) = 0$, and a is in $\mathbf{BV}(\mathbb{R}; \mathbb{R})$, then Theorem 2.2 applies ensuring the existence of solutions to

$$\left\{\begin{array}{rcl} \partial_t \rho + \partial_x q & = & \displaystyle\sum_{x \in \mathcal{I}} \left(\frac{a(x-)}{a(x+)} - 1\right) q(t, x-)\, \delta_x - \frac{1}{a(x)}\, q(t, x)\, \mu \\ \partial_t q + \partial_x P(\rho, q) & = & \displaystyle\sum_{x \in \mathcal{I}} \Xi_2\left(a(x+), a(x-), (\rho, q)(t, x-)\right) \delta_x \\ & & + \partial_1 \Xi_2\left(a(x), a(x), (\rho, q)(t, x)\right)\, \mu \end{array}\right. \tag{3.9}$$

where $\mathcal{I}$ is the set of points of discontinuity of a and, as soon as a is smooth, μ has density $\partial_x a(x)$ with respect to the Lebesgue measure. In (3.9) we also used (1.8).

As an application of Theorem 2.2, we characterize the class of conditions Ξ that yield in the limit the case of the smooth pipe, i.e., equation (3.6).

Theorem 3.2. *Let p satisfy* **(p)**, *$(\bar{\rho}, \bar{q})$ be a subsonic state and $\bar{a}$ be positive. For any Ξ_2 of class $\mathbf{C}^2$ with $\Xi_2\left(a, a, (\rho, q)\right) = 0$ and*

$$\partial_1 \Xi_2\left(a, a, (\rho, q)\right) = -\frac{1}{a}\, \frac{q^2}{\rho} \tag{3.10}$$

there exists a positive δ such that for all initial data (ρ_o, q_o) and for all $a \in \mathbf{BV}(\mathbb{R}; \mathbb{R})$ with $a' \in \mathbf{L}^1(\mathbb{R}; \mathbb{R})$ and

$$\left\|(\rho_o, q_o) - (\bar{\rho}, \bar{q})\right\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R}^2)} < \delta\,, \quad \mathrm{TV}\,(\rho_o, q_o) < \delta\,, \quad \|a - \bar{a}\|_{\mathbf{L}^\infty(\mathbb{R}, \mathbb{R})} < \delta\,, \quad \left\|a'\right\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R})} < \delta$$

	$\Psi_2(a^-,(\rho^-,q^-),a^+,(\rho^+,q^+))$	$\Xi_2(a^+,a^-,(\rho^-,q^-))$	$D_1\Xi_2(a,a,(\rho,q))$
[L]	$a^+P(\rho^+,q^+) - a^-P(\rho^-,q^-)$	$\left(\frac{a^-}{a^+}-1\right)\left(\frac{(q^-)^2}{\rho^-}+p(\rho^-)\right)$	$-\frac{1}{a}\left(\frac{q^2}{\rho}+p(\rho)\right)$
[p]	$p(\rho^+) - p(\rho^-)$	$\left(\left(\frac{a^-}{a^+}\right)^2-1\right)\frac{(q^-)^2}{\rho^-}$	$-\frac{2}{a}\frac{q^2}{\rho}$
[P]	$P(\rho^+,q^+) - P(\rho^-,q^-)$	0	0
[S]	$a^+P(\rho^+,q^+) - a^-P(\rho^-,q^-)$ $-\int_{a^-}^{a^+} p\left(R(\alpha;a^-,\rho^-,q^-)\right)\,d\alpha$	$\left(\frac{a^-}{a^+}-1\right)\left(\frac{(q^-)^2}{\rho^-}+p(\rho^-)\right)$ $+\frac{1}{a^+}\int_{a^-}^{a^+} p\left(R(\alpha;a^-,\rho^-,q^-)\right)\,\mathrm{d}\alpha$	$-\frac{1}{a}\frac{q^2}{\rho}$

Table 1: Various definitions of junction conditions, with the corresponding functions Ψ_2 from (1.2), Ξ_2 from (1.3) and its partial derivative $\partial_1\Xi_2$.

problem (3.6) *admits a solution* (ρ_*, q_*). *Moreover, there exists a sequence of piecewise constant approximations* a^h *of* a*, with* $\mathrm{TV}\,(a^h) < \delta$*, such that the corresponding solution* (ρ^h, q^h) *converges to* (ρ_*, q_*) *pointwise in time and in* $\mathbf{L}^1_{\mathbf{loc}}$ *in space. In particular, at each discontinuity point* y *of* a^h*,* (ρ^h, q^h) *satisfies the junction condition* (3.7).

The proof is deferred to § 4.4.

We now test the above condition against various junction condition found in the literature, we refer in particular to [8] for the motivations and further information of the conditions considered below. More precisely, with reference to the labelling in Table 1, we consider definition **[L]** from [6], condition **[p]** from [3, 4], condition **[P]** from [6, 7] and condition **[S]** from [11, 15]. All these conditions differ only in the second component Ξ_2, the first one being fixed as in (3.8) to comply with mass conservation.

Simple computations lead to the results in Table 1, where the map $a \to R(a; a^-, \rho^-, q^-)$ is the first component of the solution to the stationary version of (3.6), parametrized by the section a, i.e.,

$$\left\{\begin{array}{ll} \frac{\mathrm{d}}{\mathrm{d}a} q = -\frac{1}{a}\, q & \rho(a^-) = \rho^- \\ \frac{\mathrm{d}}{\mathrm{d}a}\left(P(\rho,q)\right) = -\frac{1}{a}\,\frac{q^2}{\rho} & q(a^-) = q^- \,. \end{array}\right.$$

On the basis of Theorem 3.2, we know that condition **[S]** is compatible with the smooth limit (3.6). Moreover, Theorem 2.2 and Table 1, in particular the comparison of the rightmost column with (3.10), ensure that all the other conditions do *not* converge to (3.6) in the smooth pipe limit.

Remark that substituting in **[S]** any other smooth function $R = R(a; a^-, \rho^-, q^-)$ such that $R(a^-; a^-, \rho^-, q^-) = \rho^-$ yields a new condition at the junction compatible with the smooth limit.

3.3 Full Gas Dynamics in Pipes with Varying Section

The full Euler system in a pipeline with smoothly varying section $a = a(x)$ is

$$\begin{cases} \partial_t \rho + \partial_x (\rho\, v) = -\dfrac{a'}{a}\, \rho\, v \\ \partial_t (\rho\, v) + \partial_x \left(\rho\, v^2 + p\right) = -\dfrac{a'}{a}\, \rho\, v^2 \\ \partial_t \left(\frac{1}{2}\,\rho\, v^2 + \rho\, e\right) + \partial_x \left(v \left(\frac{1}{2}\,\rho\, v^2 + \rho\, e + p\right)\right) = -\dfrac{a'}{a}\, v \left(\frac{1}{2}\,\rho\, v^2 + \rho\, e + p\right) \end{cases} \tag{3.11}$$

see, for instance [10, 15, 18, 19]. Here, x is the abscissa along the pipe, $\rho > 0$ denotes gas density, $q \in \mathbb{R}$ the momentum density, $p = p(\rho, s)$ the pressure and $e = e(\rho, s)$ the energy density and s the entropy density. These two latter functions satisfy

(E) $e \in \mathbf{C}^2(]0, +\infty[\times \mathbb{R},]0, +\infty[)$ and $\partial_s e(\rho, s) > 0$ for all $\rho > 0$ and $s \in \mathbb{R}$.

(P) $p \in \mathbf{C}^2(]0, +\infty[\times \mathbb{R},]0, +\infty[)$, $p(\rho, s) = \rho^2\, \partial_\rho e(\rho, s)$, $\partial_\rho p(\rho, s) > 0$ and $\partial^2_{\rho\rho} \left(\rho\, p(\rho, s)\right) > 0$ for all $\rho > 0$ and $s \in \mathbb{R}$.

We restrict our attention to the subsonic region where $v \in \left]0, \sqrt{\partial_\rho p(\rho, s)}\right[$.

The conditions found in the literature, see [10], imposed at a point $\bar{x}$ where the section suffers a discontinuity fit into the form

$$\begin{cases} \Delta(a\, \rho\, v)(t, \bar{x}) = 0 \\ \Delta(\rho\, v^2 + p)(t, \bar{x}) = \Xi_2 \left(a(x+), a(x-), (\rho, v, s)(t, x-)\right) \\ \Delta \left(a\, v \left(\frac{1}{2}\,\rho\, v^2 + \rho\, e + p\right)\right) (t, \bar{x}) = 0 \end{cases} \tag{3.12}$$

The conservation of mass imposed by the first equality and the conservation of energy imposed by the third equality in (3.12) amount to setting

$$\begin{aligned} \Xi_1 \left(a^+, a^-, (\rho^-, v^-, s^-)\right) &= \left(\frac{a^-}{a^+} - 1\right) \rho^-\, v^- \\ \Xi_3(a^+, a^-, u^-) &= \left(\frac{a^-}{a^+} - 1\right) \left(v^- \left(\tfrac{1}{2}\,\rho^- (v^-)^2 + \rho^-\, e^- + p^-\right)\right) \end{aligned}$$

so that

$$\begin{aligned} \partial_1 \Xi_1 \left(a, a, (\rho, v, s)\right) &= -\frac{1}{a}\, \rho\, v \\ \partial_1 \Xi_3(a, a, u) &= -\frac{1}{a} \left(v \left(\tfrac{1}{2}\,\rho\, v^2 + \rho\, e + p\right)\right) \end{aligned}$$

The second equality in (3.12) is treated in different ways in the literature, giving rise to conditions analogous to those considered in § 3.2. Indeed, Table 1 directly extends to the present full 3×3 case, simply understanding the map R as the ρ component $a \to R(a; a^-, \rho^-, v^-, s^-)$

in the solution to the stationary Cauchy Problem

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}a}(\rho\, v) = -\dfrac{1}{a}\,\rho\, v & \rho(a^-) = \rho^- \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\left(\rho\, v^2 + p\right) = -\dfrac{1}{a}\,\rho\, v^2 & v(a^-) = v^- \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\left(v\left(\frac{1}{2}\,\rho\, v^2 + \rho\, e + p\right)\right) = -\dfrac{1}{a}\, v\left(\frac{1}{2}\,\rho\, v^2 + \rho\, e + p\right) & s(a^-) = s^-\,.
\end{cases}
\tag{3.13}
$$

3.4 Balance Laws with Measure Valued Source Term

The theory developed in Section 2 allows to give a meaning to the following balance law, where the source term is *non conservative*:

$$
\partial_t u + \partial_x f(u) = \partial_\zeta G(\zeta, u)\, D\zeta \tag{3.14}
$$

where G is smooth and ζ has bounded variation. In the case G independent of u, we recover the *conservative* case (1.9). In the general, *non* conservative case, (3.14) can be given different meanings.

A choice consists in setting

$$
\Xi(z^+, z^-, u^-) = G(z^+, u^-) - G(z^-, u^-)\,, \tag{3.15}
$$

corresponding to the following condition at each point of jump:

$$
f(u^+) - f(u^-) = G(z^+, u^-) - G(z^-, u^-)\,.
$$

The framework developed in the preceding section in connection with the Cauchy Problem (1.5) comprises (3.14). Therefore, we can particularize Definition 2.1 to the general case of non conservative products of the type (3.14).

Definition 3.3. *Fix an initial datum $u_o \in \mathbf{L^1_{loc}}(\mathbb{R};\mathbb{R}^n)$. Let $\Xi \in \mathbf{C}^2(\mathcal{Z}\times\mathcal{Z}\times\Omega;\mathbb{R}^n)$ be such that*

$$
D_1\Xi(z, z, u) = D_z G(z, u)\,. \tag{3.16}
$$

Then, a map $u \in \mathbf{C}^0([0,T];\mathbf{L^1_{loc}}(\mathbb{R};\mathbb{R}^n))$ with $u(t) \in \mathbf{BV}(\mathbb{R};\mathbb{R}^n)$ and left continuous for all $t \in [0,T]$, is a Ξ–solution *to* (1.5) *if for all test function $\varphi \in \mathbf{C}^1_c(]0,T[\,\times\mathbb{R};\mathbb{R})$,*

$$
\begin{aligned}
& -\int_0^{+\infty}\int_\Omega \Big(u(t,x)\,\partial_t\varphi(t,x) + f\left(u(t,x)\right)\,\partial_x\varphi(t,x)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
= & \sum_{x\in\mathcal{I}}\int_0^{+\infty} \Xi\left(\zeta(x+),\zeta(x),u(t,x)\right)\varphi(t,x)\,\mathrm{d}t \\
& + \int_0^{+\infty}\int_{\mathbb{R}} D_z G\left(\zeta(x),u(t,x)\right)\varphi(t,x)\,D\mu(x)\;\mathrm{d}t
\end{aligned}
\tag{3.17}
$$

where $\mathcal{I}$ is the set of jump points of ζ and μ is as in (1.6)*, and moreover $u(0) = u_o$.*

This definition clearly separates those part of the solution that depend exclusively on (3.14) from those part, in the middle term in (3.17), that depend on the arbitrary choice of Ξ.

In particular, the choice (3.15) yields

$$\Xi\left(\zeta(x+),\zeta(x),u(t,x)\right) = G\left(\zeta(x+),u(t,x)\right) - G\left(\zeta(x),u(t,x)\right) \tag{3.18}$$

where we keep using the left continuous representatives. For completeness, we remark that the alternative choice $\Xi(z^+,z^-,u^-) = G(z^+,u^+) - G(z^-,u^+)$ also meets condition (3.16).

A straightforward application of Theorem 2.2 now ensures the existence of Ξ–solutions to (3.14), as soon as $G \in \mathbf{C}^2(\mathcal{Z}\times\Omega;\mathbb{R}^{n\times m})$, $\zeta \in \mathbf{BV}(\mathbb{R};\mathcal{Z})$, $\Xi \in \mathbf{C}^2(\mathcal{Z}\times\mathcal{Z}\times\Omega;\mathbb{R}^m)$ and satisfies (3.16). Moreover, these solutions are limits of *"discretized"* approximations where (1.3) is imposed to the points of jump in ζ.

4 Technical Details

Below, by $\mathcal{O}(1)$ we denote a constant depending exclusively on f and Ξ.

4.1 Preliminary Results

First, we prove a Lipschitz-type estimate on the map Ξ which we use throughout this paper.

Lemma 4.1. *Assume that* **(Ξ.1)**, **(Ξ.3)** *hold. Then,*

$$\left\|\Xi(z^+,z^-,u_2) - \Xi(z^+,z^-,u_1)\right\| = \mathcal{O}(1)\left\|z^+ - z^-\right\| \|u_2 - u_1\| . \tag{4.1}$$

Proof. Since the map $u \mapsto \Xi(z^+,z^-,u)$ is smooth, we can compute

$$\begin{aligned}
&\left\|\Xi(z^+,z^-,u_2) - \Xi(z^+,z^-,u_1)\right\| \\
&\quad\le \|u_2-u_1\| \int_0^1 \left\|D_u\Xi(z^+,z^-,u_1+s\,(u_2-u_1)) - D_u\Xi(z^-,z^-,u_1+s\,(u_2-u_1))\right\| \mathrm{d}s \\
&\quad\le \mathcal{O}(1)\left\|z^+-z^-\right\| \|u_2-u_1\| ,
\end{aligned}$$

where we used the equality $D_u\Xi(z^-,z^-,u_1+s\,(u_2-u_1)) = 0$. □

Introduce a map T related to the generalized Riemann problem.

Lemma 4.2. *Let f satisfy* **(f.1)**–**(f.3)**, *Ξ satisfy* **(Ξ.1)**, **(Ξ.3)** *and A_{i_o} be as in* (2.1). *Then, for any $\bar z \in \mathcal{Z}$ and $\bar u \in A_{i_o}$, there exists $\bar\delta > 0$ and a Lipschitz map*

$$T\colon B(\bar z;\bar\delta)^2 \times B(\bar u;\bar\delta) \to A_{i_o}$$

such that

$$\begin{cases} f(u^+) - f(u^-) = \Xi(z^+,z^-,u^-) \\ z^+,z^- \in B(\bar z;\bar\delta) \\ u^+,u^- \in B(\bar u;\bar\delta) \end{cases} \qquad \Longleftrightarrow \qquad u^+ = T(z^+,z^-,u^-) .$$

Furthermore,

$$\begin{aligned}
\left\|T(z^+,z^-,u^-) - u^-\right\| &= \mathcal{O}(1)\left\|z^+-z^-\right\| , && (4.2)\\
\left\|T(z^+,z^-,u_2) - T(z^+,z^-,u_1) - (u_2-u_1)\right\| &= \mathcal{O}(1)\left\|z^+-z^-\right\| \|u_2-u_1\| . && (4.3)
\end{aligned}$$

Proof. Since $\bar u \in A_{i_o}$, **(f.1)** and **(f.2)** ensure that the function f is locally invertible at $\bar u$. We define

$$T(z^+, z^-, u^-) = f^{-1}\left(f(u^-) + \Xi(z^+, z^-, u^-)\right). \tag{4.4}$$

By **(Ξ.1)**, **(Ξ.3)** we compute

$$\begin{aligned}
\left\|T(z^+, z^-, u^-) - u^-\right\| &= \left\|T(z^+, z^-, u^-) - f^{-1}\left(f(u^-)\right)\right\| \\
&= \mathcal{O}(1)\left\|\Xi(z^+, z^-, u^-) - \Xi(z^-, z^-, u^-)\right\| \\
&= \mathcal{O}(1)\left\|z^+ - z^-\right\|,
\end{aligned}$$

proving (4.2). Introduce the smooth map

$$b(\xi, \Delta, v) = f^{-1}\left(f(u_1 + v) + \xi + \Delta\right) - f^{-1}\left(f(u_1) + \xi\right) - v.$$

Since $b(\xi, 0, 0) = b(0, 0, v) = 0$, the estimate

$$b(\xi, \Delta, v) = \mathcal{O}(1)\left[\|\xi\| \cdot \|v\| + \|\Delta\|\right]$$

holds, see [5, § 2.9]. The left hand side of (4.3) can be written as

$$\begin{aligned}
&\left\|T(z^+, z^-, u_2) - T(z^+, z^-, u_1) - (u_2 - u_1)\right\| \\
&= \left\|b\left[\Xi(z^+, z^-, u_1), \Xi(z^+, z^-, u_2) - \Xi(z^+, z^-, u_1), u_2 - u_1\right]\right\| \\
&\le \mathcal{O}(1)\left[\left\|\Xi(z^+, z^-, u_1)\right\| \cdot \|u_2 - u_1\| + \left\|\Xi(z^+, z^-, u_2) - \Xi(z^+, z^-, u_1)\right\|\right] \\
&\le \mathcal{O}(1)\left\|z^+ - z^-\right\| \|u_2 - u_1\|.
\end{aligned}$$

□

4.2 The Case ζ Piecewise Constant

In this section, we consider the case of $\mathcal{I}$ being finite, with ζ being piecewise constant. We index the points $x \in \mathcal{I}$ so that $x_i < x_j$ if and only if $i < j$. In this case, the general Definition 2.1 reduces to the following one, often found in the literature, see for instance [9, 10, 11, 16].

Definition 4.3. *A weak solution to the Cauchy Problem (1.5) with a piecewise constant ζ is a map $u \in \mathbf{C}^0([0, +\infty[\,; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \Omega))$ with $u(t) \in \mathbf{BV}(\mathbb{R}; \Omega)$, left continuous, for all $t \ge 0$, such that for all $\varphi \in \mathbf{C}^1_c(]0, +\infty[\times \mathbb{R}; \mathbb{R})$ whose support does not intersect $[0, +\infty[\times \mathcal{I}$*

$$\int_0^{+\infty} \int_{\mathbb{R}} \left(u\, \partial_t \varphi + f(u)\, \partial_x \varphi\right) \mathrm{d}x\, \mathrm{d}t = 0, \tag{4.5}$$

$u(0) = u_o$ and for all $x \in \mathcal{I}$

$$f\left(u(t, x+)\right) - f\left(u(t, x)\right) = \Xi\left(\zeta(x+), \zeta(x), u(t, x)\right) \quad \text{for a.e. } t \in [0, +\infty[\,.$$

4.2.1 The Generalized Riemann Problem

By *Generalized Riemann Problem* we consider the Cauchy Problem (1.5) with ζ and the initial datum u_o being as follows:

$$\zeta(x) = z^- \chi_{]-\infty,0[}(x) + z^+ \chi_{]0,+\infty[}(x) \quad \text{and} \quad u_o(x) = u^\ell \chi_{]-\infty,0[}(x) + u^r \chi_{]0,+\infty[}(x)\,. \tag{4.6}$$

For $u \in A_{i_o}$, call $\sigma_i \to H_i(\sigma_i)(u)$ the Lax curve of the i-th family exiting u, see [5, § 5.2] or [12, § 9.3]. Introduce recursively the states $w_0, \ldots, w_{n+1}$ with $w_0 = u^\ell$, $w_{n+1} = u^r$ and

$$\begin{cases} w_{i+1} = H_{i+1}(\sigma_{i+1})(w_i) & \text{if } i = 0, \ldots, i_o - 1,\\ w_{i_o+1} = T(z^+, z^-, w_{i_o}) & \\ w_{i+1} = H_i(\sigma_i)(w_i) & \text{if } i = i_o + 1, \ldots, n\,. \end{cases}$$

If $z^+ - z^-$ is sufficiently small, [1, Lemma 3] ensures that the waves' sizes $(\sigma_1, \ldots, \sigma_n)$ and the states $(w_1, \ldots, w_n)$ exist, are uniquely defined and are Lipschitz continuous functions of z^-, z^+, u^ℓ, u^r, which ensures also the well posedness of the Generalized Riemann Problem (1.5), (4.6). The following notation is of use below:

$$(\sigma_1, \ldots, \sigma_n) = E(z^+, z^-, u^r, u^\ell)\,. \tag{4.7}$$

We thus write the solution u to the Generalized Riemann Problem (1.5) (4.6), in the sense of Definition 4.3, as the glueing along $x = 0$ of the Lax solutions to the (standard) Riemann Problems

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = u^\ell \chi_{]-\infty,0[}(x) + w^{i_o} \chi_{]0,+\infty[}(x), \end{cases} \qquad \begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0,x) = w^{i_o+1} \chi_{]-\infty,0[}(x) + u^r \chi_{]0,+\infty[}(x). \end{cases}$$

4.2.2 Interaction Estimates

Let $u^\ell, u^r \in \Omega$ be initial states for the Generalized Riemann Problem (1.5) (4.6). We separate the waves with negative or positive propagation speed as follows:

$$\begin{gathered} \boldsymbol{\sigma}' = (\sigma_1, \ldots, \sigma_{i_o}, 0, \ldots, 0), \qquad \boldsymbol{\sigma}'' = (0, \ldots, 0, \sigma_{i_o+1}, \ldots, \sigma_n), \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}' + \boldsymbol{\sigma}'' \in \mathbb{R}^n. \end{gathered} \tag{4.8}$$

Given two n-tuples of waves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the waves i with size α_i and j with size β_j are *approaching* whenever $i > j$ or $\min\left\{\alpha_i, \beta_j\right\} < 0$. Call $\mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ the set of these pairs.

In the following we recall several lemmas which are straightforward generalizations of results in [1].

Lemma 4.4. *Let f satisfy* **(f.1)**–**(f.3)**, *Ξ satisfy* **(Ξ.1)**, **(Ξ.3)** *and A_{i_o} be as in* (2.1). *Fix $z^+, z^- \in \mathcal{Z}$ and $u^\ell, u^r \in A_{i_o}$. Then, there exists a $\delta > 0$ such that if $z^+, z^- \in B(\bar z; \delta)$, $u^\ell, u^r \in B(\bar u; \delta)$, we have*

$$\begin{aligned} \left\| u^r - u^\ell \right\| &= \mathcal{O}(1) \left(\|\boldsymbol{\sigma}\| + \left\| z^+ - z^- \right\| \right), \\ \|\boldsymbol{\sigma}\| &= \mathcal{O}(1) \left(\left\| u^r - u^\ell \right\| + \left\| z^+ - z^- \right\| \right). \end{aligned}$$

Proof. By Lemma 4.2, we get

$$\left\|u^r - u^\ell\right\| \le \sum_{i=1}^{n+1} \|w_i - w_{i-1}\| = \mathcal{O}(1)\|\boldsymbol{\sigma}\| + \left\|T(z^+, z^-, w_{i_o}) - w_{i_o}\right\| = \mathcal{O}(1)\Big(\|\boldsymbol{\sigma}\| + \left\|z^+ - z^-\right\|\Big).$$

By the Lipschitz continuity of E as defined in (4.7), we get

$$\begin{aligned}
\|\boldsymbol{\sigma}\| &= \left\|E(z^+, z^-, u^r, u^\ell) - E(z^+, z^-, T(z^+, z^-, u^\ell), u^\ell)\right\| \\
&= \mathcal{O}(1)\left\|u^r - T(z^+, z^-, u^\ell)\right\| \\
&= \mathcal{O}(1)\left(\left\|u^r - u^\ell\right\| + \left\|u^\ell - T(z^+, z^-, u^\ell)\right\|\right) \\
&= \mathcal{O}(1)\left(\left\|u^r - u^\ell\right\| + \left\|z^+ - z^-\right\|\right)
\end{aligned}$$

completing the proof. □

Lemma 4.5 ([1, Lemma 5]). *Let f satisfy* **(f.1)**–**(f.3)** *and A_{i_o} be as in* (2.1). *For $u \in \Omega$ sufficiently close to $\bar{u} \in A_{i_o}$ and $y_1, y_2, \boldsymbol{\alpha} \in \mathbb{R}^n$ sufficiently small, we have*

$$\left\|y_2 + H(\boldsymbol{\alpha})(u) - H(\boldsymbol{\alpha})(u + y_1)\right\| = \mathcal{O}(1)\left(\|\boldsymbol{\alpha}\|\,\|y_1\| + \|y_1 - y_2\|\right).$$

Lemma 4.6. *Let f satisfy* **(f.1)**–**(f.3)**, *Ξ satisfy* **(Ξ.1)**, **(Ξ.3)** *and A_{i_o} be as in* (2.1). *For $u \in \Omega$ sufficiently close to $\bar{u} \in A_{i_o}$, $z^+, z^- \in \mathcal{Z}$ sufficiently close to $\bar{z} \in \mathcal{Z}$ and $\boldsymbol{\alpha} \in \mathbb{R}^n$ sufficiently small, we have*

$$\left\|T(z^+, z^-, H(\boldsymbol{\alpha})(u)) - H(\boldsymbol{\alpha})(T(z^+, z^-, u))\right\| = \mathcal{O}(1)\,\|\boldsymbol{\alpha}\|\,\left\|z^+ - z^-\right\|.$$

Proof. Applying Lemma 4.5 with $y_1 = T(z^+, z^-, u) - u$ and $y_2 = T(z^+, z^-, H(\boldsymbol{\alpha})(u)) - H(\boldsymbol{\alpha})(u)$ gives

$$\begin{aligned}
&\left\|T(z^+, z^-, H(\boldsymbol{\alpha})(u)) - H(\boldsymbol{\alpha})(T(z^+, z^-, u))\right\| \\
&\le \mathcal{O}(1)\left(\|\boldsymbol{\alpha}\|\left\|T(z^+, z^-, u) - u\right\| + \left\|T(z^+, z^-, H(\boldsymbol{\alpha})(u)) - H(\boldsymbol{\alpha})(u) - T(z^+, z^-, u) + u\right\|\right).
\end{aligned}$$

The result follows from Lemma 4.2. □

Lemma 4.7. *Let f satisfy* **(f.1)**–**(f.3)**, *Ξ satisfy* **(Ξ.1)**, **(Ξ.3)** *and A_{i_o} be as in* (2.1). *Fix $z^+, z^- \in \mathcal{Z}$ and $u^\ell, u^r \in A_{i_o}$. Then, there exists a $\delta > 0$ such that if $u^r, u^\ell \in B(\bar{u}; \delta)$ and $z^+, z^- \in B(\bar{z}; \delta)$. Let*

$$\begin{aligned}
u^- &= H(\boldsymbol{\alpha})(u^\ell), \\
u^r &= H(\boldsymbol{\beta}'')\left(T\left(z^+, z^-, H(\boldsymbol{\beta}')(u^-)\right)\right), \\
u^r &= H(\boldsymbol{\sigma}'')\left(T\left(z^+, z^-, H(\boldsymbol{\sigma}')(u^\ell)\right)\right),
\end{aligned}$$

with $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma} \in \mathbb{R}^n$ *and* $u^- \in B(\bar{u}; \delta)$. *Then,*

$$\sum_{i=1}^{n} |\sigma_i - \alpha_i - \beta_i| = \mathcal{O}(1) \left(\sum_{(i,j) \in \mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \left|\alpha_i \, \beta_j\right| + \left\|z^+ - z^-\right\| \sum_{i > i_o} |\alpha_i| \right), \tag{4.9}$$

where $\mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, *as above, denotes the set of approaching waves. Analogously, if*

$$\begin{aligned} u^+ &= H(\boldsymbol{\alpha''}) \left(T \left(z^+, z^-, H(\boldsymbol{\alpha'})(u^\ell) \right) \right), \\ u^r &= H(\boldsymbol{\beta})(u^+), \\ u^r &= H(\boldsymbol{\sigma}'') \left(T \left(z^+, z^-, H(\boldsymbol{\sigma}')(u^\ell) \right) \right), \end{aligned}$$

then,

$$\sum_{i=1}^{n} |\sigma_i - \alpha_i - \beta_i| = \mathcal{O}(1) \left(\sum_{(i,j) \in \mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \left|\alpha_i \beta_j\right| + \left\|z^+ - z^-\right\| \sum_{i < i_o} |\beta_i| \right). \tag{4.10}$$

Proof. It is sufficient to prove (4.9), since (4.10) is proved analogously. We set

$$\begin{aligned} \tilde{u} &= H(\boldsymbol{\alpha''} + \boldsymbol{\beta''}) \left(T \left(z^+, z^-, H(\boldsymbol{\alpha'} + \boldsymbol{\beta'})(u^\ell) \right) \right), \\ u_1 &= H(\boldsymbol{\beta}'') \circ H(\boldsymbol{\alpha}'') \left(T \left(z^+, z^-, H(\boldsymbol{\alpha}')(u^\ell) \right) \right), \\ u_2 &= H(\boldsymbol{\beta}'') \left(T \left(z^+, z^-, H(\boldsymbol{\alpha})(u^\ell) \right) \right). \end{aligned}$$

By the Lipschitz continuity of E, we obtain

$$\begin{aligned} \left\|\boldsymbol{\sigma} - (\boldsymbol{\alpha} + \boldsymbol{\beta})\right\| &= \left\|E(z^+, z^-, u^r, u^\ell) - E(z^+, z^-, \tilde{u}, u^\ell)\right\| \\ &= \mathcal{O}(1) \left\|u^r - \tilde{u}\right\| \\ &= \mathcal{O}(1) \left(\|u^r - \tilde{u} + u_1 - u_2\| + \|u_1 - u_2\| \right). \end{aligned}$$

To estimate the first term we consider the function $u^r - \tilde{u} + u_1 - u_2$ which is $\mathbf{C}^2$ w.r.t. $\boldsymbol{\alpha}, \boldsymbol{\beta}$. Moreover, we assume that there are no approaching waves and obtain

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_i, 0, \ldots, 0), \quad \boldsymbol{\beta} = (0, \ldots, 0, \beta_j, \ldots, \beta_n), \quad i \leq j.$$

The case $i = j$ is given in the case of two rarefaction waves and $\alpha_i, \beta_i \geq 0$. If $i \leq i_o$, then $\boldsymbol{\alpha}'' = 0$, $H(\boldsymbol{\alpha'} + \boldsymbol{\beta'})(u) = H(\boldsymbol{\beta'}) \circ H(\boldsymbol{\alpha'})(u)$, whence $u^r = \tilde{u}$ and $u_1 = u_2$.

If $i > i_o$, then $\boldsymbol{\beta}' = 0$ and $H(\boldsymbol{\alpha''} + \boldsymbol{\beta''})(u) = H(\boldsymbol{\beta''}) \circ H(\boldsymbol{\alpha''})(u)$, whence $u^r = u_2$ and $\tilde{u} = u_1$.

In all cases we get $u_0^+ - \tilde{u}^+ + u_1 - u_2 = 0$. Standard considerations (see e.g. [5, § 7.3], [12, § 13.3] or [21]) and Lemma 4.2 lead to

$$\|u^r - \tilde{u} + u_1 - u_2\| = \mathcal{O}(1) \sum_{(i,j) \in \mathcal{A}_{\boldsymbol{\alpha}.\boldsymbol{\beta}}} \left|\alpha_i \, \beta_j\right|$$

in the general case.

Concerning $\|u_1 - u_2\|$, we get

$$\|u_1 - u_2\| \leq \mathcal{O}(1) \left\| H(\boldsymbol{\alpha''}) \left(T(z^+, z^-, H(\boldsymbol{\alpha}')(u^\ell)) \right) - T(z^+, z^-, H(\boldsymbol{\alpha})(u^\ell)) \right\| .$$

The equality $H(\boldsymbol{\alpha})(u) = H(\boldsymbol{\alpha}'') \circ H(\boldsymbol{\alpha}')(u)$ and Lemma 4.6 with $u = H(\boldsymbol{\alpha}')(u^\ell)$ lead to

$$\|u_1 - u_2\| = \mathcal{O}(1) \left\| z^+ - z^- \right\| \sum_{i > i_o} |\alpha_i| .$$

The result follows. $\square$

Lemma 4.7 suggests that the quantity $\|z^+ - z^-\|$ is a convenient way to measure the strength of the zero–waves associated to the coupling condition. More precisely, we define the strength of the zero–wave at a junction with parameters $z^+, z^- \in \mathcal{Z}$ as $\sigma = \|z^+ - z^-\|$.

Wave-front tracking approximate solutions

We adapt the wave-front tracking techniques from [1, 5, 11, 15] to construct a sequence of approximate solutions to the Cauchy problem (1.5) and prove uniform **BV**-estimates in space. The approximate solutions converge towards a solution to the Cauchy problem with finitely many junctions. First, we define the approximations.

Definition 4.8. *Let $\zeta \in \mathbf{BV}(\mathbb{R}; \mathcal{Z})$ be piecewise constant. For $\varepsilon > 0$, a continuous map*

$$u^\varepsilon \colon [0, +\infty[\to \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \mathbb{R}^n)$$

is an ε-approximate solution to (1.5) if the following conditions hold:

- *u^ε as a function of (t, x) is piecewise constant with discontinuities along finitely many straight lines in the (t, x)-plane. There are only finitely many wave-front interactions and at most two waves interact with each other. There are four types of discontinuities: shocks (or contact discontinuities), rarefaction waves, non–physical waves and zero–waves. We distinguish these waves' indexes in the sets $\mathcal{J} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP} \cup \mathcal{ZW}$, the generic index in $\mathcal{J}$ being α.*

- *At a shock (or contact discontinuity) $x_\alpha = x_\alpha(t)$, $\alpha \in \mathcal{S}$, the traces $u^+ = u^\varepsilon(t, x_\alpha+)$ and $u^- = u^\varepsilon(t, x_\alpha-)$ are related by $u^+ = H_{i_\alpha}(\sigma_\alpha)(u^-)$ for $1 \leq i_\alpha \leq n$ and wave-strength σ_α. If the i_α-th family is genuinely nonlinear, the Lax entropy condition $\sigma_\alpha < 0$ holds and*

$$\left| \dot{x}_\alpha - \lambda_{i_\alpha}(u^+, u^-) \right| \leq \varepsilon,$$

  where $\lambda_{i_\alpha}(u^+, u^-)$ is the wave speed described by the Rankine-Hugoniot conditions.

- *For a rarefaction wave $x_\alpha = x_\alpha(t)$, $\alpha \in \mathcal{R}$ the traces are related by $u^+ = H_{i_\alpha}(\sigma_\alpha)(u^-)$ for a genuinely nonlinear family $1 \leq i_\alpha \leq n$ and wave-strength $0 < \sigma_\alpha \leq \varepsilon$. Moreover,*

$$\left| \dot{x}_\alpha - \lambda_{i_\alpha}(u^+) \right| \leq \varepsilon.$$

- *All non–physical fronts* $x = x_\alpha(t)$, $\alpha \in \mathcal{NP}$ *travel at the same speed* $\dot{x}_\alpha = \hat{\lambda}$ *with* $\hat{\lambda} > \sup_{u,i} \left|\lambda_i(u)\right|$. *The total strength of all non–physical fronts is uniformly bounded by*

$$\sum_{\alpha \in \mathcal{NP}} \left\| u^\varepsilon(t, x_\alpha+) - u^\varepsilon(t, x_\alpha-) \right\| \le \varepsilon \quad \textit{for all } t > 0\,.$$

- *Zero–waves are located at the junctions* $x_\alpha \in \mathcal{I}$. *At a zero–wave* x_α, $\alpha \in \mathcal{ZW}$, *the traces are related by the coupling condition* $u^+ = T\left(\zeta(x_\alpha+), \zeta(x_\alpha-), u^-\right)$ *for all* $t > 0$ *except at the interaction times.*

- *The initial data satisfies* $\left\| u^\varepsilon(0, \cdot) - u_o \right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \le \varepsilon$.

Next, we prove the existence of ε–approximate solutions.

Theorem 4.9. *Let* $\Omega \subseteq \mathbb{R}^n$ *be open,* f *satisfy* **(f.1)**–**(f.3)** *and* Ξ *satisfy* **(Ξ.1)**–**(Ξ.3)**. *Fix* $\bar{u} \in A_{i_o}$ *and* $\bar{z} \in \mathcal{Z}$. *Then, there exist* $\delta > 0$ *such that for all piecewise constant* $\zeta \in \mathbf{BV}(\mathbb{R}; \mathcal{Z})$ *with*

$$\zeta(\mathbb{R}) \subseteq B(\bar{z}; \delta) \qquad \textit{and} \qquad \mathrm{TV}\,(\zeta) < \delta$$

and for all initial data u_o *with*

$$u_o(\mathbb{R}) \subseteq B(\bar{u}; \delta)\,, \quad \mathrm{TV}\,(u_o) < \delta,$$

for every ε *sufficiently small there exists an* ε*–approximate solution to* (1.5) *in the sense of Definition 4.8. Moreover, the total variation in space* $\mathrm{TV}\,(u^\varepsilon(t, \cdot))$ *and the total variation in time* $\mathrm{TV}\,(u^\varepsilon(\cdot, x))$, $x \neq x_\alpha$, $\alpha \in \mathcal{ZW}$ *are bounded uniformly for* ε *sufficiently small and for every piecewise constant* ζ *with* $\mathrm{TV}\,(\zeta) < \delta$.

Proof. *Description of the wave front tracking algorithm.* For notational convenience, we drop the ε. Let ε and $\mathrm{TV}\,(\zeta)$ be sufficiently small, then we construct the approximate solution in the following way:

- To obtain piecewise constant approximate solutions, we discretize the rarefactions as in [5]. For a fixed small parameter δ_R, each rarefaction of size σ is divided into $m = [[\sigma/\delta_R]] + 1$ wave-fronts, each one with size $\sigma/m \le \delta_R$.

- Given initial data u_o, we can define a piecewise constant approximation $u(0, \cdot)$ satisfying the requirements of Definition 4.8 and

$$\mathrm{TV}\,\left(u(0, \cdot)\right) \le \mathrm{TV}\,(u_o)\,.$$

  For small t, $u(t, x)$ is constructed by solving the generalized Riemann problem at every point x_α with $\alpha \in \mathcal{ZW}$ and by solving the homogeneous Riemann Problem at every remaining discontinuity in $u(0, \cdot)$.

- At every interaction point, a new Riemann Problem arises. Notice that because of their fixed speed, two non–physical fronts cannot interact with each other, neither can the zero–waves. Moreover, by a slight modification of the speed of some waves (only among shocks, contact discontinuities and rarefactions), it is possible to achieve the property that not more than two wave-fronts interact at a point.

After each interaction time, the number of wave-fronts may increase. In order to prevent this number to become infinite in finite time, a specific treatment has been proposed for waves whose strength is below a threshold value ρ by means of a simplified Riemann solver [5, § 7.2].

Suppose that two wave–fronts of strengths σ, σ' interact at a given point (t, x). If $x \neq x_\alpha$, $\alpha \in \mathcal{ZW}$, we use the classical accurate or simplified homogeneous Riemann solver as in [5, § 7.2]. Assume now that $x = x_\alpha$, $\alpha \in \mathcal{ZW}$. We briefly recall the different situations that can occur, see [1] for more details.

- If the wave approaching the zero wave is physical and $|\sigma\,\sigma'| \geq \rho$ we use the (accurate) generalized Riemann solver.

- If the wave approaching the zero wave is physical and $\left|\sigma\,\sigma'\right| < \rho$, we use a simplified Riemann solver. Assume that the wave-front on the right is the zero–wave. Let u_l, $u_m = H_i(\sigma)(u_l)$, $u_r = T(\zeta(x_\alpha+), \zeta(x_\alpha-), u_m)$ be the states before the interaction. We define the auxiliary states

$$\tilde{u}_m = T(\zeta(x_\alpha+), \zeta(x_\alpha-), u_l)\,, \qquad \tilde{u}_r = H_i(\sigma)(\tilde{u}_m)\,.$$

  Then, three fronts propagate after the interaction: the zero–wave $(u_l, \tilde{u}_m)$, the physical front $(\tilde{u}_m, \tilde{u}_r)$ and the non–physical one $(\tilde{u}_r, u_r)$. Due to the commutation defect, we use Lemma 4.6 to ensure that the introduced error, i.e. the size of the generated non–physical wave, is of second order.

- Suppose now that the wave on the left belongs to $\mathcal{NP}$. Again we use a simplified solver: let $u_l,\ u_m,\ u_r = T(\zeta(x_\alpha+), \zeta(x_\alpha-), u_m)$ be the states before the interaction and define the new state $\tilde{u}_l = T(\zeta(x_\alpha+), \zeta(x_\alpha-), u_l)$. After the interaction time, only two fronts propagate: the zero–wave $(u_l, \tilde{u}_l)$ and the non–physical wave $(\tilde{u}_l, u_r)$. Lemma 4.2 ensures that the error we made is quadratic.

Stability of the algorithm. We recall how junctions are taken care in [1], within the Glimm functionals [14]:

$$V(t) = \sum_{\alpha \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP} \cup \mathcal{ZW}} |\sigma_\alpha|\,, \qquad Q(t) = \sum_{\alpha, \beta \in \tilde{\mathcal{A}}} |\sigma_\alpha \sigma_\beta|, \tag{4.11}$$

measuring respectively the total wave strengths and the interaction potential in $u(t, \cdot)$. Remember that if $\alpha \in \mathcal{ZW}$ then the strength of the wave located in x_α is given by $\sigma_\alpha = \left\|\zeta(x_\alpha+) - \zeta(x_a-)\right\|$. Notice that there exists a constant $C > 1$ (see Lemma 4.4) such that

$$\frac{1}{C}\left(\mathrm{TV}\,(u(t, \cdot)) + \mathrm{TV}\,(\zeta)\right) \leq V(t) \leq C\left(\mathrm{TV}\,(u(t, \cdot)) + \mathrm{TV}\,(\zeta)\right)\,.$$

Thus, according to the estimates in Lemma 4.2 and Lemma 4.7 and to the classical ones [5, Lemma 7.2], at every time τ when two waves of strengths σ, σ' interact, we get:

$$V(\tau+) - V(\tau-) \ \leq \ C\left|\sigma\,\sigma'\right|, \tag{4.12}$$
$$Q(\tau+) - Q(\tau-) \ \leq \ (C\,V(\tau-) - 1)\left|\sigma\,\sigma'\right|. \tag{4.13}$$

Therefore, if V is sufficiently small, (4.13) implies

$$Q(\tau+) - Q(\tau-) \leq -\frac{1}{2}\left|\sigma\,\sigma'\right|. \tag{4.14}$$

By (4.12) and (4.14) we can choose a constant C large enough and $\delta_* > 0$ so that (4.14) holds and the quantity

$$\Upsilon(t) = V(t) + C\,Q(t) \tag{4.15}$$

decreases at every interaction time τ provided that $V(\tau-)$ is sufficintly small. Thus, by standard arguments [1], choosing initial data u_o satisfying

$$\mathrm{TV}\,(u_o) + \mathrm{TV}\,(\zeta) \le \delta\,, \tag{4.16}$$

ensures that the ε–approximate solution satisfies for any $t \ge 0$,

$$\mathrm{TV}\,(u(t,\cdot)) + \mathrm{TV}\,(\zeta) \le \delta_*\,. \tag{4.17}$$

The same arguments used in [1] allow to control the total number of wave fronts, that the maximal strength of a rarefaction wavelet is bounded by $\mathcal{O}(1)\,\varepsilon$, that the sum of the strengths of all $\mathcal{NP}$ waves is also bounded by $\mathcal{O}(1)\,\varepsilon$ and that $t \to \mathrm{TV}\,(u(t,x))$, for $x \notin \mathcal{I}$, is bounded uniformly in ε and ζ. $\square$

Passing to the Limit $\boldsymbol{\varepsilon} \to 0$

Theorem 4.10. *Let $\Omega \subseteq \mathbb{R}^n$ be open, f satisfy* **(f.1)**–**(f.3)** *and Ξ satisfy* **(Ξ.1)**–**(Ξ.3)**. *Fix $\bar u \in A_{i_o}$ and $\bar z \in \mathcal{Z}$. Then, there exist $\delta > 0$ such that for all piecewise constant $\zeta \in \mathbf{BV}(\mathbb{R};\mathcal{Z})$ with*

$$\zeta(\mathbb{R}) \subseteq B(\bar z;\delta) \quad \text{and} \quad \mathrm{TV}\,(\zeta) < \delta$$

and for all initial data u_o with

$$u_o(\mathbb{R}) \subseteq B(\bar u;\delta)\,, \quad \mathrm{TV}\,(u_o) < \delta,$$

the Cauchy Problem (1.5) admits a solution u in the sense of Definition 2.1 enjoying the properties:

(1) *The maps $t \to \mathrm{TV}\,(u(t,\cdot))$ and $t \to \left\|(u(t,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^n)}$ are uniformly bounded and the map $x \to u(t,x)$ is left continuous, for all $t \ge 0$.*

(2) *For all $x \in \mathbb{R}$, the map $t \to u(t,x)$ admits a representative $\tilde u_x$ such that $\mathrm{TV}\,(\tilde u_x)$ is uniformly bounded.*

(3) *For all $t \ge 0$, $u(t,\cdot) \in \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R};\mathbb{R}^n)$ and the map $t \to u(t,\cdot)$ is $\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$–Lipschitz continuous.*

(4) *For all $T > 0$ and for all open interval $J \subseteq \mathbb{R}\setminus\mathcal{I}$, the map $x \to u(\cdot,x)$ is $\mathbf{L}^1([0,T];\mathbb{R}^n)$–Lipschitz continuous, with a Lipschitz constant independent of J, $\mathcal{I}$ being the set of points of jump of ζ.*

Proof. For $\varepsilon > 0$ sufficiently small, fix an ε–approximate solution u^ε. By Theorem 4.9, u^ε satisfies **(1)**–**(2)**–**(3)**–**(4)**. By Helly Theorem as extended in [5, § 2.5], there exists a map $u\colon [0,+\infty[\times\mathbb{R} \to \mathbb{R}^n$ such that, up to a subsequence, $u^\varepsilon(t,\cdot)$ converges to $u(t,\cdot)$ in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R};\mathbb{R}^n)$ for all $t \in [0,+\infty[$ and u satisfys **(1)**, **(3)**.

We now prove that u satisfies **(4)**. By possibly passing to subsequences, we may assume that $u^\varepsilon(\cdot,x) \to u(\cdot,x)$ for a.e. $x \in \mathbb{R}$ in $\mathbf{L}^1([0,T];\mathbb{R}^n)$. Since u^ε satisfies **(4)**, we may pass the Lipschitz continuity of $x \to u^\varepsilon(\cdot,x)$ to the limit $\varepsilon \to 0$ for a.e. $x \in \mathbb{R}$. The limit u is left continuous in the space variable x by **(1)**, hence u satisfies **(4)**.

We now prove that for all $x \in \mathbb{R}$, $u^\varepsilon(\cdot,x) \to u(\cdot,x)$ in $\mathbf{L}^1([0,T];\mathbb{R}^n)$. To this aim, fix an arbitrary $x \in \mathbb{R}$ and $y < x$ such that $]y,x[\subset \mathbb{R} \setminus \mathcal{I}$ and $u^\varepsilon(\cdot,y) \to u(\cdot,y)$ in $\mathbf{L}^1([0,T];\mathbb{R}^n)$. Both $x \to u^\varepsilon(\cdot,x)$ and $x \to u(\cdot,x)$ are Lipschitz continuous, hence

$$
\begin{aligned}
\int_0^T \left\| u^\varepsilon(t,x) - u(t,x) \right\| \mathrm{d}t &\leq \mathcal{O}(1)\,|x-y| + \int_0^T \left\| u^\varepsilon(t,y) - u(t,y) \right\| \mathrm{d}t \,; \\
\limsup_{\varepsilon \to 0} \int_0^T \left\| u^\varepsilon(t,x) - u(t,x) \right\| \mathrm{d}t &\leq \mathcal{O}(1)\,|x-y| \,.
\end{aligned}
$$

Letting now $y \to x$ we obtained the desired convergence.

Note that $\mathrm{TV}\left(u^\varepsilon(\cdot,x)\right)$ is bounded uniformly, so that $u(\cdot,x)$ admits a **BV** representative, proving **(2)**.

Finally, we prove that u solves (1.5). Choose $\varphi \in \mathbf{C}^1_c(]0,T[\times \mathbb{R};\mathbb{R})$ and K so that $\operatorname{spt}\varphi \subseteq]0,T[\times]-K,K[$. Then,

$$
\int_0^T \int_{-K}^{K} \left(u^\varepsilon\, \partial_t \varphi + f(u^\varepsilon)\, \partial_x \varphi\right) \mathrm{d}x\ \mathrm{d}t = \int_0^T \sum_{\alpha \in \mathcal{J}} e_{\varepsilon,\alpha}(t)\, \varphi(t, x_\alpha(t))\, \mathrm{d}t \,,
$$

where $e_{\varepsilon,\alpha}(t)$ measures the error in the Rankine–Hugoniot conditions along the discontinuity supported on $x = x_\alpha(t)$, $\alpha \in \mathcal{J}$, i.e.,

$$
e_{\varepsilon,\alpha}(t) = \dot{x}_\alpha \left(u^\varepsilon(t, x_\alpha(t)+) - u^\varepsilon(t, x_\alpha(t))\right) - \left(f(u^\varepsilon(t, x_\alpha(t)+)) - f(u^\varepsilon(t, x_\alpha(t)))\right).
$$

By Definition 4.8 and standard estimates,

$$
\sum_{\alpha \in \mathcal{J} \setminus \mathcal{ZW}} |e_{\varepsilon,\alpha}(t)| \leq \mathcal{O}(1)\, \varepsilon \,.
$$

Since the coupling condition (1.3) holds along the zero–waves $\alpha \in \mathcal{ZW}$, we obtain

$$
\left\| \int_0^T\!\!\int_{-K}^{K} \left(u^\varepsilon \partial_t \varphi + f(u^\varepsilon) \partial_x \varphi\right) \mathrm{d}x\ \mathrm{d}t + \int_0^T \sum_{\alpha \in \mathcal{ZW}} \varphi(t, x_\alpha)\, \Xi(\zeta(x_\alpha+), \zeta(x_\alpha), u^\varepsilon(t, x_\alpha))\, \mathrm{d}t \right\| \leq C\, \varepsilon. \tag{4.18}
$$

As $\varepsilon \to 0$ the first integrand above converges to the integrand on the left hand side of (4.5). Using **(Ξ.1)** and the convergence $u^\varepsilon(\cdot, x_\alpha) \to u(\cdot, x_\alpha)$ in $\mathbf{L}^1([0,T];\mathbb{R}^n)$, we prove the convergence of the second integrand in the left hand side of (4.18), obtaining

$$
\int_0^{+\infty}\!\!\int_{\mathbb{R}} \left(u\, \partial_t \varphi + f(u)\, \partial_x \varphi\right) \mathrm{d}x\ \mathrm{d}t + \int_0^{+\infty} \sum_{\alpha \in \mathcal{ZW}} \varphi(t, x_\alpha)\, \Xi(\zeta(x_\alpha+), \zeta(x_\alpha), u(t, x_\alpha))\, \mathrm{d}t = 0 \,,
$$

completing the proof. □

4.3 Convergence Towards a General ζ

Proof of Theorem 2.2. The proof consists of different steps.

Step 1: Approximation of ζ. Let $\zeta \in \mathbf{BV}(\mathbb{R};\mathcal{Z})$. Call $\mathcal{I}$ the, possibly infinite, set of points of jump in ζ. Recall that $D\zeta$ is a finite measure. By Lusin Theorem [20, Theorem 2.24], for any $h>0$, there exists a $g^h \in \mathbf{C}^0_c(\mathbb{R};\mathbb{R}^p)$ such that $\left\| g^h(x) \right\| \leq 1$ and

$$\|D\zeta\| \left(\left\{ x \in \mathbb{R} \colon g^h(x) \neq v(x) \right\} \right) < h\,. \tag{4.19}$$

Introduce points $\{x_1, \ldots, x_{N_h-1}\} \in \mathbb{R}$ such that[1]:

(i) $x_0 = -\infty$, $x_1 < -1/h$, $x_{i-1} < x_i$ for $i = 2, \ldots, N_h - 1$, $x_{N_h-1} > 1/h$ and $x_{N_h} = +\infty$.

(ii) $\sum_{x \in \mathcal{I} \setminus \mathcal{I}^h} \left\| \Delta\zeta(x) \right\| < h$ for a suitable set of points $\mathcal{I}^h$ contained in $\{x_1, x_2, \ldots, x_{N_h-1}\}$.

(iii) Whenever $x_i \in \mathcal{I}^h$, $\mathrm{TV}\left(\zeta, [x_{i-1}, x_i[\right) < h/(1 + \sharp\mathcal{I}^h)$.

(iv) $\mathrm{TV}\left(\zeta,]x_{i-1}, x_i[\right) < h$ for all $i = 1, \ldots, N_h$.

(v) $\left\| g^h\left(x'\right) - g^h\left(x''\right) \right\| < h$ for $x', x'' \in]x_{i-1}, x_i[$, $i = 1, \ldots, N_h$.

(vi) $x_i - x_{i-1} \in \left]0, h\right[$ for all $i = 2, \ldots, N_h - 1$.

Such points exist since all these properties are stable if further points are inserted. We approximate ζ by means of the piecewise constant left continuous function

$$\zeta^h(x) = \zeta\left(-\infty\right) \chi_{]-\infty, x_1]}(x) + \sum_{i=2}^{N_h-1} \zeta\left(x_{i-1}+\right)\ \chi_{]x_{i-1}, x_i]}(x) + \zeta\left(x_{N_h-1}+\right) \chi_{]x_{N_h-1}, +\infty[}(x). \tag{4.20}$$

By Theorem 4.10, there exists a solution u^h to the Cauchy Problem (1.5) with ζ substituted by ζ^h as in (4.20). We prove that as $h \to 0$ the solution u^h converges in $\mathbf{L}^1_{\mathbf{loc}}$ to a solution to (1.5), possibly up to a subsequence.

Step 2: Select a Convergent Subsequence. We claim that there exists a map u and a convergent subsequence, which we keep denoting u^h, such that

$$\begin{array}{rcll} u^h(t,\cdot) & \to & u(t,\cdot) \text{ in } \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R};\mathbb{R}^n) \text{ for all } t; & (4.21)\\ u^h(\cdot,x) & \to & u(\cdot,x) \text{ in } \mathbf{L}^1_{\mathbf{loc}}([0,+\infty[;\mathbb{R}^n) \text{ for all } x; & (4.22)\\ t & \to & u(t,\cdot) \text{ is Lipschitz continuous in } \mathbf{L}^1(\mathbb{R};\mathbb{R}^n); & (4.23)\\ \mathrm{TV}\left(u(t,\cdot)\right) & \text{is} & \text{bounded uniformly in } t\,; & (4.24)\\ x \to u(t,x) & \text{is} & \text{left continuous for all } t \geq 0\,. & (4.25) \end{array}$$

Indeed, by **(1)** and **(3)** in Theorem 4.10, we can apply Helly Theorem as presented in [5, § 2.5] obtaining the existence of a map u satisfying (4.21), (4.23), (4.24) and (4.25).

We are left with the convergence (4.22). Introduce a point $y < x$ and all the points of jump $\bar{x}_0, \ldots, \bar{x}_{M+1}$ (for a suitable $M \geq 0$) in ζ^h such that

$$-\infty \leq \bar{x}_0 < y \leq \bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_M < x \leq \bar{x}_{M+1} \leq +\infty\,.$$

[1] Everywhere, $\sharp A$ stands the (finite) cardinality of the set A.

We now estimate

$$
\begin{aligned}
\mathrm{TV}\left(\zeta^h;[y,x[\right) &= \sum_{i=1}^{M}\left\|\Delta\zeta^h\left(\bar x_i\right)\right\| = \sum_{i=1}^{M}\left\|\zeta(\bar x_i+)-\zeta(\bar x_{i-1}+)\right\| \\
&\le \left\|\zeta(\bar x_1-)-\zeta(\bar x_0+)\right\| + \left\|\zeta(\bar x_1+)-\zeta(\bar x_1-)\right\| + \sum_{i=2}^{M}\left\|\zeta(\bar x_i+)-\zeta(\bar x_{i-1}+)\right\| \\
&\le \mathrm{TV}\left(\zeta,]\bar x_0,\bar x_1[\right) + \mathrm{TV}\left(\zeta,]y,x[\right) \\
&\le h + \mathrm{TV}\left(\zeta,]y,x[\right) .
\end{aligned} \tag{4.26}
$$

where to get to the last line above we used **(iv)**.

Fix a positive T. By the triangle inequality, **(4)** in Theorem 4.10, inequality (4.26) and Lemma 4.2, since $u^h(t,\bar x_i+) = T\left(\zeta^h(\bar x_i+),\zeta^h(\bar x_i),u^h(t,x_i)\right)$,

$$
\begin{aligned}
&\int_0^T \left\|u^h(t,x)-u^h(t,y)\right\| \mathrm{d}t \\
\le\ & \int_0^T \left\|u^h(t,x)-u^h(t,\bar x_M+)\right\| \mathrm{d}t + \int_0^T \left\|u^h(t,\bar x_M+)-u^h(t,\bar x_M)\right\| \mathrm{d}t \\
&+\sum_{i=1}^{M-1}\left(\int_0^T \left\|u^h(t,\bar x_{i+1})-u^h(t,\bar x_i+)\right\| \mathrm{d}t + \int_0^T \left\|u^h(t,\bar x_i+)-u^h(t,\bar x_i)\right\| \mathrm{d}t\right) \\
&+\int_0^T \left\|u^h(t,\bar x_1)-u^h(t,y+)\right\| \mathrm{d}t + \int_0^T \left\|u^h(t,y+)-u^h(t,y)\right\| \mathrm{d}t \\
\le\ & \mathcal{O}(1)\left|x-\bar x_M\right| + \mathcal{O}(1)\left\|\Delta\zeta^h(\bar x_M)\right\| \\
&+\mathcal{O}(1)\sum_{i=1}^{M-1}\left(\left|\bar x_{i+1}-\bar x_i\right| + \left\|\Delta\zeta^h(\bar x_i)\right\|\right) + \mathcal{O}(1)\left|\bar x_1-y\right| + \mathcal{O}(1)\left\|\Delta\zeta^h(y)\right\| \\
\le\ & \mathcal{O}(1)\left(|x-y| + \mathrm{TV}\left(\zeta^h,[y,x[\right)\right) \\
\le\ & \mathcal{O}(1)\left(|x-y| + h + \mathrm{TV}\left(\zeta,[y,x[\right)\right) .
\end{aligned} \tag{4.27}
$$

Since u^h converges to u in $\mathbf{L}^1_{\mathbf{loc}}\left([0,+\infty[\times\mathbb{R},\mathbb{R}^n\right)$ too, possibly passing to a subsequence, we may assume that for a.e. $x\in\mathbb{R}$ we have $u^h(\cdot,x)\to u(\cdot,x)$ in $\mathbf{L}^1_{\mathbf{loc}}([0,+\infty[;\mathbb{R}^n)$. Pass to the limit $h\to 0$ in (4.27) and obtain that for a.e. $x,y\in\mathbb{R}$ with $y<x$,

$$
\int_0^T \left\|u(t,x)-u(t,y)\right\| \mathrm{d}t \le \mathcal{O}(1)\ \left(|x-y| + \mathrm{TV}\left(\zeta,[y,x[\right)\right) . \tag{4.28}
$$

By the left continuity of $x\to u(t,x)$ and of the right hand side of (4.28) (with respect to both x and y), the inequality (4.28) holds for *all* $x,y\in\mathbb{R}$ with $y<x$.

Fix now an arbitrary $x\in\mathbb{R}$ and choose $y\in\mathbb{R}$ with $y<x$ and such that $u^h(\cdot,y)\to u(\cdot,y)$. By the triangle inequality, (4.27) and (4.28), we have

$$
\int_0^T \left\|u^h(t,x)-u(t,x)\right\| \mathrm{d}t \le \mathcal{O}(1)\left(|x-y| + h + \mathrm{TV}\left(\zeta,[y,x[\right)\right) + \int_0^T \left\|u^h(t,y)-u(t,y)\right\| \mathrm{d}t .
$$

Hence, for almost every $y < x$,

$$\limsup_{h\to 0} \int_0^T \left\| u^h(t,x) - u(t,x) \right\| \mathrm{d}t \le \mathcal{O}(1) \left(|x-y| + \mathrm{TV}\left(\zeta, [y,x[\right) \right)$$

which proves the convergence for every $x \in \mathbb{R}$, since the latter right hand side vanishes as $y \to x^-$.

Step 3: The Limit is a Solution. Fix $\varphi \in \mathbf{C}_c^1(]0,+\infty[\times\mathbb{R};\mathbb{R})$ such that $\operatorname{spt}\varphi \subseteq [0,T] \times [-K,K]$ for suitable $T, K > 0$. Showing that the left hand side below vanishes in the limit $h \to 0$ completes the proof.

$$\begin{aligned}
&\Bigg\| - \int_0^T \int_{-K}^K \left(u\,\partial_t\varphi + f(u)\,\partial_x\varphi\right) \mathrm{d}t\,\mathrm{d}x - \sum_{x\in\mathcal{I}\,,\,|x|\le K} \int_0^T \Xi\left(\zeta(x+), \zeta(x), u(t,x)\right) \varphi(t,x)\,\mathrm{d}t \\
&\quad - \int_0^T \int_{-K}^K D^+_{v(x)} \Xi\left(\zeta(x), \zeta(x), u(t,x)\right) \varphi(t,x)\,\mathrm{d}\|\mu\|(x)\,\mathrm{d}t \Bigg\| \\
\le\ & \mathcal{E}_1^h + \mathcal{E}_2^h + \mathcal{E}_3^h + \mathcal{E}_4^h + \mathcal{E}_5^h + \mathcal{E}_6^h + \mathcal{E}_7^h + \mathcal{E}_8^h + \mathcal{E}_9^h + \mathcal{E}_{10}^h\,.
\end{aligned}$$

To this aim, consider the terms on the right hand side separately:

Term $\mathcal{E}_1^h$: By the $\mathbf{L}^1_{\mathbf{loc}}$ convergence proved in Step 2.

$$\begin{aligned}
\mathcal{E}_1^h &= \left\| - \int_0^T \int_{-K}^K \left(u\,\partial\varphi + f(u)\,\partial_x\varphi\right) \mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{-K}^K \left(u^h\,\partial\varphi + f(u^h)\,\partial_x\varphi\right) \mathrm{d}x\,\mathrm{d}t \right\| \\
&\to 0 \quad \text{as } h \to 0\,.
\end{aligned}$$

Term $\mathcal{E}_2^h$: Each u^h is a solution, hence

$$- \int_0^T \int_{-K}^K \left(u^h\,\partial_t\varphi + f(u^h)\,\partial_x\varphi\right) \mathrm{d}t\,\mathrm{d}x = \sum_{i\colon |x_i|\le K} \int_0^T \Xi\left(\zeta^h(x_i+), \zeta^h(x_i), u^h(t,x_i)\right) \varphi(t,x_i)\,\mathrm{d}t$$

so that

$$\begin{aligned}
\mathcal{E}_2^h &= \Bigg\| - \int_0^T \int_{-K}^K \left(u^h\,\partial_t\varphi + f(u^h)\,\partial_x\varphi\right) \mathrm{d}t\,\mathrm{d}x \\
&\qquad - \sum_{i\colon |x_i|\le K} \int_0^T \Xi\left(\zeta^h(x_i+), \zeta^h(x_i), u^h(t,x_i)\right) \varphi(t,x_i)\,\mathrm{d}t \Bigg\| \\
&= 0\,.
\end{aligned}$$

Term $\mathcal{E}_3^h$: Recall that by (4.20), $\zeta^h(x_i) = \zeta(x_{i-1}+)$ and $\zeta^h(x_i+) = \zeta(x_i+)$. By the Lipschitz continuity of Ξ and **(iii)**

$$\begin{aligned}
\mathcal{E}_3^h &= \Bigg\| \sum_{i\colon |x_i|\le K\,,\, x_i\in\mathcal{I}^h} \int_0^T \Xi\left(\zeta^h(x_i+), \zeta^h(x_i), u^h(t,x_i)\right)\, \varphi(t,x_i)\,\mathrm{d}t \\
&\qquad - \sum_{i\colon |x_i|\le K\,,\, x_i\in\mathcal{I}^h} \int_0^T \Xi\left(\zeta(x_i+), \zeta(x_i), u^h(t,x_i)\right)\, \varphi(t,x_i)\,\mathrm{d}t \Bigg\| \\
&\le \mathcal{O}(1) \sum_{i\colon |x_i|\le K\,,\, x_i\in\mathcal{I}^h} \left\|\zeta(x_{i-1}+) - \zeta(x_i)\right\| \\
&\le \mathcal{O}(1)\, \sharp\mathcal{I}^h\, \frac{h}{1+\sharp\mathcal{I}^h} \\
&\le \mathcal{O}(1)\, h \\
&\to 0 \quad \text{as } h \to 0\,.
\end{aligned}$$

Term $\mathcal{E}_4^h$: Recall that if $x_i \not\in \mathcal{I}$, then $\zeta\,(x_i+) = \zeta\,(x_i)$, which implies the equality $\Xi\left(\zeta(x_i+), \zeta(x_i), u^h(t,x_i)\right) = 0$. Hence, by **($\Xi$.1)**, **($\Xi$.3)** and **(ii)** we compute

$$\begin{aligned}
\mathcal{E}_4^h &= \Bigg\| \sum_{i\colon |x_i|\le K\,,\, x_i\in\mathcal{I}^h} \int_0^T \Xi\left(\zeta(x_i+), \zeta(x_i), u^h(t,x_i)\right)\, \varphi(t,x_i)\,\mathrm{d}t \\
&\qquad - \sum_{x\in\mathcal{I}, |x|\le K} \int_0^T \Xi\left(\zeta(x+), \zeta(x), u^h(t,x)\right)\, \varphi(t,x)\,\mathrm{d}t \Bigg\| \\
&\le \Bigg\| \sum_{x\in\mathcal{I}\setminus\mathcal{I}^h, |x|\le K} \int_0^T \Xi\left(\zeta(x+), \zeta(x), u^h(t,x)\right)\, \varphi(t,x)\,\mathrm{d}t \Bigg\| \\
&\le \mathcal{O}(1) \sum_{x\in\mathcal{I}\setminus\mathcal{I}^h, |x|\le K} \left\|\Delta\zeta\,(x)\right\| \\
&\le \mathcal{O}(1)\, h \\
&\to 0 \quad \text{as } h \to 0\,.
\end{aligned}$$

Term $\mathcal{E}_5^h$: Using Lemma 4.1

$$\begin{aligned}
\mathcal{E}_5^h &= \Bigg\| \sum_{x\in\mathcal{I}, |x|\le K} \int_0^T \Xi\left(\zeta(x+), \zeta(x), u^h(t,x)\right)\, \varphi(t,x)\,\mathrm{d}t \\
&\qquad - \sum_{x\in\mathcal{I}, |x|\le K} \int_0^T \Xi\left(\zeta(x+), \zeta(x), u(t,x)\right)\, \varphi(t,x)\,\mathrm{d}t \Bigg\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathcal{O}(1) \sum_{x \in \mathcal{I}} \left(\left\| \Delta\zeta(x) \right\| \int_0^T \left\| u^h(t,x) - u(t,x) \right\| \mathrm{d}t \right) \\
&\to 0 \qquad \text{as } h \to 0\,.
\end{aligned}
$$

The last limit is due to (4.22) and the convergence of the series $\sum_{x\in\mathcal{I}} \left\|\Delta\zeta(x)\right\|$. This concludes the convergence to the discrete part of the measure.

Term $\mathcal{E}_6^h$. Recall that by (4.20) we have $\zeta^h(x_i) = \zeta(x_{i-1}+)$ and $\zeta^h(x_i+) = \zeta(x_i+)$. We use below also **(ii)**:

$$
\begin{aligned}
\mathcal{E}_6^h &= \Bigg\| \sum_{i\colon |x_i|\leq K\,,\, x_i\notin\mathcal{I}^h} \int_0^T \Xi\left(\zeta^h(x_i+), \zeta^h(x_i), u^h(t,x_i)\right) \varphi(t,x_i)\,\mathrm{d}t \\
&\qquad - \sum_{i\colon |x_i|\leq K\,,\, x_i\notin\mathcal{I}^h} \int_0^T \Xi\left(\zeta(x_{i-1}+) + \mu(]x_{i-1},x_i[), \zeta(x_{i-1}+), u^h(t,x_i)\right) \varphi(t,x_i)\,\mathrm{d}t \Bigg\| \\
&\leq \mathcal{O}(1) \sum_{i\colon |x_i|\leq K\,,\, x_i\notin\mathcal{I}^h} \left\| \zeta(x_i+) - \zeta(x_{i-1}+) - \mu(]x_{i-1},x_i[) \right\| \\
&= \mathcal{O}(1) \sum_{i\colon |x_i|\leq K\,,\, x_i\notin\mathcal{I}^h} \left\| D\zeta(]x_{i-1},x_i]) - \mu(]x_{i-1},x_i[) \right\| \\
&\leq \mathcal{O}(1) \sum_{x\in\mathcal{I}\setminus\mathcal{I}^h} \left\| \Delta\zeta(x) \right\| \\
&= \mathcal{O}(1)\, h \\
&\to 0 \qquad \text{as } h \to 0\,.
\end{aligned}
$$

Term $\mathcal{E}_7^h$. Using **(iii)**,

$$
\begin{aligned}
\mathcal{E}_7^h &= \Bigg\| \sum_{i\colon |x_i|\leq K\,,\, x_i\notin\mathcal{I}^h} \int_0^T \Xi\left(\zeta(x_{i-1}+) + \mu(]x_{i-1},x_i[), \zeta(x_{i-1}+), u^h(t,x_i)\right) \varphi(t,x_i)\,\mathrm{d}t \\
&\qquad - \sum_{i\colon |x_i|\leq K} \int_0^T \Xi\left(\zeta(x_{i-1}+) + \mu(]x_{i-1},x_i[), \zeta(x_{i-1}+), u^h(t,x_i)\right) \varphi(t,x_i)\,\mathrm{d}t \Bigg\| \\
&\leq \mathcal{O}(1) \sum_{i\colon |x_i|\leq K\,,\, x_i\in\mathcal{I}^h} \mathrm{TV}\left(\zeta;]x_{i-1},x_i[\right) \\
&\leq \mathcal{O}(1)\, \sharp\mathcal{I}^h \, \frac{h}{1+\sharp\mathcal{I}^h} \\
&\leq \mathcal{O}(1)\, h \\
&\to 0 \qquad \text{as } h \to 0\,.
\end{aligned}
$$

Term $\mathcal{E}_8^h$. Introduce now $\delta_i = \|\mu\|(]x_{i-1},x_i[)$, $\mathcal{J} = \left\{i \in \{1,\ldots,N_h\} \colon \delta_i \neq 0\right\}$ and for $i \in \mathcal{J}$, let $v_i = \mu(]x_{i-1},x_i[)/\delta_i$. Below, we use **(Ξ.4)** with δ_i for t and v_i for v, and **(iv)**:

$$
\begin{aligned}
\mathcal{E}_8^h \;\leq\;& \left\| \sum_{i\colon |x_i|\leq K} \int_0^T \Xi\left(\zeta(x_{i-1}+) + \mu(]x_{i-1},x_i[), \zeta(x_{i-1}+), u^h(t,x_i)\right) \varphi(t,x_i)\,\mathrm{d}t \right. \\
& \left. - \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \int_0^T \delta_i\, D^+_{v_i}\Xi\left(\zeta(x_{i-1}+), \zeta(x_{i-1}+), u^h(t,x_i)\right) \varphi(t,x_i)\,\mathrm{d}t \right\| \\
\leq\;& \mathcal{O}(1) \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \int_0^T \left\| \Xi\left(\zeta(x_{i-1}+) + \mu(]x_{i-1},x_i[), \zeta(x_{i-1}+), u^h(t,x_i)\right) \right. \\
& \left. -\delta_i\, D^+_{v_i}\Xi\left(\zeta(x_{i-1}+), \zeta(x_{i-1}+), u^h(t,x_i)\right) \right\| \mathrm{d}t \\
\leq\;& \mathcal{O}(1) \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \sigma(\delta_i)\,\delta_i \\
\leq\;& \mathcal{O}(1)\,\sigma(h)\;\mathrm{TV}\,(\zeta) \\
\to\;& 0 \quad \text{as } h \to 0\,.
\end{aligned}
$$

Term $\mathcal{E}_9^h$. Use **(Ξ.4)** and recall that by (1.6), $v_i = (1/\delta_i)\int_{]x_{i-1},x_i[} v(y)\,\mathrm{d}\|\mu\|(y)$, while clearly $v(x) = (1/\delta_i)\int_{]x_{i-1},x_i[} v(x)\,\mathrm{d}\|\mu\|(y)$. We also use g^h, that is defined in **Step 1** and satisfies (4.19).

$$
\begin{aligned}
\mathcal{E}_9^h \;=\;& \left\| \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \int_0^T \delta_i\, D^+_{v_i}\Xi\left(\zeta(x_{i-1}+), \zeta(x_{i-1}+), u^h(t,x_i)\right) \varphi(t,x_i)\,\mathrm{d}t \right. \\
& \left. - \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \int_0^T \int_{]x_{i-1},x_i[} D^+_{v(x)}\Xi\left(\zeta(x_{i-1}+), \zeta(x_{i-1}+), u^h(t,x_i)\right) \varphi(t,x_i)\,\mathrm{d}\|\mu\|(x)\,\mathrm{d}t \right\| \\
\leq\;& \mathcal{O}(1) \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \int_{]x_{i-1},x_i[} \left\| v(x) - v_i \right\| \mathrm{d}\|\mu\|(x) \\
\leq\;& \mathcal{O}(1) \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \frac{1}{\delta_i} \int_{]x_{i-1},x_i[^2} \left\| v(x) - v(y) \right\| \mathrm{d}(\|\mu\|\otimes\|\mu\|)(x,y) \\
\leq\;& \mathcal{O}(1) \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \frac{1}{\delta_i} \int_{]x_{i-1},x_i[^2} \left[\left\| v(x) - g^h(x) \right\| + \left\| g^h(y) - v(y) \right\| \right] \mathrm{d}(\|\mu\|\otimes\|\mu\|)(x,y) \qquad (4.29) \\
& + \mathcal{O}(1) \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \frac{1}{\delta_i} \int_{]x_{i-1},x_i[^2} \left\| g^h(x) - g^h(y) \right\| \mathrm{d}(\|\mu\|\otimes\|\mu\|)(x,y)\,. \qquad (4.30)
\end{aligned}
$$

The two terms in the integral in (4.29) are estimated in the same way, using (4.19), as

$$
\sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \int_{]x_{i-1},x_i[^2} \frac{1}{\delta_i} \left\| v(x) - g^h(x) \right\| \mathrm{d}(\|\mu\|\otimes\|\mu\|)(x,y)
$$

$$
\begin{aligned}
&\leq \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \int_{]x_{i-1},x_i[} \left\| v(x) - g^h(x) \right\| \mathrm{d}\|\mu\|(x) \\
&\leq \int_{\mathbb{R}} \left\| v(x) - g^h(x) \right\| \mathrm{d}\|\mu\|(x) \\
&\leq \int_{\left\{x\in\mathbb{R}\colon v(x)\neq g^h(x)\right\}} \left(\|v(x)\| + \left\| g^h(x) \right\| \right) \mathrm{d}\|\mu\|(x) \\
&\leq 2\,h \\
&\to 0 \quad \text{as } h \to 0\,.
\end{aligned}
$$

We now estimate the term (4.30) by means of **(v)**:

$$
\begin{aligned}
& \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \frac{1}{\delta_i} \int_{]x_{i-1},x_i[^2} \left\| g^h(x) - g^h(y) \right\| \mathrm{d}(\|\mu\|\otimes\|\mu\|)(x,y) \\
\leq\; & h \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \frac{1}{\delta_i} \int_{]x_{i-1},x_i[^2} \mathrm{d}(\|\mu\|\otimes\|\mu\|)(x,y) \\
\leq\; & h \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \delta_i \\
\leq\; & h\ \mathrm{TV}\,(\zeta) \\
\to\; & 0 \quad \text{as } h \to 0\,.
\end{aligned}
$$

Term $\mathcal{E}_{10}^h$. Using **($\boldsymbol{\Xi}$.4)**

$$
\begin{aligned}
\mathcal{E}_{10}^h \;=\; & \left\| \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \int_0^T \int_{]x_{i-1},x_i[} D^+_{v(x)} \Xi \left(\zeta(x_{i-1}+), \zeta(x_{i-1}+), u^h(t,x_i) \right)\, \varphi(t,x_i)\, \mathrm{d}\|\mu\|(x)\, \mathrm{d}t \right. \\
& \left. - \int_0^T \int_{-K}^{K} D^+_{v(x)} \Xi \left(\zeta(x), \zeta(x), u(t,x) \right) \varphi(t,x)\, \mathrm{d}\|\mu\|\,(x)\, \mathrm{d}t \right\| \\
=\; & \left\| \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \int_0^T \int_{]x_{i-1},x_i[} D^+_{v(x)} \Xi \left(\zeta(x_{i-1}+), \zeta(x_{i-1}+), u^h(t,x_i) \right) \varphi(t,x_i)\, \mathrm{d}\|\mu\|(x)\, \mathrm{d}t \right. \\
& \left. - \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \int_0^T \int_{]x_{i-1},x_i[} D^+_{v(x)} \Xi \left(\zeta(x), \zeta(x), u(t,x) \right) \varphi(t,x)\, \mathrm{d}\|\mu\|\,(x)\, \mathrm{d}t \right\| \\
\leq\; & \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \int_0^T \int_{]x_{i-1},x_i[} \left\| D^+_{v(x)} \Xi \left(\zeta(x_{i-1}+), \zeta(x_{i-1}+), u^h(t,x_i) \right)\, \varphi(t,x_i) \right. \\
& \qquad \left. - D^+_{v(x)} \Xi \left(\zeta(x), \zeta(x), u(t,x) \right) \varphi(t,x) \right\| \mathrm{d}\|\mu\|\,(x)\, \mathrm{d}t \\
\leq\; & \mathcal{O}(1) \sum_{i\colon |x_i|\leq K\,,\, i\in\mathcal{J}} \int_0^T \int_{]x_{i-1},x_i[} \Big(\left\| \zeta(x_{i-1}+) - \zeta(x) \right\| + \left\| u^h(t,x_i) - u^h(t,x) \right\| \\
& \qquad + \left\| u^h(t,x) - u(t,x) \right\| + |x_i - x| \Big)\, \mathrm{d}\|\mu\|\,(x)\, \mathrm{d}t\ .
\end{aligned}
$$

Observe that

$$\begin{array}{rcll}
\left\|\zeta(x_{i-1}+)-\zeta(x)\right\| & \leq & h & \text{by } \textbf{(iv)} \\
\int_0^T \left\|u^h(t,x_i)-u^h(t,x)\right\| \mathrm{d}t & \leq & \mathcal{O}(1)\, h & \text{by (4.27)} \\
|x_i - x| & \leq & h & \text{by } \textbf{(vi)}
\end{array}$$

while by (4.22), Fubini Theorem and the Dominated Convergence Theorem,

$$\int_{\mathbb{R}} \int_0^T \left\|u^h(t,x)-u(t,x)\right\| \mathrm{d}t \; \mathrm{d}\|\mu\|\,(x) \to 0 \qquad \text{as } h \to 0\,.$$

The proof is completed. □

4.4 Proof Relative to Section 3

Proof of Theorem 3.1. It is immediate to check that **(f.1)**–**(f.3)** hold, thanks to **(p)**. Define Ξ as in (3.3). Then, conditions **(Ξ.1)** and **(Ξ.2)** follow from the assumed $\mathbf{C}^2$ regularity of K in all its variables. Condition **(Ξ.3)** follows from (3.3) and $K\left(0,(\rho,q)\right) \equiv 0$. Concerning **($\Xi$.4)**, we have

$$D_v^+ \Xi(z,z,v) = \left[\begin{array}{c} 0 \\ \partial_1 K\left(0,(\rho,q)\right)\,\|v\| \end{array}\right]$$

indeed, for v such that $\|v\| \leq 1$, we can estimate

$$\begin{array}{rl}
& \left\|K\left(t\,\|v\|,(\rho,q)\right) - \|v\|\,\partial_1 K\left(0,(\rho,q)\right)\, t\right\| \\
= & \left\|\int_0^1 \Big(\partial_1 K\left(s\,t\,\|v\|,(\rho,q)\right) - \partial_1 K\left(0,(\rho,q)\right)\Big)\, t\,\|v\|\,\mathrm{d}s\right\| \\
\leq & \|K\|_{\mathbf{C}^2([0,r]\times\Omega;\mathbb{R})}\, t^2
\end{array}$$

proving **(Ξ.4)** with $\sigma(t) = \|K\|_{\mathbf{C}^2([0,r]\times\Omega;\mathbb{R})}\, t$.

Theorem 2.2 can then be applied, exhibiting the existence of a solution in the sense of Definition 2.1.

To obtain the formulation (3.5) from (2.2), only the two terms in the right hand side of the second equations need to be considered. The first one is immediate: it only requires the substitution (3.3). Concerning the second one, recall that by (1.6), $\mathrm{d}\mu\,(x) = \Gamma''(x)\,\mathrm{d}x$, so that $\mathrm{d}\|\mu\|\,(x) = \left\|\Gamma''(x)\right\|\mathrm{d}x$, and that $v(x) = \frac{\Gamma''(x)}{\left\|\Gamma''(x)\right\|}$ for a.e. x with respect to the measure $\|\mu\|$.

Hence, since $\left\|v(x)\right\| = 1$ for a.e. x with respect to the measure $\|\mu\|$,

$$D_{v(x)}^+ \Xi\left(\Gamma'(x),\Gamma'(x),(\rho,q)\right) \mathrm{d}\|\mu\|\,(x) = \partial_1 K\left(0,(\rho,q)(x)\right)\,\left\|\Gamma''(x)\right\|\mathrm{d}x$$

completing the proof. □

Proof of Theorem 3.2. Condition **(p)** ensures that **(f.1)**–**(f.3)** hold. The choice (3.8) and the assumptions on Ξ_2 imply that **(Ξ.1)**–**(Ξ.4)** hold. Since the distributional derivative of a has neither Cantor part nor atomic part, due to (3.10) problem (3.9) reduces to (3.6). □

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