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Ph.D. Thesis in Mathematics

Universal constructions arising from quantization of Lie bialgebras

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Abstract

This Ph.D. thesis is devoted to the study of the theory of quantization of Lie bialgebras and related universal constructions.

We give a new treatment of the Drinfeld associator arising from the Knizhnik–Zamolodchikov connection, showing its main identities and properties through concrete evaluation of parallel transports with respect to flat connections along well–chosen paths. We used undergraduate Mathematics in all the reasonings, simplifying the previous treatments existing in literature.

We provide a more detailed version of P. Ševera's quantization of Lie bialgebras, giving explicit and diagrammatic proofs of all the categorical statements. We then show that such a construction is compatible with Drinfeld–Yetter modules and twists.

We give a new proof of the Enriquez–Etingof *Hensel's lemma*, which is a statement playing a key role in the proof of the invertibility of the Etingof–Kazhdan quantization functor. Our proof involves techniques of basic linear algebra and ring theory.

Finally, we present a combinatorial description of the Appel–Toledano Laredo universal Drinfeld–Yetter algebra $\mathfrak{U}^1_{\mathsf{DY}}$, which is involved in the theory of universal quantization functors. We define the set of Drinfeld–Yetter mosaics and the set of Drinfeld–Yetter looms, and we use them to give a combinatorial description of $\mathfrak{U}^1_{\mathsf{DY}}$.

Keywords: Lie bialgebras, Quantization, Monoidal categories, Hopf algebras, Associators.

Contents

1 Introduction **7**
1.1 Introduction . . . 7
1.2 Introduzione . . . 12
1.3 Introduction . . . 17

I Preliminaries 23

2 Monoidal–type categories **25**
2.1 Monoidal categories . . . 25
2.2 Braided monoidal categories . . . 29
2.3 Infinitesimally braided monoidal categories . . . 31
2.4 Turning monoidal categories into strict ones . . . 32
2.5 Some identities in monoidal–type categories . . . 34
2.6 The Drinfeld center . . . 36
2.7 Deforming infinitesimally braided monoidal categories . . . 37
2.8 The category of vector spaces . . . 39

3 Hopf monoids **41**
3.1 Monoids . . . 41
3.2 Modules . . . 43
3.3 Comonoids . . . 44
3.4 Comodules . . . 50
3.5 Bimonoids and Hopf monoids . . . 50
3.6 Left–right Yetter–Drinfeld H–modules . . . 51

4 Lie bialgebras and Drinfeld–Yetter modules **53**
4.1 Lie bialgebras . . . 53
4.2 Manin triples and the Drinfeld double . . . 56
4.3 Universal enveloping algebras . . . 62
4.4 Drinfeld–Yetter modules . . . 63
4.5 The Drinfeld–Yetter module structure of the universal enveloping algebra . . . 68
4.6 Lie bialgebra twists . . . 72
4.7 Quantization of Lie bialgebras: a short introduction . . . 76

II Quantization of Lie bialgebras 79

5 The Drinfeld associator arising from the Knizhnik–Zamolodchikov connection 81
5.1 Elementary analysis of piecewise C^∞ functions . . . 82
5.2 Formal linear ODEs . . . 84
5.3 Norms and limits . . . 87
5.4 Formal connections and parallel transport . . . 92
5.5 Flat formal connections . . . 96
5.6 The Knizhnik-Zamolodchikov connection and the Drinfeld-Kohno (Lie) algebras . . 101
5.7 The Drinfeld associator: definition and properties . . . 102
5.8 The Hexagon Equation . . . 107
5.9 The Pentagon Equation . . . 110

6 Etingof–Kazhdan quantization of finite–dimensional Lie bialgebras 121
6.1 The universal Verma modules . . . 121
6.2 The monoidal structure on the forgetful functor . . . 123
6.3 Tannaka–Krein duality and quantization of Lie bialgebras . . . 127

7 Ševera quantization of Lie bialgebras 129
7.1 M–adapted functors . . . 129
7.2 The multiplication along a comonoid . . . 131
7.3 The Hopf monoid $F(M \otimes' M)$ . . . 134
7.4 The functor of coinvariants . . . 137
7.5 Deforming M–adapted functors . . . 142
7.6 Quantization of Lie bialgebras . . . 148
7.7 Functoriality . . . 153
7.8 Quantization of twists . . . 157
7.9 Quantization of Drinfeld–Yetter modules . . . 162

III Universal constructions 175

8 PROPs 177
8.1 PROPs . . . 177
8.2 The PROP of Lie bialgebras . . . 178
8.3 The PROP of Hopf algebras . . . 179
8.4 Universal constructions . . . 179
8.5 Colored PROPs . . . 181
8.6 The colored PROP of Drinfeld–Yetter modules . . . 181
8.7 Universal quantization functors . . . 182

9 Enriquez–Etingof "Hensel" Lemma 185
9.1 Modules over rings of formal power series . . . 185
9.2 Proof of the Enriquez–Etingof "Hensel" Lemma . . . 190

10 The universal Drinfeld–Yetter algebra and its combinatorics **193**
10.1 Enriquez's universal algebras 193
10.2 Drinfeld–Yetter universal algebras 194
10.3 The universal Drinfeld–Yetter algebra $\mathfrak{U}_{\mathsf{DY}}^1$ 196
10.4 Drinfeld–Yetter mosaics 198
10.5 Drinfeld–Yetter looms 203
10.6 An explicit formula for the multiplication of $\mathfrak{U}_{\mathsf{DY}}^1$ 207
10.7 Essential Drinfeld–Yetter looms 213
10.8 Counting Drinfeld–Yetter looms 215
10.9 Some explicit computations in $\mathfrak{U}_{\mathsf{DY}}^1$ 217
10.10 Some conjectures related to the algebra $\mathfrak{U}_{\mathsf{DY}}^1$ 222

References **226**

Index of definitions **230**

Chapter 1

Introduction

1.1 Introduction

Overview

Symmetry plays a fundamental role in various areas of science, such as physics, chemistry, and mathematics. A unifying way to describe and understand symmetries is through the language of group theory. For example, in quantum mechanics, the states of a system are often represented by vectors in a Hilbert space, and transformations on these states due to symmetries are represented by unitary (sometimes anti–unitary) operators, forming certain groups. Group theory is used to analyze the properties of these structures and to understand the consequences of symmetries for quantum systems, such as the properties of the periodic table of chemical elements or the spectral lines resulting from the excitation of certain atoms or molecules.
Deformation theory concerns the understanding and description of *small* variations or modifications of mathematical structures. In this context, one would like to modify certain objects while maintaining certain given algebraic properties. The simplest case is the deformation of structures as a one-parameter family in a formal sense. An important application is the interpretation of the non-commutative associative algebra of all observables in quantum mechanics as a formal deformation (with the parameter viewed as the Planck constant $\hbar$) of the commutative algebra of observables in classical mechanics, i.e., smooth functions on a phase space (for example, a symplectic manifold) equipped with a Poisson structure, see [BFF$^+$78].
For example, the transition from a Lie algebra to its enveloping algebra (see [CE99]) can be seen as a deformation of the symmetric algebra generated by the underlying vector space of the Lie algebra. In the case of finite dimension, there is also an interpretation within the framework of quantization by deformation of this example using the well–known linear Poisson structure on the dual space of the Lie algebra (see [Gut83]). For this reason, in a broad sense, enveloping algebras are referred to as a 'quantization' of Lie algebras.
A natural mathematical structure that generalizes group theory is the theory of Hopf algebras, some of which have been called *quantum groups* by Vladimir G. Drinfeld (see [Dri86]). In this theory, one finds formal associative deformations of enveloping algebras.

The theory of quantization of Lie bialgebras has its origins in the 1980s with the work of P.P.Kulish and N.Y.Reshetikhin [KR83], in which the first *quantum group*, i.e. the deformation of the universal enveloping algebra of $\mathfrak{sl}(2)$ was discovered. A few years later, the Kulish–Reshetikhin's example was independently generalized to the case of any symmetrizable Kac–Moody algebra by V.G.Drinfeld and M.Jimbo, with the discovery of the objects that today are called Drinfeld–Jimbo quantum groups, see [Dri86] and [Jim85]. More in detail, given a Lie bialgebra $(\mathfrak{b}, [\cdot,\cdot], \delta)$ (see §4.1), we say that a topological Hopf algebra H is a quantization of $\mathfrak{b}$ (or, equivalently, a deformation of the universal enveloping algebra $\mathsf{U}(\mathfrak{b})$) if there exists an isomorphism

$$H/\hbar \cdot H \cong \mathsf{U}(\mathfrak{b}) \qquad \text{such that} \qquad \delta(x) = \frac{\Delta(\tilde{x}) - \Delta^{\mathsf{op}}(\tilde{x})}{\hbar} \quad \mod \ \hbar$$

where δ is the Lie cobracket of $\mathfrak{b}$, Δ is the comultiplication of H, $\tilde{x}$ is any lift of x in H, and $\mathsf{U}(\mathfrak{b})$ is considered with its standard Hopf algebra structure.

In the proceedings of the Workshops held in the Euler International Mathematical Institute, Saint Petersburg (at that time Leningrad) in Fall 1990 [Dri92] Drinfeld announced some unsolved problems in the theory of quantum groups. In particular, the following questions were raised among the others: Q. 1.1: *Can every Lie bialgebra be quantized?*; Q.1.2: *Does there exist a universal quantization for Lie bialgebras?*. Most of the solutions to Drinfeld's problems were provided by P.Etingof and D.Kazhdan in their series of articles [EK96][EK98][EK00a][EK00b][EK08]. In particular, a direct application of the quantization of Lie bialgebras is the deformation quantization of Poisson Lie groups (see [Šev16, §6]). We also mention for completeness the Tamarkin's approach [Tam02] by topological operads to the quantization of Lie bialgebras.

Further extensions of the Etingof–Kazhdan quantization were done more recently by B.Enriquez and G.Halbout (quantization of coboundary Lie bialgebras and quantization of quasi–Lie bialgebras) [EH10a] [EH10b], by Š.Sakáloš and P.Ševera, with the works [SŠ15][Šev16], and by A.Appel and V.Toledano Laredo [ATL18]. All the quantization constructions provided so far depend on a choice of a very complicated mathematical object called *Drinfeld associator* (see [Dri90b] and [Dri90a]), which, more precisely, is a formal power series $\Phi(A, B)$ in two non–commuting variables satisfying some algebraic properties, namely the pentagon and the hexagon equations (see Chapter 5).

Moreover, all the constructions cited above involve PROPs (product and permutation categories), that is a notion which appeared in the 1960s with the works of F.W.Lawvere and S.MacLane [Law63], [ML65]. Namely, a PROP is a $\mathbb{K}$–linear, strict symmetric monoidal category having as objects the set $\mathbb{N}^r$, where r is the *set of colors*. The generating morphisms of a PROP encode the datas of some specific algebraic–type object, such as Lie algebras or associative algebras. Universal functors, such as the universal enveloping algebra of a Lie algebra, are described with PROPs through the notion of *universal construction*. In particular, the Etingof–Kazhdan quantization of Lie bialgebras provides a universal construction $Q : \mathsf{QUE} \to \underline{\mathsf{LBA}}^{kar}[[\hbar]]$ from the PROP of quantized universal enveloping algebras to the (completed topological) PROP of Lie bialgebras, see §8.7. Furthermore, in the article [EK98] it is proved that such a functor is invertible, hence proving a *dequantization result*. A more accessible approach to this result was given afterward by B.Enriquez and P.Etingof [EE05], through a result the authors call *Hensel lemma*. A cohomological interpretation of universal quantization functors was then provided by B.Enriquez in the articles [Enr01b], [Enr01a], [Enr05] through some universal algebras, the latter being reformulated by A.Appel and V.Toledano Laredo in [ATL19] (see 10).

Main results

The main results of this thesis – which to our best knowledge are new – are the following

1. *A gentle introduction to Drinfeld associators*: in Chapter 5 we provide a self–contained proof of the main identities of the Drinfeld associator (see [Dri90b] and [Dri90a]) arising from the Knizhnik–Zamolodchikov connection [KZ84] on the complex configuration space $Y^n \subset \mathbb{C}^n$ (see Equation (5.6.2))
$$^{(n)}\Gamma_{\text{КнЗа}}(z_1, \ldots, z_n) \coloneqq \sum_{1 \leqslant i < j \leqslant n} \frac{A_{ij}}{z_i - z_j}(\mathsf{d}z_i - \mathsf{d}z_j)$$
where A_{ij} are elements satisfying the so–called *infinitesimal braid relations*, see §5.6. We used a elementary approach, that is given by concrete evaluation of parallel transports with respect to flat connections along certain paths. The originality of this part, which is an extract of [BRW23], is in the fact that we used undergraduate mathematics to prove all the results, which, although are fundamental in the theory of the quantization of Lie bialgebras, never seemed to have a sufficiently comprehensible treatment in the existing literature. In particular, given three elements A, B, C such that the sum $\Lambda \coloneqq A + B + C$ is a central element, the hexagon equation
$$e^{\hbar\Lambda/2} \;=\; e^{\hbar A/2}\;\Phi(C, A)\; e^{\hbar C/2}\;\Phi(B, C)\; e^{\hbar B/2}\;\Phi(A, B)$$
is obtained by computing the parallel transport along the six paths (5.8.7) with respect to the pulled–back one–dimensional (hence flat) connection (5.8.3). Similarly, for any elements $\{A_{ij}\}_{1 \leqslant i \neq j \leqslant 4}$ satisfying the infinitesimal braid relations, the pentagon equation
$$\Phi(A_{12}, A_{23} + A_{24})\Phi(A_{13} + A_{23}, A_{34}) = \Phi(A_{23}, A_{34})\Phi(A_{12} + A_{13}, A_{24} + A_{34})\Phi(A_{12}, A_{23})$$
follows by computing the parallel transport along the five affine paths (5.9.5) with respect to the flat connection (5.9.2).

2. *A more detailed version of P. Ševera's quantization of Lie bialgebras*: In Chapter 7 we provide a more detailed version of the Ševera's quantization of Lie bialgebras [Šev16]. Although Etingof–Kazhdan quantization has pioneered a very fertile field of research, its infinite–dimensional setting appears to be particularly intricate (contrarily to the finite–dimensional one). Ševera's construction does not have such a problem, and in addition it is easily shown to be compatible with the quantization of Drinfeld–Yetter modules and with twists (the Etingof–Kazhdan quantization is also compatible with twists, as proven with much more efforts by Enriquez and Halbout in [EH10a]). In particular, we provide explicit and diagramatic proofs of all the categorical statements, and we simplify Ševera's simplicial approach by more direct proofs using the notion of *multiplication along a comonoid object*.

3. *A more detailed proof of Enriquez–Etingof "Hensel" lemma*: In Chapter 9 we provide a more detailed proof of Enriquez–Etingof *Hensel Lemma*, see [EE05, Lemma 3.1]. Such a result was used by the authors in order to give a simpler proof of the dequantization theorem, whose proof was sketched by Etingof and Kazhdan with arguments related to the Grothendieck–Teichmüller semigroup. Our proof relies on basic linear algebra and ring theoretic arguments.

4. *Structure of the universal Drinfeld–Yetter algebra*: In Chapter 10, which is based on a forthcoming paper joint with A. Appel [AR], we provide a combinatorial description of the universal Drinfeld–Yetter algebra (which was defined in [ATL19] by A.Appel and V.Toledano Laredo)
$$\mathfrak{U}^1_{\mathsf{DY}} \coloneqq \mathsf{End}_{\mathsf{DY}}([V_1])$$

where DY is the colored PROP generated by a universal Lie bialgebra object [1] and a universal Drinfeld–Yetter [1]–module $[V_1]$, and the associative multiplication of $\mathfrak{U}^1_{\mathsf{DY}}$ is given by the composition of endomorphisms, see §8.6.
It turns out that the vector space structure of $\mathfrak{U}^1_{\mathsf{DY}}$ is isomorphic to the direct sum of all group algebras of the symmetric groups $\mathfrak{S}_n$, hence $\mathfrak{U}^1_{\mathsf{DY}}$ has a standard basis $\mathcal{B} = \{r^\sigma_n, \sigma \in \mathfrak{S}_n, n \geqslant 0\}$. Moreover, by definition, the structure constants of $\mathfrak{U}^1_{\mathsf{DY}}$ with respect to the basis $\mathcal{B}$ are integers. We define certain combinatorial objects, (the sets of all $n \times m$ Drinfeld–Yetter mosaics $\mathfrak{M}_{n,m}$ and of $n \times m$ Drinfeld–Yetter looms $\mathfrak{L}_{n,m}$) defined as tilings of an empty $n \times m$ grid with some specific tiles and according to some rules. The main result of this Chapter is the following formula giving a combinatorial description of the multiplication of $\mathfrak{U}^1_{\mathsf{DY}}$ in terms of the Drinfeld–Yetter looms:

$$r^\sigma_n \circ r^\tau_m = \sum_{L \in \mathfrak{L}_{n,m}} (-1)^{\xi(L)} r^{\tilde{\gamma}_{n,m}(\sigma,L,\tau)}_{n+m}$$

where $\xi(L)$ is a function counting the number of some specific tiles appearing in the Drinfeld–Yetter loom L, and $\tilde{\gamma}_{n,m}(\sigma, L, \tau)$ is a permutation in the symmetric group $\mathfrak{S}_{n+m}$ built up the permutations $\sigma \in \mathfrak{S}_n$, $\tau \in \mathfrak{S}_m$ and the Drinfeld–Yetter loom $L \in \mathfrak{L}_{n,m}$.

Structure of the thesis

This manuscript is structured in three parts, each of them made of three chapters.
The first part contains a quick treatment of prerequisites necessary for reading. In particular, Chapter 2 contains an introduction of monoidal, braided monoidal, and infinitesimally braided monoidal categories (also called *Cartier categories*, see [HV22, Def. 2.1] and [Car93]). We define their relative functors, and exhibit the main identities following by the axioms. We then mention the well–known Mac Lane's coherence Theorem [ML63] and his refinement due to P. Schauenburg [Sch01], allowing – in certain situations – to pass to strict monoidal categories in order to prove categorical identities (see §2.4). Finally, in 2.7 we prove the main result of this Chapter, that is the construction of the *deformed* braided monoidal category $\mathscr{C}^\Phi_\hbar$ due to Drinfeld [Dri90b, p. 1455].
Chapters 3–4 provide a concise overview of Hopf algebras, treated in full generality as objects of a braided monoidal category, and Lie bialgebras. In particular, we introduce finite–dimensional Manin triples, the Drinfeld double of a Lie bialgebra (see §4.2), and the universal enveloping algebra of a Lie algebra (see §4.3). Then in §4.4 we focus on Drinfeld–Yetter modules, which are the *linearized version* (i.e. the Lie bialgebras counterpart) of the usual Yetter–Drinfeld modules for Hopf algebras, and in §4.5 we present, given a Lie bialgebra $\mathfrak{b}$, the non–trivial infinitesimally braided monoidal structure of the category $\mathsf{DY}(\mathfrak{b})$ of Drinfeld–Yetter $\mathfrak{b}$–modules. Finally, we prove that the universal enveloping algebra $\mathsf{U}(\mathfrak{b})$ belongs to $\mathsf{DY}(\mathfrak{b})$, and in §4.7 we give a short introduction on quantization of Lie bialgebras.
In the second part of the thesis, the problem of quantization of Lie bialgebras is treated. Chapter 5, which is an extract of [BRW23], contains a new treatment of the well–known Drinfeld associator arising from the Knizhnik–Zamolodchikov connection and its main identities, the pentagon and the hexagon equations (see Equations (5.9.1) and (5.8.2)). We choose to give a pedagogical and more accessible-to-reading presentation to such an important object, which is essential for the quantization theory of Lie bialgebras. In particular, the main technique involved for the pentagon and hexagon equations is the computation of explicit parallel transports with respect to flat connections along well–chosen paths, leading to the desired algebraic identities.
Chapters 6–7 are devolved to the Etingof–Kazhdan and Ševera quantization's techniques of Lie

bialgebras. We choose to consider the Etingof–Kazhdan quantization only in the finite–dimensional case, since we think that Ševera's approach is much simpler in the infinite–dimensional setting. The two techniques contain however several reasoning in common, as we can think, in some unspecified sense, that they are the *dual* of the each other. Ševera's original paper [Šev16] is quite short and lacks of details and computations, although all the key ideas are clearly presented. We provide more details and insights through a diagrammatic approach, which allows to understand better how the categorical axioms imply all the reasonings.

Finally, the third and last part of this thesis is dedicated to universal constructions. In Chapter 8 we introduce PROPs (product and permutation categories), which are categories enclosing the information of some algebraic structures. The main PROPs we are interested in are the one of Hopf algebras (and of quantized universal enveloping algebras) and the one of Lie bialgebras. We finally mention the fact that the Etingof–Kazhdan quantization provides a universal construction, solving a problem stated by Drinfeld.

In Chapter 9 we deal with the proof by B.Enriquez and P.Etingof [EE05] of the fact that the Etingof–Kazhdan's and Ševera's universal constructions are invertible functors. In particular, the whole reasoning is based on what they call *Hensel's Lemma*. In §9.2 we give a new and more detailed proof of this statement.

The last Chapter 10, which is based on a forthcoming paper, contains a combinatorial description of the *universal Drinfeld–Yetter algebra* $\mathfrak{U}^1_{\mathsf{DY}}$, defined by A. Appel and V.Toledano Laredo in [ATL19]. Such an algebra is involved in the context of universal functors, and is a PROPic refinement of an algebra defined by B.Enriquez. We define some original combinatorial objects, namely the sets of *Drinfeld–Yetter mosaics* and of *Drinfeld–Yetter looms* (see respectively §10.4 and §10.5), with whom we describe the multiplication of $\mathfrak{U}^1_{\mathsf{DY}}$ (see Theorem 10.6.9). We finally present some explicit computations and links with other combinatorial structures, such as permutation patterns and bumpless pipedreams, see §10.9 and §10.10.

1.2 Introduzione

Panoramica

La simmetria svolge un ruolo fondamentale in varie aree della scienza, come la fisica, la chimica e la matematica. Un modo unificante per descrivere e comprendere le simmetrie è attraverso il linguaggio della teoria dei gruppi. Ad esempio, in meccanica quantistica, gli stati di un sistema sono spesso rappresentati da vettori in uno spazio di Hilbert, e le trasformazioni su questi stati dovute alle simmetrie sono rappresentate da operatori unitari (a volte anti-unitari), formando certi gruppi. La teoria dei gruppi è utilizzata per analizzare le proprietà di queste strutture e per comprendere le conseguenze delle simmetrie per i sistemi quantistici, come le proprietà della tavola periodica degli elementi chimici o le linee spettrali risultanti dall'eccitazione di certi atomi o molecole.
La teoria delle deformazioni riguarda la comprensione e la descrizione di variazioni o modifiche *piccole* delle strutture matematiche. In questo contesto, si desidera modificare certi oggetti mantenendo determinate proprietà algebriche date. Il caso più semplice è la deformazione di strutture come una famiglia a un parametro in senso formale. Un'applicazione importante è l'interpretazione dell'algebra associativa non commutativa di tutte le osservabili in meccanica quantistica come una deformazione formale (con il parametro visto come la costante di Planck $\hbar$) dell'algebra commutativa delle osservabili in meccanica classica, ovvero funzioni lisce su uno spazio delle fasi (ad esempio, una varietà simplattica) dotato di una struttura di Poisson, si veda [BFF$^+$78].
Per esempio, il passaggio da un'algebra di Lie alla sua algebra inviluppante (si veda [CE99]) può essere visto come una deformazione dell'algebra simmetrica generata dallo spazio vettoriale sottostante dell'algebra di Lie. Nel caso di dimensione finita, c'è anche un'interpretazione nell'ambito della quantizzazione per deformazione di questo esempio utilizzando la ben nota struttura di Poisson lineare sullo spazio duale dell'algebra di Lie (si veda [Gut83]). Per questa ragione, in senso lato, le algebre avvolgenti sono considerate una *quantizzazione* delle algebre di Lie.
Una struttura matematica naturale che generalizza la teoria dei gruppi è la teoria delle algebre di Hopf, alcune delle quali sono state chiamate *gruppi quantici* da Vladimir G. Drinfeld (si veda [Dri86]). In questa teoria si trovano deformazioni associative formali delle algebre inviluppanti.

La teoria della quantizzazione delle bialgebre di Lie ha origine negli anni '80 con il lavoro di P.P.Kulish e N.Y.Reshetikhin [KR83], in cui fu scoperto il primo *gruppo quantico*, ovvero la deformazione dell'algebra inviluppante universale di $\mathfrak{sl}(2)$. Pochi anni dopo, l'esempio di Kulish–Reshetikhin fu generalizzato da V.G.Drinfeld e M.Jimbo al caso di una qualsiasi algebra di Kac–Moody simmetrizzabile, con la scoperta degli oggetti che oggi vengono chiamati gruppi quantici di Drinfeld–Jimbo, si veda [Dri86] e [Jim85]. Più in dettaglio, data una bialgebra di Lie $(\mathfrak{b}, [\cdot,\cdot], \delta)$ (si veda §4.1), diciamo che un'algebra di Hopf topologica H è una quantizzazione di $\mathfrak{b}$ (o, equivalentemente, una deformazione dell'algebra inviluppante universale $\mathsf{U}(\mathfrak{b})$) se esiste un isomorfismo

$$H/\hbar \cdot H \cong \mathsf{U}(\mathfrak{b}) \qquad \text{tale che} \qquad \delta(x) = \frac{\Delta(\tilde{x}) - \Delta^{\mathsf{op}}(\tilde{x})}{\hbar}$$

dove δ è il cobracket di Lie di $\mathfrak{b}$, Δ è il coprodotto di H, $\tilde{x}$ è un qualsiasi sollevamento di x in H e $\mathsf{U}(\mathfrak{b})$ è equipaggiata con la struttura standard di algebra di Hopf.

Nei proceeding del seminario tenutosi all'Istituto Matematico Internazionale Eulero di San Pietroburgo (allora Leningrado) nell'autunno del 1990 [Dri92] Drinfeld annunciò alcuni problemi irrisolti nella teoria dei gruppi quantici. In particolare, furono sollevate, tra le altre, le seguenti domande: Q.

1.1: *È possibile quantizzare ogni bialgebra di Lie?*; Q.1.2: *Esiste una quantizzazione universale per le bialgebre di Lie?.* La maggior parte delle soluzioni ai problemi di Drinfeld é stata fornita da P.Etingof e D.Kazhdan nella loro serie di articoli [EK96][EK98][EK00a][EK00b][EK08].

Ulteriori estensioni della quantizzazione di Etingof–Kazhdan sono state fornite più recentemente da B.Enriquez e G.Halbout (quantizzazione di bialgebre di Lie di cobordo e quantizzazione di quasi–bialgebre di Lie) [EH10a] [EH10b], da Š.Sakáloš e P.Ševera, con i lavori [SŠ15][Šev16], e da A.Appel e V.Toledano Laredo [ATL18]. Tutte le costruzioni di quantizzazione fornite finora dipendono dalla scelta di un oggetto matematico piuttosto complicato chiamato *associatore di Drinfeld* (si veda [Dri90b] e [Dri90a]) che, più precisamente, è una serie di potenze formali $\Phi(A, B)$ in due variabili non commutative che soddisfano alcune proprietà algebriche, quali l'equazione del pentagono e l'equazione dell'esagono (si veda il capitolo 5).

Inoltre, tutte le costruzioni citate coinvolgono le PROP (product and permutations categories), una nozione apparsa negli anni '60 nei lavori di F.W.Lawvere e S.MacLane [Law63], [ML65]. In particolare, una PROP è una categoria $\mathbb{K}$-lineare, strettamente simmetrica monoidale, avente come oggetti l'insieme $\mathbb{N}^r$, dove r rappresenta l'*insieme di colori.* I morfismi generanti di una PROP codificano i dati di un oggetto di tipo algebrico specifico, come ad esempio algebre di Lie o algebre associative. I funtori universali, come l'algebra inviluppante universale di un'algebra di Lie, sono descritti con le PROP attraverso la nozione di *costruzione universale.* In particolare, la quantizzazione di Etingof–Kazhdan delle bialgebre di Lie fornisce una costruzione universale $Q : \mathsf{QUE} \to \underline{\mathsf{LBA}}^{kar}[[\hbar]]$ dalla PROP delle algebre inviluppanti universali quantizzate alla PROP (topologica e completata) delle bialgebre di Lie, si veda §8.7. Inoltre, nell'articolo [EK98] si dimostra che tale funtore è invertibile, dimostrando dunque un *risultato di dequantizzazione.* Un approccio più accessibile a questo risultato è stato fornito in seguito da B.Enriquez e P.Etingof [EE05], attraverso un risultato che gli autori chiamano *lemma di Hensel.* Una interpretazione coomologica dei funtori universali di quantizzazione è stata poi fornita da B.Enriquez negli articoli [Enr01b], [Enr01a], [Enr05], attraverso alcune algebre universali, le quali sono state riformulate da A.Appel e V.Toledano Laredo (si veda il capitolo 10).

Risultati principali

I principali risultati di questa tesi sono i seguenti

1. *Una gentile introduzione agli associatori di Drinfeld*: nel capitolo 5 forniamo una prova autocontenuta delle principali identità dell'associatore di Drinfeld (si veda [Dri90b] e [Dri90a]) derivante dalla connessione di Knizhnik–Zamolodchikov [KZ84] sullo spazio di configurazione complesso $Y^n \subset \mathbb{C}^n$ (si veda l'equazione (5.6.2))

$$^{(n)}\Gamma_{\text{КнЗа}}(z_1, \ldots, z_n) := \sum_{1 \leqslant i < j \leqslant n} \frac{A_{ij}}{z_i - z_j}(\mathsf{d}z_i - \mathsf{d}z_j)$$

   dove A_{ij} sono elementi che soddisfano le cosiddette *relazioni di treccia infinitesimale* (si veda §5.6). Abbiamo utilizzato un approccio elementare, dato dalla valutazione concreta di trasporti paralleli rispetto a connessioni piatte lungo determinati cammini. L'originalità di questa parte, che è un estratto di [BRW23], sta nel fatto che abbiamo usato matematica di base per dimostrare tutti i risultati che, pur essendo fondamentali nella teoria della quantizzazione delle bialgebre di Lie, non sono mai sembrati avere una trattazione sufficientemente comprensibile nella letteratura esistente. In particolare, dati tre elementi A, B, C tali che la somma $\Lambda := A + B + C$ sia un elemento centrale, l'equazione dell'esagono

$$e^{\hbar\Lambda/2} \;=\; e^{\hbar A/2}\,\Phi(C,A)\,e^{\hbar C/2}\,\Phi(B,C)\,e^{\hbar B/2}\,\Phi(A,B)$$

si ottiene calcolando il trasporto parallelo lungo i sei cammini (5.8.7) rispetto al pull–back della connessione monodimensionale (dunque piatta) (5.8.3). Analogamente, per elementi $\{A_{ij}\}_{1\leqslant i\neq j\leqslant 4}$ che soddisfano le relazioni di treccia infinitesimale, l'equazione del pentagono

$$\Phi(A_{12}, A_{23}+A_{24})\Phi(A_{13}+A_{23}, A_{34}) = \Phi(A_{23}, A_{34})\Phi(A_{12}+A_{13}, A_{24}+A_{34})\Phi(A_{12}, A_{23})$$

segue calcolando il trasporto parallelo lungo i cinque cammini affini (5.9.5) rispetto alla connessione piatta (5.9.2).

2. *Una versione più dettagliata della quantizzazione delle bialgebre di Lie di P. Ševera*: Nel capitolo 7 forniamo una versione più dettagliata della quantizzazione delle bialgebre di Lie dovuta a P. Ševera [Šev16]. Sebbene la quantizzazione di Etingof–Kazhdan abbia dato origine ad un ambito di ricerca molto fertile, la sua impostazione infinito–dimensionale appare particolarmente intricata (al contrario di quella finito–dimensionale). La costruzione di Ševera non presenta questo problema, e inoltre si dimostra facilmente essere compatibile con la quantizzazione dei moduli di Drinfeld–Yetter e con i twist (la quantizzazione di Etingof–Kazhdan è anch'essa compatibile con i twist, come dimostrato, con molti più sforzi, da Enriquez e Halbout in [EH10a]). In particolare, forniamo dimostrazioni esplicite e diagrammatiche di tutti gli enunciati categoriali e semplifichiamo l'approccio simpliciale di Ševera con dimostrazioni più dirette, utilizzando la nozione di *moltiplicazione lungo un oggetto comonoide*.

3. *Una prova più dettagliata del lemma di "Hensel" di Enriquez–Etingof*: Nel capitolo 9 forniamo una prova più dettagliata del lemma di Hensel di B.Enriquez e P.Etingof, si veda [EE05, Lemma 3.1]. Tale risultato è stato utilizzato dagli autori per fornire una prova più semplice del teorema di dequantizzazione, la cui dimostrazione è stata abbozzata da P.Etingof e D.Kazhdan con argomenti legati al semigruppo di Grothendieck–Teichmüller. La nostra dimostrazione si basa su argomenti di algebra lineare e di teoria degli anelli.

4. *Struttura dell'algebra universale di Drinfeld–Yetter*: Nel capitolo 10, che si basa su un lavoro di prossima pubblicazione congiunto con A. Appel [AR], forniamo una descrizione combinatorica dell'algebra universale di Drinfeld–Yetter (che è stata definita in [ATL19] da A.Appel e V.Toledano Laredo)

$$\mathfrak{U}^1_{\mathsf{DY}} := \mathsf{End}_{\mathsf{DY}}([V_1])$$

dove DY è la PROP colorata generata da un oggetto bialgebra di Lie universale [1] e da un modulo universale di Drinfeld–Yetter $[V_1]$ e in cui la moltiplicazione associativa di $\mathfrak{U}^1_{\mathsf{DY}}$ è data dalla composizione di endomorfismi (si veda §8.6). Si ha che la struttura di spazio vettoriale di $\mathfrak{U}^1_{\mathsf{DY}}$ è isomorfa alla somma diretta di tutte le algebre gruppo di tutti i gruppi simmetrici $\mathfrak{S}_n$, e dunque $\mathfrak{U}^1_{\mathsf{DY}}$ ha una base standard $\mathcal{B} = \{r^\sigma_n, \sigma \in \mathfrak{S}_n, n \geqslant 0\}$. Inoltre, per definizione, le costanti di struttura di $\mathfrak{U}^1_{\mathsf{DY}}$ rispetto alla base $\mathcal{B}$ sono numeri interi. Abbiamo definito alcuni oggetti combinatorici (l'insieme $\mathfrak{M}_{n,m}$ dei mosaici di Drinfeld–Yetter e l'insieme $\mathfrak{L}_{n,m}$ dei telai di Drinfeld–Yetter) definiti come riempimenti di una griglia vuota con alcune caselle specifiche e secondo alcune regole. Il risultato principale di questo capitolo è la seguente formula, che fornisce una descrizione combinatorica della moltiplicazione di $\mathfrak{U}^1_{\mathsf{DY}}$ in termini di telai di Drinfeld–Yetter:

$$r_n^\sigma \circ r_m^\tau = \sum_{L \in \mathfrak{L}_{n,m}} (-1)^{\xi(L)} r_{n+m}^{\tilde{\gamma}_{n,m}(\sigma,L,\tau)}$$

dove $\xi(L)$ è una funzione che conta il numero di alcune caselle specifiche che compaiono nel telaio Drinfeld–Yetter L e $\tilde{\gamma}_{n,m}(\sigma, L, \tau)$ è una permutazione del gruppo simmetrico $\mathfrak{S}_{n+m}$ dipendente dalle permutazioni $\sigma \in \mathfrak{S}_n$, $\tau \in \mathfrak{S}_m$ e dal telaio di Drinfeld–Yetter $L \in \mathfrak{L}_{n,m}$.

Struttura della tesi

Questo manoscritto è diviso in tre parti, ciascuna composta da tre capitoli.
La prima parte contiene una rapida trattazione dei prerequisiti necessari alla lettura. In particolare, il capitolo 2 contiene un'introduzione alle categorie monoidali, monoidali intrecciate e monoidali infinitesimamente intrecciate (dette anche *categorie di Cartier*, si vedano [HV22, Def. 2.1] e [Car93]). Definiamo dunque i loro funtori relativi ed esponiamo le principali identità derivanti dagli assiomi. Citiamo poi il noto Teorema di coerenza di Mac Lane [ML63] e il suo perfezionamento dovuto a P. Schauenburg [Sch01], che permette – in certe situazioni – di passare a categorie monoidali strette per dimostrare identità categoriali. Infine, dimostriamo il risultato principale di questo capitolo, cioè la costruzione della categoria monoidale intrecciata $\mathscr{C}_\hbar^\Phi$ dovuta a Drinfeld [Dri90b, p. 1455].
I capitoli 3–4 forniscono una panoramica concisa delle algebre di Hopf, trattate in piena generalità come oggetti di una categoria monoidale intrecciata, e delle bialgebre di Lie. In particolare, introduciamo le triple di Manin finito–dimensionali, il doppio di Drinfeld di una bialgebra di Lie e l'algebra inviluppante universale di un'algebra di Lie. Ci concentriamo dunque sui moduli di Drinfeld–Yetter, che sono la versione *linearizzata* (cioè la controparte delle bialgebre di Lie) degli usuali moduli di Yetter–Drinfeld per le algebre di Hopf, e presentiamo, data una bialgebra di Lie $\mathfrak{b}$, la struttura monoidale infinitesimamente intrecciata non banale della categoria $\mathsf{DY}(\mathfrak{b})$ di tutti i moduli di Drinfeld–Yetter su $\mathfrak{b}$. Infine, dimostriamo che l'algebra inviluppante universale $\mathsf{U}(\mathfrak{b})$ appartiene a $\mathsf{DY}(\mathfrak{b})$ e forniamo una breve introduzione sulla quantizzazione delle bialgebre di Lie.
Nella seconda parte della tesi viene trattato il problema della quantizzazione delle bialgebre di Lie. Il capitolo 5, che è un estratto di [BRW23], contiene una nuova trattazione del noto associatore di Drinfeld derivante dalla connessione Knizhnik–Zamolodchikov e delle sue principali identità, le equazioni del pentagono e dell'esagono. Abbiamo scelto di dare una presentazione pedagogica e più accessibile alla lettura di un oggetto così importante ed essenziale per la teoria della quantizzazione delle bialgebre di Lie. In particolare, la tecnica principale coinvolta per le equazioni del pentagono e dell'esagono è il calcolo di trasporti paralleli espliciti rispetto a connessioni piatte lungo cammini opportunamente scelti, che portano alle identità algebriche desiderate.
I capitoli 6–7 sono dedicati alle tecniche di quantizzazione delle bialgebre di Lie di Etingof–Kazhdan e Ševera. Abbiamo scelto di considerare la quantizzazione di Etingof–Kazhdan solo nel caso finito–dimensionale, poichè nel caso infinito–dimensionale riteniamo che l'approccio di Ševera sia molto più semplice. Le due tecniche hanno comunque diversi ragionamenti in comune, in quanto possiamo pensare, in un senso non ben precisato, che siano l'una la *duale* dell'altra. L'articolo originale di Ševera [Šev16] è piuttosto breve e manca di dettagli e calcoli, sebbene tutte le idee chiave siano chiaramente presentate. Abbiamo fornito maggiori dettagli e approfondimenti attraverso un approccio diagrammatico, che permette di capire meglio come gli assiomi categoriali implichino tutti i ragionamenti.
Infine, la terza e ultima parte di questa tesi è dedicata alle costruzioni universali. Nel capitolo 8

introduciamo le PROP (categorie di prodotti e permutazioni), che sono categorie che racchiudono le informazioni di alcune strutture algebriche. Le principali PROP a cui siamo interessati sono quella delle algebre di Hopf (e delle algebre inviluppanti universali quantizzate) e quella delle bialgebre di Lie. Infine, citiamo che la quantizzazione di Etingof–Kazhdan fornisce una costruzione universale, risolvendo un problema enunciato da Drinfeld.
Nel capitolo 9 trattiamo la dimostrazione di Enriquez e Etingof [EE05] del fatto che le costruzioni universali di Etingof–Kazhdan e Ševera sono funtori invertibili. In particolare, l'intero ragionamento si basa su quello che viene chiamato *Lemma di Hensel*. In §9.2 forniamo una nuova e più dettagliata dimostrazione di questo risultato.
L'ultimo capitolo 10, che si basa su un articolo di prossima pubblicazione, contiene una descrizione combinatorica dell'*algebra universale di Drinfeld–Yetter* $\mathfrak{U}^1_{\mathsf{DY}}$. Tale algebra appare nel contesto dei funtori universali ed è un raffinamento di natura PROPpica di un'algebra definita da Enriquez. Definiamo alcuni oggetti combinatorici originali, ovvero gli insiemi dei *mosaici di Drinfeld–Yetter* e dei *telai di Drinfeld–Yetter*, con i quali descriviamo la moltiplicazione di $\mathfrak{U}^1_{\mathsf{DY}}$. Presentiamo infine alcuni calcoli espliciti e collegamenti con altre strutture combinatorie, come i permutation patterns e i bumpless pipedreams.

1.3 Introduction

Aperçu

Symétrie joue un rôle fondamental dans plusieurs domaines de science, notamment en physique, en chimie, et en mathématiques. Une façon unifiée pour décrire et comprendre des symétries est le langage de la théorie des groupes. Par exemple, en mécanique quantique, les états d'un système sont représentés par des vecteurs non nuls dans un espace hilbertien, et les transformations de ces états par rapport à certaines symétries sont données par des sous-groupes du groupe de toutes les transformations unitaires (parfois anti-unitiaires). La théorie des groupes est utilisée pour analyser les propriétés de ces structures et pour comprendre les conséquences des symétries pour des systèmes quantiques, comme par exemple les propriétés du système périodique des éléments chimiques ou les lignes spectrales provenant de l'excitation de certains atomes ou molécules.
La théorie des déformations concerne la compréhension et description des 'petites' variations ou modifications des structures mathématiques, alors on aimerait bien modifier certains objets et maintenir certaines propriétés algébriques données. Le cas le plus simple est la déformation des structures en tant que famille à un paramétre formel. Une application importante est l'interprétation de l'algèbre associative non commutative de toutes les observables de la mécanique quantique comme une déformation formelle (dont le paramètre est vu comme la constante de Planck $\hbar$) de l'algèbre commutative des observables de la mécanique classique, c.-à-d. des fonctions de classe $\mathcal{C}^\infty$ sur un espace de phase (par exemple une variété symplectique) muni d'une structure de Poisson, voir [BFF+78].
Par exemple, le passage d'une algèbre de Lie à son algèbre enveloppante [CE99] peut être vu comme une déformation de l'algèbre symétrique engendrée par l'espace vectoriel sous-jacent de l'algèbre de Lie. Dans le cas de dimension finie, il y également une interprétation dans le cadre de la quantification par déformation de cet exemple en utilisant la structure de Poisson linéaire bien connue sur l'espace dual de l'algèbre de Lie, voir [Gut83]. Pour cette raison, au sens large, on parle des algèbres enveloppantes comme une 'quantification' des algèbres de Lie.
Une structure mathématique naturelle qui généralise la théorie des groupes est la théorie des algèbres de Hopf dont certaines ont été appelées 'groupes quantiques' par Vladimir G. Drinfeld [Dri86]. Dans cette théorie on trouve des déformation associatives formelles des algèbres enveloppantes.

La théorie de la *quantification des bigèbres de Lie* a ses origines dans les années 80 avec le travail de P.P.Kulish et N.Y.Reshetikhin [KR83] dans lequel le premier exemple d'un *groupe quantique*, i.e. de la déformation de l'algèbre enveloppante de l'algèbre de Lie $\mathfrak{sl}(2)$, a été construit. Quelques années plus tard, l'exemple de Kulish et Reshetikhin fut généralisé– indépendamment par Drinfeld et Jimbo– au cas de toute algèbre de Kac-Moody symétrisable, que l'on connaît aujourd'hui sous le nom de groupes quantiques de Drinfeld-Jimbo, [Dri86] et [Jim85]. Plus précisément, étant donné une bigèbre de Lie $\big(\mathfrak{b}, [\cdot,\cdot], \delta\big)$ (voir paragraphe §4.1) on dit qu'une algèbre de Hopf topologique H sur l'anneau $\mathbb{K}[[\hbar]]$ est une quantification de $\mathfrak{b}$ (ou de manière équivalente une déformation formelle de l'algèbre enveloppante de l'algèbre de Lie $\big(\mathfrak{b}, [\cdot,\cdot]\big)$) s'il y a un isomorphisme

$$H/(\hbar H) \cong \mathsf{U}(\mathfrak{b}) \qquad \text{tel que} \qquad \delta(x) = \frac{\Delta(x) - \Delta^{\mathrm{op}}(x)}{\hbar} \mod \hbar$$

où δ est le cocrochet de Lie de $\mathfrak{b}$, Δ est la comultiplication de H et $\mathsf{U}(\mathfrak{b})$ est considérée avec sa structure d'algèbre de Hopf cocommutative standard.

Dans les comptes rendus des ateliers organisés à l'Institut de Mathématiques Euler, Saint Peterbourg (à l'époque encore Leningrad) de l'autumn 1990 V.G.Drinfeld [Dri92] annonça quelques problèmes ouverts de la théorie des groupes quantiques. En particulier, les deux questions suivantes ont été posées: Q.1.1: *Toute bigèbre de Lie peut-elle être quantifiée?*; Q.1.2: *Existe-t-il une quantification universelle des bigèbres de Lie?*. La plupart des solutions a été trouvé par P.Etingof et D.Kazhdan dans leur série d'articles [EK96] [EK98] [EK00a] [EK00b] [EK08].

Les résultats d'Etingof-Kazhdan ont été généralisés plus récemment par B.Enriquez et G.Halbout (la quantification des bigèbres de Lie cobord et des quasi-bigèbres de Lie, voir [EH10a] [EH10b]), par Š.Sakáloš and P.Ševera, voir [SŠ15][Šev16], et par A.Appel et V. Toledano Laredo [ATL18]. Toutes ces constructions obtenues jusqu'à présent dépendent du choix d'un objet mathématique assez difficile à accéder, à savoir *l'associateur de Drinfeld*, voir [Dri90b] et [Dri90a]), qui, plus précisément est une série formelle $\Phi(A, B)$ en deux variables noncommutates satisfaisant des propriétés algébriques, à savoir les équations du pentagone et de l'hexagone (voir le chapitre 5).

De plus, toutes ces constructions ci-dessus sont exprimable par le langage des PROPs (product and permutation categories), une notion qui apparut dans les années 60 dans les travaux de W.Lawvere et S.MacLane [Law63], [ML65]. Plus précisément, une PROP est une catégorie monoïdale stricte symétrique dont l'ensemble des objets est $\mathbb{N}^r$ (r le nombre de 'couleurs'). Les morphismes générateur d'une PROP encodent les données d'un objet algébrique avec les identités, comme par exemples des algèbres ou cogèbres (co)associatives ou des algèbres ou cogèbres de Lie. Les foncteurs universels, comme par exemple l'algèbre enveloppante d'une algèbre de Lie se décrivent par des PROPs par la notion d'une construction universelle. Par exemple, la quantification des bigèbres de Lie selon Etingof-Kazhdan donne une construction universelle $Q : \mathsf{QUE} \to \underline{\mathsf{LBA}}^{kar}[[\hbar]]$ de la PROP des algèbres enveloppantes quantifiées à celle des bigèbres de Lie (topologiques complètes), voir §8.7. De plus, dans l'article [EK98] il y a une esquisse de preuve du fait qu'un tel foncteur est inversible (au niveau des PROPs) de sorte qu'il existe un *foncteur de déquantification*. Une approche plus accessible à ce résultat a été faite quelques années plus tard par B.Enriquez et P.Etingof [EE05] basé sur un Lemme de la théorie des modules sur l'anneau des séries formelles qu'ils ont appelé 'Hensel's Lemma'. Une interprétation cohomologique des foncteurs universels de quantifications a été donnée par B.Enriquez dans les articles [Enr01b], [Enr01a], [Enr05] par certaines algèbres universelles qui ont été reformulées par A.Appel et V.Toledano Laredo dans [ATL19] (voir 10).

Résultats pricipaux

Nous avons obtenu les résultats principaux suivants qui sont nouveaux au meilleur de notre connaissance:

1. *A gentle introduction to Drinfeld associators*: dans le chapitre 5 nous donnons une démonstration détaillée de l'associateur de Drinfeld et de ses deux identités ([Dri90b] et [Dri90a]) qui provient de la construction de V.G.Drinfeld utilisant la connection de Knizhnik–Zamolodchikov [KZ84] sur l'espace des configurations complexe $Y^n \subset \mathbb{C}^n$ (voir l'équation (5.6.2))

$$^{(n)}\Gamma_{\text{КнЗа}}(z_1, \ldots, z_n) := \sum_{1 \leqslant i < j \leqslant n} \frac{A_{ij}}{z_i - z_j} (\mathsf{d}z_i - \mathsf{d}z_j)$$

où A_{ij} sont des éléments d'une algèbre associative complexe unitaire qui satisfont les soi-disantes *relations de tresse infinitésimales*, voir §5.6. Nous utilisons une approche élémentaire donnée par l'évaluation concrète des transports parallèles d'une connection plate le long de certains chemins bien-choisis dans des parties ouvertes de $\mathbb{R}$ ou de $\mathbb{R}^2$. L'avantage de notre approche –comme nous le voyons– est le fait que tous les énoncés s'expriment à l'aide des techniques mathématiques qui ne dépassent jamais le niveau de Licence. Par contre, dans la littérature que nous avons étudiée il ne semble pas y avoir un traitement suffisamment compréhensible malgré l'importance fondamentale des associateurs dans la théorie de la quantification des bigèbres de Lie. Les deux identités importantes de l'associateur de Drinfeld so présentent comme suit: étant donné trois éléments A, B, C de l'algèbre tels que la somme $\Lambda := A + B + C$ est un élément central, l'équation de l'hexagone,

$$e^{\hbar\Lambda/2} \;=\; e^{\hbar A/2}\, \Phi(C, A)\, e^{\hbar C/2}\, \Phi(B, C)\, e^{\hbar B/2}\, \Phi(A, B)$$

s'obtient à l'aide du calcul du transport parallèle le long des six chemins (5.8.7) par rapport à la connexion pull–back (5.8.3) qui est complexe de dimension 1 donc plate. De manière similaire, quels que soient les éléments $\{A_{ij}\}_{1\leqslant i\neq j\leqslant 4}$ de l'algèbre satisfaisant les relations de tresse infinitésimales, l'équation du pentagone,

$$\Phi(A_{12}, A_{23} + A_{24})\Phi(A_{13} + A_{23}, A_{34}) = \Phi(A_{23}, A_{34})\Phi(A_{12} + A_{13}, A_{24} + A_{34})\Phi(A_{12}, A_{23})$$

eest déduit par le calcul du transport parallèle le long des cinq chemins (5.9.5) par rapport à la connexion plate (5.9.2). Cette partie de la thèse a été mise sur la toile, voir [BRW23], et sous-mise à publication.

2. *Une version plus détaillée de la quantification des bigèbres de Lie selon P. Ševera*: dans le chapitrer 7 nous donnons une version beaucoup plus détaillée de la quantification des bigèbres de Lie par P.Ševera [Šev16]. Bien que la quantification d'Etingof–Kazhdan ait incité –par ses méthodes catégorielles– un domaine assez fécond, leur traitement des bigèbres de Lie de dimension infinie semble assez intrigant contrairement au cas de dimension finie. Dans la construction de P.Ševera un tel problème n'existe pas, et en plus il n'est pas difficile à montrer qu'elle est compatible d'une part avec la quantification des modules des bigèbres de Lie (appelés modules de Drinfeld–Yetter) aux modules usuels de Yetter–Drinfeld de l'algèbre de Hopf, et d'autre part avec les twists (certaines modification par des cobord des cocrochets): la quantificationand d'Etingof–Kazhdan quantization est également compatible avec les twists, comme B.Enriquez et G.Halbout on montré avant avec des efforts énormes, voir [EH10a]). En particulier, nous allons donner des démonstrations détaillées et explicites par diagrammes, et nous simplifions la preuve de Ševera en enlevant les arguments simpliciaux et en juste utilisant la notion d'une *multiplication le long d'un objet comonoïdal.*

3. *Une démonstration plus détaillée du 'lemme de Hensel' selon B.Enriquez–P.Etingof* : en chapitre 9 nous donnons une démonstration plus détaillé d'un lemme important de la théorie des $\mathbb{K}[[\lambda]]$-modules qui permet de déduire l'inversibilité d'une application $\mathbb{K}[[\lambda]]$ si le permier ordre est inversible: c'est trivial si tous les modules sont topologiquement libres, mais dans la situation importante les hypothèses sont plus géneérales. Ce lemme est appelé 'Lemme de Hensel' par les auteurs, voir [EE05, Lemma 3.1], mais une relation avec le résultat bien connu en algèbre commutative qui porte le même nom n'a pas du tout été claire pour nous. Enriquez et Etingof utilisent ce lemme pour donner une démonstration plus simple d'un théorême de déquantification déjà esquissé par Etingof et Kazhdan pour pouvoir invertir un foncteur dans un contexte de PROPs. Notre démonstration du 'lemme de Hensel-Enriquez-Etingof' se fait par l'algèbre linéaire topologique élémentaire des $\mathbb{K}[[\lambda]]$-modules.

4. *Structure de l'algèbre de Drinfeld–Yetter universelle*: dans le chapitre 10 basé sur un preprint en préparation avec A. Appel, [AR], nous donnons une description combinatoire de l'algèbre de Drinfeld–Yetter universelle définie dans [ATL19] par A.Appel and V.Toledano Laredo. On pose
$$\mathfrak{U}^1_{\mathsf{DY}} := \mathsf{End}_{\mathsf{DY}}([V_1])$$
où DY est la PROP colorée engendrée par un objet universel de bigèbre de Lie [1] et un deuxième objet universel de module de Drinfeld–Yetter $[V_1]$ par rapport à [1], et la multiplication associative de $\mathfrak{U}^1_{\mathsf{DY}}$ est donnée par la composition des endomorphismes, see §8.6.
Il s'ensuit que la structure d'espace vectoriel de $\mathfrak{U}^1_{\mathsf{DY}}$ est isomorphe à la somme directe de toutes les algèbres de groupe des groupes symétriques $\mathfrak{S}_n$, alors $\mathfrak{U}^1_{\mathsf{DY}}$ a une base standard $\mathcal{B} = \{r^\sigma_n, \sigma \in \mathfrak{S}_n, n \geqslant 0\}$. Ed plus, par définition, les constantes de structure de $\mathfrak{U}^1_{\mathsf{DY}}$ par rapport à la base $\mathcal{B}$ sont des entiers. On définit certains objets combinatoires (les ensembles de tous les $n \times m$ mosaïques de Drinfeld–Yetter $\mathfrak{M}_{n,m}$ et de tous les métiers à tisser ('looms' en anglais) $n \times m$ de Drinfeld–Yetter $\mathfrak{L}_{n,m}$) définis comme des pavages d'un maillage $n \times m$ avec certains carreaux spécifiques qui satisfont quelques règles de composition. Le résultat principal de ce chapitre est la formule suivante donnant une description combinatoire de la multiplication de $\mathfrak{U}^1_{\mathsf{DY}}$ en termes des métiers à tisser de Drinfeld–Yetter:
$$r^\sigma_n \circ r^\tau_m = \sum_{L \in \mathfrak{L}_{n,m}} (-1)^{\xi(L)} r_{n+m}^{\tilde{\gamma}_{n,m}(\sigma,L,\tau)}$$
où $\xi(L)$ est une fonction qui numérote le nombre de tous les carreaux d'un certain type apparaissant dans le métier à tisser de Drinfeld–Yetter L, et $\tilde{\gamma}_{n,m}(\sigma, L, \tau)$ est une permutation dans le group symmétrique $\mathfrak{S}_{n+m}$ définie par les permutations $\sigma \in \mathfrak{S}_n$, $\tau \in \mathfrak{S}_m$ et par le métier à tisser de Drinfeld–Yetter $L \in \mathfrak{L}_{n,m}$.

Structure de la thèse

Le manuscrit est structuré en trois grandes parties dont chacune a trois chapitres.
La première partie donne une vue panoramique des prérequis nécessaires pour la lecture. En particulier, le chapitre 2 contient une introduction aux catégories monoïdales, aux catégories monoïdales tressées et aux catégories monoïdales infinitésmalement tressées (dont les dernières sont également appelées catégories de Cartier, voir [HV22, Def. 2.1] and [Car93])). On définit leurs foncteurs relatifs et leurs identités principales selon les axiomes. Nous mentionnons le théorème bien-connu de cohérence de S.Mac Lane [ML63] et un raffinement de ce résultat par P. Schauenburg [Sch01]: ceci permet –dans quelques situtations– de passer des catégories monoïdales aux catégories monoïdales strictes où tous les associateurs et uniteurs sont triviaux afin de démontrer des identités catégorielles. Finalement, on donne des détails de preuve du résultat bien connu de ce chaptire, à savoir la construction de la catégorie monoïdale tressée *deformée* $\mathscr{C}_h^\Phi$ par V.G.Drinfeld [Dri90b, p. 1455] à l'aide d'un associateur de Drinfeld.
Les chapitres 3–4 donnent un résumé concise des algèbres de Hopf, traitées en toute généralité en tant qu'objets d'une catégorie monoïdale tressée, et des bigèbres de Lie. En particulier, on rappelle des objets bien-counnus, à savoir les triplets de Manin de dimension finie, le double de Drinfeld d'une bigèbre de Lie, and l'algèbre enveloppante d'une algèbre de Lie. Ensuite on focalise sur les modules de Drinfeld–Yetter d'une bigèbre de Lie donnée, qui présentent une 'version linearisée' ou une contrepartie bigèbre de Lie des modules de Yetter–Drinfeld usuels pour les algèbres de Hopf,

and nous présentons, étant donné une bigèbre de Lie $\mathfrak{b}$, la structure de catégorie monoïdale infinitésmalement tressée non–triviale de la catégorie $\mathsf{DY}(\mathfrak{b})$ de tous les modules de Drinfeld–Yetter par rapport à $\mathfrak{b}$ selon Ševera [Šev16]. Finalement, on donne les détails de la preuve du fait que l'algèbre enveloppante $\mathsf{U}(\mathfrak{b})$ est toujours un module de Drinfeld-Yetter de manière canonique alors appartient à la catégorie $\mathsf{DY}(\mathfrak{b})$. Ensuite, nos donnons une introduction au cadre de la quantification des bigèbres de Lie.
La deuxième partie de la thèse concedrne la quantification des bigèbres de Lie. D'abord le chapitre 5 qui est extrait de la prépublication [BRW23] contient une nouvelle exposition de l'associateur de Drinfeld provenant de la connexion de Knizhnik-Zamolodchikov et ses identités principales, celle du pentagone et celle de l'hexagone. Nous avons choisi de donnner une présentation pédagogique et accessible pour un tel objet important qui est essentiel pour la théorie de la quantification des bigèbres de Lie. En particulier, la technique principale est le calcul des transports parallèles explicites d'une connexion plate le long des chemin bien-choisis, et le passage à la limite de ces chemins vers les singularités de la connexion. Ceci nous amène aux preuves des identités désirées.
Les chapitres 6–7 sont consacrés aux techniques de quantification d'Etingof-Kazhdan et de Ševera pour les bigèbres de Lie. Pour la première approche, celle d'Etingof-Kazhdan, nous ne la présentons que dans le cas des bigèbres de dimension finie pour éviter la discussion topologique assez hard. L'approche de P.Ševera marche sans distinction pour toute bigèbre de Lie uniformément et est plus simple. Mais les deux techniques ont les idées en commun qui proviennent de la théorie des représentations des algèbres de Lie, et l'apporoche de Ševera peut être vue comme un (pré)dual –astucieusement choisi– de celle d'Etingof-Kazhdan. La publication de P.Ševera de 2016, [Šev16] est extrêmement courte (14 pages) et ne donne que des esquisses de démonstrations ou caluls bien que toutes les idées soient clairement présentées. Nous avons rajouté beaucoup de détails moyennant une approche diagrammatique qui est adaptée aux techniques catégoriques afin de faciliter la compréhension comme nous espérons.
Finalement, la troisième et dernière partie de cette thèse est consacrée aux constructions universelles. Dans le chapitre 8 nous introduisons PROPs ('product and permutation categories'), qui sont –au sens large– des catégories qui encodent l'information sur certaines structures algébriques dans leurs ensembles de morphismes. Les PROPs qui nous intéressent sont celle des algèbres de Hopf (surtout celle des algèbres enveloppantes universelles quantifiées) et celle des bigèbres de Lie. On mentionne que la quantification d'Etingof–Kazhdan et celle de Ševera admet une formulation en termes de PROPs, donc une construction universelle, ainsi résolvant un problème posé par Drinfeld.
Dans le chapitre 9 nous rajoutons une preuve détaillé du 'lemme de Hensel' utilisé dans la construction de déquantification d'Enriquez-Etingof [EE05]. Dans §9.2 nous donnons les détails.
Le dernier chapitre 10 basé sur notre preprint pas encore publié contient la description combinatoire de l'algèbre de Drinfeld–Yetter universelle $\mathfrak{U}^1_{\mathsf{DY}}$. Une telle algèbre est liée au contexte des foncteurs universels et présente un raffinement PROPique d'uen algèbre définie par B.Enriquez. Nous définissons des objets combinatoires comme des mosaïques et des métiers à tisser déjà mentionnés ci-dessus à l'aide desquels la multiplication de $\mathfrak{U}^1_{\mathsf{DY}}$ peut être décrite. Finalement, on présente des calculs explicites et des liens avec d'autres structures combinatoires comme 'permutations patterns' et 'bumpless pipedreams'.

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Part I

Preliminaries

Chapter 2

Monoidal–type categories

The aim of this Chapter is to introduce the well–known concepts and main properties of monoidal, braided monoidal and infinitesimally braided monoidal categories. The reader can find further details in the book of S. Mac Lane [ML13, VII], the book of P.Etingof, S. Gelaki, D. Nikshych and V. Ostrik [EGNO15, Ch. 2,8], the book of M. Aguiar and S. Mahajan [AM10, Part I], the article of A. Joyal and R. Street [JS93], and L. Trujillo B.Sc. Thesis [Tru20]. See also the article of A. Ardizzoni, L. Bottegoni, A. Sciandra and T. Weber [ABSW23] for a recent generalization of the concept of infinitesimally braided monoidal category.

2.1 Monoidal categories

Definition 2.1.1. *A* ***monoidal category*** *is a sextuple* $(\mathscr{C}, \otimes, I, a, \ell, r)$*, where:*

- $\mathscr{C}$ *is a category;*
- $\otimes$ *is a functor* $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$*, called the* ***tensor product*** *of* $\mathscr{C}$*;*
- I *is an object of* $\mathscr{C}$*, called the* ***unit*** *of* $\mathscr{C}$*;*
- a *is a natural isomorphism* $a : \otimes(\otimes \times \mathsf{id}) \to \otimes(\mathsf{id} \times \otimes)$*, called an* ***associativity constraint;***
- ℓ *is a natural isomorphism* $\ell : \otimes(I \times \mathsf{id}) \to \mathsf{id}$*, called a* ***left unit constraint;***
- r *is a natural isomorphism* $r : \otimes(\mathsf{id} \times I) \to \mathsf{id}$*, called a* ***right unit constraint;***

such that the pentagonal diagram

$$\begin{array}{ccc} \big(X \otimes (Y \otimes Z)\big) \otimes W & \xleftarrow{a_{X,Y,Z} \otimes \mathsf{id}_W} & \big((X \otimes Y) \otimes Z\big) \otimes W \\ & & \downarrow{\scriptstyle a_{X \otimes Y, Z, W}} \\ {\scriptstyle a_{X, Y \otimes Z, W}} \downarrow & & (X \otimes Y) \otimes (Z \otimes W) \\ & & \downarrow{\scriptstyle a_{X,Y,Z \otimes W}} \\ X \otimes \big((Y \otimes Z) \otimes W\big) & \xrightarrow[\mathsf{id}_X \otimes a_{Y,Z,W}]{} & X \otimes \big(Y \otimes (Z \otimes W)\big) \end{array} \tag{2.1.1}$$

commutes for any X, Y, Z, W in $\mathsf{Obj}(\mathscr{C})$ and the triangular diagram

$$\begin{array}{ccc} (X \otimes I) \otimes Y & \xrightarrow{a_{X,I,Y}} & X \otimes (I \otimes Y) \\ & {\scriptstyle r_X \otimes \mathsf{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathsf{id}_X \otimes \ell_Y} & \\ & X \otimes Y & \end{array} \tag{2.1.2}$$

commutes for any X, Y in $\mathsf{Obj}(\mathscr{C})$.

Definition 2.1.2. *Let $(\mathscr{C}, \otimes, I, a, \ell, r)$ and $(\mathscr{C}', \otimes', I', a', \ell', r')$ be two monoidal categories.*

- *A* ***monoidal functor*** *from $\mathscr{C}$ to $\mathscr{C}'$ is a triple $(F, \varphi_0^F, \varphi_2^F)$, where $F : \mathscr{C} \to \mathscr{C}'$ is a functor, $\varphi_0^F : I' \to F(I)$ is a morphism, and φ_2^F is a natural transformation*

$$\varphi_2^F(X, Y) : F(X) \otimes' F(Y) \to F(X \otimes Y)$$

such that the hexagon

$$\begin{array}{ccc} \big(F(X) \otimes' F(Y)\big) \otimes' F(Z) & \xrightarrow{a'_{F(X),F(Y),F(Z)}} & F(X) \otimes' \big(F(Y) \otimes' F(Z)\big) \\ {\scriptstyle \varphi_2^F(X,Y) \otimes' \mathsf{id}_{F(Z)}} \downarrow & & \downarrow {\scriptstyle \mathsf{id}_{F(X)} \otimes' \varphi_2^F(Y,Z)} \\ F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\ {\scriptstyle \varphi_2^F(X \otimes Y, Z)} \downarrow & & \downarrow {\scriptstyle \varphi_2^F(X, Y \otimes Z)} \\ F\big((X \otimes Y) \otimes Z\big) & \xrightarrow{F(a_{X,Y,Z})} & F\big(X \otimes (Y \otimes Z)\big) \end{array} \tag{2.1.3}$$

and the squares

$$\begin{array}{ccc} I' \otimes' F(X) & \xrightarrow{\ell'_{F(X)}} & F(X) \\ {\scriptstyle \varphi_0^F \otimes' \mathsf{id}_{F(X)}} \downarrow & & \uparrow {\scriptstyle F(\ell_X)} \\ F(I) \otimes' F(X) & \xrightarrow{\varphi_2^F(I,X)} & F(I \otimes X) \end{array} \qquad \begin{array}{ccc} F(X) \otimes' I' & \xrightarrow{r'_{F(X)}} & F(X) \\ {\scriptstyle \mathsf{id}_{F(X)} \otimes' \varphi_0^F} \downarrow & & \uparrow {\scriptstyle F(r_X)} \\ F(X) \otimes' F(I) & \xrightarrow{\varphi_2^F(X,I)} & F(X \otimes I) \end{array} \tag{2.1.4}$$

commute for any X, Y, Z in $\mathsf{Obj}(\mathscr{C})$. If all the morphisms φ_0^F and $\varphi_2^F(X, Y)$ are invertible we say that $(F, \varphi_0^F, \varphi_2^F)$ is a ***strongly monoidal functor****.*

- *A* ***natural monoidal transformation*** *$\eta : (F, \varphi_0^F, \varphi_2^F) \to (G, \varphi_0^G, \varphi_2^G)$ between monoidal functors from $\mathscr{C}$ to $\mathscr{C}'$ is a natural transformation $\eta : F \to G$ such that the diagrams*

$$\begin{array}{ccc} F(X) \otimes' F(Y) & \xrightarrow{\varphi_2^F(X,Y)} & F(X \otimes Y) \\ {\scriptstyle \eta(X) \otimes' \eta(Y)} \downarrow & & \downarrow {\scriptstyle \eta(X \otimes Y)} \\ G(X) \otimes' G(Y) & \xrightarrow{\varphi_2^G(X,Y)} & G(X \otimes Y) \end{array} \qquad \begin{array}{ccc} & I' & \\ {\scriptstyle \varphi_0^F} \swarrow & & \searrow {\scriptstyle \varphi_0^G} \\ F(I) & \xrightarrow[\eta(I)]{} & G(I) \end{array}$$

commute for any X, Y in $\mathsf{Obj}(\mathscr{C})$.

- *A* ***comonoidal functor*** *from $\mathscr{C}$ to $\mathscr{C}'$ is a triple (F, ψ_F^0, ψ_F^2), where $F : \mathscr{C} \to \mathscr{C}'$ is a functor, $\psi_F^0 : F(I) \to I'$ is a morphism, and ψ_F^2 is a natural transformation*

$$\psi_F^2(X, Y) : F(X \otimes Y) \to F(X) \otimes' F(Y)$$

such that the hexagon

$$\begin{array}{ccc} F\big((X \otimes Y) \otimes Z\big) & \xrightarrow{F(a_{U,V,W})} & F\big(X \otimes (Y \otimes Z)\big) \\ \downarrow{\scriptstyle \psi_F^2(X \otimes Y, Z)} & & \downarrow{\scriptstyle \psi_F^2(X, Y \otimes Z)} \\ F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\ \downarrow{\scriptstyle \psi_F^2(X,Y) \otimes' \mathsf{id}_{F(Z)}} & & \downarrow{\scriptstyle \mathsf{id}_{F(X)} \otimes' \psi_F^2(Y,Z)} \\ \big(F(X) \otimes' F(Y)\big) \otimes' F(Z) & \xrightarrow{a'_{F(X),F(Y),F(Z)}} & F(X) \otimes' \big(F(Y) \otimes' F(Z)\big) \end{array} \tag{2.1.5}$$

and the squares

$$\begin{array}{ccc} F(X) & \xrightarrow{(\ell'_{F(X)})^{-1}} & I' \otimes' F(X) \\ \downarrow{\scriptstyle F(\ell_X^{-1})} & & \uparrow{\scriptstyle \psi_F^0 \otimes' \mathsf{id}_{F(U)}} \\ F(I \otimes X) & \xrightarrow{\psi_F^2(I,X)} & F(I) \otimes' F(X) \end{array} \qquad \begin{array}{ccc} F(X) & \xrightarrow{(r'_{F(X)})^{-1}} & F(X) \otimes' I' \\ \downarrow{\scriptstyle F(r_X^{-1})} & & \uparrow{\scriptstyle \mathsf{id}_{F(X)} \otimes' \psi_F^0} \\ F(X \otimes I) & \xrightarrow{\psi_F^2(X,I)} & F(X) \otimes' F(I) \end{array} \tag{2.1.6}$$

commute for any X, Y, Z in $\mathsf{Obj}(\mathscr{C})$. If all the morphisms ψ_F^0 and $\psi_F^2(X, Y)$ are invertible, we say that (F, ψ_F^0, ψ_F^2) is a ***strongly comonoidal functor****, and the triple $(F, (\psi_F^0)^{-1}, (\psi_F^2)^{-1})$ is a strongly monoidal functor.*

- *A* ***natural comonoidal transformation*** *$\theta : (F, \psi_F^0, \psi_F^2) \to (G, \psi_G^0, \psi_G^2)$ between monoidal functors from $\mathscr{C}$ to $\mathscr{C}'$ is a natural transformation $\theta : F \to G$ such that the diagrams*

$$\begin{array}{ccc} F(X \otimes Y) & \xrightarrow{\psi_F^2(X,Y)} & F(X) \otimes' F(Y) \\ \downarrow{\scriptstyle \theta(X \otimes Y)} & & \downarrow{\scriptstyle \theta(X) \otimes' \theta(Y)} \\ G(X \otimes Y) & \xrightarrow{\psi_G^2(X,Y)} & G(X) \otimes' G(Y) \end{array} \qquad \begin{array}{ccc} & I' & \\ {\scriptstyle \psi_F^0} \nearrow & & \nwarrow {\scriptstyle \psi_G^0} \\ F(I) & \xrightarrow[\theta(I)]{} & G(I) \end{array}$$

commute for any X, Y in $\mathsf{Obj}(\mathscr{C})$.

- *A* ***natural (co)monoidal isomorphism*** *is a natural (co)monoidal transformation that is also a natural isomorphism.*

- *A* ***(co)monoidal equivalence*** *between two monoidal categories $\mathscr{C}$ and $\mathscr{C}'$ is a (co)monoidal functor $F : \mathscr{C} \to \mathscr{C}'$ such that there exist a (co)monoidal functor $F' : \mathscr{C}' \to \mathscr{C}$ and two natural (co)monoidal isomorphisms $\eta : \mathsf{id}_{\mathscr{C}'} \to FF'$ and $\varepsilon : F'F \to \mathsf{id}_{\mathscr{C}}$.*

Proposition 2.1.3. *Let $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ be three monoidal categories.*

(i) If $(F, \psi_F^0, \psi_F^2) : \mathscr{C} \to \mathscr{C}'$ and $(G, \psi_G^0, \psi_G^0) : \mathscr{C}' \to \mathscr{C}''$ are two comonoidal functors, then the triple

$$\big(GF, \psi_0^G \circ G(\psi_F^0,), \psi_G^2 \circ G(\psi_F^2)\big) \tag{2.1.7}$$

is a comonoidal functor from $\mathscr{C}$ to $\mathscr{C}''$.

(ii) If $(F, \varphi_0^F, \varphi_2^F) : \mathscr{C} \to \mathscr{C}'$ and $(G, \varphi_0^G, \varphi_2^G) : \mathscr{C}' \to \mathscr{C}''$ are two monoidal functors, then the triple

$$\big(GF, G(\varphi_0^F) \circ \varphi_0^G, G(\varphi_2^F) \circ \varphi_2^G\big) \tag{2.1.8}$$

is a monoidal functor from $\mathscr{C}$ to $\mathscr{C}''$.

Proof. Let X, Y, Z be objects of $\mathscr{C}$. We have that the following diagrams commute

(1)

$$\begin{array}{ccc}
GF\big((X \otimes Y) \otimes Z\big) & \xrightarrow{G(\psi_F^2(X \otimes Y, Z))} & G\big(F(X \otimes Y) \otimes' F(Z)\big) \\
{\scriptstyle GF(a_{X,Y,Z})}\downarrow & & \downarrow{\scriptstyle G(\psi_F^2(X,Y) \otimes' \mathsf{id}_{F(Z)})} \\
GF\big(X \otimes (Y \otimes Z)\big) & & G\big((F(X) \otimes' F(Y)) \otimes' F(Z)\big) \\
{\scriptstyle G(\psi_F^2(X, Y \otimes Z))}\downarrow & & \downarrow{\scriptstyle G(a'_{F(X),F(Y),F(Z)})} \\
G\big(F(X) \otimes' F(Y \otimes Z)\big) & \xrightarrow{G(\mathsf{id}_{F(X)} \otimes' \psi_F^2(F(Y),F(Z)))} & G\big(F(X) \otimes' (F(Y) \otimes' F(Z))\big)
\end{array}$$

(2)

$$\begin{array}{ccc}
G\big(F(X \otimes Y) \otimes' F(Z)\big) & \xrightarrow{\psi_G^2(F(X \otimes Y), F(Z))} & GF(X \otimes Y) \otimes'' GF(Z) \\
{\scriptstyle G(\psi_F^2(X,Y) \otimes' \mathsf{id}_{F(Z)})}\downarrow & & \downarrow{\scriptstyle G(\psi_F^2(X,Y)) \otimes'' \mathsf{id}_{GF(Z)}} \\
G\big((F(X) \otimes' F(Y)) \otimes' F(Z)\big) & \xrightarrow{\psi_G^2(F(X) \otimes' F(Y), F(Z))} & G\big(F(X) \otimes' F(Y)\big) \otimes'' GF(Z)
\end{array}$$

(3)

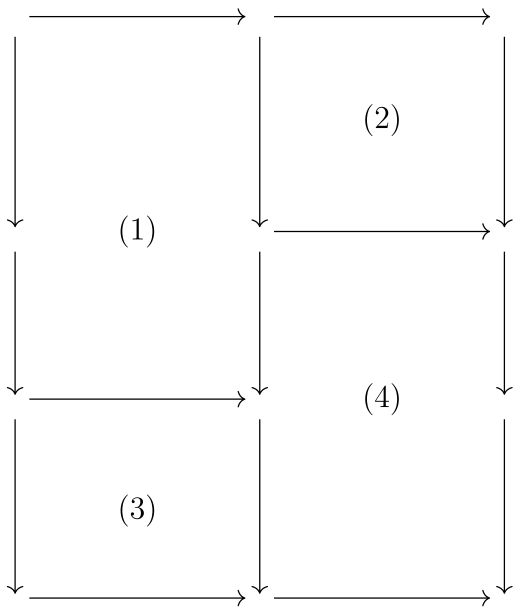
$$\begin{array}{ccc}
G\big(F(X) \otimes' F(Y \otimes Z)\big) & \xrightarrow{G(\mathsf{id}_{F(X)} \otimes' \psi_2^F(F(Y),F(Z)))} & G\big(F(X) \otimes' (F(Y) \otimes' F(Z))\big) \\
{\scriptstyle \psi_G^2(F(X), F(Y \otimes Z))}\downarrow & & \downarrow{\scriptstyle \psi_G^2(F(X), F(Y) \otimes' F(Z))} \\
GF(X) \otimes'' GF(Y \otimes Z) & \xrightarrow{\mathsf{id}_{GF(X)} \otimes'' G(\psi_F^2(Y,Z))} & GF(X) \otimes'' G\big(F(Y) \otimes' F(Z)\big)
\end{array}$$

(4)

$$\begin{array}{ccc}
G\big((F(X) \otimes' F(Y)) \otimes' F(Z)\big) & \xrightarrow{\psi_G^2(F(X) \otimes' F(Y), F(Z))} & G\big(F(X) \otimes' F(Y)\big) \otimes'' GF(Z) \\
{\scriptstyle G(a'_{F(X),F(Y),F(Z)})}\downarrow & & \downarrow{\scriptstyle \psi_G^F(F(X),F(Y)) \otimes'' \mathsf{id}_{GF(Z)}} \\
G\big(F(X) \otimes' (F(Y) \otimes' F(Z))\big) & & \big(GF(X) \otimes'' GF(Y)\big) \otimes'' GF(Z) \\
{\scriptstyle \psi_G^2(F(X), F(Y) \otimes' F(Z))}\downarrow & & \downarrow{\scriptstyle a''_{GF(X),GF(Y),GF(Z)}} \\
GF(X) \otimes'' G\big(F(Y) \otimes' F(Z)\big) & \xrightarrow{\mathsf{id}_{GF(X)} \otimes'' \psi_G^2(F(Y),F(Z))} & GF(X) \otimes'' \big(GF(Y) \otimes'' GF(Z)\big)
\end{array}$$

where the first follows from the fact that (F, ψ_F^0, ψ_F^2) is a comonoidal functor, the second and the third follow from the naturality of ψ_G^2, and the last follows from the fact that is a comonoidal functor.

The joint diagram



gives that the triple (2.1.7) satisfies (2.1.5). Finally, the fact that the triple (2.1.7) satisfies the squares (2.1.6) follows by the diagrams

$$
\begin{array}{ccccc}
GF(X) & & \xrightarrow{\quad (\ell''_{GF(X)})^{-1} \quad} & & I''\otimes'' GF(X) \\
\Big\downarrow {\scriptstyle GF(\ell_X^{-1})} & \searrow^{G((\ell'_{F(X)})^{-1})} & & & \Big\uparrow {\scriptstyle \psi_G^0\otimes \mathsf{id}_{GF(X)}} \\
 & & G\big(I'\otimes' F(X)\big) & \xrightarrow{\psi_G^2(I',F(X))} & G(I')\otimes'' GF(X) \\
 & & \Big\uparrow {\scriptstyle G(\psi_F^0\otimes' \mathsf{id}_{F(X)})} & & \Big\uparrow {\scriptstyle G(\psi_F^0)\otimes'' \mathsf{id}_{GF(X)}} \\
GF(I\otimes X) & \xrightarrow{G(\psi_F^2(I,X))} & G\big(F(I)\otimes' F(X)\big) & \xrightarrow{\psi_G^2(F(I),F(X))} & GF(I)\otimes'' GF(X)
\end{array}
$$

and

$$
\begin{array}{ccccc}
GF(X) & & \xrightarrow{\quad (r''_{GF(X)})^{-1} \quad} & & GF(X)\otimes'' I'' \\
\Big\downarrow {\scriptstyle GF(r_X^{-1})} & \searrow^{G((r'_{F(X)})^{-1})} & & & \Big\uparrow {\scriptstyle \mathsf{id}_{GF(X)}\otimes'' \psi_G^0} \\
 & & G\big(F(X)\otimes' I'\big) & \xrightarrow{\psi_G^2(F(X),I')} & GF(X)\otimes'' G(I') \\
 & & \Big\uparrow {\scriptstyle G(\mathsf{id}_{F(X)}\otimes' \psi_F^0)} & & \Big\uparrow {\scriptstyle \mathsf{id}_{GF(X)}\otimes G(\psi_F^0)} \\
GF(X\otimes I) & \xrightarrow{G(\psi_F^2(X,I))} & G\big(F(X)\otimes' F(I)\big) & \xrightarrow{\psi_G^2(F(X),F(I))} & GF(X)\otimes'' GF(I)
\end{array}
$$

where we used again the fact that (F, ψ_F^0, ψ_F^2) and (G, ψ_G^0, ψ_G^0) are comonoidal and the naturality of ψ_G^2. The proof of the second part of the statement is analogous. $\square$

2.2 Braided monoidal categories

Definition 2.2.1. *A* ***braided monoidal category*** *is a seventuple $(\mathscr{C}, \otimes, I, a, \ell, r, c)$, where $(\mathscr{C}, \otimes, I, a, \ell, r)$ is a monoidal category and c is a* ***commutativity constraint****, or* ***braiding****, i.e. a natural isomorphism $c : \otimes \to \otimes^{\mathsf{op}}$ such that the two hexagonal diagrams*

$$\begin{array}{ccccc} & X\otimes(Y\otimes Z) & \xrightarrow{c_{X,Y\otimes Z}} & (Y\otimes Z)\otimes X & \\ \nearrow^{a_{X,Y,Z}} & & & & \searrow^{a_{Y,Z,X}} \\ (X\otimes Y)\otimes Z & & & & Y\otimes(Z\otimes X) \\ \searrow_{c_{X,Y}\otimes \mathsf{id}_Z} & & & & \nearrow_{\mathsf{id}_Y\otimes c_{X,Z}} \\ & (Y\otimes X)\otimes Z & \xrightarrow[a_{Y,X,Z}]{} & Y\otimes(X\otimes Z) & \end{array} \tag{2.2.1}$$

and

$$\begin{array}{ccccc} & (X\otimes Y)\otimes Z & \xrightarrow{c_{X\otimes Y,Z}} & Z\otimes(X\otimes Y) & \\ \nearrow^{a^{-1}_{X,Y,Z}} & & & & \searrow^{a^{-1}_{Z,X,Y}} \\ X\otimes(Y\otimes Z) & & & & (Z\otimes X)\otimes Y \\ \searrow_{\mathsf{id}_X\otimes c_{Y,Z}} & & & & \nearrow_{c_{X,Z}\otimes \mathsf{id}_Y} \\ & X\otimes(Z\otimes Y) & \xrightarrow[a^{-1}_{X,Z,Y}]{} & (X\otimes Z)\otimes Y & \end{array} \tag{2.2.2}$$

commute for any X, Y, Z *in* $\mathsf{Obj}(\mathscr{C})$*. If moreover the equality* $c_{Y,X} \circ c_{X,Y} = \mathsf{id}_{X\otimes Y}$ *holds for any* X, Y *in* $\mathsf{Obj}(\mathscr{C})$ *we say that* $\mathscr{C}$ *is a* ***symmetric monoidal category****.*

Definition 2.2.2. *Let* $(\mathscr{C}, \otimes, I, a, \ell, r, c)$ *and* $(\mathscr{C}', \otimes', I', a', \ell', r', c')$ *be two braided monoidal categories.*

- *A monoidal functor* $(F, \varphi_0^F, \varphi_2^F)$ *from* $\mathscr{C}$ *to* $\mathscr{C}'$ *is said to be a* ***braided monoidal functor*** *if for any pair* (X, Y) *of objects in* $\mathscr{C}$ *the square*

$$\begin{array}{ccc} F(X)\otimes' F(Y) & \xrightarrow{\varphi_2^F(X,Y)} & F(X\otimes Y) \\ {\scriptstyle c'_{F(X),F(Y)}}\big\downarrow & & \big\downarrow{\scriptstyle F(c_{X,Y})} \\ F(Y)\otimes' F(X) & \xrightarrow{\varphi_2^F(Y,X)} & F(Y\otimes X) \end{array} \tag{2.2.3}$$

  commutes.

- *A comonoidal functor* (F, ψ_F^0, ψ_F^2) *from* $\mathscr{C}$ *to* $\mathscr{C}'$ *is said to be a* ***braided comonoidal functor*** *if for any pair* (X, Y) *of objects in* $\mathscr{C}$ *the square*

$$\begin{array}{ccc} F(X\otimes Y) & \xrightarrow{\psi_F^2(X,Y)} & F(X)\otimes' F(Y) \\ {\scriptstyle F(c_{X,Y})}\big\downarrow & & \big\downarrow{\scriptstyle c'_{F(X),F(Y)}} \\ F(Y\otimes X) & \xrightarrow{\psi_F^2(Y,X)} & F(Y)\otimes' F(X) \end{array} \tag{2.2.4}$$

  commutes.

Proposition 2.2.3. *Let* $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ *be three braided monoidal categories.*

(i) *If* $(F, \psi^0_F, \psi^2_F) : \mathscr{C} \to \mathscr{C}'$ *and* $(G, \psi^0_G, \psi^0_G) : \mathscr{C}' \to \mathscr{C}''$ *are two braided comonoidal functors, then the triple*

$$\big(GF, \psi^G_0 \circ G(\psi^0_F,), \psi^2_G \circ G(\psi^2_F)\big) \tag{2.2.5}$$

is a braided comonoidal functor from $\mathscr{C}$ *to* $\mathscr{C}''$.

(ii) *If* $(F, \varphi^F_0, \varphi^F_2) : \mathscr{C} \to \mathscr{C}'$ *and* $(G, \varphi^G_0, \varphi^G_2) : \mathscr{C}' \to \mathscr{C}''$ *are two braided monoidal functors, then the triple*

$$\big(GF, G(\varphi^F_0) \circ \varphi^G_0, G(\varphi^F_2) \circ \varphi^G_2\big) \tag{2.2.6}$$

is a braided monoidal functor from $\mathscr{C}$ *to* $\mathscr{C}''$.

Proof. Let X, Y be objects of $\mathscr{C}$. The proof of (i) follows from Proposition 2.1.3 and from the commutativity of following diagram

$$\begin{array}{ccccc}
GF(X \otimes Y) & \xrightarrow{G(\psi^2_F(X,Y))} & G(F(X) \otimes' F(Y)) & \xrightarrow{\psi^G_2(F(X),F(Y))} & GF(X) \otimes'' GF(Y) \\
{\scriptstyle GF(c_{X,Y})}\downarrow & & \downarrow{\scriptstyle G(c'_{F(X),F(Y)})} & & \downarrow{\scriptstyle c''_{GF(X),GF(Y)}} \\
GF(Y \otimes X) & \xrightarrow{G(\psi^2_F(Y,X))} & G(F(Y) \otimes' F(X)) & \xrightarrow{\psi^G_2(F(Y),F(X))} & GF(Y) \otimes'' GF(X)
\end{array}$$

where the left (resp. right) square commutes since (F, ψ^F_0, ψ^F_2) (resp. (G, ψ^G_0, ψ^G_2)) is braided comonoidal. The proof of (ii) is analogous. □

2.3 Infinitesimally braided monoidal categories

We shall sometimes use the following

Notation 2.3.1. *If* f *(resp.* g*) is a morphism (resp. an invertible morphism) in a category* $\mathscr{C}$ *and* f *and* g *are composable, we denote by* f^g *the composition* $g^{-1} \circ f \circ g$.

Definition 2.3.2. *An* ***infinitesimally braided monoidal category*** *is an eightuple* $(\mathscr{C}, \otimes, I, a, \ell, r, c, t)$*, where* $(\mathscr{C}, \otimes, I, a, \ell, r, c)$ *is a pre–additive*[1] *symmetric monoidal category and* t *is an* ***infinitesimal braiding****, i.e. a natural morphism* $t : \otimes \to \otimes$ *such that the following relations hold for any* X, Y, Z *in* $\mathsf{Obj}(\mathscr{C})$*:*

$$t_{X,Y\otimes Z} = (t_{X,Y} \otimes \mathsf{id}_Z)^{a^{-1}_{X,Y,Z}} + (\mathsf{id}_Y \otimes t_{X,Z})^{a_{Y,X,Z} \circ (c_{X,Y} \otimes \mathsf{id}_Z) \circ a^{-1}_{X,Y,Z}} \tag{2.3.1a}$$

$$c_{X,Y} \circ t_{X,Y} = t_{Y,X} \circ c_{X,Y}. \tag{2.3.1b}$$

Definition 2.3.3. *Let* $(\mathscr{C}, \otimes, I, a, \ell, r, c, t)$ *and* $(\mathscr{C}', \otimes', I', a', \ell', r', c', t')$ *be two infinitesimally braided categories.*

- *A braided monoidal functor* $(F, \varphi^F_0, \varphi^F_2)$ *from* $\mathscr{C}$ *to* $\mathscr{C}'$ *is said to be an* ***infinitesimally braided monoidal functor*** *if for any* X, Y *in* $\mathsf{Obj}(\mathscr{C})$ *the square*

$$\begin{array}{ccc}
F(X) \otimes' F(Y) & \xrightarrow{\varphi^F_2(X,Y)} & F(X \otimes Y) \\
{\scriptstyle t'_{F(X),F(Y)}}\downarrow & & \downarrow{\scriptstyle F(t_{X,Y})} \\
F(X) \otimes' F(Y) & \xrightarrow{\varphi^F_2(Y,X)} & F(X \otimes Y)
\end{array} \tag{2.3.2}$$

commutes.

[1]see [KS06, Ch. 8] for more details on pre–additive categories

- *A braided comonoidal functor* (F, ψ_F^0, ψ_F^2) *from* $\mathscr{C}$ *to* $\mathscr{C}'$ *is said to be an* ***infinitesimally braided comonoidal functor*** *if for any* X, Y *in* $\mathsf{Obj}(\mathscr{C})$ *the square*

$$\begin{array}{ccc} F(X\otimes Y) & \xrightarrow{\psi_F^2(X,Y)} & F(X)\otimes' F(Y) \\ {\scriptstyle F(t_{X,Y})}\downarrow & & \downarrow {\scriptstyle t'_{F(X),F(Y)}} \\ F(X\otimes Y) & \xrightarrow{\psi_F^2(Y,X)} & F(X)\otimes' F(Y) \end{array} \tag{2.3.3}$$

commutes.

Proposition 2.3.4. *Let* $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ *be three infinitesimally braided monoidal categories.*

(i) If $(F, \psi_F^0, \psi_F^2) : \mathscr{C} \to \mathscr{C}'$ *and* $(G, \psi_G^0, \psi_G^0) : \mathscr{C}' \to \mathscr{C}''$ *are two infinitesimally braided comonoidal functors, then the triple*

$$\big(GF, \psi_0^G \circ G(\psi_F^0,), \psi_G^2 \circ G(\psi_F^2)\big) \tag{2.3.4}$$

is an infinitesimally braided comonoidal functor from $\mathscr{C}$ *to* $\mathscr{C}''$.

(ii) If $(F, \varphi_0^F, \varphi_2^F) : \mathscr{C} \to \mathscr{C}'$ *and* $(G, \varphi_0^G, \varphi_2^G) : \mathscr{C}' \to \mathscr{C}''$ *are two infinitesimally braided monoidal functors, then the triple*

$$\big(GF, G(\varphi_0^F) \circ \varphi_0^G, G(\varphi_2^F) \circ \varphi_2^G\big) \tag{2.3.5}$$

is an infinitesimally braided monoidal functor from $\mathscr{C}$ *to* $\mathscr{C}''$.

Proof. Let X, Y be objects of $\mathscr{C}$. The proof of (i) follows from Proposition 2.2.3 and from the commutativity of following diagram

$$\begin{array}{ccccc} GF(X\otimes Y) & \xrightarrow{G(\psi_F^2(X,Y))} & G(F(X)\otimes' F(Y)) & \xrightarrow{\psi_2^G(F(X),F(Y))} & GF(X)\otimes'' GF(Y) \\ {\scriptstyle GF(t_{X,Y})}\downarrow & & \downarrow{\scriptstyle G(t'_{F(X),F(Y)})} & & \downarrow{\scriptstyle t''_{GF(X),GF(Y)}} \\ GF(X\otimes Y) & \xrightarrow{G(\psi_F^2(X,Y))} & G(F(X)\otimes' F(Y)) & \xrightarrow{\psi_2^G(F(X),F(Y))} & GF(X)\otimes'' GF(Y) \end{array}$$

where the left (resp. right) square commutes since (F, ψ_0^F, ψ_2^F) (resp. (G, ψ_0^G, ψ_2^G)) is braided comonoidal. The proof of (ii) is analogous. □

2.4 Turning monoidal categories into strict ones

Definition 2.4.1. *A monoidal category* $(\mathscr{C}, \otimes, I, a, \ell, r)$ *is said to be* ***strict*** *if the associativity and the unit constraints are all identities. A monoidal (resp. comonoidal) functor* $(F, \varphi_0^F, \varphi_2^F)$ *(resp.* (F, ψ_F^0, ψ_F^2)*) is said to be strict if the morphisms* φ_0^F, φ_2^F *(resp* ψ_F^0, ψ_F^2*) are identities of the target category.*

The following result is well known as the Mac Lane coherence's Theorem:

Theorem 2.4.2 ([ML63]). *Let* $\mathscr{C}$ *be a monoidal category. Then there exists a strict category* $\mathscr{C}^{\mathsf{str}}$ *which is monoidally equivalent to* $\mathscr{C}$.

MacLane's result is very powerful, but it requires to replace the category $\mathscr{C}$ with a strict one $\mathscr{C}^{\mathsf{str}}$ with different objects. The following result, due to P. Schauenburg [Sch01], allows to consider a strict category which preserves the objects of $\mathscr{C}$.

Theorem 2.4.3 ([Sch01])**.** *Let $\mathscr{C}$ be a monoidal category of structured sets. Then there exists a strict category $\mathscr{C}^{\mathsf{str}}$ with the same objects of $\mathscr{C}$ which is monoidally equivalent to $\mathscr{C}$.*

It turns out that all categories of algebraic objects are categories of structured sets and therefore for our purpose it is convenient to rely on Schauenburg's result. MacLane's coherence Theorem can be extended to the case of braided monoidal categories, see [JS93, §4] for more details.
Mac Lane's and Schauenburg's coherence's theorems allow to 'forget the bracketing and ignore associators', and then simplifying a large number of proofs and of reasonings in a considerable way. A possible way to make this more precise is the following:
The strongly monoidal functor $F : \mathscr{C} \to \mathscr{C}^{\mathsf{str}}$ in the above monoidal equivalence is in general not *strictly monoidal*, i.e. its monoidal structure φ_2^F is not the identity morphism, hence it does in general not map morphisms made up out of compositions of associators and unitors tensored with identity morphims to the identity morphism in view of diagram (2.1.3) for $\alpha' = 1$, but to combinations of φ_2^F and its inverses.
However, in order to justify the passage to strict categories to simplify diagrams –which is very often used in the literature– this can at least be done *partially*, for instance in the following way, see [BH23] for more details: pick a subset L of the class $\mathscr{C}$, for instance a finite set of objects nonassociatively generating all the objects in a given commutative diagram. Then form the *free* nonassociative semigroup (magma) $\mathcal{M}L$ generated by L and the unit element I of $\mathscr{C}$, and the free (associative unital) monoid $\mathcal{F}L$ generated by L with multiplication $*$. There is an obvious map $F_L : \mathcal{M}L \to \mathscr{C}$ sending the free nonassociative words in L to concrete objects in $\mathscr{C}$ concatenated by the tensor product $\otimes$ of $\mathscr{C}$, and there is the obvious morphism of magmas $\Xi_L : \mathcal{M}L \to \mathcal{F}L$ which is the identity on generators in L and maps I to the unit element of $\mathcal{F}L$. We can make $\mathcal{M}L$ into a –in general nonstrict– monoidal category by attaching to each pair of nonassociative words the Hom-set of the category $\mathscr{C}$ attached to the pair of objects in $\mathscr{C}$ via F_L: here the monoidal structure is the free nonassociative multiplication on objects and the usual tensor product of morphisms in $\mathscr{C}$. Associators and unitors are borrowed by the ones in $\mathscr{C}$. Then F_L becomes a strictly monoidal full and faithful functor. Next, on the free monoid $\mathcal{F}L$ we can attach to each pair of associative words in L the Hom-set of $\mathscr{C}$ attached to the corresponding pair of *well-bracketed words* in $\mathscr{C}$: for instance for the associative word $l_1 * l_2 * l_3 * l_4$ we take $l_1 \otimes (l_2 \otimes (l_3 \otimes l_4))$, etc. Then $\mathcal{F}L$ becomes a strict monoidal category where the tensor product is just $*$ on objects, and on morphisms it is the usual tensor product of $\mathscr{C}$ but composed with the unique –thanks to the coherence theorem– rebracketing morphisms to 'restore' the well-bracketing. It can be shown that Ξ_L can also be made into a strictly monoidal full and faithful functor by consequently using the canonical rebracketing morphisms.
In order to use this 'partial strictification' for the proof of a diagram, more precisely of the equality of two morphisms in $\mathscr{C}$ involving associators and unitors we can first lift the morphisms to $\mathcal{M}L$ (made up of all the involved objects in $\mathscr{C}$ and where composed objects in $\mathscr{C}$ should be composed objects in $\mathcal{M}L$) and map the situation to the strict category $\mathcal{F}L$ by means of the functor Ξ_L where all the associators and unitors now really become identities since the functor Ξ_L is strictly monoidal, see again diagram (2.1.3). Since the functors F_L and Ξ_L are full and faithful, the equality of the two mapped morphisms implies the equality of the two morphisms in $\mathscr{C}$, which reflects the above-mentioned strategy of 'passing to strict monoidal categories' in the literature.

2.5 Some identities in monoidal–type categories

In this Section we collect some well–known identities holding in monoidal–type categories. All the proofs can be find in [Kas12].

Proposition 2.5.1. *(*[Kas12, Lem. XI.2.2–XI.2.3]*) Let $(\mathscr{C}, \otimes, I, a, \ell, r)$ be a monoidal category. Then for any X, Y, Z, W in $\mathsf{Obj}(\mathscr{C})$ and f, g in $\mathsf{Hom}(\mathscr{C})$ we have*

$$f \otimes g = (f \otimes \mathsf{id}_{t(g)}) \circ (\mathsf{id}_{s(f)} \otimes g) = (\mathsf{id}_{t(f)} \otimes g) \circ (f \otimes \mathsf{id}_{s(g)}) \tag{2.5.1a}$$

$$\ell_X \otimes \mathsf{id}_Y = \ell_{X \otimes Y} \circ a_{I,X,Y} \tag{2.5.1b}$$

$$r_{X \otimes Y} = (\mathsf{id}_X \otimes r_Y) \circ a_{X,Y,I} \tag{2.5.1c}$$

$$\ell_{I \otimes X} = \mathsf{id}_I \otimes \ell_X \tag{2.5.1d}$$

$$r_{X \otimes I} = r_X \otimes \mathsf{id}_I \tag{2.5.1e}$$

$$\ell_I = r_I. \tag{2.5.1f}$$

Proposition 2.5.2. [Kas12, Prop. XIII.1.2 and Th. XIII.1.3] *Let $(\mathscr{C}, \otimes, I, a, \ell, r, c)$ be a braided monoidal category. Then for any X, Y, Z in $\mathsf{Obj}(\mathscr{C})$ we have*

$$r_X = \ell_X \circ c_{X,I} \tag{2.5.2a}$$

$$\ell_X = r_X \circ c_{I,X} \tag{2.5.2b}$$

$$c_{I,X} = c^{-1}_{X,I} \tag{2.5.2c}$$

and the following dodecagon identity, which is the categorical interpretation of the Yang–Baxter Equation, holds

$$\begin{aligned} &a_{Z,Y,X} \circ (c_{Y,Z} \otimes \mathsf{id}_X) \circ a^{-1}_{Y,Z,X} \circ (\mathsf{id}_Y \otimes c_{X,Z}) \circ a_{Y,X,Z} \circ (c_{X,Y} \otimes \mathsf{id}_Z) \\ = &(\mathsf{id}_Z \otimes c_{X,Y}) \circ a_{Z,X,Y} \circ (c_{X,Z} \otimes \mathsf{id}_Y) \circ a^{-1}_{X,Z,Y} \circ (\mathsf{id}_X \otimes c_{Y,Z}) \circ a_{X,Y,Z}. \end{aligned} \tag{2.5.3}$$

Proposition 2.5.3. [Kas12, pp.495-496] *Let $(\mathscr{C}, \otimes, I, a, \ell, r, c, t)$ be an infinitesimally braided monoidal category. Then for any X, Y, Z in $\mathsf{Obj}(\mathscr{C})$ we have*

$$t_{X \otimes Y, Z} = (\mathsf{id}_X \otimes t_{Y,Z})^{a_{X,Y,Z}} + (t_{X,Z} \otimes \mathsf{id}_Y)^{\circ a^{-1}_{X,Y,Z} \circ (\mathsf{id}_X \otimes c_{Y,Z}) \circ a_{X,Y,Z}} \tag{2.5.4a}$$

$$t_{X,I} = 0 \tag{2.5.4b}$$

$$t_{I,X} = 0 \tag{2.5.4c}$$

and

$$[t_{X,Y} \otimes \mathsf{id}_Z, t_{X \otimes Y, Z}] = 0 \tag{2.5.5a}$$

$$[\mathsf{id}_X \otimes t_{Y,Z}, t_{X, Y \otimes Z}] = 0. \tag{2.5.5b}$$

Proposition 2.5.3 gives us the crucial property of an infinitesimally braided monoidal category, that is, its infinitesimally braiding t give rise to a collection of natural morphisms satisfying the infinitesimally braid relations (2.5.8a), (2.5.8b), (2.5.8c). This will be the crucial feature needed in order to construct, given a Drinfeld associator (see Chapter 5), *deformed* braided monoidal categories, see §2.7. Namely, we have the following

Proposition 2.5.4. *Let $(\mathscr{C}, \otimes, I, a, \ell, r, c, t)$ be an infinitesimally braided monoidal category. For any X, Y, Z, W in $\mathsf{Obj}(\mathscr{C})$, consider the following endomorphims of $(X \otimes Y) \otimes Z$*

$$t^{12}_{X,Y,Z} \coloneqq t_{X,Y} \otimes \mathsf{id}_Z \tag{2.5.6a}$$
$$t^{23}_{X,Y,Z} \coloneqq (\mathsf{id}_X \otimes t_{Y,Z})^{a_{X,Y,Z}} \tag{2.5.6b}$$
$$t^{13}_{X,Y,Z} \coloneqq (t_{X,Z} \otimes \mathsf{id}_Y)^{a^{-1}_{X,Z,Y} \circ (\mathsf{id}_X \otimes c_{Y,Z}) \circ a_{X,Y,Z}} \tag{2.5.6c}$$
$$t^{21}_{X,Y,Z} \coloneqq (c^{-1}_{X,Y} \circ t_{Y,X} \circ c_{X,Y}) \otimes \mathsf{id}_Z \tag{2.5.6d}$$
$$t^{32}_{X,Y,Z} \coloneqq \big(\mathsf{id}_X \otimes (c^{-1}_{Y,Z} \circ t_{Z,Y} \circ c_{Y,Z})\big)^{a_{X,Y,Z}} \tag{2.5.6e}$$
$$t^{31}_{X,Y,Z} \coloneqq (c^{-1}_{X,Z} \circ t_{Z,X} \circ c_{X,Z}) \otimes \mathsf{id}_Y)^{a^{-1}_{X,Z,Y} \circ (\mathsf{id}_X \otimes c_{Y,Z}) \circ a_{X,Y,Z}} \tag{2.5.6f}$$

and the following endomorphisms of $((X \otimes Y) \otimes Z) \otimes W$

$$t^{12}_{X,Y,Z,W} \coloneqq t^{12}_{X,Y,Z} \otimes \mathsf{id}_W \tag{2.5.7a}$$
$$t^{23}_{X,Y,Z,W} \coloneqq t^{23}_{X,Y,Z} \otimes \mathsf{id}_W \tag{2.5.7b}$$
$$t^{13}_{X,Y,Z,W} \coloneqq t^{13}_{X,Y,Z} \otimes \mathsf{id}_W \tag{2.5.7c}$$
$$t^{14}_{X,Y,Z,W} \coloneqq (t^{12}_{X,W,Y,Z})^{(a^{-1}_{X,W,Y} \otimes \mathsf{id}_W) \circ a^{-1}_{X,W \otimes Y,Z} \circ (\mathsf{id}_X \otimes a^{-1}_{W,Y,Z}) \circ (\mathsf{id}_X \otimes c_{Y \otimes Z,W}) \circ a_{X,Y \otimes Z,W} \circ (a_{X,Y,Z} \otimes \mathsf{id}_W)} \tag{2.5.7d}$$
$$t^{24}_{X,Y,Z,W} \coloneqq (\mathsf{id}_X \otimes t^{13}_{Y,Z,W})^{a_{X,Y \otimes Z,W} \circ (a_{X,Y,Z} \otimes \mathsf{id}_W)} \tag{2.5.7e}$$
$$t^{34}_{X,Y,Z,W} \coloneqq \big((\mathsf{id}_X \otimes t^{23}_{Y,Z,W})^{a_{X,Y \otimes Z,W}}\big)^{a_{X,Y,Z} \otimes \mathsf{id}_W} \tag{2.5.7f}$$
$$t^{21}_{X,Y,Z,W} \coloneqq t^{21}_{X,Y,Z} \otimes \mathsf{id}_W \tag{2.5.7g}$$
$$t^{32}_{X,Y,Z,W} \coloneqq t^{32}_{X,Y,Z} \otimes \mathsf{id}_W \tag{2.5.7h}$$
$$t^{31}_{X,Y,Z,W} \coloneqq t^{31}_{X,Y,Z} \otimes \mathsf{id}_W \tag{2.5.7i}$$
$$t^{41}_{X,Y,Z,W} \coloneqq (t^{21}_{X,W,Y,Z})^{(a^{-1}_{X,W,Y} \otimes \mathsf{id}_W) \circ a^{-1}_{X,W \otimes Y,Z} \circ (\mathsf{id}_X \otimes a^{-1}_{W,Y,Z}) \circ (\mathsf{id}_X \otimes c_{Y \otimes Z,W}) \circ a_{X,Y \otimes Z,W} \circ (a_{X,Y,Z} \otimes \mathsf{id}_W)} \tag{2.5.7j}$$
$$t^{42}_{X,Y,Z,W} \coloneqq (\mathsf{id}_X \otimes t^{31}_{Y,Z,W})^{a_{X,Y \otimes Z,W} \circ (a_{X,Y,Z} \otimes \mathsf{id}_W)} \tag{2.5.7k}$$
$$t^{43}_{X,Y,Z,W} \coloneqq \big((\mathsf{id}_X \otimes t^{32}_{Y,Z,W})^{a_{X,Y \otimes Z,W}}\big)^{a_{X,Y,Z} \otimes \mathsf{id}_W} \tag{2.5.7l}$$

Then the morphisms $t^{ij}_{X,Y,Z}$ and $t^{ij}_{X,Y,Z,W}$ satisfy the infinitesimal braid relations, namely

$$t^{ij} - t^{ji} = 0 \qquad \text{for all } i, j \text{ with } \#\{i, j\} = 2 \tag{2.5.8a}$$
$$[t^{ij}, t^{ik} + t^{jk}] = 0 \qquad \text{for all } i, j, k \text{ with } \#\{i, j, k\} = 3 \tag{2.5.8b}$$
$$[t^{ij}, t^{kh}] = 0 \qquad \text{for all } i, j, k, h \text{ with } \#\{i, j, k, h\} = 4. \tag{2.5.8c}$$

Proof. It suffices to concatenate appropriately relations (2.3.1b), (2.3.1a), (2.5.4a), (2.5.5a), (2.5.5b). □

Notation 2.5.5. *We denote by α and β the following two isomorphisms in a braided monoidal category:*

$$\alpha_{X,Y,Z,W} \coloneqq a_{X \otimes Y,Z,W} \circ (a_{X,Y,Z} \otimes \mathsf{id}_W)^{-1} : (X \otimes (Y \otimes Z)) \otimes T \to (X \otimes Y) \otimes (Z \otimes T)$$
$$\beta_{X,Y,Z,W} \coloneqq \alpha_{X,Z,Y,W} \circ \big((\mathsf{id}_X \otimes c_{Y,Z}) \otimes \mathsf{id}_W\big) \circ (\alpha_{X,Y,Z,T})^{-1} : (X \otimes Y) \otimes (Z \otimes T) \to (X \otimes Z) \otimes (Y \otimes T).$$

2.6 The Drinfeld center

We shall need the notion of Drinfeld center in order to show, using Theorem 3.6.3, the compatibility of Ševera's left module and right comodule structure in §7.9. For more details we remand the reader to [Maj91], [JS91], [Kas12, XIII.4], [Müg03] and [Thu18].

Definition 2.6.1. *Let $\mathscr{C}$ be a monoidal category. The **Drinfeld center** of $\mathscr{C}$ is the category $\mathscr{Z}(\mathscr{C})$ where:*

- *Objects are pairs $(X, c_{-,X})$, where X is in $\mathsf{Obj}(\mathscr{C})$ and $c_{-,X}$ is a family of natural isomorphisms indexed by all objects of $\mathscr{C}$, $c_{Y,X} : Y \otimes X \to X \otimes Y$ such that*

$$c_{Y\otimes Z,X} = a_{X,Y,Z} \circ (c_{Y,Z} \otimes \mathsf{id}_X) \circ a^{-1}_{Y,X,Z} \circ (\mathsf{id}_Y \otimes c_{Z,X}) \circ a_{Y,Z,X}.$$

- *Morphisms $f : (X, c_{-,X}) \to (Y, c_{-,Y})$ are morphisms $f : X \to Y$ such that for any $Z \in \mathscr{C}$ the following identity holds*

$$(f \otimes \mathsf{id}_Z) \circ c_{Z,X} = c_{Z,Y} \circ (\mathsf{id}_Z \otimes f),$$

  i.e. the diagram

$$\begin{array}{ccc} Z \otimes X & \xrightarrow{\mathsf{id}_Z \otimes f} & Z \otimes Y \\ {\scriptstyle c_{Z,X}}\downarrow & & \downarrow{\scriptstyle c_{Z,Y}} \\ X \otimes Z & \xrightarrow{f \otimes \mathsf{id}_Z} & Y \otimes Z \end{array}$$

  commutes.

We collect the main properties of $\mathscr{Z}(\mathscr{C})$ in the following Theorem, see [Kas12, XIII.4.2 and XIII.4.3] for the proofs:

Theorem 2.6.2. *Let $\mathscr{C}$ be a monoidal category. Then*

(i) $\mathscr{Z}(C)$ is a braided monoidal category, where:

- *The unit object is (I, id_I).*
- *The tensor product is $(X, c_{-,X}) \otimes (Y, c_{-,Y}) := (X \otimes Y, c_{-,X\otimes Y})$, where*

$$c_{Z,X\otimes Y} = a^{-1}_{X,Y,Z} \circ (\mathsf{id}_X \otimes c_{Z,Y}) \circ a_{X,Z,Y} \circ (c_{Z,X} \otimes \mathsf{id}_Y) \circ a^{-1}_{Z,X,Y}.$$

- *The braiding is*

$$c_{V,W} : (X, c_{-,X}) \otimes (Y, c_{-,Y}) \to (Y, c_{-,Y}) \otimes (X, c_{-,X}).$$

(ii) The functor

$$\begin{aligned} \Pi : \mathscr{Z}(\mathscr{C}) &\to \mathscr{C} \\ (X, c_{-,X}) &\mapsto X \\ f &\mapsto f \end{aligned}$$

is a monoidal functor.

(iii) (Universal property of $\mathscr{Z}(\mathscr{C})$): If $\mathscr{C}'$ is another braided monoidal category and $F : \mathscr{C}' \to \mathscr{C}$ is a monoidal functor which is bijective on objects and surjective on morphisms, then there exists a unique braided monoidal functor $\mathscr{Z}(F) : \mathscr{C}' \to \mathscr{Z}(\mathscr{C})$ such that $F = \Pi \circ \mathscr{Z}(F)$.

2.7 Deforming infinitesimally braided monoidal categories

Let $(\mathscr{C}, \otimes, \mathbb{K}, a, \ell, r, c, t)$ be an (algebraic) infinitesimally braided monoidal category and $\hbar$ be a formal variable. Then we can consider a new infinitesimally braided monoidal category $(\mathscr{C}_\hbar, \bar{\otimes}, \mathbb{K}, \overline{a}, \overline{\ell}, \overline{r}, \overline{c}, \overline{t})$, where $\mathsf{Obj}(\mathscr{C}_\hbar) = \mathsf{Obj}(\mathscr{C})$, $\mathsf{Hom}_{\mathscr{C}_\hbar}(X, Y) = \mathsf{Hom}_{\mathscr{C}}(X, Y)[[\hbar]]$ (i.e. $f \in \mathsf{Hom}_{\mathscr{C}_\hbar}(X, Y)$ is a morphism of topologically free $K[[\hbar]]$–modules $f : X[[\hbar]] \to Y[[\hbar]]$), $\bar{\otimes}$ is the functor which coincides with $\otimes$ on objects and is the $\hbar$–adic completion of $\otimes$ on the morphisms, and $\overline{f}$ denotes the $\hbar$–adic completion of f. For more details on topologically free modules and on the $\hbar$–adic completion see [Kas12, XVI] and references therein. We shall keep the notation id_X for the $\hbar$–adic completion of the identity map id_X of X in $\mathsf{Obj}(\mathscr{C})$.

Definition 2.7.1. *Let $\mathbb{K}$ be a field of characteristics zero and $\hbar$ be a formal parameter. A* ***Drinfeld associator*** *is a formal power series $\Phi(A, B) \in \mathbb{K}\langle\langle A, B\rangle\rangle$ in two non–commuting variables A, B such that*

(i) For any elements $\{A_{ij}\}_{1\leqslant i,j\leqslant 4}$ satisfying the infinitesimal braid relations (2.5.8a) (2.5.8b) (2.5.8c)*, the* ***pentagon equation***

$$\Phi(A_{12}, A_{23} + A_{24})\Phi(A_{13} + A_{23}, A_{34}) = \Phi(A_{23}, A_{34})\Phi(A_{12} + A_{13}, A_{24} + A_{34})\Phi(A_{12}, A_{23}) \quad (2.7.1)$$

holds.

(ii) For any elements A, B, C and $\Lambda := A + B + C$ satisfying $[\Lambda, A] = [\Lambda, B] = [\Lambda, C] = 0$, the ***hexagon equation***

$$e^{\hbar\Lambda/2} = e^{\hbar A/2}\Phi(C, A)e^{\hbar C/2}\Phi(B, C)e^{\hbar B/2}\Phi(A, B) \quad (2.7.2)$$

holds.

(iii) Φ satisfies

$$\Phi = 1 + \mathcal{O}(\hbar^2). \quad (2.7.3)$$

(iv) For any A, B one has

$$\Phi(A, B)^{-1} = \Phi(B, A). \quad (2.7.4)$$

We shall present the construction of the Drinfeld associator arising from the Knizhnik–Zamolodchikov connection in Chapter 5.

Theorem 2.7.2. *Let $(\mathscr{C}, \otimes, \mathbb{K}, a, \ell, r, c, t)$ be an (algebraic) infinitesimally braided monoidal category, $\hbar$ be a formal variable, and Φ be a Drinfeld associator. Then $\mathscr{C}_\hbar^\Phi := (\mathscr{C}_\hbar, \bar{\otimes}, \mathbb{K}, \overline{a}^\Phi, \overline{\ell}, \overline{r}, \overline{c}^\Phi)$ is a braided monoidal category, where*

$$\overline{a}^\Phi_{X,Y,Z} = \overline{a}_{X,Y,Z} \circ \Phi(\overline{t}^{12}_{X,Y,Z}, \overline{t}^{23}_{X,Y,Z}) \quad (2.7.5a)$$
$$\overline{c}^\Phi_{X,Y} = \overline{c}_{X,Y} \circ e^{\overline{t}_{X,Y}/2} \quad (2.7.5b)$$

where $\overline{t}^{12}_{X,Y,Z}, \overline{t}^{23}_{X,Y,Z}$ are the $\hbar$–adic completions of the morphisms defined in Proposition 2.5.4.

Proof. In order to give the proof we may suppose that the category $\mathscr{C}$ is strict. First, the triangle axiom (2.1.2) holds since

$$\begin{aligned}(\mathsf{id}_X \bar{\otimes} \bar{\ell}_Y) \circ \bar{a}^{\Phi}_{X,I,Y} &= \Phi(\bar{t}_{X,I} \bar{\otimes} \mathsf{id}_Y, \mathsf{id}_X \bar{\otimes} \bar{t}_{I,Y}) \\ &\overset{(2.7.3),(2.5.4b),(2.5.4c)}{=} \mathsf{id}_{X\otimes I\otimes Y}.\end{aligned}$$

Next, we show that the pentagon axiom (2.1.1), i.e. that

$$(\mathsf{id}_X \bar{\otimes} \bar{a}^{\Phi}_{Y,Z,W}) \circ \bar{a}^{\Phi}_{X,Y\otimes Z,W} \circ (\bar{a}^{\Phi}_{X,Y,Z} \bar{\otimes} \mathsf{id}_W) = \bar{a}^{\Phi}_{X,Y,Z\otimes W} \circ \bar{a}^{\Phi}_{X\otimes Y,Z,W} \tag{2.7.6}$$

holds for any X, Y, Z, W in $\mathsf{Obj}(\mathscr{C})$. We have

$$\begin{aligned}\mathsf{id}_X \bar{\otimes} \bar{a}^{\Phi}_{Y,Z,W} &\overset{(2.7.5a)}{=} \mathsf{id}_X \bar{\otimes} \Phi(\bar{t}_{Y,Z} \bar{\otimes} \mathsf{id}_W, \mathsf{id}_Y \bar{\otimes} \bar{t}_{Z,W}) \\ &\overset{(2.5.6a),(2.5.6b)}{=} \mathsf{id}_X \bar{\otimes} \Phi(\bar{t}^{12}_{Y,Z,W}, \bar{t}^{23}_{Y,Z,W}) \\ &\overset{(2.5.7b),(2.5.7f)}{=} \Phi(\bar{t}^{23}_{X,Y,Z,W}, \bar{t}^{34}_{X,Y,Z,W})\end{aligned}$$

and using Equations (2.3.1a) and (2.5.4a) we get

$$\begin{aligned}\bar{a}^{\Phi}_{X,Y\otimes Z,W} &\overset{(2.7.5a)}{=} \Phi(\bar{t}_{X,Y\otimes Z} \bar{\otimes} \mathsf{id}_W, \mathsf{id}_X \bar{\otimes} \bar{t}_{Y\otimes Z,W}) \\ &\overset{(2.5.7a),(2.5.7c),(2.5.7e),(2.5.7f)}{=} \Phi(\bar{t}^{12}_{X,Y,Z,W} + \bar{t}^{13}_{X,Y,Z,W}, \bar{t}^{24}_{X,Y,Z,W} + \bar{t}^{34}_{X,Y,Z,W}).\end{aligned}$$

Finally, we have

$$\begin{aligned}\bar{a}^{\Phi}_{X,Y,Z} \bar{\otimes} \mathsf{id}_W &\overset{(2.7.5a)}{=} \Phi(\bar{t}_{X,Y} \bar{\otimes} \mathsf{id}_Z, \mathsf{id}_X \bar{\otimes} \bar{t}_{Y,Z}) \bar{\otimes} \mathsf{id}_W \\ &\overset{(2.5.6a),(2.5.6b)}{=} \Phi(\bar{t}^{12}_{X,Y,Z}, \bar{t}^{23}_{X,Y,Z}) \bar{\otimes} \mathsf{id}_W \\ &\overset{(2.5.7a),(2.5.7b)}{=} \Phi(\bar{t}^{12}_{X,Y,Z,W}, \bar{t}^{23}_{X,Y,Z,W}).\end{aligned}$$

Therefore the left hand side of (2.7.6) is equal to

$$\Phi(\bar{t}^{23}_{X,Y,Z,W}, \bar{t}^{34}_{X,Y,Z,W}) \Phi(\bar{t}^{12}_{X,Y,Z,W} + \bar{t}^{13}_{X,Y,Z,W}, \bar{t}^{24}_{X,Y,Z,W} + \bar{t}^{34}_{X,Y,Z,W}) \Phi(\bar{t}^{12}_{X,Y,Z,W}, \bar{t}^{23}_{X,Y,Z,W}).$$

On the other side, using Equations (2.3.1a) and (2.5.4a) we get

$$\begin{aligned}\bar{a}^{\Phi}_{X,Y,Z\otimes W} &\overset{(2.7.5a)}{=} \Phi(\bar{t}_{X,Y} \bar{\otimes} \mathsf{id}_{Z\otimes W}, \mathsf{id}_X \bar{\otimes} \bar{t}_{Y,Z\otimes W}) \\ &\overset{(2.5.7a),(2.5.7b),(2.5.7e)}{=} \Phi(\bar{t}^{12}_{X,Y,Z,W}, \bar{t}^{23}_{X,Y,Z,W} + \bar{t}^{24}_{X,Y,Z,W})\end{aligned}$$

and

$$\begin{aligned}\bar{a}^{\Phi}_{X\otimes Y,Z,W} &\overset{(2.7.5a)}{=} \Phi(\bar{t}_{X\otimes Y,Z} \bar{\otimes} \mathsf{id}_W, \mathsf{id}_{X\otimes Y} \bar{\otimes} \bar{t}_{Z,W}) \\ &\overset{(2.5.7c),(2.5.7b),(2.5.7a)}{=} \Phi(\bar{t}^{13}_{X,Y,Z,W} + \bar{t}^{23}_{X,Y,Z,W}, \bar{t}^{12}_{X,Y,Z,W}),\end{aligned}$$

showing that the left hand side of (2.7.6) is

$$\Phi(\bar{t}^{13}_{X,Y,Z,W} + \bar{t}^{23}_{X,Y,Z,W}, \bar{t}^{12}_{X,Y,Z,W}) \Phi(\bar{t}^{13}_{X,Y,Z,W} + \bar{t}^{23}_{X,Y,Z,W}, \bar{t}^{12}_{X,Y,Z,W}).$$

Therefore, the pentagon axiom (2.7.6) holds due to the pentagon equation (2.7.1). Finally, in order to prove the hexagon axiom (2.2.1) we need to show that the following identity

$$\Phi(\bar{t}^{12}_{Y,Z,X}, \bar{t}^{23}_{Y,Z,X}) \bar{c}_{X,Y\otimes Z} e^{\bar{t}_{X,Y\otimes Z}/2} \Phi(\bar{t}^{12}_{X,Y,Z}, \bar{t}^{23}_{X,Y,Z}) = (\mathsf{id}_Y \otimes \bar{c}_{X,Z} e^{\bar{t}_{X,Z}/2}) \Phi(\bar{t}^{12}_{Y,X,Z}, \bar{t}^{13}_{Y,X,Z}) (\bar{c}_{X,Y} e^{\bar{t}_{X,Y}/2} \otimes \mathsf{id}_Z) \tag{2.7.7}$$

holds for any X, Y, Z in $\mathsf{Obj}(\mathscr{C})$. Using Equation (2.3.1a) and the commutativity of the following two diagrams (which follows from the naturality of the braiding)

$$\begin{CD} X\otimes Y\otimes Z @>{c_{X,Y\otimes Z}}>> Y\otimes Z\otimes X \\ @V{t^{23}_{X,Y,Z}}VV @VV{t^{12}_{Y,Z,X}}V \\ X\otimes Y\otimes Z @>{c_{X,Y\otimes Z}}>> Y\otimes Z\otimes X \end{CD} \qquad \begin{CD} X\otimes Y\otimes Z @>{c_{X,Y\otimes Z}}>> Y\otimes Z\otimes X \\ @V{t^{13}_{X,Y,Z}}VV @VV{t^{23}_{Y,Z,X}}V \\ X\otimes Y\otimes Z @>{c_{X,Y\otimes Z}}>> Y\otimes Z\otimes X \end{CD}$$

we can rewrite the left hand side of (2.7.7) as

$$\bar{c}_{X,Y\otimes Z} \circ \Phi(\bar{t}^{23}_{X,Y,Z}, \bar{t}^{13}_{X,Y,Z}) \circ e^{(\bar{t}^{12}_{X,Y,Z} + \bar{t}^{13}_{X,Y,Z})/2} \circ \Phi(\bar{t}^{12}_{X,Y,Z}, \bar{t}^{23}_{X,Y,Z})$$

which, by defining $\Lambda_{X,Y,Z}/2 \coloneqq (\bar{t}^{12}_{X,Y,Z} + \bar{t}^{23}_{X,Y,Z} + \bar{t}^{13}_{X,Y,Z})/2$ (that commutes with $\bar{t}^{12}_{X,Y,Z}, \bar{t}^{23}_{X,Y,Z}, \bar{t}^{13}_{X,Y,Z}$) we rewrite as

$$\bar{c}_{X,Y\otimes Z} \circ e^{\Lambda_{X,Y,Z}/2} \circ \Phi(\bar{t}^{13}_{X,Y,Z}, \bar{t}^{23}_{X,Y,Z}) \circ e^{-\bar{t}^{23}_{X,Y,Z}/2} \circ \Phi(\bar{t}^{12}_{X,Y,Z}, \bar{t}^{23}_{X,Y,Z}).$$

On the other side, using the commutativity of the following diagrams (again using the naturality of the braiding)

$$\begin{CD} X\otimes Y\otimes Z @>{c_{X,Y}\otimes\mathsf{id}_Z}>> Y\otimes X\otimes Z \\ @V{t^{12}_{X,Y,Z}}VV @VV{t^{12}_{Y,X,Z}}V \\ X\otimes Y\otimes Z @>{c_{X,Y}\otimes\mathsf{id}_Z}>> Y\otimes X\otimes Z \end{CD} \qquad \begin{CD} X\otimes Y\otimes Z @>{c_{X,Y}\otimes\mathsf{id}_Z}>> Y\otimes X\otimes Z \\ @V{t^{13}_{X,Y,Z}}VV @VV{t^{23}_{Y,X,Z}}V \\ X\otimes Y\otimes Z @>{c_{X,Y}\otimes\mathsf{id}_Z}>> Y\otimes X\otimes Z \end{CD}$$

together with Equation (2.2.1) we rewrite the right hand side of (2.7.7) as

$$\bar{c}_{X,Y\otimes Z} \circ e^{\bar{t}^{13}_{X,Y,Z}/2} \circ \Phi(\bar{t}^{12}_{X,Y,Z}, \bar{t}^{13}_{X,Y,Z}) \circ e^{\bar{t}^{12}_{X,YZ}/2}.$$

Therefore, using the inverse formula (2.7.4) for the associator Φ we have that the identity (2.7.7) holds if and only if

$$e^{\Lambda_{X,Y,Z}/2} = e^{\bar{t}^{13}_{X,Y,Z}/2} \Phi(\bar{t}^{12}_{X,Y,Z}, \bar{t}^{13}_{X,Y,Z}) e^{\bar{t}^{12}_{X,Y,Z}/2} e^{\bar{t}^{23}_{X,Y,Z}/2} \Phi(\bar{t}^{13}_{X,Y,Z}, \bar{t}^{23}_{X,Y,Z})$$

which is exactly the hexagon equation (2.7.2) with $A = \bar{t}^{13}_{X,Y,Z}$, $B = \bar{t}^{23}_{X,Y,Z}$ and $C = \bar{t}^{12}_{X,Y,Z}$. The proof of the second hexagon axiom (2.2.2) follows by a similar argument. □

2.8 The category of vector spaces

We end this Chapter by presenting the standard infinitesimally braided monoidal structure of the category $\mathsf{Vect}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$.

Proposition 2.8.1. *Let $\mathbb{K}$ be a field of characteristics zero. Then the category $\mathsf{Vect}_{\mathbb{K}}$ of all vector spaces over $\mathbb{K}$ is infinitesimally symmetric monoidal, where:*

- *$\otimes$ is the usual tensor product of vector spaces.*
- *The unit is the ground field $\mathbb{K}$.*
- *The associativity constraint is the canonical linear map*

$$\begin{aligned} a_{X,Y,Z} : (X \otimes Y) \otimes Z &\to X \otimes (Y \otimes Z) \\ (x \otimes y) \otimes z &\mapsto x \otimes (y \otimes z) \end{aligned}$$

- *The left unit constraint is*

$$\begin{aligned} \ell_X : \mathbb{K} \otimes X &\to X \\ \lambda \otimes x &\mapsto \lambda x \end{aligned}$$

- *The right unit constraint is*

$$\begin{aligned} r_X : X \otimes \mathbb{K} &\to X \\ x \otimes \lambda &\mapsto \lambda x \end{aligned}$$

- *The commutativity constraint is*

$$\begin{aligned} c_{X,Y} : X \otimes Y &\to Y \otimes X \\ x \otimes y &\mapsto y \otimes x \end{aligned}$$

- *The infinitesimal braiding is*

$$\begin{aligned} t_{X,Y} : X \otimes Y &\to Y \otimes X \\ x \otimes y &\mapsto 0 \end{aligned}$$

Notation 2.8.2. *From now on we shall sometimes denote the braiding of* $\mathsf{Vect}_{\mathbb{K}}$ *by* τ.

Chapter 3

Hopf monoids

In this Chapter we fix a monoidal category $\mathscr{C}$.

3.1 Monoids

Definition 3.1.1. *A* ***monoid*** *is a triple* (A, μ, η)*, where* A *is an object of* $\mathscr{C}$ *and* $\mu : A \otimes A \to A$ *,* $\eta : I \to A$ *are morphisms (called the multiplication and the unit) such that*

(i) (associativity axiom): the pentagon

$$\begin{array}{ccc} (A \otimes A) \otimes A & \xrightarrow{\mu \otimes \mathsf{id}} & A \otimes A \\ \downarrow{\scriptstyle a_{A,A,A}} & & \\ A \otimes (A \otimes A) & & \downarrow{\scriptstyle \mu} \\ \downarrow{\scriptstyle \mathsf{id} \otimes \mu} & & \\ A & \xrightarrow{\quad \mu \quad} & A \end{array} \tag{3.1.1}$$

commutes;

(ii) (unit axiom): the diagram

$$\begin{array}{ccccc} I \otimes A & \xrightarrow{\eta \otimes \mathsf{id}_A} & A \otimes A & \xleftarrow{\mathsf{id}_A \otimes \eta} & A \otimes I \\ & {\scriptstyle \ell_A} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle r_A} & \\ & & A & & \end{array} \tag{3.1.2}$$

commutes.

If moreover $\mathscr{C}$ *is braided monoidal and the triangle*

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\quad c_{A,A} \quad} & A \otimes A \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mu} & \\ & A & \end{array} \tag{3.1.3}$$

commutes we say that (A,μ,η) is a commutative monoid. If moreover $\mathscr{C}$ is infinitesimally braided monoidal and the triangle

$$\begin{array}{ccc} A\otimes A & \xrightarrow{t_{A,A}} & A\otimes A \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & A & \end{array} \tag{3.1.4}$$

commutes we say that (A,μ,η) is an inifnitesimally braided commutative monoid.

Definition 3.1.2. *Let (A,μ,η) and (A',μ',η') be two monoids. A morphism $f : A \to A'$ is said to be a* ***morphism of monoids*** *if the diagrams*

$$\begin{array}{ccc} A\otimes A & \xrightarrow{f\otimes f} & A'\otimes A' \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu'} \\ A & \xrightarrow{f} & A' \end{array} \qquad\qquad \begin{array}{ccc} I & \xrightarrow{\eta} & A \\ & {\scriptstyle\eta'}\searrow \quad \swarrow{\scriptstyle f} & \\ & A' & \end{array} \tag{3.1.5}$$

commute.

We denote the category of all monoids of $\mathscr{C}$ by $\mathsf{Mon}(\mathscr{C})$.

Proposition 3.1.3. *Suppose that $\mathscr{C}$ is braided. Then*

(i) If (A_1,μ_1,η_1) and (A_2,μ_2,η_2) are two monoids, so is $\big(A_1\otimes A_2, (\mu_1\otimes\mu_2)\circ\beta_{A_1,A_1,A_2,A_2}, \eta_1\otimes\eta_2\big)$.

(ii) If (A_1,μ_1,η_1), (A_2,μ_2,η_2), (A_3,μ_3,η_3) and (A_4,μ_4,η_4) are four monoids, and $f : A_1 \to A_2$, $g : A_3 \to A_4$ are morphisms of monoids, then so is $(f\otimes g) : A_1\otimes A_3 \to A_2\otimes A_4$.

(iii) If (A,μ,η) is a commutative monoid, then μ is a morphism of monoids.

Note that the first two statements gives that $\mathsf{Mon}(\mathscr{C})$ is a monoidal category.

The proof follows by reversing all the arrows of Proposition 3.3.3.

Proposition 3.1.4. *Let $(\mathscr{C},\otimes,I,a,\ell,r)$ and $(\mathscr{C}',\otimes',I',a,\ell,r)$ be two monoidal categories, $(F,\varphi_0^F,\varphi_2^F)$ be a monoidal functor and (A,μ,η), (A_1,μ_1,η_1),(A_2,μ_2,η_2) be monoids in $\mathscr{C}$. Then*

(i) The triple $\big(F(A), F(\mu)\circ\varphi_2^F(A,A), F(\eta)\circ\varphi_0^F\big)$ is a monoid in $\mathscr{C}'$.

(ii) If $f : A_1 \to A_2$ is a morphism of monoids, then so is $F(f) : F(A_1) \to F(A_2)$.

(iii) If $\mathscr{C}$ and $\mathscr{C}'$ are braided and $(F,\varphi_0^F,\varphi_2^F)$ is a braided monoidal functor, then

$$\varphi_2^F(A_1,A_2) : F(A_1)\otimes' F(A_2) \to F(A_1\otimes A_2)$$

is a morphism of monoids.

The proof follows by reversing all the arrows of Proposition 3.3.4.

Remark 3.1.5. *If $\mathscr{C} = \mathsf{Vect}_{\mathbb{K}}$, then $\mathsf{Mon}(\mathscr{C})$ is the usual braided monoidal category of unital associative algebras.*

3.2 Modules

Definition 3.2.1. *Let (A,μ,η) be a monoid. A* ***left*** *A–**module** is a pair (M,μ_M), where M is an object and $\mu_M : A\otimes M \to M$ is a morphism, called the action of A on M, such that the pentagon*

$$\begin{array}{ccc} (A\otimes A)\otimes M & \xrightarrow{\mu\otimes \mathsf{id}_M} & A\otimes M \\ \downarrow{\scriptstyle a_{A,A,M}} & & \\ A\otimes(A\otimes M) & & \downarrow{\scriptstyle \mu_M} \\ \downarrow{\scriptstyle \mathsf{id}_A\otimes\mu_M} & & \\ A\otimes M & \xrightarrow[\mu_M]{} & M \end{array} \tag{3.2.1}$$

and the triangle

$$\begin{array}{ccc} I\otimes M & \xrightarrow{\eta\otimes\mathsf{id}_M} & A\otimes M \\ & {\scriptstyle \ell_M}\searrow \quad \swarrow{\scriptstyle \mu_M} & \\ & M & \end{array} \tag{3.2.2}$$

commute.

Definition 3.2.2. *Let (A,μ,η) be a monoid and let M, M' be two left A–modules. A morphism $f : M \to M'$ is said to be a morphism of left A–modules if if the following diagram commutes*

$$\begin{array}{ccc} A\otimes M & \xrightarrow{\mu_M} & M \\ \downarrow{\scriptstyle \mathsf{id}_A\otimes f} & & \downarrow{\scriptstyle f} \\ A\otimes M' & \xrightarrow{\mu_{M'}} & M' \end{array}$$

We denote the category of left A–modules by $\mathsf{Mod}(A)$.

Remark 3.2.3. *If $\mathscr{C} = \mathsf{Vect}_{\mathbb{K}}$ and A is an algebra object of $\mathscr{C}$ then $\mathsf{Mod}(A)$ is the usual category of left A–modules.*

With the same reasoning of Proposition 3.1.4 one can show the following

Proposition 3.2.4. *Let $\mathscr{C}, \mathscr{C}'$ be two monoidal categories, $(F, \varphi_0^F, \varphi_2^F)$ be a monoidal functor from $\mathscr{C}$ to $\mathscr{C}'$, (A,μ,η) be a monoid in $\mathscr{C}$ and (M,μ_M) be in $\mathsf{Mod}(A)$. Then $\big(F(M), F(\mu_M)\circ\varphi_2^F(A,M)\big)$ is in $\mathsf{Mod}(F(A))$.*

We have also the following

Definition 3.2.5. *Let (A,μ,η) be in $\mathsf{Mon}(\mathscr{C})$.*

- *We say that a pair (R,μ_R) is a* ***right*** *A–**module** if R is in $\mathsf{Obj}(\mathscr{C})$, $\mu_R : R\otimes A \to R$ is a morphism such that the pentagon*

$$\begin{array}{ccc} R\otimes(A\otimes A) & \xrightarrow{\mathsf{id}_R\otimes\mu} & R\otimes A \\ \downarrow{\scriptstyle a^{-1}_{R,A,A}} & & \\ (R\otimes A)\otimes A & & \downarrow{\scriptstyle \mu_R} \\ \downarrow{\scriptstyle \mu_R\otimes\mathsf{id}_A} & & \\ R\otimes A & \xrightarrow{\mu_R} & R \end{array}$$

and the triangle

$$\begin{array}{ccc} R \otimes I & \xrightarrow{\mathsf{id}_R \otimes \eta} & R \otimes A \\ & {\scriptstyle r_R} \searrow \quad \swarrow {\scriptstyle \mu_R} & \\ & R & \end{array}$$

commute.

- *Let A' be another monoid, and M be in $\mathsf{Obj}(\mathscr{C})$ having both a left A–module structure (M, μ_L) and a right A'–module structure (M, μ_R). Then we say that (M, μ_R, μ_L) is a A–A'–**bimodule** if the diagram*

$$\begin{array}{ccc} (A \otimes M) \otimes A' & \xrightarrow{a_{A,M,A'}} & A \otimes (M \otimes A') \\ & & \downarrow {\scriptstyle \mathsf{id}_A \otimes \mu_R} \\ {\scriptstyle \mu_L \otimes \mathsf{id}_{A'}} \downarrow & & A \otimes M \\ & & \downarrow {\scriptstyle \mu_L} \\ M \otimes A' & \xrightarrow{\mu_R} & M \end{array}$$

commutes.

3.3 Comonoids

Definition 3.3.1. *A **comonoid** is a triple (C, Δ, ε), where C is an object of $\mathscr{C}$ and $\Delta : C \to C \otimes C$, $\varepsilon : C \to \mathbb{K}$ are morphisms (called the comultiplication and the counit) such that*

(i) (coassociativity axiom): the square

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ & & \downarrow {\scriptstyle \Delta \otimes \mathsf{id}} \\ {\scriptstyle \Delta} \downarrow & & (C \otimes C) \otimes C \\ & & \downarrow {\scriptstyle a_{C,C,C}} \\ C \otimes C & \xrightarrow{\mathsf{id} \otimes \Delta} & C \otimes (C \otimes C) \end{array} \tag{3.3.1}$$

commutes;

(ii) (counit axiom): the diagram

$$\begin{array}{ccccc} I \otimes C & \xleftarrow{\varepsilon \otimes \mathsf{id}} & C \otimes C & \xrightarrow{\mathsf{id} \otimes \varepsilon} & C \otimes I \\ & {\scriptstyle \ell_C^{-1}} \nwarrow & \uparrow {\scriptstyle \Delta} & \nearrow {\scriptstyle r_C^{-1}} & \\ & & C & & \end{array} \tag{3.3.2}$$

commutes.

If moreover $\mathscr{C}$ is braided monoidal and the triangle

$$\begin{array}{ccc} C\otimes C & \xrightarrow{\quad c_{C,C}\quad} & C\otimes C \\ & {\scriptstyle\Delta}\nwarrow \qquad \nearrow{\scriptstyle\Delta} & \\ & C & \end{array} \tag{3.3.3}$$

commutes, we say that (C,Δ,ε) is a cocommutative comonoid. If moreover $\mathscr{C}$ is infinitesimally braided monoidal and the triangle

$$\begin{array}{ccc} C\otimes C & \xrightarrow{\quad t_{C,C}\quad} & C\otimes C \\ & {\scriptstyle\Delta}\nwarrow \qquad \nearrow{\scriptstyle\Delta} & \\ & C & \end{array} \tag{3.3.4}$$

commutes we say that (C,Δ,ε) is an infinitesimally braided cocommutative comonoid.

Definition 3.3.2. *Let (C,Δ,ε) and $(C',\Delta',\varepsilon')$ be two comonoids. A morphism $f: C\to C'$ is said to be a* ***morphism of comonoids*** *if the diagrams*

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\otimes f} \\ C' & \xrightarrow{\Delta'} & C'\otimes C' \end{array} \qquad\qquad \begin{array}{ccc} C & \xrightarrow{\quad\varepsilon\quad} & I \\ & {\scriptstyle f}\searrow \qquad \nearrow{\scriptstyle\varepsilon'} & \\ & C' & \end{array} \tag{3.3.5}$$

commute.

We denote the category of all comonoids of $\mathscr{C}$ by $\mathsf{Comon}(\mathscr{C})$.

Proposition 3.3.3. *Suppose that $\mathscr{C}$ is braided. Then*

(i) If $(C_1,\Delta_1,\varepsilon_1)$ and $(C_2,\Delta_2,\varepsilon_2)$ are two comonoids, so is $\big(C_1\otimes C_2, \beta_{C_1,C_1,C_2,C_2}\circ\Delta_1\otimes\Delta_2, \varepsilon_1\otimes\varepsilon_2\big)$.

(ii) If $(C_1,\Delta_1,\varepsilon_1)$, $(C_2,\Delta_2,\varepsilon_2)$, $(C_3,\Delta_3,\varepsilon_3)$ and $(C_4,\Delta_4,\varepsilon_4)$ are four comonoids, and $f: C_1\to C_2$, $g: C_3\to C_4$ are morphisms of comonoids, then so is $(f\otimes g): C_1\otimes C_3\to C_2\otimes C_4$.

(iii) If (C,Δ,ε) is a cocommutative comonoid, then Δ is a morphism of comonoids.

Note that the first two statements gives that $\mathsf{Comon}(\mathscr{C})$ is a monoidal category.

Proof. In order to give the proof we may suppose that $\mathscr{C}$ is strict.
(i): The fact that the morphism $(\beta_{C_1,C_1,C_2,C_2})\circ\Delta_1\otimes\Delta_2$ is coassociative follows from the following commutative diagram

$$\begin{array}{ccccc} C_1\otimes C_2 & \xrightarrow{\Delta_1\otimes\Delta_2} & C_1^{\otimes 2}\otimes C_2^{\otimes 2} & \xrightarrow{\beta_{C_1,C_1,C_2,C_2}} & C_1\otimes C_2\otimes C_1\otimes C_2 \\ {\scriptstyle\Delta_1\otimes\Delta_2}\downarrow & & \downarrow{\scriptstyle\Delta_1\otimes\mathsf{id}_{C_1}\otimes\Delta_2\otimes\mathsf{id}_{C_2}} & & \downarrow{\scriptstyle\Delta_1\otimes\Delta_2\otimes\mathsf{id}_{C_1\otimes C_2}} \\ C_1^{\otimes 2}\otimes C_2^{\otimes 2} & \xrightarrow{\mathsf{id}_{C_1}\otimes\Delta_1\otimes\mathsf{id}_{C_2}\otimes\Delta_2} & C_1^{\otimes 3}\otimes C_2^{\otimes 3} & \xrightarrow{\mathsf{id}_{C_1}\otimes\beta_{C_1,C_1,C_2\otimes C_2,C_2}} & C_1^{\otimes 2}\otimes C_2^{\otimes 2}\otimes C_1\otimes C_2 \\ {\scriptstyle\beta_{C_1,C_1,C_2,C_2}}\downarrow & & \downarrow{\scriptstyle\beta_{C_1,C_1\otimes C_1,C_2,C_2\otimes C_2}} & & \downarrow{\scriptstyle\beta_{C_1,C_1,C_2,C_2}\otimes\mathsf{id}_{C_1\otimes C_2}} \\ C_1\otimes C_2\otimes C_1\otimes C_2 & \xrightarrow{\mathsf{id}_{C_1\otimes C_2}\otimes\Delta_1\otimes\Delta_2} & C_1\otimes C_2\otimes C_1^{\otimes 2}\otimes C_2^{\otimes 2} & \xrightarrow{\mathsf{id}_{C_1\otimes C_2}\otimes\beta_{C_1,C_1,C_2,C_2}} & (C_1\otimes C_2)^{\otimes 3} \end{array}$$

where the top left square follows from the coassociativity of Δ_1 and Δ_2, the top right follows from Equation (2.5.1a), the bottom left follows from the naturality of the braiding c and the bottom right follows from Equation (2.5.3). Finally, the counity of $\varepsilon_1 \otimes \varepsilon_2$ is given by the commutativity of the diagram

$$\begin{array}{ccccc}
C_1 \otimes C_2 & \xleftarrow{\varepsilon_1 \otimes \varepsilon_2 \otimes \mathsf{id}_{C_1 \otimes C_2}} & (C_1 \otimes C_2)^{\otimes 2} & \xrightarrow{\mathsf{id}_{C_1 \otimes C_2} \otimes \varepsilon_1 \otimes \varepsilon_2} & C_1 \otimes C_2 \\
\uparrow & & \uparrow{\scriptstyle \beta_{C_1,C_1,C_2,C_2}} & & \uparrow \\
C_1 \otimes C_2 & \xleftarrow{\varepsilon_1 \otimes \mathsf{id}_{C_1} \otimes \varepsilon_2 \otimes \mathsf{id}_{C_2}} & C_1^{\otimes 2} \otimes C_2^{\otimes 2} & \xrightarrow{\mathsf{id}_{C_1} \otimes \varepsilon_1 \otimes \mathsf{id}_{C_2} \otimes \varepsilon_2} & C_1 \otimes C_2 \\
& \nwarrow{\scriptstyle \mathsf{id}} & \uparrow{\scriptstyle \Delta_1 \otimes \Delta_2} & \nearrow{\scriptstyle \mathsf{id}} & \\
& & C_1 \otimes C_2 & &
\end{array}$$

which follows from the naturality of the braiding c and from the counity of ε_1 and ε_2.
(ii) Following statement (i), we have that $C_1 \otimes C_3$ and $C_2 \otimes C_4$ have the following comonoid structure:

$$\big(C_1 \otimes C_3, \beta_{C_1,C_1,C_3,C_3} \circ \Delta_1 \otimes \Delta_3, \varepsilon_1 \otimes \varepsilon_3\big)$$
$$\big(C_2 \otimes C_4, \beta_{C_2,C_2,C_4,C_4} \circ \Delta_2 \otimes \Delta_4, \varepsilon_2 \otimes \varepsilon_4\big).$$

The fact that $f \otimes g$ is compatible with the comultiplications is given by the following diagram

$$\begin{array}{ccccc}
C_1 \otimes C_3 & \xrightarrow{\Delta_1 \otimes \Delta_3} & C_1 \otimes C_1 \otimes C_3 \otimes C_3 & \xrightarrow{\beta_{C_1,C_1,C_3,C_3}} & C_1 \otimes C_3 \otimes C_1 \otimes C_3 \\
{\scriptstyle f \otimes g}\downarrow & & \downarrow{\scriptstyle (f \otimes f) \otimes (g \otimes g)} & & \downarrow{\scriptstyle (f \otimes g) \otimes (f \otimes g)} \\
C_2 \otimes C_4 & \xrightarrow{\Delta_2 \otimes \Delta_4} & C_2 \otimes C_2 \otimes C_4 \otimes C_4 & \xrightarrow{\beta_{C_2,C_2,C_4,C_4}} & C_2 \otimes C_4 \otimes C_2 \otimes C_4
\end{array}$$

which follows from the fact that f and g are morphism of comonoids and from the naturality of the braiding. Finally, the fact that $f \otimes g$ is compatible with the counits follows by – again using that f and g are morphism of comonoids – the diagram

$$\begin{array}{ccc}
C_1 \otimes C_3 & \xrightarrow{\varepsilon_1 \otimes \varepsilon_3} & I \otimes I \\
{\scriptstyle f \otimes g}\downarrow & & \Vert \\
C_2 \otimes C_4 & \xrightarrow{\varepsilon_2 \otimes \varepsilon_4} & I \otimes I
\end{array}$$

(iii) Using the coassociativity and cocommutativity of C we have

$$\begin{aligned}
(\beta_{C,C,C,C}) \circ (\Delta \otimes \Delta) \circ \Delta &= (\beta_{C,C,C,C}) \circ (\mathsf{id}_{C \otimes C} \otimes \Delta) \circ (\Delta \otimes \mathsf{id}_C) \circ \Delta \\
&= (\beta_{C,C,C,C}) \circ (\mathsf{id}_{C \otimes C} \otimes \Delta) \circ (\mathsf{id}_C \otimes \Delta) \circ \Delta \\
&= (\beta_{C,C,C,C}) \circ (\mathsf{id}_C \otimes \Delta \otimes \mathsf{id}_C) \circ (\mathsf{id}_C \otimes \Delta) \circ \Delta \\
&= (\mathsf{id}_C \otimes \Delta \otimes \mathsf{id}_C) \circ (\mathsf{id}_C \otimes \Delta) \circ \Delta \\
&= (\mathsf{id}_{C \otimes C} \otimes \Delta) \circ (\mathsf{id}_C \otimes \Delta) \circ \Delta \\
&= (\mathsf{id}_{C \otimes C} \otimes \Delta) \circ (\Delta \otimes \mathsf{id}_C) \circ \Delta \\
&= (\Delta \otimes \Delta) \circ \Delta.
\end{aligned}$$

□

The next Proposition collects all the properties relating comonoids and comonoidal functors.

Proposition 3.3.4. *Let $(\mathscr{C}, \otimes, I, a, \ell, r)$ and $(\mathscr{C}', \otimes', I', a', \ell', r')$ be two monoidal categories, (F, ψ_F^0, ψ_F^2) be a comonoidal functor and (C, Δ, ε), $(C_1, \Delta_1, \varepsilon_1)$, $(C_2, \Delta_2, \varepsilon_2)$ be comonoids in $\mathscr{C}$. Then*

(i) The triple $\big(F(C), \psi_F^2(C, C) \circ F(\Delta), \psi_F^0 \circ F(\varepsilon)\big)$ is a comonoid in $\mathscr{C}'$.

(ii) If $f : C_1 \to C_2$ is a morphism of comonoids, then so is $F(f) : F(C_1) \to F(C_2)$.

(iii) If $\mathscr{C}$ and $\mathscr{C}'$ are braided monoidal categories and (F, ψ_F^0, ψ_F^2) is a braided comonoidal functor, then

$$\psi_F^2(C_1, C_2) : F(C_1 \otimes C_2) \to F(C_1) \otimes' F(C_2)$$

is a morphism of comonoids.

Proof. (i): Using the coassociativity of Δ, the naturality of ψ_F^2 and the fact that F is comonoidal we obtain the following commutative diagram

$$\begin{array}{ccccccc}
F(C) & \xrightarrow{F(\Delta)} & & & F(C\otimes C) & \xrightarrow{\psi_F^2(C,C)} & F(C)\otimes' F(C) \\
\downarrow{\scriptstyle F(\Delta)} & & & & \downarrow{\scriptstyle F(\Delta\otimes\mathsf{id})} & & \downarrow{\scriptstyle F(\Delta)\otimes'\mathsf{id}_{F(C)}} \\
F(C\otimes C) & \xrightarrow{F(\mathsf{id}\otimes\Delta)} & F(C\otimes(C\otimes C)) & \xleftarrow{F(a_{C,C,C})} & F((C\otimes C)\otimes C) & \xrightarrow{\psi_F^2(C\otimes C,C)} & F(C\otimes C)\otimes' F(C) \\
 & & & & & & \downarrow{\scriptstyle \psi_F^2(C,C)\otimes'\mathsf{id}_{F(C)}} \\
\downarrow{\scriptstyle \psi_F^2(C,C)} & & \downarrow{\scriptstyle \psi_F^2(C,C\otimes C)} & & & & (F(C)\otimes F(C))\otimes' F(C) \\
 & & & & & & \downarrow{\scriptstyle a'_{F(C),F(C),F(C)}} \\
F(C)\otimes' F(C) & \xrightarrow{\mathsf{id}\otimes' F(\Delta)} & F(C)\otimes' F(C\otimes C) & & \xrightarrow{\mathsf{id}_{F(C)}\otimes'\psi_F^2(C,C)} & & F(C)\otimes'(F(C)\otimes' F(C))
\end{array}$$

giving the coassociativity of $\psi_F^2(C, C) \circ F(\Delta)$. Similarly, using the counity of ε, the naturality of ψ_F^2 and the fact that F is comonoidal we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
I'\otimes F(C) & \xleftarrow{\psi_F^0\otimes\mathsf{id}} & F(I)\otimes F(C) & \xleftarrow{F(\varepsilon)\otimes\mathsf{id}} & F(C)\otimes F(C) & \xrightarrow{\mathsf{id}\otimes F(\varepsilon)} & F(C)\otimes F(I) & \xrightarrow{\mathsf{id}\otimes\psi_F^0} & F(C)\otimes I' \\
 & & \uparrow{\scriptstyle \psi_F^2(I,C)} & & \uparrow{\scriptstyle \psi_F^2(C,C)} & & \uparrow{\scriptstyle \psi_F^2(C,I)} & & \\
 & & F(I\otimes C) & \xleftarrow{F(\varepsilon\otimes\mathsf{id}_C)} & F(C\otimes C) & \xrightarrow{F(\mathsf{id}_C\otimes\varepsilon)} & F(C\otimes I) & & \\
 & & & \nwarrow{\scriptstyle F(\ell_C^{-1})} & \uparrow{\scriptstyle F(\Delta)} & \nearrow{\scriptstyle F(r_C^{-1})} & & & \\
 & \ell^{-1}_{F(C)} & & & F(C) & & & r^{-1}_{F(C)} &
\end{array}$$

giving the counity of $\psi_0^F \circ F(\varepsilon)$.

(*ii*): In order to give the proof we may suppose that $\mathscr{C}$ and $\mathscr{C}'$ are strict. The fact that $F(f)$ is compatible with the comultiplications is given by the following diagram

$$\begin{array}{ccccc}
F(C_1) & \xrightarrow{F(\Delta_1)} & F(C_1\otimes C_1) & \xrightarrow{\psi_F^2(C_1,C_1)} & F(C_1)\otimes' F(C_1) \\
\downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle F(f\otimes f)} & & \downarrow{\scriptstyle F(f)\otimes' F(f)} \\
F(C_2) & \xrightarrow{F(\Delta_2)} & F(C_2\otimes C_2) & \xrightarrow{\psi_F^2(C_2,C_2)} & F(C_2)\otimes' F(C_2)
\end{array}$$

which follows from the fact that f is a morphism of comonoids and from the naturality of ψ_F^2. Next, the fact that $F(f)$ is compatible with counits is given by the following diagram

$$\begin{array}{ccccc}
F(C_1) & \xrightarrow{F(\varepsilon_1)} & F(I) & \xrightarrow{\psi_F^0} & I' \\
\downarrow{\scriptstyle F(f)} & & \| & & \| \\
F(C_2) & \xrightarrow{F(\varepsilon_2)} & F(I) & \xrightarrow{\psi_F^0} & I'
\end{array}$$

(iii): In order to give the proof we may suppose that $\mathscr{C}$ and $\mathscr{C}'$ are strict. Recall that, from statement (i) of this Proposition and from statement (i) of Proposition 3.3.3 we have that $F(C_1 \otimes C_2)$ is a comonoid with comultiplication given by

$$\psi_F^2(C_1 \otimes C_2, C_1 \otimes C_2) \circ F(\beta_{C_1,C_1,C_2,C_2}) \circ F(\Delta_1 \otimes \Delta_2),$$

while $F(C_1) \otimes' F(C_2)$ is a comonoid with comultiplication given by

$$\beta'_{F(C_1),F(C_1),F(C_2),F(C_2)} \circ \big(\psi_F^2(C_1, C_1) \otimes' \psi_F^2(C_2, C_2)\big) \circ \big(F(\Delta_1) \otimes' F(\Delta_2)\big).$$

In order to obtain the commutativity of the square (3.3.5), consider the following eight diagrams

(1)

$$\begin{array}{ccc}
F(C_1 \otimes C_2) & \xrightarrow{F(\Delta_1 \otimes \Delta_2)} & F(C_1^{\otimes 2} \otimes C_2^{\otimes 2}) \\
{\scriptstyle \psi_F^2(C_1,C_2)}\big\downarrow & & \big\downarrow{\scriptstyle \psi_F^2(C_1^{\otimes 2},C_2^{\otimes 2})} \\
F(C_1) \otimes' F(C_2) & \xrightarrow{F(\Delta_1)\otimes' F(\Delta_2)} & F(C_1^{\otimes 2}) \otimes' F(C_2^{\otimes 2})
\end{array}$$

(2)

$$\begin{array}{ccc}
F(C_1^{\otimes 2} \otimes C_2^{\otimes 2}) & \xrightarrow{\psi_F^2(C_1,C_1 \otimes C_2^{\otimes 2})} & F(C_1) \otimes' F(C_1 \otimes C_2^{\otimes 2}) \\
{\scriptstyle \psi_F^2(C_1^{\otimes 2},C_2^{\otimes 2})}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{id}_{F(C_1)}\otimes' \psi_F^2(C_1,C_2^{\otimes 2})} \\
F(C_1^{\otimes 2}) \otimes' F(C_2^{\otimes 2}) & \xrightarrow{\psi_F^2(C_1,C_1)\otimes' F_{C_2^{\otimes 2}}} & F(C_1)^{\otimes' 2} \otimes' F(C_2^{\otimes 2})
\end{array}$$

(3)

$$\begin{array}{ccc}
F(C_1) \otimes' F(C_1 \otimes C_2^{\otimes 2}) & \xrightarrow{\mathsf{id}_{F(C_1)}\otimes' \psi_F^2(C_1 \otimes C_2, C_2)} & F(C_1) \otimes' F(C_1 \otimes C_1) \otimes' F(C_2) \\
{\scriptstyle \mathsf{id}_{F(C_1)}\otimes' \psi_F^2(C_1,C_2^{\otimes 2})}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{id}_{F(C_1)}\otimes' \psi_F^2(C_1,C_2)\otimes' \mathsf{id}_{F(C_2)}} \\
F(C_1)^{\otimes' 2} \otimes' F(C_2^{\otimes 2}) & \xrightarrow{\mathsf{id}_{F(C_1)^{\otimes' 2}}\otimes' \psi_F^2(C_2,C_2)} & F(C_1)^{\otimes' 2} \otimes' F(C_2)^{\otimes' 2}
\end{array}$$

(4)

$$\begin{array}{ccc}
F(C_1^{\otimes 2} \otimes C_2^{\otimes 2}) & \xrightarrow{F(\mathsf{id}_{C_1}\otimes c_{C_1,C_2}\otimes \mathsf{id}_{C_2})} & F((C_1 \otimes C_2)^{\otimes 2}) \\
{\scriptstyle \psi_F^2(C_1,C_1 \otimes C_2^{\otimes 2})}\big\downarrow & & \big\downarrow{\scriptstyle \psi_F^2(C_1,C_2 \otimes C_1 \otimes C_2)} \\
F(C_1) \otimes' F(C_1 \otimes C_2^{\otimes 2}) & \xrightarrow{\mathsf{id}_{F(C_1)}\otimes' F(c_{C_1,C_2}\otimes \mathsf{id}_{C_2})} & F(C_1) \otimes' F(C_2 \otimes C_1 \otimes C_2)
\end{array}$$

(5)

$$\begin{array}{ccc}
F(C_1) \otimes' F(C_1 \otimes C_2^{\otimes 2}) & \xrightarrow{\mathsf{id}_{F(C_1)}\otimes' F(c_{C_1,C_2}\otimes \mathsf{id}_{C_2})} & F(C_1) \otimes' F(C_2 \otimes C_1 \otimes C_1) \\
{\scriptstyle \mathsf{id}_{F(C_1)}\otimes' \psi_F^2(C_1 \otimes C_2,C_2)}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{id}_{F(C_1)}\otimes' \psi_F^2(C_2 \otimes C_1,C_2)} \\
F(C_1) \otimes' F(C_1 \otimes C_2) \otimes' F(C_2) & \xrightarrow{\mathsf{id}_{F(C_1)}\otimes' F(c_{C_1,C_2})\otimes' \mathsf{id}_{F(C_2)}} & F(C_1) \otimes' F(C_2 \otimes C_1) \otimes' F(C_2)
\end{array}$$

(6)

$$\begin{array}{ccc}
F(C_1)\otimes' F(C_1\otimes C_2)\otimes' F(C_2) & \xrightarrow{\mathsf{id}_{F(C_1)}\otimes' F(c_{C_1,C_2})\otimes'\mathsf{id}_{F(C_2)}} & F(C_1)\otimes' F(C_2\otimes C_1)\otimes' F(C_2) \\
\downarrow{\scriptstyle \mathsf{id}_{F(C_1)}\otimes'\psi_F^2(C_1,C_2)\otimes'\mathsf{id}_{F(C_2)}} & & \downarrow{\scriptstyle \mathsf{id}_{F(C_1)}\otimes'\psi_F^2(C_2,C_1)\otimes'\mathsf{id}_{F(C_2)}} \\
F(C_1)^{\otimes' 2}\otimes' F(C_2)^{\otimes' 2} & \xrightarrow{\mathsf{id}_{F(C_1)}\otimes' c_{F(C_1),F(C_2)}\otimes'\mathsf{id}_{F(C_2)}} & F(C_1)\otimes' F(C_2)\otimes' F(C_1)\otimes' F(C_2)
\end{array}$$

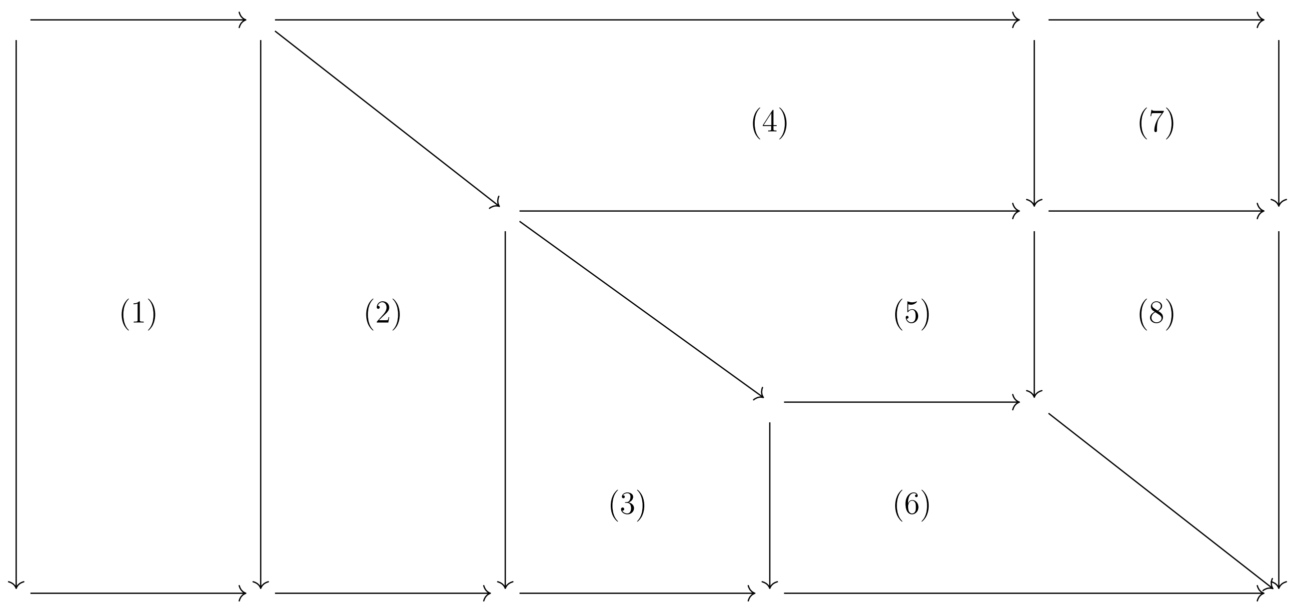
(7)

$$\begin{array}{ccc}
F((C_1\otimes C_2)^{\otimes 2}) & \xrightarrow{\psi_F^2(C_1\otimes C_2,C_1\otimes C_2)} & F(C_1\otimes C_2)^{\otimes' 2} \\
\downarrow{\scriptstyle \psi_F^2(C_1,C_2\otimes C_1\otimes C_2)} & & \downarrow{\scriptstyle \psi_F^2(C_1,C_2)\otimes'\mathsf{id}_{F(C_1\otimes C_2)}} \\
F(C_1)\otimes' F(C_2\otimes C_1\otimes C_2) & \xrightarrow{\mathsf{id}_{F(C_1)}\otimes'\psi_F^2(C_2,C_1\otimes C_2)} & F(C_1)\otimes' F(C_2)\otimes' F(C_1\otimes C_2)
\end{array}$$

(8)

$$\begin{array}{ccc}
F(C_1)\otimes' F(C_2\otimes C_1\otimes C_2) & \xrightarrow{\mathsf{id}_{F(C_1)}\otimes'\psi_F^2(C_2,C_1\otimes C_2)} & F(C_1)\otimes' F(C_2)\otimes' F(C_1\otimes C_2) \\
\downarrow{\scriptstyle \mathsf{id}_{F(C_1)}\otimes'\psi_F^2(C_2\otimes C_1,C_2)} & & \downarrow{\scriptstyle \mathsf{id}_{F(C_1)\otimes' F(C_2)}\otimes'\psi_F^2(C_1,C_2)} \\
F(C_1)\otimes' F(C_2\otimes C_1)\otimes' F(C_2) & \xrightarrow{\mathsf{id}_{F(C_1)}\otimes'\psi_F^2(C_2,C_1)\otimes'\mathsf{id}_{F(C_2)}} & F(C_1)\otimes' F(C_2)\otimes' F(C_1)\otimes' F(C_2)
\end{array}$$

all of them commuting because (F,ψ_F^0,ψ_F^2) is comonoidal. Joining them we obtain



proving that $\psi_F^2(C_1,C_2)$ satisfies the square condition (3.3.5). Next, recall from part (i) and Proposition 3.3.3 that the counit of $F(C_1)\otimes' F(C_2)$ is $(\psi_F^0\otimes'\psi_F^0)\circ(F(\varepsilon_1)\otimes' F(\varepsilon_2))$, while the counit of $F(C_1\otimes C_2)$ is $\psi_F^0\circ F(\varepsilon_1\otimes\varepsilon_2)$. The fact that the triangle (3.3.5) commutes follows by the commutativity of the following diagram

$$\begin{array}{ccccc}
F(C_1\otimes C_2) & \xrightarrow{F(\varepsilon_1\otimes\varepsilon_2)} & F(I\otimes I)=F(I) & \xrightarrow{\psi_0} & I' \\
\downarrow{\scriptstyle \psi_2(C_1,C_2)} & & \downarrow{\scriptstyle \psi_2(I,I)} & & \| \\
F(C_1)\otimes F(C_2) & \xrightarrow{F(\varepsilon_1)\otimes' F(\varepsilon_2)} & F(I)\otimes' F(I) & \xrightarrow{\psi_0\otimes\psi_0} & I'\otimes I'=I'
\end{array}$$

where we used again that (F, ψ_F^0, ψ_F^2) is comonoidal. $\square$

Remark 3.3.5. *If $\mathscr{C} = \mathsf{Vect}_{\mathbb{K}}$, then $\mathsf{Comon}(\mathscr{C})$ is the usual braided monoidal category of counital coassociative coalgebras.*

3.4 Comodules

Definition 3.4.1. *Let (C, Δ, ε) be a comonoid. A* ***right*** *C–**comodule** is a pair (N, Δ_N), where N is an object and $\Delta_N : C \to N \otimes C$ is a morphism, called the coaction of C on N, such that the square*

$$\begin{array}{ccc} N & \xrightarrow{\Delta_N} & N \otimes C \\ & & \big\downarrow{\scriptstyle \Delta_N \otimes \mathsf{id}_C} \\ \big\downarrow{\scriptstyle \Delta_N} & & (N \otimes C) \otimes C \\ & & \big\downarrow{\scriptstyle a_{N,C,C}} \\ N \otimes C & \xrightarrow{\mathsf{id}_N \otimes \Delta} & N \otimes (C \otimes C) \end{array} \tag{3.4.1}$$

and the triangle

$$\begin{array}{ccc} N \otimes C & \xrightarrow{\mathsf{id}_N \otimes \varepsilon} & I \otimes N \\ {\scriptstyle \Delta_N}\nwarrow & & \nearrow{\scriptstyle \ell_N^{-1}} \\ & N & \end{array} \tag{3.4.2}$$

commute.

Definition 3.4.2. *Let (C, Δ, ε) be a comonoid and let N, N' be two right C–comodules. A morphism $f : N \to N'$ is said to be a morphism of right C–comodules if if the following diagram commutes*

$$\begin{array}{ccc} N & \xrightarrow{f} & N' \\ {\scriptstyle \Delta_N}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_{N'}} \\ N \otimes C & \xrightarrow{f \otimes \mathsf{id}_C} & N' \otimes C \end{array}$$

We denote the category of right C–comodules by $\mathsf{Comod}(C)$.

Remark 3.4.3. *If $\mathscr{C} = \mathsf{Vect}_{\mathbb{K}}$ and C is a comonoid object of $\mathscr{C}$ then $\mathsf{Comod}(A)$ is the usual category of right C–comodules.*

With the same reasoning of Proposition 3.3.4 one can show the following

Proposition 3.4.4. *Let $\mathscr{C}, \mathscr{C}'$ be two monoidal categories, (F, ψ_F^0, ψ_F^2) be a comonoidal functor from $\mathscr{C}$ to $\mathscr{C}'$, (C, Δ, ε) be a comonoid in $\mathscr{C}$ and (N, Δ_N) be in $\mathsf{Comod}(C)$. Then the triple $\big(F(N), \psi_F^2(A, M) \circ F(\Delta_N)\big)$ is in $\mathsf{Comod}(F(C))$.*

3.5 Bimonoids and Hopf monoids

Let $\mathscr{C}$ be a braided monoidal category and H be in $\mathsf{Obj}(\mathscr{C})$ having both a monoid structure (H, μ, η) and a comonoid structure (H, Δ, ε). The following result is standard (a proof for $\mathscr{C} = \mathsf{Vect}_{\mathbb{K}}$ can be find in [Kas12, Th. III.2.2], and the generalization to any $\mathscr{C}$ is straightforward):

Proposition 3.5.1. *The following statements are equivalent*

(i) μ *and* η *are morphisms of comonoids.*

(ii) Δ *and* ε *are morphisms of monoids.*

Definition 3.5.2. *A* ***bimonoid*** *is a quintuple* $(H, \mu, \eta, \Delta, \varepsilon)$, *where* (H, μ, η) *is a monoid,* (H, Δ, ε) *is a comonoid and* μ, η *are morphisms of comonoids (equivalently,* Δ, ε *are morphisms of monoids). A morphism of bimonoids* $f : H \to H'$ *is a morphism which is both a morphism of monoids and of comonoids.*

Definition 3.5.3. *A* ***Hopf monoid*** *is a sextuple* $(H, \mu, \eta, \Delta, \varepsilon, S)$, *where the quintuple* $(H, \mu, \eta, \Delta, \varepsilon)$ *is a bimonoid and* $S : H \to H$ *is a morphism, called the* ***antipode***, *such that the following diagram commutes*

$$\begin{array}{ccccccc} & & H \otimes H & \xrightarrow{S \otimes \mathrm{id}_H} & H \otimes H & & \\ & \overset{\Delta}{\nearrow} & & & & \overset{\mu}{\searrow} & \\ H & \xrightarrow{\varepsilon} & I & & \xrightarrow{\quad\eta\quad} & & H \\ & \underset{\Delta}{\searrow} & & & & \underset{\mu}{\nearrow} & \\ & & H \otimes H & \xrightarrow{\mathrm{id}_H \otimes S} & H \otimes H & & \end{array} \tag{3.5.1}$$

A morphism of Hopf monoids $f : H \to H'$ *is a morphism of bimonoids such that* $f \circ S = S' \circ f$.

If $\mathscr{C}$ is braided, then the subcategory of $\mathscr{C}$ of all bimonoids (resp. of all Hopf monoids) is a monoidal category, that we denote by $\mathsf{Bimon}(\mathscr{C})$ (resp. $\mathsf{Hopf}(\mathscr{C})$). If H is in $\mathsf{Hopf}(\mathscr{C})$, then it is well–known that $\mathsf{Mod}(H)$ is a monoidal category. In particular, if (X, μ_X) and (Y, μ_Y) are in $\mathsf{Mod}(H)$, then $(X \otimes Y, \mu_{X\otimes Y})$ is in $\mathsf{Mod}(H)$, where

$$\mu_{X\otimes Y} \coloneqq (\mu_X \otimes \mu_Y) \circ \beta_{H,H,X,Y} \circ (\Delta \otimes \mathrm{id}_{X\otimes Y}) \tag{3.5.2}$$

where β is the morphisms interchanging the second and the third factor, opportunely composed with associators, see 2.5.5.

Remark 3.5.4. *If* $\mathscr{C} = \mathsf{Vect}_{\mathbb{K}}$, *then* $\mathsf{Bimon}(\mathscr{C})$ *(resp.* $\mathsf{Hopf}(\mathscr{C})$*) is the usual braided monoidal category of bialgebras (resp. Hopf algebras).*

3.6 Left–right Yetter–Drinfeld H–modules

The following Definition is given in [Kas12, IX.5]

Definition 3.6.1. *Let* $\mathscr{C}$ *be a braided monoidal category,* $(H, \mu, \eta, \Delta, \varepsilon, S)$ *be a Hopf monoid in* $\mathscr{C}$ *and* X *be a object of* $\mathscr{C}$ *having both a left* H*–module structure* (X, μ_X) *and a right* H*–comodule structure* (X, Δ_X). *We say that the triple* (X, μ_X, Δ_X) *is a* ***left–right Yetter–Drinfeld*** H*–*

module[1] *if the following diagram commutes*

$$
\begin{array}{ccccc}
H \otimes X & \xrightarrow{\Delta \otimes \mathsf{id}_X} & (H \otimes H) \otimes X & \xrightarrow{a_{H,H,X}} & H \otimes (H \otimes X) \\
\downarrow{\scriptstyle \Delta \otimes \Delta_X} & & & & \downarrow{\scriptstyle \mathsf{id}_H \otimes \mu_X} \\
(H \otimes H) \otimes (X \otimes H) & & & & H \otimes X \\
\downarrow{\scriptstyle \beta_{H,H,X,H}} & & & & \downarrow{\scriptstyle c_{H,X}} \\
(H \otimes X) \otimes (H \otimes H) & & & & X \otimes H \\
\downarrow{\scriptstyle \mu_X \otimes \mu} & & & & \downarrow{\scriptstyle \Delta_X \otimes \mathsf{id}_H} \\
X \otimes H & \xleftarrow{\mathsf{id}_X \otimes \mu} & X \otimes (H \otimes H) & \xleftarrow{a_{X,H,H}} & (X \otimes H) \otimes H
\end{array}
\tag{3.6.1}
$$

where β is the natural isomorphism swapping the second and third tensors, composed opportunetely with associators, see 2.5.5.
A morphism of left–right Yetter–Drinfeld H–modules is a morphism $f : X \to Y$ which is both a morphism of left H–modules and of right H–comodules.

We shall denote the category of all left–right Yetter–Drinfeld H–modules by $\mathscr{Y}\mathscr{D}(H)$.

Remark 3.6.2. *If $\mathscr{C} = \mathsf{Vect}_{\mathbb{K}}$ and H is a Hopf algebra, then any object of $\mathscr{Y}\mathscr{D}(H)$ is a left–right Yetter–Drinfeld module in the usual sense, see* [Rad93, Def.2].

Let H be a Hopf monoid and consider the monoidal category $\mathsf{Mod}(H)$ and its strictification $\mathsf{Mod}(H)^{\mathsf{str}}$. Then we can consider the strict braided monoidal category $\mathscr{Z}(\mathsf{Mod}(H))$, i.e. its Drinfeld center. We have the following very important

Theorem 3.6.3. *(*[Kas12, XIII.5.2]*) Let $(X, c_{-,X})$ be in $\mathscr{Z}(\mathsf{Mod}(H))$ and consider the morphism*

$$
\begin{aligned}
\Delta_X : X &\to X \otimes H \\
x &\mapsto c_{H,V}(1 \otimes x).
\end{aligned}
$$

Then the triple $(\Delta_X, \mu_X, \Delta_X)$ is a left–right Yetter–Drinfeld H–module

[1] In [Kas12, Def. IX.5.1] (resp. [Rad93, Def. 2]) left–right Yetter–Drinfeld H–modules are called crossed H–bimodules (resp. left quantum Yang–Baxter H–module)

Chapter 4

Lie bialgebras and Drinfeld–Yetter modules

4.1 Lie bialgebras

The following definitions are standard. For more details we remand the reader to [Hal15], [Hum12], [ES02], [Mic80], [GG78].

Definition 4.1.1. *A **Lie algebra** is a pair $(\mathfrak{g}, [\cdot,\cdot])$, where $\mathfrak{g}$ is a vector space and $[\cdot,\cdot] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ is a bilinear map, called the **Lie bracket**, satisfying the following two conditions for all $x, y, z \in \mathfrak{g}$:*

(i) (antisymmetry):

$$[x, y] = -[y, x]; \tag{4.1.1}$$

(ii) (Jacobi identity):

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{4.1.2}$$

A morphism of Lie algebras is a linear map $f : \mathfrak{g} \to \mathfrak{g}'$ such that $f([x, y]) = [f(x), f(y)]$.

Definition 4.1.2. *Let $\mathfrak{g}$ be a Lie algebra. A (left) **Lie $\mathfrak{g}$–module** is a pair (V, μ), where V is a vector space and μ is a linear map*

$$\begin{aligned} \mu : \mathfrak{g}\otimes V &\to V \\ (x, v) &\mapsto x\cdot v \end{aligned}$$

such that for any $x, y \in \mathfrak{g}$ and $v \in V$ the following identity holds

$$[x, y]\cdot v = x\cdot(y\cdot v) - y\cdot(x\cdot v). \tag{4.1.3}$$

Definition 4.1.3. *A **Lie coalgebra** is a pair $(\mathfrak{c}, \delta)$, where $\mathfrak{c}$ is a vector space and $\delta : \mathfrak{c} \to \mathfrak{c}\otimes\mathfrak{c}$ is a linear map, called the **Lie cobracket**, such that:*

(i) (antisymmetry):

$$\delta = -(\tau_{\mathfrak{c},\mathfrak{c}} \circ \delta); \tag{4.1.4}$$

(ii) (coJacobi identity):

$$(\mathsf{id}_{\mathfrak{c}\otimes\mathfrak{c}\otimes\mathfrak{c}} + \sigma + \sigma^2) \circ (\mathsf{id}_{\mathfrak{c}} \otimes \delta) \circ \delta = 0 \tag{4.1.5}$$

where σ is the cyclic permutation of $\mathfrak{S}_3$. A morphism of Lie coalgebras is a linear map $f : \mathfrak{c} \to \mathfrak{c}'$ such that $(f \otimes f) \circ \delta = \delta' \circ f$.

Notation 4.1.4. *We shall sometimes use the Sweedler's notation*

$$\delta(x) = \sum_{\langle x \rangle} x' \otimes x''.$$

Note that we can write Equation (4.1.5) in Sweedler's notation as

$$\sum_{\langle x \rangle} \sum_{\langle x'' \rangle} \Big(x' \otimes (x'')' \otimes (x'')'' + (x'')' \otimes (x'')'' \otimes x' + (x'')'' \otimes x' + (x')'' \Big) = 0. \tag{4.1.6}$$

Definition 4.1.5. *Let $\mathfrak{c}$ be a Lie coalgebra. A (right)* ***Lie $\mathfrak{c}$–comodule*** *is a pair (V, π_V^*), where V is a vector space and π_V^* is a linear map*

$$\begin{aligned} \pi_V^* : V &\to \mathfrak{c} \otimes V \\ v &\mapsto \sum_{[v]} v^{[0]} \otimes v^{[1]} \end{aligned}$$

(where $v^{[0]} \in \mathfrak{c}$ and $v^{[1]} \in V$) such that, for any $v \in V$, the following identity holds in $\mathfrak{c} \otimes \mathfrak{c} \otimes V$

$$(\delta \otimes \mathsf{id}_V) \circ \pi_V^* = (\tau_{\mathfrak{c},\mathfrak{c}} \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{c}} \otimes \pi_V^*) \circ \pi_V^* - (\mathsf{id}_{\mathfrak{c}} \otimes \pi_V^*) \circ \pi_V^* \tag{4.1.7}$$

which we can express using the Sweedler's notation as

$$\sum_{[v]} \sum_{[v^{[1]}]} \Big(\big(v^{[1]}\big)^{[0]} \otimes v^{[0]} - v^{[0]} \otimes \big(v^{[1]}\big)^{[0]} \Big) \otimes \big(v^{[1]}\big)^{[1]} - \sum_{[v]} \sum_{\langle v^{[0]} \rangle} \big(v^{[0]\prime} \otimes v^{[0]\prime\prime} \big) \otimes v^{[1]} = 0. \tag{4.1.8}$$

A morphism of right Lie $\mathfrak{c}$–comodules is a linear map $f : V \to W$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ {\scriptstyle \pi_V^*}\downarrow & & \downarrow{\scriptstyle \pi_W^*} \\ \mathfrak{c} \otimes V & \xrightarrow[\mathsf{id}_{\mathfrak{c}} \otimes f]{} & \mathfrak{c} \otimes W \end{array}$$

commutes.

Remark 4.1.6. *Note that Definition 4.1.5 coincides with the usual definition of left Lie $\mathfrak{c}^{\mathsf{op}}$–comodule, where $\mathfrak{c}^{\mathsf{op}}$ is the opposite Lie coalgebra $\mathfrak{c}^{\mathsf{op}} = (\mathfrak{c}, \tau \circ \delta = -\delta)$. It is indeed well–known that the usual notions of right $\mathfrak{c}$–comodule (see* [Maj95, p. 382]*) and of left $\mathfrak{c}^{\mathsf{op}}$–comodule are equivalent, and then we shall use – as in* [EK98, p. 6] *and* [ES02, p. 199] *– right comodules with the coaction going from V to $\mathfrak{c} \otimes V$, and not from V to $V \otimes \mathfrak{c}$.*
It is also important to underline that the linear map π_V^* is not the dual map of a linear map π_V. We use the notation π_V^* as in the articles of Etingof and Kazhdan [EK96], [EK98].

Definition 4.1.7. *A* ***Lie bialgebra*** *is a triple $(\mathfrak{b}, [\cdot,\cdot], \delta)$, where $(\mathfrak{b}, [\cdot,\cdot])$ is a Lie algebra, $(\mathfrak{b}, \delta)$ is a Lie coalgebra, and the following relation, called the* ***cocycle condition***, *is satisfied*

$$\begin{aligned} \delta([x,y]) &= x \cdot \delta(y) - y \cdot \delta(x) \\ &= \sum_{\langle y \rangle} \Big([x, y'] \otimes y'' + y' \otimes [x, y''] \Big) - \sum_{\langle x \rangle} \Big([y, x'] \otimes x'' + x' \otimes [y, x''] \Big) \\ &= \sum_{\langle x \rangle} \Big([x', y] \otimes x'' + x' \otimes [x'', y] \Big) + \sum_{\langle y \rangle} \Big([x, y'] \otimes y'' + y' \otimes [x, y''] \Big) \end{aligned} \tag{4.1.9}$$

for all $x, y \in \mathfrak{b}$.
A morphism of Lie bialgebras is a linear map $f : \mathfrak{b} \to \mathfrak{b}'$ *that is both a morphism of Lie algebras and a morphism of Lie coalgebras.*
The following result is well–known, see e.g. [Maj95, Prop. 8.1.2]*:*

Theorem 4.1.8. *The dual vector space of a finite–dimensional Lie bialgebra* $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}}, \delta_{\mathfrak{b}})$ *has a standard structure of Lie bialgebra* $(\mathfrak{b}^*, [\cdot,\cdot]_{\mathfrak{b}^*}, \delta_{\mathfrak{b}^*})$ *defined by*

$$([f,g]_{\mathfrak{b}^*}, x) = (f \otimes g, \delta_{\mathfrak{b}}(x)) \qquad \text{and} \qquad (\delta_{\mathfrak{b}^*}(f), x \otimes y) = (f, [x,y]_{\mathfrak{b}})$$

where $(\cdot,\cdot)$ *denotes the natural pairing between* $\mathfrak{b}$ *and* $\mathfrak{b}^*$.

Let $(\mathfrak{b}, [\cdot,\cdot], \delta)$ *be a finite–dimensional Lie bialgebra and let* $\{e_1, \ldots, e_n\}$ *be a basis of* $\mathfrak{b}$*. The structure's constants of* $\mathfrak{b}$ *with respect to the basis* $\{e_1, \ldots, e_n\}$ *are* $\{\alpha_{i,j}^k, \beta_k^{i,j}\}$*, where*

$$[e_i, e_j] = \sum_{k=1}^{n} \alpha_{i,j}^k e_k \qquad \text{and} \qquad \delta(e_k) = \sum_{i,j} \beta_k^{i,j} (e_i \otimes e_j)$$

for $1 \leqslant i,j,k \leqslant n$*. Note that the antisymmetry of the Lie bracket* (4.1.1) *and the antisymmetry of the Lie cobracket* (4.1.4) *are equivalent to the following conditions*

$$\alpha_{i,j}^k = -\alpha_{j,i}^k \qquad \text{and} \qquad \beta_k^{i,j} = -\beta_k^{j,i}. \tag{4.1.10}$$

The Jacobi identity (4.1.2) *is then equivalent to*

$$\sum_{s,t} \left(\alpha_{j,k}^t \alpha_{i,t}^s + \alpha_{k,i}^t \alpha_{j,t}^s + \alpha_{i,j}^t \alpha_{k,t}^s \right) = 0 \tag{4.1.11}$$

and the coJacobi identity (4.1.5) *is equivalent to*

$$\sum_{i,j,s,t} \left(\beta_k^{i,j} \beta_j^{s,t} + \beta_k^{s,j} \beta_j^{t,i} + \beta_k^{t,j} \beta_j^{i,s} \right) = 0. \tag{4.1.12}$$

In order to write the cocycle condition (4.1.9) *in terms of the structure's constants, we get*

$$\delta([e_i, e_j]) = \sum_k \alpha_{i,j}^k \delta(e_k) = \sum_{k,u,v} \alpha_{i,j}^k \beta_k^{u,v} (e_u \otimes e_v),$$

and

$$\begin{aligned} e_i \cdot \delta(e_j) &= \sum_{s,t} \beta_j^{s,t} e_i \cdot (e_s \otimes e_t) \\ &= \sum_{s,t} \beta_j^{s,t} \Big(e_s \otimes [e_i, e_t] + [e_i, e_s] \otimes e_t \Big) \\ &= \sum_{s,t,u} \beta_j^{s,t} \alpha_{i,t}^u (e_s \otimes e_u) + \sum_{s,t,v} \beta_j^{s,t} \alpha_{i,s}^v (e_v \otimes e_t), \end{aligned}$$

and

$$\begin{aligned} e_j \cdot \delta(e_i) &= \sum_{p,q} \beta_i^{p,q} e_j \cdot (e_p \otimes e_q) \\ &= \sum_{p,q} \beta_i^{p,q} \big(e_p \otimes [e_j, e_q] + [e_j, e_p] \otimes e_q \big) \\ &= \sum_{p,q,l} \beta_i^{p,q} \alpha_{j,q}^{l} (e_p \otimes e_l) + \sum_{p,q,h} \beta_i^{p,q} \alpha_{j,p}^{h} (e_h \otimes e_q). \end{aligned}$$

Therefore by specializing at $e_a \otimes e_b$ and by changing name of the variables, we obtain that the cocycle condition is satisfied if and only if

$$\sum_k \alpha_{i,j}^k \beta_k^{a,b} = \sum_k \left(\alpha_{i,k}^b \beta_j^{a,k} + \alpha_{i,k}^a \beta_j^{k,b} - \alpha_{j,k}^b \beta_i^{a,k} - \alpha_{j,k}^a \beta_i^{k,b} \right) \tag{4.1.13}$$

for any $i, j, a, b \in \{1, \dots n\}$.

4.2 Manin triples and the Drinfeld double

Definition 4.2.1. *A* ***finite–dimensional Manin triple*** *is a triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, where:*

- *$\mathfrak{g}$ is a finite–dimensional Lie algebra equipped with a non degenerate and invariant bilinear form $< \cdot, \cdot >$, that means that*

$$< [x, y], z > = < x, [y, z] > \tag{4.2.1}$$

  for all $x, y, z \in \mathfrak{g}$;

- *$\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} \cong \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces;*

- *$\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic subspaces of $\mathfrak{g}$ with respect to $< \cdot, \cdot >$, that means that $< \cdot, \cdot > |_{\mathfrak{g}_+, \mathfrak{g}_+} = 0$ and $< \cdot, \cdot > |_{\mathfrak{g}_-, \mathfrak{g}_-} = 0$.*

It is possible to define Manin triples for infinite–dimensional Lie bialgebras, see [ATL12, §4.3] *for more details.*

Lemma 4.2.2. *Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a finite–dimensional Manin triple. Then* $\dim \mathfrak{g}_+ = \dim \mathfrak{g}_-$.

Proof. Consider the map

$$\begin{aligned} f_+ : \mathfrak{g}_+ &\to \mathsf{Hom}(\mathfrak{g}_-, \mathbb{K}) = \mathfrak{g}_-^* \\ x &\mapsto < x, \cdot > . \end{aligned}$$

Since $< \cdot, \cdot >$ is non degenerate in $\mathfrak{g}$, it follows that $\ker f_+ = \{0\}$ (if there exists a non–trivial element $v \in \ker f_+$ then $< x, v > = 0$ for all $x \in \mathfrak{g}$, and then $<, >$ would be degenerate). Therefore, we have $\dim \mathfrak{g}_+ \leqslant \dim \ker f_+ + \dim \mathsf{Im} f_+ \leqslant \dim \mathfrak{g}_-^* = \dim \mathfrak{g}_-$. Similarly, applying the same argument to the map

$$\begin{aligned} f_- : \mathfrak{g}_- &\to \mathsf{Hom}(\mathfrak{g}_+, \mathbb{K}) = \mathfrak{g}_+^* \\ y &\mapsto < \cdot, y > \end{aligned}$$

we obtain $\dim \mathfrak{g}_- \leqslant \dim \mathfrak{g}_+^* = \dim \mathfrak{g}_+$. □

Proposition 4.2.3. *Let* $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ *be a finite–dimensional Manin triple and let* δ *be the Lie coalgebra structure on* $\mathfrak{g}_+$ *induced by* $\mathfrak{g}_-$. *Then* $(\mathfrak{g}_+, [\cdot,\cdot], \delta)$ *is a Lie bialgebra.*

Proof. We have to show that the cocycle condition holds. Let $\{e_1, \dots, e_n\}$ be a basis of $\mathfrak{g}_+$ and let $\{e_1^*, \dots, e_n^*\}$ be the dual basis, i.e. $e_i^* \in \mathfrak{g}_+^* \cong \mathfrak{g}_-$ and $(e_j^*, e_i) = \boldsymbol{\delta}_{ij}$, where $(\cdot,\cdot)$ denotes the natural pairing between $\mathfrak{g}_+$ and $\mathfrak{g}_+^*$, and $\boldsymbol{\delta}_{ij}$ denotes the usual Kronecker's delta. Then the cocycle condition (4.1.9) is satisfied if and only if for any r, s, k, l

$$\left(e_r^* \otimes e_s^* \;,\; \delta([e_k, e_l]) - e_k \cdot \delta(e_l) + e_l \cdot \delta(e_k)\right) = 0 \tag{4.2.2}$$

We have

$$\begin{aligned}(e_r^* \otimes e_s^*, \delta([e_k, e_l])) &= \sum_c \alpha_{k,l}^c (e_r^* \otimes e_s^*, \delta(e_c)) \\ &= \sum_{c,d,f} \alpha_{k,l}^c \beta_c^{d,f} (e_r^* \otimes e_s^*, e_d \otimes e_f) \\ &= \sum_{c,d,f} \alpha_{k,l}^c \beta_c^{d,f} \boldsymbol{\delta}_{r,d} \boldsymbol{\delta}_{s,f} \\ &= \sum_c \alpha_{k,l}^c \beta_c^{r,s},\end{aligned}$$

while

$$\begin{aligned}(e_r^* \otimes e_s^*, e_k \cdot \delta(e_l)) &= \sum_{i,j} \beta_l^{i,j} (e_r^* \otimes e_s^*, e_i \otimes [e_k, e_j] + [e_k, e_i] \otimes e_j) \\ &= \sum_{i,j,t} \beta_l^{i,j} \alpha_{k,j}^t (e_r^* \otimes e_s^*, e_i \otimes e_t) + \sum_{i,j,h} \beta_l^{i,j} \alpha_{k,i}^h (e_r^* \otimes e_s^*, e_h \otimes e_j) \\ &= \sum_{i,j,t} \beta_l^{i,j} \alpha_{k,j}^t \boldsymbol{\delta}_{r,i} \boldsymbol{\delta}_{s,t} + \sum_{i,j,h} \beta_l^{i,j} \alpha_{k,i}^h \boldsymbol{\delta}_{r,h} \boldsymbol{\delta}_{s,j} \\ &= \sum_j \beta_l^{r,j} \alpha_{k,j}^s + \sum_i \beta_l^{i,s} \alpha_{k,i}^r,\end{aligned}$$

and

$$\begin{aligned}(e_r^* \otimes e_s^*, e_l \cdot \delta(e_k)) &= \sum_{p,q} \beta_k^{p,q} (e_r^* \otimes e_s^*, e_p \otimes [e_l, e_q] + [e_l, e_p] \otimes e_q) \\ &= \sum_{p,q,a} \beta_k^{p,q} \alpha_{l,q}^a (e_r^* \otimes e_s^*, e_p \otimes e_a) + \sum_{p,q,b} \beta_k^{p,q} \alpha_{l,p}^b (e_r^* \otimes e_s^*, e_b \otimes e_q) \\ &= \sum_{p,q,a} \beta_k^{p,q} \alpha_{l,q}^a \boldsymbol{\delta}_{r,p} \boldsymbol{\delta}_{s,a} + \sum_{p,q,b} \beta_k^{p,q} \alpha_{l,p}^b \boldsymbol{\delta}_{r,b} \boldsymbol{\delta}_{s,q} \\ &= \sum_q \beta_k^{r,q} \alpha_{l,q}^s + \sum_p \beta_k^{p,s} \alpha_{l,p}^r.\end{aligned}$$

It is easy to see, by renaming variables, that condition (4.2.2) is equivalent to (4.1.13). □

Proposition 4.2.4. *Let* $(\mathfrak{b}, [\cdot,\cdot], \delta)$ *be a finite–dimensional Lie bialgebra. Then* $(\mathfrak{b} \oplus \mathfrak{b}^*, \mathfrak{b}, \mathfrak{b}^*)$ *is a finite–dimensional Manin triple.*

Proof. We have to construct a non degenerate and invariant bilinear form that satisfies the definition of a finite–dimensional Manin triple and we have to define the mixed bracket $[x, y]$ for $x \in \mathfrak{b}$ and $y \in \mathfrak{b}^*$. Consider the bilinear form given by

$$< x + y, x' + y' >:= y(x') + y'(x) \tag{4.2.3}$$

where $x, x' \in \mathfrak{b}$ and $y, y' \in \mathfrak{b}^*$. It is clear that $< \cdot, \cdot >\restriction_{\mathfrak{b},\mathfrak{b}} = 0$ and $< \cdot, \cdot >\restriction_{\mathfrak{b}^*,\mathfrak{b}^*} = 0$. We define the mixed bracket $[x, y]$ for $x \in \mathfrak{b}$, $y \in \mathfrak{b}^*$ in such a way that $< \cdot, \cdot >$ is invariant: given $\{e_1, \ldots, e_n\}$ be a basis of $\mathfrak{b}$, we have that the bilinear form (4.2.3) is invariant if and only if

$$\begin{aligned} < [e_i^*, e_j], e_k^* > &= - < [e_j, e_i^*], e_k^* > \\ &= - < e_j, [e_i^*, e_k^*] > \\ &= - \sum_t \beta_t^{i,k} < e_j, e_t^* > \\ &= -\beta_j^{i,k} \end{aligned}$$

and

$$\begin{aligned} < [e_i^*, e_j], e_k > &=< e_i^*, [e_j, e_k] > \\ &= \sum_s \alpha_{j,k}^s < e_i^*, e_s > \\ &= \alpha_{j,k}^i. \end{aligned}$$

Hence, we define the mixed bracket as

$$[e_i^*, e_j] := \sum_{k=1}^n \alpha_{j,k}^i e_k^* - \beta_j^{i,k} e_k.$$

We now have to prove that such a bracket satisfies the Jacobi identity. It is clear that the identity is satisfied for $x, x', x'' \in \mathfrak{b}$ or for $y, y', y'' \in \mathfrak{b}^*$. Then we have to prove it in the two mixed cases. Let $e_i \in \mathfrak{b}$ and $e_j^*, e_k^* \in \mathfrak{b}^*$. Then we have:

$$\begin{aligned} [e_i, [e_j^*, e_k^*]] &= \sum_t \beta_t^{j,k} [e_i, e_t^*] \\ &= - \sum_t \beta_t^{j,k} [e_t^*, e_i] \\ &= - \sum_{t,s} \beta_t^{j,k} \alpha_{i,s}^t e_s^* + \sum_{t,s} \beta_t^{j,k} \beta_i^{t,s} e_s, \end{aligned}$$

and

$$\begin{aligned} [e_k^*, [e_i, e_j^*]] &= -[e_k^*, [e_j^*, e_i]] \\ &= \sum_t \beta_i^{j,t} [e_k^*, e_t] - \sum_t \alpha_{i,t}^j [e_k^*, e_t^*] \\ &= \sum_{t,s} \beta_i^{j,t} \alpha_{t,s}^k e_s^* - \sum_{t,s} \beta_i^{j,t} \beta_t^{k,s} e_s - \sum_{t,s} \alpha_{i,t}^j \beta_s^{k,t} e_s^*, \end{aligned}$$

and

$$
\begin{aligned}
[e_j^*, [e_k^*, e_i]] &= \sum_t \alpha_{i,t}^k [e_j^*, e_t^*] - \sum_t \beta_i^{k,t} [e_j^*, e_t] \\
&= \sum_{t,s} \alpha_{i,t}^k \beta_s^{j,t} e_s^* - \sum_{t,s} \beta_i^{k,t} \alpha_{t,s}^j e_s^* + \sum_{t,s} \beta_i^{k,t} \beta_t^{j,s} e_s.
\end{aligned}
$$

Therefore, we have that

$$[e_i, [e_j^*, e_k^*]] + [e_k^*, [e_i, e_j^*]] + [e_j^*, [e_k^*, e_i]] = 0$$

if and only if

$$
\begin{cases}
\sum_{t,s} \left(-\beta_t^{j,k} \alpha_{i,s}^t + \beta_i^{j,t} \alpha_{t,s}^k - \alpha_{i,t}^j \beta_s^{k,t} + \alpha_{i,t}^k \beta_s^{j,t} - \beta_i^{k,t} \alpha_{t,s}^j \right) = 0 \\
\sum_{t,s} \left(\beta_t^{j,k} \beta_i^{t,s} e_s - \beta_i^{j,t} \beta_t^{k,s} + \beta_i^{k,t} \beta_t^{j,s} \right) = 0
\end{cases}
$$

It is easy to see, using equation (4.1.10) and renaming opportunely variables, that the first equation vanishes for the cocycle identity (4.1.13). Similarly, using equation (4.1.10) and renaming opportunely variables, it is easy to see that the second equation vanishes for the coJacobi identity (4.1.12). We therefore proved the Jacobi identity in the first mixed case.

Let $e_j, e_k \in \mathfrak{b}$ and $e_i^* \in \mathfrak{b}^*$. We have

$$
\begin{aligned}
[e_i^*, [e_j, e_k]] &= \sum_t \alpha_{j,k}^t [e_i^*, e_t] \\
&= \sum_{t,s} \alpha_{j,k}^t \alpha_{t,s}^i e_s^* - \sum_{t,s} \alpha_{j,k}^t \beta_t^{i,s} e_s,
\end{aligned}
$$

and

$$
\begin{aligned}
[e_k, [e_i^*, e_j]] &= \sum_t \alpha_{j,t}^i [e_k, e_t^*] - \sum_t \beta_j^{i,t} [e_k, e_t] \\
&= -\sum_t \alpha_{j,t}^i [e_t^*, e_k] - \sum_t \beta_j^{i,t} [e_k, e_t] \\
&= -\sum_{t,s} \alpha_{j,t}^i \alpha_{k,s}^t e_s^* + \sum_{t,s} \alpha_{j,t}^i \beta_k^{t,s} e_s - \sum_{t,s} \beta_j^{i,t} \alpha_{k,t}^s e_s,
\end{aligned}
$$

and

$$
\begin{aligned}
[e_j, [e_k, e_i^*]] &= -[e_j, [e_i^*, e_k]] \\
&= -\sum_t \alpha_{k,t}^i [e_j, e_t^*] + \sum_t \beta_k^{i,t} [e_j, e_t] \\
&= \sum_t \alpha_{k,t}^i [e_t^*, e_j] + \sum_{t,s} \beta_k^{i,t} \alpha_{j,t}^s e_s \\
&= \sum_{t,s} \alpha_{k,t}^i \alpha_{j,s}^t e_s^* - \sum_{t,s} \alpha_{k,t}^i \beta_j^{t,s} e_s + \sum_{t,s} \beta_k^{i,t} \alpha_{j,t}^s e_s.
\end{aligned}
$$

Therefore we have that

$$[e_i^*, [e_j, e_k]] + [e_k, [e_i^*, e_j]] + [e_j, [e_k, e_i^*]] = 0$$

if and only if

$$\begin{cases} \sum_{t,s}\left(-\alpha^t_{j,k}\beta^{t,s}_i + \alpha^i_{j,t}\beta^{t,s}_k - \beta^{i,t}_j\alpha^s_{k,t} - \alpha^i_{k,t}\beta^{t,s}_j + \beta^{i,t}_k\alpha^s_{j,t}\right) = 0 \\ \sum_{t,s}\left(\alpha^t_{j,k}\alpha^i_{t,s} - \alpha^i_{j,t}\alpha^t_{k,s} + \alpha^i_{k,t}\alpha^t_{j,s}\right) = 0 \end{cases}$$

It is easy to see, using equation (4.1.10) and renaming variables, that the first equation vanishes for the cocycle identity (4.1.13). Similarly, using equation (4.1.10) and renaming variables, it is easy to see that the second equation vanishes for the Jacobi identity (4.1.11). We therefore proved the Jacobi identity in the second mixed case, and this concludes the proof. □

Remark 4.2.5. *If $(\mathfrak{b}, [\cdot,\cdot], \delta)$ is a finite–dimensional Lie bialgebra and $\mathfrak{b}\oplus\mathfrak{b}^*$ is the associated Manin triple, we may write the mixed bracket in a coordinate–free way by*

$$[x, y] = -y \circ \mathsf{ad}_x + (y \otimes \mathsf{id})(\delta(x)) \tag{4.2.4}$$

for any $x \in \mathfrak{b}$ and $y \in \mathfrak{b}^*$.

According to [Max23, 1.1.2], there is no way to define morphisms of Manin triples. However, we can define isomorphisms and it is possible to show that equivalent Manin triples lead to equivalent Lie bialgebras.
Propositions 4.2.3 and 4.2.4 gives a one-to-one correspondence between finite–dimensional Lie bialgebras and Manin triples. Furthermore, Proposition 4.2.4 allows to consider the following

Definition 4.2.6. *Let $(\mathfrak{b}, [\cdot,\cdot], \delta)$ be a finite–dimensional Lie bialgebra and let $(\mathfrak{b}\oplus\mathfrak{b}^*, \mathfrak{b}, \mathfrak{b}^*)$ be the finite–dimensional Manin triple associated to $\mathfrak{b}$. The* ***Drinfeld double*** *of $\mathfrak{b}$ is the finite–dimensional Lie algebra $\mathfrak{b}\oplus\mathfrak{b}^*$, and we denote it by $\mathfrak{d}_{\mathfrak{b}}$.*

Theorem 4.2.7. *Let $\mathfrak{b}$ be a finite–dimensional Lie bialgebra. Then $\mathfrak{d}_{\mathfrak{b}}$ is a quasi–triangular Lie bialgebra. In other words, there exists $r = \sum_i s_i \otimes t_i \in \mathfrak{d}_{\mathfrak{b}} \otimes \mathfrak{d}_{\mathfrak{b}}$ such that the following three conditions are satisfied*

(i) r is a solution of the classical Yang–Baxter equation, i.e.

$$\sum_{i,j}\Big([s_i, s_j] \otimes t_i \otimes t_j + s_i \otimes [t_i, s_j] \otimes t_j + s_i \otimes s_j \otimes [t_i, t_j]\Big) = 0.$$

(ii) $r + \tau \circ r$ is $\mathfrak{b}$–invariant, i.e. the following identity holds for all $x \in \mathfrak{b}$:

$$\sum_i\Big(s_i \otimes [x, t_i] + [x, s_i] \otimes t_i + t_i \otimes [x, s_i] + [x, t_i] \otimes s_i\Big) = 0.$$

(iii) The triple $(\mathfrak{d}_{\mathfrak{b}}, [\cdot,\cdot]_{\mathfrak{d}_{\mathfrak{b}}}, \delta_{\mathfrak{d}_{\mathfrak{b}}})$ is a Lie bialgebra, where $\delta_{\mathfrak{d}_{\mathfrak{b}}}(x) := x \cdot r$.

Proof. We have that $\mathfrak{d}_{\mathfrak{b}}$ is a Lie algebra, where the mixed Lie bracket is given by (4.2.4). Set $\delta_{\mathfrak{d}_{\mathfrak{b}}} := \delta_{\mathfrak{b}} \oplus -\delta_{\mathfrak{b}^*}$. It is clear that $(\mathfrak{d}_{\mathfrak{b}}, \delta_{\mathfrak{d}_{\mathfrak{b}}})$ is a Lie coalgebra. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathfrak{b}$ and let

$$r = \sum_{j=1}^n e_j \otimes e_j^* \in \mathfrak{d}_{\mathfrak{b}} \otimes \mathfrak{d}_{\mathfrak{b}}. \tag{4.2.5}$$

For any $1 \leqslant i \leqslant n$ we have

$$\begin{aligned}
e_i \cdot r &= e_i \cdot \sum_j^n e_j \otimes e_j^* \\
&= \sum_j \big([e_i, e_j] \otimes e_j^* + e_j \otimes [e_i, e_j^*]\big) \\
&= \sum_j \big([e_i, e_j] \otimes e_j^* - e_j \otimes [e_j^*, e_i]\big) \\
&= \sum_j [e_i, e_j] \otimes e_j^* + \sum_{j,k} \beta_i^{j,k}(e_j \otimes e_k) - \sum_{j,k} \alpha_{i,k}^j (e_j \otimes e_k^*) \\
&= \sum_j [e_i, e_j] \otimes e_j^* + \sum_{j,k} \beta_i^{j,k}(e_j \otimes e_k) - \sum_k [e_i, e_k] \otimes e_k^* \\
&= \sum_{j,k} \beta_i^{j,k}(e_j \otimes e_k) \\
&= \delta_{\mathfrak{b}}(e_i)
\end{aligned}$$

and

$$\begin{aligned}
e_i^* \cdot r &= e_i^* \cdot \sum_j e_j \otimes e_j^* \\
&= \sum_j \big([e_i^*, e_j] \otimes e_j^* + e_j \otimes [e_i^*, e_j^*]\big) \\
&= \sum_{j,k} \alpha_{j,k}^i (e_k^* \otimes e_j^*) - \sum_{j,k} \beta_j^{i,k}(e_k \otimes e_j^*) + \sum_j e_j \otimes [e_i^*, e_j^*] \\
&= -\sum_{j,k} \alpha_{k,j}^i (e_k^* \otimes e_j^*) - \sum_k e_k \otimes [e_i^*, e_k^*] + \sum_j e_j \otimes [e_i^*, e_j^*] \\
&= -\sum_{j,k} \alpha_{k,j}^i (e_k^* \otimes e_j^*) \\
&= -\delta_{\mathfrak{b}^*}(e_i^*),
\end{aligned}$$

hence $\delta_{\mathfrak{d}_{\mathfrak{b}}}(x) = x \cdot r$.
(i): We have

$$\begin{aligned}
&\sum_{i,j} \Big([e_i, e_j] \otimes e_i^* \otimes e_j^* + e_i \otimes [e_i^*, e_j] \otimes e_j^* + e_i \otimes e_j \otimes [e_i^*, e_j^*]\Big) \\
&= \sum_{i,j,k} \alpha_{i,j}^k (e_k \otimes e_i^* \otimes e_j^*) + \sum_{i,j,k} \alpha_{j,k}^i (e_i \otimes e_k^* \otimes e_j^*) \\
&- \sum_{i,j,k} \beta_j^{i,k}(e_i \otimes e_k \otimes e_j^*) + \sum_{i,j,k} \beta_k^{i,j}(e_i \otimes e_j \otimes e_k) \\
&= \sum_{i,j,k} \alpha_{i,j}^k (e_k \otimes e_i^* \otimes e_j^*) - \sum_{i,j,k} \alpha_{k,j}^i (e_i \otimes e_k^* \otimes e_j^*) \\
&- \sum_{i,j,k} \beta_j^{i,k}(e_i \otimes e_k \otimes e_j^*) + \sum_{i,j,k} \beta_k^{i,j}(e_i \otimes e_j \otimes e_k) \\
&= 0.
\end{aligned}$$

(ii): For any $1 \leqslant j \leqslant n$ we have

$$\begin{aligned}
&\sum_i \big(e_i \otimes [e_j, e_i^*] + [e_j, e_i] \otimes e_i^* + e_i^* \otimes [e_j, e_i] + [e_j, e_i^*] \otimes e_i\big) \\
&= \sum_i \big(- e_i \otimes [e_i^*, e_j] + [e_j, e_i] \otimes e_i^* + e_i^* \otimes [e_j, e_i] - [e_i^*, e_j] \otimes e_i\big) \\
&= - \sum_{i,k} \alpha^i_{j,k}(e_i \otimes e_k^*) + \sum_{i,k} \beta^{i,k}_j (e_i \otimes e_k) + \sum_{i,k} \alpha^k_{j,i}(e_k \otimes e_i^*) \\
&+ \sum_{i,k} \alpha^k_{j,i}(e_i^* \otimes e_k) - \sum_{i,k} \alpha^i_{j,k}(e_k^* \otimes e_i) + \sum_{i,k} \beta^{i,k}_j (e_k \otimes e_i) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\sum_i \big(e_i \otimes [e_j^*, e_i^*] + [e_j^*, e_i] \otimes e_i^* + e_i^* \otimes [e_j^*, e_i] + [e_j^*, e_i^*] \otimes e_i\big) \\
&= \sum_{i,k} \beta^{j,i}_k (e_i \otimes e_k^*) + \sum_{i,k} \alpha^j_{i,k}(e_k^* \otimes e_i^*) - \sum_{i,k} \beta^{j,k}_i (e_k \otimes e_i^*) \\
&+ \sum_{i,k} \alpha^j_{i,k}(e_i^* \otimes e_k^*) - \sum_{i,k} \beta^{j,k}_i (e_i^* \otimes e_k) + \sum_{i,k} \beta^{j,i}_k (e_k^* \otimes e_i) \\
&= 0.
\end{aligned}$$

(iii): We have that

$$\delta([x,y]) = [x,y] \cdot r = x \cdot (y \cdot r) - y \cdot (x \cdot r) = x \cdot \delta(y) - y \cdot \delta(x).$$

Hence the cocycle condition (4.1.9) is satisfied, and this concludes the proof. □

4.3 Universal enveloping algebras

In this Section we give a brief introduction to the universal enveloping of a Lie algebra. More details can be find in [Dix96, Chapter 2], [Kas12, V.2], [Hum12, Chapter 3], and [CE99, XIII].

Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie bialgebra over a field $\mathbb{K}$ of characteristics zero. Consider the free (tensor) algebra $\mathsf{T}(\mathfrak{g}) = \oplus_{r=0}^{\infty} \mathsf{T}^r(\mathfrak{g})$ where $\mathsf{T}^0(\mathfrak{g}) \coloneqq \mathbb{K}$, $\mathsf{T}^1(\mathfrak{g}) \coloneqq \mathfrak{g}$, and (for each nonnegative integer r) $\mathsf{T}^{r+1}(\mathfrak{g}) \coloneqq \mathsf{T}^r(\mathfrak{g}) \otimes \mathfrak{g}$. The associative multiplication in $\mathsf{T}(\mathfrak{g})$ is the usual tensor product (decorated with the associators in the monoidal category $\mathsf{Vect}_{\mathbb{K}}$).

Notation 4.3.1. *We shall write $a_1 \otimes \cdots \otimes a_r$ for an element of $\mathsf{T}^r(\mathfrak{g})$ made out of the multiplication of $a_1, \ldots, a_r \in \mathfrak{g}$ with the small tensor symbol $\otimes$ for the multiplication.*

Hence $\big(\mathsf{T}(\mathfrak{g}), \otimes, 1\big)$ is an associative algebra over $\mathbb{K}$. It is free in the sense that every $\mathbb{K}$–linear map $\mathfrak{g} \to A$ where A is an arbitrary associative algebra over $\mathbb{K}$ can uniquely be extended to an morphism of algebras $\mathsf{T}(\mathfrak{g}) \to A$.

Definition 4.3.2. *Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{I}(\mathfrak{g})$ be the two–sided ideal of $\mathsf{T}(\mathfrak{g})$ generated by all elements of the form $x \otimes y - y \otimes x - [x,y]$, $x, y \in \mathfrak{g}$. The **universal enveloping algebra** $\mathsf{U}(\mathfrak{g})$ of $\mathfrak{g}$ is defined to be the quotient algebra*

$$\mathsf{U}(\mathfrak{g}) \coloneqq \mathsf{T}(\mathfrak{g})/\mathcal{I}(\mathfrak{g}).$$

We denote the canonical projection by $\pi_{\mathfrak{g}} : \mathsf{T}(\mathfrak{a}) \to \mathsf{U}(\mathfrak{a})$. We have that $\mathsf{U}(\mathfrak{g})$ carries a natural exhaustive filtration $\big(\mathsf{U}_{(n)}(\mathfrak{g})\big)_{n\in\mathbb{Z}}$, where $\mathsf{U}_{(0)}(\mathfrak{g}) = \mathbb{K}$, $\mathsf{U}_{(n)}(\mathfrak{g}) = \{0\}$ for all integers $n \leqslant -1$, and for non–negative n $\mathsf{U}_{(n)}(\mathfrak{g})$ is defined by the image under $\pi_{\mathfrak{g}}$ of the filtration submodule $\oplus_{r=0}^{n}\mathsf{T}^r(\mathfrak{g})$, see [Dix96, §2.3] for more details.
Recall that the association $\mathsf{U} : \mathfrak{g} \to \mathsf{U}(\mathfrak{g})$ is a functor from the category of all Lie algebras over $\mathbb{K}$ to the category of all associative algebras over $\mathbb{K}$, where morphisms of Lie algebras are mapped (by freeness) to morphisms of free algebras. Moreover, the functor U can be seen as a left adjoint functor of the functor $\mathfrak{L}$ which associates to any associative algebra the Lie algebra consisting of the same underlying vector spaces equipped with the commutator bracket.

Moreover, recall that $\mathsf{U}(\mathfrak{g})$ is a Hopf algebra with a cocommutative coassociative comultiplication $\Delta : \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g}) \otimes \mathsf{U}(\mathfrak{g})$, a counit $\varepsilon : \mathsf{U}(\mathfrak{g}) \to \mathbb{K}$, and an antipode $S : \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$. In order to see this, recall first that the free algebra $\mathsf{T}(\mathfrak{g})$ carries a Hopf algebra structure given by the shuffle comultiplication $\Delta_{sh} : \mathsf{T}(\mathfrak{g}) \to \mathsf{T}(\mathfrak{g}) \otimes \mathsf{T}(\mathfrak{g})$, the canonical projection $\varepsilon_{\mathsf{T}(\mathfrak{g})} : \mathsf{T}(\mathfrak{g}) \to \mathbb{K} = \mathsf{T}^0(\mathfrak{g})$, and the $\mathbb{K}$–linear map $S_{\mathsf{T}(\mathfrak{g})} : \mathsf{T}(\mathfrak{g}) \to \mathsf{T}(\mathfrak{g})$ given further down: for all $n \in \mathbb{N} \setminus \{0\}$, $x_1, \ldots, x_n \in \mathfrak{g}$ the shuffle comultiplication is given by $\Delta_{sh}(1) = 1 \otimes 1$, $\Delta_{sh}(x) = x \otimes 1 + 1 \otimes x$, and for $n \geqslant 2$:

$$\begin{aligned}\Delta_{sh}(x_1 \otimes \cdots \otimes x_n) &= (x_1 \otimes \cdots \otimes x_n) \otimes 1 + 1 \otimes (x_1 \otimes \cdots \otimes x_n) \\ &\quad + \sum_{r=1}^{n-1} \sum_{\sigma\in \mathrm{Sh}_{r,n-r}} \big(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}\big) \otimes \big(x_{\sigma(r+1)} \otimes \cdots \otimes x_{\sigma(n)}\big)\end{aligned}$$

where a *shuffle permutation* in $\mathrm{Sh}_{r,n-r} \subset \mathfrak{S}_n$ is a permutation satisfying $\sigma(1) < \sigma(2) < \cdots < \sigma(r)$ and $\sigma(r+1) < \sigma(r+2) < \cdots < \sigma(n)$. The antipode is given by

$$S_{\mathsf{T}(\mathfrak{g})}(1) \coloneqq 1, \quad S_{\mathsf{T}(\mathfrak{g})}(x_1 \otimes \cdots \otimes x_n) \coloneqq (-1)^n x_n \otimes x_{n-1} \otimes \cdots \otimes x_2 \otimes x_1.$$

By the freeness of $\mathsf{T}(\mathfrak{g})$ all the three maps Δ_{sh}, $\epsilon_{\mathsf{T}(\mathfrak{g})}$ and $S_{\mathsf{T}(\mathfrak{g})}$ are morphisms of algebras uniquely induced on generators $x \in \mathfrak{g}$ by $\Delta_{sh}(x) = x \otimes 1 + 1 \otimes x$, by $\varepsilon_{\mathsf{T}(\mathfrak{g})}(x) = 0$, and by $S_{\mathsf{T}(\mathfrak{g})}(x) = -x$. It turns out that all these maps pass to the quotient $\pi_{\mathfrak{g}} : \mathsf{T}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$ to define the corresponding maps $\Delta : \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g}) \otimes \mathsf{U}(\mathfrak{g})$, $\varepsilon : \mathsf{U}(\mathfrak{g}) \to \mathbb{K}$, and $S : \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$ since the ideal $\mathcal{I}(\mathfrak{g})$ is annihilated by $\varepsilon_{\mathsf{T}(\mathfrak{g})}$, stable by $S_{\mathsf{T}(\mathfrak{g})}$ and sent to the canonical image of $\mathcal{I}(\mathfrak{g}) \otimes 1 + 1 \otimes \mathcal{I}(\mathfrak{g})$ in $\mathsf{T}(\mathfrak{g}) \otimes \mathsf{T}(\mathfrak{g})$ by Δ_{sh}. In other words, the surjective morphism of unital algebras $\pi_{\mathfrak{g}}$ is a morphism of Hopf algebras, i.e. we have in addition

$$\Delta \circ \pi_{\mathfrak{g}} = (\pi_{\mathfrak{g}} \otimes \pi) \circ \Delta_{sh}, \quad \varepsilon \circ \pi_{\mathfrak{g}} = \varepsilon_{\mathsf{T}(\mathfrak{g})}, \quad S \circ \pi_{\mathfrak{g}} = \pi \circ S_{\mathsf{T}(\mathfrak{g})}.$$

4.4 Drinfeld–Yetter modules

Definition 4.4.1. *Let $(\mathfrak{b}, [\cdot,\cdot], \delta)$ be a Lie bialgebra. A* ***Drinfeld–Yetter*** $\mathfrak{b}$***–module***[1] *is a triple (V, π, π^*), where:*

- *the pair (V, π) is a left Lie $\mathfrak{b}$–module (see Definition 4.1.2);*
- *the pair (V, π^*) is a right Lie $\mathfrak{b}$–comodule (see Definition 4.1.5);*

[1] In [EK98] and [Šev16] Drinfeld–Yetter modules are called dimodules

- *the following compatibility condition in $\mathfrak{b}\otimes V$ is satisfied (where we omit associators for brevity):*

$$\pi^* \circ \pi = (\mathsf{id}_{\mathfrak{b}} \otimes \pi) \circ (\tau_{\mathfrak{b},\mathfrak{b}} \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \pi^*) + ([\cdot,\cdot] \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \pi^*) - (\mathsf{id}_{\mathfrak{b}} \otimes \pi) \circ (\delta \otimes \mathsf{id}_V). \quad (4.4.1)$$

A morphism of Drinfeld–Yetter modules is a map $f : V \to V'$ that is both a morphism of Lie $\mathfrak{b}$–modules and of Lie $\mathfrak{b}$–comodules.

Note that if (V, π, π^*) is a Drinfeld–Yetter $\mathfrak{b}$–module, $x \in \mathfrak{b}$ and $v \in V$, we can write Equation (4.4.1) using Sweedler's notation:

$$\pi^*(\pi(x \otimes v)) = \sum_{[v]} v^{[0]} \otimes x \cdot v^{[1]} + \sum_{[v]} [x, v^{[0]}] \otimes v^{[1]} - \sum_{\langle x \rangle} x' \otimes \pi(x'' \otimes v). \quad (4.4.2)$$

We shall denote the category of Drinfeld–Yetter modules over $\mathfrak{b}$ by $\mathsf{DY}(\mathfrak{b})$.

Remark 4.4.2. *We shall sometimes use the following alternative definition of Drinfeld–Yetter module, which is used by P. Ševera in* [Šev16]. *Let (V, π, π^*) be a Drinfeld–Yetter module and set $\tilde{\rho} := -\pi^*$. Hence, a Drinfeld–Yetter $\mathfrak{b}$–module is equivalent to a triple $(V, \pi, \tilde{\rho})$, where the pair (V, π) is a left Lie $\mathfrak{b}$–module, the pair $(V, \tilde{\rho})$ is a left Lie $\mathfrak{b}^{\mathsf{op}}$–comodule, i.e. it satisfies*

$$(\delta \otimes \mathsf{id}_V) \circ \tilde{\rho} = (\mathsf{id}_{\mathfrak{b}} \otimes \tilde{\rho}) \circ \tilde{\rho} - (\tau_{\mathfrak{b},\mathfrak{b}} \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \tilde{\rho}) \circ \tilde{\rho} \quad (4.4.3)$$

and the maps π and $\tilde{\rho}$ satisfy the following compatibility relation

$$\tilde{\rho} \circ \pi = (\mathsf{id}_{\mathfrak{b}} \otimes \pi) \circ (\tau_{\mathfrak{b},\mathfrak{b}} \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \tilde{\rho}) + ([\cdot,\cdot] \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \tilde{\rho}) + (\mathsf{id}_{\mathfrak{b}} \otimes \pi) \circ (\delta \otimes \mathsf{id}_V) \quad (4.4.4)$$

which we express in Sweedler's notation as

$$\tilde{\rho}(\pi(x \otimes v)) = \sum_{[v]} v^{[0]} \otimes x \cdot v^{[1]} + \sum_{[v]} [x, v^{[0]}] \otimes v^{[1]} + \sum_{\langle x \rangle} x' \otimes \pi(x'' \otimes v). \quad (4.4.5)$$

Proposition 4.4.3. $\mathsf{DY}(\mathfrak{b})$ *is an infinitesimally braided monoidal category, where the associativity, commutativity and unit constraints are the same of $\mathsf{Vect}_{\mathbb{K}}$, and the infinitesimally braiding is (upon a choice of sign depending on the convention used, see Remark 4.4.2)*

$$t^{\mathfrak{b}}_{V,W} = (\mathsf{id}_V \otimes \pi_W)(a_{V,\mathfrak{b},W})(\tau_{\mathfrak{b},V} \otimes \mathsf{id}_W)(\tilde{\rho}_V \otimes \mathsf{id}_W) + (\pi_V \otimes \mathsf{id}_W)(\tau_{V,\mathfrak{b}} \otimes \mathsf{id}_W)(a^{-1}_{V,\mathfrak{b},W})(\mathsf{id}_V \otimes \tilde{\rho}_W). \quad (4.4.6a)$$

$$-t^{\mathfrak{b}}_{V,W} = (\mathsf{id}_V \otimes \pi_W)(a_{V,\mathfrak{b},W})(\tau_{\mathfrak{b},V} \otimes \mathsf{id}_W)(\pi^*_V \otimes \mathsf{id}_W) + (\pi_V \otimes \mathsf{id}_W)(\tau_{V,\mathfrak{b}} \otimes \mathsf{id}_W)(a^{-1}_{V,\mathfrak{b},W})(\mathsf{id}_V \otimes \pi^*_W). \quad (4.4.6b)$$

Proof. Let $(V, \pi_V, \tilde{\rho}_V)$ and $(W, \pi_W, \tilde{\rho}_W)$ be two Drinfeld–Yetter modules. Then it is easy to see that the maps

$$\pi_{V \otimes W} := (\pi_V \otimes \mathsf{id}_W) \circ a^{-1}_{\mathfrak{b},V,W} + (\mathsf{id}_V \otimes \pi_W) \circ a_{V,\mathfrak{b},W} \circ (\tau_{\mathfrak{b},V} \otimes \mathsf{id}_W) \circ a^{-1}_{\mathfrak{b},V,W}$$
$$\tilde{\rho}_{V \otimes W} := a_{\mathfrak{b},V,W} \circ (\tilde{\rho}_V \otimes \mathsf{id}_W) + a_{\mathfrak{b},V,W} \circ (\tau_{V,\mathfrak{b}} \otimes \mathsf{id}_W) \circ a^{-1}_{V,\mathfrak{b},W} \circ (\mathsf{id}_V \otimes \tilde{\rho}_W)$$

define respectively a Drinfeld–Yetter $\mathfrak{b}$–module structure on $V \otimes W$. Next, we show that the map given in (4.4.6a) is indeed an infinitesimal braiding. We first show that each $t^{\mathfrak{b}}_{V,W}$ is a morphism of Drinfeld–Yetter $\mathfrak{b}$–modules. Let $v \in V$, $w \in W$, $x \in \mathfrak{b}$. Using Sweedler's notations we write $\tilde{\rho}_V(v) = \sum_{[v]} v^{[0]} \otimes v^{[1]}$ and $\tilde{\rho}_W(w) = \sum_{[w]} w^{[0]} \otimes w^{[1]}$, and then

$$t^{\mathfrak{b}}_{V,W}(v \otimes w) = \sum_{[w]} \left(w^{[0]} \cdot v\right) \otimes w^{[1]} + \sum_{[v]} v^{[1]} \otimes \left(v^{[0]} \cdot w\right). \quad (4.4.7)$$

Therefore, using equation (4.4.2) we get

$$\begin{aligned}
t^{\mathfrak{b}}_{V,W}(x\cdot(v\otimes w)) &= t^{\mathfrak{b}}_{V,W}(x\cdot v\otimes w)+t^{\mathfrak{b}}_{V,W}(v\otimes x\cdot w)\\
&=\sum_{[w]}\big(w^{[0]}\cdot(x\cdot v)\big)\otimes w^{[1]}+\sum_{[x\cdot v]}(x\cdot v)^{[1]})\otimes\big((x\cdot v)^{[0]}\cdot w\big)\\
&+\sum_{[x\cdot w]}\big((x\cdot w)^{[0]}\cdot v\big)\otimes(x\cdot w)^{[1]}+\sum_{[v]}v^{[1]}\otimes\big(v^{[0]}\cdot(x\cdot w)\big)\\
&=\underline{\sum_{[w]}\big(w^{[0]}.(x.v)\big)\otimes w^{[1]}}+\sum_{[v]}\big(x\cdot(v^{[1]})\big)\otimes\big(v^{[0]}\cdot w\big)\\
&+\underline{\sum_{[v]}v^{[1]}\otimes\big([x,v^{[0]}]\cdot w\big)}+\underline{\sum_{\langle x\rangle}(x''\cdot v)\otimes(x'\cdot w)}\\
&+\underline{\sum_{[v]}v^{[1]}\otimes\big(v^{[0]}\cdot(x\cdot w)\big)}+\underline{\sum_{[w]}\big([x,w^{[0]}]\cdot v\big)\otimes w^{[1]}}\\
&+\sum_{[w]}\big(w^{[0]}\cdot v\big)\otimes(x\cdot w^{[1]})+\underline{\sum_{\langle x\rangle}(x'\cdot v)\otimes(x''\cdot w)}.
\end{aligned}$$

The wavy underlined terms cancel because of the antisymmetry of the cobracket δ. The dashed underlined terms and the dotted underlined terms simplify, using Equation (4.1.3), to

$$\underline{\sum_{[w]}\big(x\cdot(w^{[0]}\cdot v)\big)\otimes w^{[1]}}\quad\text{and}\quad\underline{\sum_{[v]}v^{[1]}\otimes\big(x\cdot(v^{[0]}\cdot w)\big)}.$$

Hence we obtain that

$$\begin{aligned}
t^{\mathfrak{b}}_{V,W}(x\cdot(v\otimes w)) &= \sum_{[w]}\big(x.(w^{[0]}.v)\big)\otimes w^{[1]}+\sum_{[w]}\big(w^{[0]}\cdot v\big)\otimes(x\cdot w^{[1]})\\
&+\sum_{[v]}\big(x\cdot(v^{[1]})\big)\otimes\big(v^{[0]}\cdot w\big)+\sum_{[v]}v^{[1]}\otimes\big(x\cdot(v^{[0]})\cdot w\big)\\
&= x\cdot t^{\mathfrak{b}}_{V,W}(v\otimes w)
\end{aligned}$$

i.e. that each $t^{\mathfrak{b}}_{V,W}$ is a morphism of Lie $\mathfrak{b}$-modules. Next, for $v\in V$ and $w\in W$ we compute

$$\begin{aligned}
\tilde{\rho}_{V,W}\circ t^{\mathfrak{b}}_{V,W}(v\otimes w) &= \underline{\underline{\sum_{[w]}\sum_{[w^{[0]}\cdot v]}\big(w^{[0]}\cdot v\big)^{[0]}\otimes\Big(\big(w^{[0]}\cdot v\big)^{[1]}\otimes w^{[1]}\Big)}}+\sum_{[w]}\sum_{[w^{[1]}]}\big(w^{[1]}\big)^{[0]}\otimes\Big(\big(w^{[0]}\cdot v\big)\otimes\big(w^{[1]}\big)^{[1]}\Big)\\
&+\sum_{[v]}\sum_{[v^{[1]}]}\big(v^{[1]}\big)^{[0]}\otimes\Big(\big(v^{[1]}\big)^{[1]}\otimes\big(v^{[0]}\cdot w\big)\Big)+\underline{\underline{\sum_{[v]}\sum_{[v^{[0]}\cdot w]}\big(v^{[0]}\cdot w\big)^{[0]}\otimes\Big(v^{[1]}\otimes\big(v^{[0]}\cdot w\big)^{[1]}\Big)}}.
\end{aligned}$$

Upon using Equation (4.4.5) we compute the sum S of the two double underlined terms

$$\begin{aligned} S \;=\; & \underset{\rule{0pt}{0pt}}{\underbrace{\sum_{[w]}\sum_{[v]}\left[w^{[0]},v^{[0]}\right]\otimes\left(v^{[1]}\otimes w^{[1]}\right)}} + \sum_{[w]}\sum_{[v]} v^{[0]}\otimes\left((w^{[0]}\cdot v^{[1]})\otimes w^{[1]}\right) \\ & + \sum_{[w]}\sum_{\langle w^{[0]}\rangle} w^{[0]\prime}\otimes\left((w^{[0]\prime\prime}\cdot v)\otimes w^{[1]}\right) \\ & + \underbrace{\sum_{[v]}\sum_{[w]}\left[v^{[0]},w^{[0]}\right]\otimes\left(v^{[1]}\otimes w^{[1]}\right)} + \sum_{[v]}\sum_{[w]} w^{[0]}\otimes\left(v^{[1]}\otimes(v^{[0]}\cdot w^{[1]})\right) \\ & + \sum_{[v]}\sum_{\langle v^{[0]}\rangle} v^{[0]\prime}\otimes\left(v^{[1]}\otimes(v^{[0]\prime\prime}\cdot w)\right) \end{aligned}$$

and the sum of the two wavy underlined terms cancels thanks to the antisymmetry of the Lie bracket. We thus get

$$\begin{aligned} \pi^*_{V,W}\circ t^{\mathfrak{b}}_{V,W}(v\otimes w) = & \underline{\sum_{[w]}\sum_{[v]} v^{[0]}\otimes\left((w^{[0]}\cdot v^{[1]})\otimes w^{[1]}\right)} + \underline{\sum_{[w]}\sum_{\langle w^{[0]}\rangle} w^{[0]\prime}\otimes\left((w^{[0]\prime\prime}\cdot v)\otimes w^{[1]}\right)} \\ & + \underline{\sum_{[v]}\sum_{[w]} w^{[0]}\otimes\left(v^{[1]}\otimes(v^{[0]}\cdot w^{[1]})\right)} + \underline{\sum_{[v]}\sum_{\langle v^{[0]}\rangle} v^{[0]\prime}\otimes\left(v^{[1]}\otimes(v^{[0]\prime\prime}\cdot w)\right)} \\ & + \underline{\sum_{[w]}\sum_{[w^{[1]}]} (w^{[1]})^{[0]}\otimes\left((w^{[0]}\cdot v)\otimes(w^{[1]})^{[1]}\right)} + \underline{\sum_{[v]}\sum_{[v^{[1]}]} (v^{[1]})^{[0]}\otimes\left((v^{[1]})^{[1]}\otimes(v^{[0]}\cdot w)\right)}. \end{aligned} \tag{4.4.8}$$

On the other hand we compute

$$\begin{aligned} \mathsf{id}_{\mathfrak{b}}\otimes t^{\mathfrak{b}}_{V,W}(v\otimes w) = & (\mathsf{id}_b\otimes t^b_{V,W})\left(\sum_{[v]} v^{[0]}\otimes\left(v^{[1]}\otimes w\right) + \sum_{[w]} w^{[0]}\otimes\left(v\otimes w^{[1]}\right)\right) \\ = & \underline{\sum_{[v]}\sum_{[w]} v^{[0]}\otimes\left((w^{[0]}\cdot v^{[1]})\otimes w^{[1]}\right)} + \underline{\sum_{[v]}\sum_{[v^{[1]}]} v^{[0]}\otimes\left((v^{[1]})^{[1]}\otimes((v^{[1]})^{[0]}\cdot w)\right)} \\ & + \underline{\sum_{[w]}\sum_{[v]} w^{[0]}\otimes\left(v^{[1]}\otimes(w^{[0]}\cdot w^{[1]})\right)} + \underline{\sum_{[w]}\sum_{[w^{[1]}]} w^{[0]}\otimes\left(((w^{[1]})^{[0]}\cdot v)\otimes((w^{[1]})^{[1]})\right)} \end{aligned} \tag{4.4.9}$$

Therefore, in the difference of the two preceding equations, (4.4.9) − (4.4.8), the normally underlined terms cancel, and the dashed and dotted underlined terms also cancel, respectively, thanks to Equation (4.4.3). It follows that each $t^{\mathfrak{b}}_{V,W}$ is a morphism of Drinfeld–Yetter $\mathfrak{b}$–modules.
Next we show that $t^{\mathfrak{b}}$ is natural: let $(V', \pi_{V'}, \tilde{\rho}_{V'})$ and $(W', \pi_{W'}, \tilde{\rho}_{W'})$ two other Drinfeld–Yetter $\mathfrak{b}$–modules, and let $\phi : V \to V'$, $\psi : W \to W'$ two morphisms of Drinfeld–Yetter $\mathfrak{b}$–modules. We

compute the naturality of the second summand of Equation (4.4.6a):

$$\begin{aligned}
&(\pi_{V'}\otimes \mathsf{id}_{W'})\circ(\tau_{V',\mathfrak{b}}\otimes \mathsf{id}_{W'})\circ a^{-1}_{V',\mathfrak{b},W'}\circ(\mathsf{id}_{V'}\otimes\tilde\rho_{W'})\circ(\phi\otimes\psi)\\
&=(\pi_{V'}\otimes \mathsf{id}_{W'})\circ(\tau_{V',\mathfrak{b}}\otimes \mathsf{id}_{W'})\circ((\phi\otimes\mathsf{id}_{\mathfrak{b}})\otimes\psi)\circ a^{-1}_{V,\mathfrak{b},W}\circ(\mathsf{id}_{V}\otimes\tilde\rho_{W})\\
&=(\pi_{V'}\otimes \mathsf{id}_{W'})\circ((\mathsf{id}_{\mathfrak{b}}\otimes\phi)\otimes\psi)\circ(\tau_{V,\mathfrak{b}}\otimes \mathsf{id}_{W})\circ a^{-1}_{V,\mathfrak{b},W}\circ(\mathsf{id}_{V}\otimes\tilde\rho_{W})\\
&=(\phi\otimes\psi)\circ(\pi_{V}\otimes \mathsf{id}_{W})\circ(\tau_{V,\mathfrak{b}}\otimes \mathsf{id}_{W})\circ a^{-1}_{V,\mathfrak{b},W}\circ(\mathsf{id}_{V}\otimes\tilde\rho_{W})
\end{aligned}$$

where we used the definition of morphism of Drinfeld–Yetter modules and the naturality of the associativity and of the commutativity constraints. The first summand in Equation (4.4.6a) is shown to be natural in an analogous manner. This shows naturality of $t^{\mathfrak{b}}$.
Next, we show the symmetry property (2.3.1b) for $t^{\mathfrak{b}}$: for all $v\in V$ and $w\in W$ we get

$$\begin{aligned}
(t^{\mathfrak{b}}_{W,V}\circ\tau_{V,W})(v\otimes w)&=t^{b}_{W,V}(w\otimes v)\\
&=\sum_{[v]}\left(v^{[0]}\cdot w\right)\otimes v^{[1]}+\sum_{[w]}w^{[1]}\otimes\left(w^{[0]}\cdot v\right)\\
&=\beta_{V_1,V_2}\left(\sum_{[w]}\left(w^{[0]}\cdot v\right)\otimes w^{[1]}+\sum_{[v]}v^{[1]}\otimes\left(v^{[0]}\cdot w\right)\right)\\
&=(\tau_{V,W}\circ t^{\mathfrak{b}}_{V,W})(v\otimes w)
\end{aligned}$$

Finally, we show the property (2.3.1a) for $t^{\mathfrak{b}}$. Given U,V,W in $\mathsf{DY}(\mathfrak{b})$, $u\in U$, $v\in V$ and $w\in W$ we get

$$\begin{aligned}
t^{\mathfrak{b}}_{U\otimes V,W}((u\otimes v)\otimes w)&=\sum_{[w]}\left(w^{[0]}\cdot(u\otimes v)\right)\otimes w^{[1]}+\sum_{[u\otimes v]}(u\otimes v)^{[1]}\otimes\left((u\otimes v)^{[0]}\cdot w\right)\\
&=\underbrace{\sum_{[w]}\left((w^{[0]}\cdot u)\otimes v\right)\otimes w^{[1]}}_{\text{dashed}}+\underbrace{\sum_{[w]}\left(u\otimes(w^{[0]}\cdot v)\right)\otimes w^{[1]}}_{\text{dotted}}\\
&\quad+\underbrace{\sum_{[u]}\left(u^{[1]}\otimes v\right)\otimes\left(u^{[0]}\cdot w\right)}_{\text{dashed}}+\underbrace{\sum_{[v]}\left(u\otimes v^{[1]}\right)\otimes\left(v^{[0]}\cdot w\right)}_{\text{dotted}},
\end{aligned}$$

and it easily seen that the sum of the dashed underlined terms give the second summand in Equation (4.4.6a), and the sum of the dotted underlined terms give the first summand in Equation (4.4.6a). □

We end this Section with the following

Theorem 4.4.4. *Let $\mathfrak{b}$ be a finite–dimensional Lie bialgebra, V a vector space and $\pi:\mathfrak{b}\otimes V\to V$ and $\pi^*:V\to\mathfrak{b}\otimes V$ be two linear maps. Then the triple (V,π,π^*) is a Drinfeld–Yetter $\mathfrak{b}$–module if and only if the maps π,π^* induce a left Lie $\mathfrak{d}_{\mathfrak{b}}$–module structure on V.*

Proof. Let $\rho:\mathfrak{b}\to\mathsf{Hom}_{\mathbb{K}}(V,V)$ be the linear map defined by $\rho(x):=\pi(x\otimes\cdot)$ and consider the following linear map

$$\begin{aligned}
\vartheta:\mathfrak{d}_{\mathfrak{b}}&\to\mathsf{Hom}_{\mathbb{K}}(V,V)\\
\mathfrak{b}\ni x&\mapsto\rho(x)\\
\mathfrak{b}^*\ni y&\mapsto(y\otimes\mathsf{id}_V)\circ\pi^*
\end{aligned}$$

We claim that (V,ϑ) is a left Lie $\mathfrak{d}_\mathfrak{b}$–module if and only if (V,π,π^*) is a Drinfeld–Yetter $\mathfrak{b}$–module. By definition, (V,ϑ) is a left Lie $\mathfrak{d}_\mathfrak{b}$–module if and only if the following identity holds for any $x,y\in\mathfrak{d}_\mathfrak{b}$ and $v\in V$:

$$\vartheta(x)\big(\vartheta(y)(v)\big)-\vartheta(y)\big(\vartheta(x)(v)\big)-\vartheta([x,y])(v)=0. \tag{4.4.10}$$

We compute the left hand side of (4.4.10) in three cases:

- $x,y\in\mathfrak{b}$: it is clear that (4.4.10) holds if and only if (V,π) is a left Lie $\mathfrak{b}$–module.
- $x\in\mathfrak{b}$ and $y\in\mathfrak{b}^*$: recalling the formula of the mixed bracket of $\mathfrak{d}_\mathfrak{b}$, we have

$$\begin{aligned}\mathsf{LHS}\text{ of (4.4.10)} &= \vartheta(x)\big(\vartheta(y)(v)\big)-\vartheta(y)\big(\vartheta(x)(v)\big)-\vartheta([x,y])(v)\\ &=\rho(x)\big((y\otimes\mathsf{id}_V)(\pi^*(v))\big)-(y\otimes\mathsf{id}_V)\big(\pi^*(\rho(x)(v))\big)\\ &+\vartheta(y\circ\mathsf{ad}_x)(v)-\vartheta\big((y\otimes\mathsf{id}_\mathfrak{b})(\delta(x))\big)(v)\\ &=(y\otimes\mathsf{id}_V)\big(\mathsf{id}_\mathfrak{b}\otimes\rho(x)\big)(\pi^*(v))-(y\otimes\mathsf{id}_V)\big(\pi^*(\rho(x)(v))\big)\\ &+(y\circ\mathsf{ad}_x\otimes\mathsf{id})(\pi^*(v))-\rho\big((y\otimes\mathsf{id}_\mathfrak{b})(\delta(x))\big)(v)\\ &=(y\otimes\mathsf{id}_V)\big((\mathsf{id}_\mathfrak{b}\otimes\pi)\circ(12)\circ(\mathsf{id}_\mathfrak{b}\otimes\pi^*)\big)(x\otimes v)-(y\otimes\mathsf{id}_V)(\pi^*\circ\pi)(x\otimes v)\\ &+(y\otimes\mathsf{id}_V)\big(([\cdot,\cdot]\circ\mathsf{id}_V)\circ(\mathsf{id}_\mathfrak{b}\otimes\pi^*)\big)(x\otimes v)-(y\otimes\mathsf{id}_V)\big((\mathsf{id}_\mathfrak{b}\otimes\pi)\circ(\delta\otimes\mathsf{id}_V)\big)(x\otimes v).\end{aligned}$$

  Therefore (4.4.10) holds if and only if the compatibility relation (4.4.1) is satisfied.

- $x,y\in\mathfrak{b}^*$: recalling the formula of the Lie bracket of the dual of a finite–dimensional Lie coalgebra, we have

$$\begin{aligned}\mathsf{LHS}\text{ of (4.4.10)} &= \vartheta(x)\big(\vartheta(y)(v)\big)-\vartheta(y)\big(\vartheta(x)(v)\big)-\vartheta([x,y])(v)\\ &=(x\otimes\mathsf{id}_V)\big(\pi^*((y\otimes\mathsf{id}_V)(\pi^*(v)))\big)-(y\otimes\mathsf{id}_V)\big(\pi^*((x\otimes\mathsf{id}_V)(\pi^*(v)))\big)\\ &-([x,y]\otimes\mathsf{id}_V)(\pi^*(v))\\ &=(y\otimes x\otimes\mathsf{id}_V)(\mathsf{id}_\mathfrak{b}\otimes\pi^*)(\pi^*(v))-(x\otimes y\otimes\mathsf{id}_V)(\mathsf{id}_\mathfrak{b}\otimes\pi^*)(\pi^*(v))\\ &+(x\otimes y\otimes\mathsf{id}_V)(\delta\otimes\mathsf{id}_V)(\pi^*(v))\\ &=(x\otimes y\otimes\mathsf{id}_V)(\tau\otimes\mathsf{id}_V)(\mathsf{id}_\mathfrak{b}\otimes\pi^*)(\pi^*(v))-(x\otimes y\otimes\mathsf{id}_V)(\mathsf{id}_\mathfrak{b}\otimes\pi^*)(\pi^*(v))\\ &+(x\otimes y\otimes\mathsf{id}_V)(\delta\otimes\mathsf{id}_V)(\pi^*(v))\end{aligned}$$

  Therefore (4.4.10) holds if and only if (V,π^*) is a right Lie $\mathfrak{b}$–comodule, i.e. if Equation (4.1.7) holds.

□

In particular, it can be shown that the Theorem above realizes an equivalence of monoidal categories $\mathsf{DY}(\mathfrak{b})\cong\mathsf{Mod}(\mathfrak{d}_\mathfrak{b})$, inducing an infinitesimally braided monoidal structure on the latter category.

4.5 The Drinfeld–Yetter module structure of the universal enveloping algebra

In this Section we present, given a Lie bialgebra $\mathfrak{b}$, the Drinfeld–Yetter $\mathfrak{b}$–module structure of $\mathsf{U}(\mathfrak{b})$. We shall need the following

Lemma 4.5.1. *Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra, and let $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ be a linear map satisfying Equations (4.1.4) and (4.1.9). Let V, W be two left Lie $\mathfrak{g}$–modules together with two maps $\pi_V^* : V \to \mathfrak{g} \otimes V$ and $\pi_W^* : \mathfrak{g} \to \mathfrak{g} \otimes W$ both satisfying Equation (4.4.1). Then for any morphism of left Lie $\mathfrak{g}$–modules $\phi : V \to W$ the obstruction map*

$$\Xi_{V,W}^{\phi} \coloneqq \pi_W^* \circ \phi - (\mathsf{id}_{\mathfrak{g}} \otimes \phi) \circ \pi_V^* : V \to \mathfrak{g} \otimes W \tag{4.5.1}$$

is a morphism of Lie $\mathfrak{g}$–modules.

Proof. For any $x \in \mathfrak{g}$ and $v \in V$ we have, using Equation (4.4.2)

$$\begin{aligned}
\Xi_{V,W}^{\phi}(x \cdot v) &= \Xi_{V,W}^{\phi}(\pi_V(x \otimes v)) = (\pi_W^* \circ \phi)(\pi_V(x \otimes v)) - \big((\mathsf{id}_{\mathfrak{g}} \otimes \phi) \circ \pi_V^*\big)(\pi_V(x \otimes v)) \\
&= \pi_W^*(\pi_W(x \otimes \phi(v))) - \big((\mathsf{id}_{\mathfrak{g}} \otimes \phi) \circ \pi_V^*\big)(\pi_V(x \otimes v)) \\
&= \sum_{[v]} \phi(v)^{[0]} \otimes x \cdot \phi(v)^{[1]} + \sum_{[v]} [x, \phi(v)^{[0]}] \otimes \phi(v)^{[1]} - \sum_{\langle x \rangle} x' \otimes x'' \cdot \phi(v) \\
&- \sum_{[v]} v^{[0]} \otimes \phi(x \cdot v^{[1]}) - \sum_{[v]} [x, v^{[0]}] \otimes \phi(v^{[1]}) + \sum_{\langle x \rangle} x'' \otimes \phi(x'' \cdot v) \\
&= x \cdot \big(\pi_W^* \circ \phi\big)(v) - \sum_{\langle x \rangle} x' \otimes x'' \cdot \phi(v) \\
&- x \cdot \big((\mathsf{id}_{\mathfrak{g}} \otimes \phi) \circ \pi_v^*\big)(v) + \sum_{\langle x \rangle} x'' \otimes \phi(x'' \cdot v) \\
&= x \cdot \Xi_{V,W}^{\phi}(v).
\end{aligned}$$

□

We can now prove the main result of this Section:

Theorem 4.5.2. *Let $(\mathfrak{b}, [\cdot,\cdot], \delta)$ be a Lie bialgebra. Then*

(i) There exists a unique Drinfeld–Yetter $\mathfrak{b}$–module structure on $\mathsf{U}(\mathfrak{b})$ such that for all $x \in \mathfrak{b}$ and $u \in \mathsf{U}(\mathfrak{b})$

$$\pi(x \otimes u) = xu \quad \text{and} \quad \pi^*(1) = 0. \tag{4.5.2}$$

In particular, its right Lie coaction satisfies $\pi^*(x) = -\delta(x)$ for all $x \in \mathfrak{b}$.

(ii) The comultiplication $\Delta : \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g}) \otimes \mathsf{U}(\mathfrak{g})$ is a morphism of Drinfeld–Yetter $\mathfrak{b}$–modules.

(iii) $\mathsf{U}(\mathfrak{b})$ is an infinitesimally braided cocommutative comonoid in $\mathsf{DY}(\mathfrak{b})$, i.e. $t^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})} \circ \Delta = 0$.

(iv) If $(\mathfrak{b}', [\cdot,\cdot]', \delta')$ is another Lie bialgebra and $\varphi : \mathfrak{b} \to \mathfrak{b}'$ is a morphism of Lie bialgebras, the induced morphism of Hopf algebras $\mathsf{U}(\varphi) : \mathsf{U}(\mathfrak{b}) \to \mathsf{U}(\mathfrak{b}')$ satisfies

$$\mathsf{U}(\varphi)(x \cdot u) = \varphi(x) \cdot \mathsf{U}(\varphi)(u) \qquad \textit{and} \qquad (\pi')^*(\mathsf{U}(\varphi)(x)) = (\varphi \otimes \mathsf{U}(\varphi))(\pi^*(u)). \tag{4.5.3}$$

Proof. (*i*): We first prove uniqueness. Let π_1^* and π_2^* be two linear maps $\mathsf{U}(\mathfrak{b}) \to \mathfrak{b} \otimes \mathsf{U}(\mathfrak{b})$ satisfying Equations (4.5.2), (4.1.7) and (4.4.1) and set $\phi \coloneqq \pi_1^* - \pi_2^*$. Then we have $\phi(1) = \pi_1^*(1) - \pi_2^*(1) =$

$0 - 0 = 0$. Next, for any $x \in \mathfrak{b}$ we have

$$\begin{aligned}
\phi(x) &= \pi_1^*(x) - \pi_2^*(x) \\
&= \pi_1^*(x \cdot 1) - \pi_2^*(x \cdot 1) \\
&= \pi_1^*(\pi(x \otimes 1)) - \pi_2^*(\pi(x \otimes 1)) \\
&= \big((\mathsf{id}_{\mathfrak{b}} \otimes \pi) \circ (\tau_{\mathfrak{b},\mathfrak{b}} \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \pi_1^*) + ([\cdot,\cdot] \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \pi_1^*) - (\mathsf{id}_{\mathfrak{b}} \otimes \pi) \circ (\delta \otimes \mathsf{id}_V)\big)(x \otimes 1) \\
&- \big((\mathsf{id}_{\mathfrak{b}} \otimes \pi) \circ (\tau_{\mathfrak{b},\mathfrak{b}} \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \pi_2^*) + ([\cdot,\cdot] \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \pi_2^*) - (\mathsf{id}_{\mathfrak{b}} \otimes \pi) \circ (\delta \otimes \mathsf{id}_V)\big)(x \otimes 1) \\
&= 0.
\end{aligned}$$

Therefore, by induction on the lenght $n(u)$ of u induced by the standard filtration of $\mathsf{U}(\mathfrak{b})$ we have, denoting $x_1 \cdot x \coloneqq x_1 \cdot x_2 \cdots x_{n+1}$ for any $x_1, \dots, x_{n+1} \in \mathfrak{b}$:

$$\begin{aligned}
\phi(x_1 \cdot x) &= \pi_1^*(x_1 \cdot x) - \pi_2^*(x_1 \cdot x) \\
&= \pi_1^*(\pi(x_1 \otimes x)) - \pi_2^*(\pi(x_1 \otimes x)) \\
&= \big((\mathsf{id}_{\mathfrak{b}} \otimes \pi) \circ (\tau_{\mathfrak{b},\mathfrak{b}} \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \pi_1^*) + ([\cdot,\cdot] \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \pi_1^*) - (\mathsf{id}_{\mathfrak{b}} \otimes \pi) \circ (\delta \otimes \mathsf{id}_V)\big)(x_1 \otimes x) \\
&- \big((\mathsf{id}_{\mathfrak{b}} \otimes \pi) \circ (\tau_{\mathfrak{b},\mathfrak{b}} \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \pi_2^*) + ([\cdot,\cdot] \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \pi_2^*) - (\mathsf{id}_{\mathfrak{b}} \otimes \pi) \circ (\delta \otimes \mathsf{id}_V)\big)(x_1 \otimes x) \\
&= 0,
\end{aligned}$$

giving the uniqueness. In order to prove existence, we use the notation $\tilde{\rho} = -\pi^*$, see 4.4.2. For each integer $n \in \mathbb{N}$ we define a $\mathbb{K}$–linear map $\check{\rho}_n : \mathsf{T}^n(\mathfrak{b}) \to \mathfrak{b} \otimes \mathsf{U}(\mathfrak{b})$ in the following way for all $x, x_1, \dots, x_n, x_{n+1} \in \mathfrak{b}$:

$$\begin{aligned}
\check{\rho}_0(1) &\coloneqq 0 \\
\check{\rho}_1(x) &\coloneqq \delta(x) \\
\check{\rho}_2(x_1 \otimes x_2) &\coloneqq \sum_{\langle x_2 \rangle} [x_1, x_2'] \otimes x_2'' + \sum_{\langle x_2 \rangle} x_2' \otimes (x_1 x_2'') + \sum_{\langle x_1 \rangle} x_1' \otimes (x_1'' x_2) \\
\check{\rho}_{n+1}\big(x_1 \otimes \cdots \otimes x_{n+1}\big) &\coloneqq x_1 \cdot \big(\check{\rho}_n\big(x_2 \otimes \cdots \otimes x_{n+1}\big)\big) + \sum_{\langle x_1 \rangle} x_1' \otimes (x_1'' x_2 \cdots x_{n+1}).
\end{aligned}$$

All these maps are well–defined by the universal property of the tensor product since (by induction over n) the corresponding right hand sides are multilinear in the variables $x_1, \dots, x_{n+1}$. Since $\mathsf{T}(\mathfrak{b})$ is a direct sum of all the $\mathsf{T}^n(\mathfrak{b})$'s the above sequence of maps defines a well-defined $\mathbb{K}$–linear map $\check{\rho} : \mathsf{T}(\mathfrak{b}) \to \mathfrak{b} \otimes \mathsf{U}(\mathfrak{b})$ whose restriction to each submodule $\mathsf{T}^n(\mathfrak{b})$ equals $\check{\rho}_n$. Clearly, the map $\check{\rho}$ satisfies the following equation for all $x \in \mathfrak{b}$ and $v \in \mathsf{T}(\mathfrak{b})$ (where $\pi_{\mathfrak{b}} : \mathsf{T}(\mathfrak{b}) \to \mathsf{U}(\mathfrak{b})$ denotes the canonical projection):

$$\check{\rho}(x \otimes v) = x \cdot (\check{\rho}(v)) + \sum_{\langle x \rangle} x' \otimes \big(x'' \pi_{\mathfrak{b}}(v)\big). \tag{4.5.4}$$

In order to prove that $\check{\rho}$ vanishes of the ideal $\mathcal{I}(\mathfrak{b})$ we shall first compute the following for all $x, y \in \mathfrak{b}$

and $v \in \mathsf{T}(\mathfrak{b})$

$$
\begin{aligned}
&\check{\rho}\Big(\big(x \otimes y - y \otimes x - [x,y]\big) \otimes v\Big) \\
\overset{(4.5.4)}{=}\;& x \cdot \check{\rho}(y \otimes v) + \sum_{\langle x \rangle} x' \otimes \big(x'' y \pi_{\mathfrak{b}}(v)\big) - y \cdot \check{\rho}(x \otimes v) - \sum_{\langle y \rangle} y' \otimes \big(y'' x \pi_{\mathfrak{b}}(v)\big) \\
& -[x,y] \cdot \check{\rho}(v) - \sum_{\langle [x,y] \rangle} [x,y]' \otimes \big([x,y]'' \pi_{\mathfrak{b}}(v)\big) \\
\overset{(4.5.4)}{=}\;& \underline{x \cdot \big(y \cdot \check{\rho}(v)\big)} + \sum_{\langle y \rangle} [x,y'] \otimes \big(y'' \pi_{\mathfrak{b}}(v)\big) + \sum_{\langle y \rangle} y' \otimes \big(x y'' \pi_{\mathfrak{b}}(v)\big) + \sum_{\langle x \rangle} x' \otimes \big(x'' y \pi_{\mathfrak{b}}(v)\big) \\
& \underline{-y \cdot \big(x \cdot \check{\rho}(v)\big)} - \sum_{\langle x \rangle} [y,x'] \otimes \big(x'' \pi_{\mathfrak{b}}(v)\big) - \sum_{\langle x \rangle} x' \otimes \big(y x'' \pi_{\mathfrak{b}}(v)\big) - \sum_{\langle y \rangle} y' \otimes \big(y'' x \pi_{\mathfrak{b}}(v)\big) \\
& \underline{-[x,y] \cdot \check{\rho}(v)} - \sum_{\langle [x,y] \rangle} [x,y]' \otimes \big([x,y]'' \pi_{\mathfrak{b}}(v)\big) \\
=\;& \sum_{\langle y \rangle} \Big([x,y'] \otimes \big(y'' \pi_{\mathfrak{b}}(v)\big) + y' \otimes \big([x,y''] \pi_{\mathfrak{b}}(v)\big)\Big) - \sum_{\langle x \rangle} \Big([y,x'] \otimes \big(x'' \pi_{\mathfrak{b}}(v)\big) + x' \otimes \big([y,x''] \pi_{\mathfrak{b}}(v)\big)\Big) \\
& - \sum_{\langle [x,y] \rangle} [x,y]' \otimes \big([x,y]'' \pi_{\mathfrak{b}}(v)\big) \\
\overset{(4.1.9)}{=}\;& 0
\end{aligned}
$$

where the underlined terms vanish thanks to Equation (4.1.3). A general element of $\mathcal{I}(\mathfrak{b})$ is a linear combination of elements of the form $X = p \otimes \big(x \otimes y - y \otimes x - [x,y]\big) \otimes v$, with $x, y \in \mathfrak{b}$ and $p, v \in \mathsf{T}(\mathfrak{b})$. In case $p = 1$ the preceding computation shows that $\check{\rho}(X) = 0$. In case $p = z \otimes q$ with $z \in \mathfrak{b}$ and $q \in \mathsf{T}(\mathfrak{b})$ we get by eqn (4.5.4), writing $w = q \otimes \big(x \otimes y - y \otimes x - [x,y]\big) \otimes v$

$$
\check{\rho}(X) = \check{\rho}(z \otimes w) = z \cdot \check{\rho}(w) + \sum_{\langle z \rangle} z' \otimes \big(z'' \pi_{\mathfrak{b}}(w)\big) = z \cdot \check{\rho}(w)
$$

since $w \in \mathcal{I}(\mathfrak{b}) = \ker \pi_{\mathfrak{b}}$. The preceding equation allows to prove by induction over the word length (or the tensor degree) of p that $\check{\rho}$ vanishes on $\mathcal{I}(\mathfrak{b})$. It follows that the map passes to the quotient to define a unique $\mathbb{K}$–linear map $\tilde{\rho} : \mathsf{U}(\mathfrak{b}) \to \mathfrak{b} \otimes \mathsf{U}(\mathfrak{b})$ such that

$$
\check{\rho} = \tilde{\rho} \circ \pi_{\mathfrak{b}}. \tag{4.5.5}
$$

It follows for all $x \in \mathfrak{b}$ and $v \in \mathsf{T}(\mathfrak{b})$:

$$
\tilde{\rho}\big(x \pi_{\mathfrak{b}}(v)\big) \;=\; \check{\rho}(x \otimes v) \overset{(4.5.4)}{=} x \cdot \big(\check{\rho}(v)\big) + \sum_{\langle x \rangle} x' \otimes \big(x'' \pi_{\mathfrak{b}}(v)\big) = x \cdot \big(\tilde{\rho}(\pi_{\mathfrak{b}}(v))\big) + \sum_{\langle x \rangle} x' \otimes \big(x'' \pi_{\mathfrak{b}}(v)\big)
$$

which proves Equation (4.4.4) since $\pi_{\mathfrak{b}}$ is surjective.
In order to prove Equation (4.4.3) for $\tilde{\rho}$ consider the following map

$$
\Psi := (\mathsf{id}_{\mathfrak{b}} \otimes \tilde{\rho}) \circ \tilde{\rho} - (\tau_{\mathfrak{b},\mathfrak{b}} \otimes \mathsf{id}_V) \circ (\mathsf{id}_{\mathfrak{b}} \otimes \tilde{\rho}) \circ \tilde{\rho} - (\delta \otimes \mathsf{id}_V) \circ \tilde{\rho} : \mathsf{U}(\mathfrak{b}) \to \mathfrak{b} \otimes \mathfrak{b} \otimes \mathsf{U}(\mathfrak{b}).
$$

It follows from a lengthy but elementary computation – using the fact that $\tilde{\rho}$ satisfies Equation (4.4.4) and the coJacobi identity (4.1.5) – that Ψ is a morphism of left Lie $\mathfrak{b}$–modules. Since

$\Psi(1) = 0$ (which follows by $\tilde\rho(1) = 0$), and since every $u \in \mathsf{U}(\mathfrak{b})$ is a finite sum of words $1, x_1 \cdots x_n$ where n is a positive integer and $x_1, \dots, x_n \in \mathfrak{b}$ we get

$$\Psi(x_1 \cdots x_n) = (x_1 \cdots x_n).\Psi(1) = 0.$$

Hence $\Psi = 0$ and $\big(\mathsf{U}(\mathfrak{b}), \pi, \tilde\rho\big)$ is a Drinfeld–Yetter $\mathfrak{b}$–module with respect to the convention 4.4.2.
(ii): It is well-known that the comultiplication Δ of $\mathsf{U}(\mathfrak{b})$ is a morphism of left $\mathfrak{b}$–modules: indeed, for all $x \in \mathfrak{b}$ and $u \in \mathsf{U}(\mathfrak{b})$ we have

$$\Delta(x \cdot u) = \Delta(x)\Delta(u) = (x \otimes 1 + 1 \otimes x)\Delta(u) = x \cdot \Delta(u).$$

In order to prove that Δ is a morphism of right Lie $\mathfrak{b}$–comodules, consider the Drinfeld–Yetter $\mathfrak{b}$–modules $\mathsf{U}(\mathfrak{b})$ and $\mathsf{U}(\mathfrak{b}) \otimes \mathsf{U}(\mathfrak{b})$. Then by Lemma 4.5.1 the obstruction map $\Xi^{\Delta}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})\otimes\mathsf{U}(\mathfrak{b})}$ is a morphism of left Lie $\mathfrak{b}$–modules. Moreover, we have

$$\Xi^{\Delta}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})\otimes\mathsf{U}(\mathfrak{b})}(u) = u \cdot \Xi^{\Delta}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})\otimes\mathsf{U}(\mathfrak{b})}(1) = 0,$$

proving the claim.
(iii): We have that $(t^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})} \circ \Delta)(1) = t^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}(1 \otimes 1) = 0$. Since both $t^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}$ and Δ are morphisms of left Lie $\mathfrak{b}$–modules, we have that the composition $t^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})} \circ \Delta$ is so. Therefore, for any $u \in \mathsf{U}(\mathfrak{b})$ we have $(t^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})} \circ \Delta)(u) = u \cdot ((t^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})} \circ \Delta)(1)) = 0$.
(iv): It is well–known that the map $\mathsf{U}(\varphi)$ –by the observation that the ideal $\mathcal{I}(\mathfrak{b})$ is mapped to the corresponding ideal $\mathcal{I}(\mathfrak{b}')$– satisfies the first of the two equations (4.5.3). In order to prove the second, consider the difference $\varPi := \big(\varphi \otimes \mathsf{U}(\varphi)\big) \circ \tilde\rho - \tilde\rho' \circ \mathsf{U}(\varphi)$. As in the proof of Lemma 4.5.1 we have that this difference is *morphism of Lie algebra modules* in the sense that for all $x \in \mathfrak{b}$ and $u \in \mathsf{U}(\mathfrak{b})$: since $\big(\varphi \otimes \mathsf{U}(\varphi)\big) \circ \delta = \delta' \circ \varphi$ we have $\varPi(x \cdot u) = \varphi(x) \cdot \varPi(u)$. It follows that $\varPi(u) = \mathsf{U}(\varphi)(u) \cdot \varPi(1) = 0$ since $\pi^*(1) = 0$, $(\pi')^*(1') = 0$, and $\mathsf{U}(\varphi)(1) = 1'$. □

4.6 Lie bialgebra twists

Notation 4.6.1. *For a vector space V, we shall denote by* Alt *the following maps*

$$\mathrm{Alt}_n(x_1 \otimes \cdots \otimes x_n) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sign}(\sigma)(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}).$$

In particular, $\mathrm{Alt}_2(x_1 \otimes x_2) = x_1 \otimes x_2 - x_2 \otimes x_1$. *Note that if $x_1 \otimes x_2 \otimes x_3$ is antsymmetric, we have*

$$\frac{1}{2}\mathrm{Alt}_3(x_1 \otimes x_2 \otimes x_3) = x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_3 \otimes x_1 + x_3 \otimes x_1 \otimes x_2.$$

For example, the coJacobi condition (4.1.5) *can be written as* $\frac{1}{2}\mathrm{Alt}_3 \circ (\delta \otimes \mathsf{id}) \circ \delta = 0$.

Notation 4.6.2. *Let $\mathfrak{b}$ be a Lie bialgebra and let $j \in \Lambda^2(\mathfrak{b})$. We shall denote by* $\mathrm{CYB}(j, j)$ *the following well-known* ***classical Yang-Baxter term***[2] $\mathrm{CYB}(j, j) \in \Lambda^3(\mathfrak{b})$*:*

$$\mathrm{CYB}(j, j) := \frac{1}{2}\sum_j \sum_j \mathrm{Alt}_3\Big([j^1, j'] \otimes j^2 \otimes j''\Big) = [j_{12}, j_{13}] + [j_{12}, j_{23}] + [j_{13}, j_{23}] \tag{4.6.1}$$

where we used Sweedler's notations

$$j = \sum_j j' \otimes j'' = \sum_j j^1 \otimes j^2.$$

[2]Many authors denote CYB(j,j) by $[[j, j]]$

Notation 4.6.3. *For an element x of a Lie algebra $\mathfrak{g}$, we denote by $\mathsf{ad}_x^{(n)}$ the left Lie $\mathfrak{g}$–module structure of $\mathfrak{g}^{\otimes n}$ given by tensoring n times the adjoint action $\mathsf{ad}_x(\cdot) = [x,\cdot]$. Note that we can rewrite the cocycle condition* (4.1.9)

$$\mathsf{ad}_x^{(2)}(\delta(y)) - \mathsf{ad}_y^{(2)}(\delta(x)) - \delta([x,y]) = 0. \tag{4.6.2}$$

Recall the definition of a Lie bialgebra twist:

Definition 4.6.4. *Let $(\mathfrak{b},[\cdot,\cdot],\delta)$ be a Lie bialgebra. We say that $j \in \Lambda^2(\mathfrak{b})$ is a **Lie bialgebra twist** if*

$$\frac{1}{2}\sum_j \mathrm{Alt}_3\left(\left(\delta(j')\right)\otimes j''\right) - \mathrm{CYB}(j,j) = 0. \tag{4.6.3}$$

Twisting Lie bialgebras produces new Lie bialgebras, as explained in the following

Proposition 4.6.5. *Let $(\mathfrak{b},[\cdot,\cdot],\delta)$ be a Lie bialgebra and $j \in \Lambda^2(\mathfrak{b})$ be a Lie bialgebra twist. Consider the linear map*

$$\begin{aligned}\delta_j : \mathfrak{b} &\to \mathfrak{b}\otimes\mathfrak{b}\\ x &\mapsto \delta(x) + \mathsf{ad}_x^{(2)}(j).\end{aligned}$$

Then the triple $(\mathfrak{b},[\cdot,\cdot],\delta_j)$ is a Lie bialgebra.

Proof. We compute for all $x \in \mathfrak{b}$:

$$\begin{aligned}
\frac{1}{2}\mathrm{Alt}_3\left((\delta_j\otimes \mathsf{id}_{\mathfrak{b}})\circ\left(\delta_j(x)\right)\right) &= \frac{1}{2}\mathrm{Alt}_3\left((\delta\otimes \mathsf{id}_{\mathfrak{b}})\circ\left(\delta(x)\right)\right)\\
&\quad+\frac{1}{2}\mathrm{Alt}_3\left((\delta\otimes \mathsf{id}_{\mathfrak{b}})\circ\left(\mathsf{ad}_x^{(2)}(j)\right)\right) + \frac{1}{2}\mathrm{Alt}_3\left(\sum_{\langle x\rangle}\left(\mathsf{ad}_{x'}^{(2)}(j)\right)\otimes x''\right)\\
&\quad+\frac{1}{2}\mathrm{Alt}_3\left(\left(\mathsf{ad}^{(2)}(j)\otimes \mathsf{id}_{\mathfrak{b}}\right)\circ\left(\mathsf{ad}_x^{(2)}(j)\right)\right)\\
&= 0 + \frac{1}{2}\sum_j \mathrm{Alt}_3\left(\left(\delta\left([x,j']\right)\right)\otimes j''\right) + \frac{1}{2}\sum_j \mathrm{Alt}_3\left(\left(\delta(j')\right)\otimes\left([x,j'']\right)\right)\\
&\quad+\frac{1}{2}\sum_j\sum_{\langle x\rangle}\mathrm{Alt}_3\Big(\left([x',j']\right)\otimes j''\otimes x'' + j'\otimes\left([x',j'']\right)\otimes x''\Big)\\
&\quad+\frac{1}{2}\sum_j\sum_j\mathrm{Alt}_3\Big(\left[[x,j'],j^1\right]\otimes j^2\otimes j'' + j^1\otimes\left[[x,j'],j^2\right]\otimes j''\\
&\qquad\qquad+[j',j^1]\otimes j^2\otimes[x,j''] + j^1\otimes[j',j^2]\otimes[x,j'']\Big).
\end{aligned}$$

And using Equation (4.1.9) and the antisymmetry of δ and j we thus obtain

$$\frac{1}{2}\mathrm{Alt}_3\left((\delta_j\otimes \mathsf{id}_{\mathfrak{b}})\circ\left(\delta_j(x)\right)\right) = \mathsf{ad}_x^{(3)}\Big(\frac{1}{2}\sum_j \mathrm{Alt}_3(\delta(j')\otimes j'') - \mathrm{CYB}(j,j)\Big)$$

where the right hand side vanishes since j is a Lie bialgebra twist. ☐

Notation 4.6.6. *From now on if $\mathfrak{b}$ is a Lie bialgebra and j is a Lie bialgebra twist, we shall denote the twisted Lie bialgebra by $\mathfrak{b}_j$.*

Lemma 4.6.7. *Let $\mathfrak{b}$ be a Lie bialgebra, j be a Lie bialgebra twist and (V, π, π^*) be a Drinfeld–Yetter $\mathfrak{b}$–module. Then the triple (V, π, π_j^*) is a Drinfeld–Yetter $\mathfrak{b}_j$–module, where*

$$\pi_j^*(v) = \pi^*(v) + \sum_j j' \otimes \pi(j'' \otimes v).$$

Equivalently, if $(V, \pi, \tilde{\rho})$ is a Drinfeld–Yetter $\mathfrak{b}$–module (with the convention given in 4.4.2) then the triple $(V, \pi, \tilde{\rho}_j)$ is a Drinfeld–Yetter $\mathfrak{b}_j$–module, where

$$\tilde{\rho}_j(v) = \tilde{\rho}(v) - \sum_j j' \otimes \pi(j'' \otimes v).$$

Proof. We have to prove that (V, π, π_j^*) satisfies Equations (4.1.7) and (4.4.1). For any $x \in \mathfrak{b}$ and $v \in V$ we have

$$\begin{aligned}
&\big(\pi_j^* \circ \pi - (\mathsf{id}_\mathfrak{b} \otimes \pi) \circ (\tau_{\mathfrak{b},\mathfrak{b}} \otimes \mathsf{id}_V) \circ (\mathsf{id}_\mathfrak{b} \otimes \pi_j^*) - ([\cdot,\cdot] \otimes \mathsf{id}_V) \circ (\mathsf{id}_\mathfrak{b} \otimes \pi_j^*) + (\mathsf{id}_\mathfrak{b} \otimes \pi) \circ (\delta_j \otimes \mathsf{id}_V)\big)(x \otimes v)\\
=&\big(\pi^* \circ \pi - (\mathsf{id}_\mathfrak{b} \otimes \pi) \circ (\tau_{\mathfrak{b},\mathfrak{b}} \otimes \mathsf{id}_V) \circ (\mathsf{id}_\mathfrak{b} \otimes \pi^*) - ([\cdot,\cdot] \otimes \mathsf{id}_V) \circ (\mathsf{id}_\mathfrak{b} \otimes \pi^*) + (\mathsf{id}_\mathfrak{b} \otimes \pi) \circ (\delta \otimes \mathsf{id}_V)\big)(x \otimes v)\\
&+\sum_j j' \otimes j'' \cdot (x \cdot v) - \sum_j j' \otimes x \cdot (j'' \cdot v) - \sum_j [x, j'] \otimes j'' \cdot v + \sum_j ([x, j'] \otimes j'' \cdot v + j' \otimes [x, j''] \cdot v)\\
=&0 - \sum_j j' \otimes [x, j''] \cdot v - \sum_j [x, j'] \otimes j'' \cdot v + \sum_j [x, j'] \otimes j'' \cdot v + \sum_j j' \otimes [x, j''] \cdot v\\
=&0
\end{aligned}$$

proving that (V, π, π_j^*) satisfies the compatibility condition (4.4.1) for $\mathfrak{b}_j$. Next, set

$$\begin{aligned}
f_j : V &\to \mathfrak{b} \otimes V\\
v &\mapsto \sum_j j' \otimes \pi(j'' \otimes v).
\end{aligned}$$

For any $v \in V$, the relation (4.1.7) give rise to eight terms:

$$\begin{aligned}
&\big((\delta_j \otimes \mathsf{id}_V) \circ \pi_j^* - (\tau_{\mathfrak{b},\mathfrak{b}} \otimes \mathsf{id}_V) \circ (\mathsf{id}_\mathfrak{b} \otimes \pi_j^*) \circ \pi_j^* + (\mathsf{id}_\mathfrak{b} \otimes \pi_j^*) \circ \pi_j^*\big)(v)\\
=&\big((\delta_j \otimes \mathsf{id}_V) \circ \pi_j^* + (\mathrm{Alt}_2 \otimes \mathsf{id}_V) \circ (\mathsf{id}_\mathfrak{b} \otimes \pi_j^*) \circ \pi_j^*\big)(v)\\
=&((\delta \otimes \mathsf{id}_V) \circ \pi^*)(v) + ((\mathrm{Alt}_2 \otimes \mathsf{id}_V) \circ (\mathsf{id}_\mathfrak{b} \otimes \pi^*) \circ \pi^*)(v)\\
&+((\mathsf{ad}^{(2)}(j) \otimes \mathsf{id}_V) \circ \pi^*)(v) + ((\delta \otimes \mathsf{id}_V) \circ f_j)(v) + ((\mathsf{ad}^{(2)}(j) \otimes \mathsf{id}_V) \circ f_j)(v)\\
&+((\mathrm{Alt}_2 \otimes \mathsf{id}_V) \circ (\mathsf{id}_\mathfrak{b} \otimes f_j) \circ \pi^*)(v) + ((\mathrm{Alt}_2 \otimes \mathsf{id}_V) \circ (\mathsf{id}_\mathfrak{b} \otimes \pi^*) \circ f_j)(v)\\
&+((\mathrm{Alt}_2 \otimes \mathsf{id}_V) \circ (\mathsf{id}_\mathfrak{b} \otimes f_j) \circ f_j)(v).
\end{aligned}$$

The first two terms vanish in view of Equation (4.1.7). The other give

$$\big((\mathsf{ad}^{(2)}(j)\otimes \mathsf{id}_V)\circ\pi^*\big)(v) = \underset{\sim\sim\sim}{\sum_{[v]}\sum_j [v^{[0]}, j']\otimes j''\otimes v^{[1]}} + \underline{\sum_{[v]}\sum_j j'\otimes [v^{[0]}, j'']\otimes v^{[1]}}$$

$$\big((\delta\otimes \mathsf{id}_V)\circ f_j\big)(v) = \sum_j\sum_{j'} (j')^{\langle 1\rangle}\otimes (j')^{\langle 2\rangle}\otimes j''\cdot v$$

$$\big((\mathsf{ad}^{(2)}(j)\otimes \mathsf{id}_V)\circ f_j\big)(v) = \sum_j\sum_j [j', j^1]\otimes j^2\otimes j''\cdot v + \sum_j\sum_j j^1\otimes [j', j^2]\otimes j''\cdot v$$

$$\big((\mathrm{Alt}_2\otimes \mathsf{id}_V)\circ(\mathsf{id}_{\mathfrak{b}}\otimes f_j)\circ\pi^*\big)(v) = \underline{\underline{\sum_{[v]}\sum_j v^{[0]}\otimes j'\otimes j''\cdot v^{[1]}}} - \underset{- - - -}{\sum_{[v]}\sum_j j'\otimes v^{[0]}\otimes j''\cdot v^{[1]}}$$

$$\begin{aligned}\big((\mathrm{Alt}_2\otimes \mathsf{id}_V)\circ(\mathsf{id}_{\mathfrak{b}}\otimes \pi^*)\circ f_j\big)(v) &= \underset{- - - -}{\sum_j\sum_{[v]} j'\otimes v^{[0]}\otimes j''\cdot v^{[1]}} + \underline{\sum_j\sum_{[v]} j'\otimes [j'', v^{[0]}]\otimes v^{[1]}}\\ &+ \sum_j\sum_{j''} j'\otimes (j'')^{\langle 1\rangle}\otimes (j'')^{\langle 2\rangle}\cdot v - \underline{\underline{\sum_j\sum_{[v]} v^{[0]}\otimes j'\otimes j''\cdot v^{[1]}}}\\ &- \underset{\sim\sim\sim}{\sum_j\sum_{[v]} [j'', v^{[0]}]\otimes j'\otimes v^{[1]}} - \sum_j\sum_{j''} (j'')^{\langle 1\rangle}\otimes j'\otimes (j'')^{\langle 2\rangle}\cdot v\end{aligned}$$

$$\begin{aligned}\big((\mathrm{Alt}_2\otimes \mathsf{id}_V)\circ(\mathsf{id}_{\mathfrak{b}}\otimes f_j)\circ f_j\big)(v) &= (\mathrm{Alt}_2\otimes \mathsf{id}_V)\left(\sum_j\sum_j j'\otimes j^1\otimes j^2\cdot (j''\cdot v)\right)\\ &= \sum_j\sum_j j'\otimes j^1\otimes [j^2, j'']\cdot v.\end{aligned}$$

The terms underlined in the same way cancel each other out. The remaning ones gives

$$(\mathsf{id}_{\mathfrak{b}\otimes\mathfrak{b}}\otimes\pi(\cdot, v))\left(\frac{1}{2}\sum_j \mathrm{Alt}_3\left(\big(\delta(j')\big)\otimes j''\right) - \mathrm{CYB}(j,j)\right)$$

which vanishes due to (4.6.3). ☐

Proposition 4.6.8. *Let $(\mathfrak{b}, [\cdot,\cdot], \delta)$ be a Lie bialgebra and j be a Lie bialgebra twist. Then*

$$\begin{aligned}\mathscr{J}_j : \mathsf{DY}(\mathfrak{b}) &\to \mathsf{DY}(\mathfrak{b}_j)\\ (V, \pi, \pi^*) &\mapsto (V, \pi, \pi^*_j)\\ f &\mapsto f\end{aligned}$$

is an invertible infinitesimally braided monoidal functor, whose inverse is $\mathscr{J}_{-j}$.

Proof. The only non–straightforward thing to check in order to prove that $\mathscr{J}_j$ is a functor is that, for any $f\in \mathsf{Hom}_{\mathsf{DY}(\mathfrak{b})}(V,W)$, then we have $f\in \mathsf{Hom}_{\mathsf{DY}(\mathfrak{b}_j)}(V,W)$. This is the case since for any $v\in V$ we have

$$\begin{aligned}(\mathsf{id}_{\mathfrak{b}}\otimes f)\circ(\pi^*_{Vj})(v) &= (\mathsf{id}_{\mathfrak{b}}\otimes f)\circ(\pi^*_V)(v) + \sum_j j'\otimes f(j''\cdot v)\\ &= \pi^*_W(f(v)) + \sum_j j'\otimes j''\cdot f(v)\\ &= \pi^*_{Wj}(f(v)).\end{aligned}$$

Next we define the monoidal structure of $\mathscr{I}_j$. Recall that the ground field is a Drinfeld–Yetter $\mathfrak{b}$–module with trivial π and π^*. We set $\psi^0_{\mathscr{I}_j} = \mathsf{id}_{\mathbb{K}}$ and $\psi^2_{\mathscr{I}_j}(V,W) = \mathsf{id}_{V\otimes W}$. Let V, W be in $\mathsf{Obj}(\mathsf{DY}(\mathfrak{b}))$. Then we have two Drinfeld–Yetter $\mathfrak{b}_j$–module structures on the tensor product $V\otimes W$: the first given by $(\pi^*_V)_j \otimes (\pi^*_W)_j$ and the second given by $(\pi^*_{V\otimes W})_j$. We show that $(\pi^*_V)_j \otimes (\pi^*_W)_j = (\pi^*_{V\otimes W})_j$: for any $v \in V$ and $w \in W$ we have

$$
\begin{aligned}
\big((\pi^*_V)_j \otimes (\pi^*_W)_j\big)(v\otimes w) &= \big((\pi^*_V)_j \otimes \mathsf{id}_W) + (\tau_{V,\mathfrak{b}_j} \otimes \mathsf{id}_W) \circ (\mathsf{id}_V \otimes (\pi^*_W)_j\big)(v\otimes w) \\
&= \big((\pi^*_V)_j \otimes \mathsf{id}_W) + (\tau_{V,\mathfrak{b}_j} \otimes \mathsf{id}_W) \circ (\mathsf{id}_V \otimes (\pi^*_W)_j\big)(v\otimes w) \\
&= \sum_v v^{[0]} \otimes v^{[1]} \otimes w + \sum_j j' \otimes (j'' \cdot v) \otimes w + \\
&+ \sum_{[w]} w^{[0]} \otimes v \otimes w^{[1]} + \sum_j j' \otimes v \otimes (j'' \cdot w) \\
&= (\pi^*_{V\otimes W})_j (v\otimes w).
\end{aligned}
$$

It clear that $\mathscr{I}_j$ is a symmetric monoidal functor. We finally show that is also infinitesimally braided monoidal: using that j is antisymmetric we get

$$
\begin{aligned}
\big(-t^{\mathfrak{b}_j}_{\mathscr{I}_j(V),\mathscr{I}_j(W)}\big)(v\otimes w) &= \sum_{[v]} v^{[1]} \otimes (v^{[0]} \cdot w) + \sum_{[w]} (w^{[0]} \cdot v) \otimes w^{[1]} \\
&- \sum_j (j'' \cdot v) \otimes (j' \cdot v) - \sum_j (j' \cdot v) \otimes (j'' \cdot w) \\
&= \sum_{[v]} v^{[1]} \otimes (v^{[0]} \cdot w) + \sum_{[w]} (w^{[0]} \cdot v) \otimes w^{[1]} + 0 \\
&= (-t^{\mathfrak{b}}_{V,W}).
\end{aligned}
$$

□

Corollary 4.6.9. *$\mathscr{I}_{-j}$ is an infinitesimally braided comonoidal functor.*

Remark 4.6.10. *Let $\mathfrak{b}$ be a Lie bialgebra and j be a twist. Consider the Drinfeld–Yetter $\mathfrak{b}$–module structure of $\mathsf{U}(\mathfrak{b})$ given by Theorem 4.5.2. Then we can endow $\mathsf{U}(\mathfrak{b})$ with another Drinfeld–Yetter $\mathfrak{b}$–module structure, which is uniquely determined by imposing relations* (4.1.7) *and* (4.4.1) *and the condition $\pi^*(1) = j$. We shall denote such a Drinfeld–Yetter $\mathfrak{b}$–module structure by $\mathsf{U}(\mathfrak{b})_j \in \mathsf{DY}(\mathfrak{b})$.*

4.7 Quantization of Lie bialgebras: a short introduction

The following definitions and results are standard. For a complete discussion on topologically free modules, we remand the reader to [Köt69], [War89], [Bou89], [Kas12], and for more details on preliminary notions on the theory of quantization of Lie bialgebras we refer to [ES02], [Maj95], [CP95] and [KS12].
Fix a formal parameter $\hbar$ and a field $\mathbb{K}$ of characteristics zero.

Definition 4.7.1. *A* ***topological Hopf algebra*** *is a a sextuple $(H, \mu, \eta, \Delta, \varepsilon, S)$, where H is a* *topologically free* $\mathbb{K}[[\hbar]]$*–module (see §9.1), and*

$$\mu : H\bar{\otimes}H \to H$$
$$\eta : \mathbb{K}[[\hbar]] \to H$$
$$\Delta : H \to H\bar{\otimes}H$$
$$\varepsilon : H \to \mathbb{K}[[\hbar]]$$
$$S : H \to H$$

are $\mathbb{K}[[\hbar]]$*–linear morphisms satisfying the axioms of Hopf algebra in the category* $\mathsf{TopFree}_{\mathbb{K}}$ *of all topologically free* $\mathbb{K}[[\hbar]]$*–modules, where* $\bar{\otimes}$ *is the* $\hbar$*–adic completion of the usual tensor product over* $\mathsf{Vect}_{\mathbb{K}}$*..*

Example 4.7.2. *Let* $(H_0, \mu_0, \eta_0, \Delta_0, \varepsilon_0, S_0)$ *be a Hopf algebra over* $\mathbb{K}$*. The trivial topological Hopf algebra associated to* $(H_0, \mu_0, \eta_0, \Delta_0, \varepsilon_0, S_0)$ *is* $(H, \overline{\mu}, \overline{\eta}, \overline{\Delta}, \overline{\varepsilon}, \overline{S})$*, where* H *is the topologically free module associated to the vector space* H *and* $mu_0, \eta_0, \Delta_0, \varepsilon_0, S_0)$ *is* $(H, \overline{\mu}, \overline{\eta}, \overline{\Delta}, \overline{\varepsilon}, \overline{S})$ *are the unique* $\mathbb{K}[[\hbar]]$*–linear maps such that, for any* $a, a' \in H_0$*, and* $f \in \mathbb{K}[[\hbar]]$*:*

$$\overline{\mu}(a\bar{\otimes}a') = \mu_0(a \otimes a')$$
$$\overline{\eta}(f) = f \cdot \eta_0(1)$$
$$\overline{\Delta}(a) = \Delta_0(a)$$
$$\overline{\varepsilon}(a) = \varepsilon_0(a)$$
$$\overline{S}(a) = S_0(a)$$

i.e. the unique $\hbar$*–linear extensions of the maps* $\mu_0, \eta_0, \Delta_0, \varepsilon_0, S_0$*.*

We are interested in deformations of universal enveloping algebras:

Definition 4.7.3. *Let* $(H, \mu, \eta, \Delta, \varepsilon, S)$ *be a Hopf algebra over* $\mathbb{K}$ *and let* $(H_\hbar, \mu_\hbar, \eta_\hbar, \Delta_\hbar, \varepsilon_\hbar, S_\hbar)$ *be a topological Hopf algebra.*

- *We say that* H_h *is a* ***deformation*** *of* H *if*
  - *(1)* $H_\hbar/(\hbar \cdot H_\hbar) \cong H[[\hbar]]$ *as topologically free modules.*
  - *(2)* $\mu_\hbar = \mu \mod \hbar$*.*
  - *(3)* $\Delta_\hbar = \Delta \mod \hbar$*.*
- *A deformation* $H_\hbar$ *of a universal enveloping algebra of a Lie algebra* $\mathfrak{g}$ *is called a* ***quantized universal enveloping algebra****.*
- *If* $H_\hbar$ *and* $H'_\hbar$ *are two deformations of* H*, we say that they are* ***equivalent deformations*** *if there exists an isomorphism of topologically free Hopf algebras over* $f : H_\hbar \to H'_\hbar$ *which is the identity modulo* $\hbar$

We shall denote the category of all quantized universal enveloping algebras by QUAlg. It has to be mentioned that any deformation is equivalent to one in which the unit and the counit are the trivial ones. Furthermore, any deformation of a bialgebra uniquely extends to a deformation of a Hopf algebra.

Proposition 4.7.4. *Let H be a quantized enveloping algebra with $H/(\hbar \cdot H) \cong \mathsf{U}(\mathfrak{g})$. Then the Lie algebra $\mathfrak{g}$ is equipped with a bialgebra structure defined by*

$$\delta(x) = \frac{\Delta(\tilde{x}) - \Delta^{\mathsf{op}}(\tilde{x})}{\hbar} \quad mod\ \hbar \tag{4.7.1}$$

where $\tilde{x}$ is any lifting of x to H through the isomorphism $H/(\hbar \cdot H) \cong \mathsf{U}(\mathfrak{g})$.

A proof of this very important result can be find in [ES02, p. 78]. We can then introduce the following

Definition 4.7.5. *Let $\mathfrak{b}$ be a Lie bialgebra. We say that a topological Hopf algebra H is a* ***quantization*** *of $\mathfrak{b}$ if there is an isomorphism $H/(\hbar \cdot H) \cong \mathsf{U}(\mathfrak{b})$ such that* (4.7.1) *holds.*

There is a functor, called the **semiclassical functor** $\mathsf{SC} : \mathsf{QUAlg} \to \mathsf{LieBialg}$ from the category of quantized universal enveloping algebras to the category of Lie bialgebras, assigning to any H the Lie bialgebra $\mathfrak{b} = \mathsf{Prim}(H/\hbar \cdot H)$, where Prim denotes the set of *primitive elements.* The problem of quantization of Lie bialgebras is to define its adjoint functor.

Part II

Quantization of Lie bialgebras

Chapter 5

The Drinfeld associator arising from the Knizhnik–Zamolodchikov connection

This Chapter, which is an extract of [BRW23], contains a pedagogical approach to the work by V.G.Drinfeld about the associator Φ (constructed by means of the Knizhnik–Zamolodchikov connections) satisfying the hexagon and pentagon equations, see [Dri90b] and [Dri90a].
Since its introduction, Drinfeld's associator has seen many important applications such as the solution of the problem of the quantization of Lie bialgebras by Etingof and Kazhdan [EK96] (1996) (see also Ševera's work [Šev16] (2016)), Tamarkin's approach [Tam99] (1999) to Kontsevich's formality theorem in deformation quantization [Kon03] (1997), and the solution of the problem of the quantization of Lie quasibialgebras by Enriquez and Halbout [EH10a] and [EH10b] in 2010, see also [SŠ15]. Moreover, regarding the associator as a formal power series in the free algebra generated by two distinct elements, its coefficients are directly related to multiple zeta values, see e.g. [ES02, p.209-213], thus establishing an important link to number theory.
We are going to give a detailed and self–contained account of the definition of the particular associator coming from the Knizhnik–Zamolodchikov connections and the proof of the hexagon and pentagon identities. The problem of finding solutions to such identities is quite non-trivial: for instance, the naive choice $\Phi(A, B) = 1$ would solve the pentagon equation, but for non-commuting A_{13}, A_{23} it would clearly not solve the hexagon equation.
We shall only mention the following nonexhaustive list of important results: there exist rational associators, see [Dri90b], [Dri90a], and the work by Bar-Natan, [BN98] (1998). Other associators have been constructed linked to the Kashiwara-Vergne conjecture by Alekseev, Enriquez, Torossian [AET10] (2010). Moreover, non–trivial solutions of the pentagon equations automatically satisfy a certain hexagon equation, see Furusho's work [Fur10].
Drinfeld's original method consists in the comparison of different global solutions (in certain simply connected regions of $\mathbb{R}^N$) of the linear system (a first order linear partial differential equation) defined by the Knizhnik–Zamolodchikov connection (see [KZ84]) with respect to their pole structure at certain complex or real hyperplanes, referred to as *monodromy*, see also Kohno's work [Koh85]. This approach is partially motivated by the theory of complex differential equations with singularities, compare for instance [CL55] or [Del70]. From a point of view of differential geometry this amounts to the computation of covariantly constant sections of a trivial vector bundle with respect to a flat connection which are uniquely determined by their value at a given point.
We have chosen a slightly more elementary method: since the value of a covariantly constant section (the solutions of the linear system) at some point x can be defined by the parallel transport along a continuous piecewise smooth path joining a reference point to x, we find it reasonable to focus

on formal parallel transports (in the algebra of all formal power series in a given associative unital complex algebra, $\mathcal{A}[[\lambda]]$), i.e. first order ordinary linear differential equations along concrete continuous piecewise smooth paths in explicitly given contractible regions of $\mathbb{R}$ and $\mathbb{R}^2$ with respect to some flat formal connection deriving from the Knizhnik–Zamolodchikov one. However, in Drinfeld's approach the 'reference point' is –in some sense– 'at a singularity', and in order to capture that we use the well–known regularization procedure: first, we make paths within the domain depend on a strictly positive 'small' parameter δ such that in the limit $\delta \to 0$ these δ–dependent paths c_δ would be pushed to the boundary of these regions where the connection becomes singular. Then we compute the δ–dependent parallel transports within the domain where as usual a composition of paths corresponds to multiplication of the corresponding parallel transports in $\mathcal{A}[[\lambda]]$. It will turn out that each such parallel transport $W^{(c_\delta)}$ factorizes in a multiplication of invertible formal power series as

$$W^{(c_\delta)} = S^{(c_\delta)} G^{(c_\delta)} H^{(c_\delta)} \tag{5.0.1}$$

where $S^{(c_\delta)}$ is 'singular', i.e. diverging –in powers of $|\ln(\delta)|$– for $\delta \to 0$, $G^{(c_\delta)}$ is 'good', i.e. converging to a wanted term for $\delta \to 0$, and $H^{(c_\delta)}$ is 'harmless', i.e. converging to 1 for $\delta \to 0$ where the terms proportional to λ^n, $n > 0$, tend to zero dominated by a 'power law' δ^β, $\beta > 0$. Harmless terms will turn out to be stable by conjugation with singular terms. Parallel transports along different (composed) paths having the same initial and final points will be equal due to the flatness of the used connection –for instance the famous Knizhnik-Zamolodchikov connection– thereby inducing algebraic identities: in all the important identities the singular terms cancel out for all strictly positive δ, and the remaining terms give the wanted identities in the limit $\delta \to 0$.

5.1 Elementary analysis of piecewise C^∞ functions

Definition 5.1.1. *Let $a, b \in \mathbb{R}$, m be a non–negative integer and*

$$D = \{a = a_0 < a_1 < \cdots < a_m < a_{m+1} := b\}.$$

The space of all ***piecewise $\mathcal{C}^\infty$-functions on*** *$[a,b]$* ***with singular set*** *D is*

$$\begin{aligned}\mathcal{C}^\infty_D([a,b],\mathbb{C}) \ :=\ \Big\{ f \in \mathcal{C}^\infty([a,b]\setminus D,\mathbb{C}) \ \Big| \ & \forall\ r, i \in \mathbb{N} \text{ with } 1 \leqslant i \leqslant m+1 : \\ & \lim_{\epsilon\downarrow 0} f^{(r)}(a_i - \epsilon) \text{ exists and } \ \forall\ 0 \leqslant i \leqslant m : \lim_{\epsilon\downarrow 0} f^{(r)}(a_i + \epsilon) \text{ exists}\Big\}\end{aligned} \tag{5.1.1}$$

where $\epsilon \downarrow 0$ means that only strictly positive real numbers ϵ are considered in the limit. We shall refer to D as the (potential) singular set *and to its open dense complement $[a,b]\setminus D$ as the* regular set.

Remark 5.1.2. *Note that*

(i) $\mathcal{C}^\infty_D([a,b],\mathbb{C})$ is a commutative algebra.

(ii) If $D' \subseteq D$, the canononical injection $\mathcal{C}^\infty_D([a,b],\mathbb{C}) \hookrightarrow \mathcal{C}^\infty_{D'}([a,b],\mathbb{C})$ is a morphism of algebras.

(iii) Since the usual derivative (defined only on the regular set $[a,b]\setminus D$) is compatible with left- and right-sided limits, induces a derivation of algebras

$$\mathcal{C}^\infty_D([a,b],\mathbb{C}) \to \mathcal{C}^\infty_D([a,b],\mathbb{C}) : f \mapsto \frac{\mathsf{d}f}{\mathsf{d}s}. \tag{5.1.2}$$

We shall need the following subalgebra of $\mathcal{C}^\infty_D([a,b],\mathbb{C})$:

$$\mathcal{C}^\infty_D([a,b],\mathbb{C})^0 := \big\{f \in \mathcal{C}^\infty_D([a,b],\mathbb{C}) \;\big|\; f \text{ extends to a continuous function } [a,b] \to \mathbb{C}\big\}. \tag{5.1.3}$$

We shall also need to compose these piecewise $\mathcal{C}^\infty$–functions: given another closed interval $[a',b']$ and a finite subset $D' = \{a' = a'_0 < a'_1 < \cdots < a'_{m'} < a'_{m'+1} = b'\}$, we say that $\theta \in \mathcal{C}^\infty_{D'}([a',b'],\mathbb{C})$ is **compatible** with $[a,b]$ and D if the following condition is satisfied:

$$\theta\big([a',b'] \setminus D\big) \subset [a,b] \subset \mathbb{R} \quad \text{and} \quad \theta|^{-1}(D) \text{ is finite subset of } [a',b'] \tag{5.1.4}$$

where $\theta|$ denotes the $\mathcal{C}^\infty$–function $[a',b'] \setminus D' \to [a,b]$ outside its singular set. It is immediate that the composition $f \circ \theta$ is a well–defined function on $[a',b'] \setminus \big(D' \cup \theta|^{-1}(D)\big)$, and the iterated chain rule (also called Faa di Bruno Theorem, see e.g. [AF88, p.291, equation (3)]) shows that it is a $\mathcal{C}^\infty$–function. The left–side and right–side limits of the r–th derivative $f \circ \theta$ at the singular points in $D' \cup \theta|^{-1}(D)$ exist which easily follows from the continuity of the continuous extensions $\theta^{(r)}_j : [a'_j, a'_{j+1}] \to [a,b]$ of the restriction of $\theta^{(r)}$ to $]a'_j, a'_{j+1}[$ for all integers $r \geqslant 0$ and $0 \leqslant j \leqslant m'$. Hence, we can define the composition

$$f \circ \theta \in \mathcal{C}^\infty_{D' \cup \theta|^{-1}(D)}\big([a',b'],\mathbb{C}\big). \tag{5.1.5}$$

It is not hard to see that the chain rule works for this composition and differentiation (5.1.2).
Next, we need to use the well–known *Riemann integral*: note that for every element $f \in \mathcal{C}^\infty_D([a,b],\mathbb{C})$ and $\alpha, \beta \in [a,b]$ we can define the Riemann integral

$$I^\beta_\alpha(f) := \begin{cases} \int_\alpha^\beta \hat{f}(s)ds & \text{if } \alpha \leqslant \beta \\ -\int_\beta^\alpha \hat{f}(s)ds & \text{if } \alpha \geqslant \beta \end{cases} \tag{5.1.6}$$

where $\hat{f}$ is any extension of f from $[a,b] \setminus D$ to $[a,b]$ (for instance $\hat{f}(a_i) = 0$ for all $0 \leqslant i \leqslant N+1$: it is well–known that any such extension is Riemann integrable and that the integral does not depend on the extension, that is which values of $\hat{f}$ are chosen at the singular points contained in the domain of integration, see e.g. [Lan96, p.273]. Recall Chasles's rule for any $f \in \mathcal{C}^\infty_D([a,b],\mathbb{C})$:

$$\forall\, \alpha, \beta, \gamma \in [a,b] : \quad I^\gamma_\alpha(f) = I^\beta_\alpha(f) + I^\gamma_\beta(f).$$

Recall that the complex-valued function $[a,b] \to \mathbb{C}$ defined for every $f \in \mathcal{C}^\infty_D([a,b],\mathbb{C})$ by

$$I_\alpha(f)(s) := I^s_\alpha(f)$$

is the usual *primitive of* f. We resume the following properties of the primitive which are well–known variants of the fundamental theorem of calculus and standard integration techniques:

Theorem 5.1.3. *For any $f, g \in C^\infty_D\big([a,b],\mathbb{C}\big)$ and $h \in C^\infty_D\big([a,b],\mathbb{C}\big)^0$ the following holds:*

(i) For any $\alpha \in [a,b]$ the primitive I_α defines a $\mathbb{C}$–linear map $C^\infty_D\big([a,b],\mathbb{C}\big) \to C^\infty_D\big([a,b],\mathbb{C}\big)^0$ whence $I_\alpha(f)$ is always continuous. Moreover,

$$I_\alpha(f)(\alpha) = 0. \tag{5.1.7}$$

(ii) Fundamental theorem of calculus: for the derivatives (in the sense of (5.1.2)) we get

$$\frac{\mathsf{d}I_\alpha(f)}{\mathsf{d}s} = f \quad \text{and} \quad I_\alpha\left(\frac{\mathsf{d}h}{\mathsf{d}s}\right) = h - h(\alpha). \tag{5.1.8}$$

Moreover, any element $F \in C_D^\infty\big([a,b],\mathbb{C}\big)^0$ satisfying $\frac{\mathsf{d}F}{\mathsf{d}s} = f$ and $F(\alpha) = 0$ is equal to $I_\alpha(f)$.

(iii) Let $\theta \in \mathcal{C}_{D'}^\infty\big([a',b'],\mathbb{R}\big)^0$ such that θ is compatible with $[a,b]$ and D. Then the composition $f \circ \theta$ is in $\mathcal{C}_{D' \cup \theta|^{-1}(D)}^\infty\big([a',b'],\mathbb{C}\big)$. Moreover, for each $\alpha' \in [a',b']$ there is the usual 'change-of-variables-rule'

$$I_{\theta(\alpha')}(f) \circ \theta = I_{\alpha'}\left((f \circ \theta)\frac{\mathsf{d}\theta}{\mathsf{d}s}\right) \tag{5.1.9}$$

where both sides of the preceding equation are elements of $\mathcal{C}_{D' \cup \theta|^{-1}(D)}^\infty\big([a',b'],\mathbb{C}\big)^0$.

See [Lan96, p.272-274] for the proof of all the statements.

Definition 5.1.4. *We shall call a triple $(\theta,[a',b'],D')$ consisting of a continuous map $\theta : [a',b'] \to [a,b]$ satisfying the hypotheses of statement (iii) of the preceding Theorem 5.1.3 a* ***continuous piecewise $\mathcal{C}^\infty$ reparametrization*** *of $\big([a,b],D\big)$.*

5.2 Formal linear ODEs

In this Section we review a particular case of the general theory described in Chen's classical work [Che61, p.110-115].

Let $\mathcal{A}$ be a complex algebra and λ be a formal parameter. Then we may consider the following topological algebra

$$\Big(\mathcal{C}_D^\infty\big([a,b],\mathbb{C}\big) \otimes \mathcal{A}\Big)[[\lambda]]. \tag{5.2.1}$$

Note that each element F in this algebra can canonically be considered as a function from $[a,b] \setminus D$ to $\mathcal{A}[[\lambda]]$, and we shall sometimes use the notation $s \mapsto F(s) = \sum_{r=0}^\infty F_r(s)\lambda^r$. Tensoring the usual derivative $\frac{\mathsf{d}}{\mathsf{d}s}$ with the identity map on $\mathcal{A}$ and extending on formal power series in the usual componentwise way we get a derivative of the algebra (5.2.1) which is again a derivation of algebras. We shall denote it by the same symbol $\frac{\mathsf{d}}{\mathsf{d}s}$. In a completely analogous way we can extend the Riemann integral I_α^β and the primitive I_α where we shall continue to use the same symbols. It is obvious that I_α^β takes its values in $\mathcal{A}[[\lambda]]$ and that all the statements of Theorem 5.1.3 remain true when f, g and h are replaced by elements in the corresponding algebras (5.2.1).

Fix $Y \in \Big(\mathcal{C}_D^\infty\big([a,b],\mathbb{C}\big) \otimes \mathcal{A}\Big)[[\lambda]]$. We consider the following formal linear ordinary differential equation (formal linear ODE)

$$\left\{\begin{array}{rcl} \dfrac{\mathsf{d}\omega}{\mathsf{d}s} & = & \lambda Y \omega \\ \omega(\alpha) & = & \omega_\alpha \end{array}\right. \tag{5.2.2}$$

where $\alpha \in [a,b]$, ω_α is an element of $\mathcal{A}[[\lambda]]$, and we look for solutions

$$\omega \in \Big(\mathcal{C}_D^\infty\big([a,b],\mathbb{C}\big)^0 \otimes \mathcal{A}\Big)[[\lambda]],$$

whence ω is required to be continuous and piecewise $\mathcal{C}^\infty$. The theory of existence and uniqueness of these formal linear ODEs is well–known to be much simpler than the one of the usual differential equations. Indeed, first there is the usual reformulation in terms of integral equations known from usual ODE theory: suppose first that ω is a continuous piecewise $\mathcal{C}^\infty$ solution of (5.2.2). Taking primitives on both sides gives –using in Theorem 5.1.3 the second equation of (5.1.8)– the integral equation

$$\omega = \omega_\alpha + \lambda I_\alpha\big(Y\omega\big) \tag{5.2.3}$$

where ω_α is considered as the constant function on $[a,b]$ with value ω_α. On the other hand, if the continuous piecewise $\mathcal{C}^\infty$ element ω is a solution of the formal integral equation (5.2.3), then $\omega(\alpha) = \omega_\alpha$ by (5.1.7), and differentiation of the integral equation –using the first equation of (5.1.8)– gives the formal linear ODE (5.2.2).

Solving the formal integral equation (5.2.3) is quite simple due to the presence of the factor λ in front of Y: consider the following $\mathbb{C}[[\lambda]]$-linear maps

$$L_Y : \Big(\mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big)^0 \otimes \mathcal{A}\Big)[[\lambda]] \to \Big(\mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big) \otimes \mathcal{A}\Big)[[\lambda]]$$
$$F \mapsto L_Y(F) := YF$$

and

$$I_\alpha : \Big(\mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big) \otimes \mathcal{A}\Big)[[\lambda]] \to \Big(\mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big)^0 \otimes \mathcal{A}\Big)[[\lambda]].$$

Then the composition $I_\alpha \circ L_Y$ is a well–defined $\mathbb{C}[[\lambda]]$–linear endomorphism of the $\mathbb{C}[[\lambda]]$–module $\Big(\mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big)^0 \otimes \mathcal{A}\Big)[[\lambda]]$. Hence, the formal integral equation (5.2.3) can be rewritten as

$$\big(\mathsf{id} - \lambda I_\alpha \circ L_Y\big)(\omega) = \omega_\alpha, \quad \text{hence} \quad \omega = \big(\mathsf{id} - \lambda I_\alpha \circ L_Y\big)^{-1}(\omega_\alpha) = \sum_{r=0}^{\infty} \lambda^r \big(I_\alpha \circ L_Y\big)^{\circ r}(\omega_\alpha) \tag{5.2.4}$$

since it is obvious –thanks to the presence of the factor λ– that the formal series $\mathsf{id} - \lambda I_\alpha \circ L_Y$ is always invertible in the algebra of all $\mathbb{C}[[\lambda]]$–linear endomorphisms of $\Big(\mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big)^0 \otimes \mathcal{A}\Big)[[\lambda]]$ seen as a $\mathbb{C}[[\lambda]]$–module by the usual geometric series formula. Note that the formula (5.2.4) is very often written out in terms of iterated integrals:

$$\omega(s) = \omega_\alpha + \sum_{r=1}^{\infty} \lambda^r \Bigg(\int_\alpha^s \Bigg(\hat{Y}(s_1) \int_\alpha^{s_1} \Bigg(\hat{Y}(s_2) \int_\alpha^{s_2} \Bigg(\cdot\cdot \int_\alpha^{s_{r-1}} \hat{Y}(s_r) ds_r \Bigg) \cdot\cdot ds_3 \Bigg) ds_2 \Bigg) ds_1 \Bigg) \omega_\alpha \tag{5.2.5}$$

where $\hat{Y}$ denotes any extension of Y from $[a,b] \setminus D$ to $[a,b]$. We shall write $W_{\cdot\alpha} := (s \mapsto W_{s\alpha})$ for the particular solution ω of the formal ODE (5.2.2) with initial condition $\omega_\alpha = 1$, the unit element of the algebra $\Big(\mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big)^0 \otimes \mathcal{A}\Big)[[\lambda]]$, hence

$$\begin{cases} \dfrac{\mathsf{d}W_{\cdot\alpha}}{\mathsf{d}s} &= \lambda Y W_{\cdot\alpha} \\ W_{\cdot\alpha}(\alpha) = W_{\alpha\alpha} &= 1 \end{cases} \tag{5.2.6}$$

and refer to it as the **fundamental solution** of the formal ODE (5.2.2) normalized at α, see e.g. [CL55, p.69]. Moreover, we shall refer to the value of the fundamental solution $W_{\cdot\alpha}$ at $\beta \in [a,b]$,

$$W_{\beta\alpha} := W_{\cdot\alpha}(\beta) \in \mathcal{A}[[\lambda]] \tag{5.2.7}$$

as the **propagator** (from α to β).

We collect some properties of the above formal linear ODEs in the following

Proposition 5.2.1. *We have the following:*

(i) Every formal linear ODE (5.2.2) has a unique (continuous) solution ω given by the formulas (5.2.4) or (5.2.5). It can be expressed by the fundamental solution $W_{\cdot\alpha}$ normalized at α in the following way:

$$\omega = W_{\cdot\alpha}\omega_\alpha. \tag{5.2.8}$$

(ii) Groupoid properties: Every fundamental solution $W_{\cdot\alpha}$ has only invertible values in $\mathcal{A}[[\lambda]]$, and for all $\alpha, \beta, \gamma \in [a, b]$ we have the following identities for the propagators

$$W_{\alpha\alpha} = 1, \quad W_{\gamma\beta}W_{\beta\alpha} = W_{\gamma\alpha}, \quad W_{\alpha\beta} = W_{\beta\alpha}^{-1}. \tag{5.2.9}$$

(iii) Reparametrization: Let $a', b' \in \mathbb{R}$ with $a' < b'$, let D' be a finite set with $\{a', b'\} \subset D' \subset [a', b']$, let $\theta \in \mathcal{C}^\infty_{D'}\big([a', b'], \mathbb{R}\big)^0$ be a continuous piecewise $\mathcal{C}^\infty$ reparametrization of $\big([a, b], D\big)$. Let $\alpha' \in [a', b']$, and let $W_{\cdot\theta(\alpha')}$ the fundamental solution of (5.2.6) normalized at $\theta(\alpha')$. Then $W'_{\cdot\alpha'} := W_{\cdot\theta(\alpha')} \circ \theta$ is a fundamental solution normalized at α' of the formal linear ODE

$$\frac{\mathsf{d}W'_{\cdot\alpha'}}{\mathsf{d}s'} = \big(Y \circ \theta\big)\frac{\mathsf{d}\theta}{\mathsf{d}s'}W'_{\cdot\alpha'}, \quad \text{hence} \ \ \forall\ \alpha, \beta \in [a, b]: \quad W'_{\beta'\alpha'} = W_{\theta(\beta')\theta(\alpha')} \in \mathcal{A}[[\lambda]]. \tag{5.2.10}$$

Moreover, the propagator $W_{\beta\alpha}$ only depends on the values of Y between α and β.

(iv) Factorization: Let $Y = Y_0 + Z$ with $Y_0, Z \in \Big(\mathcal{C}^\infty_D\big([a, b], \mathbb{C}\big) \otimes \mathcal{A}\Big)[[\lambda]]$, and let $W_{\cdot\alpha}$ and $U_{\cdot\alpha}$ be the fundamental solutions normalized at α for the formal linear ODE's

$$\frac{\mathsf{d}W_{\cdot\alpha}}{\mathsf{d}s} = \lambda Y W_{\cdot\alpha} \quad \text{and} \quad \frac{\mathsf{d}U_{\cdot\alpha}}{\mathsf{d}s} = \lambda Y_0 U_{\cdot\alpha}.$$

Then $W_{\cdot\alpha}$ factorizes in the following way:

$$W_{\cdot\alpha} = U_{\cdot\alpha}\Xi_{\cdot\alpha} \quad \text{where} \quad \frac{\mathsf{d}\Xi_{\cdot\alpha}}{\mathsf{d}s} = \lambda\left(U_{\cdot\alpha}^{-1} Z U_{\cdot\alpha}\right)\Xi_{\cdot\alpha} \quad \text{and} \quad \Xi_{\alpha\alpha} = 1. \tag{5.2.11}$$

(v) Suppose that $YI_\alpha(Y) = I_\alpha(Y)Y$. Then the fundamental solution $W_{\cdot\alpha}$ of the formal linear ODE (5.2.6) is explicitly given by

$$W_{\cdot\alpha} = e^{\lambda I_\alpha(Y)}. \tag{5.2.12}$$

Proof. (*i*): Existence and uniqueness follow from the considerations in (5.2.4), and (5.2.8) can be read off (5.2.5).
(*ii*): Again by (5.2.4) and (5.2.5) it is immediate that $W_{\cdot\alpha}$ is a formal series in the associative unital algebra $\Big(\mathcal{C}^\infty_D\big([a, b], \mathbb{C}\big)^0 \otimes \mathcal{A}\Big)[[\lambda]]$ whose constant term is 1, hence it is invertible by a similar geometric series argument. The first equation of (5.2.9) is part of the definition (5.2.6). For the second note that $W_{\cdot\beta}$ and $W_{\cdot\alpha}$ satisfy the same formal linear ODE with initial condition (at β) 1 and $W_{\beta\alpha}$, respectively. Hence, by (5.2.8) we get $W_{\cdot\alpha} = W_{\cdot\beta}W_{\beta\alpha}$ which gives the second equation of (5.2.9) upon choosing $s = \gamma$. The third equation of (5.2.9) follows from the first and the second upon setting $\alpha = \gamma$.
(*iii*): Equation (5.2.10) is an easy consequence of the chain rule and equation (5.2.2). The last statement follows either directly from the iterated integral form (5.2.5) or by choosing –assuming that $\alpha \leqslant \beta$ without loss of generality– $[a', b'] = [\alpha, \beta]$ and $\theta : [\alpha, \beta] \to [a, b]$ the canonical injection.

(*iv*): Using the well–known formula $\frac{\mathsf{d}(U_{\cdot\alpha}^{-1})}{\mathsf{d}s} = -U_{\cdot\alpha}^{-1}\frac{\mathsf{d}U_{\cdot\alpha}}{\mathsf{d}s}U_{\cdot\alpha}^{-1}$ gives the result upon differentiating $\Xi_{\cdot\alpha} = U_{\cdot\alpha}^{-1}W_{\cdot\alpha}$.
(*v*): For each positive integer r, differentiating the rth power of $I_\alpha(Y)^r$ we get $rYI_\alpha(Y)^{r-1}$ thanks to the hypothesis $YI_\alpha(Y) = I_\alpha(Y)Y$ which shows the result when differentiating the exponential series $e^{\lambda I_\alpha(Y)} = \sum_{r=0}^{\infty}\frac{\lambda^r}{r!}I_\alpha(Y)^r$. □

5.3 Norms and limits

We shall have to discuss limits of solutions of formal linear ODEs given by elements Y of the algebra (5.2.1) depending on a parameter δ in some subset $J \subset \mathbb{R}^\ell$, and we are interested in limits when $\delta \to \delta_0$ where δ_0 is an accumulation point of J. Since the complex associative unital algebra $\mathcal{A}$ is completely arbitrary, we have to include a discussion to make sense of these limits. Recall that every complex vector space has at least one norm: in fact, let $\mathsf{B} := (e_i)_{i\in\mathfrak{S}}$ be a basis for E labeled by the set $\mathfrak{S}$: every vector $\xi \in E$ is a linear combination $\xi = \sum_{i\in\mathfrak{S}} x_i e_i$ where all the $x_i \in \mathbb{C}$ and the subset of $\mathfrak{S}$ for which $x_i \neq 0$ is finite. Define

$$||\xi||_{\mathsf{B}} = ||\xi|| := \max\big\{|x_i| \;\big|\; i \in \mathfrak{S}\big\}, \tag{5.3.1}$$

and the norm properties are easy to check directly. Having a norm allows us to define *limits*: more precisely, for a given positive integer ℓ let $J \subset \mathbb{R}^\ell$ be a non-empty set, and let $\mathrm{Fun}(J, E)$ denote the complex vector space of all maps $J \to E$. Fix a norm $||\ ||$ on E, some norm $|\ |$ on $\mathbb{R}^\ell$, and a function $f \in \mathrm{Fun}(J, E)$. Let δ_0 be an accumulation point of J. For any $\zeta \in E$ recall the following definition of a limit:

$$\lim_{\delta\to\delta_0} f(\delta) = \zeta \text{ w.r.t. } ||\ || \qquad \text{iff} \qquad \forall\, \epsilon \in \mathbb{R},\ \epsilon > 0\ \exists\, \epsilon' \in \mathbb{R},\ \epsilon' > 0 :\ \forall\, \delta \in J : \\ \text{if } |\delta - \delta_0| < \epsilon' \ \text{ then } \ ||f(\delta) - \zeta|| < \epsilon. \tag{5.3.2}$$

As usual, if the limit exists, it is unique. However, the existence of the limit a priori depends on the norms $|\ |$ and $||\ ||$ used. Recall that two norms $||\ ||$ and $||\ ||'$ on E are called *equivalent* if:

$$\exists\, C_1, C_2 \in \mathbb{R},\ C_1, C_2 > 0\ \ \forall\, \xi \in E : \quad C_1||\xi|| \leqslant ||\xi||' \leqslant C_2||\xi||.$$

Hence, if the norms $||\ ||$ and $||\ ||'$ on E are equivalent and if the norms $|\ |$ and $|\ |'$ on $\mathbb{R}^\ell$ are equivalent, it is easy to see that in (5.3.2) the statement using $||\ ||$ and $|\ |$ is equivalent to the one using $||\ ||'$ and $|\ |'$: in this case the limit does not depend on the norms used. In general, two given norms on a complex vector space are not equivalent; however, in the case of a finite–dimensional vector space it is well–known that any two norms are equivalent, see e.g. [Lan96, p.145, Thm.4.3.]. This always applies to the norms $|\ |$ and $|\ |'$ on $\mathbb{R}^\ell$ in statement (5.3.2), but in general not to the norms $||\ ||$ and $||\ ||'$ on E. In the following, the relevant limits will always take place in finite–dimensional subspaces, thereby insuring that the computation of limits will not depend on the norms chosen. More precisely, consider the following algebra

$$\Big(\mathrm{Fun}\big(J, \mathbb{C}\big) \otimes \mathcal{A}\Big)[[\lambda]]. \tag{5.3.3}$$

Each element F of this algebra is a formal power series $F = \sum_{r=0}^{\infty}\lambda^r F_r$ where each component F_r is an element of $\mathrm{Fun}\big(J, \mathbb{C}\big) \otimes \mathcal{A}$, hence can be considered as a map $J \to \mathcal{A}$, and choosing a norm $||\ ||$ on $\mathcal{A}$ and a norm $|\ |$ on $\mathbb{R}^\ell \supset J$ –among all the equivalent ones– we can consider limits

$F_r \to \xi_r \in \mathcal{A}$ for each non–negative integer r seperately in the sense of definition (5.3.2). For any $\xi = \sum_{r=0}^{\infty} \lambda^r \xi_r \in \mathcal{A}[[\lambda]]$ we thus define limits componentwise for each F in the algebra (5.3.3):

$$\lim_{\delta\to\delta_0} F(\delta) = \xi \text{ w.r.t. } || \, || \quad \text{iff} \quad \forall\, r \in \mathbb{N}: \; \lim_{\delta\to\delta_0} F_r(\delta) = \xi_r \text{ w.r.t. } || \, ||. \tag{5.3.4}$$

We enumerate some important properties of limits in the algebra (5.3.3) in the following

Proposition 5.3.1. *Let $J \subset \mathbb{R}^\ell$ as above, let $\delta_0 \in \mathbb{R}^\ell$ be an accumulation point of J, and let $F = \sum_{r=0}^{\infty} \lambda^r F_r$ be an element of $\Big(\mathrm{Fun}\big(J,\mathbb{C}\big) \otimes \mathcal{A}\Big)[[\lambda]]$. Then*

(i) For each $r \in \mathbb{N}$ there is a finite–dimensional subspace $V_r^{(F)} = V_r$ of $\mathcal{A}$ such that for each $r \in \mathbb{N}$

$$F_r \in \mathrm{Fun}\big(J,\mathbb{C}\big) \otimes V_r. \tag{5.3.5}$$

(ii) Let $|| \, ||$ and $|| \, ||'$ be two norms on the complex vector space $\mathcal{A}$, and let $\xi = \sum_{r=0}^{\infty} \lambda^r \xi_r \in \mathcal{A}[[\lambda]]$. Then the statement

$$\lim_{\delta\to\delta_0} F(\delta) = \xi \text{ w.r.t. } || \, || \quad \text{is equivalent to} \quad \lim_{\delta\to\delta_0} F(\delta) = \xi \text{ w.r.t. } || \, ||',$$

hence limits in the algebra (5.3.3) do not depend on the norms used.

(iii) Let $\tilde{F} = \sum_{r=0}^{\infty} \lambda^r \tilde{F}_r$ be another element of $\Big(\mathrm{Fun}\big(J,\mathbb{C}\big) \otimes \mathcal{A}\Big)[[\lambda]]$, and let $\tilde{\xi} \in \mathcal{A}[[\lambda]]$ such that $\lim_{\delta\to\delta_0} \tilde{F}(\delta) = \tilde{\xi}$ with respect to any norm on $\mathcal{A}$. Then for all $\alpha, \beta \in \mathbb{C}$:

$$\lim_{\delta\to\delta_0} \Big(\alpha F(\delta) + \beta \tilde{F}(\delta)\Big) = \alpha \lim_{\delta\to\delta_0} F(\delta) + \beta \lim_{\delta\to\delta_0} \tilde{F}(\delta) = \alpha\xi + \beta\tilde{\xi} \tag{5.3.6}$$

and

$$\lim_{\delta\to\delta_0} \Big(F(\delta)\tilde{F}(\delta)\Big) = \Big(\lim_{\delta\to\delta_0} F(\delta)\Big) \Big(\lim_{\delta\to\delta_0} \tilde{F}(\delta)\Big) = \xi\tilde{\xi}. \tag{5.3.7}$$

Proof. (i): By definition of the algebraic tensor product each F_r is a finite sum $F_{r1} \otimes A_{r1} + \cdots + F_{rN_r} \otimes A_{rN_r}$ where N_r is a non–negative integer, $F_{r1}, \ldots, F_{rN_r}$ are functions $J \to \mathbb{C}$, and $A_{r1}, \ldots, A_{rN_r}$ are elements of $\mathcal{A}$. Defining V_r as the complex linear hull of $A_{r1}, \ldots, A_{rN_r}$ proves the statement.
(ii): We shall prove a slightly more general statement: for each non–negative integer r let V_r' be another finite–dimensional subspace of $\mathcal{A}$ such that (5.3.5) is satisfied. Then each F_r clearly is an element of $\mathrm{Fun}\big(J,\mathbb{C}\big) \otimes \big(V_r \cap V_r'\big)$. We can enlarge each V_r, V_r' by at most one dimension to include ξ_r. From the definition of the limit (5.3.2), it is clear that it suffices to look at the restrictions of the norms $|| \, ||$ and $|| \, ||'$ on $\mathcal{A}$ to the finite–dimensional subspaces V_r, V_r' and $V_r \cap V_r'$: the restriction of the norm $|| \, ||$ to $V_r \cap V_r'$ is equivalent to the restriction of the norm $|| \, ||'$ thanks to the finite dimension of $V_r \cap V_r'$ which shows that the limit statements with respect to the norms $|| \, ||$ and $|| \, ||'$ are equivalent.
(iii): The first equation (5.3.6) is the usual statement that in any normed vector space addition and scalar multiplication are continuous. All the limits do not depend on the norms (take for instance for each $r \in \mathbb{N}$ the finite-dimensional vector space $V_r + \tilde{V}_r$) thanks to the preceding statement (ii). The second statement (5.3.7) is slightly more involved since the normed vector space $\big(\mathcal{A}, || \, ||\big)$ is in general not a normed algebra in the sense that $||AA'|| \leqslant ||A|| \; ||A'||$ for all $A, A' \in \mathcal{A}$. We shall

first prove an intermediate estimate: for each $r \in \mathbb{N}$ pick a finite–dimensional subspace $\tilde{V}_r$ such that $\tilde{F}_r \in \mathrm{Fun}(J, \mathbb{C}) \otimes \tilde{V}_r$ (which is possible thanks to statement (i)). We have for each $r \in \mathbb{N}$

$$\begin{aligned}\Big(F(\delta)\tilde{F}(\delta)\Big)_r &= \sum_{u=0}^{r} F_u(\delta)\tilde{F}_{r-u}(\delta) \in \sum_{u=0}^{r} V_u \tilde{V}_{r-u} \\ &\subset \big(V_0 + \cdots + V_r\big)\big(\tilde{V}_0 + \cdots + \tilde{V}_r\big) =: V_{(r)}\tilde{V}_{(r)}.\end{aligned} \tag{5.3.8}$$

Clearly, the subspaces $V_{(r)}$ and $\tilde{V}_{(r)}$ of $\mathcal{A}$ are finite–dimensional. Consider the restriction of the algebra multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ to the finite–dimensional vector space $V_{(r)} \otimes \tilde{V}_{(r)}$: the image $\mu\Big(V_{(r)} \otimes \tilde{V}_{(r)}\Big) = V_{(r)}\tilde{V}_{(r)}$ is again a finite–dimensional subspace of $\mathcal{A}$. Choosing a basis $\{e_1, \ldots, e_M\}$ of the finite–dimensional subspace $V_{(r)} + \tilde{V}_{(r)} + V_{(r)}\tilde{V}_{(r)}$ of $\mathcal{A}$ which is compatible with the subspaces, $V_{(r)}$, $\tilde{V}_{(r)}$, and $V_{(r)}\tilde{V}_{(r)}$, expanding the elements $A \in V_{(r)}$, $\tilde{A} \in \tilde{V}_{(r)}$ in that basis, and using the norm $|| \; ||$ as in (5.3.1) by extending the chosen basis to all of $\mathcal{A}$ we get the intermediate estimate

$$\exists\, C_{V_{(r)}\tilde{V}_{(r)}} \in \mathbb{R}, C_{V_{(r)}\tilde{V}_{(r)}} \geqslant 0 \;\; \forall\, A \in V_{(r)} \; \forall\, \tilde{A} \in \tilde{V}_{(r)} : \quad ||A\tilde{A}|| \leqslant C_{V_{(r)}\tilde{V}_{(r)}} ||A||\, ||\tilde{A}||.$$

This shows that the restriction of the multiplication to $V_{(r)} \otimes \tilde{V}_{(r)}$ is a continuous map onto its image $V_{(r)}\tilde{V}_{(r)}$: this fact together with (5.3.8) proves the statement (5.3.7). □

For the rest of this Section we choose the maximum norm $| \; |$ on $\mathbb{R}^\ell$, see (5.3.1) w.r.t. the canonical basis, and suppose that

$$\emptyset \neq J \subset \big\{\delta \in \mathbb{R}^\ell \setminus \{0\} \;\big|\; |\delta| \leqslant 1/4\big\} \quad \text{and} \quad 0 \text{ is an accumulation point of J.} \tag{5.3.9}$$

We shall now distinguish three important subsets $\mathcal{L}$, $\mathcal{B}$, and $\mathcal{H}$ of the algebra (5.3.3): we shall refer to them as the set of all **at most logarithmically divergent**, **bounded** and **harmless elements**, respectively: for an element $F = \sum_{r=0}^{\infty} F_r \lambda^r$ of $\Big(\mathrm{Fun}\,(J, \mathbb{C}) \otimes \mathcal{A}\Big)[[\lambda]]$ we say

$$\begin{array}{lllll} F \in \mathcal{L} & \text{iff} \;\; \forall r \in \mathbb{N} & \exists C_r, \alpha_r \in \mathbb{R},\; C_r \geqslant 0,\; \alpha_r > 0 & \forall \delta \in J : \;\; ||F_r(\delta)|| & \leqslant \;\; C_r |\ln(|\delta|)|^{\alpha_r}, \\ F \in \mathcal{B} & \text{iff} \;\; \forall r \in \mathbb{N} & \exists C_r \in \mathbb{R},\; C_r \geqslant 0, & \forall \delta \in J : \;\; ||F_r(\delta)|| & \leqslant \;\; C_r, \\ F \in \mathcal{H} & \text{iff} \;\; \forall r \in \mathbb{N} & \exists C_r, \beta_r \in \mathbb{R},\; C_r \geqslant 0,\; \beta_r > 0 & \forall \delta \in J : \;\; ||F_r(\delta)|| & \leqslant \;\; C_r |\delta|^{\beta_r}. \end{array} \tag{5.3.10}$$

In all the subsequent computations all the terms which we shall deal with are at most logarithmically divergent in the above sense. Let $\mathcal{G} := 1 + \lambda\Big(\mathrm{Fun}\,(J, \mathbb{C}) \otimes \mathcal{A}\Big)[[\lambda]]$, and define the following subsets

$$\mathcal{G}_{\mathcal{L}} := 1 + \lambda\mathcal{L}, \;\; \mathcal{G}_{\mathcal{B}} := 1 + \lambda\mathcal{B}, \;\; \mathcal{G}_{\mathcal{H}} := 1 + \lambda\mathcal{H}. \tag{5.3.11}$$

and refer to them as the *at most logarithmically divergent, bounded and harmless subgroups* of the group $\mathcal{G}$, respectively. These terms become clear in the following

Proposition 5.3.2. *With the above hypotheses we have the following statements for the algebra (5.3.3):*

(i) The definition (5.3.10) does not depend on the chosen norm.

(ii) $\mathcal{L}$ and $\mathcal{B}$ are unital subalgebras over $\mathbb{C}[[\lambda]]$ of $\Big(\mathrm{Fun}\,(J, \mathbb{C}) \otimes \mathcal{A}\Big)[[\lambda]]$, and $\mathcal{H}$ is a two-sided ideal of $\mathcal{L}$. There are the following inclusions:

$$\mathcal{L} \supset \mathcal{B} \supset \mathcal{H}. \tag{5.3.12}$$

(iii) For all $H \in \mathcal{H}$: $\lim_{\delta \to 0} H(\delta) = 0$.

(iv) $\mathcal{G}$ is a subgroup of the group of all invertible elements of $\Big(\mathrm{Fun}\,(J, \mathbb{C}) \otimes \mathcal{A}\Big)[[\lambda]]$, *and $\mathcal{G}_{\mathcal{L}} \supset \mathcal{G}_{\mathcal{B}} \supset \mathcal{G}_{\mathcal{H}}$ are subgroups of $\mathcal{G}$, where $\mathcal{G}_{\mathcal{H}}$ is a normal subgroup of $\mathcal{G}_{\mathcal{L}}$, i.e. it is stable by conjugations with all elements in $\mathcal{G}_{\mathcal{L}}$.*

(v) For all $\Psi \in \mathcal{G}_{\mathcal{H}}$: $\lim_{\delta \to 0} \Psi(\delta) = 1$.

Before giving the proof of this Proposition we shall recall some elementary inequalities in the following

Lemma 5.3.3. *For all $\delta \in \,]0, 1/4]$ and $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 0$, we have the following inequalities*

$$\delta \leqslant \frac{1}{2} \leqslant |\ln(\delta)|, \tag{5.3.13}$$

$$|\ln(\delta)|^{\alpha}\delta^{\beta} \leqslant \left(\frac{2\alpha}{\beta}\right)^{\alpha} \delta^{\beta/2}. \tag{5.3.14}$$

Proof. Recall the following elementary inequalities for every real number t such that $0 < t \leqslant 1$

$$1 \leqslant \frac{1}{t} \quad \text{hence} \quad 1 - \delta = \int_{\delta}^{1} dt \leqslant \int_{\delta}^{1} \frac{1}{t} dt = -\ln(\delta) = |\ln(\delta)|$$

which proves (5.3.13) upon noting that $\delta \leqslant 1/4 < 1/2 < 1 - \delta$. Moreover, since $\frac{1}{t} \leqslant \left(\frac{1}{t}\right)^{1+\frac{\beta}{2\alpha}}$ we get

$$|\ln(\delta)| = -\ln(\delta) = \int_{\delta}^{1} \frac{1}{t} dt \leqslant \int_{\delta}^{1} \left(\frac{1}{t}\right)^{1+\frac{\beta}{2\alpha}} dt = \frac{2\alpha}{\beta}\left(\delta^{-\beta/(2\alpha)} - 1\right) \leqslant \frac{2\alpha}{\beta}\delta^{-\beta/(2\alpha)},$$

from which –upon multiplying both sides of this inequality by $\delta^{\beta/\alpha}$ and then raising to the power of α– we deduce the result (5.3.14). □

Proof (of Proposition 5.3.2):
(i): Since each F_r takes its values in a finite dimensional vector space the restriction of any other norm to this subspace is equivalent to the norm $||\ ||$: this would only change the 'C-constants' of the definition, but not the criterion to be an element of $\mathcal{L}$, $\mathcal{B}$ or $\mathcal{H}$.
(ii): It is immediate that $\mathcal{L}$, $\mathcal{B}$ and $\mathcal{H}$ are complex vector spaces: if F, F' are in one of the three subsets then by means of the upper bounds of the norms of each $F_r(j)$ and $F'_r(j)$ we get an upper bound of the norm of each $zF_r + z'F'_r$ (where $z, z' \in \mathbb{C}$) by passing to twice the maximum of the two constants $|z|C_r, |z'|C_r$ and to the maximum of the exponents α_r, α'_r of $|\ln(|\delta|)|$ –the latter being > 1– (resp. to the minimum of the exponents β_r, β'_r of $|\delta|$ –the latter being < 1). Next, for the multiplication of FF' we have that each $\big(F(\delta)F'(\delta)\big)_r$ $(r \in \mathbb{N})$ is equal to the sum $\sum_{u=0}^{r} F_u(\delta)F'_{r-u}(\delta)$. Suppose first that $F, F' \in \mathcal{L}$. Since by Proposition 5.3.1 for each $\delta \in J$ every $F_u(\delta)$ is an element of some finite–dimensional subspace V_u (only depending on F_u) and every $F'_{r-u}(\delta)$ is an element of some other finite–dimensional subspace V'_{r-u} (only depending on F_{r-u}) it follows as in the proof of Proposition 5.3.1, equation (5.3.8) that –upon setting $V_{(r)} = V_0 + \cdots + V_r$ and $V'_{(r)} = V_0 + \cdots + V'_r$– the following estimate holds for all $\delta \in J$

$$||\big(F(\delta)F'(\delta)\big)_r|| \leqslant \sum_{u=0}^{r} C_u C'_{r-u} C_{V_{(r)}V'_{(r)}} |\ln(|\delta|)|^{\alpha_u + \alpha'_{r-u}} \leqslant C|\ln(|\delta|)|^{\alpha}$$

where C is $r+1$ times the maximum of all the triple multiplications of the 'C–constants' and α is the maximum of all the numbers $\alpha_u + \alpha'_{r-u}$. This is done in an analogous way for $\mathcal{B}$ and $\mathcal{H}$ proving that $\mathcal{L}$, $\mathcal{B}$ and $\mathcal{H}$ are closed under multiplication. Evidently, $\mathbb{C}[[\lambda]]$ belongs to $\mathcal{L}$ and $\mathcal{B}$, hence $\mathcal{L}$ and $\mathcal{B}$ in particular are $\mathbb{C}[[\lambda]]$–submodules and unital associative algebras. The inclusion (5.3.12) follows at once from inequality (5.3.13). Finally, for any $F \in \mathcal{L}$ and $F' \in \mathcal{H}$ it is shown in a similar way as above that FF' and $F'F$ are in $\mathcal{H}$ upon using the second inequality (5.3.14). This shows that $\mathcal{H}$ is a two–sided ideal of $\mathcal{L}$ and hence also a $\mathbb{C}[[\lambda]]$–submodule.
(iii) – *(v)*: immediately follow from the upper bounds defining $\mathcal{H}$.
(iv): We only have to observe that every element $1 + \lambda F$ (where F is in a $\mathbb{C}[[\lambda]]$–subalgebra) always has an inverse, namely the well–known geometric series $\sum_{r=0}^{\infty}(-\lambda)^r F^r$, for which the terms of positive order are all in the given subalgebra, which proves that $\mathcal{G}_\mathcal{L}$, $\mathcal{G}_\mathcal{B}$ and $\mathcal{G}_\mathcal{H}$ are subgroups of $\mathcal{G}$. The normality of $\mathcal{G}_\mathcal{H}$ follows from the fact that $\mathcal{H}$ is a two-sided ideal of $\mathcal{L}$. □

We shall now apply these limit considerations to the term Y appearing in a formal linear ODE, see (5.2.2). Y normally belongs to the algebra (5.2.1). In order to incorporate limits we shall make Y dependent on the parameter δ in the set $J \subset \mathbb{R}^\ell$, see (5.3.9), i.e. we consider

$$Y \in \Big(\mathrm{Fun}\Big(J, \mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big)\Big) \otimes \mathcal{A}\Big)[[\lambda]]. \tag{5.3.15}$$

Hence, each element Y is a formal power series $\sum_{r=0}^{\infty} Y_r \lambda^r$, where each Y_r is a (non unique) finite sum $Y_r = Y_{r1} \otimes A_{r1} + \cdots + Y_{rN_r} \otimes A_{rN_r}$ where N_r is an non–negative integer, $A_{r1}, \ldots, A_{rN_r} \in \mathcal{A}$ and $Y_{r1}, \ldots, Y_{rN_r}$ are functions on J with values in $\mathcal{C}^\infty_D([a,b],\mathbb{C})$ (see (5.1.1)). It makes sense to consider the formal linear ODE (5.2.2) for these J-dependent Y:

Proposition 5.3.4. *Let Y be an element of the algebra (5.3.15).*
For each $\alpha \in [a,b]$ there exists a unique element

$$W_{\cdot\alpha} \in \Big(\mathrm{Fun}\Big(J, \mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big)^0\Big) \otimes \mathcal{A}\Big)[[\lambda]] \tag{5.3.16}$$

satisfying the formal linear J–dependent ODE (5.2.6) with respect to the parameter $s \in [a,b]$ for each $\delta \in J$ such that $W_{\alpha\alpha} = 1$. We shall call $W_{\cdot\alpha}$ the fundamental solution of the formal linear J–dependent ODE (5.2.6) normalized at α. Moreover, for each $\beta \in [a,b]$ the J–dependent propagator $W_{\beta\alpha} := W_{\cdot\alpha}(s=\beta)$ satisfies

$$W_{\beta\alpha} \in \Big(\mathrm{Fun}\,(J,\mathbb{C}) \otimes \mathcal{A}\Big)[[\lambda]]. \tag{5.3.17}$$

Proof. This is done in complete analogy to the treatment in §5.2 where we can literally follow (5.2.2), the integral equation (5.2.3) –the primitive I_α being extended to the algebra occurring in (5.3.15) by first composing it with the functions of J with values in $\mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big)$ on the left tensor factor, then tensoring with the identity on $\mathcal{A}$, and finally extending componentwise– and the iterated integrals equation (5.2.5). This proves the existence of a unique fundamental solution $W_{\cdot\alpha}$ as in (5.3.16). Evaluating s at β gives (5.3.17). □

The following Lemma will be the key criterion later on to prove that certain factors in a propagator are in the harmless group $\mathcal{G}_\mathcal{H}$. First, as usual, having fixed a norm $||\ ||$ on $\mathcal{A}$ we shall denote by the same symbol $||\ ||$ the map

$$\begin{aligned} \mathrm{Fun}\Big(J, \mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big)\Big) \otimes \mathcal{A} \ &\to\ \mathrm{Fun}\Big(J, \mathcal{C}^0_D\big([a,b],\mathbb{R}\big)\Big) : \\ \Big((\delta,s) \mapsto G(\delta,s)\Big) \ &\mapsto\ \Big((\delta,s) \mapsto ||G(\delta,s)||\Big) \end{aligned}$$

for all $s \in [a,b] \setminus D$. Writing G in $\mathrm{Fun}\Big(J, \mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big)\Big) \otimes \mathcal{A}$ in a basis $((e_i)_{i\in\mathfrak{S}})$ of $\mathcal{A}$ as $G = G_0 e_0 + \cdots + G_N e_N$ we get the well–known estimate –using the monotonicity of the Riemann integral– for any $\alpha \leqslant s \leqslant \beta \in [a,b]$:

$$\left|\left|\int_\alpha^s \hat{G}(\delta, s_1)\mathsf{d}s_1\right|\right| \leqslant (\beta-\alpha)\sup\Big\{||\hat{G}(\delta,s_1)|| \;\Big|\; s_1 \in [0,1]\Big\} \tag{5.3.18}$$

where we have written $\hat{G}$ for any extension of the function G from $J \times \big([a,b]\setminus D\big)$ to $J \times [a,b]$.

Lemma 5.3.5. *Let $Y = \sum_{r=0}\lambda^r Y_r$ be an element of the algebra (5.3.15). Fix an arbitrary norm $||\ ||$ on $\mathcal{A}$ and two elements $\alpha, \beta \in [a,b]$. Consider the following three conditions on Y referred to as upper bounds for Y: there is an extension $\hat{Y}$ of Y to $[a,b]$ such that for each non–negative integer r:*

$$\begin{array}{llll}
(L): & \exists C_r, \alpha_r \in \mathbb{R}, C_r \geqslant 0, \alpha_r > 0 & \forall \delta \in J: \sup\Big\{||\hat{Y}_r(\delta,s)|| \;\Big|\; s\in[a,b]\Big\} \leqslant & C_r|\ln(|\delta|)|^{\alpha_r}, \\
(B): & \exists C_r \in \mathbb{R}, C_r \geqslant 0, & \forall \delta \in J: \sup\Big\{||\hat{Y}_r(\delta,s)|| \;\Big|\; s\in[a,b]\Big\} \leqslant & C_r, \\
(H): & \exists C_r, \beta_r \in \mathbb{R}, C_r \geqslant 0, \beta_r > 0 & \forall \delta \in J: \sup\Big\{||\hat{Y}_r(\delta,s)|| \;\Big|\; s\in[a,b]\Big\} \leqslant & C_r|\delta|^{\beta_r}.
\end{array} \tag{5.3.19}$$

Then the three conditions do not depend on the norms used. Moreover, if condition (L) (resp. (B) resp. (H)) is satisfied then the propagator $W_{\beta\alpha}$ (see (5.3.17)) for Y belongs to the subgroup $\mathcal{G}_{\mathcal{L}}$ (resp. $\mathcal{G}_{\mathcal{B}}$ resp. $\mathcal{G}_{\mathcal{H}}$) of $\mathcal{G}$.

Proof. The norm independence follows from the fact that the norms will always be restricted to finite–dimensional subspaces of $\mathcal{A}$. Concerning the second statement, we first do the case $\alpha \leqslant \beta$: writing out the propagator $W_{\beta\alpha}$ in terms of iterated integrals as in equation (5.2.5) (for $s = \beta$) it can be seen by an easy induction using the estimate (5.3.18) that each iterated integral has as upper bound a multiplication of integrals of the form $\delta \mapsto \int_\alpha^\beta \big|\big|\hat{Y}_i(\delta,s)\big|\big|\mathsf{d}s$ where each such integral has an upper bound by the last inequality of (5.3.18) and thus the desired upper bound according to the conditions (L), (B) or (H). Passing to suitable maxima of multiplications of constants of 'type C', to suitable maxima of sums of exponents of 'type α_i', and to suitable minima of exponents of 'type β' we get the desired upper bounds for (5.3.10). The case $\alpha \geqslant \beta$ is done in a completely analogous manner by using the rule (5.1.6). $\square$

5.4 Formal connections and parallel transport

Definition 5.4.1. *Let $N \geqslant 1$ be an integer, and let $U \subset \mathbb{R}^N$ be a non-empty open subset. A **formal connection** Γ on U consists of N elements $\Gamma_1, \ldots, \Gamma_N \in \big(C^\infty(U,\mathbb{C})\otimes\mathcal{A}\big)[[\lambda]]$.*

We shall consider each Γ_i as a function $\Gamma_i : U \to \mathcal{A}[[\lambda]]$ by evaluating at $x \in U$. Every element Γ_i in the algebra $\big(C^\infty(U,\mathbb{C})\otimes\mathcal{A}\big)[[\lambda]]$ is thus a formal power series $\Gamma_i = \sum_{r=0}^\infty \Gamma_{ir}\lambda^r$ such that each non–negative integer r the component Γ_{ir} is a finite sum of terms of the form $f \otimes A$ with $f \in \mathcal{C}^\infty\big(U,\mathbb{C}\big)$ and $A \in \mathcal{A}$. A very common notation borrowed from differential geometry is $\Gamma = \sum_{i=1}^N \Gamma_i \mathsf{d}x_i$, which relates to differential forms (connection 1–forms). We shall, however, use a sign convention for Γ which is different from the one used in differential geometry to avoid additional signs.
Next, we consider continuous piecewise $\mathcal{C}^\infty$–paths $c : [a,b] \to U$, hence

$$c \in \mathcal{C}^\infty_D\big([a,b],U\big)^0. \tag{5.4.1}$$

This means that each real component $c_1, \ldots, c_N$ of c is an element of $\mathcal{C}^\infty_D\big([a,b],\mathbb{R}\big)^0$ and that for each $s \in [a,b]$ the value $c(s)$ lies in $U \subset \mathbb{R}^N$. The most elementary paths are *line segments*, i.e. given two points $\xi, \eta \in \mathbb{R}^N$ we can consider the affine path joining the initial point ξ and the final point η which is defined by

$$c_{\eta\leftarrow\xi} = c : [0,1] \to \mathbb{R}^N : s \mapsto (1-s)\xi + s\eta. \tag{5.4.2}$$

In case ξ, η are elements of the open subset U it has of course to be checked whether all the values of the affine path also lie in U. If this is the case then it is clear that there is $\epsilon \in \mathbb{R}$, $\epsilon > 0$, such that the right hand side of (5.4.2) makes sense as a $\mathcal{C}^\infty$-function from the larger open interval $]-\epsilon, 1+\epsilon[$ to U.

Returning to general continuous piecewise smooth paths, we can associate to each such path c defined in (5.4.1) the element $Y := \Gamma^{(c)}$ for a formal linear ODE by

$$\Gamma^{(c)} := \sum_{i=1}^{N} \big(\Gamma_i \circ c\big) \frac{\mathsf{d}c_i}{\mathsf{d}s} \quad \in \quad \Big(\mathcal{C}^\infty_D\big([a,b],\mathbb{C}\big) \otimes \mathcal{A}\Big)[[\lambda]].$$

Fix $\alpha, \beta \in [a,b]$, then we can consider the formal linear ODE (5.2.6) for the choice $Y = \Gamma^{(c)}$ and its particular solution ${}^{\Gamma}W^{(c)}_{\cdot\alpha}$ normalized at α. In differential geometry the propagator ${}^{\Gamma}W^{(c)}_{\beta\alpha} = W_{\beta\alpha} \in \mathcal{A}[[\lambda]]$, see (5.2.7), is called the **parallel transport** from $c(\alpha)$ to $c(\beta)$ along the path c (with respect to the connection Γ). Since parallel transports are propagators all the statements of Proposition 5.2.1 are true for parallel transports. Note that for a constant path $c_\xi(s) = \xi \in U$ for all $s \in [a,b]$, the element $\Gamma^{(c_\xi)} = 0$, and the parallel transport is reduced to the unit element of $\mathcal{A}$.

Next, we shall need the composition of two continuous piecewise smooth paths $c_1 : [0,1] \to U$ with singular set D_1 and $c_2 : [0,1] \to U$ with singular set D_2 which are compatible in the groupoid sense $c_2(0) = c_1(1)$. Recall the classical definition from algebraic topology of the composed path $c_2 * c_1 : [0,1] \to U$ with singular set $D_{12} := \big(\frac{1}{2}D_1\big) \cup \big(\frac{1}{2}D_2 + \frac{1}{2}\big)$:

$$\text{if } c_1(1) = c_2(0): \qquad (c_2 * c_1)(s) := \begin{cases} c_1(2s) & \text{if } \ 0 \leqslant s \leqslant \frac{1}{2}, \\ c_2(2s-1) & \text{if } \ \frac{1}{2} \leqslant s \leqslant 1. \end{cases} \tag{5.4.3}$$

It is evident from the definition that $c_2 * c_1$ is continuous and piecewise smooth with singular set D_{12}. The crucial fact is that composition may create new singularities at the point $s = \frac{1}{2}$ for the higher k–fold derivatives of the path for $k \geqslant 1$.

Definition 5.4.2. *Let N' be a positive integer, let U' be a non–empty open subset of $\mathbb{R}^{N'}$, and let $\Theta : U' \to U$ be a $\mathcal{C}^\infty$–map. For any formal connection Γ on U define the* ***pulled–back connection*** *$\Gamma' := \Theta^*\Gamma$ on U' defined by*

$$\forall\, j \in \mathbb{N},\ 1 \leqslant j \leqslant N' : \quad \big(\Theta^*\Gamma\big)_j := \sum_{i=1}^{N} \big(\Gamma_i \circ \Theta\big) \frac{\partial \Theta_i}{\partial x'_j}. \tag{5.4.4}$$

We now state how the above operations on formal connections and paths translate to parallel transports:

Theorem 5.4.3. *Let $U \subset \mathbb{R}^N$ be a non–empty open subset and let Γ be a formal connection defined on U. Let $c : [a,b] \to U$ and $c_1, c_2 : [0,1] \to U$ be continuous piecewise smooth paths. Then we have the following:*

(i) Let θ be a continuous piecewise $\mathcal{C}^\infty$–reparametrization of $\big([a,b],D\big)$. Then

$$\Gamma^{(c\circ\theta)} = \big(\Gamma^{(c)}\circ\theta\big)\frac{\mathsf{d}\theta}{\mathsf{d}s'}, \quad \text{and} \quad {}^{\Gamma}W^{(c)}_{\theta(\beta')\theta(\alpha')} = {}^{\Gamma}W^{(c\circ\theta)}_{\beta'\alpha'} \quad \in \quad \mathcal{A}[[\lambda]]. \tag{5.4.5}$$

This shows that reparametrizations (in the sense of (5.1.4)) do not change parallel transport as long as the initial and final points remain the same.

(ii) In the previous statement suppose that $a'=a$, $b'=b$, $D'=D$, $\theta(\alpha)=\beta$ and $\theta(\beta)=\alpha$. Then we get the well–known inversion formula

$${}^{\Gamma}W^{(c\circ\theta)}_{\beta\alpha} = \left({}^{\Gamma}W^{(c)}_{\beta\alpha}\right)^{-1}. \tag{5.4.6}$$

(iii) The parallel transport along the composed path $c_2 * c_1 : [0,1] \to U$, see (5.4.3) is given as follows

$${}^{\Gamma}W^{(c_2*c_1)}_{10} \quad = \quad {}^{\Gamma}W^{(c_2)}_{10}\;{}^{\Gamma}W^{(c_1)}_{10}. \tag{5.4.7}$$

(iv) Let $U'\subset\mathbb{R}^{N'}$ be a non–empty open subset, let $\Theta : U'\to U$ be a $\mathcal{C}^\infty$–map, let $\Gamma'=\Theta^*\Gamma$ be the pulled-back formal connection, and let $c' : [a,b]\to U'$ be a continuous piecewise smooth path. Then for all $\alpha,\beta\in[a,b]$

$$(\Theta^*\Gamma)^{(c')} = \Gamma^{(\Theta\circ c')} \quad \text{and} \quad {}^{\Theta^*\Gamma}W^{(c')}_{\beta\alpha} = {}^{\Gamma}W^{(\Theta\circ c')}_{\beta\alpha}. \tag{5.4.8}$$

Proof. (i): The formula for $\Gamma^{(c\circ\theta)}$ is straightforward and the equation for the parallel transport is deduced from (5.2.10).
(ii): This is an immediate consequence of (5.4.5) and the last equation of (5.2.9).
(iii): Define the two smooth reparametrizations $\theta_1,\theta_2 : [0,1]\to[0,1]$ given by $\theta_1(s)=\frac{1}{2}s$ and $\theta_2(s)=\frac{1}{2}s+\frac{1}{2}$. We have (suppressing the symbol Γ)

$$W^{(c_2*c_1)}_{10} \overset{(5.2.9)}{=} W^{(c_2*c_1)}_{1\frac{1}{2}}\,W^{(c_2*c_1)}_{\frac{1}{2}0} \overset{(5.4.5)}{=} W^{((c_2*c_1)\circ\theta_2)}_{10}\,W^{((c_2*c_1)\circ\theta_1)}_{10} \quad = \quad W^{(c_2)}_{10}\,W^{(c_1)}_{10}.$$

(iv): This is straightforward from the definitions and the chain rule for partial derivatives. □

Example 5.4.4. *The affine inversion j of the interval $[a,b]$ given by $\iota(s)=a+b-s$ for all $s\in[a,b]$ serves as such a reparametrization for the choice $\alpha=a$ and $\beta=b$ for an inversion in (ii).*

Remark 5.4.5. *It is well–known that composition of paths is in general not associative, i.e. if $c_3 : [0,1]\to U$ is a third path with $c_2(1)=c_3(0)$ then in general $c_3*(c_2*c_1)\neq(c_3*c_2)*c_1$, but the corresponding multiplication of parallel transports does not depend on the bracketing, i.e.*

$$W^{(c_3*(c_2*c_1))}_{10} \overset{(5.4.7)}{=} W^{(c_3)}_{10}\,W^{(c_2)}_{10}\,W^{(c_1)}_{10} \overset{(5.4.7)}{=} W^{((c_3*c_2)*c_1)}_{10}.$$

It turns out that certain connections are formulated by *complex coordinates* which allow for much more compact computations: we do not have to go into the detail of general holomorphic connections, since for our purpose it suffices to study complex rational ones. More precisely, let $U\subset\mathbb{C}^N$ be a non-empty open set. Recall that a complex rational function in N complex variables $z=(z_1,\ldots,z_N)$ defined on U is a quotient $f(z)=\frac{g(z)}{h(z)}$ where $f,g\in\mathbb{C}[z_1,\ldots,z_N]$, hence are complex polynomials in N variables such that g is different from the zero polynomial, and the zeros of g all belong to

$\mathbb{C}^N \setminus U$. Hence, the function $z \mapsto \frac{g(z)}{h(z)}$ is a well-defined function on U. Decomposing each complex variable in real and imaginary part as usual,

$$z_1 = x_1 + iy_1, \ldots, z_N = x_N + iy_N \quad \text{or} \quad z = x + iy,$$

it is clear that each complex rational function f is a particular complex-valued rational function in $2N$ real variables $x_1, \ldots, x_N, y_1, \ldots, y_N =: (x, y)$ which we can write in the following way

$$f(z) = f(x + iy) =: \check{f}(x, y) =: f^{(1)}(x, y) + if^{(2)}(x, y) \tag{5.4.9}$$

with unique real rational functions $f^{(1)}, f^{(2)}$. Hence, each complex rational function is a $\mathcal{C}^\infty$-function in the real variables. Let $\mathbb{C}_U(z)$ denote the set of all complex rational functions which are well-defined on U. It is easy to check that they form a complex unital subalgebra of $\mathcal{C}^\infty(U, \mathbb{C})$. Recall the following well–known rules for the complex derivatives for all integers $1 \leqslant j \leqslant N$:

$$\left(\frac{\partial f}{\partial z_j}\right)^\vee = \frac{1}{2}\left(\frac{\partial \check{f}}{\partial x_j} - i\frac{\partial \check{f}}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial \check{f}}{\partial x_j} + i\frac{\partial \check{f}}{\partial y_j} = 0, \quad \text{hence} \quad \frac{\partial \check{f}}{\partial x_j} = \left(\frac{\partial f}{\partial z_j}\right)^\vee = -i\frac{\partial \check{f}}{\partial y_j} \tag{5.4.10}$$

for all complex rational functions f where the first two equations are easy to check on polynomials and the second is nothing but the well–known 'holomorphicity condition' $\partial f / \partial \bar{z}_j = 0$ for the complex conjugate variables. Next, let Γ be a formal connection on U which is complex rational in the following way:

$$\Gamma(z) = \sum_{j=1}^N \Gamma_j(z)\mathsf{d}z_j \quad \text{and} \quad \forall\, 1 \leqslant j \leqslant N: \ \ \Gamma_j \in \Big(\mathbb{C}_U(z) \otimes \mathcal{A}\Big)[[\lambda]]. \tag{5.4.11}$$

We can rewrite this expression in the $2N$ real x and y coordinates:

$$\begin{aligned}
\Gamma(z) &= \sum_{j=1}^N \Gamma_j(z)\mathsf{d}z_j = \sum_{j=1}^N \Gamma_j(x + iy)(\mathsf{d}x_j + i\mathsf{d}y_j) = \sum_{j=1}^N \check{\Gamma}_j(x, y)\mathsf{d}x_j + \sum_{j=1}^N i\check{\Gamma}_j(x, y)\mathsf{d}y_j \\
&=: \sum_{j=1}^N \check{\Gamma}_j^{[1]}(x, y)\mathsf{d}x_j + \sum_{j=1}^N \check{\Gamma}_j^{[2]}(x, y)\mathsf{d}y_j =: \check{\Gamma}(x, y).
\end{aligned} \tag{5.4.12}$$

and get an ordinary formal connection with components $\check{\Gamma}_j^{[1]} = \check{\Gamma}_j$ in the x_j–directions, and $\check{\Gamma}_j^{[2]} = i\check{\Gamma}_j$ in the y_j–directions.

Let N' be a positive integer, let $U' \subset \mathbb{C}^{N'}$ be a non-empty open set, let $z' = x' + iy' = (z'_1, \ldots, z'_{N'})$ be complex coordinates, and let $\Theta_1, \ldots \Theta_N : U' \to \mathbb{C}$ be complex rational functions such that the map $\Theta = (\Theta_1, \ldots, \Theta_N) : U' \to \mathbb{C}^N$ takes its values in U. We shall write

$$\Theta(z') = \Theta(x' + iy') = \check{\Theta}(x', y') = \Theta^{(1)}(x', y') + i\Theta^{(2)}(x', y') \tag{5.4.13}$$

where $\Theta^{(1)}, \Theta^{(2)} : U' \to \mathbb{R}^N$ are real rational functions.

Proposition 5.4.6. *Let Γ be a complex rational connection on $U \subset \mathbb{C}^N$, and define the complex pullback $\Theta^*\Gamma$ by formula (5.4.4) with x'_j replaced by z'_j. Then*

$$(\Theta^*\Gamma)^\vee = \check{\Theta}^*\check{\Gamma}. \tag{5.4.14}$$

Proof. We compute using (5.4.12):

$$\left(\check{\Theta}^*\check{\Gamma}\right)(x',y') = \sum_{j=1}^{N'}\sum_{i=1}^{N}\Bigg(\check{\Gamma}_i^{[1]}\left(\Theta^{(1)}(x',y'),\Theta^{(2)}(x',y')\right)\frac{\partial\Theta_i^{(1)}}{\partial x'_j}(x',y')\mathsf{d}x'_j$$
$$+\check{\Gamma}_i^{[1]}\left(\Theta^{(1)}(x',y'),\Theta^{(2)}(x',y')\right)\frac{\partial\Theta_i^{(1)}}{\partial y'_j}(x',y')\mathsf{d}y'_j$$
$$+\check{\Gamma}_i^{[2]}\left(\Theta^{(1)}(x',y'),\Theta^{(2)}(x',y')\right)\frac{\partial\Theta_i^{(2)}}{\partial x'_j}(x',y')\mathsf{d}x'_j$$
$$+\check{\Gamma}_i^{[2]}\left(\Theta^{(1)}(x',y'),\Theta^{(2)}(x',y')\right)\frac{\partial\Theta_i^{(2)}}{\partial y'_j}(x',y')\mathsf{d}y'_j\Bigg),$$

hence with (5.4.12) and (5.4.13) we get

$$\left(\check{\Theta}^*\check{\Gamma}\right)(x',y') \overset{(5.4.9)}{=} \sum_{j=1}^{N'}\sum_{i=1}^{N}\Gamma_i\big(\Theta(z')\big)\left(\frac{\partial\check{\Theta}_i}{\partial x'_j}(x',y')\mathsf{d}x'_j+\frac{\partial\check{\Theta}_i}{\partial y'_j}(x',y')\mathsf{d}y'_j\right)$$
$$\overset{(5.4.10)}{=} \sum_{j=1}^{N'}\sum_{i=1}^{N}\left(\Gamma_i\circ\Theta\right)^{\vee}(x',y')\left(\frac{\partial\Theta_i}{\partial z'_j}\right)^{\vee}(x',y')\mathsf{d}z'_j = \left(\sum_{j=1}^{N'}\left(\Theta^*\Gamma\right)_j\mathsf{d}z'_j\right)^{\vee}(x',y'),$$

which proves the Proposition. □

5.5 Flat formal connections

Definition 5.5.1. *Let N be a positive integer, $U\subset\mathbb{R}^N$ a non-empty open subset, and Γ a formal connection on U. Γ is called* ***flat*** *if the following conditions hold:*

$$\forall\ i,j\in\mathbb{N},\ 1\leqslant i,j\leqslant N:\quad 0=\frac{\partial\Gamma_i}{\partial x_j}-\frac{\partial\Gamma_j}{\partial x_i}+\lambda\Big(\Gamma_i\Gamma_j-\Gamma_j\Gamma_i\Big).\tag{5.5.1}$$

Remark 5.5.2. *Any formal connection on an open set of $\mathbb{R}^1$ is flat.*

Moreover, complex rational flatness is equivalent to flatness in the following sense:

Proposition 5.5.3. *Let $U\subset\mathbb{C}^N$ be an open set and let Γ be a formal connection which is complex rational in the sense of (5.4.11). Let $\check{\Gamma}$ be the formal connection in the sense of (5.4.12). Then Γ is flat in the complex sense, i.e.*

$$\forall\ i,j\in\mathbb{N},\ 1\leqslant i,j\leqslant N:\quad 0=\frac{\partial\Gamma_i}{\partial z_j}-\frac{\partial\Gamma_j}{\partial z_i}+\lambda\Big(\Gamma_i\Gamma_j-\Gamma_j\Gamma_i\Big)\tag{5.5.2}$$

if and only if $\check{\Gamma}$ is flat in the usual 'real sense', see (5.5.1) for $2N$ real variables (x,y).

Proof. We denote the right–hand side of (5.5.2) by R_{ij}. Equation (5.4.10) allows to replace complex derivatives $\partial/\partial z_i$ by the real ones, and thanks to (5.4.12) $\check{R}_{ij}$ equals

$$\frac{\partial\check{\Gamma}_i^{[1]}}{\partial x_j}-\frac{\partial\check{\Gamma}_j^{[1]}}{\partial x_i}+\lambda\big[\check{\Gamma}_i^{[1]},\check{\Gamma}_j^{[1]}\big] = -\left(\frac{\partial\check{\Gamma}_i^{[2]}}{\partial y_j}-\frac{\partial\check{\Gamma}_j^{[2]}}{\partial y_i}+\lambda\big[\check{\Gamma}_i^{[2]},\check{\Gamma}_j^{[2]}\big]\right)$$
$$= -i\left(\frac{\partial\check{\Gamma}_i^{[2]}}{\partial x_j}-\frac{\partial\check{\Gamma}_j^{[1]}}{\partial y_i}+\lambda\big[\check{\Gamma}_i^{[2]},\check{\Gamma}_j^{[1]}\big]\right)$$

which exactly gives the components of the right hand side of (5.5.1) for $\check{\Gamma}$ whence the result. $\square$

Remark 5.5.4. *Any complex rational formal connection on an open subset of* $\mathbb{C}^1$ *is flat.*

Moreover, we mention the following well–known result:

Proposition 5.5.5. *Let* N, N' *be positive integers, let* $U \subset \mathbb{R}^N$ *and* $U' \subset \mathbb{R}^{N'}$ *be non–empty open subsets, let* $\Theta : U' \to U$ *be a* $\mathcal{C}^\infty$*–map, and let* Γ *be a flat formal connection on* U*. Then the pulled–back formal connection* $\Gamma' := \Theta^*\Gamma$*, see (5.4.4), is also flat.*

Proof. We get

$$\begin{aligned}
\frac{\partial \Gamma'_u}{\partial x'_v} - \frac{\partial \Gamma'_v}{\partial x'_u} + \lambda\Big(\Gamma'_u\Gamma'_v - \Gamma'_v\Gamma'_u\Big) &= \sum_{i=1}^N \frac{\partial\left((\Gamma_i \circ \Theta)\frac{\partial \Theta_i}{\partial x'_u}\right)}{\partial x'_v} - \sum_{j=1}^N \frac{\partial\left((\Gamma_j \circ \Theta)\frac{\partial \Theta_j}{\partial x'_v}\right)}{\partial x'_u} \\
&\quad + \lambda \sum_{i,j=1}^N \Big((\Gamma_i \circ \Theta)(\Gamma_j \circ \Theta) - (\Gamma_j \circ \Theta)(\Gamma_i \circ \Theta)\Big)\frac{\partial \Theta_i}{\partial x'_u}\frac{\partial \Theta_j}{\partial x'_v} \\
&= \sum_{i=1}^N \left(\frac{\partial^2 \Theta_i}{\partial x'_v \partial x'_u} - \frac{\partial^2 \Theta_i}{\partial x'_u \partial x'_v}\right)(\Gamma_i \circ \Theta) \\
&\quad + \sum_{i,j=1}^N \left(\left(\frac{\partial \Gamma_i}{\partial x_j} - \frac{\partial \Gamma_j}{\partial x_i} + \lambda\Big(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i\Big)\right) \circ \Phi\right)\frac{\partial \Theta_i}{\partial x'_u}\frac{\partial \Theta_j}{\partial x'_v} \\
&= 0 + 0 = 0,
\end{aligned}$$

thanks to the chain rule, Schwartz's rule and to the flatness of Γ whence Γ' is flat. $\square$

The significance of flat connections is the following well–known result about the path–independence of parallel transports:

Theorem 5.5.6. *Let* $N \geqslant 1$*,* $U \subset \mathbb{R}^N$ *be a non–empty open subset and* Γ *be a flat formal connection. Let* $p, q \in U$*,* $\epsilon \in \mathbb{R}$*,* $\epsilon > 0$*, and* $c_0, c_1 :]a - \epsilon, b + \epsilon[\to U$ *be two smooth paths such that*

(i) $c_0(a) = p = c_1(a)$*,* $c_0(b) = q = c_1(b)$*;*

(ii) there exists a smooth homotopy F *between* c_0 *and* c_1*: more precisely, there is an open subset* $\mathcal{O} \subset \mathbb{R}^2$ *with* $]a - \epsilon, b + \epsilon[\times] - \epsilon, 1 + \epsilon[\subset \mathcal{O}$*, and* $F : \mathcal{O} \to U$ *is a* $\mathcal{C}^\infty$*-map satisfying*

$$\begin{array}{lcccc}
\forall\ s \in]a - \epsilon, b + \epsilon[& : & F(s, 0) = c_0(s) & \text{and} & F(s, 1) = c_1(s), \\
\forall\ t \in] - \epsilon, 1 + \epsilon[& : & F(a, t) = p & \text{and} & F(b, t) = q.
\end{array} \tag{5.5.3}$$

Then the parallel transport with respect to Γ *from* p *to* q *along* c_0 *is equal to the parallel transport along* c_1 *with respect to* Γ *from* p *to* q*:* ${}^\Gamma W_{ba}^{(c_0)} = {}^\Gamma W_{ba}^{(c_1)}$*.*

Proof. Consider the smooth map $\Gamma^{(F)} \in \Big(\mathcal{C}^\infty(\mathcal{O}, \mathbb{C}) \otimes \mathcal{A}\Big)[[\lambda]]$ given by

$$\Gamma^{(F)}(s, t) := \sum_{i=1}^N \Gamma_i\big(F(s, t)\big)\frac{\partial F_i}{\partial s}(s, t). \tag{5.5.4}$$

For each $(s,t) \in \mathcal{O}$ let $W^{(F)}_{\cdot a}(s,t)$ denote the parallel transport from p to $F(s,t)$ along the smooth path $s \to F(s,t)$, i.e. $W^{(F)}_{\cdot a}$ (we suppress the symbol Γ attached to W in this proof) satisfies the differential equation

$$\frac{\partial W^{(F)}_{\cdot a}}{\partial s} = \lambda \Gamma^{(F)} W^{(F)}_{\cdot a}. \tag{5.5.5}$$

Since Γ and F are smooth, it follows that $(s,t) \mapsto W^{(F)}_{\cdot a}$ is smooth, and so it is an element of $\big(C^\infty(\mathcal{O},\mathbb{C}) \otimes A\big)[[\lambda]]$: indeed since $W^{(F)}_{\cdot a}$ is made out of iterated integrals (in the s–direction) the claim follows from the usual rule of differentiation of integrals depending on a parameter:

$$\frac{\partial}{\partial t}\int_a^s f(t,s')\mathsf{d}s' = \int_a^s \frac{\partial f}{\partial t}(t,s')\mathsf{d}s'.$$

Differentiating equation (5.5.5) with respect to t, and using equation (5.5.4), we get –upon using the Schwartz rule that all partial derivatives commute–

$$\begin{aligned}
\frac{\partial^2 W^{(F)}_{\cdot a}}{\partial t \partial s} &= \lambda \frac{\partial}{\partial t}\left(\sum_{i=1}^N (\Gamma_i \circ F)\frac{\partial F_i}{\partial s} W^{(F)}_{\cdot a}\right)\\
&= \lambda \sum_{i=1}^N \sum_{j=1}^N \left(\frac{\partial \Gamma_i}{\partial x_j}\circ F\right)\frac{\partial F_j}{\partial t}\frac{\partial F_i}{\partial s}W^{(F)}_{\cdot a} + \lambda \sum_{i=1}^N (\Gamma_i \circ F)\frac{\partial^2 F_i}{\partial t \partial s}W^{(F)}_{\cdot a} + \lambda \sum_{i=1}^N (\Gamma_i \circ F)\frac{\partial F_i}{\partial s}\frac{\partial W^{(F)}_{\cdot a}}{\partial t}\\
&\overset{(5.5.1)}{=} \underline{\lambda \sum_{i=1}^N \sum_{j=1}^N \left(\frac{\partial \Gamma_j}{\partial x_i}\circ F\right)\frac{\partial F_j}{\partial t}\frac{\partial F_i}{\partial s}W^{(F)}_{\cdot a}}\\
&\quad - \lambda^2 \sum_{i=1}^N \sum_{j=1}^N (\Gamma_i \circ F)(\Gamma_j \circ F)\frac{\partial F_j}{\partial t}\frac{\partial F_i}{\partial s}W^{(F)}_{\cdot a} + \underline{\lambda^2 \sum_{i=1}^N \sum_{j=1}^N (\Gamma_j \circ F)(\Gamma_i \circ F)\frac{\partial F_j}{\partial t}\frac{\partial F_i}{\partial s}W^{(F)}_{\cdot a}}\\
&\quad + \underline{\lambda \sum_{i=1}^N (\Gamma_i \circ F)\frac{\partial^2 F_i}{\partial t \partial s}W^{(F)}_{\cdot a}} + \lambda \sum_{i=1}^N (\Gamma_i \circ F)\frac{\partial F_i}{\partial s}\frac{\partial W^{(F)}_{\cdot a}}{\partial t}\\
&\overset{(5.5.5)}{=} \underline{\lambda \frac{\partial}{\partial s}\left(\sum_{i=1}^N (\Gamma_i \circ F)\frac{\partial F_i}{\partial t}W^{(F)}_{\cdot a}\right)} + \lambda \Gamma^{(F)}\left(\frac{\partial W^{(F)}_{\cdot a}}{\partial t} - \lambda \sum_{j=1}^N (\Gamma_j \circ F)\frac{\partial F_j}{\partial t}W^{(F)}_{\cdot a}\right).
\end{aligned}$$

Hence, setting

$$H := \frac{\partial W^{(F)}_{\cdot a}}{\partial t} - \lambda \sum_{j=1}^N (\Gamma_j \circ F)\frac{\partial F_j}{\partial t}W^{(F)}_{\cdot a} \;\in\; \Big(\mathcal{C}^\infty(\mathcal{O},\mathbb{C}) \otimes \mathcal{A}\Big)[[\lambda]],$$

the preceding equation gives us the formal linear ODE

$$\frac{\partial H}{\partial s} = \lambda \Gamma^{(F)} H \tag{5.5.6}$$

with initial condition at $s = a$ for each $t \in \mathcal{O}''$ with $\mathcal{O}'' = \{t \in \mathbb{R} \mid (a,t) \in \mathcal{O}\}$, note that $[0,1] \subset \mathcal{O}''$:

$$H(a,t) = \frac{\partial W^{(F)}_{aa}(t)}{\partial t} - \lambda \sum_{j=1}^N \Gamma_j\big(F(a,t)\big)\frac{\partial F_j}{\partial t}(a,t)W^{(F)}_{aa}(t) \overset{(5.5.3)}{=} \frac{\partial 1}{\partial t}(t) - \lambda \sum_{j=1}^N \Gamma_j(p)\frac{\partial p}{\partial t}(t) = 0.$$

Hence the formal linear ODE (5.5.6) has the unique solution $H(s,t) = 0 \;\forall (s,t) \in \mathcal{O} \supset [a,b]\times[0,1]$. It follows by the definition of H that there is the following formal linear ODE with respect to t:

$$\frac{W^{(F)}_{\cdot a}}{\partial t}(s,t) = \lambda \sum_{j=1}^N \Gamma_j\big(F(s,t)\big)\frac{\partial F_j}{\partial t}(s,t)W^{(F)}_{\cdot a}(s,t),$$

and since there is no derivative with respect to s in this equation, we can set $s = b$ and get

$$\frac{W_{ba}^{(F)}}{\partial t}(t) = \lambda \sum_{j=1}^{N} \Gamma_j\big(F(b,t)\big)\frac{\partial F_j}{\partial t}(b,t) W_{ba}^{(F)}(t) \overset{(5.5.3)}{=} \lambda \sum_{j=1}^{N} \Gamma_j\big(q\big)\frac{\partial q}{\partial t}(t) W_{ba}^{(F)}(t) = 0.$$

It follows that the parallel transport $W_{ba}^{(F)}$ does not depend on t, hence in particular

$$W_{ba}^{(c_0)} = W_{ba}^{(F)}(t=0) = W_{ba}^{(F)}(t=1) = W_{ba}^{(c_1)}.$$

□

We shall give a Corollary to the preceding Theorem which will cover all the cases we shall discuss later: we need to establish first a relation between continuous piecewise smooth paths (which will turn up while doing composition of paths), and overall smooth paths: specializing to $[a,b] = [0,1]$ (which will be only parameter interval in the sequel) we can prove the following

Corollary 5.5.7. *Let $N \in \mathbb{N} \setminus \{0\}$, $U \subset \mathbb{R}^N$ be a non–empty open set, Γ a flat formal connection on U, and $c_1, c_2 : [0,1] \to U$ two* continuous piecewise smooth paths having the same initial point p and final point q. *Suppose that there is an open set $U' \subset \mathbb{R}^N$ and a point $\varpi \in U'$ such that c_1 and c_2 take all their values in $U' \subset U$ and which is star–shaped around $\varpi \in U'$. Then the parallel transports $p \to q$ along c_1 and along c_2 are equal. In particular, the parallel transport along any continuous piecewise smooth loop $c_3 : [0,1] \to U' \subset U$ is trivial, i.e. equal to $1 \in \mathcal{A}$.*

We first need the following well–known Smoothing Lemma which is a technical tool allowing for smoothing reparametrizations: it will only be needed in the proof of Corollary 5.5.7:

Lemma 5.5.8 (Smoothing Lemma). *Let $c : [a,b] \to U \subset \mathbb{R}^N$ be a continuous piecewise smooth path with potential singular set D. Then there exists a smooth map $\theta : \mathbb{R} \to [a,b]$ such that its restriction to $[a,b]$ is strictly monotonous and surjective (hence has a continuous inverse on $[a,b]$), it induces the identity map on D, and whose higher derivatives all vanish at the points of D. Moreover, the composition $c \circ \theta : \mathbb{R} \to [a,b]$ (in the sense of (5.1.5)) is an everywhere well–defined smooth map all of whose higher derivatives (for $r \geqslant 1$) vanish at all points of D. For all $s \leqslant a$ the map $c \circ \theta$ takes the constant value $c(a)$, and for all $s \geqslant b$ it takes the constant value $c(b)$. In particular, for every positive ϵ the restriction of $c \circ \theta$ to $[a-\epsilon, b+\epsilon]$ yields a reparametrized path which is smooth.*

Proof. Let $\rho : \mathbb{R} \to \mathbb{R}$ be the following function

$$\rho(s) := \begin{cases} e^{-1/s} & \text{if } s > 0, \\ 0 & \text{if } s \leqslant 0. \end{cases}$$

It is well–known and not hard to see that ρ is $\mathcal{C}^\infty$, has all of its higher derivatives equal to zero at 0 and only strictly positive values for $s > 0$. Define the function $\vartheta : \mathbb{R} \to \mathbb{R}$ by

$$\vartheta(s) := \sum_{i=0}^{m} \big(a_{i+1} - a_i\big) \frac{\rho(s - a_i)\rho(a_{i+1} - s)}{\int_{a_i}^{a_{i+1}} \rho(s' - a_i)\rho(a_{i+1} - s')\mathsf{d}s'}$$

and $\theta : \mathbb{R} \to \mathbb{R}$ by the primitive of ϑ:

$$\theta(s) := a + \int_a^s \vartheta(s')\mathsf{d}s'.$$

Then all the properties of θ follow from the fact that $a_{i+1} - a_i > 0$ and that the smooth function $s \mapsto \rho(s - a_i)\rho(a_{i+1} - s)$ is strictly positive on $]a_i, a_{i+1}[$ and zero outside $]a_i, a_{i+1}[$. Moreover, it is clear that the composition $c \circ \theta$ is smooth on $[a, b] \setminus D$ as a composition of smooth maps. By the iterated chain rule it follows that all the higher derivatives of $c \circ \theta$ tend to zero at the points of D since all the higher derivatives of θ go to zero at these points whereas all the higher left–side and right–side derivatives of c remain bounded. □

Proof. (of Corollary 5.5.7):
Let $c : [0, 1] \to U' \subset U$ be the continuous piecewise smooth loop with $c(0) = c(1) = p$ defined by the composition $(c_2 \circ \iota) * c_1$ where $\iota : [0, 1] \to [0, 1]$ is the interval inversion $\iota(s) = 1 - s$. Furthermore, let $d : [0, 1] \to U'$ be the affine path joining ϖ with p, i.e. $d(s) = (1 - s)\varpi + sp$. Let $\check{c} : [0, 1] \to U'$ be the piecewise smooth path $\check{c} := \big(d \circ \iota\big) * (c * d)$. Clearly, $\check{c}$ is a continuous piecewise smooth loop based at ϖ. Choose a smooth reparametrization θ of the path $\check{c}$ in the sense of the preceding Lemma 5.5.8. Recall that θ is a smooth map $\mathbb{R} \to [0, 1]$ with $\theta(s) = 0$ for all $s \leqslant 0$ and $\theta(s) = 1$ for all $s \geqslant b$. Thanks to (5.4.5) and to the fact that $\theta(0) = 0$, $\theta(1) = 1$ we have the following equality of parallel transports

$$ {}^{\Gamma}W_{10}^{(\check{c}\circ\theta)} = {}^{\Gamma}W_{10}^{(\check{c})} = {}^{\Gamma}W_{10}^{((d\circ\iota)*(c*d))} \overset{(5.4.7)}{=} {}^{\Gamma}W_{10}^{(d)^{-1}}\, {}^{\Gamma}W_{10}^{(c)}\, {}^{\Gamma}W_{10}^{(d)}. \tag{5.5.7} $$

Next, the map $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}^N$ defined by

$$ \tilde{F}(s, t) = (1 - t)\check{c}\big(\theta(s)\big) + tq $$

is clearly smooth, hence in particular continuous, whence the inverse image $\mathcal{O} := \tilde{F}^{-1}(U')$ is an open subset of $\mathbb{R}^2$ which contains the rectangle $[0, 1] \times [0, 1]$ thanks to the hypothesis that U' is star-shaped around ϖ and that all the points of the loop c and hence of $\check{c}$ are in U'. By compactness of $[0, 1]$ there is $\epsilon > 0$ such that the open rectangle $] - \epsilon, 1 + \epsilon[\times] - \epsilon, 1 + \epsilon[$ is still contained in $\mathcal{O}$ thanks to the Heine–Borel Theorem. Let $F : \mathcal{O} \to U' \subset \mathcal{U} \subset \mathbb{R}^N$ denote the restriction of $\tilde{F}$ to $\mathcal{O}$. Then F clearly satisfies all the hypotheses of Theorem 5.5.6: for all $s \in] - \epsilon, 1 + \epsilon[$ we have $F(s, 0) = \check{c}\big(\theta(s)\big) =: c_{(0)}(s)$ and $F(s, 1) = \varpi =: c_{(1)}(s)$ (the constant loop at ϖ), and of course for all $t \in] - \epsilon, 1 + \epsilon[$ we get $F(0, t) = \varpi = F(1, t)$. By Theorem 5.5.6 we get

$$ {}^{\Gamma}W_{10}^{(\check{c}\circ\theta)} = {}^{\Gamma}W_{10}^{(c_{(1)})} = 1 $$

since $c_{(1)}$ is the constant loop whence $\Gamma^{(c_{(1)})} = 0$. Thanks to equation (5.5.7) we get

$$ 1 = {}^{\Gamma}W_{10}^{(c)} = \Big({}^{\Gamma}W_{10}^{(c_2)}\Big)^{-1}\, {}^{\Gamma}W_{10}^{(c_1)} $$

which proves the statement. The case of a continuous piecewise smooth loop c_3 based at p is a particular case of the preceding statement upon choosing the constant loop c_4 at p as a second path. □

5.6 The Knizhnik-Zamolodchikov connection and the Drinfeld-Kohno (Lie) algebras

Definition 5.6.1. *Let $n \geqslant 2$ be an integer and let A_{ij}, $1 \leqslant i \neq j \leqslant n$ be elements of $\mathcal{A}$. We say that the elements A_{ij} satisfy the **infinitesimal braid relations** if*

$$A_{ij} - A_{ji} = 0 \qquad \forall\ 1 \leqslant i \neq j \leqslant n, \tag{5.6.1a}$$

$$[A_{ij} + A_{ik}, A_{jk}] = 0 \qquad \forall i,j,k \in \{1,\dots,n\} \ \ \text{such that } \#\{i,j,k\} = 3, \tag{5.6.1b}$$

$$[A_{ij}, A_{kl}] = 0 \qquad \forall i,j,k,l \in \{1,\dots,n\} \ \ \text{such that } \#\{i,j,k,l\} = 4. \tag{5.6.1c}$$

The following definition appears in [Koh87, p.142, eqn (1.1.4)] and [Koh87, p.146, eqn (1.3.2)]:

Definition 5.6.2. *Let $n \geqslant 2$ be an integer and let $\mathbb{K}$ be a field of characteristic zero. The **n–th Drinfeld–Kohno algebra** is the $\mathbb{K}$–algebra $\mathcal{T}_n$ generated by $n^2 - n$ elements t_{ij}, $1 \leqslant i \neq j \leqslant n$, subject to the infinitesimal braid relations. Similarly, the **n–th Drinfeld–Kohno Lie algebra** is the $\mathbb{K}$–Lie algebra $\mathfrak{t}_n$ generated by $n^2 - n$ generators t_{ij}, $1 \leqslant i \neq j \leqslant n$, subject to the relations (5.6.1a), (5.6.1b), and (5.6.1c), seen as Lie brackets in the free Lie algebra generated by the t_{ij}.*

Definition 5.6.3. *For each integer $n \geqslant 2$ we define the n–th **ordered configuration space** as*

$$Y_n := \{ z \in \mathbb{C}^n \mid \forall\ i,j \in \mathbb{N},\ 1 \leqslant i,j \leqslant n : \ \text{if } i \neq j \text{ then } z_i \neq z_j \}. \tag{5.6.2}$$

Recall that the usual permutation of coordinates defines a right action of the permutation group $\mathfrak{S}_n$ on $\mathbb{C}^n$ given by $z = (z_1,\dots,z_n) \mapsto (z_{\sigma(1)},\dots,z_{\sigma(n)}) =: z\sigma$ for each permutation $\sigma \in \mathfrak{S}_n$. This right action preserves Y_n on which it acts freely, and the quotient $X_n := Y_n/\mathfrak{S}_n$ is a complex n–dimensional manifold called the n–th (unordered) configuration space. The fundamental groups of X_n and of Y_n are well–known to be isomorphic to the braid group of n strands, B_n, and to the pure braid group of n strands, $P_n \subset B_n$, respectively.
The following well–known (formal) connection is very important, see [KZ84]:

Definition 5.6.4. *Let $n \geqslant 2$ and $\mathcal{A}$ be a complex algebra containing $n(n-1)/2$ elements $A_{ij} = A_{ji}$ (indexed by $1 \leqslant i \neq j \leqslant n$) satisfying the infinitesimal braid relations (5.6.1b) and (5.6.1c). The formal **Knizhnik–Zamolodchikov** (КнЗа)**-connection** ${}^{(n)}\Gamma_{\text{КнЗа}}$ on Y_n (with respect to $\mathcal{A}$) is*

$${}^{(n)}\Gamma_{\text{КнЗа}}(z_1,\dots,z_n) := \sum_{1 \leqslant i < j \leqslant n} \frac{A_{ij}}{z_i - z_j}(\mathsf{d}z_i - \mathsf{d}z_j). \tag{5.6.3}$$

Clearly, the КнЗа-connection is complex rational in the sense of (5.4.11). We have the following

Theorem 5.6.5. *For all integers $n \geqslant 2$ the Knizhnik–Zamolodchikov connection is (formally) flat.*

A very detailed proof of this statement can be found in [Kas12, p.452-454].

Remark 5.6.6. *Note that:*

(i) For all integers $n \geqslant 2$ the КнЗа–connection ${}^{(n)}\Gamma_{\text{КнЗа}}$ is invariant by all pull–backs with respect to translations $T_v : Y_n \to Y_n : z \mapsto z + (v,v,\dots,v)$ (for all $v \in \mathbb{C}$) and with respect to all complex homotheties $H_p : Y_n \to Y_n$ given by $z \mapsto pz$ for all $p \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, in the sense that $T_v^*\left({}^{(n)}\Gamma_{\text{КнЗа}}\right) = {}^{(n)}\Gamma_{\text{КнЗа}}$ and $H_p^*\left({}^{(n)}\Gamma_{\text{КнЗа}}\right) = {}^{(n)}\Gamma_{\text{КнЗа}}$.

(ii) *For any integers $1 \leqslant i \neq j \leqslant n$, if 'particle i is near to particle j', (i.e. the distance $|z_i - z_j|$ becomes 'very small') then the term containing A_{ij} in the* КнЗа*-connection will be 'very large' compared to the others: this intuition will motivate the choice of paths in the following sections.*

(iii) *For $n = 2$ and $n = 3$ there are the following isomorphisms of open sets of $\mathbb{C}^2$ and of $\mathbb{C}^3$ which are given by explicit bijective complex rational maps:*

$$Y_2 \;\cong\; \mathbb{C}^\times \times \mathbb{C} \quad \text{and} \quad Y_3 \;\cong\; \mathbb{C}^{\times\times} \times \mathbb{C}^\times \times \mathbb{C}$$

where $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ and $\mathbb{C}^{\times\times} := \mathbb{C} \setminus \{0,1\}$. In fact, the invertible linear map $(z_1, z_2) \mapsto (z_1 - z_2, z_1 + z_2)$ gives the first isomorphism. For $n = 3$ the invertible rational map $Y_3 \to \mathbb{C}^{\times\times} \times \mathbb{C}^\times \times \mathbb{C}$ given by

$$(z_1, z_2, z_3) \mapsto \left(\frac{z_2 - z_1}{z_3 - z_1}, z_3 - z_1, z_1\right) \quad \text{with inverse} \quad \vartheta : (z, v, w) \mapsto (w, zv + w, v + w) \tag{5.6.4}$$

defines an isomorphism concerning Y_3, see also [Dri90b, p.1453] *or* [Kas12, p.469, eqn (7.3)]. *An elementary computation shows that for $n = 3$ the pullback of ${}^{(3)}\Gamma_{\text{КнЗа}}$ with respect to the rational map $\vartheta : \mathbb{C}^{\times\times} \times \mathbb{C}^\times \times \mathbb{C} \to Y_3$, see (5.6.4), is equal to*

$$\left(\vartheta^* \left({}^{(3)}\Gamma_{\text{КнЗа}}\right)\right)(z, v, w) = \left(\frac{A_{12}}{z} + \frac{A_{23}}{z - 1}\right) \mathsf{d}z \; + \; \frac{A_{12} + A_{13} + A_{23}}{v} \mathsf{d}v \tag{5.6.5}$$

where we have used Proposition 5.4.6 and equations (5.4.4) and (5.4.14). Note further that the right action of the permutation group $\mathfrak{S}_3$ on Y_3 can be transferred to $\mathbb{C}^{\times\times} \times \mathbb{C}^\times \times \mathbb{C}$ and projected to $\mathbb{C}^{\times\times}$ by means of the maps (5.6.4). This gives the following maps on $\mathbb{C}^{\times\times}$:

$$\tau_{12}(z) = \tfrac{z}{z-1}, \quad \tau_{23}(z) = \tfrac{1}{z}, \quad \tau_{13}(z) = 1 - z, \quad \zeta(z) = \tfrac{1}{1-z}, \quad \zeta^{-1}(z) = \zeta\big(\zeta(z)\big) = \tfrac{z-1}{z} \tag{5.6.6}$$

where τ_{ij} denotes the transposition exchanging i and j, and $\zeta = (132) \in \mathfrak{S}_3$.

5.7 The Drinfeld associator: definition and properties

In this Section the Drinfeld associator is treated: we are not following the usual definition, but use the statement of [Kas12, p.465, Lemma XIX.6.3] as a definition. The parallel transport we are interested in is denoted there by $G_a(1 - a)$ with $a = \delta$. From now on we set $\lambda = \frac{\hbar}{2\pi i}$.
Set $U :=]0, 1[$ and $J :=]0, 1/4]$. For any $A, B \in \mathcal{A}$ define the formal connection

$$\Gamma(B, A)(x) := \left(\frac{1}{x} A + \frac{1}{x - 1} B\right) \mathsf{d}x \tag{5.7.1}$$

on U which is a flat formal connection, see §5.5, because U is one–dimensional. Note that the interval inversion $\iota :]0, 1[\to]0, 1[$ defined by

$$\iota(x) = 1 - x \tag{5.7.2}$$

is well–defined and smooth, and it is easy to compute the pulled–back connection

$$\iota^* \Gamma(B, A) = \Gamma(A, B). \tag{5.7.3}$$

For all $\delta, \epsilon \in J$ we define the affine path $c_{(\delta\epsilon)} : [0,1] \to U$ from δ to $1-\epsilon$, namely

$$c_{(\delta\epsilon)}(s) := (1-s)\delta + s(1-\epsilon) = \delta + s(1-\delta-\epsilon). \tag{5.7.4}$$

Then

$$\Gamma(B,A)^{(c_{(\delta\epsilon)})}(s) = \frac{1-\delta-\epsilon}{\delta + s(1-\delta-\epsilon)} A + \frac{1-\delta-\epsilon}{\delta - 1 + s(1-\delta-\epsilon)} B. \tag{5.7.5}$$

We are interested in the parallel transport ${}^{\Gamma(B,A)}W_{10}^{(c_{(\delta\epsilon)})}$ along the path $c_{(\delta\epsilon)}$ from δ to $1-\epsilon$. Setting $J' := J \times J$ it follows from the general theory described in §5.3 that the map $(\delta,\epsilon) \mapsto {}^{\Gamma(B,A)}W_{10}^{(c_{(\delta\epsilon)})}$ is an element of the algebra $\Big(\mathrm{Fun}\big(J',\mathbb{C}\big) \otimes \mathcal{A}\Big)[[\lambda]]$, see (5.3.17). It can be expressed in terms of iterated integrals in the following way: we make a change of variables $u := \delta + s(1-\delta-\epsilon)$, and we set $A_0 := A$, $A_1 := B$. Hence,

$$\begin{aligned}{}^{\Gamma(B,A)}W_{10}^{(c_{(\delta\epsilon)})} &= 1 + \sum_{r=1}^{\infty} \lambda^r \sum_{i_1,\ldots,i_r=0}^{1} \\ &\left(\int_\delta^{1-\epsilon} \frac{1}{u_1 - i_1} \left(\int_\delta^{u_1} \frac{1}{u_2 - i_2} \left(\cdots \left(\int_\delta^{u_{r-1}} \frac{1}{u_r - i_r} \mathrm{d}u_r\right) \cdots\right) \mathrm{d}u_2\right) \mathrm{d}u_1\right) \\ &A_{i_1} \cdots A_{i_r}\end{aligned} \tag{5.7.6}$$

It is to be expected that the preceding expression becomes singular whenever $\delta \to 0$ or $\epsilon \to 0$: in order to see this assume for a moment that A and B commute. Clearly, $\Gamma(B,A)^{(c_{(\delta\epsilon)})}$ commutes with its primitive, and a straightforward computation following formula (5.2.12) of Proposition 5.2.1 gives

$$\text{if } AB = BA \quad \text{then} \quad {}^{\Gamma(B,A)}W_{10}^{(c_{(\delta\epsilon)})} = e^{\lambda \ln(\epsilon) B} e^{\lambda\big(\ln(1-\epsilon)A - \ln(1-\delta)B\big)} e^{-\lambda \ln(\delta) A} \tag{5.7.7}$$

showing that the divergences of the parallel transport are the left and the right factors and are logarithmic for $\delta \to 0$ or $\epsilon \to 0$ in that particular case whereas the middle factor converges to 1. Returning to the general case, in order to capture the singular terms we shall break the computation in two parts separated by the mid–point 1/2: consider the following 'exponential half–paths' $\tilde{c}_{(1,\delta)}, \tilde{c}_{(2,\epsilon)} : [0,1] \to U$ defined by

$$\begin{array}{rcll} \tilde{c}_{(1,\delta)}(s) & := & \frac{1}{2} e^{\ln(2\delta)(1-s)} & \text{joining} \quad \delta \;\to\; \frac{1}{2}, \\ \tilde{c}_{(2,\epsilon)}(s) & := & 1 - \frac{1}{2} e^{\ln(2\epsilon)s} & \text{joining} \quad \frac{1}{2} \;\to\; 1-\epsilon. \end{array} \tag{5.7.8}$$

Hence, the composed path $\tilde{c}_{(2,\epsilon)} * \tilde{c}_{(1,\delta)}$ is continuous and piecewise smooth with singular set $D = \{0, 1/2, 1\}$ and joins $\delta \to 1-\epsilon$. The following continuous piecewise smooth reparametrization $\gamma : [0,1] \to [0,1]$ (with singular set $\{0,1/2,1\}$) obviously links the affine path $c_{(\delta\epsilon)}$ with $\tilde{c}_{(2,\epsilon)} * \tilde{c}_{(1,\delta)}$:

$$\gamma(s) := \begin{cases} \frac{\frac{1}{2} e^{\ln(2\delta)(1-2s)} - \delta}{1-\delta-\epsilon} & \text{if } 0 \leqslant s \leqslant \frac{1}{2}, \\ \frac{1 - \frac{1}{2} e^{\ln(2\delta)(2s-1)} - \delta}{1-\delta-\epsilon} & \text{if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases} \qquad \text{hence} \quad c_{(\delta\epsilon)} \circ \gamma = \tilde{c}_{(2,\epsilon)} * \tilde{c}_{(1,\delta)}. \tag{5.7.9}$$

Using the interval inversion ι as a continuous piecewise smooth reparametrization $[0,1] \to [0,1]$ given by (5.7.2) we can write

$$\tilde{c}_{(1,\delta)} = \iota \circ \tilde{c}_{(2,\delta)} \circ \iota.$$

Since parallel transport is independent on reparametrizations, see (5.4.5), we get

$$
\begin{aligned}
{}^{\Gamma(B,A)}W_{10}^{(c_{(\delta\epsilon)})} &= {}^{\Gamma(B,A)}W_{10}^{(c_{(\delta\epsilon)}\circ\gamma)} \overset{(5.7.9)}{=} {}^{\Gamma(B,A)}W_{10}^{(\tilde{c}_{(2,\epsilon)}*\tilde{c}_{(1,\delta)})} \\
&\overset{(5.4.7)}{=} {}^{\Gamma(B,A)}W_{10}^{(\tilde{c}_{(2,\epsilon)})}\ {}^{\Gamma(B,A)}W_{10}^{(\iota\circ\tilde{c}_{(2,\delta)}\circ\iota)} \\
&\overset{(5.4.8),(5.7.3)(5.4.6)}{=} {}^{\Gamma(B,A)}W_{10}^{(\tilde{c}_{(2,\epsilon)})}\ \left({}^{\Gamma(A,B)}W_{10}^{(\tilde{c}_{(2,\delta)})}\right)^{-1}.
\end{aligned} \tag{5.7.10}
$$

It follows that it suffices to compute the parallel transport along the exponential half-path $\tilde{c}_{(2,\epsilon)}$, the parallel transport along the other half $\tilde{c}_{(1,\delta)}$ follows from the symmetry and an exchange of A and B. The choice of the exponential function in the path $\tilde{c}_{(2,\epsilon)}$ becomes clear when computing

$$
\Gamma(B,A)^{(\tilde{c}_{(2,\epsilon)})}(s) = \ln(2\epsilon)B + \frac{-\ln(2\epsilon)}{2e^{-\ln(2\epsilon)s}-1}A, \tag{5.7.11}
$$

and we see that the term in front of B does not depend on s.

Lemma 5.7.1. *We have the following factorization of the parallel transport* $s \mapsto {}^{\Gamma(B,A)}W_{s0}^{(\tilde{c}_{(2,\epsilon)})}$ *in the algebra* $\Big(\mathrm{Fun}\big(]0,1/4],\mathcal{C}^{\infty}_{\{0,1\}}([0,1],\mathbb{C})\big)\otimes\mathcal{A}\Big)[[\lambda]]$

$$
{}^{\Gamma(B,A)}W_{s0}^{(\tilde{c}_{(2,\epsilon)})} = e^{\lambda\ln(\epsilon)sB}\psi_\epsilon(B,A)(s) \tag{5.7.12}
$$

where $(s,\epsilon)\mapsto\psi_\epsilon(B,A)(s)$ *is in the group* $\mathcal{G}_{\mathcal{B}}$ *of bounded terms (with respect to* (s,ϵ)*, see (5.3.10) and (5.3.11)). We set* $\psi_\epsilon(B,A) := \psi_\epsilon(B,A)(1)$*. Moreover, there is a well–defined element* $\psi(B,A)\in\mathcal{A}[[\lambda]]$ *such that the following limit exists*

$$
\lim_{\epsilon\to 0}\psi_\epsilon(B,A)(s) = \begin{cases} 1 & \text{if } s=0, \\ \psi(B,A)\ \in\ \mathcal{A}[[\lambda]] & \text{if } 0<s\leqslant 1, \end{cases} \tag{5.7.13}
$$

in the sense of limits discussed in §5.3, see (5.3.2), (5.3.4) and Proposition 5.3.1.

Proof. In (5.7.11) we set

$$
Y_\epsilon(s) := \ln(2\epsilon)B \quad\text{and}\quad Z_\epsilon(s) := \frac{-\ln(2\epsilon)}{2e^{-\ln(2\epsilon)s}-1}A
$$

and use the factorization statement (5.2.11):

$$
{}^{\Gamma(B,A)}W_{s0}^{(\tilde{c}_{(2,\epsilon)})} = U_{s0}^{(\epsilon)}\Xi_{s0}^{(\epsilon)}. \tag{5.7.14}
$$

Clearly, the formal linear ODE $\mathsf{d}U_{\cdot 0}^{(\epsilon)}/\mathsf{d}s = \lambda Y_\epsilon U_{\cdot 0}^{(\epsilon)}$ with initial condition 1 is trivially given by the exponential function $U_{s0}^{(\epsilon)} = e^{\lambda\ln(2\epsilon)sB}$, and we have to solve the formal linear ODE with initial condition 1,

$$
\frac{\mathsf{d}\Xi_{s0}^{(\epsilon)}}{\mathsf{d}s}(s) = e^{-\lambda\ln(2\epsilon)sB}Z_\epsilon(s)e^{\lambda\ln(2\epsilon)sB}\Xi_{s0}^{(\epsilon)} = e^{-\lambda\ln(2\epsilon)s\,\mathsf{ad}_B}\big(Z_\epsilon(s)\big)\Xi_{s0}^{(\epsilon)}
$$

where $\mathsf{ad}_B:\mathcal{A}\to\mathcal{A}$ denotes the usual adjoint map $\xi\mapsto B\xi-\xi B$, and we have used the well–known identity that conjugation with exponentials is the exponential of ad which is standard in Lie group theory, see e.g. [KMS93, p.38, Cor.4.25]. We can compute the solution $\Xi_{s0}^{(\epsilon)}$ in terms of iterated integrals, see (5.2.5), where the following abbreviations make computations easier: set $\nu := -\ln(2\epsilon)$

and $\tau := \nu s, \tau_i := \nu s_i \quad \forall\, i \in \mathbb{N}$. Since $0 < 2\epsilon \leqslant 1/2 < 1$ it follows that $\nu > 0$ and that the limit $\lim_{\epsilon\to 0}$ corresponds to $\lim_{\nu\to+\infty}$. Then Ξ_ϵ is given by the following expression:

$$\Xi^{(\epsilon)}_{s0} = 1 + \sum_{r=1}^{\infty} \lambda^r \sum_{\ell_1,\ldots,\ell_r=0}^{\infty} \frac{\lambda^{\ell_1+\cdots+\ell_r}}{\ell_1!\cdots\ell_r!} \mathsf{ad}_B^{\ell_1}(A)\cdots\mathsf{ad}_B^{\ell_r}(A) \underbrace{\left(\int_0^{\nu s} g_1 \left(\int_0^{\tau_1} g_2 \left(\cdots\left(\int_\delta^{\tau_{r-1}} g_r \mathsf{d}\tau_r\right)\cdots\right)\mathsf{d}\tau_2\right)\mathsf{d}\tau_1\right)}_{:=I_{r,\ell_1,\ldots,\ell_r}(s,\nu)} \tag{5.7.15}$$

where $g_i := \frac{\tau_i^{\ell_i}}{2e^{\tau_i}-1}$. We shall prove that for all non–negative integers $r, \ell_1, \ldots, \ell_r$ with $r \geqslant 1$ and all $s \in [0,1]$ the iterated real integral $I_{r,\ell_1,\ldots,\ell_r}(s,\nu)$ at the end of (5.7.15) converges to a non–negative real number for $\nu \to +\infty$: this will prove that the limit $\lim_{\epsilon\to 0}\Xi_\epsilon(s)$ exists. In case $s = 0$ this is of course obvious since all these integrals vanish. For $s > 0$, the crucial observation is that all the real numbers $\tau_1, \ldots, \tau_r$ are non–negative whence all the functions $\tau_i \mapsto \frac{\tau_i^{\ell_i}}{2e^{\tau_i}-1}$, $i \in \mathbb{N}\setminus\{0\}$, take non–negative values on the interval $[0, \nu s]$. Thanks to the monotonicity of the Riemann integral it follows that enlarging ν makes the interval $[0, \nu s]$ bigger which in turn makes the value of the iterated integral larger: hence the function $[\ln(2), +\infty[\ \mapsto\ [0, +\infty[$ given by $\nu \mapsto I_{r,\ell_1,\ldots,\ell_r}(s,\nu)$ is strictly increasing. By the well–known principle stating that every increasing bounded sequence of real numbers converges it suffices to show that all the integrals $I_{r,\ell_1,\ldots,\ell_r}(s,\nu)$ admit an upper bound independent on all $s \in [0,1]$ and $\nu \in [\ln(2), +\infty[$: indeed, the elementary inequality $e^{\tau_i} - 1 \geqslant 0$ for all positive integer i (since $\tau_i \geqslant 0$) implies

$$\forall\, i \in \mathbb{N}\setminus\{0\}: \quad 2e^{\tau_i} - 1 = e^{\tau_i} + e^{\tau_i} - 1 \geqslant e^{\tau_i}, \quad \text{hence} \quad \frac{\tau_i^{\ell_i}}{2e^{\tau_i}-1} \leqslant \tau_i^{\ell_i} e^{-\tau_i},$$

and the integral $I_{r,\ell_1,\ldots,\ell_r}(s,\nu)$ can be bounded by

$$I_{r,\ell_1,\ldots,\ell_r}(s,\nu) \leqslant \left(\int_0^{\nu s} \tau_1^{\ell_1} e^{-\tau_1}\mathsf{d}\tau_1\right)\cdots\left(\int_0^{\nu s} \tau_r^{\ell_r} e^{-\tau_r}\mathsf{d}\tau_r\right) \leqslant \ell_1!\cdots\ell_r!$$

thanks to the well–known integral (for all non–negative integers n)

$$\int_0^\infty \tau^n e^{-\tau}\mathsf{d}\tau = n!.$$

This shows that the limit $\lim_{\epsilon\to 0}\Xi_\epsilon(s)$ exists and does not depend on $0 < s \leqslant 1$. Using the factorization equation (5.7.14), the trivial fact that $\ln(2\epsilon) = \ln(2) + \ln(\epsilon)$ and defining

$$\psi_\epsilon(B,A)(s) := e^{\lambda \ln(2) s B}\Xi_\epsilon(s)$$

shows the factorization equation (5.7.12) and the limit (5.7.13). In particular, it implies that $\epsilon \mapsto \psi_\epsilon(B,A)$ is bounded, i.e. it is an element of $\mathcal{G}_{\mathcal{B}}$. □

This Lemma –together with the factorization equation (5.7.10)– has the following consequence

Theorem 5.7.2. *The parallel transport* ${}^{\Gamma(B,A)}W_{10}^{(c_{(\delta\epsilon)})}$ *along the path* $c_{(\delta\epsilon)}$*, see (5.7.4), factorizes in the following way*

$${}^{\Gamma(B,A)}W_{10}^{(c_{(\delta\epsilon)})} = e^{\lambda\ln(\epsilon)B}\Phi_{\delta,\epsilon}(A,B)e^{-\lambda\ln(\delta)A}, \tag{5.7.16}$$

with

$$\Phi_{\delta,\epsilon}(A,B) := \psi_\epsilon(B,A)\,(\psi_\delta(A,B))^{-1}\,. \tag{5.7.17}$$

The following limit exists,

$$\lim_{(\delta,\epsilon)\to(0,0)} \Phi_{\delta,\epsilon}(A,B) := \Phi(A,B) \;\in\; \mathcal{A}[[\lambda]], \tag{5.7.18}$$

and is called the ***Drinfeld associator*** *with respect to* $A, B \in \mathcal{A}$*.*

Proposition 5.7.3. *We have the following properties of the Drinfeld associator:*

(i) $\Phi_{\delta,\epsilon}(A,B)^{-1} = \Phi_{\epsilon,\delta}(B,A)$*, hence*

$$\Phi(A,B)^{-1} = \Phi(B,A). \tag{5.7.19}$$

(ii) $\Phi(A,B) - 1 \;\in\; \lambda^2\mathcal{A}[[\lambda]]$.

(iii) Let $\Lambda, \Lambda' \in \mathcal{A}$ *be central for* A, B *in the sense that*

$$[\Lambda, A] = 0 = [\Lambda', A] \quad \text{and} \quad [\Lambda, B] = 0 = [\Lambda', B] \quad \text{and} \quad [\Lambda, \Lambda'] = 0.$$

Then

$$\Phi(A+\Lambda, B+\Lambda') = \Phi(A,B). \tag{5.7.20}$$

Proof. (i): immediately follows from the definitions (5.7.17) and (5.7.18).
(ii): computing the coefficient of λ^1 of (5.7.16) we get from the right hand side

$$\ln(\epsilon)B + \left(\Phi_{\delta,\epsilon}(A,B)\right)_1 - \ln(\delta)A$$

and from the left hand side the integral (compare (5.7.6))

$$\int_\delta^{1-\epsilon} \frac{du}{u}A - \int_\delta^{1-\epsilon} \frac{du}{1-u}B = \ln(1-\epsilon)A - \ln(\delta)A + \ln(\epsilon)B - \ln(1-\delta)B$$

showing $\left(\Phi_{\delta,\epsilon}(A,B)\right)_1 = \ln(1-\epsilon)A - \ln(1-\delta)B$. Therefore, the claim easily follows from the fact that $\ln(1-x) \to 0$ if $x \to 0$ whence $\lim_{(\delta,\epsilon)\to(0,0)} \left(\Phi_{\epsilon,\delta}(B,A)\right)_1 = 0$.
(iii): Since the connection $\Gamma(B+\Lambda', A+\Lambda)$ evaluated on the path $c_{(\delta\epsilon)}$, $\Gamma(B+\Lambda', A+\Lambda)^{(c_{(\delta\epsilon)})}$, see (5.7.5) is equal to $\Gamma(\Lambda',\Lambda)^{(c_{(\delta\epsilon)})} + \Gamma(B,A)^{(c_{(\delta\epsilon)})}$ we can use the factorization statement (5.2.11) with $Y = \Gamma(\Lambda',\Lambda)^{(c_{(\delta\epsilon)})}$, $Z = \Gamma(B,A)^{(c_{(\delta\epsilon)})}$, and the fact that Λ and Λ' commute with all words in $\mathcal{A}$ whose letters are A, B, Λ or Λ' (hence $U_{\cdot 0}^{-1} Z U_{\cdot 0} = Z$ in (5.2.11)) we can use (5.2.12) and (5.7.7) to conclude that

$$\Phi_{\delta,\epsilon}(A+\Lambda, B+\Lambda') = e^{\lambda\left(\ln(1-\epsilon)\Lambda - \ln(1-\delta)\Lambda'\right)}\,\Phi_{\delta,\epsilon}(A,B)$$

hence, passing to the limit we conclude. □

Remark 5.7.4. *Note that if* $\mathcal{A}$ *carries the structure of a bialgebra and* $A, B \in \mathcal{A}$ *are primitive elements then* $\Phi(A,B)$ *is a (formally) grouplike element. In particular, when* $\mathcal{A} = \mathbb{C}\langle A, B\rangle$ *the Drinfeld associator is a formal exponential series whose exponent is an element of the formal power series with coefficients in the free Lie algebra generated by two elements.*

5.8 The Hexagon Equation

Let $A, B, C \in \mathcal{A}$ such that the sum $\Lambda := A + B + C$ commutes with all the three, i.e.

$$A\Lambda = \Lambda A,\ B\Lambda = \Lambda B,\ C\Lambda = \Lambda C. \tag{5.8.1}$$

We shall prove the **Hexagon Equation** for the Drinfeld associator, i.e.

$$e^{\lambda\pi i\Lambda} \;=\; e^{\lambda\pi iA}\ \Phi(C,A)\ e^{\lambda\pi iC}\ \Phi(B,C)\ e^{\lambda\pi iB}\ \Phi(A,B). \tag{5.8.2}$$

Let $U := \mathbb{C}^{\times\times} := \mathbb{C}\setminus\{0,1\}$ and consider the complex version of the connection $\Gamma(B,A)$, see (5.7.1), i.e.

$$\Gamma(B,A) := \left(\frac{1}{z}A + \frac{1}{z-1}B\right)\mathsf{d}z \tag{5.8.3}$$

which is flat, see Proposition 5.5.3. Recall the rational maps $\zeta, \zeta\circ\zeta = \zeta^{-1} : \mathbb{C}^{\times\times} \to \mathbb{C}^{\times\times}$ defined by $\zeta(z) = \frac{1}{1-z}$ and $\zeta^{-1}(z) = \frac{z-1}{z}$ coming from the cyclic permutations in Y_3, see (5.6.6). We compute the pull-backs of the connection (5.8.3): using Proposition 5.4.6 we get, upon setting $\tilde{C} := -A - B = C - \Lambda$,

$$\begin{aligned}(\zeta^*\Gamma(B,A))(z) &= \Gamma(B,A)\left(\frac{1}{1-z}\right)\frac{1}{(1-z)^2} = \frac{1}{1-z}A + \frac{1}{z(1-z)}B \\ &= \frac{1}{z}B + \frac{-A-B}{z-1} = \Gamma(\tilde{C},B)(z).\end{aligned} \tag{5.8.4}$$

Iterating this formula (recall that $\zeta\circ\zeta = \zeta^{-1}$) gives

$$\left(\left(\zeta^{-1}\right)^*\Gamma(B,A)\right)(z) = \Gamma(A,\tilde{C})(z). \tag{5.8.5}$$

We shall now consider the parallel transport with respect to the connection $\Gamma(B,A)$ along a continuous piecewise smooth loop c_δ depending on a parameter $\delta \in J$ based at the point $\delta \in \mathbb{C}^{\times\times}$, which is the composition of six paths,

$$c_\delta := c_{(\mathrm{VI},\delta)} * \Big(c_{(\mathrm{V},\delta)} * \Big(c_{(\mathrm{IV},\delta)} * \big(c_{(\mathrm{III},\delta)} * \big(c_{(\mathrm{II},\delta)} * c_{(\mathrm{I},\delta)}\big)\big)\Big)\Big) \tag{5.8.6}$$

given by

$$\begin{array}{llclcl} c_{(\mathrm{I},\delta)}(s) := (1-s)\delta + s(1-\delta) &=& \delta + s(1-2\delta) & \text{joining} & \delta \to 1-\delta, \\ c_{(\mathrm{II},\delta)}(s) := \frac{1-\frac{\delta}{2}-\frac{\delta}{2}e^{i\pi s}}{1-\frac{\delta}{2}+\frac{\delta}{2}e^{i\pi s}} &=& 1 - \frac{\delta}{\left(1-\frac{\delta}{2}\right)e^{-i\pi s}+\frac{\delta}{2}} & \text{joining} & 1-\delta \to \frac{1}{1-\delta}, \\ c_{(\mathrm{III},\delta)}(s) := \zeta\left(c_{(\mathrm{I},\delta)}(s)\right) &=& \frac{1}{1-\delta-s(1-2\delta)} & \text{joining} & \frac{1}{1-\delta} \to \frac{1}{\delta}, \\ c_{(\mathrm{IV},\delta)}(s) := \zeta\left(c_{(\mathrm{II},\delta)}(s)\right) &=& \frac{1}{2} + \left(\frac{1}{\delta}-\frac{1}{2}\right)e^{-i\pi s} & \text{joining} & \frac{1}{\delta} \to -\frac{1}{\delta}+1, \\ c_{(\mathrm{V},\delta)}(s) := \zeta\left(\zeta\left(c_{(\mathrm{I},\delta)}(s)\right)\right) &=& \frac{\delta-1+s(1-2\delta)}{\delta+s(1-2\delta)} & \text{joining} & -\frac{1}{\delta}+1 \to -\frac{\delta}{1-\delta}, \\ c_{(\mathrm{VI},\delta)}(s) := \zeta\left(\zeta\left(c_{(\mathrm{II},\delta)}(s)\right)\right) &=& \frac{\delta}{-\left(1-\frac{\delta}{2}\right)e^{-i\pi s}+\frac{\delta}{2}} & \text{joining} & -\frac{\delta}{1-\delta} \to \delta. \end{array} \tag{5.8.7}$$

It is easy to check that all the six paths take all their values in the lower half plane (including the x–axis and excluding 0 and 1).

The singular set D for the loop c_δ is thus equal to $\left\{0, \frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\right\}$. The following geometric description of the paths, see Figure 5.1, may perhaps clarify the whole procedure: the three 'odd'

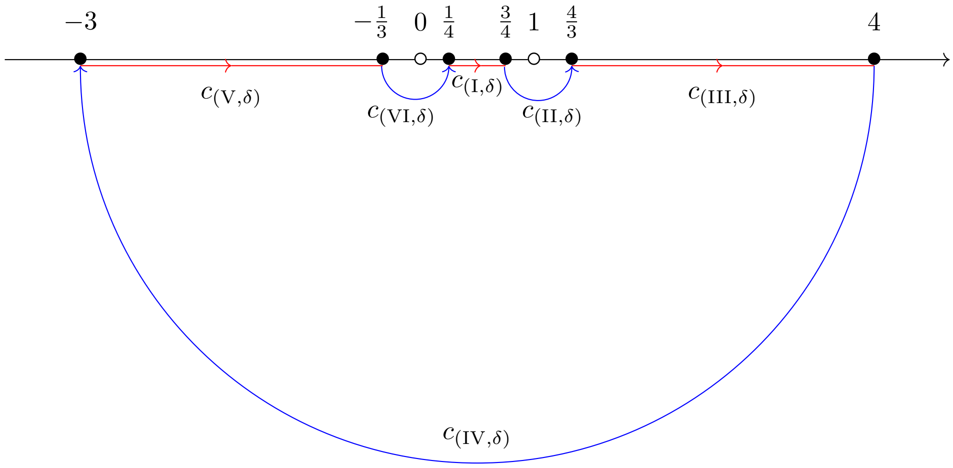


Figure 5.1: The paths (5.8.7) at $\delta = 1/4$

paths $c_{(\mathrm{I},\delta)}$, $c_{(\mathrm{III},\delta)}$, and $c_{(\mathrm{V},\delta)}$ parametrize the closed intervals $[\delta, 1-\delta]$, $\zeta\big([\delta, 1-\delta]\big) = [1/(1-\delta), 1/\delta]$, and $\zeta^2\big([\delta, 1-\delta]\big) = [-(1/\delta)+1, -\delta/(1-\delta)]$, respectively, all along the x–axis. Note that $c_{(\mathrm{III},\delta)}$ and $c_{(\mathrm{V},\delta)}$ are not affine paths in the sense of (5.4.2). The three 'even' paths $c_{(\mathrm{II},\delta)}$, $c_{(\mathrm{IV},\delta)}$, and $c_{(\mathrm{VI},\delta)}$ parametrize lower half circles with centres $1 + \frac{\delta^2}{2-2\delta}$, $\frac{1}{2}$, and $-\frac{\delta^2}{2-2\delta}$, respectively, having radii $\frac{2\delta-\delta^2}{2-2\delta}$, $\frac{1}{\delta} - \frac{1}{2}$, and $\frac{2\delta-\delta^2}{2-2\delta}$, respectively, as can be checked by a lengthy, but elementary computation. $c_{(\mathrm{II},\delta)}$ and $c_{(\mathrm{VI},\delta)}$ are traced counterclockwise (where the parametrization is not uniform), and $c_{(\mathrm{IV},\delta)}$ is traced clockwise with uniform parametrization.

Remark 5.8.1. *Note that the whole picture of the six paths in the doubly punctured plane has a reflection symmetry (the map* $\mathsf{s} : z \mapsto 1 - \bar{z}$*) with respect to the straight line* $x = \frac{1}{2}$*. Hence, with the usual interval inversion* ι *of the interval* $[0,1]$, $\iota(s) = 1 - s$*, it is easy to see that the following holds by using the concrete formulas* (5.8.7)*:* $\mathsf{s} \circ c_{(\mathrm{I},\delta)} \circ \iota = c_{(\mathrm{I},\delta)}$, $\mathsf{s} \circ c_{(\mathrm{II},\delta)} \circ \iota = c_{(\mathrm{VI},\delta)}$, $\mathsf{s} \circ c_{(\mathrm{III},\delta)} \circ \iota = c_{(\mathrm{V},\delta)}$, *and* $\mathsf{s} \circ c_{(\mathrm{IV},\delta)} \circ \iota = c_{(\mathrm{IV},\delta)}$.

We shall now compute the parallel transports along the six paths. First, it is immediate that the parallel transport ${}^{\Gamma(B,A)}W_{10}^{(c_{(\mathrm{I},\delta)})}$ coincides with the parallel transport ${}^{\Gamma(B,A)}W_{10}^{(c_{\delta\delta})}$ of the preceding Section, see (5.7.6): this can be seen by using the smooth injection $]0,1[\to \mathbb{C}^{\times\times}$ to pull back the connection (5.8.3) to the interval. Using the formulas (5.8.4) and (5.8.5) and the fact that $c_{(\mathrm{III},\delta)} = \zeta \circ c_{(\mathrm{I},\delta)}$ and $c_{(\mathrm{V},\delta)} = \zeta \circ \zeta \circ c_{(\mathrm{I},\delta)}$, see ((5.8.7)) we get –upon using (5.4.8)– the following formulas

$$
\begin{aligned}
{}^{\Gamma(B,A)}W_{10}^{(c_{(\mathrm{I},\delta)})} &= {}^{\Gamma(B,A)}W_{10}^{(c_{\delta\delta})} \overset{(5.7.16)}{=} e^{\lambda \ln(\delta) B}\Phi_{\delta,\delta}(A,B)e^{-\lambda\ln(\delta)A},\\
{}^{\Gamma(B,A)}W_{10}^{(c_{(\mathrm{III},\delta)})} &\overset{(5.8.4)}{=} {}^{\Gamma(\tilde{C},B)}W_{10}^{(c_{\delta\delta})} \overset{(5.7.16)}{=} e^{\lambda \ln(\delta) \tilde{C}}\Phi_{\delta,\delta}(B,\tilde{C})e^{-\lambda\ln(\delta)B},\\
{}^{\Gamma(B,A)}W_{10}^{(c_{(\mathrm{V},\delta)})} &\overset{(5.8.5)}{=} {}^{\Gamma(A,\tilde{C})}W_{10}^{(c_{\delta\delta})} \overset{(5.7.16)}{=} e^{\lambda \ln(\delta) A}\Phi_{\delta,\delta}(\tilde{C},A)e^{-\lambda\ln(\delta)\tilde{C}}.
\end{aligned}
\tag{5.8.8}
$$

For the even paths we can proceed in exactly the same way: using the fact that $c_{(\mathrm{IV},\delta)} = \zeta \circ c_{(\mathrm{II},\delta)}$ and $c_{(\mathrm{VI},\delta)} = \zeta \circ \zeta \circ c_{(\mathrm{I},\delta)}$, see (5.8.7) we get –upon using (5.4.8)– the following formulas:

$$
{}^{\Gamma(B,A)}W_{10}^{(c_{(\mathrm{IV},\delta)})} = {}^{\Gamma(\tilde{C},B)}W_{10}^{c_{(\mathrm{II},\delta)}} \quad \text{and} \quad {}^{\Gamma(B,A)}W_{10}^{(c_{(\mathrm{VI},\delta)})} = {}^{\Gamma(A,\tilde{C})}W_{10}^{c_{(\mathrm{II},\delta)}},
\tag{5.8.9}
$$

hence it suffices to compute ${}^{\Gamma(B,A)}W_{10}^{(c_{(\mathrm{II},\delta)})}$.

Lemma 5.8.2. *The parallel transport* ${}^{\Gamma(B,A)}W_{10}^{(c_{(\mathrm{II},\delta)})}$ *along the path* $c_{(\mathrm{II},\delta)}$ *factorizes in the following way*

$$ {}^{\Gamma(B,A)}W_{10}^{(c_{(\mathrm{II},\delta)})} = e^{\lambda i\pi B} H^{(\delta)}(B,A) \tag{5.8.10}$$

where the element $\delta \mapsto H^{(\delta)}(B,A)$ *of* $\Big(\mathrm{Fun}\,(]0,1/4],\mathbb{C})\otimes\mathcal{A}\Big)[[\lambda]]$ *is a harmless group term, see (5.3.10), (5.3.11), and Proposition 5.3.2.*

Proof. We compute

$$\begin{aligned}
\Gamma(B,A)^{(c_{(\mathrm{II},\delta)})}(s) &= -\, i\pi \frac{\delta\left(1-\frac{\delta}{2}\right)e^{i\pi s}}{\left(1-\frac{\delta}{2}\right)^2 - \frac{\delta^2}{4}e^{i2\pi s}} A + i\pi \frac{1-\frac{\delta}{2}}{1-\frac{\delta}{2}+\frac{\delta}{2}e^{i\pi s}} B \\
&= \underbrace{i\pi B}_{=:Y_0(s)} \;\; \underbrace{-i\pi\delta\left(\frac{\frac{1}{2}e^{i\pi s}}{1-\frac{\delta}{2}+\frac{\delta}{2}e^{i\pi s}}B + \frac{\left(1-\frac{\delta}{2}\right)e^{i\pi s}}{\left(1-\frac{\delta}{2}\right)^2-\frac{\delta^2}{4}e^{i2\pi s}}A\right)}_{=:Z^{(\delta)}(s)}.
\end{aligned} \tag{5.8.11}$$

We can now apply the factorization statement (iv) of Proposition 5.2.1, see (5.2.11): clearly, the fundamental solution $U_{\cdot 0}(s)$ of the formal linear ODE $\mathsf{d}U_{\cdot 0}/\mathsf{d}s = \lambda Y_0 U_{\cdot 0}$ is simply given by the formal exponential $U_{\cdot 0}(s) = e^{\lambda i\pi s B}$, and the parallel transport ${}^{\Gamma(B,A)}W_{\cdot 0}^{(c_{(\mathrm{II},\delta)})}$ thus factorizes thanks to (5.2.11) as follows for all $s\in[0,1]$

$$ {}^{\Gamma(B,A)}W_{s0}^{(c_{(\mathrm{II},\delta)})} \;=\; e^{\lambda i\pi s B}\; H^{(\delta)}(B,A)(s)$$

where $H^{(\delta)}(B,A)(s)$ is a fundamental solution for the formal linear ODE

$$\begin{aligned}
\frac{\mathsf{d}H^{(\delta)}(B,A)}{\mathsf{d}s}(s) &= \lambda e^{-\lambda i\pi s B} Z^{(\delta)}(s) e^{\lambda i\pi s B} H^{(\delta)}(B,A)(s) \\
&= \lambda \underbrace{\left(e^{-\lambda i\pi s \mathsf{ad}_B}\left(Z^{(\delta)}(s)\right)\right)}_{=:\tilde{Z}^{(\delta)}(s)} H^{(\delta)}(B,A)(s)
\end{aligned}$$

with initial condition $H^{(\delta)}(B,A)(0) = 1$. We shall make the upper bound test (5.3.19) for $\tilde{Z}^{(\delta)}(s)$, see Lemma 5.3.5: writing $\tilde{Z}^{(\delta)}(s) = \sum_{r=0}^{\infty}\lambda^r \tilde{Z}_r^{(\delta)}(s)$ we get –upon using an arbitrary norm $||\;||$ on the complex vector space $\mathcal{A}$– for each $r\in\mathbb{N}$ upon setting $d_{r0} := 1$ if $r=0$ and $d_{r0} := 0$ otherwise:

$$\begin{aligned}
\left|\left|\tilde{Z}_r^{(\delta)}(s)\right|\right| \quad &= \quad \left|\left|\frac{i^r\pi^r s^r}{r!}\mathsf{ad}_B^{\circ r}\left(Z^{(\delta)}(s)\right)\right|\right| \\
&\overset{(5.8.11)}{=} \quad \frac{\pi^{r+1}s^r}{r!}\delta\left|\left|\frac{d_{r0}\,\frac{1}{2}e^{i\pi s}}{1-\frac{\delta}{2}+\frac{\delta}{2}e^{i\pi s}}B + \frac{\left(1-\frac{\delta}{2}\right)e^{i\pi s}}{\left(1-\frac{\delta}{2}\right)^2-\frac{\delta^2}{4}e^{i2\pi s}}\mathsf{ad}_B^{\circ r}\,(A)\right|\right| \\
&\leqslant \quad \underbrace{\frac{\pi^{r+1}}{r!}\left(||B|| + 2\,||\mathsf{ad}_B^{\circ r}\,(A)||\right)\delta}_{=:\check{C}_r}
\end{aligned}$$

where we have used $|e^{i\tau}| = 1$ for each real number τ, and the elementary lower bounds

$$\begin{aligned}
\left|1-\frac{\delta}{2}+\frac{\delta}{2}e^{i\pi s}\right| \quad &\geqslant \quad 1-\frac{\delta}{2}-\left|\frac{\delta}{2}e^{i\pi s}\right| = 1-\delta \geqslant \frac{1}{2}, \\
\left|\left(1-\frac{\delta}{2}\right)^2 - \frac{\delta^2}{4}e^{i2\pi s}\right| \quad &\geqslant \quad \left(1-\frac{\delta}{2}\right)^2 - \left|\frac{\delta^2}{4}e^{i2\pi s}\right| = 1-\delta \geqslant \frac{1}{2}.
\end{aligned}$$

for the denominators. It follows that $\delta \mapsto H^{(\delta)}(B,A)$ is a harmless group term. □

We now need to put the loop c_δ in a star-shaped open set U' of $\mathbb{C}^{\times\times}$ because $\mathbb{C}^{\times\times}$ is not star-shaped (it is not even simply connected). Define

$$U' := \mathbb{C} \setminus \left\{ \frac{1-i}{2} + t(\gamma + i) \in \mathbb{C} \;\middle|\; t \in \mathbb{R},\ t \geqslant \frac{1}{2},\ \gamma \in \{-1, 1\} \right\}. \tag{5.8.12}$$

Theorem 5.8.3 (Hexagon Equation)**.** *Let $A, B, C \in \mathcal{A}$ be three elements satisfying (5.8.1). Then the Hexagon equation (5.8.2) for the Drinfeld associator holds.*

Proof. Since the composed loop c_δ, see (5.8.6), is contained in the star–shaped open subset U', see (5.8.12), we can apply Corollary 5.5.7 to conclude that the parallel transport around the loop c_δ along the flat connection $\Gamma(B, A)$, see (5.8.3), is equal to 1. Abbreviating ${}^{\Gamma(B,A)}W_{10}^{(c)}$ by $W_{10}^{(c)}$ for any piecewise smooth path $c : [0, 1] \to U'$ we get

$$\begin{aligned}
1 = W_{10}^{(c_\delta)} &\overset{(5.4.7)}{=} W_{10}^{(c_{(\mathrm{VI},\delta)})} W_{10}^{(c_{(\mathrm{V},\delta)})} W_{10}^{(c_{(\mathrm{IV},\delta)})} W_{10}^{(c_{(\mathrm{III},\delta)})} W_{10}^{(c_{(\mathrm{II},\delta)})} W_{10}^{(c_{(\mathrm{I},\delta)})} \\
&\overset{(5.8.9),(5.8.10),(5.8.8)}{=} e^{\lambda i\pi A}\ H^{(\delta)}(A, \tilde{C})\quad e^{\lambda \ln(\delta) A}\ \Phi_{\delta,\delta}(\tilde{C}, A)\ e^{-\lambda \ln(\delta)\tilde{C}} \\
&\qquad e^{\lambda i\pi \tilde{C}}\ H^{(\delta)}(\tilde{C}, B)\quad e^{\lambda \ln(\delta)\tilde{C}}\ \Phi_{\delta,\delta}(B, \tilde{C})\ e^{-\lambda \ln(\delta) B} \\
&\qquad e^{\lambda i\pi B}\ H^{(\delta)}(B, A)\quad e^{\lambda \ln(\delta) B}\ \Phi_{\delta,\delta}(A, B)\ e^{-\lambda \ln(\delta) A}.
\end{aligned} \tag{5.8.13}$$

Clearly, the three singular terms $\delta \mapsto e^{\lambda \ln(\delta) A}$, $\delta \mapsto e^{\lambda \ln(\delta) B}$, and $\delta \mapsto e^{\lambda \ln(\delta)\tilde{C}}$ belong to the at most logarithmically diverging group terms, $\mathcal{G}_{\mathcal{L}}$, see (5.3.11) for $J =]0, 1/4]$. Hence, the following three conjugations again define harmless group terms according to statement (iv) of Proposition 5.3.2:

$$\begin{aligned}
\tilde{H}^{(\delta)}(A, \tilde{C}) &:= e^{-\lambda \ln(\delta) A}\ H^{(\delta)}(A, \tilde{C})\ e^{\lambda \ln(\delta) A}, \\
\tilde{H}^{(\delta)}(\tilde{C}, B) &:= e^{-\lambda \ln(\delta)\tilde{C}}\ H^{(\delta)}(\tilde{C}, B)\ e^{\lambda \ln(\delta)\tilde{C}}, \\
\tilde{H}^{(\delta)}(B, A) &:= e^{-\lambda \ln(\delta) B}\ H^{(\delta)}(B, A)\ e^{\lambda \ln(\delta) B}.
\end{aligned}$$

Rewriting (5.8.13) by means of these harmless group terms we see that the three singular terms mentioned above, $e^{\lambda \ln(\delta) A}$, $e^{\lambda \ln(\delta) B}$, and $e^{\lambda \ln(\delta)\tilde{C}}$, cancel out, and we are left with the following identity:

$$1 = e^{\lambda i\pi A}\ \tilde{H}^{(\delta)}(A, \tilde{C})\ \Phi_{\delta,\delta}(\tilde{C}, A)\ e^{\lambda i\pi \tilde{C}}\ \tilde{H}^{(\delta)}(\tilde{C}, B)\ \Phi_{\delta,\delta}(B, \tilde{C})\ e^{\lambda i\pi B}\ \tilde{H}^{(\delta)}(B, A)\ \Phi_{\delta,\delta}(A, B).$$

Passing to the limit $\delta \to 0$ we get –thanks to the limit rules (5.3.7), the definition of the Drinfeld associator (5.7.18), and the fact that harmless group terms tend to 1 for $\delta \to 0$ (see statement (v) of Proposition 5.3.2)– the following equation (recall that $\tilde{C} = C - \Lambda$)

$$1 = e^{\lambda\pi i A}\Phi(C - \Lambda, A)e^{\lambda\pi i(C-\Lambda)}\Phi(B, C - \Lambda)e^{\lambda\pi i B}\Phi(A, B).$$

This equation immediately results in the Hexagon equation (5.8.2) thanks to the fact that Λ commutes with A, B and C whence $\Phi(C - \Lambda, A) = \Phi(C, A)$, $\Phi(B, C - \Lambda) = \Phi(B, C)$ by (5.7.20). □

5.9 The Pentagon Equation

Let $A_{12} = A_{21}, A_{13} = A_{31}, A_{14} = A_{41}, A_{23} = A_{32}, A_{24} = A_{42}, A_{34} = A_{43} \in \mathcal{A}$ be six elements satisfying the infinitesimal braid relations (5.6.1b) and (5.6.1c). We shall prove the **Pentagon Equation** for the Drinfeld associator, i.e.

$$\Phi(A_{12}, A_{23} + A_{24})\Phi(A_{13} + A_{23}, A_{34}) = \Phi(A_{23}, A_{34})\Phi(A_{12} + A_{13}, A_{24} + A_{34})\Phi(A_{12}, A_{23}) \tag{5.9.1}$$

We shall represent each side of the Pentagon Equation by the parallel transport along the composition of three paths, $c_{(\mathrm{III},\delta)} * (c_{(\mathrm{II},\delta)} * c_{(\mathrm{I},\delta)})$, for the right hand side, and along the composition of two paths $c_{(\mathrm{V},\delta)} * c_{(\mathrm{IV},\delta)}$ for the left hand side, both having the same initial and final points.
Consider the open set $U := \{x \in \mathbb{R}^4 \mid x_1 < x_2 < x_3 < x_4\} \subset Y_4$. Following Drinfeld, we shall use the pull–back of the КнЗа-connection ${}^{(4)}\Gamma_{\text{КнЗа}}$ from Y_4 to U (with respect to the canonical injection $U \to Y_4$) which is still flat, see Theorem 5.6.5 and Proposition 5.5.5. Since the КнЗа–connections are invariant under simultaneous translations, we assume that the first coordinate x_1 of all the paths is fixed to be 0. Next, the fact that A_{14} does not occur in the Pentagon equation may lead us to the ansatz that the difference $x_4 - x_1$ should remain constant; on the other hand the fact that there are terms in the Pentagon Equation not containing 1 and not containing 4 suggests that x_4 should be 'far away from $x_1 = 0$', hence we set $x_1 = 0$ and $x_4 = 1$. Define the open full triangle $U' := \{(x_2, x_3) \in \mathbb{R}^2 \mid 0 < x_2 < x_3 < 1\}$. Note that U' is invariant under the involutive diffeomorphism $\Theta : U' \to U' : (x_2, x_3) \mapsto (1 - x_3, 1 - x_2)$. Moreover, it is easy to see that U' is convex and star–shaped around $\left(\frac{1}{2}, \frac{2}{3}\right)$. Using the injection $i : U' \to U \to Y_4 : (x_2, x_3) \mapsto (0, x_2, x_3, 1)$ we can pull back the КнЗа–connection ${}^{(4)}\Gamma_{\text{КнЗа}}$ to U'. An easy computation gives

$$\begin{aligned}
\Gamma(x_2, x_3) &:= \Gamma\big((A_{ij})\big)(x_2, x_3) \\
&:= \left(i^* \left({}^{(4)}\Gamma_{\text{КнЗа}}\right)\right)(x_2, x_3) \\
&= \left(\frac{1}{x_2}A_{12} + \frac{1}{x_2 - x_3}A_{23} + \frac{1}{x_2 - 1}A_{24}\right)\mathsf{d}x_2 \\
&\quad + \left(\frac{1}{x_3}A_{13} - \frac{1}{x_2 - x_3}A_{23} + \frac{1}{x_3 - 1}A_{34}\right)\mathsf{d}x_3.
\end{aligned} \tag{5.9.2}$$

Clearly, Γ is (formally) flat according to Theorem 5.6.5 and Proposition 5.5.5, but its (formal) flatness can easily be computed directly from formula (5.9.2). Moreover, it is easy to compute that

$$\Theta^*\big(\Gamma\big((A_{ij})\big)\big) = \Gamma\big((A_{\sigma(i)\sigma(j)})\big) \quad \text{with} \quad \sigma = (14)(23) \in \mathfrak{S}_4. \tag{5.9.3}$$

Next, we would like to substantiate in terms of paths the five 'zones' which Drinfeld mentiones in his articles, see cf. [Dri90b, p.1454] or [Dri90a, p.834, line 3,4] (where the fifth zone in [?Dri90] has been forgotten in the English translation, see the original article in Russian language for a complete description): here certain pairs of coordinates are 'very close' to each others, others are 'medium close' and still others are 'far' which is expressed in terms of inequalities using the symbol $\lll$: using the real number $\delta \in J$ –which is meant to be sent to zero– we interpret –as a rule of thumb– 'very close' as $\approx \delta^2$, 'medium close' as around $\approx \delta$, and 'far' as ≈ 1. Inspired by the picture [Kas12, p.478, Fig.8.2.] we first use the following subdivision of the interval $]0, 1[$, in which we imagine that both x_2 and x_3 'move' between the selected positions,

$$0 < \delta^2 < \delta - \delta^2 < \delta < 1 - \delta < 1 - \delta + \delta^2 < 1 - \delta^2 < 1,$$

and associate the following five points in U' (as part of the (x_2, x_3) plane) as an interpretation of the five zones (recall that $x_1 = 0$ and $x_4 = 1$):

$$\begin{array}{lllllll}
\text{zone 1}: & "x_2 - x_1 \lll & x_3 - x_1 & \lll x_4 - x_1" & \text{interpreted as} & (\delta^2, \delta) & =: p_1, \\
\text{zone 2}: & "x_3 - x_2 \lll & x_3 - x_1 & \lll x_4 - x_1" & \text{interpreted as} & (\delta - \delta^2, \delta) & =: p_2, \\
\text{zone 3}: & "x_3 - x_2 \lll & x_4 - x_2 & \lll x_4 - x_1" & \text{interpreted as} & (1 - \delta, 1 - \delta + \delta^2) & =: p_3, \\
\text{zone 4}: & "x_4 - x_3 \lll & x_4 - x_2 & \lll x_4 - x_1" & \text{interpreted as} & (1 - \delta, 1 - \delta^2) & =: p_4, \\
\text{zone 5}: & "x_2 - x_1 \lll & & \lll x_4 - x_1 & \text{and} & & \\
 & \;\; x_4 - x_3 \lll & & \lll x_4 - x_1" & \text{interpreted as} & (\delta^2, 1 - \delta^2) & =: p_5.
\end{array} \tag{5.9.4}$$

There is thus the following ansatz for the following five affine paths subsequently joining the above five points by the unique line segments between them, see (5.4.2), where $\iota : [0,1] \to [0,1]$ denotes the usual interval inversion $s \mapsto 1-s$:

$$\begin{aligned} c_{(\mathrm{I},\delta)} &:= c_{p_2 \leftarrow p_1}, \quad c_{(\mathrm{II},\delta)} := c_{p_3 \leftarrow p_2} = \Theta \circ c_{(\mathrm{II},\delta)} \circ \iota, \quad c_{(\mathrm{III},\delta)} := c_{p_4 \leftarrow p_3} = \Theta \circ c_{(\mathrm{I},\delta)} \circ \iota, \\ c_{(\mathrm{IV},\delta)} &:= c_{p_5 \leftarrow p_1}, \quad c_{(\mathrm{V},\delta)} := c_{p_4 \leftarrow p_5} = \Theta \circ c_{(\mathrm{IV},\delta)} \circ \iota. \end{aligned} \tag{5.9.5}$$

which can be depicted in Figure 5.2 describing a non–regular pentagon whose vertices are the five 'zone' points (5.9.4) and whose edges are the images of the five affine paths (5.9.5).

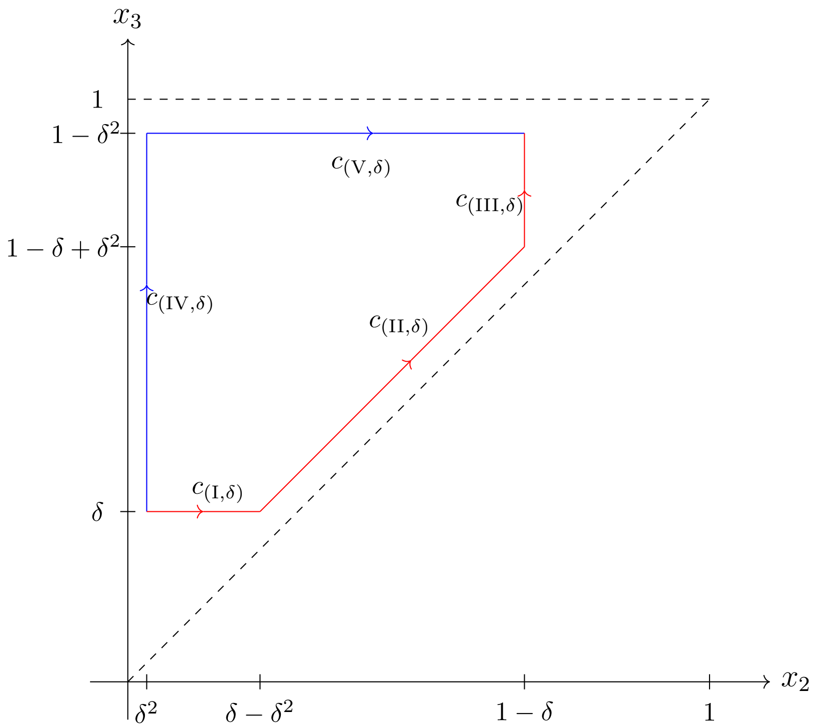


Figure 5.2: The paths (5.9.5) in the x_2-x_3-plane

Note that the symmetry Θ –which is a symmetry of the pentagon– is the reflection with respect to the straight line whose equation is $x_3 = 1 - x_2$. Note furthermore that the second path is the only path where both coordinates x_2, x_3 are moving, but their distance is kept constant (inspired by the observation of the absence of A_{23} in the middle factor on the right hand side of the Pentagon Equation.

We are now going to compute the parallel transports along the five paths of (5.9.5) with respect to the connection Γ, see (5.9.2). Thanks to (5.9.3), (5.4.6), and (5.4.8) it suffices to compute the parallel transports $W_{10}^{(c_{(\mathrm{I},\delta)})}$, $W_{10}^{(c_{(\mathrm{II},\delta)})}$, and $W_{10}^{(c_{(\mathrm{IV},\delta)})}$ –we henceforth suppress the symbol Γ attached to W–, the other two will be immediate from the above identities for symmetries and reparametrizations of parallel transports. We get the following Lemma:

Lemma 5.9.1. *With the above notations, we get the following results for the five parallel transports* *for all $\delta \in J$:*

$$\begin{array}{rcl}
W_{10}^{(c_{(\mathrm{I},\delta)})} & = & e^{\lambda \ln(\delta) A_{23}}\ \psi_\delta(A_{23}, A_{12})\ H^{(\mathrm{I},\delta)}\ \psi_\delta(A_{12}, A_{23})^{-1}\ e^{-\lambda \ln(\delta) A_{12}}, \\
W_{10}^{(c_{(\mathrm{II},\delta)})} & = & e^{\lambda \ln(\delta)(A_{24}+A_{34})}\ \psi_\delta(A_{24}+A_{34}, A_{12}+A_{23})\ H^{(\mathrm{II},\delta)}\ \psi_\delta(A_{12}+A_{13}, A_{24}+A_{34})^{-1} \\
 & & e^{-\lambda \ln(\delta)(A_{12}+A_{13})}, \\
W_{10}^{(c_{(\mathrm{III},\delta)})} & = & e^{\lambda \ln(\delta) A_{34}}\ \psi_\delta(A_{34}, A_{23})\ H^{(\mathrm{III},\delta)}\ \psi_\delta(A_{23}, A_{34})^{-1}\ e^{-\lambda \ln(\delta) A_{23}}, \\
W_{10}^{(c_{(\mathrm{IV},\delta)})} & = & e^{\lambda \ln(\delta^2) A_{34}}\ \psi_{\delta^2}(A_{34}, A_{13}+A_{23})\ H^{(\mathrm{IV},\delta)}\ \psi_\delta(A_{13}+A_{23}, A_{34})^{-1} \\
 & & e^{-\lambda \ln(\delta)(A_{13}+A_{23})}, \\
W_{10}^{(c_{(\mathrm{V},\delta)})} & = & e^{\lambda \ln(\delta)(A_{23}+A_{24})}\ \psi_\delta(A_{23}+A_{24}, A_{12})\ H^{(\mathrm{V},\delta)}\ \psi_{\delta^2}(A_{12}, A_{23}+A_{24})^{-1} \\
 & & e^{-\lambda \ln(\delta^2) A_{12}}.
\end{array} \tag{5.9.6}$$

where the ψ–terms are defined in (5.7.12), see also (5.7.13) and (5.7.17), and the terms $\delta \mapsto H^{(i,\delta)}$ for $i = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}$ are harmless group terms, see (5.3.10), (5.3.11), and Proposition 5.3.2. Note that the appearance of δ^2 in the terms $W_{10}^{(c_{(\mathrm{IV},\delta)})}$ and $W_{10}^{(c_{(\mathrm{V},\delta)})}$ is crucial for Theorem 5.9.2.

Proof. In order to compute the parallel transports we shall again use the half exponential paths already used in §5.7, of the type (5.7.8). More precisely, for each of the three paths $c_{(\mathrm{I},\delta)}, c_{(\mathrm{II},\delta)}, c_{(\mathrm{IV},\delta)}$ –recall that the parallel transport along $c_{(\mathrm{III},\delta)}$ and $c_{(\mathrm{V},\delta)}$ can be computed using the symmetry indicated in (5.9.5)– we choose a mid-point on the corresponding line segment in U', and we replace each affine path $c_{(i,\delta)}$ by a composition of two exponential half paths tracing the same line segment, $\tilde{c}_{(i,2,\delta)} * \big(\check{c}_{(i,1,\delta)} \circ \iota\big)$ where $i = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}$, $\iota : [0,1] \to [0,1] : s \mapsto 1-s$ is the usual interval inversion, $\check{c}_{(i,1,\delta)}$ joins the midpoint to the initial point, and $\tilde{c}_{(i,2,\delta)}$ joins the midpoint to the final point. Since $c_{(i,\delta)}$ and $\tilde{c}_{(i,2,\delta)} * \big(\check{c}_{(i,1,\delta)} \circ \iota\big)$ just differ by a reparametrization they induce the same parallel transport, see (5.4.5), which implies

$$W_{10}^{(c_{(i,\delta)})} = W_{10}^{(\tilde{c}_{(i,2,\delta)})} \Big(W_{10}^{(\check{c}_{(i,1,\delta)})}\Big)^{-1} \tag{5.9.7}$$

where the inversion fomula (5.4.6) has been used. Hence, we choose the following paths (where $\hat{\delta} := \delta(1-\delta/2)$)

$$\begin{array}{rclcccc}
\check{c}_{(\mathrm{I},1,\delta)}(s) & := & \big(\frac{\delta}{2} e^{\ln(2\delta)s}, \delta\big) & \text{joining} & \big(\frac{\delta}{2}, \delta\big) & \to & (\delta^2, \delta)\,, \\
\tilde{c}_{(\mathrm{I},2,\delta)}(s) & := & \big(\delta - \frac{\delta}{2} e^{\ln(2\delta)s}, \delta\big) & \text{joining} & \big(\frac{\delta}{2}, \delta\big) & \to & (\delta - \delta^2, \delta)\,, \\
\check{c}_{(\mathrm{II},1,\delta)}(s) & := & \Theta\big(\tilde{c}_{(\mathrm{II},2,\delta)}(s)\big) & \text{joining} & \Big(\frac{1-\delta^2}{2}, \frac{1+\delta^2}{2}\Big) & \to & (\delta - \delta^2, \delta), \\
\tilde{c}_{(\mathrm{II},2,\delta)}(s) & := & \Big(1 - \frac{1}{2} e^{\ln(2\hat{\delta})s}\Big)(1,1) + \big(-\frac{\delta^2}{2}, \frac{\delta^2}{2}\big) & \text{joining} & \Big(\frac{1-\delta^2}{2}, \frac{1+\delta^2}{2}\Big) & \to & (1-\delta, 1-\delta+\delta^2)\,, \\
\check{c}_{(\mathrm{IV},1,\delta)}(s) & := & \big(\delta^2, \frac{1}{2} e^{\ln(2\delta)s}\big) & \text{joining} & \big(\delta^2, \frac{1}{2}\big) & \to & (\delta^2, \delta), \\
\tilde{c}_{(\mathrm{IV},2,\delta)}(s) & := & \Big(\delta^2, 1 - \frac{1}{2} e^{\ln(2\delta^2)s}\Big) & \text{joining} & \big(\delta^2, \frac{1}{2}\big) & \to & (\delta^2, 1-\delta^2)\,.
\end{array} \tag{5.9.8}$$

When we compute $\Gamma^{(c_{(i,u,\delta)})}(s)$ (for $i = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}$ and $u = 1, 2$) we shall see further down that we always get the following form –writing d for the paths $\check{c}_{(i,1,\delta)}$ or $\tilde{c}_{(i,2,\delta)}$:

$$\Gamma^{(d)}(s) = \underbrace{\ln(2\epsilon(\delta))B + \frac{-\ln(2\epsilon(\delta))}{2e^{-\ln(2\epsilon(\delta))s} - 1} A}_{=:Y_0(B,A)(s,\epsilon)} + \underbrace{\sum_{1 \leqslant i < j \leqslant 4} f_{ij}(s,\delta) A_{ij},}_{=:Z(s,\delta)} \tag{5.9.9}$$

–which is well–known from §5.7, (5.7.11)– where $\epsilon : J \to J$ is a monomial of δ of the form $\epsilon(\delta) = \delta^\ell$ where ℓ is a positive integer, A, B are certain linear combinations of the algebra elements A_{ij}, and f_{ij}

are real–valued functions of $s \in [0,1]$ and $\delta \in J$. From the general factorization statement (5.2.11) and the solution (5.7.12) of Lemma 5.7.1 we have the factorization

$$W_{s0}^{(c_{(i,u,\delta)})} = \underbrace{e^{\lambda \ln(\epsilon) s B} \; \psi_\epsilon(B,A)(s)}_{=: U_{s0}^{(\epsilon)}} \; H_{s0}^{(i,u,\delta)}, \tag{5.9.10}$$

where $s \mapsto H_{s0}^{(i,u,\delta)}$ is a fundamental solution to the formal linear ODE

$$\frac{\mathsf{d} H_{s0}^{(i,u,\delta)}}{\mathsf{d}s} = \lambda U_{s0}^{(\epsilon)^{-1}} \; Z(s,\delta) \; U_{s0}^{(\epsilon)} \; H_{s0}^{(i,u,\delta)} =: \lambda \tilde{Z}(s,\delta) \; H_{s0}^{(i,u,\delta)}.$$

We shall show later that all these factors $\delta \mapsto H_{10}^{(i,u,\delta)}$ are harmless group terms: first, we shall prove that it suffices to show that for each $1 \leqslant i < j \leqslant 4$ there are non–negative real numbers C_{ij} and $\beta_{ij} > 0$ such that

$$\forall \; s \in [0,1], \; \forall \; \delta \in J: \quad |f_{ij}(s,\delta)| \leqslant C_{ij} \delta^{\beta_{ij}}. \tag{5.9.11}$$

Indeed, we check that the preceding condition (5.9.11) implies the estimate of type (H), (5.3.19): write $\psi_\epsilon(B,A)(s) = \sum_{r=0}^{\infty} \psi_r(s,\epsilon)\lambda^r$ and its inverse $\psi_\epsilon(B,A)(s)^{-1}$ as $\sum_{r=0}^{\infty} \hat{\psi}_r(s,\epsilon)\lambda^r$ where of course $\psi_0(s,\epsilon) = 1 = \hat{\psi}_0(s,\epsilon)$. Fix a norm $|| \; ||$ on the complex vector space $\mathcal{A}$. Since $s = |s| \leqslant 1$ and for each non–negative integer r there are positive real constants C_r and $\hat{C}_r$ with $||\psi_r(s,\epsilon)|| \leqslant C_r$ and $||\hat{\psi}_r(s,\epsilon)|| \leqslant \hat{C}_r$ independent on s, ϵ thanks to Lemma 5.7.1 we get for each non–negative integer r

$$\begin{aligned}
\left|\left|\tilde{Z}(s,\delta)_r\right|\right| &= \left|\left|\left(e^{\lambda \ln(\epsilon) s \mathrm{ad}_B} \left(\sum_{u,v=0}^{\infty} \psi_u(s,\epsilon) Z(s,\delta) \hat{\psi}_v(s,\epsilon) \lambda^{u+v} \right)\right)_r\right|\right| \\
&= \left|\left| \sum_{v,w=0}^{r} \sum_{1 \leqslant i<j \leqslant 4} f_{ij}(s,\delta) \frac{(\ln(\epsilon))^w s^w}{w!} f_{ij}(s,\delta) \, (\mathrm{ad}_B)^{\circ w} \left(\psi_{r-v-w}(s,\epsilon) A_{ij} \hat{\psi}_v(s,\epsilon) \right) \right|\right| \\
&\leqslant \sum_{w=0}^{r} \sum_{1 \leqslant i<j \leqslant 4} |f_{ij}(s,\delta)| \frac{\ell^w |\ln(\delta)|^w}{w!} \left|\left| \sum_{v=0}^{r} (\mathrm{ad}_B)^{\circ w} \left(\psi_{r-v-w}(s,\epsilon) A_{ij} \hat{\psi}_v(s,\epsilon) \right) \right|\right| \\
&\overset{(5.9.11)}{\leqslant} \sum_{w=0}^{r} \sum_{1 \leqslant i<j \leqslant 4} C_{ij} \delta^{\beta_{ij}} |\ln(\delta)|^w C'_w \overset{(5.3.14)}{\leqslant} C''_r \delta^{\beta_r}
\end{aligned}$$

where the non–negative real constant C'_w $(0 \leqslant w \leqslant r)$ is an upper bound for the finite sum over v of bounded algebra elements, $0 < \beta_r$ is the minimum of all $\beta_{ij}/2$, $1 \leqslant i < j \leqslant 4$, coming from inequality (5.3.14), and the non–negative real number C''_r is the maximum of all appearing non–negative multiplicative upper bounds. This proves the last inequality in (5.3.19) and, according to Lemma 5.3.5, the fact that each $\delta \mapsto H_{10}^{(i,u,\delta)}$ is a harmless group term.

In the following we prove the criterion (5.9.11) for each path where the following elementary inequality will occur quite often:

$$\forall \; s \in [0,1] \; \forall \; \gamma \in \,]0,1[: \quad 1 \leqslant e^{-\ln(\gamma)s} \leqslant \frac{1}{\gamma}. \tag{5.9.12}$$

Recall that the logarithms $\ln(2\delta)$, $\ln(2\hat{\delta})$, and $\ln\left(1 - \frac{\delta}{2}\right)$ are non-positive numbers.

I. An elementary computation gives the following formulas for $\Gamma^{(\check{c}_{(\mathrm{I},1,\delta)})}$ and $\Gamma^{(\tilde{c}_{(\mathrm{I},2,\delta)})}$ showing that they are of the form (5.9.9) with $\epsilon = \delta$, with $B = A_{12}$, $A = A_{23}$ for the first path, and with $B = A_{23}$, $A = A_{12}$ for the second path:

$$\begin{aligned}
\Gamma^{(\check{c}_{(\mathrm{I},1,\delta)})}(s) &= \ln(2\delta)A_{12} + \frac{-\ln(2\delta)}{2e^{-\ln(2\delta)s}-1}A_{23} + \delta\underbrace{\frac{(-\ln(2\delta))}{2e^{-\ln(2\delta)s}-\delta}}_{=:f_{24}^{(\mathrm{I},1,\delta)}(s)}A_{24},\\
\Gamma^{(\tilde{c}_{(\mathrm{I},2,\delta)})}(s) &= \ln(2\delta)A_{23} + \frac{-\ln(2\delta)}{2e^{-\ln(2\delta)s}-1}A_{12} + \delta\underbrace{\frac{\ln(2\delta)}{2(1-\delta)e^{-\ln(2\delta)s}+\delta}}_{=:f_{24}^{(\mathrm{I},2,\delta)}(s)}A_{24}.
\end{aligned}$$

For both denominators in the expressions for $f^{(\mathrm{I},1,\delta)}(s)$ and $f^{(\mathrm{I},2,\delta)}(s)$ the inequality (5.9.12) gives us the obvious lower bound $2-\delta > 1$ (for $s = 0$), hence

$$\begin{aligned}
\left|f_{24}^{(\mathrm{I},1,\delta)}(s)\right| &\leqslant \delta\big(\ln(2) + |\ln(\delta)|\big) \overset{(5.3.14)}{\leqslant} 3\delta^{1/2},\\
\left|f_{24}^{(\mathrm{I},2,\delta)}(s)\right| &\leqslant \delta\big(\ln(2) + |\ln(\delta)|\big) \overset{(5.3.14)}{\leqslant} 3\delta^{1/2},
\end{aligned}$$

thanks to $\ln(2) \leqslant 1$ and $\delta \leqslant \delta^{1/2}$ for all $\delta \in]0,1]$. By the criterion (5.9.11) the terms $\delta \mapsto H^{(\mathrm{I},1,\delta))}$ and $\delta \mapsto H^{(\mathrm{I},2,\delta))}$ in the factorization equation (5.9.10) are thus harmless group terms. The factorization equation (5.9.10) and (5.9.7) prove the first equation in (5.9.6) upon setting $H^{(\mathrm{I},\delta))} := H^{(\mathrm{I},2,\delta))}\big(H^{(\mathrm{I},1,\delta))}\big)^{-1}$.

II: An elementary, but lengthy computation gives the following formula for $\Gamma^{(\tilde{c}_{(\mathrm{II},2,\delta)})}$ showing that it is of the form (5.9.9) with $\epsilon = \delta$, $B = A_{24} + A_{34}$, and $A = A_{12} + A_{13}$:

$$\begin{aligned}
\Gamma^{(\tilde{c}_{(\mathrm{II},2,\delta)})} &= \ln(2\delta)\,(A_{24}+A_{34}) + \frac{-\ln(2\delta)}{2e^{-\ln(2\delta)s}-1}\,(A_{12}+A_{13})\\
&+ \underbrace{\frac{\ln\left(1-\frac{\delta}{2}\right) - \delta^2\ln(2\delta)e^{-\ln(2\hat{\delta})s}}{1+\delta^2 e^{-\ln(2\hat{\delta})s}}}_{=:f_{24}^{(\mathrm{II},2,\delta)}(s)}A_{24} + \underbrace{\frac{\ln\left(1-\frac{\delta}{2}\right) + \delta^2\ln(2\delta)e^{-\ln(2\hat{\delta})s}}{1-\delta^2 e^{-\ln(2\hat{\delta})s}}}_{=:f_{34}^{(\mathrm{II},2,\delta)}(s)}A_{34}\\
&+ \underbrace{\frac{\ln\left(1-\frac{\delta}{2}\right) - \ln(2\hat{\delta})2e^{-\ln(2\delta)s} + \ln(2\delta)(2-\delta^2)e^{-\ln(2\hat{\delta})s}}{\left((2-\delta^2)e^{-\ln(2\hat{\delta})s}-1\right)\left(2e^{-\ln(2\delta)s}-1\right)}}_{=:f_{12}^{(\mathrm{II},2,\delta)}(s)}A_{12}\\
&+ \underbrace{\frac{\ln\left(1-\frac{\delta}{2}\right) - \ln(2\hat{\delta})2e^{-\ln(2\delta)s} + \ln(2\delta)(2+\delta^2)e^{-\ln(2\hat{\delta})s}}{\left((2+\delta^2)e^{-\ln(2\hat{\delta})s}-1\right)\left(2e^{-\ln(2\delta)s}-1\right)}}_{=:f_{13}^{(\mathrm{II},2,\delta)}(s)}A_{13}
\end{aligned}$$

We shall now prove the upper bound (5.9.11) for the four functions $f_{24}^{(\mathrm{II},2,\delta)}$, $f_{34}^{(\mathrm{II},2,\delta)}$, $f_{12}^{(\mathrm{II},2,\delta)}$, and $f_{13}^{(\mathrm{II},2,\delta)}$. Note first the following elementary inequality for all $\delta \in J$

$$\left|\ln\left(1-\frac{\delta}{2}\right)\right| = -\ln\left(1-\frac{\delta}{2}\right) \leqslant \delta. \tag{5.9.13}$$

Indeed, for all $0 < x \leqslant 1$ we have $\frac{1}{x} \leqslant \frac{1}{x^2}$, hence

$$-\ln\left(1-\frac{\delta}{2}\right) \;=\; \int_{1-\frac{\delta}{2}}^{1} \frac{1}{x}\mathsf{d}x \;\leqslant\; \int_{1-\frac{\delta}{2}}^{1} \frac{1}{x^2}\mathsf{d}x \;=\; \frac{\delta}{2-\delta} \;\leqslant\; \delta.$$

For $f_{24}^{(\mathrm{II},2,\delta)}$ and $f_{34}^{(\mathrm{II},2,\delta)}$ we can bound both denominators by $1/2$ from below thanks to the lower bound 1 in (5.9.12). In the numerators the exponential function $e^{-\ln(2\hat{\delta})s}$ has an upper bound $\frac{1}{2\hat{\delta}} = \frac{1}{2\delta\left(1-\frac{\delta}{2}\right)} \leqslant \frac{1}{\delta}$ by (5.9.12). Hence, both functions have the following upper bounds (where we also use the inequality (5.9.13)): for all $s \in [0,1]$ and $\delta \in J$

$$\left\{ \begin{array}{l} \left|f_{24}^{(\mathrm{II},2,\delta)}(s)\right| \\ \left|f_{34}^{(\mathrm{II},2,\delta)}(s)\right| \end{array} \right. \;\leqslant\; 2\delta + 2\ln(2\delta)\delta^2\frac{1}{\delta} \overset{(5.3.14)}{\leqslant} 8\delta^{\frac{1}{2}},$$

where the inequalities $\delta \leqslant \delta^{\frac{1}{2}}$ and $\ln(2) \leqslant 1$ have been used. It follows that the criterion (5.9.11) holds for $f_{24}^{(\mathrm{II},2,\delta)}$ and $f_{34}^{(\mathrm{II},2,\delta)}$.
For $f_{12}^{(\mathrm{II},2,\delta)}$ and $f_{13}^{(\mathrm{II},2,\delta)}$ note first that their numerators can be expressed in the following form where we have extracted a factor $\frac{1}{2}e^{-\ln(2\delta)s}$:

$$\frac{1}{2}e^{-\ln(2\delta)s}\left(-2\ln\left(1-\frac{\delta}{2}\right)\left(2-e^{\ln(2\delta)s}\right) + 4\ln(2\delta)\left(e^{-\ln\left(1-\frac{\delta}{2}\right)s}-1\right) - 2g\ln(2\delta)\delta^2 e^{-\ln\left(1-\frac{\delta}{2}\right)s}\right).$$

with $g \in \{-1, 1\}$. On the other hand, in the denominators of $f_{12}^{(\mathrm{II},2,\delta)}$ and $f_{13}^{(\mathrm{II},2,\delta)}$ we can bound both left factors from below by $\frac{1}{2}$ thanks to the lower bound 1 in (5.9.12), and both right factors from below by $e^{-\ln(2\delta)s}$. It follows that for all $s \in [0,1]$ and $\delta \in J$

$$\begin{aligned} \left\{ \begin{array}{l} \left|f_{12}^{(\mathrm{II},2,\delta)}(s)\right| \\ \left|f_{13}^{(\mathrm{II},2,\delta)}(s)\right| \end{array} \right. \;&\leqslant\; 2\left|\ln\left(1-\frac{\delta}{2}\right)\right|\left(2-e^{\ln(2\delta)s}\right) + 4|\ln(2\delta)|\left(e^{-\ln\left(1-\frac{\delta}{2}\right)s}-1\right) \\ &\quad +2|\ln(2\delta)|\delta^2 e^{-\ln\left(1-\frac{\delta}{2}\right)s} \\ &\leqslant\; 4\delta + 4\big(\ln(2)+|\ln(\delta)|\big)\left(e^{-\ln\left(1-\frac{\delta}{2}\right)s}-1\right) + 4\big(\ln(2)+|\ln(\delta)|\big)\delta^2. \end{aligned}$$

Here we have used the inequality (5.9.13), the fact that $e^{\ln(2\delta)s} \geqslant 2\delta$ and $e^{-\ln\left(1-\frac{\delta}{2}\right)s} \leqslant 1/(1-\delta/2) \leqslant 2$ thanks to the upper bound in (5.9.12). Finally, we get, again by (5.9.12) for $\gamma = \left(1-\frac{\delta}{2}\right)$,

$$e^{-\ln\left(1-\frac{\delta}{2}\right)s} - 1 \leqslant \frac{1}{1-\frac{\delta}{2}} - 1 = \frac{\delta}{2-\delta} \leqslant \delta,$$

and again by the inequality (5.3.14) we get the final upper bound

$$\left|f_{12}^{(\mathrm{II},2,\delta)}(s)\right| \leqslant 24\delta^{\frac{1}{2}} \quad \text{and} \quad \left|f_{13}^{(\mathrm{II},2,\delta)}(s)\right| \leqslant 24\delta^{\frac{1}{2}}$$

since $\ln(2) \leqslant 1$ and $\delta \leqslant \delta^{\frac{1}{2}}$. It follows that the criterion (5.9.11) holds for $f_{12}^{(\mathrm{II},2,\delta)}$ and $f_{13}^{(\mathrm{II},2,\delta)}$.
These upper bounds show that the parallel transport along the path $\tilde{c}_{(\mathrm{II},2,\delta)}$ factorizes according to (5.9.10):

$$W_{10}^{(c_{(\mathrm{II},2,\delta)})} \;=\; e^{\lambda\ln(\delta)(A_{24}+A_{34})}\;\psi_\delta(A_{24}+A_{34}, A_{12}+A_{13})\;H^{(\mathrm{II},2,\delta)}, \tag{5.9.14}$$

where $\delta \mapsto H^{(\mathrm{II},2,\delta)}$ is a harmless term. Using (5.9.3) and the second equation of (5.9.5) we can conclude that the parallel transport $W_{10}^{(\check{c}_{(\mathrm{II},1,\delta)})}$ is given by formula (5.9.14) with the index change induced by the inversion permutation σ, see (5.9.3). Passing to the inverse of $W_{10}^{(\check{c}_{(\mathrm{II},1,\delta)})}$ and using formula (5.9.7) we get the proof of the second equation of (5.9.6) upon setting $H^{(\mathrm{II},\delta))} := H^{(\mathrm{II},2,\delta))}\big(H^{(\mathrm{II},1,\delta))}\big)^{-1}$.

III: Due to the symmetry $c_{(\mathrm{III},\delta)} = \Theta \circ c_{(\mathrm{I},\delta)} \circ \iota$ we get the third formula of (5.9.6) by taking the first one, applying the inversion permutation σ, see (5.9.3), and passing to the inverse.

IV: An elementary computation gives the following formulas for $\Gamma^{(\check{c}_{(\mathrm{IV},1,\delta)})}$ and $\Gamma^{(\tilde{c}_{(\mathrm{IV},2,\delta)})}$ showing that they are of the form (5.9.9) with $\epsilon = \delta$, with $B = A_{13} + A_{23}$, $A = A_{34}$ for the first path, and with $\epsilon = \delta^2$, $B = A_{34}$, $A = A_{13} + A_{23}$ for the second path:

$$\begin{aligned}
\Gamma^{(\check{c}_{(\mathrm{IV},1,\delta)})}(s) &= \ln(2\delta)\,(A_{13} + A_{23}) + \frac{-\ln(2\delta)}{2e^{-\ln(2\delta)s} - 1} A_{34} + \underbrace{\frac{2\ln(2\delta)\delta^2 e^{-\ln(2\delta)s}}{1 - 2\delta^2 e^{-\ln(2\delta)s}}}_{=: f_{23}^{(\mathrm{IV},1,\delta)}(s)} A_{23}, \\
\Gamma^{(\tilde{c}_{(\mathrm{IV},2,\delta)})}(s) &= \ln(2\delta^2) A_{34} + \frac{-\ln(2\delta^2)}{2e^{-\ln(2\delta^2)s} - 1}\,(A_{13} + A_{23}) \\
&\quad - \underbrace{\frac{2\ln(2\delta^2)\delta^2 e^{-\ln(2\delta^2)s}}{\left(2e^{-\ln(2\delta^2)s} - 1\right)\left(2(1-\delta^2)e^{-\ln(2\delta^2)s} - 1\right)}}_{=: f_{23}^{(\mathrm{IV},2,\delta)}(s)} A_{23}.
\end{aligned}$$

The denominator of $\left|f_{23}^{(\mathrm{IV},1,\delta)}(s)\right|$ can be bounded from below by 1/2 upon using the upper bound in inequality (5.9.12) for $\gamma = 2\delta$. In the numerator we get the upper bound $\frac{1}{2\delta}$ for the exponential function, again thanks to (5.9.12), hence

$$\left|f_{23}^{(\mathrm{IV},1,\delta)}(s)\right| \leqslant 2(\ln(2) + |\ln(\delta)|)\delta \overset{(5.3.14)}{\leqslant} 6\delta^{\frac{1}{2}}, \tag{5.9.15}$$

hence the criterion (5.9.11) holds for $f_{23}^{(\mathrm{IV},1,\delta)}(s)$.
Next, the right factor in the denominator of $\left|f_{23}^{(\mathrm{IV},2,\delta)}(s)\right|$ can be bounded from below by 1/2 upon using the lower bound in inequality (5.9.12) for $\gamma = 2\delta^2$. We can bound the left factor in that denominator from below by $e^{-\ln(2\delta^2)s}$, hence

$$\left|f_{23}^{(\mathrm{IV},2,\delta)}(s)\right| \leqslant 4\big(\ln(2) + 2|\ln(\delta)|\big)\delta^2 \overset{(5.3.14)}{\leqslant} 12\delta, \tag{5.9.16}$$

hence the criterion (5.9.11) holds for $f_{23}^{(\mathrm{IV},2,\delta)}(s)$. Both upper bounds (5.9.15) and (5.9.16) prove that both parallel transports $W_{10}^{(\check{c}_{(\mathrm{IV},1,\delta)})}$ and $W_{10}^{(\tilde{c}_{(\mathrm{IV},2,\delta)})}$ factorize in the way described in (5.9.10) with harmless group terms $\delta \mapsto H^{(\mathrm{IV},1,\delta)}$ and $\delta \mapsto H^{(\mathrm{IV},2,\delta)}$, respectively. This proves the fourth parallel transport equation in (5.9.6) upon setting $H^{(\mathrm{IV},\delta))} := H^{(\mathrm{IV},2,\delta))}\big(H^{(\mathrm{IV},1,\delta))}\big)^{-1}$.

V: Due to the symmetry $c_{(\mathrm{IV},\delta)} = \Theta \circ c_{(\mathrm{I},\delta)} \circ \iota$ we get the fifth formula of (5.9.6) by taking the fourth one, applying the inversion permutation σ, see (5.9.3), and passing to the inverse. □

By means of these informations we can prove the Pentagon Equation:

Theorem 5.9.2. *The Pentagon Equation* (5.9.1) *for the Drinfeld associator holds.*

Proof. According to Corollary 5.5.7 we have the following equation of parallel transports along the paths (5.9.5) because U' is star-shaped around $\left(\frac{1}{2},\frac{2}{3}\right)$, and the two composed paths $c_{(\mathrm{V},\delta)} * c_{(\mathrm{IV},\delta)}$ and $c_{(\mathrm{III},\delta)} * \left(c_{(\mathrm{II},\delta)} * c_{(\mathrm{I},\delta)}\right)$ are both continuous and piecewise smooth and have the same initial point (δ^2,δ) and final point $(1-\delta, 1-\delta^2)$:

$$W^{(c_{(\mathrm{V},\delta)})}\ W^{(c_{(\mathrm{IV},\delta)})} \ = \ W^{(c_{(\mathrm{III},\delta)})}\ W^{(c_{(\mathrm{II},\delta)})}\ W^{(c_{(\mathrm{I},\delta)})}. \tag{5.9.17}$$

In view of the length of the formulas (5.9.6) of the preceding Lemma 5.9.1 we define the following abbreviations where ϵ, ϵ' are monomials in δ (in practice δ or δ^2), A, B are certain linear combinations of the elements $A_{ij} = A_{ji} \in \mathcal{A}$ for $1 \leqslant i < j \leqslant 4$, and i is an element of $\{\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV},\mathrm{V}\}$:

$$\Phi_{\epsilon,\epsilon'}\left(A, B, H^{(i,\delta)}\right) \ := \ \psi_{\epsilon'}(B,A)\ H^{(i,\delta)}\ \psi_{\epsilon}(A,B)^{-1}. \tag{5.9.18}$$

We recall the relevant commutation relations for the elements A_{ij} coming from the conditions (5.6.1b) and (5.6.1c):

$$[A_{12}, A_{34}] \ = 0, \tag{5.9.19}$$
$$[A_{12}, A_{13}+A_{23}] \ = 0 = \ [A_{12}, A_{12}+A_{13}+A_{23}]\,, \tag{5.9.20}$$
$$[A_{23}, A_{12}+A_{13}] \ = 0 = \ [A_{23}, A_{12}+A_{13}+A_{23}]\,, \tag{5.9.21}$$
$$[A_{23}, A_{24}+A_{34}] \ = 0 = \ [A_{23}, A_{23}+A_{24}+A_{34}]\,, \tag{5.9.22}$$
$$[A_{34}, A_{23}+A_{24}] \ = 0 = \ [A_{34}, A_{23}+A_{24}+A_{34}]\,. \tag{5.9.23}$$

Moreover, recall that if $A \in \mathcal{A}$ commutes with $B_1, \ldots, B_N \in \mathcal{A}$ then the formal exponential $e^{\lambda\gamma A}$ ($\gamma \in \mathbb{C}$) commutes with any formal series whose coefficients consist of noncommutative polynomials in $B_1, \ldots, B_N \in \mathcal{A}$, hence in particular

$$[A,B] = 0 \quad \text{implies} \quad e^{\lambda\gamma A}e^{\lambda\gamma' B} = e^{\lambda(\gamma A+\gamma' B)} = e^{\lambda\gamma' B}e^{\lambda\gamma A}. \tag{5.9.24}$$

As in the proof of the Hexagon Equation (5.8.2) we denote the conjugation $L_\delta H^{(i,\delta)} L_\delta^{-1}$ of a harmless term $\delta \mapsto H^{(i,\delta)}$ by an at most logarithmically divergent term $\delta \mapsto L_\delta$ (as elements in $\big(\mathrm{Fun}(]0,1/4],\mathbb{C}) \otimes \mathcal{A}\big)[[\lambda]]$) by $\tilde{H}^{(i,\delta)}$. Note also that

$$L_\delta\ \Phi_{\epsilon,\epsilon'}\left(A, B, H^{(i,\delta)}\right)\ L_\delta^{-1} \ = \ \Phi_{\epsilon,\epsilon'}\left(L_\delta A L_\delta^{-1}, L_\delta B L_\delta^{-1}, \tilde{H}^{(i,\delta)}\right). \tag{5.9.25}$$

We compute the left hand side of (5.9.17) and try to 'push' the 'singular terms' of the form $e^{\lambda \ln(\epsilon) A}$ (with $\epsilon = \delta$ or $\epsilon = \delta^2$) to the left and to the right: here (5.9.24) will be used:

$$\begin{aligned}
W_{10}^{(c_{(\mathrm{V},\delta)})}\ W_{10}^{(c_{(\mathrm{IV},\delta)})} &= e^{\lambda\ln(\delta)(A_{23}+A_{24})}\ \Phi_{\delta^2,\delta}\left(A_{12}, A_{23}+A_{24}, H^{(\mathrm{V},\delta)}\right)\ e^{-\lambda\ln(\delta^2)A_{12}} \\
&\qquad e^{\lambda\ln(\delta^2)A_{34}}\ \Phi_{\delta,\delta^2}\left(A_{13}+A_{23}, A_{34},\ H^{(\mathrm{IV},\delta)}\right)\ e^{-\lambda\ln(\delta)(A_{13}+A_{23})} \\
&= e^{\lambda\ln(\delta)(A_{23}+A_{24})}\ \Phi_{\delta^2,\delta}\left(A_{12}, A_{23}+A_{24}, H^{(\mathrm{V},\delta)}\right)\ e^{\lambda\ln(\delta^2)A_{34}} \\
&\qquad e^{\lambda\ln(\delta^2)A_{34}}\ \Phi_{\delta,\delta^2}\left(A_{13}+A_{23}, A_{34},\ H^{(\mathrm{IV},\delta)}\right)\ e^{-\lambda\ln(\delta)(A_{13}+A_{23})} \\
&= e^{\lambda\ln(\delta)(A_{23}+A_{24}+2A_{34})}\ \Phi_{\delta^2,\delta}\left(A_{12}, A_{23}+A_{24}, \tilde{H}^{(\mathrm{V},\delta)}\right) \\
&\qquad \Phi_{\delta,\delta^2}\left(A_{13}+A_{23}, A_{34},\ \tilde{H}^{(\mathrm{IV},\delta)}\right)\ e^{-\lambda\ln(\delta)(2A_{12}+A_{13}+A_{23})},
\end{aligned} \tag{5.9.26}$$

where the first equality follows by (5.9.6), the second by (5.9.19), and the third by (5.9.19),(5.9.23),(5.9.20), (5.9.25),(5.9.24). Next, we compute the right hand side of (5.9.17) in a similar way:

$$
\begin{aligned}
&W_{10}^{(c_{(\mathrm{III},\delta)})}\, W_{10}^{(c_{(\mathrm{II},\delta)})}\, W_{10}^{(c_{(\mathrm{I},\delta)})} \\
&\overset{(5.9.6)}{=} \quad e^{\lambda \ln(\delta) A_{34}}\ \Phi_{\delta,\delta}\left(A_{23}, A_{34},\ H^{(\mathrm{III},\delta)}\right)\ e^{-\lambda \ln(\delta) A_{23}} \\
&\qquad e^{\lambda \ln(\delta)(A_{24}+A_{34})}\ \Phi_{\delta,\delta}\left(A_{12}+A_{13}, A_{24}+A_{34}, H^{(\mathrm{II},\delta)}\right)\ e^{-\lambda \ln(\delta)(A_{12}+A_{13})} \\
&\qquad e^{\lambda \ln(\delta) A_{23}}\ \Phi_{\delta,\delta}\left(A_{12}, A_{23}, H^{(\mathrm{I},\delta)}\right)\ e^{-\lambda \ln(\delta) A_{12}} \\
&\overset{(5.9.22),(5.9.21)}{=} \quad e^{\lambda \ln(\delta) A_{34}}\ \Phi_{\delta,\delta}\left(A_{23}, A_{34},\ H^{(\mathrm{III},\delta)}\right)\ e^{\lambda \ln(\delta)(A_{24}+A_{23}+A_{34})} \\
&\qquad e^{-\lambda \ln(\delta) 2A_{23}}\ \Phi_{\delta,\delta}\left(A_{12}+A_{13}, A_{24}+A_{34}, H^{(\mathrm{II},\delta)}\right)\ e^{\lambda \ln(\delta) 2A_{23}} \\
&\qquad e^{-\lambda \ln(\delta)(A_{12}+A_{13}+A_{23})}\ \Phi_{\delta,\delta}\left(A_{12}, A_{23}, H^{(\mathrm{I},\delta)}\right)\ e^{-\lambda \ln(\delta) A_{12}} \\
&\overset{(5.9.22),(5.9.23),(5.9.20),(5.9.21)(5.9.25)}{=} \quad e^{\lambda \ln(\delta)(A_{23}+A_{24}+2A_{34})}\ \Phi_{\delta,\delta}\left(A_{23}, A_{34},\ \tilde{H}^{(\mathrm{III},\delta)}\right) \\
&\qquad \Phi_{\delta,\delta}\left(A_{12}+A_{13}, A_{24}+A_{34}, \tilde{H}^{(\mathrm{II},\delta)}\right) \\
&\qquad \Phi_{\delta,\delta}\left(A_{12}, A_{23}, \tilde{H}^{(\mathrm{I},\delta)}\right)\ e^{-\lambda \ln(\delta)(2A_{12}+A_{13}+A_{23})}.
\end{aligned}
\tag{5.9.27}
$$

A comparison of the preceding equations (5.9.26) and (5.9.27) immediately shows that the singular terms $e^{\lambda \ln(\delta)(A_{23}+A_{24}+2A_{34})}$ and $e^{-\lambda \ln(\delta)(2A_{12}+A_{13}+A_{23})}$ cancel out in (5.9.17), leaving only multiplications of terms of type (5.9.18) which tend to the desired multiplication of Drinfeld associators yielding the Pentagon Equation (5.9.1) in the limit $\delta \to 0$ thanks to the limit rules (5.3.7), the definition of the Drinfeld associator (5.7.18), and the fact that harmless group terms tend to 1 for $\delta \to 0$ (see statement (v) of Proposition 5.3.2). $\square$

Chapter 6

Etingof–Kazhdan quantization of finite–dimensional Lie bialgebras

In this Chapter we give a brief description of the Etingof–Kazhdan quantization of finite–dimensional Lie bialgebras [EK96]. The key idea of Etingof and Kazhdan is, given a Lie bialgebra $\mathfrak{b}$, to define a monoidal structure J on the deformed forgetful functor $(\mathscr{O}^{\mathfrak{d}_\mathfrak{b}})^{\Phi}_{\hbar} : (\mathsf{Mod}(\mathfrak{d}_\mathfrak{b}))^{\Phi}_{\hbar} \to \mathsf{TopFree}_{\mathbb{K}}$, giving rise to an equivalence of monoidal categories. By Tannaka–Krein duality, the twist F_J associated to J allows to construct a non–trivial topological Hopf algebra on $\mathsf{U}(\mathfrak{d}_\mathfrak{b})$, which will be the quantization of the Drinfeld double of $\mathfrak{b}$. They then find a Hopf subalgebra quantizing the Lie bialgebra $\mathfrak{b}$. Such a construction turns out to be also compatible with R–matrices, giving a quantization of any classical solution r of the classical Yang–Baxter equation, see [EK96, §5].
In this Chapter we shall focus particularly on the construction on the monoidal structure in the case of finite–dimensional Lie bialgebras. For more details on the functoriality of the construction, on the case of infinite–dimensional Lie bialgebras and on its PROPic description, we remand to [EK96], [EK98], [ES02] and [ATL18].

6.1 The universal Verma modules

Let $\mathfrak{b}$ be a Lie bialgebra, $\mathfrak{d}_\mathfrak{b}$ be its Drinfeld double, $\hbar$ be formal parameter and Φ be a Drinfeld associator. Recall that the categories $\mathsf{DY}(\mathfrak{b})$ and $\mathsf{Mod}(\mathfrak{d}_\mathfrak{b})$ are monoidally equivalent; hence, since $\mathsf{DY}(\mathfrak{b})$ is infinitesimally braided, we have that also $\mathsf{Mod}(\mathfrak{d}_\mathfrak{b})$ is so. We can thus consider the corresponding deformed category $(\mathsf{Mod}(\mathfrak{d}_\mathfrak{b}))^{\Phi}_{\hbar}$. Set $\mathfrak{b}_+ := \mathfrak{b}$ and $\mathfrak{b}_- := \mathfrak{b}^*$.
We now introduce the key objects of the Etingof–Kazhdan quantization:

Definition 6.1.1. *The **universal Verma modules** of $\mathfrak{d}_\mathfrak{b}$ are the following objects of $(\mathsf{Mod}(\mathfrak{d}_\mathfrak{b}))^{\Phi}_{\hbar}$*

$$M_{\pm} := \mathsf{U}(\mathfrak{d}_\mathfrak{b}) \otimes_{\mathsf{U}(\mathfrak{b}_{\pm})} c_{\pm} = \frac{\mathsf{U}(\mathfrak{d}_\mathfrak{b}) \otimes c_{\pm}}{I_{\pm}}, \tag{6.1.1}$$

where $c_{\pm}$ is the trivial irreducible $\mathfrak{b}_{\pm}$–module of dimension 1 and $I_{\pm}$ is the two–sided ideal of $\mathsf{U}(\mathfrak{d}_\mathfrak{b}) \otimes c_{\pm}$ generated by all elements $u \otimes u_{\pm} \cdot \lambda_{\pm} - u \cdot u_{\pm} \otimes \lambda_{\pm}$, where $u \in \mathsf{U}(\mathfrak{d}_\mathfrak{b})$, $u_{\pm} \in \mathsf{U}(\mathfrak{b}_{\pm})$, $\lambda_{\pm} \in c_{\pm}$.

The following Lemma gives the vector space structure of $M_{\pm}$:

Lemma 6.1.2. *Let $[u \otimes \lambda]$ denote the equivalence class of $u \otimes \lambda$ in the quotient* (6.1.1)*. Then the maps*

$$
\begin{aligned}
\psi_- : \mathsf{U}(\mathfrak{b}_+) &\to M_- \\
u_+ &\mapsto [u_+ \otimes 1] \\
\psi_+ : \mathsf{U}(\mathfrak{b}_-) &\to M_+ \\
u_- &\mapsto [u_- \otimes 1]
\end{aligned}
$$

are isomorphisms of vector spaces.

Proof. We prove the statement for ψ_-; the one for ψ_+ is analogous. The map ψ_- is clearly injective. In order to show that it is surjective, let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{b}_+$ and $\{y_1, \ldots, y_n\}$ be a basis of $\mathfrak{b}_-$. Let $[u \otimes \lambda] \in M_-$. It follows by the Poincaré–Birkhoff–Witt theorem that u has the following form

$$u = \sum_{i \in I} x_1^{\ell_{i,1}} \cdots x_n^{\ell_{i,n}} \cdot y_1^{k_{i,1}} \cdots y_n^{k_{i,n}}.$$

Note that, since the left action of $\mathsf{U}(\mathfrak{b}_-)$ on $\mathsf{U}(\mathfrak{d}_\mathfrak{b})$ commutes with the right action of $\mathsf{U}(\mathfrak{b}_-)$ on c_-, if at least one of the $k_{i,j}$ is positive then $[u \otimes 1] = 0$ in M_-. Therefore, supposing that $k_{i,j} = 0$ for all $j = 1, \ldots, n$, we have

$$[u \otimes \lambda] = \sum_{i \in I} [x_1^{\ell_{i,1}} \cdots x_n^{\ell_{i,n}} \otimes \lambda] = \psi_-\left(\sum_{i \in I} x_1^{\ell_{i,1}} \cdots x_n^{\ell_{i,n}} \otimes \lambda\right) = \psi_-\left(\lambda \sum_{i \in I} x_1^{\ell_{i,1}} \cdots x_n^{\ell_{i,n}} \otimes 1\right).$$

□

The previous Lemma allows to endow the vector spaces $M_\pm$ with a coalgebra structure, induced by the one of $\mathsf{U}(\mathfrak{b}_\mp)$. We shall denote by $\Delta^\pm$ and $\varepsilon^\pm$ the usual comultiplication and counit of $\mathsf{U}(\mathfrak{b}_\pm)$, and by $\Delta_\pm$ and $\varepsilon_\pm$ the comultiplication and counit of $M_\pm$, i.e. $\Delta_\pm := (\psi_\pm \otimes \psi_\pm) \circ \Delta^\pm \circ \psi_\pm^{-1}$ and $\varepsilon_\pm := \varepsilon^\pm \circ \psi_\pm^{-1}$. Setting $1_\pm := \psi_\pm(1) = [1 \otimes 1] \in M_\pm$, we get $\Delta_\pm(1_\pm) = ((\psi_\pm \otimes \psi_\pm) \circ \Delta^\pm)(1) = (\psi_\pm \otimes \psi_\pm)(1 \otimes 1) = 1_\pm \otimes 1_\pm$.

Remark 6.1.3. *Note that the vectors $1_\pm \otimes 1_\pm$ are $\mathfrak{b}_\pm$–invariant: for $x_\pm \in \mathfrak{b}_\pm$ we have that*

$$
\begin{aligned}
[\Delta^\pm(x_\pm), 1_\pm \otimes 1_\pm] &= [x_\pm \otimes 1 + 1 \otimes x_\pm, 1_\pm \otimes 1_\pm] \\
&= x_\pm \cdot 1_\pm \otimes 1_\pm + 1_\pm \otimes x_\pm \cdot 1_\pm \\
&= x_\pm \cdot [1 \otimes 1] \otimes 1_\pm + 1_\pm \otimes x_\pm \cdot [1 \otimes 1] \\
&= 0.
\end{aligned}
$$

The following Proposition is a crucial step for the Etingof–Kazhdan quantization (see [EK96, 2.3]):

Proposition 6.1.4. $(M_\pm, \overline{\Delta}_\pm, \overline{\varepsilon}_\pm)$ *are comonoids in* $(\mathsf{Mod}(\mathfrak{d}_\mathfrak{b}))_h^\Phi$.

Proof. We first show that $\overline{\Delta}_\pm$ is coassociative, i.e. that the diagram

$$
\begin{array}{ccccc}
M_\pm & \xleftarrow{\ \overline{\Delta}_\pm\ } & M_\pm & \xrightarrow{\ \overline{\Delta}_\pm\ } & M_\pm \\
{\scriptstyle \mathsf{id}\bar{\otimes}\overline{\Delta}_\pm}\downarrow & & & & \downarrow{\scriptstyle \overline{\Delta}_\pm\bar{\otimes}\mathsf{id}} \\
M_\pm \bar{\otimes} (M_\pm \bar{\otimes} M_\pm) & & \xleftarrow[\ a^\Phi_{M_\pm, M_\pm, M_\pm}\ ]{} & & (M_\pm \bar{\otimes} M_\pm) \bar{\otimes} M_\pm
\end{array}
$$

commutes. Since $(\mathsf{U}(\mathfrak{b}_\pm), \Delta^\pm, \varepsilon^\pm)$ is a coalgebra in $\mathsf{Vect}_\mathbb{K}$, we have

$$(\mathsf{id} \otimes \Delta^\pm) \circ \Delta^\pm = a_{\mathsf{U}(\mathfrak{b}_\pm),\mathsf{U}(\mathfrak{b}_\pm),\mathsf{U}(\mathfrak{b}_\pm)} \circ (\Delta^\pm \otimes \mathsf{id}) \circ \Delta^\pm.$$

Therefore, we have

$$(\mathsf{id} \otimes \Delta_\pm) \circ \Delta_\pm = a_{M_\pm,M_\pm,M_\pm} \circ (\Delta_\pm \otimes \mathsf{id}) \circ \Delta_\pm,$$

where a is the trivial associativity constraint. Hence it suffices to show that

$$\overline{a}^\Phi_{M_\pm,M_\pm,M_\pm} = \overline{a}_{M_\pm,M_\pm,M_\pm}$$

on the image of $(\overline{\Delta}_\pm \bar{\otimes} \mathsf{id}) \circ \overline{\Delta}_\pm$. This can be easily seen in the case of $1_\pm$: we have

$$(\overline{\Delta}_\pm \bar{\otimes} \mathsf{id}) \circ \overline{\Delta}_\pm(1_\pm) = (1_\pm \bar{\otimes} 1_\pm) \bar{\otimes} 1_\pm$$

and

$$\begin{aligned}
&\overline{a}^\Phi_{M_\pm,M_\pm,M_\pm}((1_\pm \bar{\otimes} 1_\pm) \bar{\otimes} 1_\pm) \\
&= \overline{a}_{M_\pm,M_\pm,M_\pm} \circ \big(\Phi(\bar{t}_{M_\pm,M_\pm} \bar{\otimes} \mathsf{id}_{M_\pm}, (\overline{a}_{M_\pm,M_\pm,M_\pm})^{-1} \circ (\mathsf{id}_{M_\pm} \bar{\otimes} \bar{t}_{M_\pm,M_\pm}) \circ \overline{a}_{M_\pm,M_\pm,M_\pm}((1_\pm \bar{\otimes} 1_\pm) \bar{\otimes} 1_\pm)\big) \\
&= \overline{a}_{M_\pm,M_\pm,M_\pm}\big((1 + \mathcal{O}(\hbar^2))((1_\pm \bar{\otimes} 1_\pm) \bar{\otimes} 1_\pm)\big) \\
&= \overline{a}_{M_\pm,M_\pm,M_\pm}\big((1_\pm \bar{\otimes} 1_\pm) \bar{\otimes} 1_\pm)\big) \\
&= 1_\pm \bar{\otimes} (1_\pm \bar{\otimes} 1_\pm)
\end{aligned}$$

where the third equality follows from the fact that $1_\pm$ is $\mathfrak{b}_\pm$–invariant. Next, let $[u_\pm \otimes 1]$ be any element of $M_\pm$. By the definition of $M_\pm$ we have $[u_\mp \otimes 1] = u_\mp \cdot 1_\pm$ for any $u_\mp \in \mathsf{U}(\mathfrak{b}_\mp)$ (in other words, the module $M_\pm$ is freely generated over $\mathsf{U}(\mathfrak{b}_\mp)$ by the vector $1_\pm$). Since all the maps involved are $\mathsf{U}(\mathfrak{d}_b)$–linear, we have

$$\big((\overline{\Delta}_\pm \bar{\otimes} \mathsf{id}) \circ \overline{\Delta}_\pm\big)(u_\mp \cdot 1_\pm) = u_\mp \cdot \big((\overline{\Delta}_\pm \bar{\otimes} \mathsf{id}) \circ \overline{\Delta}_\pm\big)(1_\pm) = u_\mp \cdot \big((1_\pm \bar{\otimes} 1_\pm) \bar{\otimes} 1_\pm\big)$$

and

$$\overline{a}^\Phi_{M_\pm,M_\pm,M_\pm}\big(u_\mp \cdot ((1_\pm \bar{\otimes} 1_\pm) \bar{\otimes} 1_\pm)\big) = u_\mp \cdot \overline{a}^\Phi_{M_\pm,M_\pm,M_\pm}\big((1_\pm \bar{\otimes} 1_\pm) \bar{\otimes} 1_\pm\big) = u_\mp \cdot \big(1_\pm \bar{\otimes} (1_\pm \bar{\otimes} 1_\pm)\big).$$

Finally, the counit axiom

$$(\bar{\varepsilon}_\pm \bar{\otimes} \mathsf{id}_{M_\pm}) \circ \overline{\Delta}_\pm = (\mathsf{id}_{M_\pm} \bar{\otimes} \bar{\varepsilon}_\pm) \circ \overline{\Delta}_\pm$$

directly follows from the counity of $\mathsf{U}(\mathfrak{b}_\mp)$, since the identities do not involve associators. □

6.2 The monoidal structure on the forgetful functor

Consider the forgetful functor

$$\begin{aligned}
\mathscr{O}^{\mathfrak{d}_\mathfrak{b}} : \mathsf{Mod}(\mathfrak{d}_\mathfrak{b}) &\to \mathsf{Vect}_K \\
V &\mapsto V \\
f &\mapsto f.
\end{aligned}$$

The aim of this Section is to define a monoidal structure on the deformed functor

$$(\mathscr{O}^{\mathfrak{d}_\mathfrak{b}})^\Phi_\hbar : (\mathsf{Mod}(\mathfrak{d}_\mathfrak{b}))^\Phi_\hbar \to \mathsf{TopFree}_\mathbb{K}.$$

First we need the following result (see [EK96, 2.1]):

Lemma 6.2.1. *The assignment $1 \to 1_+ \otimes 1_-$ extends to an isomorphism of left Lie $\mathfrak{d}_{\mathfrak{b}}$–modules*

$$\phi : \mathsf{U}(\mathfrak{d}_{\mathfrak{b}}) \to M_+ \otimes M_-.$$

Corollary 6.2.2. *We have that $(\mathscr{O}^{\mathfrak{d}_{\mathfrak{b}}})^{\Phi}_{\hbar}$ is represented by $M_+ \bar{\otimes} M_-$, i.e.*

$$(\mathscr{O}^{\mathfrak{d}_{\mathfrak{b}}})^{\Phi}_{\hbar}(V) \cong \mathsf{Hom}_{(\mathsf{Mod}(\mathfrak{d}_{\mathfrak{b}}))^{\Phi}_{\hbar}}(M_+ \bar{\otimes} M_-, V).$$

From now on we shall denote by F the functor $\mathsf{Hom}_{(\mathsf{Mod}(\mathfrak{d}_{\mathfrak{b}}))^{\Phi}_{\hbar}}(M_+ \bar{\otimes} M_-, -)$ and we will do all the reasonings with F instead of $(\mathscr{O}^{\mathfrak{d}_{\mathfrak{b}}})^{\Phi}_{\hbar}$. We now construct a monoidal structure on F:

Proposition 6.2.3. *For V and W in $\mathsf{Obj}((\mathsf{Mod}(\mathfrak{d}_{\mathfrak{b}}))^{\Phi}_{\hbar})$, consider the map*

$$\begin{aligned} J_{V,W} : F(V) \bar{\otimes} F(W) &\to F(V \bar{\otimes} W) \\ v \bar{\otimes} w &\mapsto J_{V,W}(v \bar{\otimes} w) \end{aligned}$$

defined by

$$J_{V,W}(v \bar{\otimes} w) = (v \bar{\otimes} w) \circ \overline{\beta}^{\Phi}_{M_+,M_+,M_-,M_-} \circ (\overline{\Delta}_+ \bar{\otimes} \overline{\Delta}_-)$$

where $\beta_{M_+,M_+,M_-,M_-} : (M_+ \bar{\otimes} M_+) \bar{\otimes} (M_- \bar{\otimes} M_-) \to (M_+ \bar{\otimes} M_-) \otimes (M_+ \bar{\otimes} M_-)$ is a natural isomorphism given by the permutation of the second and the third components, composed opportunetely with associators (see 2.5.5). Then the triple (F, id, J) is a monoidal functor.

Proof. Suppose for simplicity that all the involved categories are strict. We are going to give three interpretations of the same proof: one algebraic, one diagramatic and one pictorial.
We have to check that J satisfies the axioms of Definition 2.1.2. We have that diagrams (2.1.4) trivially hold and have the following form

$$\begin{array}{ccc} \mathbb{K}[[\hbar]] \bar{\otimes} V & \xrightarrow{\bar{\ell}_V} & V \\ \mathsf{id} \downarrow & & \uparrow \bar{\ell}_V \\ \mathbb{K}[[\hbar]] \bar{\otimes} V & \xrightarrow{J_{\mathbb{K}[[\hbar]],V} = \mathsf{id}} & \mathbb{K}[[\hbar]] \bar{\otimes} V \end{array} \qquad \begin{array}{ccc} V \bar{\otimes} \mathbb{K}[[\hbar]] & \xrightarrow{\bar{r}_V} & V \\ \mathsf{id} \downarrow & & \uparrow \bar{r}_V \\ V \bar{\otimes} \mathbb{K}[[\hbar]] & \xrightarrow{J_{V,\mathbb{K}[[\hbar]]} = \mathsf{id}} & V \bar{\otimes} \mathbb{K}[[\hbar]] \end{array}$$

It remains to check the relation arising from diagram (2.1.3): for three left Lie $\mathfrak{d}_{\mathfrak{b}}$–modules U, V, W and elements $u \in F(U)$, $v \in F(V)$ and $w \in F(W)$ we have to prove that

$$\big(J_{U \bar{\otimes} V,W} \circ (J_{U,V} \bar{\otimes} \mathsf{id}_W)\big)(u \bar{\otimes} v \bar{\otimes} w) = \big(J_{U,V \bar{\otimes} W} \circ (\mathsf{id}_U \bar{\otimes} J_{V,W})\big)(u \bar{\otimes} v \bar{\otimes} w). \tag{6.2.1}$$

Set $c_{+,-} \coloneqq \overline{c}^{\Phi}_{M_+,M_-}$, $\mathsf{id}_+ \coloneqq \mathsf{id}_{M_+}$ and generalize this notation for tensor product and for M_- (e.g. $c_{+,--}$ denotes $\overline{c}^{\Phi}_{M_+,M_- \bar{\otimes} M_-}$ and id_- denotes id_{M_-}). We have that the left hand side of (6.2.1) is equal to

$$(u \bar{\otimes} v \bar{\otimes} w) \circ (\mathsf{id}_+ \bar{\otimes} c_{+,-} \bar{\otimes} \mathsf{id}_- \bar{\otimes} \mathsf{id}_+ \bar{\otimes} \mathsf{id}_-) \circ (\overline{\Delta}_+ \bar{\otimes} \overline{\Delta}_- \bar{\otimes} \mathsf{id}_+ \bar{\otimes} \mathsf{id}_-) \circ (\mathsf{id}_+ \bar{\otimes} c_{+,-} \bar{\otimes} \mathsf{id}_-) \circ (\overline{\Delta}_+ \bar{\otimes} \overline{\Delta}_-) \tag{6.2.2}$$

while the right hand side of (6.2.1) is equal to

$$(u \bar{\otimes} v \bar{\otimes} w) \circ (\mathsf{id}_+ \bar{\otimes} \mathsf{id}_- \bar{\otimes} \mathsf{id}_+ \bar{\otimes} c_{+-} \bar{\otimes} \mathsf{id}_-) \circ (\mathsf{id}_+ \bar{\otimes} \mathsf{id}_- \bar{\otimes} \overline{\Delta}_+ \bar{\otimes} \overline{\Delta}_-) \circ (\mathsf{id}_+ \bar{\otimes} c_{+-} \bar{\otimes} \mathsf{id}_-)(\overline{\Delta}_+ \bar{\otimes} \circ \overline{\Delta}_-). \tag{6.2.3}$$

For the functoriality of the braiding, we have

$$(\overline{\Delta}_+ \bar{\otimes} \overline{\Delta}_- \bar{\otimes} \mathsf{id}_+ \bar{\otimes} \mathsf{id}_-) \circ (\mathsf{id}_+ \bar{\otimes} c_{+,-} \bar{\otimes} \mathsf{id}_-) = (\mathsf{id}_{++} \bar{\otimes} c_{+,--} \bar{\otimes} \mathsf{id}_-) \circ (\overline{\Delta}_+ \bar{\otimes} \mathsf{id}_+ \bar{\otimes} \overline{\Delta}_- \bar{\otimes} \mathsf{id}_-) \tag{6.2.4a}$$

$$(\mathsf{id}_+ \bar{\otimes} \mathsf{id}_- \bar{\otimes} \overline{\Delta}_+ \bar{\otimes} \overline{\Delta}_-) \circ (\mathsf{id}_+ \bar{\otimes} c_{+,-} \bar{\otimes} \mathsf{id}_-) = (\mathsf{id}_+ \bar{\otimes} c_{++,-} \bar{\otimes} \mathsf{id}_{--}) \circ (\mathsf{id}_+ \bar{\otimes} \overline{\Delta}_+ \bar{\otimes} \mathsf{id}_- \otimes \overline{\Delta}_-) \tag{6.2.4b}$$

and applying (6.2.4a) to (6.2.2) and (6.2.4b) to (6.2.3) we have that (6.2.2) and (6.2.3) are respectively equal to

$$(u\bar\otimes v\bar\otimes w)\circ(\mathsf{id}_+\bar\otimes c_{+,-}\bar\otimes\mathsf{id}_{-+-})\circ(\mathsf{id}_{++}\bar\otimes c_{+,--}\bar\otimes\mathsf{id}_-)\circ(\overline{\Delta}_+\bar\otimes\mathsf{id}_+\bar\otimes\overline{\Delta}_-\bar\otimes\mathsf{id}_-)\circ(\overline{\Delta}_+\bar\otimes\overline{\Delta}_-) \tag{6.2.5a}$$

$$(u\bar\otimes v\bar\otimes w)\circ(\mathsf{id}_{+-+}\bar\otimes c_{+,-}\bar\otimes\mathsf{id}_-)\circ(\mathsf{id}_+\bar\otimes c_{++,-}\bar\otimes\mathsf{id}_{--})\circ(\mathsf{id}_+\bar\otimes\overline{\Delta}_+\bar\otimes\mathsf{id}_-\bar\otimes\overline{\Delta}_-)\circ(\overline{\Delta}_+\bar\otimes\overline{\Delta}_-) \tag{6.2.5b}$$

By the fact that $M_\pm$ are comonoids (see Proposition 6.1.4), we have that

$$(\overline{\Delta}_+\bar\otimes\mathsf{id}_+\bar\otimes\overline{\Delta}_-\bar\otimes\mathsf{id}_-)\circ(\overline{\Delta}_+\bar\otimes\overline{\Delta}_-)=(\mathsf{id}_+\bar\otimes\overline{\Delta}_+\bar\otimes\mathsf{id}_-\bar\otimes\overline{\Delta}_-)\circ(\overline{\Delta}_+\bar\otimes\overline{\Delta}_-).$$

Moreover, using the hexagon equations (2.2.1) and (2.2.2) we get

$$(c_{+,-}\bar\otimes\mathsf{id}_-\bar\otimes\mathsf{id}_+)\circ(\mathsf{id}_+\bar\otimes c_{+,--})=(c_{+,-}\bar\otimes\mathsf{id}_-\bar\otimes\mathsf{id}_+)\circ(\mathsf{id}_+\bar\otimes\mathsf{id}_-\bar\otimes c_{+,-})\circ(\mathsf{id}_+\bar\otimes c_{+,-}\bar\otimes\mathsf{id}_-) \tag{6.2.6a}$$

$$(\mathsf{id}_-\bar\otimes\mathsf{id}_+\bar\otimes c_{+,-})\circ(\mathsf{id}_+\bar\otimes c_{+,--})=(\mathsf{id}_-\bar\otimes\mathsf{id}_+\bar\otimes c_{+,-})\circ(c_{+,-}\bar\otimes\mathsf{id}_+\bar\otimes\mathsf{id}_-)\circ(\mathsf{id}_+\bar\otimes c_{+,-}\bar\otimes\mathsf{id}_-) \tag{6.2.6b}$$

and by (2.5.1a) we have that

$$(c_{+,-}\bar\otimes\mathsf{id}_-\bar\otimes\mathsf{id}_+)\circ(\mathsf{id}_+\bar\otimes\mathsf{id}_-\bar\otimes c_{+,-})=(\mathsf{id}_-\bar\otimes\mathsf{id}_+\bar\otimes c_{+,-})\circ(c_{+,-}\bar\otimes\mathsf{id}_+\bar\otimes\mathsf{id}_-).$$

Therefore Equation (6.2.1) holds and this concludes the algebraic proof. From a diagramatic point of view, Equation (6.2.1) is described by the commutativity of the diagram

$$\begin{array}{ccc}
F(U)\bar\otimes F(V)\bar\otimes F(W) & \xrightarrow{J_{U,V}\bar\otimes\mathsf{id}} & F(U\bar\otimes V)\bar\otimes F(W) \\
{\scriptstyle \mathsf{id}\bar\otimes J_{V,W}}\downarrow & & \downarrow{\scriptstyle J_{U\bar\otimes V,W}} \\
F(U)\bar\otimes F(U\bar\otimes W) & \xrightarrow{J_{U,V\bar\otimes W}} & F(U\bar\otimes V\bar\otimes W)
\end{array}$$

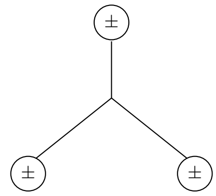
which, using the definition of J, is equivalent to the commutativity of the following diagram

$$\begin{array}{ccccc}
M_+\bar\otimes M_- & \xrightarrow{\overline{\Delta}_+\bar\otimes\overline{\Delta}_-} & M_+^{\bar\otimes 2}\bar\otimes M_-^{\bar\otimes 2} & \xrightarrow{\mathsf{id}_+\bar\otimes c_{+,-}\bar\otimes\mathsf{id}_-} & (M_+\bar\otimes M_-)^{\bar\otimes 2} \\
{\scriptstyle \overline{\Delta}_+\bar\otimes\overline{\Delta}_-}\downarrow & & & & \downarrow{\scriptstyle \overline{\Delta}_+\bar\otimes\overline{\Delta}_-\bar\otimes\mathsf{id}_{+-}} \\
M_+^{\bar\otimes 2}\bar\otimes M_-^{\bar\otimes 2} & & & & M_+^{\bar\otimes 2}\bar\otimes M_-^{\bar\otimes 2}\bar\otimes M_+\bar\otimes M_- \\
{\scriptstyle \mathsf{id}_+\bar\otimes c_{+,-}\bar\otimes\mathsf{id}_-}\downarrow & & & & \downarrow{\scriptstyle \mathsf{id}_+\bar\otimes c_{+,-}\bar\otimes\mathsf{id}_{-+-}} \\
(M_+\bar\otimes M_-)^{\bar\otimes 2} & \xrightarrow{\mathsf{id}_{+-}\bar\otimes\overline{\Delta}_+\bar\otimes\overline{\Delta}_-} & M_+\bar\otimes M_-\bar\otimes M_+^{\bar\otimes 2}\bar\otimes M_-^{\bar\otimes 2} & \xrightarrow{\mathsf{id}_{+-+}\otimes c_{+,-}\bar\otimes\mathsf{id}_-} & (M_+\bar\otimes M_-)^{\bar\otimes 3}
\end{array}$$

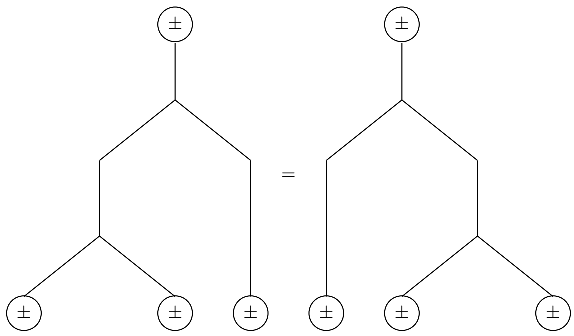
In order to show its commutativity, it suffices to translate in diagrams the algebraic expressions above: we obtain

$$\begin{array}{ccccc}
M_+\bar\otimes M_- & \xrightarrow{\overline{\Delta}_+\bar\otimes\overline{\Delta}_-} & M_+^{\bar\otimes 2}\bar\otimes M_-^{\bar\otimes 2} & \xrightarrow{\mathsf{id}_+\bar\otimes c_{+,-}\bar\otimes\mathsf{id}_-} & (M_+\bar\otimes M_-)^{\bar\otimes 2} \\
{\scriptstyle \overline{\Delta}_+\bar\otimes\overline{\Delta}_-}\downarrow & & \downarrow{\scriptstyle \overline{\Delta}_+\bar\otimes\mathsf{id}_+\bar\otimes\overline{\Delta}_-\bar\otimes\mathsf{id}_-} & & \downarrow{\scriptstyle \overline{\Delta}_+\bar\otimes\overline{\Delta}_-\bar\otimes\mathsf{id}_{+-}} \\
M_+^{\bar\otimes 2}\bar\otimes M_-^{\bar\otimes 2} & \xrightarrow{\mathsf{id}_+\bar\otimes\overline{\Delta}_+\bar\otimes\mathsf{id}_-\bar\otimes\overline{\Delta}_-} & M_+^{\bar\otimes 3}\otimes M_-^{\bar\otimes 3} & \xrightarrow{\mathsf{id}_{++}\bar\otimes c_{+,--}\bar\otimes\mathsf{id}_-} & M_+^{\bar\otimes 2}\bar\otimes M_-^{\bar\otimes 2}\bar\otimes M_+\bar\otimes M_- \\
{\scriptstyle \mathsf{id}_+\bar\otimes c_{+,-}\bar\otimes\mathsf{id}_-}\downarrow & & \downarrow{\scriptstyle \mathsf{id}_+\bar\otimes c_{++,-}\bar\otimes\mathsf{id}_-} & & \downarrow{\scriptstyle \mathsf{id}_+\bar\otimes c_{+,-}\bar\otimes\mathsf{id}_{-+-}} \\
(M_+\bar\otimes M_-)^{\bar\otimes 2} & \xrightarrow{\mathsf{id}_{+-}\bar\otimes\overline{\Delta}_+\bar\otimes\overline{\Delta}_-} & M_+\bar\otimes M_-\bar\otimes M_+^{\bar\otimes 2}\bar\otimes M_-^{\bar\otimes 2} & \xrightarrow{\mathsf{id}_{+-+}\bar\otimes c_{+,-}\bar\otimes\mathsf{id}_-} & (M_+\bar\otimes M_-)^{\bar\otimes 3}
\end{array}$$

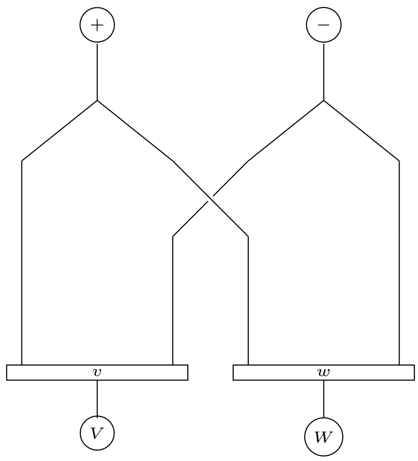
where the top left square commutes for the cocommutativity of $\overline{\Delta}_{\pm}$, the top right and bottom left squares commute for the naturality of the braiding, and finally the bottom right square commutes for the properties of the braiding. Next, we sketch a pictorial proof, using a top–to–bottom convention. We represent the morphisms $\overline{\Delta}_{\pm}$ by



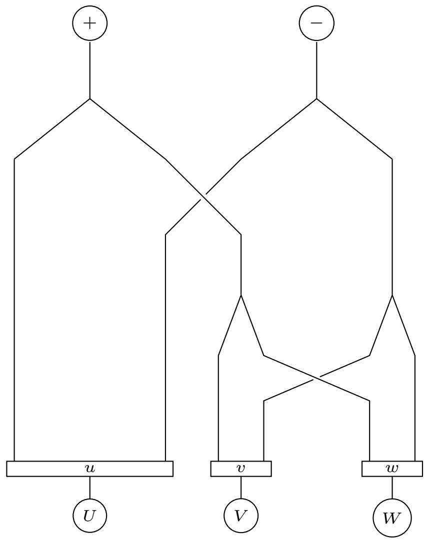
The coassociativity of $\overline{\Delta}_{\pm}$ is represented by the equality



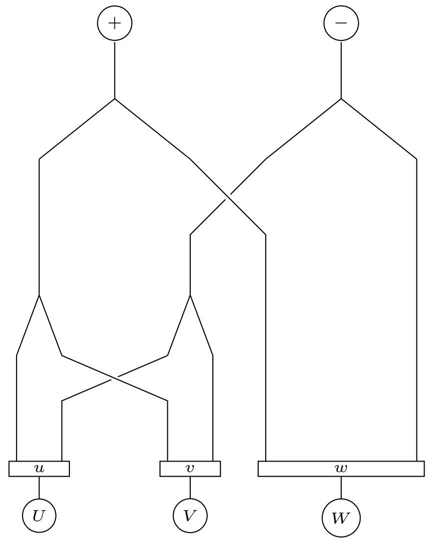
while $J_{V,W}(v\bar{\otimes}w)$ is represented by



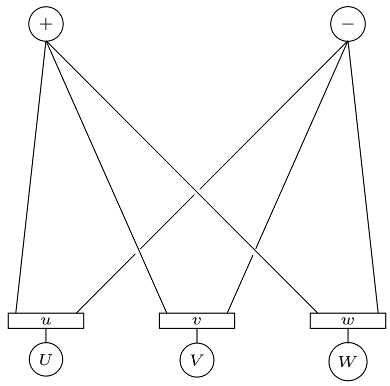
The terms $J_{U,V\bar{\otimes}W} \circ (\mathsf{id}_U \bar{\otimes} J_{V,W})$ and $J_{U\bar{\otimes}V,W} \circ (J_{U,V} \bar{\otimes} \mathsf{id}_W)$ are represented respectively by



and



and, by the coassociativity of $M_\pm$ and the properties of the braiding, both are equivalent to the picture



concluding the pictorial proof. □

6.3 Tannaka–Krein duality and quantization of Lie bialgebras

Recall the Tannaka–Krein duality for bialgebras (see [ES02, Ch. 18] for more details):

Proposition 6.3.1. *Let A be a $\mathbb{K}$–algebra and let $F : \mathsf{Mod}(A) \to \mathsf{Vect}_{\mathbb{K}}$ be the forgetful functor. Then any monoidal structure $(\mathsf{Mod}(A), \tilde{\otimes}, \mathbb{K}, \tilde{a}, \tilde{\ell}, \tilde{r})$ on $\mathsf{Mod}(A)$ together with a monoidal structure $(F, \varphi_0^F, \varphi_2^F)$ on F equips A with a bialgebra structure such that $(\mathsf{Mod}(A), \tilde{\otimes}, \mathbb{K}, \tilde{a}, \tilde{\ell}, \tilde{r})$ is monoidally equivalent to $(\mathsf{Mod}(A), \otimes, \mathbb{K}, a, \ell, r)$.*

Therefore, recalling that $\mathsf{Mod}(\mathfrak{d}_\mathfrak{b}) = \mathsf{Mod}(\mathsf{U}(\mathfrak{d}_\mathfrak{g}))$ and combining Propositions 6.2.3 and 6.3.1 gives a non–trivial topological bialgebra structure on $\mathsf{U}(\mathfrak{d}_\mathfrak{b})$, which we denote by $\mathsf{U}_\hbar(\mathfrak{d}_\mathfrak{b})^{\mathsf{EK}}$. Moreover, there exists a gauge transformation F_J such that $J_{V,W}(v\bar{\otimes}w) = F_J^{-1}(v\bar{\otimes}w)$, see [Kas12, XV.3] for more details. Hence the topological Hopf algebra $\mathsf{U}_\hbar(\mathfrak{d}_\mathfrak{b})^{\mathsf{EK}}$ is obtained by twisting the trivial one with F_J.

Theorem 6.3.2. *(*[EK96, Prop. 3.6]*) $\mathsf{U}_\hbar(\mathfrak{d}_\mathfrak{b})^{\mathsf{EK}}$ is a quantization of $\mathsf{U}(\mathfrak{d}_\mathfrak{b})$*

This shows how to quantize the Drinfeld double of any finite–dimensional Lie bialgebra. In particular, we have that $\mathsf{U}_\hbar(\mathfrak{d}_\mathfrak{b})^{\mathsf{EK}} \cong \mathsf{Hom}_{(\mathsf{DY}(\mathfrak{d}_\mathfrak{b}))^\Phi_\hbar}(M_+\bar{\otimes} M_-, M_+\bar{\otimes} M_-)$, see [EK96, Lem. 4.1].
Next, in order to define the quantization of $\mathfrak{b}$, consider the object $F(M_-)$. Then Etingof and Kazhdan proved the following

Theorem 6.3.3. *(*[EK96, Th. 4.7]*) $F(M_-)$ has the following topological bialgebra structure:*

- *The multiplication is $x \circ (\mathsf{id}_+\bar{\otimes} y) \circ \overline{a}^\Phi_{M_+,M_+,M_-} \circ F(\overline{\Delta}_+\bar{\otimes}\mathsf{id}_-)$.*
- *The unit is $a \mapsto \big(1_+\bar{\otimes}1_- \mapsto a \cdot 1_-\big)$.*
- *The comultiplication is $J^{-1}_{M_-,M_-} \circ F(\overline{\Delta}_-)$.*
- *The counit is $F(\overline{\varepsilon}_-)$.*

Moreover, $F(M_-)$ is a quantization of the Lie bialgebra $\mathfrak{b}$.

Chapter 7

Ševera quantization of Lie bialgebras

In this Chapter we exhibit the quantization technique of Lie bialgebras provided by P. Ševera in [Šev16].

7.1 M–adapted functors

Definition 7.1.1. [Šev16, Def.1] *Let $(\mathscr{C}, \otimes, I, a, \ell, r, c)$ and $(\mathscr{D}, \otimes', I', a', \ell', r', c')$ be two braided monoidal categories, let $(M, \Delta_M, \varepsilon_M)$ be a comonoid in $\mathscr{D}$ and let (F, ψ_F^0, ψ_F^2) be a braided comonoidal functor from $\mathscr{D}$ to $\mathscr{C}$. We say that F is M–**adapted** if for any X, Y in $\mathsf{Obj}(\mathscr{D})$ the morphisms*

$$\chi_M \coloneqq \psi_F^0 \circ F(\varepsilon_M) \tag{7.1.1a}$$
$$\gamma_{X,Y}^M \coloneqq \psi_F^2(X \otimes' M, M \otimes' Y) \circ F(\alpha'_{X,M,M,Y}) \circ F\big((\mathsf{id}_X \otimes' \Delta_M) \otimes' \mathsf{id}_Y\big) \tag{7.1.1b}$$

are invertible[1].

Proposition 7.1.2. *Let $\mathscr{C}, \mathscr{D}$ be braided monoidal categories, $F : \mathscr{D} \to \mathscr{C}$ be a braided comonoidal functor and $(M, \Delta_M, \varepsilon_M)$ be a comonoid in $\mathscr{D}$. Then*

(i) The functor $M \otimes' - : \mathscr{D} \to \mathscr{D}$, $X \mapsto M \otimes' X$ together with

$$\psi_\otimes^2(X, Y) = \beta'_{M,M,X,Y} \circ (\Delta_M \otimes' \mathsf{id}_{X \otimes' Y})$$
$$\psi_\otimes^0 = r_{I'} \circ (\varepsilon_M \otimes' \mathsf{id}_{I'}) = \ell_{I'} \circ (\varepsilon_M \otimes' \mathsf{id}_{I'})$$

is a comonoidal functor.

(ii) F is M–adapted if and only if the functor

$$F_M : \mathscr{D} \xrightarrow{M \otimes' -} \mathscr{D} \xrightarrow{F} \mathscr{C}$$

is strongly comonoidal.

Proof. (i): Suppose that $\mathscr{C}$ and $\mathscr{D}$ are strict. Let X, Y, Z in $\mathsf{Obj}(\mathscr{D})$. Using the coassociativity of Δ_M, the naturality of c', Equation (2.5.1a) and the strict counterparts of (2.2.1) (2.2.2) we have that the diagrams

[1] In Ševera's paper [Šev16] $\gamma_{X,Y}^M$ is denoted by $\tau_{X,Y}^{(M)}$.

(1)

$$\begin{array}{ccc}
M \otimes' X \otimes' Y \otimes' Z & \xrightarrow{\Delta_M \otimes' \mathsf{id}_{X \otimes' Y \otimes' Z}} & M^{\otimes' 2} \otimes' X \otimes' Y \otimes' Z \\
{\scriptstyle \Delta_M \otimes' \mathsf{id}_{X \otimes' Y \otimes' Z}} \big\downarrow & & \big\downarrow {\scriptstyle \Delta_M \otimes' \mathsf{id}_{M \otimes' X \otimes' Y \otimes' Z}} \\
M^{\otimes' 2} \otimes' X \otimes' Y \otimes' Z & \xrightarrow{\mathsf{id}_M \otimes' \Delta_M \otimes' \mathsf{id}_{X \otimes' Y \otimes' Z}} & M^{\otimes' 3} \otimes' X \otimes' Y \otimes' Z
\end{array}$$

(2)

$$\begin{array}{ccc}
M^{\otimes' 2} \otimes' X \otimes' Y \otimes' Z & \xrightarrow{\mathsf{id}_M \otimes' \Delta_M \otimes' \mathsf{id}_{X \otimes' Y \otimes' Z}} & M^{\otimes' 3} \otimes' X \otimes' Y \otimes' Z \\
{\scriptstyle \mathsf{id}_M \otimes' c'_{M,X} \otimes' \mathsf{id}_{Y \otimes' Z}} \big\downarrow & & \big\downarrow {\scriptstyle \mathsf{id}_M \otimes' c'_{M^{\otimes' 2},X} \otimes' \mathsf{id}_{Y \otimes' Z}} \\
M \otimes' X \otimes' M \otimes' Y \otimes' Z & \xrightarrow{\mathsf{id}_{M \otimes' X} \otimes' \Delta_M \otimes' \mathsf{id}_{Y \otimes' Z}} & M \otimes' X \otimes' M^{\otimes' 2} \otimes' Y \otimes' Z
\end{array}$$

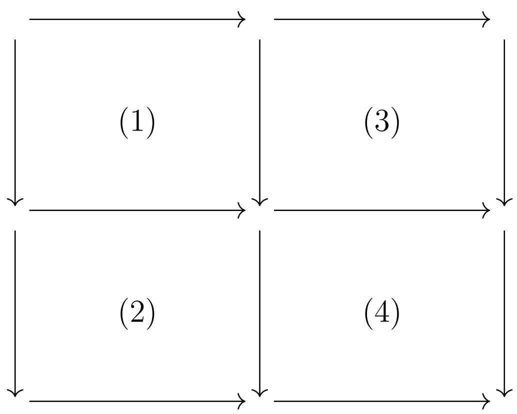
(3)

$$\begin{array}{ccc}
M^{\otimes' 2} \otimes' X \otimes' Y \otimes' Z & \xrightarrow{\mathsf{id}_M \otimes' c'_{M,X \otimes' Y} \otimes' \mathsf{id}_Z} & M \otimes' X \otimes' Y \otimes' M \otimes' Z \\
{\scriptstyle \Delta_M \otimes' \mathsf{id}_{M \otimes' X \otimes' Y \otimes' Z}} \big\downarrow & & \big\downarrow {\scriptstyle \Delta_M \otimes' \mathsf{id}_{X \otimes' Y \otimes' M \otimes' Z}} \\
M^{\otimes' 3} \otimes' X \otimes' Y \otimes' Z & \xrightarrow{\mathsf{id}_{M \otimes' M} \otimes' c'_{M,X \otimes' Y} \otimes' \mathsf{id}_Z} & M^{\otimes' 2} \otimes' X \otimes' Y \otimes' M \otimes' Z
\end{array}$$

(4)

$$\begin{array}{ccc}
M^{\otimes' 3} \otimes' X \otimes' Y \otimes' Z & \xrightarrow{\mathsf{id}_M^{\otimes' 2} \otimes' c'_{M,X \otimes' Y} \otimes' \mathsf{id}_Z} & M^{\otimes' 2} \otimes' X \otimes' Y \otimes' M \otimes' Z \\
{\scriptstyle \mathsf{id}_M \otimes' c'_{X,M^{\otimes' 2}} \otimes' \mathsf{id}_{Y \otimes' Z}} \big\downarrow & & \big\downarrow {\scriptstyle \mathsf{id}_M \otimes' c'_{M,X} \otimes' \mathsf{id}_{M \otimes' Y \otimes' Z}} \\
M \otimes' X \otimes' M^{\otimes' 2} \otimes' Y \otimes' Z & \xrightarrow{\mathsf{id}_{M \otimes' X \otimes' M} \otimes' c'_{M,Y} \otimes' \mathsf{id}_Z} & M \otimes' X \otimes' M \otimes' Y \otimes' M \otimes' Z
\end{array}$$

commute. The joint diagram



shows that the triple $(\otimes' -, \psi^0_\otimes, \psi^2_\otimes)$ satisfies the strict counterpart of (2.1.5). Finally, the strict counterpart of the fact that $(\otimes' -, \psi^0_\otimes, \psi^2_\otimes)$ satisfies the two squares (2.1.6) gives the equalities

$$(\varepsilon_M \otimes' \mathsf{id}_M) \circ \Delta_M = \mathsf{id}_M$$
$$(\mathsf{id}_M \otimes' \varepsilon_M) \circ \Delta_M = \mathsf{id}_M$$

which hold since $(M, \Delta_M, \varepsilon_M)$ is a comonoid in $\mathscr{D}$.
(ii): Using the naturality of a', c', ψ^2_F and the coherence Theorem 2.4.2 we have that for any X, Y

in $\mathsf{Obj}(\mathscr{D})$ the diagram

$$\begin{array}{ccccc}
F\big(M\otimes'(X\otimes' Y)\big) & \xleftarrow{F(a'_{M,X,Y})} & F\big((M\otimes' X)\otimes' Y\big) & \xleftarrow{F(c'_{X,M}\otimes'\mathsf{id}_Y)} & F\big((X\otimes' M)\otimes' Y\big) \\
\downarrow{\scriptstyle F(\Delta_M\otimes'\mathsf{id}_{X\otimes' Y})} & & \downarrow{\scriptstyle F((\Delta_M\otimes'\mathsf{id}_X)\otimes'\mathsf{id}_Y)} & & \downarrow{\scriptstyle F((\mathsf{id}_X\otimes'\Delta_M)\otimes'\mathsf{id}_W Y)} \\
F\big((M\otimes' M)\otimes'(X\otimes' Y)\big) & \xleftarrow{F(a'_{M\otimes' M,X,Y})} & F\big(((M\otimes' M)\otimes' X)\otimes' Y\big) & \xleftarrow{F(c'_{X,M\otimes' M}\otimes\mathsf{id}_Y)} & F\big((X\otimes'(M\otimes' M))\otimes' Y\big) \\
\downarrow{\scriptstyle F(\beta'_{M,M,X,Y})} & & & & \downarrow{\scriptstyle F(\alpha'_{X,M,M,Y})} \\
F\big((M\otimes' X)\otimes'(M\otimes' Y)\big) & & \xleftarrow{F(c'_{M,X}\otimes'\mathsf{id}_{M\otimes Y})} & & F\big((X\otimes' M)\otimes'(M\otimes' Y)\big) \\
\downarrow{\scriptstyle \psi^2_F(M\otimes' X,M\otimes' Y)} & & & & \downarrow{\scriptstyle \psi^F_2(X\otimes' M,M\otimes' Y)} \\
F(M\otimes' X)\otimes F(M\otimes' Y) & & \xleftarrow{F(c'_{M,X})\otimes\mathsf{id}_{F(M\otimes' Y)}} & & F(X\otimes' M)\otimes F(M\otimes Y)
\end{array}$$

commutes, leading to the identity

$$\psi^2_F\circ F(\psi^2_\otimes)=\big(F(c'_{M,X})\otimes\mathsf{id}_{F(M\otimes' Y)}\big)\circ\gamma^M_{X,Y}\circ\big(F(a'_{M,X,Y})\circ F(c'_{X,M}\otimes'\mathsf{id}_Y)\big)^{-1}.$$

Similarly, using the naturality of r' we have that the diagram

$$\begin{array}{ccc}
F(M\otimes' I') & \xrightarrow{F(r'_M)} & F(M) \\
\downarrow{\scriptstyle F(\varepsilon_M\otimes'\mathsf{id}_{I'})} & & \downarrow{\scriptstyle F(\varepsilon_M)} \\
F(I'\otimes' I') & \xrightarrow{F(r'_{I'})} & F(I') \\
\downarrow{\scriptstyle F(r'_{I'})} & & \\
F(I') & & \downarrow{\scriptstyle \psi^0_F} \\
\downarrow{\scriptstyle \psi^0_F} & & \\
I & \xleftarrow[\mathsf{id}_I]{} & I
\end{array}$$

commutes, leading to the identity

$$\psi^0_F\circ F(\psi^0_\otimes)=\chi_M\circ F(r'_M).$$

Hence, the morphism $\gamma^M_{X,Y}$ (resp. χ_M) is invertible if and only if the morphism $\psi^2_F\circ F(\psi^2_\otimes)$ (resp. the morphism $\psi^0_F\circ F(\psi^0_\otimes)$) is invertible, i.e. when $M\otimes'-\circ F$ is strongly comonoidal. $\square$

7.2 The multiplication along a comonoid

Definition 7.2.1. *Let $(\mathscr{C},\otimes,I,a,\ell,r,c)$ and $(\mathscr{D},\otimes',I',a',\ell',r',c')$ be two braided monoidal categories, $(M,\Delta_M,\varepsilon_M)$ be a comonoid of $\mathscr{D}$, (F,ψ^0_F,ψ^0_F) be a M–adapted functor from $\mathscr{D}$ to $\mathscr{C}$. For any objects X,Y of $\mathscr{D}$ we define the* ***multiplication map of*** *X* ***and*** *Y* ***along*** *M as the map*

$$\mu^M_{X,Y}\coloneqq F(r_X\otimes'\mathsf{id}_Y)\circ F\big((\mathsf{id}_X\otimes'\varepsilon_M)\otimes'\mathsf{id}_Y\big)\circ(\gamma^M_{X,Y})^{-1}:F(X\otimes' M)\otimes F(M\otimes' Y)\to F(X\otimes' Y)$$

Proposition 7.2.2. *Let $(\mathscr{C},\otimes,I,a,\ell,r,c)$ and $(\mathscr{D},\otimes',I',a',\ell',r',c')$ be two braided monoidal categories, (F,ψ^0_F,ψ^2_F) be a comonoidal functor from $\mathscr{D}$ to $\mathscr{C}$, $(M,\Delta_M,\varepsilon_M)$ and $(N,\Delta_N,\varepsilon_N)$ be two* *comonoids in $\mathscr{D}$ such that F is both M–adapted and N–adapted. Then for any objects X, Y of $\mathscr{D}$ the following diagram commutes*

$$\begin{array}{ccc}
\big(F(X\otimes' M)\otimes F(M\otimes' N)\big)\otimes F(N\otimes' Y) & \xrightarrow{a_{F(X\otimes' M),F(M\otimes' N),F(N\otimes' Y)}} & F(X\otimes' M)\otimes\big(F(M\otimes' N)\otimes F(N\otimes' Y)\big) \\
\downarrow{\scriptstyle \mu^M_{X,N}\otimes \mathsf{id}_{F(N\otimes' Y)}} & & \downarrow{\scriptstyle \mathsf{id}_{F(X\otimes' M)}\otimes \mu^N_{M,Y}} \\
F(X\otimes' N)\otimes F(N\otimes' Y) & & F(X\otimes' M)\otimes F(M\otimes' Y) \\
{\scriptstyle \mu^N_{X,Y}}\searrow & & \swarrow{\scriptstyle \mu^M_{X,Y}} \\
& F(X\otimes' Y) &
\end{array}$$

Proof. Suppose for simplicity that both $\mathscr{C}$ and $\mathscr{D}$ are strict. Using the naturality of ψ^2_F, the strict counterpart of (2.1.5), and Equation (2.5.1a) gives that the following diagrams commute

(1)

$$\begin{array}{ccc}
F(X\otimes' M)\otimes F(M\otimes' N)\otimes F(N\otimes' Y) & \xleftarrow{\mathsf{id}_{F(X\otimes' M)}\otimes\psi^2_F(M\otimes' N,N\otimes' Y)} & F(X\otimes' M)\otimes F(M\otimes' N^{\otimes' 2}\otimes' Y) \\
\uparrow{\scriptstyle \psi^2_F(M^{\otimes' 2},M^{\otimes' 2})\otimes\mathsf{id}_{F(N\otimes' Y)}} & & \uparrow{\scriptstyle \psi^2_F(X\otimes' M,M\otimes' N^{\otimes' 2}\otimes' Y)} \\
F(X\otimes' M^{\otimes' 2}\otimes' N)\otimes F(N\otimes' Y) & \xleftarrow{\psi^2_F(X\otimes' M^{\otimes' 2}\otimes' N,N\otimes' Y)} & F(X\otimes' M^{\otimes' 2}\otimes' N^{\otimes' 2}\otimes' Y)
\end{array}$$

(2)

$$\begin{array}{ccc}
F(X\otimes' M^{\otimes' 2}\otimes' N)\otimes F(N\otimes' Y) & \xleftarrow{\psi^2_F(X\otimes' M^{\otimes' 2}\otimes' N,N\otimes' Y)} & F(X\otimes' M^{\otimes' 2}\otimes' N^{\otimes' 2}\otimes' Y) \\
\uparrow{\scriptstyle F(\mathsf{id}_X\otimes'\Delta_M\otimes'\mathsf{id}_F)\otimes\mathsf{id}_{F(N\otimes' Y)}} & & \uparrow{\scriptstyle F(\mathsf{id}_X\otimes'\Delta_M\otimes'\mathsf{id}_{N^{\otimes' 2}\otimes' Y})} \\
F(X\otimes' M\otimes' N)\otimes F(N\otimes' Y) & \xleftarrow{\psi^2_F(X\otimes' M\otimes' N,N\otimes' Y)} & F(X\otimes' M^{\otimes' 2}\otimes' N^{\otimes' 2}\otimes Y)
\end{array}$$

(3)

$$\begin{array}{ccc}
F(X\otimes' M\otimes' N)\otimes F(N\otimes' Y) & \xleftarrow{\psi^2_F(X\otimes' M\otimes' N,N\otimes' Y)} & F(X\otimes' M^{\otimes' 2}\otimes' N^{\otimes' 2}\otimes Y) \\
\downarrow{\scriptstyle F(\mathsf{id}_X\otimes'\varepsilon_M\otimes\mathsf{id}_N)\otimes\mathsf{id}_{F(n\otimes' Y)}} & & \downarrow{\scriptstyle F(\mathsf{id}_X\otimes'\varepsilon_M\otimes\mathsf{id}_{N^{\otimes' 2}\otimes' Y})} \\
F(X\otimes' N)\otimes F(N\otimes' Y) & \xleftarrow{\psi^2_F(X\otimes N,N\otimes' Y)} & F(X\otimes' N^{\otimes' 2}\otimes' Y)
\end{array}$$

(4)

$$\begin{array}{ccc}
F(X\otimes' M)\otimes F(M\otimes' N^{\otimes' 2}\otimes' Y) & \xleftarrow{F(\mathsf{id}_{X\otimes' M})\otimes F(\mathsf{id}_M\otimes'\Delta_N\otimes' Y)} & F(X\otimes' M)\otimes F(M\otimes' N\otimes' Y) \\
\uparrow{\scriptstyle \psi^2_F(X\otimes' M,M\otimes' N^{\otimes' 2}\otimes' Y)} & & \uparrow{\scriptstyle \psi^2_F(X\otimes' M,M\otimes' N\otimes' Y)} \\
F(X\otimes' M^{\otimes' 2}\otimes' N^{\otimes' 2}\otimes' Y) & \xleftarrow{F(\mathsf{id}_{X\otimes' M^{\otimes' 2}}\otimes'\Delta_N\otimes'\mathsf{id}_Y)} & F(X\otimes' M^{\otimes' 2}\otimes' N\otimes' Y)
\end{array}$$

(5)

$$\begin{array}{ccc}
F(X\otimes' M^{\otimes' 2}\otimes' N^{\otimes' 2}\otimes' Y) & \xleftarrow{F(\mathsf{id}_{X\otimes' M^{\otimes' 2}}\otimes'\Delta_N\otimes'\mathsf{id}_Y)} & F(X\otimes' M^{\otimes' 2}\otimes' N\otimes' Y) \\
\uparrow{\scriptstyle F(\mathsf{id}_X\otimes'\Delta_M\otimes'\mathsf{id}_{N^{\otimes' 2}\otimes' Y})} & & \uparrow{\scriptstyle F(\mathsf{id}_X\otimes'\Delta_M\otimes'\mathsf{id}_{N\otimes' Y})} \\
F(X\otimes' M^{\otimes' 2}\otimes' N^{\otimes' 2}\otimes Y) & \xleftarrow{F(\mathsf{id}_{X\otimes' M}\otimes'\Delta_N\otimes'\mathsf{id}_Y)} & F(X\otimes' M\otimes' N\otimes' Y)
\end{array}$$

(6)

$$\begin{array}{ccc}
F(X \otimes' M^{\otimes' 2} \otimes' N^{\otimes 2} \otimes Y) & \xleftarrow{F(\mathsf{id}_{X \otimes' M} \otimes' \Delta_N \otimes' \mathsf{id}_Y)} & F(X \otimes' M \otimes' N \otimes' Y) \\
{\scriptstyle F(\mathsf{id}_X \otimes' \varepsilon_M \otimes \mathsf{id}_{N^{\otimes' 2} \otimes' Y})} \big\downarrow & & \big\downarrow {\scriptstyle F(\mathsf{id}_X \otimes' \varepsilon_M \otimes \mathsf{id}_{N \otimes' Y})} \\
F(X \otimes' N^{\otimes' 2} \otimes' Y) & \xleftarrow{F(\mathsf{id}_X \otimes' \Delta_N \otimes' \mathsf{id}_Y)} & F(X \otimes' N \otimes' Y)
\end{array}$$

(7)

$$\begin{array}{ccc}
F(X \otimes' M) \otimes F(M \otimes' N \otimes' Y) & \xrightarrow{\mathsf{id}_{F(X \otimes' M)} \otimes F(\mathsf{id}_X \otimes' \varepsilon_N \otimes' \mathsf{id}_F)} & F(X \otimes' M) \otimes F(M \otimes' Y) \\
{\scriptstyle \psi_F^2(X \otimes' M, M \otimes' N \otimes' Y)} \big\uparrow & & \big\uparrow {\scriptstyle \psi_F^2(X \otimes' M, M \otimes' Y)} \\
F(X \otimes' M^{\otimes' 2} \otimes' N \otimes' Y) & \xrightarrow{F(\mathsf{id}_{X \otimes' M^{\otimes' 2}} \otimes' \varepsilon_N \otimes' \mathsf{id}_Y)} & F(X \otimes' M^{\otimes' 2} \otimes Y)
\end{array}$$

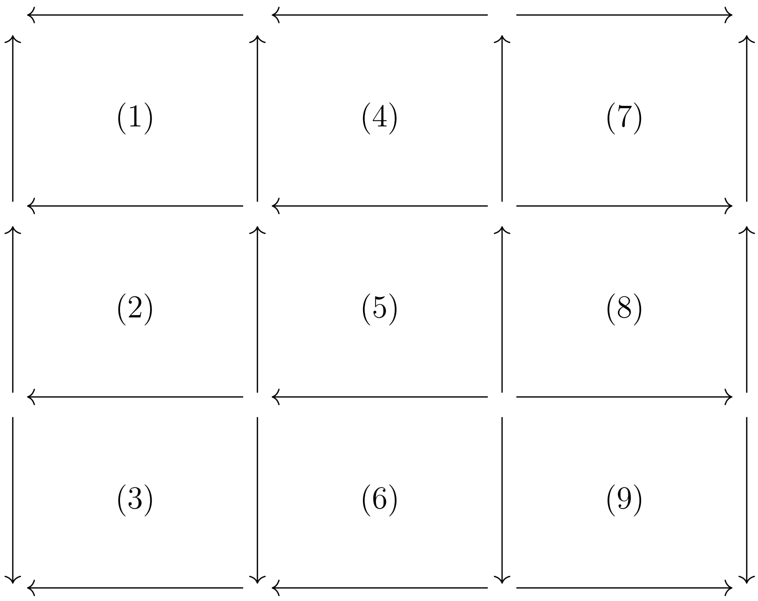
(8)

$$\begin{array}{ccc}
F(X \otimes' M^{\otimes' 2} \otimes' N \otimes' Y) & \xrightarrow{F(\mathsf{id}_{X \otimes' M^{\otimes' 2}} \otimes' \varepsilon_N \otimes' \mathsf{id}_Y)} & F(X \otimes' M^{\otimes' 2} \otimes Y) \\
{\scriptstyle F(\mathsf{id}_X \otimes' \Delta_M \otimes' \mathsf{id}_{N \otimes' Y})} \big\uparrow & & \big\uparrow {\scriptstyle F(\mathsf{id}_X \otimes' \Delta_M \otimes' \mathsf{id}_Y)} \\
F(X \otimes' M \otimes' N \otimes' Y) & \xrightarrow{F(\mathsf{id}_{X \otimes' M} \otimes' \varepsilon_N \otimes' \mathsf{id}_Y)} & F(X \otimes' M \otimes' Y)
\end{array}$$

(9)

$$\begin{array}{ccc}
F(X \otimes' M \otimes' N \otimes' Y) & \xrightarrow{F(\mathsf{id}_{X \otimes' M} \otimes' \varepsilon_N \otimes' \mathsf{id}_Y)} & F(X \otimes' M \otimes' Y) \\
{\scriptstyle F(\mathsf{id}_X \otimes' \varepsilon_M \otimes' \mathsf{id}_{M \otimes' Y})} \big\downarrow & & \big\downarrow {\scriptstyle F(\mathsf{id}_X \otimes' \varepsilon_M \otimes' \mathsf{id}_Y)} \\
F(X \otimes' N \otimes' Y) & \xrightarrow{F(\mathsf{id}_X \otimes' \varepsilon_N \otimes' \mathsf{id}_Y)} & F(X \otimes' Y)
\end{array}$$

The joint diagram



proves the statement. □

Remark 7.2.3. *The Proposition above has the following important consequences:*

- *Setting $M = X = N = Y$ gives that $\big(F(M \otimes' M), \mu^M_{M,M}\big)$ is a monoid, see Theorem 7.3.1 for the definition of the unit.*
- *Setting $M = X = N$ gives that $\big(F(M \otimes' Y), \mu^M_{M,Y}\big)$ is a left $F(M \otimes' M)$–module.*
- *Setting $M = Y = N$ gives that $\big(F(X \otimes' M), \mu^M_{X,M}\big)$ is a right $F(M \otimes' M)$–module.*
- *Setting $X = M$ and $N = Y$ gives that $F(M \otimes' N)$ is a $F(M \otimes' M)$–$F(N \otimes' N)$ bimodule.*

7.3 The Hopf monoid $F(M \otimes' M)$

Theorem 7.3.1. [Šev16, Th.1] *Let $\mathscr{C}, \mathscr{D}$ be two braided monoidal categories, $(M, \Delta_M, \varepsilon_M)$ a cocommutative comonoid object in $\mathscr{D}$ and $(F, \psi^0_F, \psi^0_F) : \mathscr{D} \to \mathscr{C}$ be a M–adapted functor. Then $F(M \otimes' M)$ is a Hopf monoid, where*

(i) The multiplication is

$$\mu^M_{M,M} = F(r'_M \otimes \mathsf{id}_M) \circ F\big((\mathsf{id}_M \otimes' \varepsilon_M) \otimes' \mathsf{id}_M\big) \circ (\gamma^M_{M,M})^{-1}. \tag{7.3.1}$$

(ii) The unit is

$$F(\Delta_M) \circ (\chi_M)^{-1}. \tag{7.3.2}$$

(iii) The comultiplication is

$$\psi^2_F(M \otimes' M, M \otimes' M) \circ F(\beta'_{M,M,M,M}) \circ F(\Delta_M \otimes' \Delta_M). \tag{7.3.3}$$

(iv) The counit is

$$\psi^0_F \circ F(r'_{I'}) \circ F(\varepsilon_M \otimes' \varepsilon_M). \tag{7.3.4}$$

(v) The antipode is

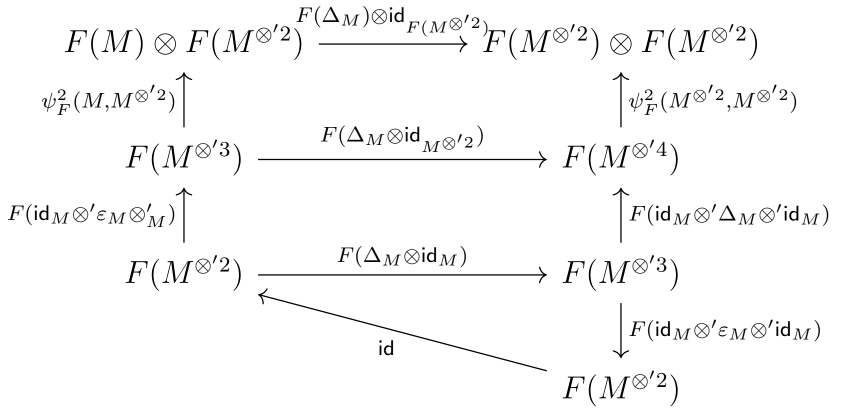
$$F(c'_{M,M}). \tag{7.3.5}$$

Proof. In order to give the proof we may suppose that $\mathscr{C}$ and $\mathscr{D}$ are strict.
(i): The fact that $\mu^M_{M,M}$ is an associative multiplication for $F(M \otimes' M)$ follows directly by Proposition 7.2.2 setting $X = Y = N = M$.
(ii): Using the naturality of ψ^2_F and the fact that $(M, \Delta_M, \varepsilon_M)$ is a comonoid in $\mathscr{D}$, we obtain that the following four diagrams commute

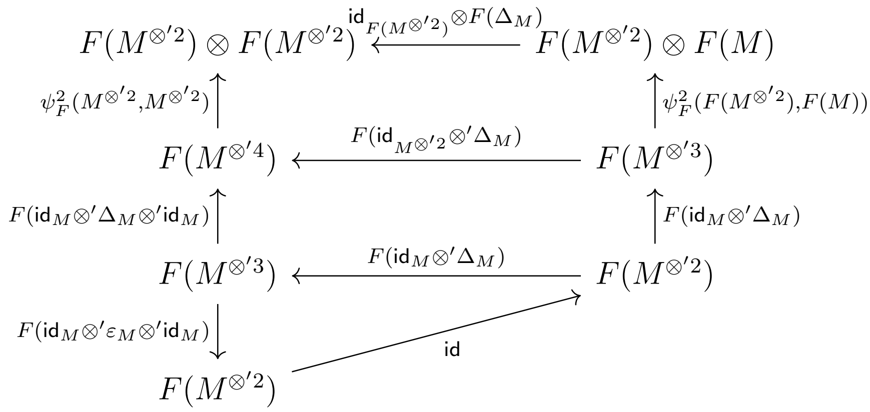
(1)

$$\begin{array}{ccccc}
I \otimes F(M^{\otimes' 2}) & \xleftarrow{\psi^0_F \otimes \mathsf{id}_{F(M^{\otimes' 2})}} & F(I') \otimes F(M^{\otimes' 2}) & \xleftarrow{F(\varepsilon_M) \otimes \mathsf{id}_{F^{\otimes' 2}}} & F(M) \otimes F(M^{\otimes' 2}) \\
 & \nwarrow^{\mathsf{id}} & \uparrow {\scriptstyle \psi^2_F(I', M^{\otimes' 2})} & & \uparrow {\scriptstyle \psi^2_F(M, M^{\otimes' 2})} \\
 & & F(I' \otimes' M^{\otimes' 2}) & \xleftarrow{F(\varepsilon_M \otimes \mathsf{id}_{M^{\otimes' 2}})} & F(M^{\otimes' 3}) \\
 & & & \nwarrow^{\mathsf{id}} & \uparrow {\scriptstyle F(\mathsf{id}_M \otimes' \varepsilon_M \otimes'_M)} \\
 & & & & F(M^{\otimes' 2})
\end{array}$$

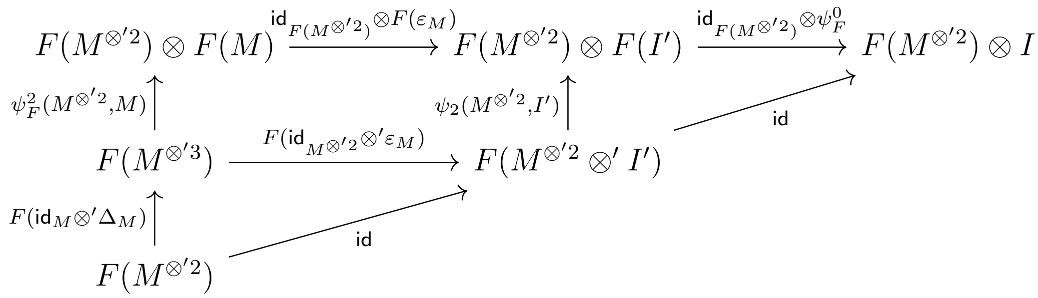
(2)



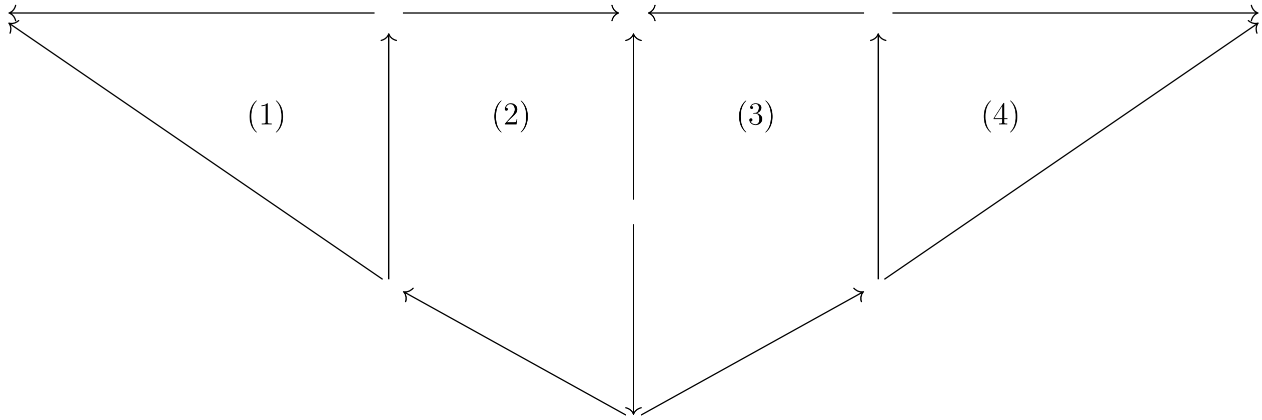
(3)



(4)



The joint diagram



gives that $F(\Delta_M) \circ (\chi_M)^{-1}$ satisfies the unit axiom.
(iii)–(iv): The triple

$$\left(F(M \otimes' M), \psi_F^2(M \otimes' M, M \otimes' M) \circ F(\beta'_{M,M,M,M}) \circ F(\Delta_M \otimes' \Delta_M), \psi_F^0 \circ F(r'_{I'}) \circ F(\varepsilon_M \otimes' \varepsilon_M)\right)$$

is a comonoid in $\mathscr{C}$ follows directly by Propositions 3.3.3 and 3.3.4.
Next, recal the following four facts:

- For any braided monoidal category $\mathscr{C}$, we have that $\mathsf{Comon}(\mathscr{C})$ is a monoidal category: hence if $(C_1, \Delta_1, \varepsilon_1)$, $(C_2, \Delta_2, \varepsilon_2)$, $(C_3, \Delta_3, \varepsilon_3)$, $(C_4, \Delta_4, \varepsilon_4)$ are four comonoids and $\alpha : C_1 \to C_2$, $\gamma : C_3 \to C_4$ are morphisms of comonoids, then $\alpha \otimes \gamma$ is a morphism of comonoids (see statement (ii) of Proposition 3.3.3).

- If (F, ψ_F^0, ψ_F^2) is a comonoidal functor, $(C_1, \Delta_1, \varepsilon_1)$, $(C_2, \Delta_2, \varepsilon_2)$ are two comonoids and $\alpha : C_1 \to C_2$ is a morphism of comonoids, then so is $F(\alpha)$ (see statement (ii) of Proposition 3.3.4).

- If (F, ψ_F^0, ψ_F^2) is a comonoidal functor and $(C_1, \Delta_1, \varepsilon_1)$, $(C_2, \Delta_2, \varepsilon_2)$ are two comonoids, then $\psi_F^2(C_1, C_2)$ is a morphism of comonoids (see statement (iii) of Proposition 3.3.4).

- If (C, Δ, ε) is a cocommutative comonoid then $\Delta : C \to C \otimes C$ is a morphism of comonoids (see statement (iii) of Proposition 3.3.3).

Therefore, since both the multiplication and the unit of $F(M \otimes' M)$ are built up on compositions of comonoid morphisms, they are so, i.e. $F(M \otimes' M)$ is a bimonoid in $\mathscr{C}$.
(v): Note first that

$$\begin{aligned}
\eta_{F(M\otimes' M)} \circ \varepsilon_{F(M\otimes' M)} &= F(\Delta_M) \circ (\chi_M)^{-1} \circ \psi_0^F \circ F(\varepsilon_M \otimes' \varepsilon_M) \\
&= F(\Delta_M) \circ \left(\psi_0^F \circ F(\varepsilon_M)\right)^{-1} \circ \psi_0^F \circ F(\varepsilon_M \otimes' \varepsilon_M) \\
&= F(\Delta_M) \circ \left(\psi_0^F \circ F(\varepsilon_M)\right)^{-1} \circ \psi_0^F \circ F(\varepsilon_M) \circ F(\varepsilon_M \otimes' \mathsf{id}_M) \\
&= F(\Delta_M) \circ F(\varepsilon_M \otimes' \mathsf{id}_M) \\
&= F(\varepsilon_M \otimes' \Delta_M)
\end{aligned}$$

and

$$\begin{aligned}
\eta_{F(M\otimes' M)} \circ \varepsilon_{F(M\otimes' M)} &= F(\Delta_M) \circ (\chi_M)^{-1} \circ \psi_0^F \circ F(\varepsilon_M \otimes' \varepsilon_M) \\
&= F(\Delta_M) \circ \left(\psi_0^F \circ F(\varepsilon_M)\right)^{-1} \circ \psi_0^F \circ F(\varepsilon_M \otimes' \varepsilon_M) \\
&= F(\Delta_M) \circ \left(\psi_0^F \circ F(\varepsilon_M)\right)^{-1} \circ \psi_0^F \circ F(\varepsilon_M) \circ F(\mathsf{id}_M \otimes' \varepsilon_M) \\
&= F(\Delta_M) \circ F(\mathsf{id}_M \otimes' \varepsilon_M) \\
&= F(\Delta_M \otimes' \varepsilon_M).
\end{aligned}$$

Moreover, using the naturality of ψ_F^2, the naturality of the braiding c' and Equations (2.5.1a), (2.2.1), (2.2.1), and (2.5.3) we have that the diagrams

$$
\begin{array}{ccc}
F(M^{\otimes' 2}) & \xrightarrow{F(\Delta_M \otimes' \varepsilon_M)} & F(M^{\otimes' 2}) \\
{\scriptstyle F(\Delta_M \otimes' \mathsf{id}_M)}\downarrow & & \downarrow{\scriptstyle F(\mathsf{id}_M \otimes' \varepsilon_M \otimes' \mathsf{id}_M)} \\
F(M^{\otimes' 3}) & \xrightarrow{F(\mathsf{id}_M \otimes' c'_{M,M})} & F(M^{\otimes' 3}) \\
{\scriptstyle F(\mathsf{id}_{M^{\otimes' 2}} \otimes' \Delta_M)}\downarrow & & \\
F(M^{\otimes' 4}) & & \downarrow{\scriptstyle F(\mathsf{id}_M \otimes' \Delta_M \otimes' \mathsf{id}_M)} \\
{\scriptstyle F(\mathsf{id}_M \otimes' c'_{M,M} \otimes' \mathsf{id}_M)}\downarrow & & \\
F(M^{\otimes' 4}) & \xrightarrow{F(\mathsf{id}_{M^{\otimes' 2}} \otimes' c'_{M,M})} & F(M^{\otimes' 4}) \\
{\scriptstyle \psi^2_F(M^{\otimes' 2}, M^{\otimes' 2})}\downarrow & & \downarrow{\scriptstyle \psi^2_F(M^{\otimes' 2}, M^{\otimes' 2})} \\
F(M^{\otimes' 2})^{\otimes 2} & \xrightarrow{\mathsf{id}_{F(M^{\otimes' 2})} \otimes F(c'_{M,M})} & F(M^{\otimes' 2})^{\otimes 2}
\end{array}
$$

and

$$
\begin{array}{ccc}
F(M^{\otimes' 2}) & \xrightarrow{F(\varepsilon_M \otimes' \Delta_M)} & F(M^{\otimes' 2}) \\
{\scriptstyle F(\mathsf{id}_M \otimes' \Delta_M)}\downarrow & & \downarrow{\scriptstyle F(\mathsf{id}_M \otimes' \varepsilon_M \otimes' \mathsf{id}_M)} \\
F(M^{\otimes' 3}) & \xrightarrow{F(c'_{M,M} \otimes' \mathsf{id}_M)} & F(M^{\otimes' 3}) \\
{\scriptstyle F(\Delta_M \otimes' \mathsf{id}_{M^{\otimes' 2}})}\downarrow & & \\
F(M^{\otimes' 4}) & & \downarrow{\scriptstyle F(\mathsf{id}_M \otimes' \Delta_M \otimes' \mathsf{id}_M)} \\
{\scriptstyle F(\mathsf{id}_M \otimes' c'_{M,M} \otimes' \mathsf{id}_M)}\downarrow & & \\
F(M^{\otimes' 4}) & \xrightarrow{F(c'_{M,M} \otimes' \mathsf{id}_{M^{\otimes' 2}})} & F(M^{\otimes' 4}) \\
{\scriptstyle \psi^2_F(M^{\otimes' 2}, M^{\otimes' 2})}\downarrow & & \downarrow{\scriptstyle \psi^2_F(M^{\otimes' 2}, M^{\otimes' 2})} \\
F(M^{\otimes' 2})^{\otimes 2} & \xrightarrow{F(c'_{M,M}) \otimes \mathsf{id}_{F(M^{\otimes' 2})}} & F(M^{\otimes' 2})^{\otimes 2}
\end{array}
$$

commute. Hence we have that

$$
\begin{aligned}
\mu_{F(M\otimes' M)} \circ (F(c'_{M,M}) \otimes \mathsf{id}_{F(M\otimes' M)}) \circ \Delta_{F(M\otimes' M)} &= F(\varepsilon_M \otimes' \Delta_M) = \eta_{F(M\otimes' M)} \circ \varepsilon_{F(M\otimes' M)} \\
\mu_{F(M\otimes' M)} \circ (\mathsf{id}_{F(M\otimes' M)} \otimes F(c'_{M,M})) \circ \Delta_{F(M\otimes' M)} &= F(\Delta_M \otimes' \varepsilon_M) = \eta_{F(M\otimes' M)} \circ \varepsilon_{F(M\otimes' M)}
\end{aligned}
$$

proving the claim. □

7.4 The functor of coinvariants

For any Lie bialgebra $\mathfrak{b}$ consider the functor of coinvariants

$$
\begin{aligned}
F^{\mathfrak{b}} : \mathsf{DY}(\mathfrak{b}) &\to \mathsf{Vect}_{\mathbb{K}} \\
V &\mapsto \frac{V}{\mathfrak{b} \cdot V} \\
f : V \to W &\mapsto F^{\mathfrak{b}}(f) : F^{\mathfrak{b}}(V) \to F^{\mathfrak{b}}(W)
\end{aligned}
$$

where $F^{\mathfrak{b}}(f)$ is the morphism making commutative the following diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle p_V}\downarrow & & \downarrow{\scriptstyle p_W} \\ \frac{V}{\mathfrak{b}\cdot V} & \xrightarrow{F^{\mathfrak{b}}(f)} & \frac{W}{\mathfrak{b}\cdot W} \end{array}$$

where p_V and p_W are the canonical projections to the quotients.
Consider also the forgetful functor

$$\begin{aligned} \mathscr{O}^{\mathfrak{b}} : \mathsf{DY}(\mathfrak{b}) &\to \mathsf{Vect}_{\mathbb{K}} \\ V &\mapsto V \\ f &\mapsto f \end{aligned}$$

which has a canonical strongly comonoidal structure given by the identity maps $\psi^0_{\mathscr{O}^{\mathfrak{b}}} = \mathsf{id}_{\mathbb{K}}$ and $\psi^2_{\mathscr{O}^{\mathfrak{b}}}(V, W) = \mathsf{id}_{V\otimes W}$.

Proposition 7.4.1. *$F^{\mathfrak{b}}$ is an infinitesimally braided comonoidal functor.*

Proof. For any V, W in $\mathsf{DY}(\mathfrak{b})$, $v \in V$ and $w \in W$ consider the map

$$\begin{aligned} \psi : V \otimes W &\to \frac{V}{\mathfrak{b}\cdot V} \otimes \frac{W}{\mathfrak{b}\cdot W} \\ v \otimes w &\mapsto p_V(v) \otimes p_W(w). \end{aligned}$$

Recall that for any $x \in \mathfrak{b}$ we have $x\cdot(v\otimes w) = x\cdot v\otimes w + v\otimes x\cdot w$. This implies that $x\cdot(v\otimes w) \in \ker\psi$, and therefore ψ induces a map

$$\begin{aligned} \psi^2_{F^{\mathfrak{b}}}(V, W) : \frac{V\otimes W}{\mathfrak{b}\cdot(V\otimes W)} &\to \frac{V}{\mathfrak{b}\cdot V} \otimes \frac{W}{\mathfrak{b}\cdot W} \\ [v \otimes w] &\mapsto p_V(v) \otimes p_W(w) = [v] \otimes [w]. \end{aligned}$$

where $[v\otimes w]$ (resp. $[v]$ resp. $[w]$) denotes the equivalence class of $v\otimes w$ (resp. v resp. w) inside the quotient $\frac{V\otimes W}{\mathfrak{b}\cdot(V\otimes W)}$ (resp. $\frac{V}{\mathfrak{b}\cdot V}$ resp. $\frac{W}{\mathfrak{b}\cdot W}$). We have that $\psi^2_{F^{\mathfrak{b}}}$ is natural: for any $f \in \mathsf{Hom}_{\mathsf{DY}(\mathfrak{b})}(V, V')$ and $g \in \mathsf{Hom}_{\mathsf{DY}(\mathfrak{b})}(W, W')$ we have

$$\begin{aligned} \Big(\big(F^{\mathfrak{b}}(f) \otimes F^{\mathfrak{b}}(g)\big) \circ \big(\psi^2_{F^{\mathfrak{b}}}(V, W)\big)\Big)([v\otimes w]) &= \big(F^{\mathfrak{b}}(f) \otimes F^{\mathfrak{b}}(g)\big)(p_V(v) \otimes p_W(w)) \\ &= F^{\mathfrak{b}}(f)(p_V(v)) \otimes F^{\mathfrak{b}}(g)(p_W(w)) \\ &= p_{V'}(f(v)) \otimes p_{W'}(g(w)) \\ &= (p_{V'} \otimes p_{W'})(f\otimes g)(v\otimes w) \\ &= \Big(\big(\psi^2_{F^{\mathfrak{b}}}(V', W')\big) \circ \big(F^{\mathfrak{b}}(f\otimes g)\big)\Big)([v\otimes w]). \end{aligned}$$

Next, consider the following linear map

$$\begin{aligned} \psi^0_{F^{\mathfrak{b}}} : \frac{\mathbb{K}}{\mathfrak{b}\cdot\mathbb{K}} &\to \mathbb{K} \\ [\lambda] &\mapsto \lambda. \end{aligned}$$

We show that $(F^{\mathfrak{b}}, \psi^0_{F^{\mathfrak{b}}}, \psi^2_{F^{\mathfrak{b}}})$ is a braided comonoidal functor: for any U, V, W in $\mathsf{Obj}(\mathsf{DY}(\mathfrak{b}))$ and for any $u \in U$, $v \in V$, $w \in W$ we have

$$\begin{aligned}
&\big((\mathsf{id}_{F^{\mathfrak{b}}(U)} \otimes \psi^2_{F^{\mathfrak{b}}}(V,W)) \circ (\psi^2_{F^{\mathfrak{b}}}(U, V\otimes W)) \circ F^{\mathfrak{b}}(a_{U,V,W})\big)([(u\otimes v)\otimes w]) \\
&= \big((\mathsf{id}_{F^{\mathfrak{b}}(U)} \otimes \psi^2_{F^{\mathfrak{b}}}(V,W)) \circ (\psi^2_{F^{\mathfrak{b}}}(U, V\otimes W))\big)([u\otimes (v\otimes w)]) \\
&= (\mathsf{id}_{F^{\mathfrak{b}}(U)} \otimes \psi^2_{F^{\mathfrak{b}}}(V,W))([u]\otimes [v\otimes w]) \\
&= [u]\otimes ([v]\otimes [w])
\end{aligned}$$

and

$$\begin{aligned}
&\big(a_{F^{\mathfrak{b}}(U),F^{\mathfrak{b}}(V),F^{\mathfrak{b}}(W)} \circ \psi^2_{F^{\mathfrak{b}}}(U,V) \otimes \mathsf{id}_{F^{\mathfrak{b}}(W)} \circ \psi_{F^{\mathfrak{b}}}(U\otimes V, W)\big)([(u\otimes v)\otimes w]) \\
&= \big(a_{F^{\mathfrak{b}}(U),F^{\mathfrak{b}}(V),F^{\mathfrak{b}}(W)} \circ \psi^2_{F^{\mathfrak{b}}}(U,V) \otimes \mathsf{id}_{F^{\mathfrak{b}}(W)}\big)([u\otimes v]\otimes [w]) \\
&= a_{F^{\mathfrak{b}}(U),F^{\mathfrak{b}}(V),F^{\mathfrak{b}}(W)}(([u]\otimes [v])\otimes [w]) \\
&= [u]\otimes ([v]\otimes [w]),
\end{aligned}$$

showing that $(F^{\mathfrak{b}}, \psi^0_{F^{\mathfrak{b}}}, \psi^2_{F^{\mathfrak{b}}})$ satisfies Equation (2.1.5). Furthermore, we have

$$\begin{aligned}
\big((\psi^0_{F^{\mathfrak{b}}} \otimes \mathsf{id}_{F^{\mathfrak{b}}(U)}) \circ \psi^2_{F^{\mathfrak{b}}}(\mathbb{K}, U) \circ F^{\mathfrak{b}}(\ell_U^{-1})\big)([u]) &= \big((\psi^0_{F^{\mathfrak{b}}} \otimes \mathsf{id}_{F^{\mathfrak{b}}(U)}) \circ \psi^2_{F^{\mathfrak{b}}}(\mathbb{K}, U)\big)([1\otimes u]) \\
&= (\psi^0_{F^{\mathfrak{b}}} \otimes \mathsf{id}_{F^{\mathfrak{b}}(U)})([1]\otimes [u]) \\
&= 1\otimes [u] \\
&= \ell^{-1}_{F^{\mathfrak{b}}(U)}([u])
\end{aligned}$$

and

$$\begin{aligned}
\big((\mathsf{id}_{F^{\mathfrak{b}}(U)} \otimes \psi^0_{F^{\mathfrak{b}}}) \circ \psi^2_{F^{\mathfrak{b}}}(U, \mathbb{K}) \circ F^{\mathfrak{b}}(r_U^{-1})\big)([u]) &= \big((\mathsf{id}_{F^{\mathfrak{b}}(U)} \otimes \psi^0_{F^{\mathfrak{b}}}) \circ \psi^2_{F^{\mathfrak{b}}}(U, \mathbb{K})\big)([u\otimes 1]) \\
&= (\mathsf{id}_{F^{\mathfrak{b}}(U)} \otimes \psi^0_{F^{\mathfrak{b}}})([u]\otimes [1]) \\
&= [u]\otimes 1 \\
&= r^{-1}_{F^{\mathfrak{b}}(U)}([u])
\end{aligned}$$

showing that $(F^{\mathfrak{b}}, \psi^0_{F^{\mathfrak{b}}}, \psi^2_{F^{\mathfrak{b}}})$ satisfies Equation (2.1.6). The fact that $(F^{\mathfrak{b}}, \psi^0_{F^{\mathfrak{b}}}, \psi^2_{F^{\mathfrak{b}}})$ is braided comonoidal follows from the following equality

$$\begin{aligned}
\big(\psi^2_{F^{\mathfrak{b}}}(W,V) \circ F^{\mathfrak{b}}(\tau_{V,W})\big)([v\otimes w]) &= \psi^2_{F^{\mathfrak{b}}}(W,V)([w\otimes v]) \\
&= [w]\otimes [v] \\
&= \psi^2_{F^{\mathfrak{b}}}([w\otimes v]) \\
&= \big(\psi^2_{F^{\mathfrak{b}}} \circ \tau_{F^{\mathfrak{b}}(V),F^{\mathfrak{b}}(W)}\big)([v\otimes w])
\end{aligned}$$

which shows that $(F^{\mathfrak{b}}, \psi^0_{F^{\mathfrak{b}}}, \psi^2_{F^{\mathfrak{b}}})$ satisfies Equation (2.2.4). In order to show that $(F^{\mathfrak{b}}, \psi^0_{F^{\mathfrak{b}}}, \psi^2_{F^{\mathfrak{b}}})$ is infinitesimally braided comonoidal, note that for any V in $\mathsf{DY}(\mathfrak{b})$, $x \in \mathfrak{b}$, $v \in V$ one has

$$p_V \circ \pi_V(x\otimes v) = 0. \tag{7.4.1}$$

Denoting by t^0 the trivial infinitesimal braiding of $\mathsf{Vect}_{\mathbb{K}}$ (see §2.8), we have

$$t^0_{F^{\mathfrak{b}}(V),F^{\mathfrak{b}}(W)} \circ \psi^2_{F^{\mathfrak{b}}}(V,W) = 0$$

and

$$
\begin{aligned}
\big(\psi^2_{F^\mathfrak{b}}(V,W)\circ F^\mathfrak{b}(t^\mathfrak{b}_{V,W})\big)([v\otimes w]) &= \psi^2_{F^\mathfrak{b}}(V,W)([t^\mathfrak{b}_{V,W}(v\otimes w)])\\
&= \psi^2_{F^\mathfrak{b}}(V,W)\big(\big[\big((\mathsf{id}_V\otimes\pi_W)(\tau_{\mathfrak{b},V}\otimes\mathsf{id}_W)(\tilde\rho_V\otimes\mathsf{id}_W)\big)(v\otimes w)\big]\big)\\
&+ \psi^2_{F^\mathfrak{b}}(V,W)\big(\big[\big((\pi_V\otimes\mathsf{id}_W)(\tau_{V,\mathfrak{b}}\otimes\mathsf{id}_W)(\mathsf{id}_V\otimes\tilde\rho_W)\big)(v\otimes w)\big]\big)\\
&= \big((p_V\otimes p_W)(\mathsf{id}_V\otimes\pi_W)(\tau_{\mathfrak{b},V}\otimes\mathsf{id}_W)(\tilde\rho_V\otimes\mathsf{id}_W)\big)(v\otimes w)\\
&+ \big((p_V\otimes p_W)(\pi_V\otimes\mathsf{id}_W)(\tau_{V,\mathfrak{b}}\otimes\mathsf{id}_W)(\mathsf{id}_V\otimes\tilde\rho_W)\big)(v\otimes w)\\
&= 0
\end{aligned}
$$

where the last equality follows from the fact that compositions $p_V\circ\pi_V$ and $p_W\circ\pi_W$ are the zero maps (see Equation (7.4.1)). Hence both compositions $t^0_{F^\mathfrak{b}(V),F^\mathfrak{b}(W)}\circ\psi^2_{F^\mathfrak{b}}(V,W)$ and $\psi^2_{F^\mathfrak{b}}(V,W)\circ F^\mathfrak{b}(t^\mathfrak{b}_{V,W})$ are the zero map, showing that $(F^\mathfrak{b},\psi^0_{F^\mathfrak{b}},\psi^2_{F^\mathfrak{b}})$ satisfies Equation (2.3.3). □

We shall need the following

Lemma 7.4.2. *Let $\mathfrak{b}$ be a Lie bialgebra, V in $\mathsf{DY}(\mathfrak{b})$, $u\in\mathsf{U}(\mathfrak{b})$ and $v\in V$. Then the following identity holds in $F^\mathfrak{b}(\mathsf{U}(\mathfrak{b})\otimes V)$:*

$$[u\otimes v]=[1\otimes(S_0(u)\cdot v)]. \tag{7.4.2}$$

Proof. The proof is by induction on the lenght $n(u)$ of u induced by the standard filtration of $\mathsf{U}(\mathfrak{b})$. If $n(u)=0$ we have $u=1$ and there is nothing to prove. For $n(u)=m+1$ we have

$$
\begin{aligned}
[1\otimes S_0(u_1\cdots u_{m+1})\cdot v] &= [1\otimes S_0(u_2\cdots u_{m+1})\cdot S_0(u_1)\cdot v]\\
&= [u_2\cdots u_{m+1}\otimes S_0(u_1)\cdot v]\\
&= -[u_2\cdots u_{m+1}\otimes u_1\cdot v]\\
&= -[u_2\cdots u_{m+1}\otimes u_1\cdot v]+[u\otimes v]-[u\otimes v]\\
&= [u\otimes v]-[u_1\cdot(u_2\cdots u_{m+1}\otimes v)]\\
&= [u\otimes v]
\end{aligned}
$$

where the first identity follows from the fact that S_0 is an anti–morphism of algebras, the second follows by induction, and the third follows from the fact that for any $x\in\mathfrak{b}$ we have $S_0(x)=-x$. □

Proposition 7.4.3. *$F^\mathfrak{b}$ is $\mathsf{U}(\mathfrak{b})$–adapted.*

Proof. By part *(ii)* of Proposition 7.1.2 it suffices to show that the functor

$$G^\mathfrak{b}:\mathsf{DY}(\mathfrak{b})\xrightarrow{\mathsf{U}(\mathfrak{b})\otimes}\mathsf{DY}(\mathfrak{b})\xrightarrow{F^\mathfrak{b}}\mathsf{Vect}_\mathbb{K}.$$

is strongly comonoidal. By part *(i)* of Proposition 2.1.3 we have that the maps

$$
\begin{aligned}
\psi^2_{G^\mathfrak{b}}(V,W) &= \psi^2_{F^\mathfrak{b}}\big(\mathsf{U}(\mathfrak{b})\otimes V,\mathsf{U}(\mathfrak{b})\otimes W\big)\circ F^\mathfrak{b}(\beta_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),V,W})\circ F^\mathfrak{b}(\Delta_0\otimes\mathsf{id}_{V\otimes W})\\
\psi^0_{G^\mathfrak{b}} &= \psi^0_{F^\mathfrak{b}}\circ F^\mathfrak{b}(\ell_\mathbb{K})\circ F^\mathfrak{b}(\varepsilon_0\otimes\mathsf{id}_\mathbb{K})
\end{aligned}
$$

gives to $G^\mathfrak{b}$ a comonoidal structure. Consider the forgetful functor $\mathscr{O}^\mathfrak{b}:\mathsf{DY}(\mathfrak{b})\to\mathsf{Vect}_\mathbb{K}$, with its strongly comonoidal structure given by $\psi^0_{\mathscr{O}^\mathfrak{b}}=\mathsf{id}$ and $\psi^2_{\mathscr{O}^\mathfrak{b}}=\mathsf{id}$. If we show that the funtors $G^\mathfrak{b}$ and

$\mathscr{O}^{\mathfrak{b}}$ are naturally comonoidally isomorphic then the claim is proved. For any V in $\mathsf{DY}(\mathfrak{b})$ consider the map

$$\begin{aligned}\tilde{\zeta}_V : \mathsf{U}(\mathfrak{b}) \otimes V &\to V\\ u \otimes v &\mapsto S_0(u) \cdot v.\end{aligned}$$

For any $x \in \mathfrak{b}$, $u \in \mathsf{U}(\mathfrak{b})$ and $v \in V$ we have $(p_V \circ \tilde{\zeta}_V)(x \cdot u \otimes v) = 0 = (p_V \circ \tilde{\zeta})(u \otimes x \cdot v)$. This implies that $x \cdot (u \otimes v) \in \ker \tilde{\zeta}_V$, and therefore $\tilde{\zeta}_V$ induces a map

$$\begin{aligned}\zeta_V : \frac{\mathsf{U}(\mathfrak{b}) \otimes V}{\mathfrak{b} \cdot (\mathsf{U}(\mathfrak{b}) \otimes V)} &\to V\\ [u \otimes v] &\mapsto S_0(u) \cdot v\end{aligned} \tag{7.4.3}$$

We have that the map

$$\begin{aligned}\theta_V : V &\to \frac{\mathsf{U}(\mathfrak{b}) \otimes V}{\mathfrak{b} \cdot (\mathsf{U}(\mathfrak{b}) \otimes V)}\\ v &\mapsto [1 \otimes v]\end{aligned} \tag{7.4.4}$$

is the inverse of ζ_V, since $\zeta_V(\theta_V(V)) = \zeta_V([1 \otimes v]) = S_0(1) \cdot v = v$ and

$$\theta_V(\zeta_V([u \otimes v])) = \theta_V(S_0(u) \cdot v) = [1 \otimes S_0(u) \cdot v] \overset{(7.4.2)}{=} [u \otimes v].$$

Therefore, $\zeta : G^{\mathfrak{b}} \to \mathscr{O}^{\mathfrak{b}}$ is a natural isomorphism. We have

$$\begin{aligned}
&\Big((\zeta_V \otimes \zeta_W) \circ (\psi^2_{G_{\mathfrak{b}}})\Big)([u \otimes (v \otimes w)])\\
&= \Big((\zeta_V \otimes \zeta_W) \circ \psi^2_{F^{\mathfrak{b}}}(\mathsf{U}(\mathfrak{b}) \otimes V, \mathsf{U}(\mathfrak{b}) \otimes W) \circ F^{\mathfrak{b}}(\beta_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),V,W}) \circ F^{\mathfrak{b}}(\Delta_0 \otimes \mathsf{id}_{V \otimes W})\Big)([u \otimes (v \otimes w)])\\
&= \Big((\zeta_V \otimes \zeta_W) \circ \psi^2_{F^{\mathfrak{b}}}(\mathsf{U}(\mathfrak{b}) \otimes V, \mathsf{U}(\mathfrak{b}) \otimes W) \circ F^{\mathfrak{b}}(\beta_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),V,W}) \circ F^{\mathfrak{b}}(\Delta_0 \otimes \mathsf{id}_{V \otimes W})\Big)([1 \otimes (S_0(u) \cdot (v \otimes w))])\\
&= \left((\zeta_V \otimes \zeta_W) \circ \psi^2_{F^{\mathfrak{b}}}(\mathsf{U}(\mathfrak{b}) \otimes V, \mathsf{U}(\mathfrak{b}) \otimes W) \circ F^{\mathfrak{b}}(\beta_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),V,W}) \circ F^{\mathfrak{b}}(\Delta_0 \otimes \mathsf{id}_{V \otimes W})\right)\left(\sum_{(S_0(u))} [1 \otimes (S_0(u))' \cdot v \otimes (S_0(u))'' \cdot w]\right)\\
&= \sum_{S_0(u)} \left((\zeta_V \otimes \zeta_W) \circ \psi^2_{F^{\mathfrak{b}}}(\mathsf{U}(\mathfrak{b}) \otimes V, \mathsf{U}(\mathfrak{b}) \otimes W) \circ F^{\mathfrak{b}}(\beta_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),V,W})\right)\left([1 \otimes 1 \otimes (S_0(u))' \cdot v \otimes (S_0(u))'' \cdot w]\right)\\
&= \sum_{(S_0(u))} \left((\zeta_V \otimes \zeta_W) \circ \psi^2_{F^{\mathfrak{b}}}(\mathsf{U}(\mathfrak{b}) \otimes V, \mathsf{U}(\mathfrak{b}) \otimes W)\right)\left([1 \otimes (S_0(u))' \cdot v \otimes 1 \otimes (S_0(u))'' \cdot w]\right)\\
&= \sum_{(S_0(u))} (\zeta_V \otimes \zeta_W)([1 \otimes (S_0(u))' \cdot v] \otimes [1 \otimes (S_0(u))'' \cdot w])\\
&= \sum_{(S_0(u))} \big(S_0(1) \cdot (S_0(u))' \cdot v \otimes S_0(1) \cdot (S_0(u))'' \cdot w\big)\\
&= \sum_{(S_0(u))} \big((S_0(u))' \cdot v \otimes (S_0(u))'' \cdot w\big)\\
&= S_0(u) \cdot (v \otimes w)\\
&= (\zeta_{V \otimes W})(\psi_2^{\mathscr{O}_{\mathfrak{b}}})([u \otimes (v \otimes w)])
\end{aligned}$$

and

$$
\begin{aligned}
\zeta_{\mathbb{K}}([u \otimes \lambda]) &= S_0(u) \cdot \lambda \\
&= \psi^0_{F^\flat}([S_0(u) \cdot \lambda]) \\
&= \psi^0_{F^\flat} \circ F^\flat(\ell_{\mathbb{K}})([1 \otimes S_0(u) \cdot \lambda]) \\
&= \psi^0_{F^\flat} \circ F^\flat(\ell_{\mathbb{K}}) \circ F^\flat(\varepsilon_0 \otimes \mathsf{id}_{\mathbb{K}})([1 \otimes S_0(u) \cdot \lambda]) \\
&= \psi^0_{F^\flat} \circ F^\flat(\ell_{\mathbb{K}}) \circ F^\flat(\varepsilon_0 \otimes \mathsf{id}_{\mathbb{K}})([u \otimes \lambda]) \\
&= \psi^0_{G^\flat}([u \otimes \lambda]).
\end{aligned}
$$

Therefore , $\zeta : G^\flat \to \mathscr{O}^\flat$ is a natural comonoidal isomorphism, concluding the proof. □

7.5 Deforming M–adapted functors

Recall from §2.7 that if $\mathscr{C}$ is an infinitesimally braided monoidal category, $\hbar$ is a formal parameter and Φ is a Drinfeld associator, we can construct a deformed braided monoidal category $\mathscr{C}^\Phi_\hbar$. In this Section we show that any infinitesimally braided comonoidal functor induces a braided comonoidal functor between the deformed categories.

Proposition 7.5.1. [Šev16, Prop.2] *Let $\mathscr{C}$,$\mathscr{D}$ be two infinitesimally braided monoidal categories, $\hbar$ be a formal parameter and Φ be a Drinfeld associator. Then*

(i) If $(M, \Delta_M, \varepsilon_M)$ is an infinitesimally cocommutative comonoid in $\mathscr{D}$, then $(M, \overline{\Delta}_M, \overline{\varepsilon}_M)$ is a cocommutative comonoid in $\mathscr{D}^\Phi_\hbar$.

(ii) If $F : \mathscr{D} \to \mathscr{C}$ is an infinitesimally braided comonoidal functor, then $F^\Phi_\hbar : \mathscr{D}^\Phi_\hbar \to \mathscr{C}^\Phi_\hbar$ is a braided comonoidal functor.

(iii) If $F : \mathscr{D} \to \mathscr{C}$ is M–adapted, then $F^\Phi_\hbar : \mathscr{D}^\Phi_\hbar \to \mathscr{C}^\Phi_\hbar$ is M–adapted

Proof. (i): the fact that the triple $(M, \overline{\Delta}_M, \overline{\varepsilon}_M)$ is a comonoid is straightforward. We have

$$
\begin{aligned}
\overline{c}^\Phi_{M,M} \circ \overline{\Delta}_M &= \overline{c}_{M,M} \circ e^{\overline{t}_{M,M}/2} \circ \overline{\Delta}_M \\
&= \overline{c}_{M,M} \circ \overline{\Delta}_M \circ e^{\overline{t}_{M,M}/2} \\
&= \overline{c}_{M,M} \circ e^{\overline{t}_{M,M}/2} \\
&= \overline{c}^\Phi_{M,M}.
\end{aligned}
$$

Where the second (resp. third) equality follows from the fact that M is infinitesimally cocommutative (resp. cocommutative) in $\mathscr{D}$.
(ii): The fact that $(F^\Phi_\hbar, \overline{\psi}^0_F, \overline{\psi}^2_F)$ satisfies (2.1.6) follows directly by the fact that (F, ψ^0_F, ψ^2_F) is comonoidal. Next, the fact that $(F^\Phi_\hbar, \overline{\psi}^0_F, \overline{\psi}^2_F)$ satisfies (2.1.5) follows from the commutativity of the diagrams

$$
\begin{array}{ccccc}
F((X\bar{\otimes}Y)\bar{\otimes}Z) & \xrightarrow{F(\overline{t}^{12}_{X,Y,Z})} & F((X\bar{\otimes}Y)\bar{\otimes}Z) & \xrightarrow{F(\overline{a}_{X,Y,Z})} & F(X\bar{\otimes}(Y\bar{\otimes}Z)) \\
{\scriptstyle \overline{\psi}^2_F(X\bar{\otimes}Y,Z)}\downarrow & & \downarrow{\scriptstyle \overline{\psi}^2_F(X\bar{\otimes}Y,Z)} & & \downarrow{\scriptstyle \overline{\psi}^2_F X,Y\bar{\otimes}Z)} \\
F(X\bar{\otimes}Y)\bar{\otimes}F(Z) & \xrightarrow{F(\overline{t}_{X,Y})\bar{\otimes}\mathsf{id}_{F(Z)}} & F(X\bar{\otimes}Y)\bar{\otimes}F(Z) & & F(X)\bar{\otimes}F(Y\bar{\otimes}Z) \\
{\scriptstyle \overline{\psi}^2_F(X,Y)\bar{\otimes}\mathsf{id}_{F(Z)}}\downarrow & & \downarrow{\scriptstyle \overline{\psi}^2_F(X,Y)\bar{\otimes}\mathsf{id}_{F(Z)}} & & \downarrow{\scriptstyle \mathsf{id}_{F(X)}\bar{\otimes}\overline{\psi}^2_F(Y,Z)} \\
F(X)\bar{\otimes}F(Y)\bar{\otimes}F(Z) & \xrightarrow{\overline{t}^{12}_{F(X),F(Y),F(Z)}} & (F(X)\bar{\otimes}F(Y))\bar{\otimes}F(Z) & \xrightarrow{\overline{a}_{F(X),F(Y),F(Z)}} & F(X)\bar{\otimes}(F(Y)\bar{\otimes}F(Z))
\end{array}
$$

and

$$
\begin{array}{ccccc}
F((X\bar{\otimes}Y)\bar{\otimes}Z) & \xrightarrow{F(\bar{a}_{X,Y,Z})} & F(X\bar{\otimes}(Y\bar{\otimes}Z)) & \xrightarrow{F(\bar{t}^{23}_{X,Y,Z})} & F(X\bar{\otimes}(Y\bar{\otimes}Z)) \\
{\scriptstyle \overline{\psi}^2_F(X\bar{\otimes}Y,Z)}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\psi}^2_F(X,Y\bar{\otimes}Z)} & & \big\downarrow{\scriptstyle \overline{\psi}^2_F(X,Y\bar{\otimes}Z)} \\
F(X\bar{\otimes}Y)\bar{\otimes}F(Z) & & F(X)\bar{\otimes}F(Y\bar{\otimes}Z) & \xrightarrow{\mathsf{id}_{F(X)}\bar{\otimes}F(\bar{t}_{Y,Z})} & F(X)\bar{\otimes}F(Y\bar{\otimes}Z) \\
{\scriptstyle \overline{\psi}^2_F(X,Y)\bar{\otimes}\mathsf{id}_{F(X)}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{id}_{F(X)}\bar{\otimes}\overline{\psi}^2_F(Y,Z)} & & \big\downarrow{\scriptstyle \mathsf{id}_{F(X)}\bar{\otimes}\overline{\psi}^2_F(Y,Z)} \\
(F(X)\bar{\otimes}F(Y))\bar{\otimes}F(Z) & \xrightarrow{\bar{a}_{F(X),F(Y),F(Z)}} & F(X)\bar{\otimes}(F(Y)\bar{\otimes}F(Z)) & \xrightarrow{\bar{t}^{23}_{F(X),F(Y),F(Z)}} & F(X)\bar{\otimes}(F(Y)\bar{\otimes}F(Z))
\end{array}
$$

both following from the naturality of ψ^2_F and from the fact that (F, ψ^0_F, ψ^2_F) is comonoidal. Finally, the fact that $(F^\Phi_\hbar, \overline{\psi}^0_F, \overline{\psi}^2_F)$ is braided comonoidal follows from the following diagram

$$
\begin{array}{ccc}
F(X\bar{\otimes}Y) & \xrightarrow{\overline{\psi}^2_F(X,Y)} & F(X)\bar{\otimes}F(Y) \\
{\scriptstyle F(e^{\bar{t}_{X,Y}/2})}\big\downarrow & & \big\downarrow{\scriptstyle e^{\bar{t}_{F(X),F(Y)}/2}} \\
F(X\bar{\otimes}Y) & \xrightarrow{\overline{\psi}^2_F(Y,X)} & F(X)\bar{\otimes}F(Y) \\
{\scriptstyle F(\bar{c}_{X,Y})}\big\downarrow & & \big\downarrow{\scriptstyle \bar{c}_{F(X),F(Y)}} \\
F(Y\bar{\otimes}X) & \xrightarrow{\overline{\psi}^2_F(Y,X)} & F(Y)\bar{\otimes}F(X)
\end{array}
$$

where the first (resp. second) square commutes since (F, ψ^0_F, ψ^2_F) is infinitesimally braided (resp. braided) comonoidal.
(iii) is straightforward. □

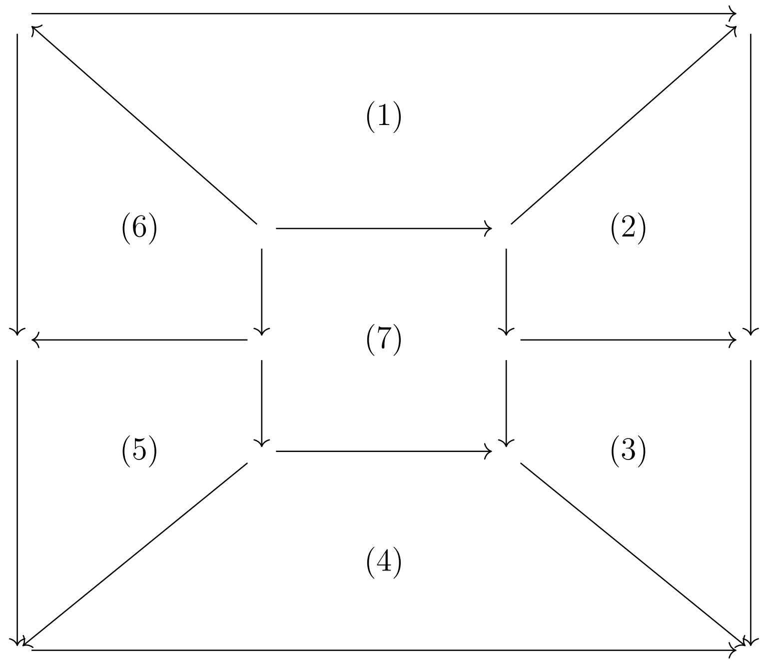
Next, let $\mathfrak{b}$ be a Lie bialgebra, $F^\mathfrak{b}$ be the functor of coinvariants, $G^\mathfrak{b} := F^\mathfrak{b} \circ \mathsf{U}(\mathfrak{b})\otimes -$, $\hbar$ be a formal parameter and Φ be a Drinfeld associator. Using Propositions 2.7.2 and 7.5.1 we get *deformed* braided comonoidal functors $(F^\mathfrak{b})^\Phi_\hbar$ and $(G^\mathfrak{b})^\Phi_\hbar$. Recall also that $G^\mathfrak{b}$ is naturally comonoidally isomorphic to the forgetful functor $\mathscr{O}^\mathfrak{b}$ through the natural isomorphism ζ of Equation (7.4.3), see Proposition 7.4.3 for more details. The corresponding natural transformation $\overline{\zeta} : (G^\mathfrak{b})^\Phi_\hbar \to (\mathscr{O}^\mathfrak{b})^\Phi_\hbar$ is also a natural isomorphism, since concides with ζ on objects and is the $\hbar$–adic completion on morphisms (hence invertible, since their first order term is invertible).

Lemma 7.5.2. *Let $\mathscr{O}^\mathfrak{b}$ the forgetful functor, and consider the following maps for any V, W in $\mathsf{Obj}\big((\mathsf{DY}(\mathfrak{b}))^\Phi_\hbar\big)$:*

$$
\begin{aligned}
\psi^2_{(\mathscr{O}^\mathfrak{b})^\Phi_\hbar}(V,W) &:= (\overline{\zeta}_V \bar{\otimes} \overline{\zeta}_W) \circ \psi^2_{(G^\mathfrak{b})^\Phi_\hbar}(V,W) \circ (\overline{\zeta}_{V\bar{\otimes}W})^{-1} \\
\psi^0_{(\mathscr{O}^\mathfrak{b})^\Phi_\hbar} &:= \psi^0_{(G^\mathfrak{b})^\Phi_\hbar} \circ (\overline{\zeta}_{\mathbb{K}[[\hbar]]})^{-1}
\end{aligned}
$$

Then the triple $\big((\mathscr{O}^\mathfrak{b})^\Phi_\hbar, \psi^0_{(\mathscr{O}^\mathfrak{b})^\Phi_\hbar}, \psi^2_{(\mathscr{O}^\mathfrak{b})^\Phi_\hbar}\big)$ is a strongly comonoidal functor between the deformed categories $\big(\mathsf{DY}(\mathfrak{b})\big)^\Phi_\hbar$ and $(\mathsf{Vect}_\mathbb{K})^\Phi_\hbar$.

Proof. It is clear that the morphisms $\psi^0_{(\mathscr{O}^\mathfrak{b})^\Phi_\hbar}$ and $\psi^2_{(\mathscr{O}^\mathfrak{b})^\Phi_\hbar}(V,W)$ are invertible, since are composition of invertible maps. The fact that the triple $\big((\mathscr{O}^\mathfrak{b})^\Phi_\hbar, \psi^0_{(\mathscr{O}^\mathfrak{b})^\Phi_\hbar}, \psi^2_{(\mathscr{O}^\mathfrak{b})^\Phi_\hbar}\big)$ satisfies the commutativity of the diagram (2.1.5) follows by – using the naturality of $\overline{\zeta}$, the definition of $\psi^2_{(\mathscr{O}^\mathfrak{b})^\Phi_\hbar}(V,W)$, the naturality of the associativity constraint and the fact that $\big((G^{\mathfrak{b}})^{\Phi}_{\hbar}, \psi^0_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}}, \psi^2_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}}\big)$ is comonoidal – the following diagram



whose seven subdiagrams are the following

(1)

$$\begin{array}{ccc}
(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}\big((U\bar{\otimes}V)\bar{\otimes}W\big) & \xrightarrow{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(a_{U,V,W})} & (\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}\big(U\bar{\otimes}(V\bar{\otimes}W)\big) \\
\uparrow{\scriptstyle \overline{\zeta}_{(U\bar{\otimes}V)\bar{\otimes}W}} & & \uparrow{\scriptstyle \overline{\zeta}_{U\bar{\otimes}(V\bar{\otimes}W)}} \\
(G^{\mathfrak{b}})^{\Phi}_{\hbar}\big((U\bar{\otimes}V)\bar{\otimes}W\big) & \xrightarrow{(G^{\mathfrak{b}})^{\Phi}_{\hbar}(a_{U,V,W})} & (G^{\mathfrak{b}})^{\Phi}_{\hbar}\big(U\bar{\otimes}(V\bar{\otimes}W)\big)
\end{array}$$

(2)

$$\begin{array}{ccc}
(G^{\mathfrak{b}})^{\Phi}_{\hbar}\big(U\bar{\otimes}(V\bar{\otimes}W)\big) & \xrightarrow{\overline{\zeta}_{U\bar{\otimes}(V\bar{\otimes}W)}} & (\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}\big(U\bar{\otimes}(V\bar{\otimes}W)\big) \\
\downarrow{\scriptstyle \psi^2_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}}(U,V\bar{\otimes}W)} & & \downarrow{\scriptstyle \psi^2_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(U,V\bar{\otimes}W)} \\
(G^{\mathfrak{b}})^{\Phi}_{\hbar}(U)\bar{\otimes}(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V\bar{\otimes}W) & \xrightarrow{\overline{\zeta}_U\bar{\otimes}\overline{\zeta}_{V\bar{\otimes}W}} & (\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(U)\bar{\otimes}(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V\bar{\otimes}W)
\end{array}$$

(3)

$$\begin{array}{ccc}
(G^{\mathfrak{b}})^{\Phi}_{\hbar}(U)\bar{\otimes}(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V\bar{\otimes}W) & \xrightarrow{\overline{\zeta}_U\bar{\otimes}\overline{\zeta}_{V\bar{\otimes}W}} & (\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(U)\bar{\otimes}(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V\bar{\otimes}W) \\
\downarrow{\scriptstyle \mathsf{id}\bar{\otimes}\psi^2_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}}(V,W)} & & \downarrow{\scriptstyle \mathsf{id}\bar{\otimes}\psi^2_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(V,W)} \\
(G^{\mathfrak{b}})^{\Phi}_{\hbar}(U)\bar{\otimes}\big((G^{\mathfrak{b}})^{\Phi}_{\hbar}(V)\bar{\otimes}(G^{\mathfrak{b}})^{\Phi}_{\hbar}(W)\big) & \xrightarrow{\overline{\zeta}_U\bar{\otimes}(\overline{\zeta}_V\bar{\otimes}\overline{\zeta}_W)} & (\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(U)\bar{\otimes}\big((\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V)\bar{\otimes}(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(W)\big)
\end{array}$$

(4)

$$\begin{array}{ccc}
\big((G^{\mathfrak{b}})^{\Phi}_{\hbar}(U)\bar{\otimes}(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V)\big)\bar{\otimes}(G^{\mathfrak{b}})^{\Phi}_{\hbar}(W) & \xrightarrow{\overline{a}^{\Phi}_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}(U),(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V),(G^{\mathfrak{b}})^{\Phi}_{\hbar}(W)}} & (G^{\mathfrak{b}})^{\Phi}_{\hbar}(U)\bar{\otimes}\big((G^{\mathfrak{b}})^{\Phi}_{\hbar}(V)\bar{\otimes}(G^{\mathfrak{b}})^{\Phi}_{\hbar}(W)\big) \\
\downarrow{\scriptstyle (\overline{\zeta}_U\bar{\otimes}\overline{\zeta}_V)\bar{\otimes}\overline{\zeta}_W} & & \downarrow{\scriptstyle \overline{\zeta}_U\bar{\otimes}(\overline{\zeta}_V\bar{\otimes}\overline{\zeta}_W)} \\
\big((\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(U)\bar{\otimes}(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V)\big)\bar{\otimes}(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(W) & \xrightarrow{\overline{a}^{\Phi}_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(U),(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V),(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(W)}} & (\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(U)\bar{\otimes}\big((\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V)\bar{\otimes}(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(W)\big)
\end{array}$$

(5)

$$\begin{array}{ccc}
(\mathscr{O}^\flat)^\Phi_\hbar(U\bar\otimes V)\bar\otimes(\mathscr{O}^\flat)^\Phi_\hbar(W) & \xleftarrow{\;\bar\zeta_{U\bar\otimes V}\ \bar\otimes\bar\zeta_W\;} & (G^\flat)^\Phi_\hbar(U\bar\otimes V)\bar\otimes(G^\flat)^\Phi_\hbar(W) \\
{\scriptstyle \psi^2_{(\mathscr{O}^\flat)^\Phi_\hbar}(U,V)\bar\otimes\mathsf{id}}\downarrow & & \downarrow{\scriptstyle \psi^2_{(G^\flat)^\Phi_\hbar}(U,V)\bar\otimes\mathsf{id}} \\
\big((\mathscr{O}^\flat)^\Phi_\hbar(U)\bar\otimes(\mathscr{O}^\flat)^\Phi_\hbar(V)\big)\bar\otimes(\mathscr{O}^\flat)^\Phi_\hbar(W) & \xleftarrow{\;(\bar\zeta_U\bar\otimes\bar\zeta_V)\bar\otimes\bar\zeta_W\;} & \big((G^\flat)^\Phi_\hbar(U)\bar\otimes(G^\flat)^\Phi_\hbar(V)\big)\bar\otimes(G^\flat)^\Phi_\hbar(W)
\end{array}$$

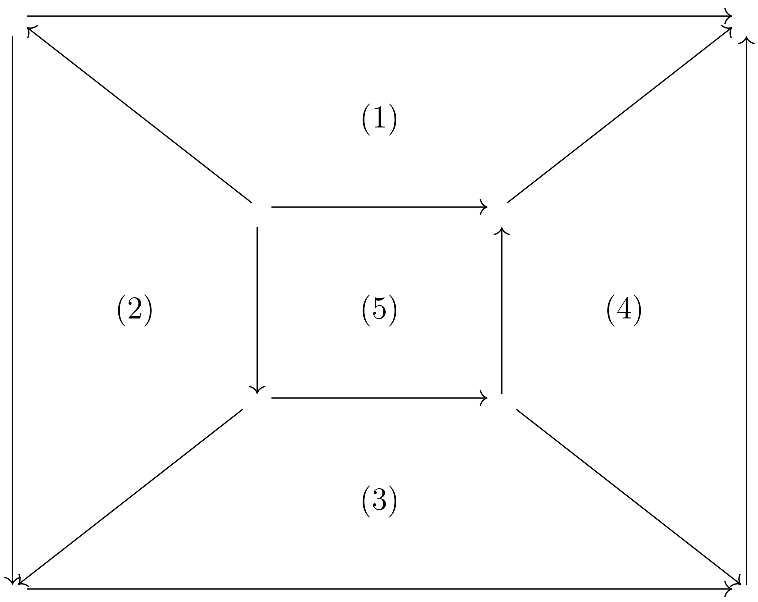
(6)

$$\begin{array}{ccc}
(\mathscr{O}^\flat)^\Phi_\hbar\big((U\bar\otimes V)\otimes W\big) & \xleftarrow{\;\bar\zeta_{(U\bar\otimes V)\bar\otimes W}\;} & (G^\flat)^\Phi_\hbar\big((U\otimes V)\bar\otimes W\big) \\
{\scriptstyle \psi^2_{(\mathscr{O}^\flat)^\Phi_\hbar}(U\bar\otimes V,W)}\downarrow & & \downarrow{\scriptstyle \psi^2_{(G^\flat)^\Phi_\hbar}(U\bar\otimes V,W)} \\
(\mathscr{O}^\flat)^\Phi_\hbar(U\bar\otimes V)\bar\otimes(\mathscr{O}^\flat)^\Phi_\hbar(W) & \xleftarrow{\;\bar\zeta_{U\bar\otimes V}\bar\otimes\bar\zeta_W\;} & (G^\flat)^\Phi_\hbar(U\bar\otimes V)\bar\otimes(G^\flat)^\Phi_\hbar(W)
\end{array}$$

(7)

$$\begin{array}{ccc}
(G^\flat)^\Phi_\hbar\big((U\bar\otimes V)\bar\otimes W\big) & \xrightarrow{\;(G^\flat)^\Phi_\hbar(a_{U,V,W})\;} & (G^\flat)^\Phi_\hbar\big(U\bar\otimes(V\bar\otimes W)\big) \\
{\scriptstyle \psi^2_{(G^\flat)^\Phi_\hbar}(U\bar\otimes V,W)}\downarrow & & \downarrow{\scriptstyle \psi^2_{(G^\flat)^\Phi_\hbar}(U,V\bar\otimes W)} \\
(G^\flat)^\Phi_\hbar(U\bar\otimes V)\bar\otimes(G^\flat)^\Phi_\hbar(W) & & (G^\flat)^\Phi_\hbar(U)\bar\otimes(G^\flat)^\Phi_\hbar(V\bar\otimes W) \\
{\scriptstyle \psi^2_{(G^\flat)^\Phi_\hbar}(U,V)\bar\otimes\mathsf{id}}\downarrow & & \downarrow{\scriptstyle \mathsf{id}\bar\otimes\psi^2_{(G^\flat)^\Phi_\hbar}(V,W)} \\
\big((G^\flat)^\Phi_\hbar(U)\bar\otimes(G^\flat)^\Phi_\hbar(V)\big)\bar\otimes(G^\flat)^\Phi_\hbar(W) & \xrightarrow{\;\overline{a}^\Phi_{(G^\flat)^\Phi_\hbar(U),(G^\flat)^\Phi_\hbar(V),(G^\flat)^\Phi_\hbar(W)}\;} & (G^\flat)^\Phi_\hbar(U)\bar\otimes\big((G^\flat)^\Phi_\hbar(V)\bar\otimes(G^\flat)^\Phi_\hbar(W)\big)
\end{array}$$

Finally, the fact that the triple $\big((\mathscr{O}^\flat)^\Phi_\hbar, \mathsf{id}_{\mathbb{K}[[\hbar]]}, \psi^2_{(\mathscr{O}^\flat)^\Phi_\hbar}\big)$ satisfies the commutativity of the first of the two diagrams (2.1.6) (the commutativity of the second can be shown in an analogous way) follows by – using the naturality of $\overline{\zeta}$ and of $\overline{\ell}$, the definitions of $\psi^2_{(\mathscr{O}^\flat)^\Phi_\hbar}(V,W)$ and $\psi^0_{(\mathscr{O}^\flat)^\Phi_\hbar}$, and the fact that $\big((G^\flat)^\Phi_\hbar, \psi^0_{(G^\flat)^\Phi_\hbar}, \psi^2_{(G^\flat)^\Phi_\hbar}\big)$ is comonoidal – the following diagram



whose five subdiagrams are the following

(1)

$$\begin{CD}
(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V) @>{(\bar{\ell}^{-1}_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V)}}>> \mathbb{K}[[\hbar]]\bar{\otimes}(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V)\\
@A{\overline{\zeta_V}}AA @AA{\bar{\zeta}_{\mathbb{K}[[\hbar]]\bar{\otimes}V}}A\\
(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V) @>{\bar{\ell}^{-1}_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V)}}>> \mathbb{K}[[\hbar]]\bar{\otimes}(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V)
\end{CD}$$

(2)

$$\begin{CD}
(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V) @<{\overline{\zeta_V}}<< (G^{\mathfrak{b}})^{\Phi}_{\hbar}(V)\\
@V{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(\bar{\ell}_V^{-1})}VV @VV{(G^{\mathfrak{b}})^{\Phi}_{\hbar}(\bar{\ell}_V^{-1})}V\\
(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathbb{K}[[\hbar]]\bar{\otimes}V) @<{\bar{\zeta}_{\mathbb{K}[[\hbar]]\bar{\otimes}V}}<< (G^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathbb{K}[[\hbar]]\bar{\otimes}V)
\end{CD}$$

(3)

$$\begin{CD}
(G^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathbb{K}[[\hbar]]\bar{\otimes}V) @>{\psi^2_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathbb{K}[[\hbar]],V)}>> (G^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathbb{K}[[\hbar]])\bar{\otimes}G^{\mathfrak{b}}(V)\\
@V{\bar{\zeta}_{\mathbb{K}[[\hbar]]\bar{\otimes}V}}VV @VV{\bar{\zeta}_{\mathbb{K}[[\hbar]]}\bar{\otimes}\bar{\zeta}_V}V\\
(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathbb{K}[[\hbar]]\bar{\otimes}V) @>{\psi^2_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathbb{K}[[\hbar]],V)}>> (\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathbb{K}[[\hbar]])\bar{\otimes}(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V)
\end{CD}$$

(4)

$$\begin{CD}
\mathbb{K}[[\hbar]]\bar{\otimes}(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V) @>{\mathsf{id}\bar{\otimes}\bar{\zeta}_V}>> \mathbb{K}[[\hbar]]\bar{\otimes}(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V)\\
@A{\psi^0_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}}\bar{\otimes}\mathsf{id}}AA @AA{\psi^0_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}\bar{\otimes}\mathsf{id}}A\\
(G^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathbb{K}[[\hbar]])\bar{\otimes}(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V) @>{\bar{\zeta}_{\mathbb{K}[[\hbar]]}\bar{\otimes}\bar{\zeta}_V}>> (\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathbb{K}[[\hbar]])\bar{\otimes}(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(V)
\end{CD}$$

(5)

$$\begin{CD}
(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V) @>{\bar{\ell}^{-1}_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V)}}>> \mathbb{K}[[\hbar]]\bar{\otimes}(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V)\\
@V{(G^{\mathfrak{b}})^{\Phi}_{\hbar}(\bar{\ell}_V^{-1})}VV @VV{\psi^0_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}}\bar{\otimes}\mathsf{id}}V\\
(G^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathbb{K}[[\hbar]]\bar{\otimes}V) @>{\psi^2_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathbb{K}[[\hbar]],V)}>> (G^{\mathfrak{b}})^{\Phi}_{\hbar}(K[[\hbar]])\bar{\otimes}(G^{\mathfrak{b}})^{\Phi}_{\hbar}(V)
\end{CD}$$

□

Remark 7.5.3. *Recalling Proposition 2.1.3, for any* $\lambda \in \mathbb{K}[[\hbar]]$ *we have*

$$\begin{aligned}
\psi^0_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(\lambda) &= \big(\psi^0_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}} \circ (\bar{\zeta}_{\mathbb{K}[[\hbar]]})^{-1}\big)(\lambda)\\
&= \Big(\psi^0_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}} \circ (F^{\mathfrak{b}})^{\Phi}_{\hbar}(\bar{\ell}_{\mathbb{K}[[\hbar]]}) \circ (F^{\mathfrak{b}})^{\Phi}_{\hbar}(\bar{\varepsilon}_M \bar{\otimes} \mathsf{id}_{\mathbb{K}[[\hbar]]}) \circ (\bar{\zeta}_{\mathbb{K}[[\hbar]]})^{-1}\Big)(\lambda)\\
&= \Big(\psi^0_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}} \circ (F^{\mathfrak{b}})^{\Phi}_{\hbar}(\bar{\ell}_{\mathbb{K}[[\hbar]]}) \circ (F^{\mathfrak{b}})^{\Phi}_{\hbar}(\bar{\varepsilon}_M \bar{\otimes} \mathsf{id}_{\mathbb{K}[[\hbar]]})\Big)([1 \otimes \lambda])\\
&= \Big(\psi^0_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}} \circ (F^{\mathfrak{b}})^{\Phi}_{\hbar}(\bar{\ell}_{\mathbb{K}[[\hbar]]})\Big)([\bar{\varepsilon}_M(1) \otimes \lambda])\\
&= \psi^0_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}([\lambda])\\
&= \lambda
\end{aligned}$$

i.e. $\psi^0_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}} = \mathsf{id}_{\mathbb{K}[[\hbar]]}$.

We shall need the following result in order to compute semiclassical limits:

Proposition 7.5.4. *For any* V, W *in* $\mathsf{Obj}((\mathsf{DY}(\mathfrak{b}))^{\Phi}_{\hbar})$*,* $v \in V$ *and* $w \in W$ *we have*

$$\psi^2_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(V,W)(v\bar{\otimes}w) \mod \hbar^2 = \frac{1}{2}\Big(v\otimes w - \hbar\sum_{[v]} v^{[1]}\otimes v^{[0]}\cdot w\Big). \tag{7.5.1}$$

Proof. We first compute the deformed *middle four interchange map* $\overline{\beta}^{\Phi}_{U,V,W,Z}$ modulo $\hbar^2$ (see 2.5.5). Recalling that $\Phi = 1 + \mathcal{O}(\hbar^2)$, we have, for any X, Y, Z, W in $\mathsf{Obj}\big((\mathsf{DY}(\mathfrak{b}))^{\Phi}_{\hbar}\big)$, $x \in X$, $y \in Y$, $z \in Z$ and $w \in W$ and setting $v := (x\bar{\otimes}y)\bar{\otimes}(w\bar{\otimes}z)$:

$$\begin{aligned}
&\overline{\beta}^{\Phi}_{X,Y,Z,W}(v) \mod \hbar^2 = \Big(\overline{\alpha}_{X,Z,Y,W}\circ\big((\mathsf{id}_X\bar{\otimes}\overline{c}^{\Phi}_{Y,Z})\bar{\otimes}\mathsf{id}_W\big)\circ(\overline{\alpha}_{X,Y,Z,T})^{-1}\Big)(v) \mod \hbar^2\\
&= \Big(\overline{a}^{\Phi}_{X\bar{\otimes}Z,Y,W}\circ(\overline{a}^{\Phi}_{X,Z,Y}\bar{\otimes}\mathsf{id}_W)^{-1}\circ\big((\mathsf{id}_X\bar{\otimes}\overline{c}_{Y,Z}\circ e^{\bar{t}_{Y,Z}/2})\bar{\otimes}\mathsf{id}_W\big)\circ(\overline{a}^{\Phi}_{X\bar{\otimes}Y,Z,W}\circ\big((\overline{a}^{\Phi}_{X,Y,Z})^{-1}\bar{\otimes}\mathsf{id}_W)\big)\Big)(v) \mod \hbar^2\\
&= \frac{1}{2}\Big((x\otimes z)\otimes(y\otimes w) + \hbar\sum_{[z]}(x\otimes z^{[0]}\cdot y)\otimes(z^{[1]}\bar{\otimes}w) + \hbar\sum_{[y]}(x\otimes y^{[0]})\otimes(y^{[1]}\cdot z\otimes w)\Big)
\end{aligned}$$

where the last equality follows from the fact that the deformed constraints satisfy $\overline{a}^{\Phi} \mod \hbar^2 = a$ and $\overline{c}^{\Phi}_{X,Y} \mod \hbar^2 = c_{X,Y}\circ\frac{1}{2}(\mathsf{id}_{X\otimes Y} + \hbar t^{\mathfrak{b}}_{X,Y})$. Next, we compute

$$\begin{aligned}
&\psi^2_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(V,W)(v\bar{\otimes}w) \mod \hbar^2\\
&= \big((\overline{\zeta}_V\bar{\otimes}\overline{\zeta}_W)\circ\psi^2_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}}(V,W)\circ(\overline{\zeta}_{V\bar{\otimes}W})^{-1}\big)(v\bar{\otimes}w) \mod \hbar^2\\
&= \Big((\overline{\zeta}_V\bar{\otimes}\overline{\zeta}_W)\circ\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b})\bar{\otimes}V,\mathsf{U}(\mathfrak{b})\bar{\otimes}W)\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\beta}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),V,W} \mod \hbar^2)\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\Delta}_0\bar{\otimes}\mathsf{id}_{V\bar{\otimes}W})\\
&\circ(\overline{\zeta}_{V\bar{\otimes}W})^{-1}\Big)(v\bar{\otimes}w)\\
&= \Big((\overline{\zeta}_V\bar{\otimes}\overline{\zeta}_W)\circ\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b})\bar{\otimes}V,\mathsf{U}(\mathfrak{b})\bar{\otimes}W)\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\beta}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),V,W} \mod \hbar^2)\Big)([(1\bar{\otimes}1)\bar{\otimes}(v\bar{\otimes}w)])\\
&= \frac{1}{2}\Big((\overline{\zeta}_V\bar{\otimes}\overline{\zeta}_W)\circ\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b})\bar{\otimes}V,\mathsf{U}(\mathfrak{b})\bar{\otimes}W)\Big)([(1\bar{\otimes}v)\bar{\otimes}(1\bar{\otimes}w)])\\
&+ \frac{1}{2}\hbar\sum_{[1]}\Big((\overline{\zeta}_V\bar{\otimes}\overline{\zeta}_W)\circ\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b})\bar{\otimes}V,\mathsf{U}(\mathfrak{b})\bar{\otimes}W)\Big)([(1\bar{\otimes}1^{[0]}\cdot v)\bar{\otimes}(1^{[1]}\bar{\otimes}w)])\\
&+ \frac{1}{2}\hbar\sum_{[v]}\Big((\overline{\zeta}_V\bar{\otimes}\overline{\zeta}_W)\circ\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b})\bar{\otimes}V,\mathsf{U}(\mathfrak{b})\bar{\otimes}W)\Big)([(1\bar{\otimes}v^{[1]})\bar{\otimes}(v^{[0]}\cdot 1\bar{\otimes}w)])\\
&= \frac{1}{2}\Big(\overline{\zeta}_V(1\bar{\otimes}v)\bar{\otimes}\overline{\zeta}_W(1\bar{\otimes}w) + 0 + \hbar\sum_{[v]}\big(\overline{\zeta}_V(1\bar{\otimes}v^{[1]})\bar{\otimes}\overline{\zeta}_W(v^{[0]}\bar{\otimes}w)\big)\Big)\\
&= \frac{1}{2}\Big(v\bar{\otimes}w + \hbar\sum_{[v]}\big(v^{[1]}\bar{\otimes}S_0(v^{[0]}\cdot w)\big)\Big)\\
&= \frac{1}{2}\Big(v\bar{\otimes}w - \hbar\sum_{[v]}\big(v^{[1]}\bar{\otimes}v^{[0]}\cdot w\big)\Big)
\end{aligned}$$

where the fifth equality follows from the fact that $\pi^*_{\mathsf{U}(\mathfrak{b})}(1) = 0$ and the last follows from the fact that $v^{[0]}$ is primitive. □

7.6 Quantization of Lie bialgebras

We now present the Ševera's quantization of Lie bialgebras. Given a Lie bialgebra $\mathfrak{b}$, consider the infinitesimally braided monoidal categories $\mathsf{Vect}_{\mathbb{K}}$ and $\mathsf{DY}(\mathfrak{b})$, see §2.8 and §4.4 for more details. The functor of coinvariants $F^{\mathfrak{b}}$ (see §7.4) is $\mathsf{U}(\mathfrak{b})$–adapted, and therefore by Theorem 7.3.1 there exists a Hopf algebra structure on $F^{\mathfrak{b}}(\mathsf{U}(\mathfrak{b})\otimes\mathsf{U}(\mathfrak{b})) = \frac{\mathsf{U}(\mathfrak{b})\otimes\mathsf{U}(\mathfrak{b})}{\mathfrak{b}\cdot(\mathsf{U}(\mathfrak{b})\otimes\mathsf{U}(\mathfrak{b}))}$. Next, let $\hbar$ be a formal parameter and Φ be a Drinfeld associator. Deforming $\mathsf{Vect}_{\mathbb{K}}$ and $\mathsf{DY}(\mathfrak{b})$ as in §2.7 gives two braided monoidal categories $(\mathsf{Vect}_{\mathbb{K}})^{\Phi}_{\hbar} = \mathsf{TopFree}_{\mathbb{K}}$ and $(\mathsf{DY}(\mathfrak{b}))^{\Phi}_{\hbar}$. Moreover, by Theorem 7.5 we have that the induced deformed comonoidal functor $(F^{\mathfrak{b}})^{\Phi}_{\hbar} : (\mathsf{DY}(\mathfrak{b}))^{\Phi}_{\hbar} \to \mathsf{TopFree}_{\mathbb{K}}$ is $\mathsf{U}(\mathfrak{b})$–adapted, inducing a topological Hopf algebra structure on $(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{U}(\mathfrak{b})\otimes\mathsf{U}(\mathfrak{b})) = (G^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{U}(\mathfrak{b}))$. Recall also that the functors $(G^{\mathfrak{b}})^{\Phi}_{\hbar}$ and $(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}$ are naturally comonoidally isomorphic. Therefore, we have a natural topological Hopf algebra structure on $(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{U}(\mathfrak{b})) = \mathsf{U}(\mathfrak{b})$, which we shall denote by $\check{\mathsf{U}}(\mathfrak{b})$.
We first introduce the following

Lemma 7.6.1. *Let* $(\mathfrak{b}, [\cdot,\cdot], \delta)$ *be a Lie bialgebra and set* $\theta \coloneqq [\cdot,\cdot]\circ\delta : \mathfrak{b}\to\mathfrak{b}$. *Then*

(i) θ *is a Lie biderivation, i.e. it satisfies*

$$\theta([x,y]) = [\theta(x),y] + [x,\theta(y)] \tag{7.6.1a}$$
$$\delta\circ\theta = (\theta\otimes\mathsf{id}_{\mathfrak{b}} + \mathsf{id}_{\mathfrak{b}}\otimes\theta)\circ\delta. \tag{7.6.1b}$$

(ii) There is a unique algebra derivation $D_\theta : \mathsf{U}(\mathfrak{b}) \to \mathsf{U}(\mathfrak{b})$ *such that for any* $x\in\mathfrak{b}$ *one has* $D_\theta(x) = \theta(x)$.

Proof. (i) Using the Jacobi identity and the cocycle condition we get

$$\begin{aligned}
\theta([x,y]) &= [\cdot,\cdot]\circ\delta([x,y])\\
&= [\cdot,\cdot]\circ\Big(\sum_{\langle x\rangle}[x',y]\otimes x'' + x'\otimes[x'',y] + \sum_{\langle y\rangle}[x,y']\otimes y'' + y'\otimes[x,y'']\Big)\\
&= \sum_{\langle x\rangle}\big([[x',y],x''] + [x',[x'',y]]\big) + \sum_{\langle y\rangle}\big([[x,y'],y''] + [y',[x,y'']]\big)\\
&= \sum_{\langle x\rangle}[[x',x''],y] + \sum_{\langle y\rangle}[x,[y',y'']]\\
&= [\theta(x),y] + [x,\theta(y)].
\end{aligned}$$

Similarly, using the coJacobi identity and the cocycle condition we get

$$\begin{aligned}
\delta\circ\theta(x) &= (\delta\circ[\cdot,\cdot]\circ\delta)(x)\\
&= \sum_{\langle x\rangle}\delta([x',x''])\\
&= \sum_{\langle x,x''\rangle} x'\cdot\big((x'')'\otimes(x'')''\big) - \sum_{\langle x,x'\rangle} x''\cdot\big((x')'\otimes(x')''\big)\\
&= \sum_{\langle x,x'\rangle}\big([x',(x'')']\otimes(x'')'' + (x'')'\otimes[x',(x'')'']\big) - \sum_{\langle x,x'\rangle}\big([x'',(x')']\otimes(x')'' + (x')'\otimes[x'',(x')'']\big)\\
&= \big((\theta\otimes\mathsf{id}_{\mathfrak{b}} + \mathsf{id}_{\mathfrak{b}}\otimes\theta)\circ\delta\big)(x)
\end{aligned}$$

(ii): Consider the following map

$$\hat{D}_\theta : \mathsf{T}(\mathfrak{b}) \to \mathsf{T}(\mathfrak{b})$$

defined for any $n \in \mathbb{N}$ and $x, x_1, \dots, x_n \in \mathfrak{b}$ by

$$\begin{aligned}
\hat{D}_\theta(1) &= 0 \\
\hat{D}_\theta(x) &= \theta(x) \\
\hat{D}_\theta(x_1 \otimes \cdots \otimes x_n) &= \sum_{r=1}^{n} (x_1 \otimes \cdots \otimes x_{r-1}) \otimes \theta(x_r) \otimes x_{r+1} \otimes \cdots \otimes x_n.
\end{aligned}$$

Clearly, $\hat{D}_\theta$ is a derivation of free algebras. Moreover, for any $x, y \in \mathfrak{b}$ we have

$$\begin{aligned}
&\hat{D}_\theta(x \otimes y - y \otimes x - [x, y]) \\
&= \theta(x) \otimes y + x \otimes \theta(y) - \theta(y) \otimes x - y \otimes \theta(x) - [\theta(x), y] - [x, \theta(y)] \\
&= \big(\theta(x) \otimes y - y \otimes \theta(x) - [\theta(x), y]\big) + \big(x \otimes \theta(y) - \theta(y) \otimes x - [x, \theta(y)]\big)
\end{aligned}$$

Hence $\hat{D}_\theta(\mathcal{I}(\mathfrak{b})) \subset \mathcal{I}(\mathfrak{b})$, i.e. $\hat{D}_\theta$ passes to the quotient giving rise to a well–defined derivation $D_\theta : \mathsf{U}(\mathfrak{b}) \to \mathsf{U}(\mathfrak{b})$. □

We can now prove the main result of this Chapter:

Theorem 7.6.2. [Šev16, Th.2] *The topological Hopf algebra* $\check{\mathsf{U}}(\mathfrak{b})$*is a quantization of* $\mathfrak{b}$.

Proof. First recall that as a topologically free module we have $\check{\mathsf{U}}(\mathfrak{b}) = \mathsf{U}(\mathfrak{b})$. Next, we look at the coalgebra structure of $\check{\mathsf{U}}(\mathfrak{b})$. By definition,

$$\check{\mathsf{U}}(\mathfrak{b}) = \overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\big((G^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{U}(\mathfrak{b}))\big) = \overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\big((F^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{U}(\mathfrak{b}) \otimes \mathsf{U}(\mathfrak{b}))\big)$$

and therefore the coalgebra structure of $\check{\mathsf{U}}(\mathfrak{b})$ is the one induced by $(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{U}(\mathfrak{b}) \otimes \mathsf{U}(\mathfrak{b}))$ through $\overline{\zeta}$, i.e.

$$\begin{aligned}
\check{\Delta} &:= \overline{\zeta}_{\mathsf{U}(\mathfrak{b})} \bar{\otimes} \overline{\zeta}_{\mathsf{U}(\mathfrak{b})} \circ \Delta_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{U}(\mathfrak{b}) \bar{\otimes} \mathsf{U}(\mathfrak{b}))} \circ \overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})} \\
\check{\varepsilon} &:= \varepsilon_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{U}(\mathfrak{b}) \bar{\otimes} \mathsf{U}(\mathfrak{b}))} \circ \overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})}.
\end{aligned}$$

We are going to prove that for any $u \in \check{\mathsf{U}}(\mathfrak{b})$ the following formula holds

$$\check{\Delta}(u) \mod \hbar^2 = \Delta_0(u) + \hbar/2 \Delta_1(u),$$

where $\Delta_1(1) = 0$ and for any $n \in \mathbb{N}$, $x_1, \dots, x_n \in \mathfrak{b}$

$$\Delta_1(x_1 \cdots x_n) = \sum_{r=1}^{n} \Delta_0(x_1 \cdots x_{r-1}) \delta(x_r) \Delta_0(x_{r+1} \cdots x_n).$$

Denoting by ψ_1 the term proportional to $\hbar$ in the formal series $\psi_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}), \mathsf{U}(\mathfrak{b}))$, we get

$$\Delta_1(u) = \psi_1(\Delta_0(u)) = \sum_{(u)} \psi_1(u^{(1)} \bar{\otimes} u^{(2)}) = -\sum_{(u)} \sum_{[u^{(1)}]} (u^{(1)})^{[1]} \bar{\otimes} \big((u^{(1)})^{[0]} u^{(2)}\big).$$

For $u=1$ we obtain $\Delta_1(1)=0$ in view of the fact that $\tilde{\rho}(1)=0$. For $u=x\in\mathfrak{b}$ we get

$$\Delta_1(x) = -\sum_{[x]} x^{[1]}\otimes x^{[0]}1 - \sum_{[1]} 1^{[1]}\otimes 1^{[0]}x = -\sum_{\langle x\rangle} x''\bar{\otimes} x' - 0 = \sum_{\langle x\rangle} x'\bar{\otimes} x'' = \delta(x).$$

For any $x\in\mathfrak{b}$ and $u\in\mathsf{U}(\mathfrak{b})$ we have

$$\begin{aligned}
\Delta_1(xu) &= -\sum_{(xu)}\sum_{[(xu)^{(1)}]} \big((xu)^{(1)}\big)^{[1]}\bar{\otimes}\big((xu)^{(1)}\big)^{[0]}(xu)^{(2)} \\
&= -\sum_{(u)}\sum_{[(xu)^{(1)}]} \big(x(u)^{(1)}\big)^{[1]}\bar{\otimes}\big(x(u)^{(1)}\big)^{[0]}(xu)^{(2)} - \sum_{(u)}\sum_{[u^{(1)}]}(u^{(1)})^{[1]}\bar{\otimes}(u^{(1)})^{[0]}xu^{(2)} \\
&= -\sum_{(u)}\sum_{[u^{(1)}]} x(u^{(1)})^{[1]}\bar{\otimes}(u^{(1)})^{[0]}u^{(2)} - \sum_{(u)}\sum_{[u^{(1)}]} x(u^{(1)})^{[1]}\bar{\otimes}[x,(u^{(1)})^{[0]}]u^{(2)} \\
&\quad -\sum_{(u)}\sum_{[u^{(1)}]}(u^{(1)})^{[1]}\bar{\otimes}(u^{(1)})^{[0]}xu^{(2)} - \sum_{(u)}\sum_{\langle x\rangle}(x''u^{(1)})\bar{\otimes}(x'u^{(2)}) \\
&= -(x\bar{\otimes}1+1\bar{\otimes}x)\sum_{(u)}\sum_{[u^{(1)}]}(u^{(1)})^{[1]}\bar{\otimes}\big((u^{(1)})^{[0]}u^{(2)}\big) + \sum_{(u)}\sum_{\langle x\rangle}(x'u^{(1)})\bar{\otimes}(x''u^{(2)}) \\
&= \Delta_0(x)\Delta_1(u)+\delta(x)\Delta_0(u),
\end{aligned}$$

where we used Equation (4.4.4). Finally, by induction on the standard filtration of the universal enveloping algebra we have

$$\begin{aligned}
\Delta_1(x_1\cdot x_2\cdots x_{n+1}) &= \Delta_0(x_1)\Delta_1(x_2\cdots x_{n+1}) + \Delta_1(x_1)\Delta_0(x_2\cdots x_{n+1}) \\
&= \Delta_0(x_1)\sum_{r=2}^{n+1}\Delta_0(x_2\cdots x_{r-1})\delta(x_r)\Delta_0(x_{r+1}\cdots x_{n+1}) + \Delta_1(x_1)\Delta_0(x_2\cdots x_{n+1}) \\
&= \sum_{r=1}^{n+1}\Delta_0(x_1\cdots x_{r-1})\delta(x_r)x_{r+1}\cdots x_{n+1}.
\end{aligned}$$

Next, we look at the antipode of $\check{\mathsf{U}}(\mathfrak{b})$. We have

$$\begin{aligned}
\check{S} &:= \overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\tau}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})} \\
&= \overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\tau}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}\circ e^{\overline{t}^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}/2})\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})} \\
&= \overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\tau}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})}\circ\Big(\sum_{r=0}^{\infty}\frac{\hbar^r}{2(r!)}\big(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}((\overline{t}^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})^r)\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})}\big)\Big) \\
&= \overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\tau}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})}\circ\Big(\sum_{r=0}^{\infty}\frac{\hbar^r}{2(r!)}\big(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{t}^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})^r\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})}\big)\Big) \\
&= \overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\tau}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})}\circ\Big(\sum_{r=0}^{\infty}\frac{\hbar^r}{2(r!)}\big(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{t}^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})}\big)^r\Big) \\
&= \overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\tau}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})}\circ e^{(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{t}^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})})/2}.
\end{aligned}$$

For any $u,v\in\mathsf{U}(\mathfrak{b})$ we have

$$t^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}(u\otimes v) = \sum_{[u]} u^{[1]}\otimes u^{[0]}v + \sum_{[v]} v^{[0]}u\otimes v^{[1]},$$

and therefore

$$\big(\zeta_{\mathsf{U}(\mathfrak{b})} \circ F^{\mathfrak{b}}(t^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}) \circ \zeta^{-1}_{\mathsf{U}(\mathfrak{b})}\big)(v) = \big(\zeta_{\mathsf{U}(\mathfrak{b})} \circ F^{\mathfrak{b}}(t^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})\big)([1 \otimes v])$$
$$= \zeta_{\mathsf{U}(\mathfrak{b})}\Big(0 + \sum_{[v]} v^{[0]} \otimes v^{[1]}\Big)$$
$$= \sum_{[v]} S_0(v^{[0]})v^{[1]}.$$

Set $v = x_1 \cdots x_n$, where $n \in \mathbb{N}$ and $x_i \in \mathfrak{b}$ and recall that for any $u \in \mathsf{U}(b)$ and $y \in \mathfrak{b}$ we have $u \cdot y = \sum_{(u)} u^{(1)} y S(u^{(2)})$, see [Kas12, Prop.IX.3.1]. Then we get

$$\begin{aligned}
\sum_{[v]} S_0(v^{[0]})v^{[1]} &= \sum_{r=1}^{n} \sum_{\langle x_r \rangle} \sum_{(x_1,\dots,x_{r-1})} S_0\big((x_1 \cdots x_{r-1})^{(1)} \cdot x'_r\big)(x_1 \cdots x_{r-1})^{(2)} x''_r x_{r+1} \cdots x_n \\
&= \sum_{r=1}^{n} \sum_{\langle x_r \rangle} \sum_{(x_1,\dots,x_{r-1})} S_0\big((x_1 \cdots x_{r-1})^{(1)} \cdot x'_r S_0(x_1 \cdots x_{r-1})^{(2)}\big)(x_1 \cdots x_{r-1})^{(3)} x''_r x_{r+1} \cdots x_n \\
&= -\sum_{r=1}^{n} \sum_{\langle x_r \rangle} \sum_{(x_1,\dots,x_{r-1})} (x_1 \cdots x_{r-1})^{(2)} \cdot x'_r S_0(x_1 \cdots x_{r-1})^{(1)}(x_1 \cdots x_{r-1})^{(3)} x''_r x_{r+1} \cdots x_n \\
&= -\sum_{r=1}^{n} \sum_{\langle x_r \rangle} \sum_{(x_1,\dots,x_{r-1})} (x_1 \cdots x_{r-1})^{(1)} \cdot x'_r S_0(x_1 \cdots x_{r-1})^{(2)}(x_1 \cdots x_{r-1})^{(3)} x''_r x_{r+1} \cdots x_n \\
&= -\sum_{r=1}^{n} \sum_{\langle x_r \rangle} \sum_{(x_1,\dots,x_{r-1})} (x_1 \cdots x_{r-1})^{(1)} \cdot x'_r 1\varepsilon_0((x_1 \cdots x^{[2]}_{r-1}))(x_1 \cdots x_{r-1})^{(3)} x''_r x_{r+1} \cdots x_n \\
&= -\sum_{r=1}^{n} \sum_{\langle x_r \rangle} x_1 \cdots x_{r-1} x'_r x''_r x_{r+1} \cdots x_n = -\sum_{r=1}^{n} \sum_{\langle x_r \rangle} x_1 \cdots x_{r-1}[x'_r, x''_r] x_{r+1} \cdots x_n \\
&= -\frac{1}{2} \sum_{r=1}^{n} \sum_{\langle x_r \rangle} x_1 \cdots x_{r-1} \theta(x_r) x_{r+1} \cdots x_n = -\frac{1}{2} D_\theta(x_1 \cdots x_n) = -\frac{1}{2} D_\theta(v).
\end{aligned}$$

Moreover, we have

$$\big(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})} \circ (F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\tau}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}) \circ \overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})}\big)(u) = \big(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})} \circ (F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\tau}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})\big)([1 \otimes u])$$
$$= \overline{\zeta}_{\mathsf{U}(\mathfrak{b})}([u \otimes u]) = \overline{S}_0(u).$$

And then we have

$$\check{S} = \overline{S}_0 \circ e^{-\hbar/2 D_\theta}.$$

Next, we look at the algebra structure of $\mathsf{U}_\hbar(\mathfrak{b})^{\check{S}}$, which is given by the following multiplication and unit:

$$\check{\mu} := \overline{\zeta}_{\mathsf{U}(\mathfrak{b})} \circ \mu_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{U}(\mathfrak{b}) \bar{\otimes} \mathsf{U}(\mathfrak{b}))} \circ (\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})} \bar{\otimes} \overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})})$$
$$\check{\eta} := \overline{\zeta}_{\mathsf{U}(\mathfrak{b})} \circ \eta_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{U}(\mathfrak{b}) \bar{\otimes} \mathsf{U}(\mathfrak{b}))}$$

We are going to show that

$$\check{\mu} = \overline{\mu}_0 \circ (\overline{S}_0 \bar{\otimes} \mathsf{id}_{\mathsf{U}(\mathfrak{b})}) \circ \big(\psi^2_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}), \mathsf{U}(\mathfrak{b}))\big)^{-1} \circ \big((\check{S})^{-1} \bar{\otimes} \mathsf{id}_{\mathsf{U}(\mathfrak{b})}\big). \tag{7.6.2}$$

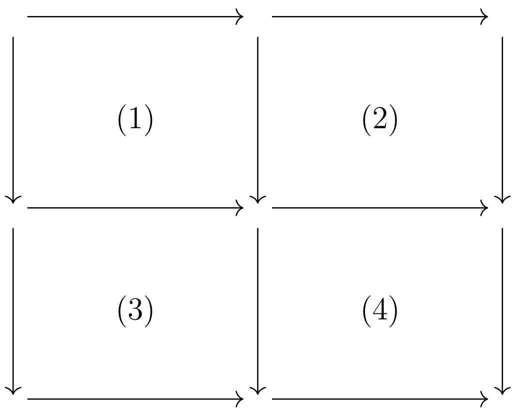
In order to do so, we compute

$$
\begin{aligned}
&(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\overline{\zeta}_{\mathsf{U}(\mathfrak{b})})\circ\overline{\gamma^{\mathsf{U}(\mathfrak{b})}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}}\\
&=(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\overline{\zeta}_{\mathsf{U}(\mathfrak{b})})\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\tau}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}\bar{\otimes}\mathsf{id}_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{U}(\mathfrak{b}))})\circ\psi^{2}_{(G^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}))\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{a}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})\\
&\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}((\overline{\tau}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})^{-1}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\\
&=\big((\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\tau}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}))\bar{\otimes}\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\big)\circ(\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})})\circ\psi^{2}_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}))\\
&\circ\overline{\zeta}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}\big(\overline{a}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}\circ(\overline{\tau}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})^{-1}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}\big)\\
&=(S^{\check{\mathrm{S}}}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\circ\psi^{2}_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}))\circ\overline{\zeta}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}\big(\overline{a}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}\circ(\overline{\tau}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})^{-1}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}\big).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\check{\mu}&:=\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ\mu_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b}))}\circ(\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})})\\
&=\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F)^{\Phi}_{\hbar}(\overline{r}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\circ(F)^{\Phi}_{\hbar}((\mathsf{id}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\overline{\varepsilon}_0)\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\circ(\overline{\gamma^{\mathsf{U}(\mathfrak{b})}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}})^{-1}\circ\big((\overline{\zeta}_{\mathsf{U}(\mathfrak{b})})^{-1}\bar{\otimes}(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})})^{-1}\big)\\
&=\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}\big(\underbrace{(\overline{r}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\circ((\mathsf{id}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\overline{\varepsilon}_0)\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\circ(\overline{\tau}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\circ\overline{a}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}}\big)\\
&\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})}\circ(\psi^{2}_{(\mathscr{O}^{\mathfrak{b}})}\big(\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})\big))^{-1}\circ\big((\check{S})^{-1}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}\big).
\end{aligned}
$$

In order to compute the wavy underlined morphism we consider the following diagram:



whose four subdiagrams are

(1)

$$
\begin{array}{ccc}
\mathsf{U}(\mathfrak{b})\bar{\otimes}\big(\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})\big) & \xrightarrow{(\overline{a}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})^{-1}} & \big(\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})\big)\bar{\otimes}\mathsf{U}(\mathfrak{b})\\
{\scriptstyle \overline{\varepsilon}_0\bar{\otimes}(\mathsf{id}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})})}\downarrow & & \downarrow{\scriptstyle (\overline{\varepsilon}_0\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}}\\
\mathbb{K}[[\hbar]]\bar{\otimes}\big(\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})\big) & \xrightarrow{(\overline{a}^{\Phi}_{\mathbb{K}[[\hbar]],\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})^{-1}} & \big(\mathbb{K}[[\hbar]]\bar{\otimes}\mathsf{U}(\mathfrak{b})\big)\bar{\otimes}\mathsf{U}(\mathfrak{b})
\end{array}
$$

(2)

$$
\begin{array}{ccc}
\big(\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})\big)\bar{\otimes}\mathsf{U}(\mathfrak{b}) & \xrightarrow{\overline{\tau}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}} & \big(\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})\big)\bar{\otimes}\mathsf{U}(\mathfrak{b})\\
{\scriptstyle (\overline{\varepsilon}_0\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}}\downarrow & & \downarrow{\scriptstyle (\mathsf{id}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\overline{\varepsilon}_0)\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}}\\
\big(\mathbb{K}[[\hbar]]\bar{\otimes}\mathsf{U}(\mathfrak{b})\big)\bar{\otimes}\mathsf{U}(\mathfrak{b}) & \xrightarrow{\overline{\tau}^{\Phi}_{\mathbb{K}[[\hbar]],\mathsf{U}(\mathfrak{b})}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}} & \big(\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathbb{K}[[\hbar]]\big)\bar{\otimes}\mathsf{U}(\mathfrak{b})
\end{array}
$$

(3)

$$\begin{array}{ccc}
\mathbb{K}[[\hbar]]\bar{\otimes}\big(\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})\big) & \xrightarrow{(\overline{a}^{\Phi}_{\mathbb{K}[[\hbar]],\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})^{-1}} & \big(\mathbb{K}[[\hbar]]\bar{\otimes}\mathsf{U}(\mathfrak{b})\big)\bar{\otimes}\mathsf{U}(\mathfrak{b}) \\
{\scriptstyle \overline{\ell}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\ell}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}} \\
\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b}) & = & \mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})
\end{array}$$

(4)

$$\begin{array}{ccc}
\big(\mathbb{K}[[\hbar]]\bar{\otimes}\mathsf{U}(\mathfrak{b})\big)\bar{\otimes}\mathsf{U}(\mathfrak{b}) & \xrightarrow{\overline{\tau}^{\Phi}_{\mathbb{K}[[\hbar]],\mathsf{U}(\mathfrak{b})}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}} & \big(\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathbb{K}[[\hbar]]\big)\bar{\otimes}\mathsf{U}(\mathfrak{b}) \\
{\scriptstyle \overline{\ell}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{r}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}} \\
\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b}) & = & \mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})
\end{array}$$

and commute respectively in view of the naturality of the associativity constraint, the naturality of the braiding, the compatibility of the braiding with the unit constraints and the triangle axiom. Hence

$$\overline{r}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\circ\big((\mathsf{id}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\overline{\varepsilon}_0)\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}\big)\circ(\overline{\tau}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\circ\overline{a}^{\Phi}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}=\overline{\ell}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})}\circ(\overline{\varepsilon}_0\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})})$$

and

$$\check{\mu}=\underset{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}{\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}\big(\overline{\ell}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})}\circ(\overline{\varepsilon}_0\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})})\big)\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})}}\circ(\psi^2_{(\mathscr{O}^{\mathfrak{b}})}(\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})))^{-1}\circ\big((\check{S})^{-1}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}\big).$$

Finally, the dotted underlined term can be computed explicitely: for any $u,v\in\mathsf{U}(\mathfrak{b})$ we have

$$\begin{aligned}
&\big(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}\big(\overline{\ell}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})}\circ(\check{\varepsilon}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})})\big)\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})}\big)(u\otimes v)\\
&=\big(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}\big(\overline{\ell}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})}\circ(\check{\varepsilon}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})})\big)\big)([1\bar{\otimes}(u\bar{\otimes}v)])\\
&=\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}([u\otimes v])=\overline{S}_0(u)v\\
&=\big(\overline{\mu}_0\circ(\overline{S}_0\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\big)(u\bar{\otimes}v).
\end{aligned}$$

We finally get

$$\check{\mu}=\overline{\mu}_0\circ(\overline{S}_0\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})})\circ\psi_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}\big(\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})\big)\circ\big((\check{S})^{-1}\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})}\big)$$

and then in particular

$$\check{\mu}\mod\hbar=\overline{\mu}_0.$$

□

7.7 Functoriality

The aim of this Section is to prove the functoriality of the Ševera's quantization.

Lemma 7.7.1. *Let* $(\mathfrak{b},[\cdot,\cdot],\delta)$ *and* $(\mathfrak{b}',[\cdot,\cdot]',\delta')$ *be two Lie bialgebras and* $\varphi:\mathfrak{b}\to\mathfrak{b}'$ *be a morphism of Lie bialgebras. Then the corresponding morphism of Hopf algebras* $\mathsf{U}(\varphi):\mathsf{U}(\mathfrak{b})\to\mathsf{U}(\mathfrak{b}')$ *has the following properties:*

(i) $\mathscr{O}^{\mathfrak{b}'}(t^{\mathfrak{b}'}_{\mathsf{U}(\mathfrak{b}'),\mathsf{U}(\mathfrak{b}')})\circ\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)=\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\circ\mathscr{O}^{\mathfrak{b}}(t^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})}).$

(ii) $\psi^2_{(\mathscr{O}^{\mathfrak{b}'})^\Phi_\hbar}(\mathsf{U}(\mathfrak{b}'),\mathsf{U}(\mathfrak{b}')) \circ \big(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\big) = \big(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\big) \circ \psi^2_{(\mathscr{O}^{\mathfrak{b}})^\Phi_\hbar}\big(\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})\big).$

(iii) If θ (resp. θ') is the canonical Lie biderivation of $\mathfrak{b}$ (resp. $\mathfrak{b}'$), we have $\theta' \circ \varphi = \varphi \circ \theta$

(iv) $\mathsf{U}(\varphi) \circ D_\theta = D_{\theta'} \circ \mathsf{U}(\varphi)$.

Proof. (i): For any $u, v \in \mathsf{U}(\mathfrak{b})$ we have – using that $\mathsf{U}(\varphi)$ concides with ϕ on $\mathfrak{b}$ and $\mathsf{U}(\varphi)(1) = 1'$ –

$$\begin{aligned}
&\big(\mathscr{O}^{\mathfrak{b}'}(t^{\mathfrak{b}'}_{\mathsf{U}(\mathfrak{b}'),\mathsf{U}(\mathfrak{b}')}) \circ \mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)(u\otimes v) = \mathscr{O}^{\mathfrak{b}'}(t^{\mathfrak{b}'}_{\mathsf{U}(\mathfrak{b}'),\mathsf{U}(\mathfrak{b}')})\big(\mathsf{U}(\varphi)(u)\otimes\mathsf{U}(\varphi)(v)\big)\\
&= \sum_{[\mathsf{U}(\varphi)(u)]}\big(\mathsf{U}(\varphi)(u)\big)^{[1]}\otimes\big(\mathsf{U}(\varphi)(u)\big)^{[0]}\mathsf{U}(\varphi)(v) + \sum_{[\mathsf{U}(\varphi)(v)]}\big(\mathsf{U}(\varphi)(v)\big)^{[0]}\mathsf{U}(\varphi)(u)\otimes\big(\mathsf{U}(\varphi)(v)\big)^{[1]}\\
&= \sum_{[u]}\mathsf{U}(\varphi)(u^{[1]})\otimes\big(\mathsf{U}(\varphi)(u^{[0]})\big)\mathsf{U}(\varphi)(v) + \sum_{[v]}\mathsf{U}(\varphi)(v^{[0]})\mathsf{U}(\varphi)(u)\otimes\mathsf{U}(\varphi)(v^{[1]})\\
&= \sum_{[u]}\mathsf{U}(\varphi)(u^{[1]})\otimes\mathsf{U}(\varphi)(u^{[0]}v) + \sum_{[v]}\mathsf{U}(\varphi)(v^{[0]}u)\otimes\mathsf{U}(\varphi)(v^{[1]})\\
&= \big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\circ\mathscr{O}^{\mathfrak{b}}(t^{\mathfrak{b}}_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})
\end{aligned}$$

(ii): Recall that

$$\begin{aligned}
\psi^2_{(\mathscr{O}^{\mathfrak{b}})^\Phi_\hbar}\big(\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})\big) &= \big(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\big)\circ\psi^2_{(F^{\mathfrak{b}})^\Phi_\hbar}\big(\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})\big)\circ(F^{\mathfrak{b}})^\Phi_\hbar(\overline{\beta}^\Phi_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})\\
&\circ(F^{\mathfrak{b}})^\Phi_\hbar(\overline{\Delta}_0\bar{\otimes}\mathsf{id}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})})\circ\overline{\zeta}^{-1}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})}
\end{aligned}$$

hence for any $u, v \in \mathsf{U}(\mathfrak{b})$

$$\begin{aligned}
\psi^2_{(\mathscr{O}^{\mathfrak{b}})^\Phi_\hbar}\big(\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})\big)(u\otimes v) = \Big(&\big(\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\bar{\otimes}\overline{\zeta}_{\mathsf{U}(\mathfrak{b})}\big)\circ\psi^2_{(F^{\mathfrak{b}})^\Phi_\hbar}\big(\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})\big)\\
&\circ(F^{\mathfrak{b}})^\Phi_\hbar(\overline{\beta}^\Phi_{\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b})})\Big)\big([(1\bar{\otimes}1)\bar{\otimes}(u\bar{\otimes}v)]\big).
\end{aligned}$$

Set $M := \mathsf{U}(\mathfrak{b})$ and $M' := \mathsf{U}(\mathfrak{b}')$. We first look at the deformed *middle four interchange* $\overline{\beta}^\Phi$, see 2.5.5. Note first that we can write β –using the notation of 2.3.1 – as

$$\beta_{M,M,M,M} = \big((\mathsf{id}_M\otimes(\tau_{M,M}\otimes\mathsf{id}_M))^{(\mathsf{id}_M\otimes a^{-1}_{M,M,M})}\big)^{a_{M,M,M\otimes M}}.$$

In order to consider its deformed version, consider the following maps $(M\otimes M)\otimes(M\otimes M)\to(M\otimes M)\otimes(M\otimes M)$:

$$\begin{aligned}
\tau_{12} &:= \tau_{M,M}\otimes\mathsf{id}_{M\otimes M}\\
\tau_{23} &:= \beta_{M,M,M,M}\\
\tau_{34} &:= \mathsf{id}_{M\otimes M}\otimes\tau_{M,M}\\
t_{12} &:= t^{b}_{M,M}\otimes\mathsf{id}_{M\otimes M}\\
t_{13} &:= \tau_{23}\circ t^{\mathfrak{b}}_{12}\circ\tau_{23}\\
t_{14} &:= \tau_{34}t^{\mathfrak{b}}_{13}\circ\tau_{34}\\
t_{23} &:= \tau_{12}t^{\mathfrak{b}}_{13}\circ\tau_{12}\\
t_{24} &:= \tau_{34}\circ t^{\mathfrak{b}}_{23}\circ\tau_{34}\\
t_{34} &:= \mathsf{id}_{M\otimes M}\circ t^{\mathfrak{b}}_{M,M}.
\end{aligned}$$

Hence we have

$$\overline{\beta}^{\Phi}_{M,M,M,M} = \Phi(\overline{t}_{23}+\overline{t}_{24},\overline{t}_{12}) \circ \Phi(\overline{t}_{23},\overline{t}_{24}) \circ \overline{\tau}_{23} \circ e^{\overline{t}_{23}/2} \circ \Phi(\overline{t}_{24},\overline{t}_{23}) \circ \Phi(\overline{t}_{12},\overline{t}_{23}+\overline{t}_{24}). \tag{7.7.1}$$

Next, recall the natural transformation $p : \mathscr{O}^{\mathfrak{b}} \to F^{\mathfrak{b}}$ defined by the canonical projection satisfies

$$\psi^2_{F^{\mathfrak{b}}} \circ p_{V\otimes W} = p_V \otimes p_W.$$

We shall denote by p' the analogue natural transformation $p' : \mathscr{O}^{\mathfrak{b}'} \to F^{\mathfrak{b}'}$. In particular, we have

$$\begin{aligned}\big(\mathsf{U}(\varphi)\circ\zeta_M\circ p_{M\otimes M}\big)(u\otimes v) &= \big(\mathsf{U}(\varphi)\circ\zeta_M\big)([u\otimes v])\\ &= \mathsf{U}(\varphi)(S_0(u)v) = \mathsf{U}(\varphi)(S_0(u))\mathsf{U}(\varphi)(v)\\ &= S_0'(\mathsf{U}(\varphi)(u))\mathsf{U}(\varphi)(v) = \zeta'_{M'}\big([\mathsf{U}(\varphi)(u)\otimes\mathsf{U}(\varphi(v))]\big)\\ &= \big(\zeta'_{M'}\circ p'_{M'\otimes M'}\circ\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)(u\otimes v).\end{aligned}$$

Also, we have that

$$\Big(\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\otimes\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\Big)\circ\tau_{ij} = \tau'_{ij}\circ\Big(\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\otimes\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\Big)$$
$$\Big(\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\otimes\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\Big)\circ\mathscr{O}^{\mathfrak{b}}(t_{ij}) = \mathscr{O}^{\mathfrak{b}'}(t'_{ij})\circ\Big(\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\otimes\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\Big)$$

and hence

$$\begin{aligned}&\Big(\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\otimes\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\Big)\circ\mathscr{O}^{\mathfrak{b}}(t_{i_1j_1}\circ\cdots\circ t_{i_Nj_N})\\ &= \Big(\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\otimes\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\Big)\circ\mathscr{O}^{\mathfrak{b}}(t_{i_1j_1})\circ\cdots\circ\mathscr{O}^{\mathfrak{b}}(t_{i_Nj_N})\\ &= \mathscr{O}^{\mathfrak{b}'}(t'_{i_1j_1})\circ\cdots\circ\mathscr{O}^{\mathfrak{b}'}(t'_{i_Nj_N})\circ\Big(\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\otimes\big(\mathsf{U}(\varphi)\otimes\mathsf{U}(\varphi)\big)\Big).\end{aligned}$$

Therefore, since for each power of the formal parameter $\hbar$ the right hand side of Equation (7.7.1) is a finite composition of terms $t_{i_1j_1}\circ\cdots\circ t_{i_Nj_N}$ we get for any $u,v\in M$:

$$\begin{aligned}&\Big(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\Big)\circ\psi^2_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(M,M)\\ &= \Big(\Big(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\Big)\circ(\overline{\zeta}_M\bar{\otimes}\overline{\zeta}_M)\circ\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}(M\bar{\otimes}M,M\bar{\otimes}M)\Big)\Big([\overline{\beta}^{\Phi}_{M,M,M,M}((1\bar{\otimes}1)\bar{\otimes}(u\bar{\otimes}v))]\Big)\\ &= \Big(\Big(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\Big)\circ(\overline{\zeta}_M\bar{\otimes}\overline{\zeta}_M)\circ\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}(M\bar{\otimes}M,M\bar{\otimes}M)\circ\overline{p}_{(M\bar{\otimes}M)\bar{\otimes}(M\bar{\otimes}M)}\Big)\Big(\overline{\beta}^{\Phi}_{M,M,M,M}((1\bar{\otimes}1)\bar{\otimes}(u\bar{\otimes}v))\Big)\\ &= \Big(\Big(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\Big)\circ(\overline{\zeta}_M\bar{\otimes}\overline{\zeta}_M)\circ(\overline{p}_{M\bar{\otimes}M}\bar{\otimes}\overline{p}_{M\bar{\otimes}M})\Big)\Big(\overline{\beta}^{\Phi}_{M,M,M,M}((1\bar{\otimes}1)\bar{\otimes}(u\bar{\otimes}v))\Big)\\ &= \Big((\overline{\zeta}'_{M'}\bar{\otimes}\overline{\zeta}'_{M'})\circ(\overline{p}'_{M'\bar{\otimes}M'}\bar{\otimes}\overline{p}'_{M'\bar{\otimes}M'})\circ\Big(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\Big)\Big)\Big(\overline{\beta}^{\Phi}_{M,M,M,M}((1\bar{\otimes}1)\bar{\otimes}(u\bar{\otimes}v))\Big)\\ &= \Big((\overline{\zeta}'_{M'}\bar{\otimes}\overline{\zeta}'_{M'})\circ(\overline{p}'_{M'\bar{\otimes}M'}\bar{\otimes}\overline{p}'_{M'\bar{\otimes}M'})\circ\overline{\beta}'^{\Phi}_{M',M',M',M'}\Big)((1'\bar{\otimes}1')\bar{\otimes}(\overline{\mathsf{U}(\varphi)}(u)\bar{\otimes}\overline{\mathsf{U}(\varphi)}(v)))\\ &= \Big((\overline{\zeta}'_{M'}\bar{\otimes}\overline{\zeta}'_{M'})\circ\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}(M'\bar{\otimes}M',M'\bar{\otimes}M')\circ\overline{p}'_{(M'\bar{\otimes}M')\bar{\otimes}(M'\bar{\otimes}M')}\circ\psi^2_{(\mathscr{O}^{\mathfrak{b}'})^{\Phi}_{\hbar}}(\overline{\beta}'^{\Phi}_{M',M',M',M'})\Big)\\ &\qquad\Big((1'\bar{\otimes}1')\bar{\otimes}(\overline{\mathsf{U}(\varphi)}(u)\bar{\otimes}\overline{\mathsf{U}(\varphi)}(v))\Big)\\ &= \Big((\overline{\zeta}'_{M'}\bar{\otimes}\overline{\zeta}'_{M'})\circ\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}(M'\bar{\otimes}M',M'\bar{\otimes}M')\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\zeta}'_{M'}\bar{\otimes}\overline{\zeta}'_{M'})\circ(F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\Delta}'_0\bar{\otimes}\mathsf{id}_{M'\bar{\otimes}M'})\circ(\overline{\zeta}'_{M'\bar{\otimes}M'})^{-1}\Big)\\ &\qquad\Big(\Big(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\Big)(u\otimes v)\Big)\\ &= \psi^2_{(\mathscr{O}^{\mathfrak{b}'})^{\Phi}_{\hbar}}(M',M')\circ\Big(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\Big)\end{aligned}$$

proving the claim.
(iii): Using the fact that φ is a morphism of Lie algebras and of Lie coalgebras we get

$$\varphi \circ \theta = \varphi \circ [\cdot,\cdot] \circ \delta = [\cdot,\cdot]'(\varphi \otimes \varphi) \circ \delta = [\cdot,\cdot]' \circ \delta' \circ \varphi = \theta' \circ \varphi.$$

(iv): The claim holds trivially for $u = 1$, since $D_\theta(1) = 0$, $\mathsf{U}(\varphi)(1) = 1'$ and $D_{\theta'}(1') = 0$. For any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathfrak{b}$ we have – using that $\mathsf{U}(\varphi)$ is a morphism of algebras, the fact that $\mathsf{U}(\varphi)$ concides with φ on $\mathfrak{b}$ and statement (iii) –

$$\begin{aligned}
\big(\mathsf{U}(\varphi) \circ D_\theta\big)(x_1 \cdots x_n) &= \sum_{r=1}^{n} \mathsf{U}(\varphi)\big(x_1 \cdots x_{r-1}\theta(x_i)x_{i+1} \cdots x_n\big) \\
&= \sum_{r=1}^{n} \varphi(x_1) \cdots \varphi(x_{r-1})\varphi(\theta(x_r))\varphi(x_{r+1})\varphi(x_n) \\
&= \sum_{r=1}^{n} \varphi(x_1) \cdots \varphi(x_{r-1})\theta'(\varphi(x_r))\varphi(x_{r+1})\varphi(x_n) \\
&= D_{\theta'}\big(\varphi(x_1) \cdots \varphi(x_n)\big) \\
&= (D_{\theta'} \circ \mathsf{U}(\varphi))(x_1 \cdots x_n).
\end{aligned}$$

□

Theorem 7.7.2. *Let* LieBialg *and* QUAlg *denote respectively the categories of Lie bialgebras and of quantized universal enveloping algebras. Then the assignment*

$$\begin{aligned}
\check{Q} : \mathsf{LieBialg} &\to \mathsf{QUAlg} \\
(\mathfrak{b}, [\cdot,\cdot], \delta) &\mapsto \big(\check{\mathsf{U}}(\mathfrak{b}), \check{\mu}, \check{\eta}, \check{\Delta}, \check{\varepsilon}, \check{S}\big) \\
\varphi : \mathfrak{b} \to \mathfrak{b}' &\mapsto \overline{\mathsf{U}(\varphi)} : \check{\mathsf{U}}(\mathfrak{b}) \to \check{\mathsf{U}}(\mathfrak{b}').
\end{aligned}$$

is a functor.

Proof. We already shown in the previous Section that $(\check{\mathsf{U}}(\mathfrak{b}), \check{\mu}, \check{\eta}, \check{\Delta}, \check{\varepsilon}, \check{S})$ is a topological Hopf algebra quantizing $\mathfrak{b}$. We prove that $\overline{\mathsf{U}(\varphi)}$ is a morphism of Hopf algebras. We have

$$\begin{aligned}
\overline{\mathsf{U}(\varphi)} \circ \check{S} &= \overline{\mathsf{U}(\varphi)} \circ \overline{S}_0 \circ e^{-\hbar/2D_\theta} \\
&= \overline{S'}_0 \circ \overline{\mathsf{U}(\varphi)} \circ e^{-\hbar/2D_\theta} \\
&= \overline{S'}_0 \circ e^{-\hbar/2D_{\theta'}} \circ \overline{\mathsf{U}(\varphi)} \\
&= \check{S}' \circ \overline{\mathsf{U}(\varphi)}.
\end{aligned}$$

hence $\overline{\mathsf{U}(\varphi)}$ intertwines the antipodes. Since the units and the counits are not deformed, they are intertwined in view of the property of the functor U. Next, using statement (ii) of the previous lemma, we get

$$\begin{aligned}
\overline{\mathsf{U}(\varphi)} \circ \check{\mu} &= \overline{\mathsf{U}(\varphi)} \circ \big(\overline{\mu}_0 \circ (\overline{S}_0 \bar{\otimes} \mathsf{id}_{\mathsf{U}(\mathfrak{b})}) \circ \psi_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}), \mathsf{U}(\mathfrak{b})) \circ ((\check{S})^{-1} \bar{\otimes} \mathsf{id}_{\mathsf{U}(\mathfrak{b})})\big) \\
&= \overline{\mu}'_0 \circ \big(\overline{\mathsf{U}(\varphi)} \bar{\otimes} \overline{\mathsf{U}(\varphi)}\big) \circ (\overline{S}_0 \bar{\otimes} \mathsf{id}_{\mathsf{U}(\mathfrak{b})}) \circ \psi_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}), \mathsf{U}(\mathfrak{b})) \circ ((\check{S})^{-1} \bar{\otimes} \mathsf{id}_{\mathsf{U}(\mathfrak{b})}) \\
&= \overline{\mu}'_0 \circ (\overline{S'}_0 \bar{\otimes} \mathsf{id}_{\mathsf{U}(\mathfrak{b}')}) \circ \big(\overline{\mathsf{U}(\varphi)} \bar{\otimes} \overline{\mathsf{U}(\varphi)}\big) \circ \psi_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}), \mathsf{U}(\mathfrak{b})) \circ ((\check{S})^{-1} \bar{\otimes} \mathsf{id}_{\mathsf{U}(\mathfrak{b})}) \\
&= \overline{\mu}'_0 \circ (\overline{S'}_0 \bar{\otimes} \mathsf{id}_{\mathsf{U}(\mathfrak{b}')}) \circ \psi_{(\mathscr{O}^{\mathfrak{b}'})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}'), \mathsf{U}(\mathfrak{b}')) \circ \big(\overline{\mathsf{U}(\varphi)} \bar{\otimes} \overline{\mathsf{U}(\varphi)}\big) \circ ((\check{S})^{-1} \bar{\otimes} \mathsf{id}_{\mathsf{U}(\mathfrak{b})}) \\
&= \overline{\mu}'_0 \circ (\overline{S'}_0 \bar{\otimes} \mathsf{id}_{\mathsf{U}(\mathfrak{b}')}) \circ \psi_{(\mathscr{O}^{\mathfrak{b}'})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}'), \mathsf{U}(\mathfrak{b}')) \circ ((\check{S}')^{-1} \bar{\otimes} \mathsf{id}_{\mathsf{U}(\mathfrak{b}')}) \circ \big(\overline{\mathsf{U}(\varphi)} \bar{\otimes} \overline{\mathsf{U}(\varphi)}\big) \\
&= \check{\mu}' \circ \big(\overline{\mathsf{U}(\varphi)} \bar{\otimes} \overline{\mathsf{U}(\varphi)}\big)
\end{aligned}$$

and

$$
\begin{aligned}
\left(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\right)\circ\check{\Delta} &= \left(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\right)\circ\psi^2_{(\mathscr{O}^{\mathfrak{b}})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}),\mathsf{U}(\mathfrak{b}))\circ\overline{\Delta}_0\\
&= \psi^2_{(\mathscr{O}^{\mathfrak{b}'})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}'),\mathsf{U}(\mathfrak{b}'))\circ\left(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\right)\circ\overline{\Delta}_0\\
&= \psi^2_{(\mathscr{O}^{\mathfrak{b}'})^{\Phi}_{\hbar}}(\mathsf{U}(\mathfrak{b}'),\mathsf{U}(\mathfrak{b}'))\circ\overline{\Delta'}_0\circ\left(\overline{\mathsf{U}(\varphi)}\bar{\otimes}\overline{\mathsf{U}(\varphi)}\right)
\end{aligned}
$$

Hence $\overline{\mathsf{U}(\varphi)}$ is a morphism of topological Hopf algebras. Therefore, since U is a functor (and then $\mathsf{U}(\varphi\circ\phi)=\mathsf{U}(\varphi)\circ\mathsf{U}(\phi)$) we get that $\check{Q}$ is a functor. □

7.8 Quantization of twists

Let $\mathfrak{b}$ be a Lie bialgebra, $j\in\Lambda^2(\mathfrak{b})$ a Lie bialgebra twist (see §4.6) and consider the comonoids $\mathsf{U}(\mathfrak{b})$ and $\mathsf{U}(\mathfrak{b})_j$ of $\mathsf{DY}(\mathfrak{b})$, see Theorem 4.5.2 and Remark 4.6.10. The functor of coinvariants of §7.4 is both $\mathsf{U}(\mathfrak{b})$–adapted and $\mathsf{U}(\mathfrak{b})_j$–adapted, in view of Proposition 7.4.3 and of the fact that $\mathsf{U}(\mathfrak{b})_j$ carries the same comonoid structure of $\mathsf{U}(\mathfrak{b})$. Set $M=\mathsf{U}(\mathfrak{b})$, $N=\mathsf{U}(\mathfrak{b})_j$, $H=F^{\mathfrak{b}}(M\otimes M)$, $H_j=F^{\mathfrak{b}}(N\otimes N)$, and $B=F^{\mathfrak{b}}(M\otimes N)$. Then in view of the considerations of Remark 7.2.3 we have that B is a H–H_j–bimodule. Recall also that B has a comonoid structure whose comultiplication and counit are respectively

$$
\begin{aligned}
\Delta_B &= \psi^2_{F^{\mathfrak{b}}}(M\otimes M,N\otimes N)\circ F^{\mathfrak{b}}(\beta_{M,M,N,N})\circ F^{\mathfrak{b}}(\Delta_M\otimes\Delta_N)\\
\varepsilon_B &= \psi^0_{F^{\mathfrak{b}}}\circ F^{\mathfrak{b}}(\ell_{\mathbb{K}}\circ(\varepsilon_M\otimes\varepsilon_N)).
\end{aligned}
$$

From now on we shall denote with the same symbols the deformed objects $M,N\in\mathsf{DY}(\mathfrak{b})^{\Phi}_{\hbar}$, $H,H_j,B\in\mathsf{TopFree}_{\mathbb{K}}$. We need the following

Lemma 7.8.1. *The left action and right action*

$$
\begin{aligned}
\mu^M_{M,N} &: F^{\mathfrak{b}}(M\otimes M)\otimes F^{\mathfrak{b}}(M\otimes N)=H\otimes B\to F(M\otimes N)=B\\
\mu^N_{M,N} &: F^{\mathfrak{b}}(M\otimes N)\otimes F^{\mathfrak{b}}(N\otimes N)=B\otimes H_j\to F(M\otimes N)=B
\end{aligned}
$$

are morphisms of coalgebras.

Proof. The proof relies on the following four principles (which are the same used to prove that $F(M\otimes' M)$ is a bimonoid in Theorem 7.3.1):

- For any braided monoidal category $\mathscr{C}$, we have that $\mathsf{Comon}(\mathscr{C})$ is a monoidal category: hence if $(C_1,\Delta_1,\varepsilon_1)$, $(C_2,\Delta_2,\varepsilon_2)$, $(C_3,\Delta_3,\varepsilon_3)$, $(C_4,\Delta_4,\varepsilon_4)$ are four comonoids and $\alpha:C_1\to C_2$, $\gamma:C_3\to C_4$ are morphisms of comonoids, then $\alpha\otimes\gamma$ is a morphism of comonoids (see statement *(ii)* of Proposition 3.3.3).

- If (F,ψ^0_F,ψ^2_F) is a comonoidal functor, $(C_1,\Delta_1,\varepsilon_1)$, $(C_2,\Delta_2,\varepsilon_2)$ are two comonoids and $\alpha:C_1\to C_2$ is a morphism of comonoids, then so is $F(\alpha)$ (see statement *(ii)* of Proposition 3.3.4).

- If (F,ψ^0_F,ψ^2_F) is a comonoidal functor and $(C_1,\Delta_1,\varepsilon_1)$, $(C_2,\Delta_2,\varepsilon_2)$ are two comonoids, then $\psi^2_F(C_1,C_2)$ is a morphism of comonoids (see statement *(iii)* of Proposition 3.3.4).

- If (C,Δ,ε) is a cocommutative comonoid then $\Delta : C \to C\otimes C$ is a morphism of comonoids (see statement (iii) of Proposition 3.3.3).

Hence, recalling that

$$\mu^M_{X,Y} := F(r_X \otimes' \mathsf{id}_Y)\circ F((\mathsf{id}_X \otimes' \varepsilon_M)\otimes' \mathsf{id}_Y)\circ (\gamma^M_{X,Y})^{-1}$$

and

$$\gamma^M_{X,Y} := \psi^2_F(X\otimes' M, M\otimes' Y)\circ F(\alpha'_{X,M,M,Y})\circ F((\mathsf{id}_X\otimes' \Delta_M)\otimes' \mathsf{id}_Y)$$

we have that the maps $\mu^M_{M,N}, \mu^N_{M,N}$ are built up of compositions of morphisms of coalgebras, and then they are so. □

Notation 7.8.2. *We shall use – as P. Ševera – the following shortcuts for all $a\in H$, $c\in H_j$, $b\in B$:*

$$\mu^M_{M,N}(a\otimes b) := a\cdot b$$
$$\mu^N_{M,N}(b\otimes c) := b\cdot c.$$

For any $n\in\mathbb{N}$, we have that $B^{\otimes n}$ is a $H^{\otimes n}$–$H_j^{\otimes}$–bimodule given by the usual left and right actions

$$(a_1\otimes\cdots\otimes a_n)\cdot(b_1\otimes\cdots b_n) := (a_1\cdot b_1)\otimes\cdots\otimes(a_n\cdot b_n)$$
$$(b_1\otimes\cdots\otimes b_n)\cdot(c_1\otimes\cdots c_n) := (b_1\cdot c_1)\otimes\cdots\otimes(b_n\cdot c_n)$$

Next, note that the topologically Hopf algebras H and H_j have the same unit element

$$1_B = 1_H = 1_{H_j} = [1\otimes 1]$$

where $[1\otimes 1]$ denotes the equivalence class of $1\otimes 1\in \mathsf{U}(\mathfrak{b})\otimes\mathsf{U}(\mathfrak{b})$ inside the quotient $\frac{\mathsf{U}(\mathfrak{b})\otimes\mathsf{U}(\mathfrak{b})}{\mathfrak{b}\cdot(\mathsf{U}(\mathfrak{b})\otimes\mathsf{U}(\mathfrak{b}))}$. However, as opposed to H and H_j, B is just a coalgebra (and not an algebra), and moreover the element 1_B is not even grouplike with respect to the comultiplicagion Δ_B. This simple observation will be the key idea for the whole reasoning. First we need the following

Lemma 7.8.3. *Consider the following maps*

$$\Lambda : H\to B$$
$$a\mapsto a\cdot 1_B$$
$$P : H_j \to B$$
$$c\mapsto 1_B\cdot c$$

Then their $\hbar$–adic completions $\overline{\Lambda}$ and $\overline{P}$ are invertible, and in particular

$$\overline{\Lambda} \mod \hbar = \overline{P} \mod \hbar = \mathsf{id}_{\mathsf{U}(\mathfrak{b})}.$$

Proof. We compute the zeroth order of the maps $\gamma^M_{M,N}$ and $\gamma^N_{M,N}$. Let $u,v\in\mathsf{U}(\mathfrak{b})$. Then we get – ignoring the completion of the deformed associator –

$$(\gamma^M_{M,N})_0([1\otimes(u\otimes v)]) = \sum_{(u)}[1\otimes u^{(1)}]\otimes[u^{(2)}\otimes v] = \sum_{(u)}[1\otimes u^{(1)}]\otimes[1\otimes S(u^{(2)}v)]$$

Next, for any $\tilde{u}, \tilde{v} \in \mathsf{U}(\mathfrak{b})$ define

$$g([1 \otimes \tilde{u}] \otimes [1 \otimes \tilde{v}]) := \sum_{(\tilde{u})} [1 \otimes \tilde{u}^{(1)} \otimes \tilde{u}^{(2)} \tilde{v}].$$

Hence we have

$$\begin{aligned} g\big((\gamma^M_{M,N})_0([1 \otimes (u \otimes v)])\big) &= \sum_{(u)} g\big([1 \otimes u^{(1)}] \otimes [1 \otimes S(u^{(2)} v)]\big) \\ &= \sum_{(u)} g\big(1 \otimes u^{(1)} \otimes u^{(2)} S(u^{(3)}) v]\big) \\ &= \sum_{(u)} g\big(1 \otimes u^{(1)} \otimes \varepsilon(u^{(2)}) v]\big) \\ &= [1 \otimes u \otimes v] \end{aligned}$$

and

$$\begin{aligned} (\gamma^M_{M,N})_0\big(g([1 \otimes u] \otimes [1 \otimes v])\big) &= \sum_{[u]} (\gamma^M_{M,N})_0\big([1 \otimes u^{(1)} \otimes u^{(2)} v]\big) \\ &= \sum_{(u)} [1 \otimes u^{(1)}] \otimes [u^2 \otimes u^{(3)} v] \\ &= \sum_{(u)} [1 \otimes u^{(1)}] \otimes [1 \otimes S(u^{(2)}) u^{(3)} v] \\ &= [1 \otimes u] \otimes [1 \otimes v] \end{aligned}$$

showing that

$$\big((\gamma^M_{M,N})_0\big)^{-1} = g = \big((\gamma^N_{M,N})_0\big)^{-1}.$$

Finally, we have

$$\begin{aligned} \overline{\Lambda}_0([1 \otimes u]) &= \big((F^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{id}_M \bar{\otimes} \overline{\varepsilon}_M \bar{\otimes} \mathsf{id}_N) \circ (\gamma^M_{M,N})_0\big)^{-1}\big)([1 \otimes u] \otimes [1 \otimes 1]) \\ &= (F^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{id}_M \bar{\otimes} \overline{\varepsilon}_M \bar{\otimes} \mathsf{id}_N)\Big(\sum_{(u)} [1 \otimes u^{(1)} \otimes u^{(2)}]\Big) \\ &= \sum_{(u)} [1 \otimes \varepsilon_M(u^{(1)}) u^{(2)}] = [1 \otimes u] \end{aligned}$$

and similarly

$$\overline{P}_0([1 \otimes u]) = [1 \otimes u]$$

proving the statement. □

We now prove the main result of this Section

Proposition 7.8.4. [Šev16, Rmk.9] *We have that*

(i) *The element* $J := (\overline{\Lambda} \bar{\otimes} \overline{\Lambda})^{-1}(\Delta_B(1_B))$ *is a Hopf algebra twist, i.e. it satisfies*

$$\big((\mathsf{id}_H \bar{\otimes} \overline{\Delta}_H)(J)\big)(1_H \bar{\otimes} J) = \big((\overline{\Delta}_H \bar{\otimes} \mathsf{id}_H)(J)\big)(J \bar{\otimes} 1_H) \tag{7.8.1a}$$

$$(\overline{\varepsilon}_H \bar{\otimes} \mathsf{id}_H)(J) = (\mathsf{id}_H \bar{\otimes} \overline{\varepsilon}_H)(J) = 1. \tag{7.8.1b}$$

(ii) The map

$$\begin{aligned} I : H &\to H_j \\ a &\mapsto P^{-1}(\Lambda(a)) \end{aligned}$$

is an isomorphism of topological Hopf algebras and satisfies

$$\Delta_{H_j}(I(a)) = (I \bar{\otimes} I)\big(J^{-1}\Delta_H(a)J\big). \tag{7.8.2}$$

(iii) The element $\overline{\zeta}_{M\otimes M}(J)$ is a quantization of the Lie bialgebra twist j, i.e.

$$\overline{\zeta}_{M\otimes M}(J) = 1 + \hbar j + \mathcal{O}(\hbar^2).$$

Proof. (i): consider the Sweedler's notation

$$J = \sum_J J' \bar{\otimes} J''.$$

Then –using Lemma 7.8.1 – we have

$$\begin{aligned} \big((\Delta_B \bar{\otimes} \mathsf{id}_B) \circ \Delta_B\big)(1_B) &= (\Delta_B \bar{\otimes} \mathsf{id}_B)\Big(\sum_J (J' \bar{\otimes} J'') \cdot (1_B \bar{\otimes} 1_B)\Big) \\ &= \sum_J (\Delta_B \otimes \mathsf{id}_B)(J' \cdot 1_B \bar{\otimes} J'' \cdot 1_B) \\ &= \sum_J \Delta_B(J' \cdot 1_B) \bar{\otimes} J'' \cdot 1_B \\ &= \sum_J \big(\Delta_H(J')) \cdot \Delta_B(1_B)\big) \bar{\otimes} J'' \cdot 1_B \\ &= \sum_J \big((\Delta_H(J')J) \cdot 1_B \bar{\otimes} 1_B\big) \bar{\otimes} J'' \cdot 1_B \\ &= \sum_J \big((\Delta_H(J') \bar{\otimes} J'')(J \bar{\otimes} 1_H)\big)(1_B \bar{\otimes} 1_B \bar{\otimes} 1_B) \end{aligned}$$

and likewise

$$\big((\mathsf{id}_B \bar{\otimes} \Delta_B) \circ \Delta_B\big)(1_B) = \sum_J \big((J' \bar{\otimes} \Delta_H(J''))(\mathsf{id}_H \bar{\otimes} J)\big)(1_B \bar{\otimes} 1_B \bar{\otimes} 1_B).$$

Therefore, Equation (7.8.1a) follows using the coassociativity of Δ_B and the fact that $\overline{\Lambda} \bar{\otimes} \overline{\Lambda} \bar{\otimes} \overline{\Lambda}$ is invertible. The proof of Equation (7.8.1b) follows by a similar argument (by using the counity of ε_H).

(ii): We first note that I is composition of invertible maps, and then is invertible too (see Lemma 7.8.3). For any $a, a' \in H$ we have

$$(aa') \cdot 1_B = 1_B \cdot I(aa') = a \cdot (a' \cdot 1_B) = (a \cdot 1_B) \cdot I(a') = (1_B \cdot I(a)) \cdot I(a) = 1_B \cdot (I(a)I(a'))$$

hence $P(I(aa')) = P(I(a)I(a'))$, and since P is invertible (see Lemma 7.8.3) we have $I(aa') = I(a)I(a')$. Moreover, we have $1_B \cdot 1_{H_j} = 1_B = 1_H \cdot 1_B = 1_B \cdot I(1_H)$ hence $P(1_{H_j}) = P(I(1_H))$, and using again the fact that P is invertible gives $I(1_H) = 1_{H_j}$, showing that I is a morphism of algebras. Next, for any $a \in H$ we have – using Lemma 7.8.1 –

$$\varepsilon_B(a \cdot 1_B) = \varepsilon_H(a)\varepsilon_B(1_B) = \varepsilon_H(a)$$

and on the other side

$$\varepsilon_B(a \cdot 1_B) = \varepsilon_B(1_B \cdot I(a)) = \varepsilon_B(1_B)\varepsilon_{H_j}(I(a)) = \varepsilon_{H_j}(I(a))$$

giving $\varepsilon_{H_j} \circ I = \varepsilon_H$. Finally, for any $a \in H$ we have – using Lemma 7.8.1 –

$$\begin{aligned}\Delta_B(a \cdot 1_B) &= \Delta_H(a) \cdot \Delta_B(1_B)\\ &= \Delta_H(a) \cdot (J \cdot (1_B \bar{\otimes} 1_B))\\ &= (\Delta_H(a)J)(1_B \bar{\otimes} 1_B)\end{aligned}$$

and on the other side

$$\begin{aligned}\Delta_B(a \cdot 1_B) &= \Delta_B(1_B \cdot I(a))\\ &= \Delta_B(1_B) \cdot \Delta_{H_j}(I(a))\\ &= \big(J \cdot (1_B \bar{\otimes} 1_B)\big) \cdot \Delta_{H_j}(I(a))\\ &= \big(J(I^{-1} \bar{\otimes} I^{-1})(\Delta_{H_j}(I(a)))\big) \cdot (1_B \bar{\otimes} 1_B)\end{aligned}$$

showing that I is an isomorphism of coalgebras satisfying Equation (7.8.2).
(iii): It is clear that $\Delta_B \mod \hbar = \overline{\Delta}_0$, i.e. the usual comultiplication of the universal enveloping algebra $\mathsf{U}(\mathfrak{b})$, and that $\mu^M_{M,N}(u \otimes v) \mod \hbar = uv$. Therefore, $\Delta_B(1) \mod \hbar = 1_B \otimes 1_B$, and then $J \mod \hbar = 1_H \otimes 1_H$. Next, we compute the order 1 term: we have

$$\begin{aligned}(\Delta_B(1_B))_1 &= (\Delta_B)_1(1_B)\\ &= \Big(\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}(M\bar{\otimes}N, M\bar{\otimes}N)} \circ F^{\mathfrak{b}}(\overline{\beta}^{\Phi}_{M,M,N,N})_1 \circ (F^{\mathfrak{b}})^{\Phi}_{\hbar}(\overline{\Delta}_0 \bar{\otimes} \overline{\Delta}_0)\Big)[1 \otimes 1]\\ &= \Big(\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}(M\bar{\otimes}N, M\bar{\otimes}N)} \circ F^{\mathfrak{b}}(\overline{\beta}^{\Phi}_{M,M,N,N})_1\Big)([(1 \otimes 1) \otimes (1 \otimes 1)])\\ &= \psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}(M\bar{\otimes}N, M\bar{\otimes}N)}\big([1_M \otimes t^{\mathfrak{b}}_{M,N}(1_M \otimes 1_N) \otimes 1_N]\big)\end{aligned}$$

where we used that the first order of a Drinfeld associator is zero, and we have ignored the completion of the undeformed associator. Using that $\tilde{\rho}_{\mathsf{U}(\mathfrak{b})}(1) = 0$ and $\tilde{\rho}_{\mathsf{U}(\mathfrak{b})_j}(1) = -j$ we get

$$\begin{aligned}t_{M,N}(1_M \otimes 1_N) &= \sum_{[1_M]} 1_M^{[1]} \otimes 1_M^{[0]} \cdot 1_N + \sum_{[1_N]} 1_N^{[0]} \cdot 1_M \otimes 1_N^{[1]}\\ &= 0 + \sum_{[1_N]} 1_N^{[0]} \otimes 1_N^{[1]}\\ &= -j\end{aligned}$$

Hence

$$\begin{aligned}J_1 &= \sum_j \big([1 \otimes j'] \otimes [j'' \otimes 1]\big)\\ &= -\sum_j \big([1 \otimes j'] \otimes [1 \otimes S_0(j'')]\big)\\ &= +\sum_j \big([1 \otimes j'] \otimes [1 \otimes j'']\big).\end{aligned}$$

Hence

$$\overline{\zeta}_{\mathsf{U}(\mathfrak{b})\bar{\otimes}\mathsf{U}(\mathfrak{b})}\Big(\sum_j \big([1\otimes j']\otimes[1\otimes j'']\big)\Big)=j.$$

□

Remark 7.8.5. *The previous proof was shortly provided by P. Ševera, considerably simplifying the analogous result for the Etingof–Kazhdan quantization due to Enriquez and Halbout* [EH10a].

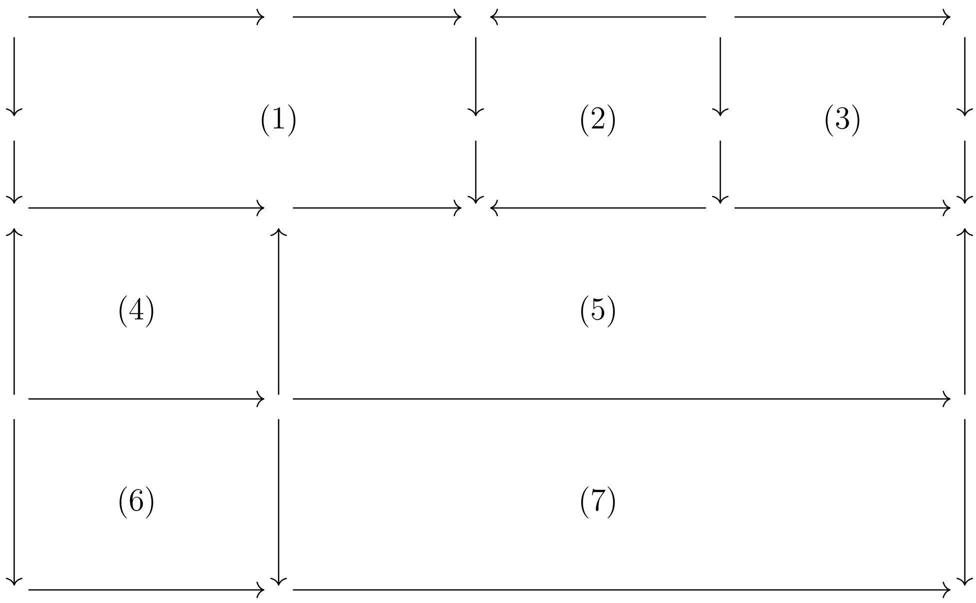
7.9 Quantization of Drinfeld–Yetter modules

Lemma 7.9.1. *Let $\mathscr{D}$ be a braided monoidal category, $(M,\Delta_M,\varepsilon_M)$ be a cocommutative comonoid in $\mathscr{D}$, X be an object of $\mathscr{D}$, and $\Delta_{M\otimes X}: M\otimes X\to (M\otimes X)\otimes(M\otimes M)$ given by the composition*

$$\Delta_{M\otimes X}\coloneqq \beta_{M,M,X,M}\circ \mathsf{id}_{M\otimes M}\otimes c^{-1}_{X,M}\circ\alpha_{M,M,M,X}\circ a_{M,M,M}\otimes\mathsf{id}_X\circ(\Delta_M\otimes\mathsf{id}_M)\otimes\mathsf{id}_X\circ\Delta_M\otimes\mathsf{id}_X. \quad (7.9.1)$$

Then $(M\otimes X,\Delta_{M\otimes X})$ is in $\mathsf{Comod}(M\otimes M)$, i.e. is a right $M\otimes M$–comodule.

Proof. In order to give the proof we may suppose that that $\mathscr{D}$ is strict. The fact that the pair $(M\otimes X,\Delta_{M\otimes X})$ satisfies the commutativity of the diagram (3.4.1) follows by considering the joint diagram



whose seven subdiagrams are the following

(1)

$$\begin{array}{ccccc}
M\otimes X & \xrightarrow{\Delta_M\otimes\mathsf{id}_X} & M^{\otimes 2}\otimes X & \xrightarrow{\Delta_M\otimes_{M\otimes X}} & M^{\otimes 3}\otimes X\\
{\scriptstyle\Delta_M\otimes\mathsf{id}_X}\downarrow & & & & \downarrow{\scriptstyle\Delta_M\otimes\mathsf{id}_{M\otimes M\otimes X}}\\
M^{\otimes 2}\otimes X & & & & M^{\otimes 4}\otimes X\\
{\scriptstyle\Delta_M\otimes\mathsf{id}_{M\otimes X}}\downarrow & & & & \downarrow{\scriptstyle\Delta_M\otimes\mathsf{id}_{M\otimes M\otimes M\otimes X}}\\
M^{\otimes 3}\otimes X & \xrightarrow{\mathsf{id}_M\otimes\Delta_M\otimes\Delta_M\otimes\mathsf{id}_X} & M^{\otimes 5}\otimes X & \xrightarrow{\mathsf{id}_{M\otimes M}\otimes c_{M,M}\otimes\mathsf{id}_{M\otimes X}} & M^{\otimes 5}\otimes X
\end{array}$$

(2)

$$\begin{array}{ccc}
M^{\otimes 3} \otimes X & \xleftarrow{\mathsf{id}_{M\otimes M}\otimes c_{X,M}} & M^{\otimes 2} \otimes X \otimes M \\
{\scriptstyle \Delta_M \otimes \mathsf{id}_{M\otimes M\otimes X}}\downarrow & & \downarrow{\scriptstyle \Delta_M \otimes \mathsf{id}_{M\otimes X\otimes M}} \\
M^{\otimes 4} \otimes X & & M^{\otimes 3} \otimes X \otimes M \\
{\scriptstyle \Delta_M \otimes \mathsf{id}_{M^{\otimes 3}\otimes X}}\downarrow & & \downarrow{\scriptstyle \Delta_M \otimes \mathsf{id}_{M\otimes M\otimes X\otimes M}} \\
M^{\otimes 5} \otimes X & \xleftarrow{\mathsf{id}_{M^{\otimes 4}}\otimes c_{X,M}} & M^{\otimes 4} \otimes X \otimes M
\end{array}$$

(3)

$$\begin{array}{ccc}
M^{\otimes 2} \otimes X \otimes M & \xrightarrow{\mathsf{id}_M\otimes c_{M,X}\otimes \mathsf{id}_M} & M \otimes X \otimes M^{\otimes 2} \\
{\scriptstyle \Delta_M \otimes \mathsf{id}_{M\otimes X\otimes M}}\downarrow & & \downarrow{\scriptstyle \Delta_M \otimes \mathsf{id}_{X\otimes M^{\otimes 2}}} \\
M^{\otimes 3} \otimes X \otimes M & & M^{\otimes 2} \otimes X \otimes M^{\otimes 2} \\
{\scriptstyle \Delta_M \otimes \mathsf{id}_{M^{\otimes 2}\otimes X\otimes M}}\downarrow & & \downarrow{\scriptstyle \Delta_M \otimes \mathsf{id}_{M\otimes X\otimes M^{\otimes 2}}} \\
M^{\otimes 4} \otimes X \otimes M & \xrightarrow{\mathsf{id}_{M^{\otimes 3}}\otimes c_{M,X}\otimes \mathsf{id}_M} & M^{\otimes 3} \otimes X \otimes M^{\otimes 2}
\end{array}$$

(4)

$$\begin{array}{ccc}
M^{\otimes 3} \otimes X & \xrightarrow{\mathsf{id}_M\otimes \Delta_M\otimes \Delta_M\otimes \mathsf{id}_X} & M^{\otimes 5} \otimes X \\
{\scriptstyle \mathsf{id}_{M\otimes M}\otimes c_{X,M}}\uparrow & & \uparrow{\scriptstyle \mathsf{id}_{M^{\otimes 3}}\otimes c_{X,M\otimes M}} \\
M^{\otimes 2} \otimes X \otimes M & \xrightarrow{\mathsf{id}_M\otimes \Delta_M\otimes \mathsf{id}_X\otimes \Delta_M} & M^{\otimes 3} \otimes X \otimes M
\end{array}$$

(5)

$$\begin{array}{ccccccc}
M^{\otimes 5} \otimes X & \xrightarrow{\mathsf{id}_{M^{\otimes 2}}\otimes c_{M,M}\otimes \mathsf{id}_{M\otimes X}} & M^{\otimes 5} \otimes X & \xleftarrow{\mathsf{id}_{M^{\otimes 4}}\otimes c_{M,X}} & M^{\otimes 4} \otimes X \otimes M & \xrightarrow{\mathsf{id}_{M^{\otimes 3}}\otimes c_{M,X}\otimes \mathsf{id}_M} & M^{\otimes 3} \otimes X \otimes M^{\otimes 2} \\
{\scriptstyle \mathsf{id}_{M^{\otimes 3}}\otimes c_{X,M\otimes M}}\uparrow & & & & & & \uparrow{\scriptstyle \mathsf{id}_{M^{\otimes 2}}\otimes c_{X,M}\otimes \mathsf{id}_{M^{\otimes 2}}} \\
M^{\otimes 3} \otimes X \otimes M^{\otimes 2} & & & \xrightarrow{\mathsf{id}_{M\otimes M}\otimes c_{M,X\otimes M}\otimes \mathsf{id}_M} & & & M^{\otimes 2} \otimes X \otimes M^{\otimes 3}
\end{array}$$

(6)

$$\begin{array}{ccc}
M^{\otimes 2} \otimes X \otimes M & \xrightarrow{\mathsf{id}_M\otimes \Delta_M\otimes \mathsf{id}_X\otimes \Delta_M} & M^{\otimes 3} \otimes X \otimes M^{\otimes 2} \\
{\scriptstyle \mathsf{id}_M\otimes c_{M,X}\otimes \mathsf{id}_M}\downarrow & & \downarrow{\scriptstyle \mathsf{id}_M\otimes c_{M\otimes M,X}\otimes \mathsf{id}_{M\otimes M}} \\
M \otimes X \otimes M^{\otimes 2} & \xrightarrow{\mathsf{id}_{M\otimes X}\otimes \Delta_M\otimes \Delta_M} & M \otimes X \otimes M^{\otimes 4}
\end{array}$$

(7)

$$\begin{array}{ccc}
M^{\otimes 3} \otimes X \otimes M^{\otimes 2} & \xrightarrow{\mathsf{id}_{M\otimes M}\otimes c_{M,X\otimes M}\otimes \mathsf{id}_M} & M^{\otimes 2} \otimes X \otimes M^{\otimes 3} \\
{\scriptstyle \mathsf{id}_M\otimes c_{M\otimes M,X}\otimes \mathsf{id}_{M\otimes M}}\downarrow & & \downarrow{\scriptstyle \mathsf{id}_M\otimes c_{M,X}\otimes \mathsf{id}_{M^{\otimes 3}}} \\
M \otimes X \otimes M^{\otimes 4} & \xrightarrow{\mathsf{id}_{M\otimes X\otimes M}\otimes c_{M,M}\otimes \mathsf{id}_M} & M \otimes X \otimes M^{\otimes 4}
\end{array}$$

here the first follows from the commutativity and coassociativity of Δ_M, the second and the third follow from Equation (2.5.1a), the fourth and the sixth follow from the naturality of the braiding, the fifth from Equation (2.5.3), and the seventh by Equation (2.2.1).

We now show that $(M \otimes X, \Delta_{M\otimes X})$ satisfies (the strict counterpart of) the commutativity of the diagram (3.4.2), i.e. that $(\mathsf{id}_{M\otimes X} \otimes \varepsilon_{M\otimes M}) \circ \Delta_{M\otimes X} = \mathsf{id}_{M\otimes X}$. We have

$$\begin{aligned}
(\mathsf{id}_{M\otimes X} \otimes \varepsilon_{M\otimes M})\Delta_{M\otimes X} &= (\mathsf{id}_{M^{\otimes 2}} \otimes \varepsilon_M \otimes \varepsilon_M)(\mathsf{id}_M \otimes c_{M,X} \otimes \mathsf{id}_M)(\mathsf{id}_{M^{\otimes 2}} \otimes c^{-1}_{M,X})(\Delta^{(2)}_M \otimes \mathsf{id}_X) \\
&= (\mathsf{id}_M \otimes ((\mathsf{id}_X \otimes \varepsilon_M) \circ c_{M,X}) \otimes \varepsilon_M)(\mathsf{id}_{M^{\otimes 2}} \otimes c^{-1}_{M,X})(\Delta^{(2)}_M \otimes \mathsf{id}_X) \\
&= (\mathsf{id}_M \otimes (c_{I,X} \circ (\varepsilon \otimes \mathsf{id}_X)) \otimes \varepsilon_M)(\mathsf{id}_{M^{\otimes 2}} \otimes c^{-1}_{M,X})(\Delta^{(2)}_M \otimes \mathsf{id}_X) \\
&= (\mathsf{id}_M \otimes \varepsilon_M \otimes \mathsf{id}_X \otimes \varepsilon_M)(\mathsf{id}_{M^{\otimes 2}} \otimes c^{-1}_{M,X})(\Delta^{(2)}_M \otimes \mathsf{id}_X) \\
&= ((\mathsf{id}_M \otimes \varepsilon)(c^{-1}_{X,I}(\varepsilon_M \otimes \mathsf{id}_X)))(\Delta^{(2)}_M \otimes \mathsf{id}_X) \\
&= (\mathsf{id}_M \otimes \varepsilon_M \otimes \varepsilon_M \otimes \mathsf{id}_X)(\Delta^{(2)}_M \otimes \mathsf{id}_X) \\
&= \mathsf{id}_M \otimes \mathsf{id}_X = \mathsf{id}_{M\otimes X}
\end{aligned}$$

where we used the naturality of the braiding, Equation (2.5.1a), and the identities $c_{I,X} = \mathsf{id}_X = c_{X,I}$ and $(\varepsilon_M \otimes \mathsf{id}_M)\Delta_M = \mathsf{id}_M$. □

Remark 7.9.2. *Let $\mathscr{C}, \mathscr{D}$ be two braided monoidal categories, $(M, \Delta_M, \varepsilon_M)$ a cocommutative comonoid object in $\mathscr{D}$ and $(F, \psi^0_F, \psi^0_F) : \mathscr{D} \to \mathscr{C}$ be a M–adapted functor. Then for any object X of $\mathscr{C}$, we have both a left $F(M \otimes' M)$–module and right $F(M \otimes' M)$–comodule structure on the object $F(M \otimes' X)$, where:*

- *The left $F(M \otimes' M)$–module structure is the multiplication of X, M along the comonoid M $\mu^M_{X,M}$, see §7.2.*
- *The right $F(M \otimes' M)$–comodule structure is given by $\psi^2_F(M \otimes' X, M \otimes' M) \circ F(\Delta_{M\otimes' X})$, see Lemma 7.9.1 and Proposition 3.4.4.*

We shall prove that such left module and right comodule structures satisfy the compatibility relation (3.6.1). Let $\mathscr{C}, \mathscr{D}$ be two braided monoidal categories, $(M, \Delta_M, \varepsilon_M)$ a cocommutative comonoid object in $\mathscr{D}$ and $(F, \psi^0_F, \psi^0_F) : \mathscr{D} \to \mathscr{C}$ be a M–adapted functor. Denote by G the (braided and strongly comonoidal) functor given by the composition $G = F \circ M\otimes'$–. Let X, Y be in $\mathsf{Obj}(\mathscr{D})$ and set $V := F(M \otimes' X) = G(X)$, $W := F(M \otimes' Y) = G(Y)$ and $H := F(M \otimes' M) = G(M)$. Consider the natural morphism given by the composition

$$\tilde{c}_{X,Y} := \psi^2_G(Y, X) \circ F\big(\mathsf{id}_M \otimes' (c'_{Y,X})^{-1}\big) \circ \big(\psi^2_G(X, Y)\big)^{-1} : V \otimes W \to W \otimes V \tag{7.9.2}$$

i.e. such that the following diagram commutes

$$\begin{array}{ccc}
V \otimes W & \xrightarrow{\tilde{c}_{X,Y}} & W \otimes V \\
{\scriptstyle \psi^2_G(X,Y)}\uparrow & & \uparrow{\scriptstyle \psi^2_G(Y,X)} \\
Z & \xrightarrow{F(\mathsf{id}_M \otimes' (c'_{X,Y})^{-1})} & F(M \otimes' (Y \otimes' X))
\end{array}$$

Proposition 7.9.3. *Under the previous hypotheses and notation, we have*

(i) The morphism $G(c'_{X,Y}) : F(M \otimes' (X \otimes' Y)) \to F(M \otimes' (Y \otimes' X))$ is a morphism of left H–modules.

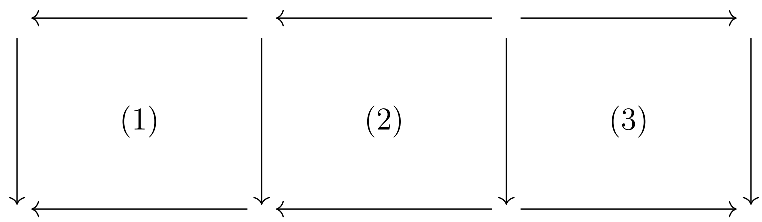
(ii) The morphism $\psi^2_G(X, Y) : F(M \otimes' (X \otimes' Y)) \to F(M \otimes' X) \otimes F(M \otimes' Y)$ is a morphism of left H–modules.

(iii) *The morphism* $\tilde{c}_{X,Y} : V \otimes W \to W \otimes V$ *is a morphism of left* H*–modules.*

(iv) *The following identity holds*

$$\tilde{c}_{M,Y} \circ (\eta_{F(M\otimes' M)} \otimes \mathsf{id}_W) \circ \ell_W^{-1} = \Delta_W. \tag{7.9.3}$$

Proof. In order to give the proof we may suppose that $\mathscr{C}$ and $\mathscr{D}$ are strict.
(i): the claim follows by considering the diagram



whose three subdiagrams are the following

(1)

$$\begin{array}{ccc}
F(M\otimes' M)\otimes F(M\otimes' X\otimes' Y) & \xleftarrow{\psi_F^2(M\otimes' M, M\otimes' X\otimes' Y)} & F(M^{\otimes' 3}\otimes' X\otimes' Y) \\
{\scriptstyle \mathsf{id}_{F(M\otimes' M)}\otimes F(\mathsf{id}_M\otimes' c'_{X,Y})}\downarrow & & \downarrow{\scriptstyle F(\mathsf{id}_{M^{\otimes' 3}}\otimes c'_{X,Y})} \\
F(M\otimes' M)\otimes F(M\otimes' Y\otimes' X) & \xleftarrow{\psi_F^2(M\otimes' M, M\otimes' Y\otimes' X)} & F(M^{\otimes' 3}\otimes' Y\otimes' X)
\end{array}$$

(2)

$$\begin{array}{ccc}
F(M^{\otimes' 3}\otimes' X\otimes' Y) & \xleftarrow{F(\mathsf{id}_M\otimes'\Delta_M\otimes'\mathsf{id}_{X\otimes' Y})} & F(M^{\otimes' 2}\otimes' X\otimes' Y) \\
{\scriptstyle F(\mathsf{id}_{M^{\otimes' 3}}\otimes c'_{X,Y})}\downarrow & & \downarrow{\scriptstyle F(\mathsf{id}_{M^{\otimes' 2}}\otimes c'_{X,Y})} \\
F(M^{\otimes' 3}\otimes' Y\otimes' X) & \xleftarrow{F(\mathsf{id}_M\otimes'\Delta_M\otimes'\mathsf{id}_{Y\otimes' X})} & F(M^{\otimes' 2}\otimes' Y\otimes' X)
\end{array}$$

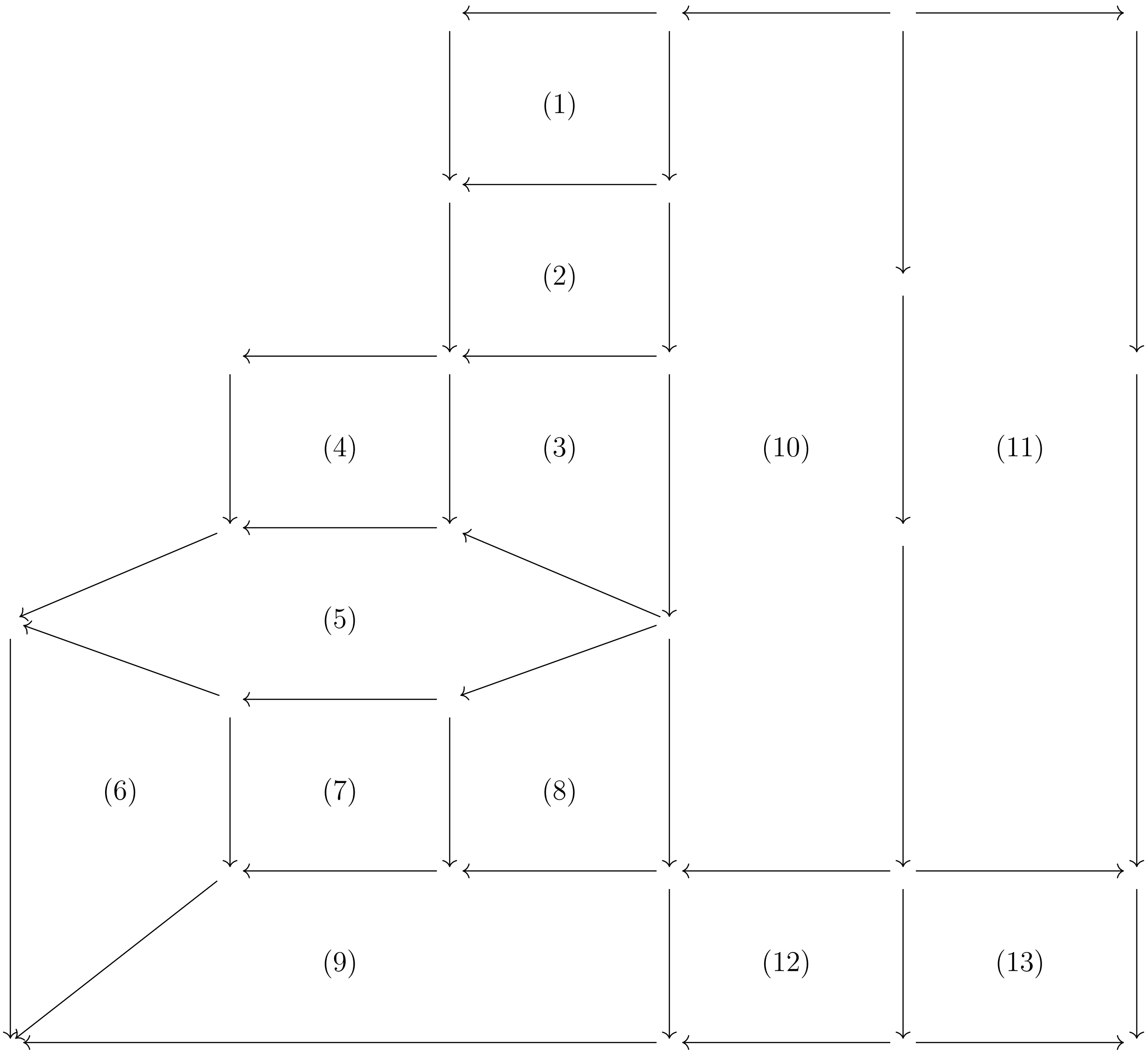
(3)

$$\begin{array}{ccc}
F(M^{\otimes' 2}\otimes' X\otimes' Y) & \xrightarrow{F(\mathsf{id}_M\otimes'\varepsilon_M\otimes'\mathsf{id}_{X\otimes' Y})} & F(M\otimes' X\otimes' Y) \\
{\scriptstyle F(\mathsf{id}_{M^{\otimes' 2}}\otimes c'_{X,Y})}\downarrow & & \downarrow{\scriptstyle F(\mathsf{id}_M\otimes' c'_{X,Y})} \\
F(M^{\otimes' 2}\otimes' Y\otimes' X) & \xrightarrow{F(\mathsf{id}_M\otimes'\varepsilon_M\otimes'\mathsf{id}_{Y\otimes' X})} & F(M\otimes' Y\otimes' X)
\end{array}$$

here the first follows from the naturality of ψ_F^2, while the second and the third follow by Equation (2.5.1a).
(ii): Set $Z \coloneqq F(M\otimes' X\otimes' Y)$. Recalling the module structure of $V\otimes W$ given by Equation (3.5.2), we have to show that the following diagram commutes

$$\begin{array}{ccc}
H\otimes Z & \xrightarrow{\mu_Z} & Z \\
{\scriptstyle \Delta_H\otimes\psi_G^2(X,Y)}\downarrow & & \\
H\otimes H\otimes V\otimes W & & \downarrow{\scriptstyle \psi_G^2(X,Y)} \\
{\scriptstyle \beta_{H,H,V,W}}\downarrow & & \\
H\otimes V\otimes H\otimes W & \xrightarrow{\mu_V\otimes\mu_W} & V\otimes W
\end{array}$$

Recalling that the comonoidal structure of G is given by Proposition 2.1.3, we obtain a diagram with the following shape



The following eight diagrams commute thanks to the naturality of ψ_F^2:

(1)

$$\begin{array}{ccc} H \otimes Z & \xleftarrow{\psi_F^2(M\otimes' M, M\otimes' X\otimes' Y)} & F(M^{\otimes' 3} \otimes' X \otimes' Y) \\ \Big\downarrow {\scriptstyle F(\Delta_M^{(2)})\otimes F(\Delta_M \otimes' \mathsf{id}_{X\otimes' Y})} & & \Big\downarrow {\scriptstyle F(\Delta_M^{(3)} \otimes' \mathsf{id}_{X\otimes' Y})} \\ H^{\otimes 2} \otimes F(M^{\otimes' 2} \otimes' X \otimes' Y) & \xleftarrow{\psi_F^2(M^{\otimes' 4}, M^{\otimes' 2}\otimes' X\otimes' Y)} & F(M^{\otimes' 6} \otimes' X \otimes' Y) \end{array}$$

(2)

$$\begin{array}{ccc} H^{\otimes 2} \otimes F(M^{\otimes' 2} \otimes' X \otimes' Y) & \xleftarrow{\psi_F^2(M^{\otimes' 4}, M^{\otimes' 2}\otimes' X\otimes' Y)} & F(M^{\otimes' 6} \otimes' X \otimes' Y) \\ \Big\downarrow {\scriptstyle F(\beta'_{M,M,M,M})\otimes F(\mathsf{id}_{M^{\otimes' 2}\otimes' X\otimes' Y})} & & \Big\downarrow {\scriptstyle F(\beta'_{M,M,M,M} \otimes' \mathsf{id}_{M^{\otimes' 2}\otimes' X\otimes' Y})} \\ H^{\otimes 2} \otimes F(M^{\otimes' 2} \otimes' X \otimes' Y) & \xleftarrow{\psi_F^2(M^{\otimes' 4}, M^{\otimes' 2}\otimes' X\otimes' Y)} & F(M^{\otimes' 6} \otimes' X \otimes' Y) \end{array}$$

(3)

$$\begin{array}{ccc}
H \otimes H \otimes F(M^{\otimes' 2} \otimes' X \otimes' Y) & \xleftarrow{\psi_F^2(M\otimes' M, M\otimes' M)\otimes' \mathrm{id}_{M^{\otimes' 2}\otimes' X \otimes' Y}} & H^{\otimes 2} \otimes F(M^{\otimes' 2} \otimes' X \otimes' Y) \\
{\scriptstyle \mathrm{id}_{H\otimes H}\otimes' F(\beta'_{M,M,X,Y})}\downarrow & & \downarrow{\scriptstyle F(\mathrm{id}_{M^{\otimes' 4}}\otimes' \beta'_{M,M,X,Y})} \\
H \otimes H \otimes F(M \otimes' X \otimes' M \otimes' Y) & \xleftarrow{\psi_F^2(M\otimes' M, M\otimes' M)\otimes' \mathrm{id}_{M\otimes' X\otimes' M\otimes' Y}} & H^{\otimes 2} \otimes F(M \otimes' X \otimes' M \otimes' Y)
\end{array}$$

(4)

$$\begin{array}{ccc}
H^{\otimes 2} \otimes F(M^{\otimes' 2} \otimes' X \otimes' Y) & \xleftarrow{\psi_F^2(M^{\otimes' 4}, M^{\otimes' 2}\otimes' X\otimes' Y)} & F(M^{\otimes' 6} \otimes' X \otimes' Y) \\
{\scriptstyle F(\mathrm{id}_{M^{\otimes' 4}}\otimes' \beta'_{M,M,X,Y})}\downarrow & & \downarrow{\scriptstyle F(\mathrm{id}_{M^{\otimes' 4}}\otimes' \beta'_{M,M,X,Y})} \\
H^{\otimes 2} \otimes F(M \otimes' X \otimes' M \otimes' Y) & \xleftarrow{\psi_F^2(M^{\otimes' 4}, M\otimes' X\otimes' M\otimes' Y)} & F(M^{\otimes' 5} \otimes' X \otimes' M \otimes' Y)
\end{array}$$

(7)

$$\begin{array}{ccc}
H \otimes F(M^{\otimes' 3} \otimes' X) \otimes W & \xleftarrow{\mathrm{id}_H \otimes \psi_F^2(M^{\otimes' 3}\otimes' X, M\otimes' Y)} & H \otimes F(M^{\otimes' 3} \otimes' X \otimes' M \otimes' Y) \\
{\scriptstyle \mathrm{id}_H\otimes F(c'_{M\otimes' M, M\otimes' X})\otimes \mathrm{id}_W}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_H\otimes F(c'_{M\otimes' M, M\otimes' X}\otimes' \mathrm{id}_{M\otimes' Y})} \\
H \otimes F(M \otimes' X \otimes' M^{\otimes' 2}) \otimes W & \xleftarrow{\mathrm{id}_H \otimes \psi_F^2(M\otimes' X\otimes' M^{\otimes' 2}, M\otimes' Y)} & H \otimes F(M \otimes' X \otimes' M^{\otimes' 2} \otimes' M \otimes' Y)
\end{array}$$

(8)

$$\begin{array}{ccc}
H \otimes F(M^{\otimes' 3} \otimes' X \otimes' M \otimes' Y) & \xleftarrow{\psi_F^2(M^{\otimes' 2}, M^{\otimes' 3}\otimes' X\otimes' M\otimes' Y)} & F(M^{\otimes' 5} \otimes' X \otimes' M \otimes' Y) \\
{\scriptstyle \mathrm{id}_H\otimes F(c'_{M\otimes' M, M\otimes' X}\otimes' \mathrm{id}_{M\otimes' Y})}\downarrow & & \downarrow{\scriptstyle F(\mathrm{id}_M\otimes \beta'_{M,M^{\otimes' 2},M\otimes' X,M}\otimes' \mathrm{id}_Y)} \\
H \otimes F(M \otimes' X \otimes' M^{\otimes' 2} \otimes' M \otimes' Y) & \xleftarrow{\psi_F^2(M^{\otimes' 2}, M\otimes' X\otimes' M^{\otimes' 3}\otimes' Y)} & F(M^{\otimes' 3} \otimes' X \otimes' M^{\otimes' 3} \otimes' Y)
\end{array}$$

(12)

$$\begin{array}{ccc}
F(M^{\otimes' 3} \otimes X \otimes' M^{\otimes' 3} \otimes' Y) & \xleftarrow{F(\mathrm{id}_M\otimes' \Delta_M \otimes' \mathrm{id}_{X\otimes' M}\otimes \Delta_M \otimes' \mathrm{id}_Y)} & F(M^{\otimes' 2} \otimes X \otimes' M^{\otimes' 2} \otimes' Y) \\
{\scriptstyle \psi_F^2(M^{\otimes' 3}\otimes' X, M^{\otimes' 3}\otimes' Y)}\downarrow & & \downarrow{\scriptstyle \psi_F^2(M^{\otimes' 2}\otimes' X, M^{\otimes' 2}\otimes' Y)} \\
F(M^{\otimes' 3} \otimes' X) \otimes F(M^{\otimes' 3} \otimes' Y) & \xleftarrow{F(\mathrm{id}_M\otimes' \Delta_M\otimes' \mathrm{id}_X)\otimes F(\mathrm{id}_M\otimes' \Delta_M\otimes' \mathrm{id}_Y)} & F(M^{\otimes' 2} \otimes' X) \otimes F(M^{\otimes' 2} \otimes' Y)
\end{array}$$

(13)

$$\begin{array}{ccc}
F(M^{\otimes' 2} \otimes X \otimes' M^{\otimes' 2} \otimes' Y) & \xleftarrow{F(\mathrm{id}_M\otimes' \varepsilon_M \otimes' \mathrm{id}_{X\otimes' M}\otimes' \varepsilon_M \otimes' \mathrm{id}_Y)} & F(M \otimes' X \otimes' M \otimes' Y) \\
{\scriptstyle \psi_F^2(M^{\otimes' 2}\otimes' X, M^{\otimes' 2}\otimes' Y)}\downarrow & & \downarrow{\scriptstyle \psi_F^2(M\otimes' X, M\otimes' Y)} \\
F(M^{\otimes' 2} \otimes' X) \otimes F(M^{\otimes' 2} \otimes' Y) & \xleftarrow{F(\mathrm{id}_M\otimes' \varepsilon_M\otimes' X)\otimes F(\mathrm{id}_M\otimes' \varepsilon_M\otimes' \mathrm{id}_X)} & V \otimes W
\end{array}$$

while, using the fact that (F, ψ_F^0, ψ_F^2) is a braided comonoidal functor, we have that the following three diagrams commute:

(5)

$$\begin{array}{ccc}
H\otimes H\otimes F(M\otimes' X\otimes' M\otimes' Y) & \xleftarrow{\psi_F^2(M^{\otimes'2},M^{\otimes'2})\otimes'\mathsf{id}_{F(M\otimes' X\otimes' M\otimes' Y)}} & F(M^{\otimes'4})\otimes F(M\otimes' X\otimes' M\otimes' Y) \\
\Big\downarrow{\scriptstyle \mathsf{id}_{H\otimes H}\otimes\psi_F^2(M\otimes' X,M\otimes' Y)} & & {\scriptstyle \psi_F^2(M^{\otimes'4},M\otimes' X\otimes' M\otimes' Y)}\Big\uparrow \\
H\otimes H\otimes V\otimes W & & F(M^{\otimes'5}\otimes' X\otimes' M\otimes' Y) \\
\Big\uparrow{\scriptstyle \mathsf{id}_H\otimes\psi_F^2(M^{\otimes'2},M\otimes' Y)\otimes\mathsf{id}_W} & & {\scriptstyle \psi_F^2(M^{\otimes'2},M^{\otimes'3}\otimes' X\otimes' M\otimes' Y)}\Big\downarrow \\
H\otimes F(M^{\otimes'3}\otimes' X)\otimes W & \xleftarrow{\mathsf{id}_H\otimes\psi_F^2(M^{\otimes'3}\otimes' X,M\otimes' Y)} & H\otimes F(M^{\otimes'3}\otimes' X\otimes' M\otimes' Y)
\end{array}$$

(6)

$$\begin{array}{ccc}
H\otimes H\otimes V\otimes W & \xleftarrow{\mathsf{id}_H\otimes\psi_F^2(M^{\otimes'2},M\otimes' X)\otimes\mathsf{id}_W} & H\otimes F(M^{\otimes'3}\otimes' X)\otimes W \\
{\scriptstyle \beta_{H,H,V,W}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathsf{id}_H\otimes c'_{M\otimes' M,M\otimes' X}\otimes\mathsf{id}_W} \\
H\otimes V\otimes H\otimes W & \xleftarrow{\mathsf{id}_H\otimes\psi_F^(M\otimes' X,M^{\otimes'2})\otimes\mathsf{id}_W} & H\otimes F(M\otimes' X\otimes' M^{\otimes'2})\otimes W
\end{array}$$

(9)

$$\begin{array}{ccc}
H\otimes F(M\otimes' X\otimes' M^{\otimes'3}\otimes' Y) & \xleftarrow{\psi_F^2(M^{\otimes'2},M\otimes' X\otimes' M^{\otimes'3}\otimes' Y)} & F(M^{\otimes'3}\otimes' X\otimes' M^{\otimes'3}\otimes' Y) \\
{\scriptstyle \mathsf{id}_H\otimes F(M\otimes' X\otimes' M^{\otimes'2},M\otimes' Y)}\Big\downarrow & & \\
H\otimes F(M\otimes' X\otimes' M^{\otimes'3}) & & \Big\downarrow{\scriptstyle \psi_F^2(M^{\otimes'3}\otimes' X,M^{\otimes'3}\otimes' Y)} \\
{\scriptstyle \mathsf{id}_H\otimes\psi_{M\otimes' X,M^{\otimes'2}}\otimes\mathsf{id}_W}\Big\downarrow & & \\
H\otimes V\otimes H\otimes W & \xleftarrow{\psi_F^2(M^{\otimes'2},M\otimes' X)\otimes\psi_F^2(M^{\otimes'2},M\otimes' Y)} & F(M^{\otimes'3}\otimes' X)\otimes F(M^{\otimes'3}\otimes' Y)
\end{array}$$

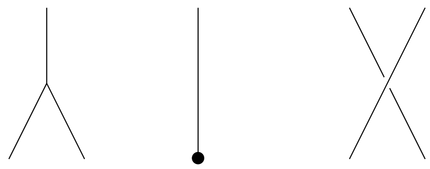
Finally, we treat the subdiagrams (10) and (11) with a diagramatic approach. The diagram (10) reads

$$\begin{array}{ccc}
F(M^{\otimes'3}\otimes' X\otimes' Y) & \xleftarrow{F(\mathsf{id}_M\otimes'\Delta_M\otimes'\mathsf{id}_{X\otimes' Y})} & F(M^{\otimes'2}\otimes' X\otimes' Y) \\
{\scriptstyle F(\Delta_M^{(3)}\otimes\mathsf{id}_{X\otimes' Y})}\Big\downarrow & & \Big\downarrow{\scriptstyle F(\Delta_M\otimes'\Delta_M\otimes\mathsf{id}_{X\otimes' Y})} \\
F(M^{\otimes'6}\otimes' X\otimes' Y) & & F(M^{\otimes'4}\otimes' X\otimes' Y) \\
{\scriptstyle F(\beta'_{M,M,M,M}\otimes'\mathsf{id}_{M^{\otimes'2}\otimes' X\otimes' Y})}\Big\downarrow & & \Big\downarrow{\scriptstyle F(\mathsf{id}_M\otimes' c'_{M,M}\otimes' c'_{M,X}\otimes'\mathsf{id}_Y)} \\
F(M^{\otimes'6}\otimes' X\otimes' Y) & & F(M^{\otimes'3}\otimes' X\otimes' M\otimes' Y) \\
{\scriptstyle \mathsf{id}_{M^{\otimes'4}}\otimes'\beta'_{M,M,X,X}}\Big\downarrow & & \\
F(M^{\otimes'5}\otimes' X\otimes' M\otimes' Y) & & \Big\downarrow{\scriptstyle F(\mathsf{id}_{M^{\otimes'2}}\otimes' c'_{M,X}\otimes'\mathsf{id}_{M\otimes' X})} \\
{\scriptstyle F(\mathsf{id}_{M^{\otimes'2}}\otimes' c'_{M^{\otimes'2},M\otimes' X}\otimes'\mathsf{id}_{M\otimes' Y})}\Big\downarrow & & \\
F(M^{\otimes'3}\otimes' X\otimes' M^{\otimes'3}\otimes' Y) & \xleftarrow{F(\mathsf{id}_M\otimes'\Delta_M\otimes'\mathsf{id}_{X\otimes' M}\otimes'\Delta_M\otimes'\mathsf{id}_Y)} & F(M^{\otimes'2}\otimes' X\otimes' M^{\otimes'2}\otimes' Y)
\end{array}$$

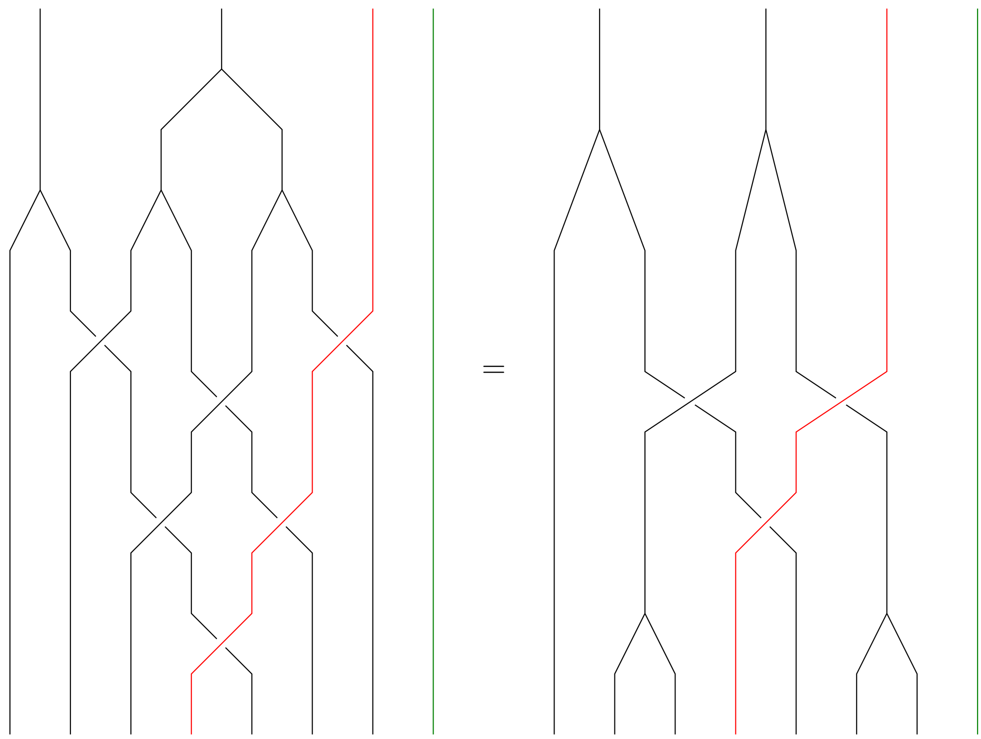
while the diagram (11) reads

$$\begin{array}{ccc}
F(M^{\otimes' 2} \otimes' X \otimes' Y) & \xrightarrow{F(\mathsf{id}_M \otimes' \varepsilon_M \otimes' \mathsf{id}_{X \otimes' Y})} & Z \\
\downarrow {\scriptstyle F(\Delta_M \otimes' \Delta_M \otimes \mathsf{id}_{X \otimes' Y})} & & \downarrow {\scriptstyle F(\Delta_M \otimes' \mathsf{id}_{X \otimes' Y})} \\
F(M^{\otimes' 4} \otimes' X \otimes' Y) & & F(M^{\otimes' 2} \otimes' X \otimes' Y) \\
\downarrow {\scriptstyle F(\mathsf{id}_M \otimes' c'_{M,M} \otimes' c'_{M,X} \otimes' \mathsf{id}_Y)} & & \\
F(M^{\otimes' 3} \otimes' X \otimes' M \otimes' Y) & & \downarrow {\scriptstyle F(\mathsf{id}_M \otimes' c'_{M,X} \otimes' \mathsf{id}_Y)} \\
\downarrow {\scriptstyle F(\mathsf{id}_{M^{\otimes' 2}} \otimes' c'_{M,X} \otimes' \mathsf{id}_{M \otimes' X})} & & \\
F(M^{\otimes' 2} \otimes' X \otimes' M^{\otimes' 2} \otimes' Y) & \xrightarrow{F(\mathsf{id}_M \otimes' \varepsilon_M \otimes' \mathsf{id}_{X \otimes' M} \otimes' \varepsilon_M \otimes' \mathsf{id}_Y)} & F(M \otimes' X \otimes' M \otimes' Y)
\end{array}$$

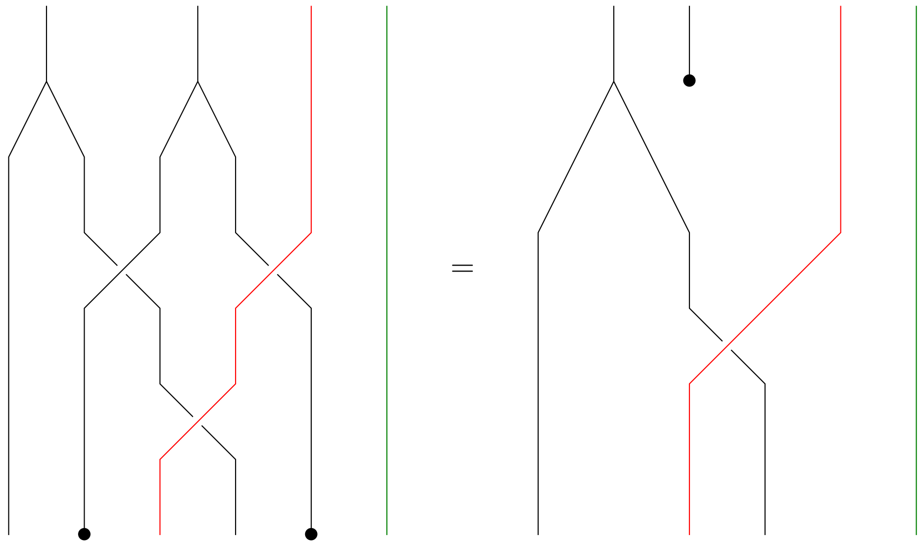
Let us denote the maps Δ_M, ε_M and c' of $\mathscr{D}$ respectively with the pictures



which are read from up to down, see [Kas12, XIV.1] for more details on notations. Then, denoting maps of M (resp. of X resp. Y) with black (resp. red resp. green) strings, we have that the commutativity of diagram (10) is represented by the equality of pictures



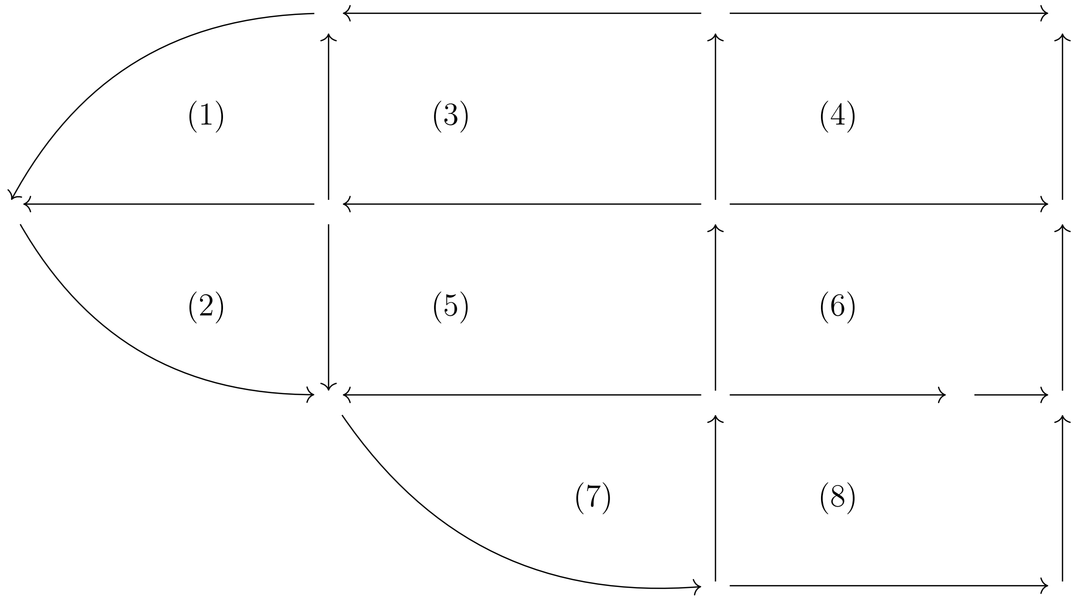
following by the naturality of c', the cocommutativity and the coassociativity of Δ_M, and the Hexagon Equation (2.2.1). Similarly, the commutativity of diagram (11) is represented by the equality of pictures



following by the naturality of c' and the fact that ε_M is a counit for Δ_M.
(iii): follows directly by statements (i) and (ii) and by the definition of $\tilde{c}$ (7.9.2).
(iv): We first prove the identity

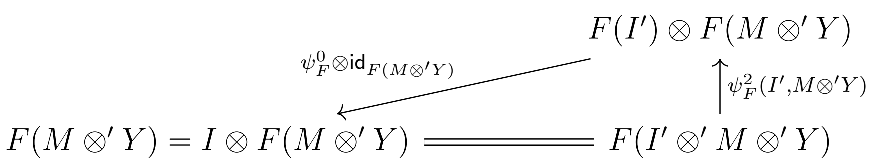
$$(\psi_G^2(M,Y))^{-1} \circ (\eta_{F(M\otimes M)} \otimes \mathsf{id}_W) = F(\Delta_M \otimes \mathsf{id}_Y) \tag{7.9.4}$$

which is given by the commutativity of the diagram



whose eight subdiagrams – using that (F, ψ_F^0, ψ_F^2) is comonoidal, the naturality of ψ_F^2 and c', and the fact that $(M, \Delta_M, \varepsilon_M)$ is a cocommutative coassociative counital comonoid – are

(1)



(2)

$$\begin{array}{ccc} F(M\otimes' Y)=I\otimes F(M\otimes' Y) & = = = = & F(I'\otimes' M\otimes' Y) \\ & \searrow\!\!\!\!\searrow & \| \\ & & F(M'\otimes' I'\otimes' Y) \end{array}$$

(3)

$$\begin{array}{ccc} F(I')\otimes F(M\otimes' Y) & \xleftarrow{F(\varepsilon_M)\otimes \mathsf{id}_{F(M\otimes' Y)}} & F(M)\otimes F(M\otimes' Y) \\ {\scriptstyle \psi^2_F(I',M\otimes' Y)}\uparrow & & \uparrow{\scriptstyle \psi^2_F(M,M\otimes' Y)} \\ F(I'\otimes' M\otimes' Y) & \xleftarrow{F(\varepsilon_M\otimes' \mathsf{id}_{M\otimes' Y})} & F(M^{\otimes' 2}\otimes' Y) \end{array}$$

(4)

$$\begin{array}{ccc} F(M)\otimes F(M\otimes' Y) & \xrightarrow{F(\Delta_M)\otimes \mathsf{id}_{F(M\otimes' Y)}} & F(M^{\otimes' 2})\otimes F(M\otimes' Y) \\ {\scriptstyle \psi^2_F(M,M\otimes' Y)}\uparrow & & \uparrow{\scriptstyle \psi^2_F(M^{\otimes' 2},M\otimes' Y)} \\ F(M^{\otimes' 2}\otimes' Y) & \xrightarrow{F(\Delta_M\otimes' \mathsf{id}_{M\otimes' Y})} & F(M^{\otimes' 3}\otimes' Y) \end{array}$$

(5)

$$\begin{array}{ccc} F(I'\otimes' M\otimes' Y) & \xleftarrow{F(\varepsilon_M\otimes' \mathsf{id}_{M\otimes' Y})} & F(M^{\otimes' 2}\otimes' Y) \\ \| & & \uparrow{\scriptstyle F(c'_{M,M}\otimes' \mathsf{id}_Y)} \\ F(M\otimes' I'\otimes' Y) & \xleftarrow{F(\mathsf{id}_M\otimes' \varepsilon_M\otimes' \mathsf{id}_Y)} & F(M^{\otimes' 2}\otimes' Y) \end{array}$$

(6)

$$\begin{array}{ccccc} F(M^{\otimes' 2}\otimes' Y) & & \xrightarrow{F(\Delta_M\otimes \mathsf{id}_{M\otimes' Y})} & & F(M^{\otimes' 3}\otimes' Y) \\ {\scriptstyle F(c'_{M,M}\otimes' \mathsf{id}_Y)}\uparrow & & & & \uparrow{\scriptstyle F(\mathsf{id}_M\otimes' c_{M,M}\otimes' \mathsf{id}_Y)} \\ F(M^{\otimes' 2}\otimes' Y) & \xrightarrow{F(\mathsf{id}_M\otimes' \Delta_M\otimes' \mathsf{id}_Y)} & F(M^{\otimes' 3}\otimes' Y) & \xrightarrow{F(c_{M,M}\otimes \mathsf{id}_{F\otimes' Y})} & F(M^{\otimes' 3}\otimes' Y) \end{array}$$

(7)

$$\begin{array}{ccc} F(M\otimes' I'\otimes' Y) & \xleftarrow{F(\mathsf{id}_M\otimes' \varepsilon_M\otimes' \mathsf{id}_Y)} & F(M^{\otimes' 2}\otimes' Y) \\ & \searrow\!\!\!\!\searrow & \uparrow{\scriptstyle F(\Delta_M\otimes' \mathsf{id}_Y)} \\ & & F(M\otimes' Y) \end{array}$$

(8)

$$\begin{array}{ccccc} F(M^{\otimes' 2}\otimes' Y) & \xrightarrow{F(\mathsf{id}_M\otimes' \Delta_M\otimes' \mathsf{id}_Y)} & F(M^{\otimes' 3}\otimes' Y) & \xrightarrow{F(c_{M,M}\otimes \mathsf{id}_{F\otimes' Y})} & F(M^{\otimes' 3}\otimes' Y) \\ {\scriptstyle F(\Delta_M\otimes' \mathsf{id}_Y)}\uparrow & & & & \uparrow{\scriptstyle F(\Delta_M\otimes' \mathsf{id}_{M\otimes' Y})} \\ F(M\otimes' Y) & & \xrightarrow{F(\Delta_M\otimes' \mathsf{id}_Y)} & & F(M^{\otimes' 2}\otimes' Y) \end{array}$$

We now can prove Equation (7.9.3). We have

$$
\begin{aligned}
&\tilde{c}_{M,Y} \circ (\eta_{F(M\otimes' M)} \otimes \mathsf{id}_W) \\
&= \psi^2_G(Y,M) \circ F\big(\mathsf{id}_M \otimes' (c'_{M,Y})^{-1}\big) \circ \big(\psi^2_G(M,Y)\big)^{-1} \circ (\eta_{F(M\otimes' M)} \otimes \mathsf{id}_W) \\
&= \psi^2_G(Y,M) \circ F\big(\mathsf{id}_M \otimes' (c'_{M,Y})^{-1}\big) \circ F(\Delta_M \otimes \mathsf{id}_Y) \\
&= \psi^2_F(M\otimes Y, M\otimes M) \circ F(\beta'_{M,M,Y,M}) \circ F(\Delta_M \otimes' \mathsf{id}_{Y\otimes' M}) \circ F\big(\mathsf{id}_M \otimes' (c'_{M,Y})^{-1}\big) \circ F(\Delta_M \otimes \mathsf{id}_Y) \\
&= F(\Delta_M \otimes' \mathsf{id}_{Y\otimes' M}) \circ F(\Delta_{M\otimes' Y}) \\
&= \Delta_{F(M\otimes' Y)}
\end{aligned}
$$

where the first equality follows by the definition of $\tilde{c}$, the second by Equation (7.9.4), the third by Proposition 2.1.3, and the fourth (using the naturality of c') by Equation (7.9.1). □

Corollary 7.9.4. *Under the previous hypotheses and notations, for any object X of $\mathscr{C}$, $F(M\otimes' X)$ is a left–right Drinfeld–Yetter $F(M\otimes' M)$–module.*

Proof. The claim follows directly by the previous statement and by Theorem 3.6.3. □

Next, by computing the semiclassical limit, we obtain the compatibility of Ševera's quantization with Drinfeld–Yetter modules. Let $\mathfrak{b}$ be a Lie bialgebra and (V, π_V, π_V^*) be a Drinfeld–Yetter $\mathfrak{b}$–module and set $M = \mathsf{U}(\mathfrak{b})$. We have showed above that $(F^{\mathfrak{b}}(M\otimes V), \mu^M_{M,V}, \Delta_{F^{\mathfrak{b}}(M\otimes V)})$ is a left–right Yetter–Drinfeld $F^{\mathfrak{b}}(M\otimes M)$–module. Consider its $\hbar$–adic completion $\big((F^{\mathfrak{b}})^{\Phi}_{\hbar}(M\otimes V), \overline{\mu^M_{M,V}}, \overline{\Delta}_{F^{\mathfrak{b}}(M\otimes V)}\big)$ in the usual deformed category where associativity constraints and braidings are deformed. Let us compute $\overline{\mu^M_{M,V}}$ mod $\hbar$. Recall that

$$\overline{\mu^M_{M,V}} = F(\mathsf{id}_M \bar{\otimes} (\varepsilon_M \bar{\otimes} \mathsf{id}_V)) \circ (\overline{\gamma^M_{M,V}})^{-1}.$$

Note that, since the first order term of the Drinfeld associator is zero, we have $(\overline{\mu^M_{M,V}})_0 = 0$. In order to compute the term proportional to $\hbar$, using the same computations of Lemma 7.8.3 we get

$$(\overline{\mu^M_{M,V}})_1 = [1\otimes u\cdot v],$$

here $u\cdot v$ denotes the canonical $\mathsf{U}(\mathfrak{b})$ left action induced by the left Lie algebra action π_V. Next, for any $v\in V$ we compute

$$
\begin{aligned}
\big(\overline{\Delta}_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}(M\otimes V)}\big)_0([1\otimes v]) &= \big(\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}(M\bar{\otimes}M, M\bar{\otimes}V) \circ (F^{\mathfrak{b}})^{\Phi}_{\hbar}(\mathsf{id}_M \otimes (\overline{\tau}^{\Phi}_{M,V})_0) \circ (\overline{\Delta}^{(2)}_0 \otimes \mathsf{id}_V)\big)([1\otimes v]) \\
&= \psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}(M\bar{\otimes}M, M\bar{\otimes}V)([1\bar{\otimes}v\bar{\otimes}1\otimes 1]) = [1\bar{\otimes}v]\bar{\otimes}[1\bar{\otimes}1]
\end{aligned}
$$

and then

$$\overline{\zeta}_{M\otimes V}\big(\big(\overline{\Delta}_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}(M\otimes V)}\big)_0([v\bar{\otimes}1])\big) = v\bar{\otimes}1.$$

Next, we compute the first order term. Recalling that

$$(\overline{\tau}^{\Phi}_{V,W})_1 = \frac{1}{2}\big((\tau_{V,W}) \circ t^{\mathfrak{b}}_{V,W}\big) \qquad \text{and} \qquad (\overline{\tau}^{\Phi}_{W,V})^{-1}_1 = -\frac{1}{2}\big(t^{\mathfrak{b}}_{W,V} \circ (\tau_{V,W})\big)$$

we have, for any $v \in V$:

$$\begin{aligned}
\left(\overline{\Delta}_{M\otimes V}\right)_1([1\otimes v]) &= \frac{1}{2}\Big((\mathsf{id}_M\bar{\otimes}(\tau_{M,V}\circ t^{\mathfrak{b}}_{M,V})\bar{\otimes}\mathsf{id}_M)\circ(\mathsf{id}_{M\otimes M}\bar{\otimes}\tau_{M,V})\circ(\overline{\Delta}_0^{(2)}\bar{\otimes}\mathsf{id}_V)\Big)(1\bar{\otimes}v)\\
&-\frac{1}{2}\Big((\mathsf{id}_M\bar{\otimes}\tau_{M,V}\bar{\otimes}\mathsf{id}_M)\circ(\mathsf{id}_{M\otimes M}\bar{\otimes}(t^{\mathfrak{b}}_{V,M}\circ\tau_{M,V}))\circ(\overline{\Delta}_0^{(2)}\bar{\otimes}\mathsf{id}_V)\Big)(1\bar{\otimes}v)\\
&=\frac{1}{2}\Big((\mathsf{id}_M\bar{\otimes}\tau_{M,V}\bar{\otimes}\mathsf{id}_M)\Big)\big((1\bar{\otimes}t_{M,V}(1\bar{\otimes}v)\bar{\otimes}1)-(1\bar{\otimes}1\bar{\otimes}t_{V,M}(1\otimes 1))\big)\\
&=\frac{1}{2}\Big(\sum_{[v]}(1\bar{\otimes}v^{[1]}\bar{\otimes}v^{[0]}\bar{\otimes}1-1\bar{\otimes}v^{[1]}\bar{\otimes}1\bar{\otimes}v^{[0]})\Big)
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\overline{\Delta}_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}(M\otimes V)}\right)_1([1\otimes v]) &= \frac{1}{2}\psi^2_{(F^{\mathfrak{b}})^{\Phi}_{\hbar}}(M\otimes V, M\otimes V)\Big(\sum_{[v]}([1\bar{\otimes}v^{[1]}\bar{\otimes}v^{[0]}\bar{\otimes}1]-[1\bar{\otimes}v^{[1]}\bar{\otimes}1\bar{\otimes}v^{[0]}])\Big)\\
&=\frac{1}{2}\sum_{[v]}([1\bar{\otimes}v^{[1]}]\bar{\otimes}[v^{[0]}\bar{\otimes}1]-[1\bar{\otimes}v^{[1]}]\bar{\otimes}[1\bar{\otimes}v^{[0]}])\\
&=\frac{1}{2}\sum_{[v]}([1\bar{\otimes}v^{[1]}]\bar{\otimes}[1\bar{\otimes}S_0(v^{[0]})]-[1\bar{\otimes}v^{[1]}]\bar{\otimes}[1\bar{\otimes}v^{[0]}])\\
&=-\sum_{[v]}[1\bar{\otimes}v^{[1]}]\bar{\otimes}[1\bar{\otimes}v^{[0]}].
\end{aligned}$$

Part III

Universal constructions

Chapter 8

PROPs

In this Chapter we introduce the concept of PROP, which appeared for the first time in [ML65]. PROPs are used in order to encode the data of algebraic structures by modeling them in a $\mathbb{K}$–linear strict symmetric monoidal category. The universal functors between algebraic structures are formalized through the concept of *universal construction.* All the PROPs of our interests will be quotients of the free PROP (see [Val07] for more details) subject to generators and relations, depending on the algebraic structure we are considering. Richer algebraic structures, such as Drinfeld–Yetter modules over a Lie bialgebra, can be encoded in a extended notion of PROP, namely colored PROPs. For further details on PROPs we remand to [EK98, §1.1–1.2], [ES02, Chapter 20], [EE05, §2], [ATL18, §6].

8.1 PROPs

Definition 8.1.1. *Let $\mathbb{K}$ be a field of characteristics zero. A $\mathbb{K}$–linear* ***PROP*** *(product and permutation category) is a $\mathbb{K}$–linear, strict, symmetric monoidal category whose objects are indexed by non–negative integers and whose tensor product is given by $[n] \otimes [m] = [n+m]$. In particular, the unit object with respect to the tensor product is $[0]$ and $[1]^{\otimes n} = [n]$.*
A morphism of PROPs is a strongly monoidal functor $F : \mathsf{P} \to \mathsf{Q}$ which is the identity on the objects and whose monoidal structure is the trivial one, i.e.

$$F([n]_{\mathsf{P}}) \otimes F([m]_{\mathsf{P}}) = [n]_{\mathsf{Q}} \otimes [m]_{\mathsf{Q}} = [n+m]_{\mathsf{Q}} = F([n+m]_{\mathsf{P}}).$$

Remark 8.1.2. *Note that, if P is a PROP, we have that for any $n \geqslant 0$ there is an action*

$$\mathbb{K}[\mathfrak{S}_n] \to \mathsf{Hom}_{\mathsf{P}}([n],[n]).$$

We shall call such morphisms permutation morphisms and we shall denote them with the related permutation. In all examples of our interest, we have that such action is faithful.

It is possible to define PROPs in a more generic setting, i.e. giving to P an enrichment over any symmetric monoidal category $\mathcal{V}$ (see [Kel82] for more details on enrichments). An important case will be where $\mathcal{V}$ is the category of all topologically free $\mathbb{K}[[\hbar]]$–modules, and in this case we say that P is a topological PROP.

8.2 The PROP of Lie bialgebras

Definition 8.2.1. *The PROP* LBA *of Lie bialgebras is the PROP generated by morphisms Lie cobracket* $\delta : [1] \to [2]$ *and Lie bracket* $\mu : [2] \to [1]$ *with the following five relations*

$$\mu \circ (\mathsf{id}_{[2]} + (12)) = 0 \tag{8.2.1a}$$

$$\mu \circ (\mu \otimes \mathsf{id}_{[1]}) \circ (\mathsf{id}_{[3]} + (123) + (312)) = 0 \tag{8.2.1b}$$

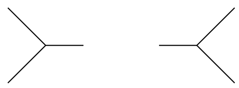
$$(\mathsf{id}_{[2]} + (12)) \circ \delta = 0 \tag{8.2.1c}$$

$$(\mathsf{id}_{[3]} + (123) + (312)) \circ (\delta \otimes \mathsf{id}_{[1]}) \circ \delta = 0 \tag{8.2.1d}$$

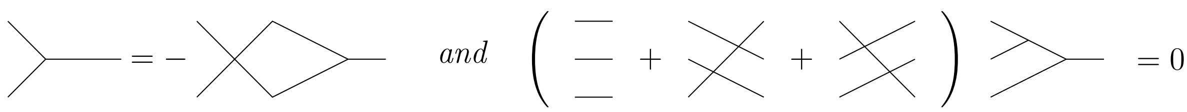
$$\delta \circ \mu - (\mathsf{id}_{[2]} - (12)) \circ (\mathsf{id}_{[1]} \otimes \mu) \circ (\delta \otimes \mathsf{id}_{[1]}) \circ (\mathsf{id}_{[2]} - (12)) = 0 \tag{8.2.1e}$$

that respectively represent the antisymmetry of the bracket, the Jacobi identity, the antisymmetry of the cobracket, the coJacobi identity and the cocycle identity.
Similarly, the PROP LA *of Lie algebras is the PROP generated by* $\mu : [1] \to [2]$ *with relations* (8.2.1a) *and* (8.2.1b)*, while the PROP* LCA *of Lie coalgebras is the PROP generated by* $\delta : [2] \to [1]$ *with relations* (8.2.1c) *and* (8.2.1d)*.*

Notation 8.2.2. *We shall represent the Lie bracket morphism and the Lie cobracket morphism of* LBA *respectively with the diagrams*

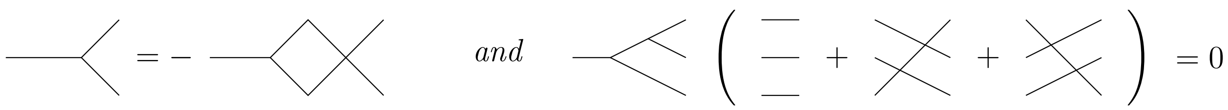


which are read from left to right. According to this pictorial representation, we represent the Lie algebra axioms (the antisymmetry of the bracket and the Jacobi rule) respectively with



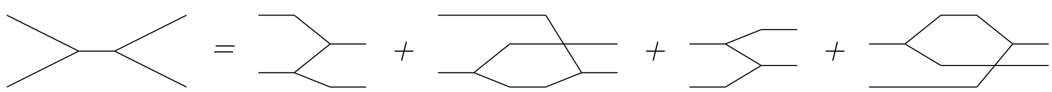
(8.2.2)

while the Lie coalgebra axioms (the antisymmetry of the cobracket and the coJacobi rule) are represented respectively by



(8.2.3)

and the cocycle condition is represented by



(8.2.4)

The same left-to-right notation can be find in [ES02, 8.2] *and* [ATL19, 5.10]*. Some authors prefer a up-to-down notation, see e.g.* [ES02, 19.1.4] *and a down-to-up notation, see e.g.* [Šev16, p. 1567].

8.3 The PROP of Hopf algebras

Definition 8.3.1. *The PROP* HA *of Hopf algebras is the PROP generated by the following morphisms (the universal multiplication, unit, comultiplication, unit, antipode and inverse antipode)*

$$\begin{aligned} m &: [2] \to [1] \\ \eta &: [0] \to [1] \\ \Delta &: [1] \to [2] \\ \varepsilon &: [1] \to [0] \\ S &: [1] \to [1] \\ S^{-1} &: [1] \to [1] \end{aligned}$$

subject to the following relations encoding all the usual axioms of Hopf algebras

$$\mu \circ (\mu \otimes \mathsf{id}_{[1]}) = \mu \circ (\mathsf{id}_{[1]} \otimes \mu) \tag{8.3.1a}$$
$$\mu \circ (\eta \otimes \mathsf{id}_{[1]}) = \mu \circ (\mathsf{id}_{[1]} \otimes \eta) = \mathsf{id}_{[1]} \tag{8.3.1b}$$
$$(\Delta \otimes \mathsf{id}_{[1]}) \circ \Delta = (\mathsf{id}_{[1]} \otimes \Delta) \circ \Delta \tag{8.3.1c}$$
$$(\mathsf{id}_{[1]} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathsf{id}_{[1]}) \circ \Delta = \mathsf{id}_{[1]} \tag{8.3.1d}$$
$$\Delta \circ \mu = (\mu \otimes \mu) \circ (23) \circ (\Delta \otimes \Delta) \tag{8.3.1e}$$
$$\eta \otimes \eta = \Delta \circ \eta \tag{8.3.1f}$$
$$\varepsilon \otimes \varepsilon = \varepsilon \circ \mu \tag{8.3.1g}$$
$$\varepsilon \circ \eta = 0 \tag{8.3.1h}$$
$$\mu \circ (S \otimes \mathsf{id}) \circ \Delta = \mu \circ (\mathsf{id} \otimes S) \circ \Delta = \eta \circ \varepsilon \tag{8.3.1i}$$
$$S \circ S^{-1} = S^{-1} \circ S = \mathsf{id}_{[1]}. \tag{8.3.1j}$$

Similarly, the PROP AA *of associative algebras is the one generated by* m, η *as above subject to relations* (8.3.1a), (8.3.1b)*; the PROP* CC *of coassociative coalgebras is the one generated by* Δ, ε *as above subject to relations* (8.3.1c), (8.3.1d)*; the PROP* BA *of bialgebras is the one generated by* $m, \eta, \Delta, \varepsilon$ *as above subject to relations* (8.3.1a)–(8.3.1h)*.*

For a pictorial representation of the generating morphisms and relations, see [ES02, p. 72].

8.4 Universal constructions

Definition 8.4.1. *Let* $b = \{b_n\}_{n\geqslant 2}$ *be a family of bracketings, i.e. every* b_n *is a fixed way of bracketing of a* n*–tensor (among all the possible* C_n *ones, where* C_n *is the* $(n-1)$*–th Catalan number). Let* P *be a PROP and* $\mathscr{C}$ *be a symmetric monoidal category. A* ***linear algebraic structure of type*** P *on an object* X *in* $\mathsf{Obj}(\mathscr{C})$ *is a symmetric monoidal functor* $F_X : \mathsf{P} \to \mathscr{C}$ *such that* $F_X([n]) = X_{b_n}^{\otimes n}$*.*

Example 8.4.2. *Fix a field* $\mathbb{K}$ *and a family of bracketings* $b = \{b_n\}_{n\geqslant 2}$*. Consider* AA *the PROP of associative algebras and* $\mathsf{Vect}_{\mathbb{K}}$ *the category of all vector spaces. Then* A *is a* $\mathbb{K}$*–algebra if and only if there exists a symmetric monoidal functor* $F_A : \mathsf{AA} \to \mathsf{Vect}_{\mathbb{K}}$ *such that* $F([1]) = A$*.*

Definition 8.4.3. *Let* $\mathsf{P}_1, \mathsf{P}_2$ *be two PROPS. A* ***universal construction*** *from* P_1 *to* P_2 *is a strict symmetric functor* $F : \mathsf{P}_2 \to \mathsf{P}_1$*.*

The idea of this Definition is that a universal construction gives a description of the generating morphisms of P_2 in terms of the ones of P_1. This will be clearer by considering the following

Example 8.4.4. *Consider the PROPs* AA *and* LA *of associative algebras and of Lie algebras. The following functor is a universal construction*

$$\begin{aligned} \mathsf{Lie} : \mathsf{LA} &\to \mathsf{AA} \\ [1]_{\mathsf{LA}} &\mapsto [1]_{\mathsf{AA}} \\ \mu &\mapsto m - m \circ (12) \end{aligned}$$

The fact that any associative algebra A has a natural Lie algebra structure with $[a,b] = ab - ba$ is thus described by the composition of the symmetric monoidal functors $F_A \circ \mathsf{Lie} : \mathsf{LA} \to \mathsf{Vect}_{\mathbb{K}}$.

In order to describe the functor *universal enveloping algebra* in a PROPic way, one needs the following

Definition 8.4.5. *Let $\mathscr{C}$ be a category. The* ***Karoubi envelope***[1] *of $\mathscr{C}$ is the category $\mathscr{C}^{kar}$ whose objects are pairs (X, π), where $X \in \mathscr{C}$ and $\pi : X \to X$ is an idempotent morphism, and whose morphisms are*

$$\mathsf{Hom}_{\mathscr{C}^{kar}}((X,\pi),(Y,\rho)) = \{f \in \mathsf{Hom}_{\mathscr{C}}(X,Y) \mid \rho \circ f = f = f \circ \pi\}.$$

In the Karoubi envelope of a category one has that every idempotent splits. Moreover, $\mathscr{C}^{kar}$ is the category containing $\mathscr{C}$ which is universal with respect the property that every idempotent is a split idempotent, see [BS01, Lem. 1.8] and references therein for more details. In particular, if P is a PROP and P^{kar} is its Karoubi envelope, we can consider in P^{kar} the object $[n]_{\mathsf{P}^{kar}} := ([n], \frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma)$ for any $n \in \mathbb{N}$. If therefore $\underline{\mathsf{P}^{kar}}$ is the closure of P^{kar} with respect to all infinite inductive limits, one can consider the object

$$S[1] := \bigoplus_{n \geqslant 0} \left([n], \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma \right) \in \underline{\mathsf{P}^{kar}}$$

which is the universal symmetric algebra. Its multiplication is given explicitely in [ATL18, 6.5]. Since for any Lie algebra $\mathfrak{g}$ there is an isomorphism of coalgebras $S(\mathfrak{g}) \cong \mathsf{U}(\mathfrak{g})$, one can define the universal enveloping algebra functor in a PROPic way through a universal construction

$$\begin{aligned} \mathsf{U} : \mathsf{AA} &\to \underline{\mathsf{LA}}^{kar} \\ [1]_{\mathsf{AA}} &\mapsto S[1] \\ m &\mapsto m_0 \end{aligned}$$

see [ATL18, 6.6] for more details. Note that the usual comultiplication of the Universal enveloping algebra can be described in a PROPic way as

$$\Delta_0([1]_{\underline{\mathsf{LBA}}^{kar}}) := \mathsf{id}_{[1]_{\underline{\mathsf{LBA}}^{kar}}} \otimes \mathsf{id}_{[0]_{\underline{\mathsf{LBA}}^{kar}}} + \mathsf{id}_{[0]_{\underline{\mathsf{LBA}}^{kar}}} \otimes \mathsf{id}_{[1]_{\underline{\mathsf{LBA}}^{kar}}}$$

which uniquely extends to any $[n]_{\underline{\mathsf{LBA}}^{kar}}$

[1] In [BS01] he Karoubi completion is called idempotent completion

8.5 Colored PROPs

Definition 8.5.1. *Let $\mathbb{K}$ be a field of characteristic zero. A* ***colored PROP*** *is a $\mathbb{K}$–linear, strict, symmetric monoidal category whose objects are finite sequences over a set A. In other words, in a colored PROP P one has*

$$\mathsf{Obj}(\mathsf{P}) = \coprod_{n \geqslant 0} \mathsf{A}^n.$$

Here the tensor product of two elements is the concatenation of sequences, and the unit with respect to the tensor product is the empty sequence. The set A is said to be the set of colors of P.

Note that any PROP is a colored PROP with $A = \{*\}$. Using colored PROPs one can describe richer algebraic structures in a categorical framework.

8.6 The colored PROP of Drinfeld–Yetter modules

Definition 8.6.1. *Let $n \geqslant 1$. The n–th* ***Drinfeld–Yetter PROP*** *is the colored PROP DY^n generated by $n+1$ objects $[1]$ and $\{[V_k]\}_{k=1,\dots,n}$ and by $2n+2$ morphisms*

$$\mu : [2] \to [1]$$
$$\delta : [1] \to [2]$$
$$\pi_k : [1] \otimes [V_k] \to [V_k]$$
$$\pi_k^* : [V_k] \to [1] \otimes [V_k]$$

such that the triple $([1], \mu, \delta)$ satisfies relations (8.2.1a)–(8.2.1e) and for any $k = 1, \dots, n$ the triple $([V_k], \pi_k, \pi_k^*)$ is a Drinfeld–Yetter module over $[1]$, i.e. the following conditions are satisfied

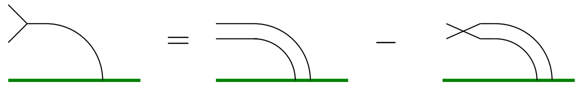
$$\pi_k \circ (\mu \otimes \mathsf{id}_{[V_k]}) = \pi_k \circ (\mathsf{id}_{[1]} \otimes \pi_k) - \pi_k \circ (\mathsf{id}_{[1]} \otimes \pi_k) \circ (21) \tag{8.6.1a}$$

$$(\delta \otimes \mathsf{id}_{[V_k]}) \circ \pi_k^* = (21) \circ (\mathsf{id}_{[1]} \otimes \pi_k^*) \circ \pi_k^* - (\mathsf{id}_{[1]} \otimes \pi_k^*) \circ \pi_k^* \tag{8.6.1b}$$

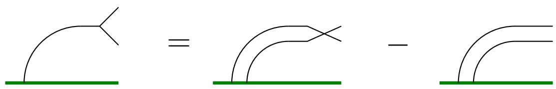
$$\pi_k^* \circ \pi_k = (\mathsf{id}_{[1]} \otimes \pi_k) \circ (12) \circ (\mathsf{id}_{[1]} \otimes \pi_k^*) + (\mu \otimes \mathsf{id}_{[V_k]}) \circ (\mathsf{id}_{[1]} \otimes \pi_k^*) - (\mathsf{id}_{[1]} \otimes \pi_k) \circ (\delta \otimes \mathsf{id}_{[V_k]}). \tag{8.6.1c}$$

We shall represent the generating morphisms of the category DY^1 with the following diagrams. We represent $\mathsf{id}_{[1]}$ with a horizontal line, $\mathsf{id}_{[V_k]}$ with a horizontal green bold line, and the morphisms $\mu, \delta, \pi_1, \pi_1^*$ respectively by the diagrams

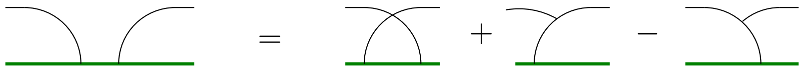
which are read from left to right. The fact that the triple $([1], \mu, \delta)$ is a Lie bialgebra object in DY^1 is then represented by the diagrams (8.2.2), (8.2.3), (8.2.4). Finally, relations (8.6.1a) (8.6.1b) (8.6.1c) are respectively represented by the following three pictorial identities



(8.6.2)



(8.6.3)



(8.6.4)

From now on, we shall refer to relations (8.6.2), (8.6.3) and (8.6.4) as respectively the action rule, the coaction rule and the Drinfeld–Yetter rule, and we shall denote single strings with a thin line and multiple strings with a bold line. We shall need the following

Definition 8.6.2. *We say that a morphism of* DY^1 *is* ***normally ordered*** *if all coactions precede all actions and all cobrackets precede all brackets.*

Remark 8.6.3. *Note that, if* ϕ *is a non–normally ordered morphism of* DY^1*, one can use the Drinfeld–Yetter, action and coaction rules in order to get a sum of normally ordered elements of* $\mathsf{Hom}_{\mathsf{DY}^1}$*. This reasoning will be the key idea of Proposition 10.3.3.*

8.7 Universal quantization functors

Definition 8.7.1. *(*[EK98, p.5],[EE05, p.6]*) The PROP* QUE *of quantized universal enveloping algebras is the topological PROP generated by the following morphisms*

$$\begin{aligned} m &: [2] \to [1] \\ \eta &: [0] \to [1] \\ \Delta &: [1] \to [2] \\ \varepsilon &: [1] \to [0] \\ \delta &: [1] \to [2] \end{aligned}$$

subject to the following relations

$$\begin{aligned} \mu \circ (\mu \otimes \mathsf{id}_{[1]}) &= \mu \circ (\mathsf{id}_{[1]} \otimes \mu) \\ \mu \circ (\eta \otimes \mathsf{id}_{[1]}) &= \mu \circ (\mathsf{id}_{[1]} \otimes \eta) = \mathsf{id}_{[1]} \\ (\Delta \otimes \mathsf{id}_{[1]}) \circ \Delta &= (\mathsf{id}_{[1]} \otimes \Delta) \circ \Delta \\ (\mathsf{id}_{[1]} \otimes \varepsilon) \circ \Delta &= (\varepsilon \otimes \mathsf{id}_{[1]}) \circ \Delta = \mathsf{id}_{[1]} \\ \Delta \circ \mu &= (\mu \otimes \mu) \circ (23) \circ (\Delta \otimes \Delta) \\ \eta \otimes \eta &= \Delta \circ \eta \\ \varepsilon \otimes \varepsilon &= \varepsilon \circ \mu \\ \varepsilon \circ \eta &= 0 \\ \Delta - (12) \circ \Delta - \hbar\delta &= 0 \\ (\mathsf{id}_{[1]} - \eta \circ \varepsilon)^{\otimes n} \circ \Delta^{(n)} &= 0 \end{aligned}$$

where $\forall n \in \mathbb{N}$ *we set* $\Delta^{(n)} := (\Delta \otimes \mathsf{id}_{[n-2]}) \circ (\Delta \otimes \mathsf{id}_{[n-3]}) \cdots \circ (\Delta \otimes \mathsf{id}_{[1]}) \circ \Delta : [1] \to [n]$.

Definition 8.7.2. *A* ***universal quantization functor of Lie bialgebras*** *is a universal construction*

$$Q : \mathsf{QUE} \to \underline{\mathsf{LBA}}^{kar}[[\hbar]]$$

such that

$$\begin{aligned} Q([1]_{\mathsf{QUE}}) &= S[[1]_{\mathsf{LBA}}] \\ Q(m) &= m_0 \mod \hbar \\ Q(\Delta) &= \Delta_0 \mod \hbar \\ Q(\Delta - (12) \circ \Delta) &= \hbar\delta \mod \hbar^2 \end{aligned}$$

where m_0 (resp. Δ_0) is the multiplication (resp. comultiplication) of the (universal) universal enveloping algebra $\mathsf{U}([1]_{\mathsf{LBA}})$.

The Etingof–Kazhdan quantization technique provides a universal quantization functor, see [EK98, §1] and [ATL18, §6.7– §6.17].

Chapter 9

Enriquez–Etingof "Hensel" Lemma

In the second article of their series [EK98], Etingof and Kazhdan prove that their universal quantization functor is invertible. In particular, their proof rely on the Grothendieck–Teichmüller semigroup (see [Dri90a] [Sch97] [Mer21] for more details).
In the article [EE05], Enriquez and Etingof give a simpler proof of the invertibility of the Etingof–Kazhdan functor, relying on a so–called *Hensel's Lemma*. In this Chapter we provide a detailed proof of the Enriquez–Etingof *Hensel Lemma*.

9.1 Modules over rings of formal power series

In this Section we recall some standard definitions and properties of formal power series and topologically free modules, see [Kas12, p.385-390].

Definition 9.1.1. *Let $\mathbb{K}$ be a field and $\hbar$ be a formal parameter. The* ***ring of all formal power series*** *$R := \mathbb{K}[[\hbar]]$ with coefficients in $\mathbb{K}$ is*

$$\mathbb{K}[[\hbar]] := \left\{ \sum_{r=0}^{\infty} a_r \hbar^r \,\middle|\, a_0, a_1, \ldots \in \mathbb{K} \right\}.$$

equipped with componentwise addition and the Cauchy multiplication. For a given vector space V over $\mathbb{K}$, the ***topologically free module*** *generated by V is*

$$V[[\hbar]] := \left\{ \sum_{r=0}^{\infty} v_r \hbar^r \,\middle|\, v_0, v_1, \ldots \in V \right\}.$$

which is a R–module by means of the Cauchy multiplication.

Remark 9.1.2. *Note that $R = \mathbb{K}[[\hbar]]$ is a filtered ring by the powers $\hbar^r R$, $r \in \mathbb{N}\backslash\{0\}$, of its maximal ideal $I = \hbar R$: $V[[\hbar]]$, i.e.*

$$R = R_{(0)} \supset I = \hbar R = R_{(1)} \supset \cdots \supset I^r = \hbar^r R = R_{(r)} \supset \cdots$$

Moreover each R–module E carries a descending filtration $\left(E_{(n)}\right)_{n\in\mathbb{N}}$ derived from the descending filtration $\left(R_{(n)}\right)_{n\in\mathbb{N}}$, compare [Bou89, p.163]*, i.e. $\forall n \in \mathbb{N}$: $E_{(n)} := \hbar^n E = I^n E$, which we shall call the $\hbar$–adic filtration of the R–module E. It is obvious that every R-linear map $g : E \to E'$ preserves the above derived filtrations, i.e. $\forall\ n \in \mathbb{N}$ we have $g(E_n) \subset E'_{(n)}$.*

Recall the well–known canonical topology on the module E, the so-called $\hbar$–**adic topology**, is associated to this filtration, see e.g. [Bou89, p.171-173] for the general theory. More precisely, it is defined as follows: for each $x \in E$ and each nonnegative integer m set

$$V_{x,m} := x + E_{(m)}. \tag{9.1.1}$$

For any $x, y \in E$ and nonnegative integers m, n we have

$$V_{x,m} \cap V_{y,n} = \begin{cases} \emptyset & \text{if } x - y \notin E_{(\min\{m,n\})}, \\ V_{x,m} & \text{if } x - y \in E_{(\min\{m,n\})} \text{ and } m = \max\{m,n\}, \\ V_{y,n} & \text{if } x - y \in E_{(\min\{m,n\})} \text{ and } n = \max\{m,n\}. \end{cases} \tag{9.1.2}$$

which can easily be deduced from the fact that $V_{x,m} \cap V_{y,n} \neq \emptyset$ if and only if there are elements $z \in E$, $v_m \in E_{(m)}$ and $v_n \in E_{(n)}$ such that $z = x + v_m = y + v_n$ from which the three conditions on the right hand side of equation (9.1.2) can easily be deduced. This implies that the family of subsets $(V_{x,m})_{(x,m)\in E\times\mathbb{N}}$ is closed under finite intersections. It follows that the familiy of all arbitrary unions of the sets $V_{x,m}$ together with the empty set forms a topology on E which is equal to the $\hbar$–adic topology. We shall refer to the sets $x + E_{(n)}$ as the basic open sets of the topology and they form a base of the $\hbar$–adic topology.

Notation 9.1.3. *For each integer $m \geqslant 0$ we denote by E_m the following factor module*

$$E_m := E/E_{(m+1)}$$

and by π^E_m the canonical projection

$$\pi^E_m : E \to E_m.$$

Clearly, for any R–linear map $g : E \to E'$ and for any nonnegative integer n there is an induced map $g_n : E_n \to E'_n$ such that the following diagram commutes since $g(E_{(n)}) \subset E'_{(n)}$:

$$\begin{array}{ccc} E & \xrightarrow{\;g\;} & E' \\ {\scriptstyle \pi^E_n}\downarrow & & \downarrow{\scriptstyle \pi^{E'}_n} \\ E_n & \xrightarrow{\;g_n\;} & E'_n \end{array} \tag{9.1.3}$$

In particular, for $n = 0$ the induced map g_0 can be considered as a $\mathbb{K}$–linear map $E_0 \to E'_0$. Clearly, for another R-linear map $h : E' \to E''$ we have

$$(h \circ g)_n = h_n \circ g_n \tag{9.1.4}$$

for all nonnegative integeres n. We have the following

Lemma 9.1.4. *Let E, E' be two R–modules (equipped with their $\hbar$-adic topologies), let $F \subset E$ a R–submodule, let E/F be the factor module (with its $\hbar$-adic topology) and denote by $\pi : E \to E/F$ the canonical projection. Then we have the following:*

(i) E is a Hausdorff topological space iff its $\hbar$–adic filtration is separated, i.e.

$$\bigcap_{n\in\mathbb{N}} E_{(n)} = \{0\}. \tag{9.1.5}$$

(ii) Each R–linear map $f : E \to E'$ is continuous.

(iii) Each R–linear sujective map $f : E \to E'$ is open.

(iv) Each R–linear isomorphism $f : E \to E'$ is a homeomorphism.

(v) The $\hbar$–adic topology on E/F is equal to the quotient topology, i.e. to the family of all subsets of E/F whose inverse images under the projection π are open in E.

Proof. (i): Suppose first that the $\hbar$–adic filtration is separated. Let $x, y \in E$ with $x \neq y$. Hence $x - y \neq 0$ and there is a non–negative integer n such that $x - y \notin E_{(n)}$. According to (9.1.2) the two open sets $V_{x,n}$ and $V_{y,n}$ have empty intersection, and clearly $x \in V_{x,n}$, and $y \in V_{y,n}$. It follows that the topological space E is Hausdorff. Conversely, suppose that the $\hbar$–adic filtration is not separated. Let $z \in \bigcap_{n\in\mathbb{N}} E_{(n)}$ a non–zero element. If there were open sets U_z and V_0 with $z \in U_z$, $0 \in V_0$, and $U_z \cap V_0 = \emptyset$, then in particular there would exist two nonnegative integers m, n and two basic open sets of the form $z + E_{(m)}$ and $E_{(n)}$ such that $\big(z + E_{(m)}\big) \cap E_{(n)} = \emptyset$. By assumption, $z \in E_{(k)}$ for all non–negative integers k, hence $z \in E_{(m)}$, and $z + E_{(m)} = E_{(m)}$, hence $\big(z + E_{(m)}\big) \cap E_{(n)} = E_{(m)} \cap E_{(n)} \ni 0$ so this intersection cannot be empty, and therefore E is not Hausdorff.
(ii): It suffices to show that the inverse image of any open set of the form $x' + E'_{(n+1)} \subset E'$ is an open set in E. Indeed for all $x \in E$

$$x \in f^{-1}\big(x' + E'_{(n)}\big) \;\Leftrightarrow\; f(x) \in x' + E'_{(n+1)} \;\Leftrightarrow\; \pi_n^{E'}\big(f(x)\big) = \pi_n^{E'}(x') \;\Leftrightarrow\; f_n\big(\pi_n^E(x)\big) = \pi_n^{E'}(x'),$$

hence for each $x \in f^{-1}\big(x' + E'_{(n+1)}\big)$ it is clear that $x + v_{(n+1)} \in f^{-1}\big(x' + E'_{(n+1)}\big)$ for all $v_{(n)} \in E_{(n+1)} = \ker(\pi_n^E)$, and we get

$$f^{-1}\big(x' + E'_{(n+1)}\big) = \bigcup_{x\in f^{-1}\left(x' + E'_{(n+1)}\right)} \big(x + E_{(n+1)}\big),$$

which is a union of open sets of E, proving the continuity of f.
(iii): For any $x \in E$ and $n \in \mathbb{N}$ we get

$$f\left(x + E_{(n+1)}\right) = f(x) + \hbar^{n+1} f(E) = f(x) + \hbar^{n+1} E' = f(x) + E'_{(n+1)}$$

hence, since every open set is a union of the basic open sets $x + E_{(n+1)}$ and direct images preserve unions it follows that the image of every open set is open.
(iv): Is a direct consequence of (ii).
(v): By definition of the quotient topology, the map π is continuous with respect to the quotient topology on E/F, and since π is R–linear, then for (ii) it is continuous with respect to the $\hbar$-adic topology on E/F. Since the quotient topology is the finest topology on E/F such that π is continuous we can infer that the $\hbar$–adic topology is a subfamily of the quotient topology. On the other hand, let $U \subset E/F$ be an open in the quotient topology. Its inverse image $f^{-1}(U)$ is an open set of the $\hbar$-adic topology of E, hence there is a set $\mathfrak{S}$ such that for each $s \in \mathfrak{S}$ there is an element $x_s \in E$ and a non–negative integer n_s such that

$$\pi^{-1}(U) = \bigcup_{s\in\mathfrak{S}} \left(x_s + E_{(n_s)}\right).$$

Since π is surjective we have $\pi\big(\pi^{-1}(U)\big) = U$, and therefore we get

$$U = \pi\big(\pi^{-1}(U)\big) = \bigcup_{s\in\mathfrak{S}} \big(\pi(x_s) + \pi(E_{(n_s)})\big) = \bigcup_{s\in\mathfrak{S}} \big(\pi(x_s) + (E/F)_{(n_s)}\big)$$

hence U belongs to the $\hbar$-adic topology of E/F. □

We shall need the following two facts:

Lemma 9.1.5. *Let E be an R–module, $A \subset E$ be a subset and let $W \subset E$ be a submodule. Then:*

(i) The closure of A is given by

$$\overline{A} = \bigcap_{n=0}^{\infty} \big(A + E_{(n)}\big)\,. \tag{9.1.6}$$

(ii) The quotient module E/W is Hausdorff if and only if $W \subset E$ is closed.

Proof. (i): Clearly $A \subset \overline{A}$. We first show that $\overline{A}$ is closed: indeed, let $B := E \setminus \overline{A}$. Since

$$\overline{A} = \bigcap_{n=0}^{\infty} \bigcup_{a\in A} \big(a + E_{(n)}\big)$$

we have

$$B = E \setminus \overline{A} = \bigcup_{n=0}^{\infty} \bigcap_{a\in A} \big(E \setminus \big(a + E_{(n)}\big)\big) = \big\{y \in E \mid \exists\, n \in \mathbb{N}\ \forall\, a \in A:\ y \notin a + E_{(n)}\big\}\,. \tag{9.1.7}$$

For all $z_n \in E_{(n)}$ whe have that $y \notin a + E_{(n)}$ implies $y + z_n \notin a + E_{(n)}$ (since $z_n \in E_{(n)}$), whence for each $y \in B$ there is $n \in \mathbb{N}$ such that the basic open set $y + E_{(n)} \subset B$ (and so B) is open. Therefore $\overline{A}$ is closed. Now let $C \subset E$ be a closed subset containing A and let $z \in E \setminus C$. Then there is $n \in \mathbb{N}$ such that $z + E_{(n)} \subset E \setminus C \subset E \setminus A$. Hence it follows that

$$\forall\, z \in E \setminus C\ \exists\, n \in \mathbb{N}\ \forall\, a \in A : a \notin z + E_{(n)} \quad \Leftrightarrow \quad \forall\, z \in E \setminus C\ \exists\, n \in \mathbb{N}\ \forall\, a \in A : z \notin a + E_{(n)}$$

hence $E \setminus C \subset E \setminus \overline{A}$, i.e. $\overline{A} \subset C$, proving that $\overline{A}$ is the smallest closed subset of E containing A, hence the closure of A.
(ii): Suppose that E/W is Hausdorff. Then the singleton set $\{0\}$ is a closed subset of E/W, and its inverse image with respect to the canonical projection $\pi^{-1}\big(\{0\}\big) = W$ is closed (since π is continuous, see part (ii) of Lemma 9.1.4).
Conversely, suppose that W is closed. We check whether the $\hbar$-adic filtration $\big((E/W)_{(n)}\big)_{n\in\mathbb{N}}$ of E/W is separated: observing that for all $n \in \mathbb{N}$ we have $(E/W)_{(n)} = \pi\big(E_{(n)}\big)$ we can infer using the surjectivity of π:

$$\{0\} = \bigcap_{n\in\mathbb{N}} \pi\big(E_{(n)}\big) \quad \Leftrightarrow \quad W = \pi^{-1}\big(\{0\}\big) = \bigcap_{n\in\mathbb{N}} \pi^{-1}\left(\pi\big(E_{(n)}\big)\right) = \bigcap_{n\in\mathbb{N}} \big(W + E_{(n)}\big) \overset{(9.1.6)}{=} \overline{W}.$$

Since by assumption W is closed we have $W = \overline{W}$, and then the claim follows. □

Remark 9.1.6. *Note that we did not assume in the second statement that E was Hausdorff: note that the intersection $\bigcap_{n\in\mathbb{N}} E_{(n)}$ is the closure of the singleton $\{0\}$ and is thus contained in every closed submodule of E.*

Next, we recall that for two nonnegative integers $m \leqslant n$ there is a unique canonical R–linear map $p_{mn} : E_n \to E_m$ such that the following triangle commutes:

$$\begin{array}{ccc} & E & \\ {\scriptstyle \pi_n^E}\swarrow & & \searrow{\scriptstyle \pi_n^E} \\ E_n & \xrightarrow{\;\;p_{mn}\;\;} & E_m \end{array} \tag{9.1.8}$$

We can thus form the inverse limit in the category of R-modules

$$\tilde{E} := \lim_{\infty \leftarrow n} E_n = \left\{ \xi = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} E_n \;\middle|\; \forall\, m, n \in \mathbb{N} : \text{ if } m \leq n, \text{ then } x_m = p_{mn}(x_n) \right\},$$

where

$$\prod_{n\in\mathbb{N}} E_n := \left\{ \xi : \mathbb{N} \to \coprod_{n\in\mathbb{N}} E_n \;\middle|\; \forall\, n \in \mathbb{N} : \; \xi(n) =: \xi_n \in E_n \right\}$$

and $\coprod_{n\in\mathbb{N}} E_n$ denotes the disjoint union of the sets E_n. Recall the canonical morphism of R–modules $p^E : E \to \tilde{E} : x \mapsto \big(\pi_n^E(x)\big)_{n\in\mathbb{N}}$ whose values are well–defined elements in the submodule $\lim_{\infty\leftarrow n} E_n$ of $\prod_{n\in\mathbb{N}} E_n$ thanks to the triangle diagram (9.1.8). Its kernel is given by the closure of $\{0\}$, see Equation (9.1.5).

Definition 9.1.7. *A R–module E is said to be* ***complete*** *if p^E is surjective.*

Remark 9.1.8. *Note that p^E is an isomorphism if and only if E is Hausdorff and complete.*

Definition 9.1.9. *A R-module E is called a* ***topologically free R-module*** *if there is a $\mathbb{K}$-vector space V and an R-linear isomorphism $f : V[[\hbar]] \to E$. We denote the category of all topologically free $\mathbb{K}$–modules by $\mathsf{TopFree}_K$.*

Remark 9.1.10. *A topologically free module E is Hausdorff and complete since*

$$V[[\hbar]]_{(n)} = \Big\{ \sum_{r=n}^{\infty} v_r \hbar^r \mid v_n, v_{n+1}, \ldots, \in V \Big\}$$

and $V[[\hbar]]_n \cong \{\sum_{r=0}^{n} v_r\hbar^r \mid v_0, \ldots, v_n \in V\}$ (as $\mathbb{K}$–vector spaces). Moreover, there is a vector space isomorphism $V \cong E_0 = E/\hbar E$.

Lemma 9.1.11. *Let V, W be two $\mathbb{K}$–vector spaces, and let E' be a complete R–module. Then there is a natural isomorphism of R–modules*

$$\mathrm{Hom}_R\big(V[[\hbar]], E'\big) \;\cong\; \mathrm{Hom}_{\mathbb{K}}\big(V, E'\big) \tag{9.1.9}$$

defined by restriction of any R-linear map $V[[\hbar]] \to E'$ to $V \subset V[[\hbar]]$. In particular

$$\mathrm{Hom}_R\big(V[[\hbar]], W[[\hbar]]\big) \;\cong\; \mathrm{Hom}_{\mathbb{K}}\big(V, W\big)[[\hbar]]. \tag{9.1.10}$$

For a proof, see [Kas12, Prop. XVI.2.3].
We mention a last result which will be useful:

Lemma 9.1.12. *Let $\mathcal{A}$ be an algebra over $\mathbb{K}$. Then $\mathcal{A}[[\hbar]]$ carries a structure of algebra over $R = \mathbb{K}[[\hbar]]$. In particular, for any $a \in \mathcal{A}[[\hbar]]$ the element $1 + \hbar a$ is invertible in $\mathcal{A}[[\hbar]]$ where the inverse is given by the ususal geometric series formula*

$$(1 + \hbar a)^{-1} = \sum_{r=0}^{\infty} (-\hbar)^r a^r.$$

9.2 Proof of the Enriquez–Etingof "Hensel" Lemma

This Section is devoted to the proof of the following Theorem which Enriquez and Etingof called *Hensel's Lemma* in [EE05, Lemma 3.1]. In the following we set $R = \mathbb{K}[[\hbar]]$.

Theorem 9.2.1. *Let E be a topologically free R-module, and let $Z \subset E$ be a closed R-submodule. Denote by $N := E/Z$ the quotient module. Finally, let M be a vector space over $\mathbb{K}$, and let $f : N \to M[[\hbar]]$ be a R-linear map such that its 0-component $f_0 : N_0 = N/(\hbar N) \to M = M[[\hbar]]_0 = M[[\hbar]]/(\hbar M[[\hbar]])$ is an isomorphism of $\mathbb{K}$–vector spaces. Then:*

(i) f is an isomorphism of R-modules.

(ii) N is topologically free.

(iii) Z is isomorphic to a topologically free module $V''[[\hbar]]$ with $V'' \subset E_0$.

Before giving the proof, let us see why such a result give rise to a dequantization statement. Let $Q : \underline{\mathsf{P}}_1 \to \underline{\mathsf{P}}_2$ be a certain *quantization functor* between two topological PROPs. Applying Theorem 9.2.1 to the collection of maps $Q_{n,m} : \mathsf{Hom}_{\underline{\mathsf{P}}_1}([n], [m]) \to \mathsf{Hom}_{\underline{\mathsf{P}}_2}([n], [m])$, we obtain that $Q : \underline{\mathsf{P}}_1 \to \underline{\mathsf{P}}_2$ is an isomorphism of PROPs. For more details we refer to [EE05].

Proof. We denote by $\tau : E \to E/Z = N$ the canonical projection. By applying an isomorphism of topologically free modules we can suppose that $E = V[[\hbar]]$ for the $\mathbb{K}$–vector space $V = E_0$. Moreover, let $F : V[[\hbar]] \to M[[\hbar]]$ be the composed map $F = f \circ \tau$. First, we show that $\tau_0 : V = E_0 \to N_0$ is surjective: indeed, let $y \in N_0$. Since the projection $\pi_0^N : N \to N_0$ is surjective there is $\eta \in N$ with $y = \pi_0^N(\eta)$. Since $\tau : E \to N$ is surjective there is $\xi \in E$ such that $\eta = \tau(\xi)$, hence

$$y = \pi_0^N\big(\tau(\xi)\big) \overset{(9.1.3)}{=} \tau_0\big(\pi_0^E(\xi)\big) =: \tau_0(v)$$

with $\pi_0^E(\xi) =: v \in V = E_0$, and τ_0 is surjective. It follows that

$$F_0 \overset{(9.1.4)}{=} f_0 \circ \tau_0 : \; V = E_0 \to M = M[[\hbar]]_0$$

is surjective as a composition of surjective maps. Define $V'' := \ker(F_0) \subset V$, and choose a complementary $\mathbb{K}$-subspace to V'', i.e. $V' \subset V$ such that $V = V' \oplus V''$. Let p' (resp. p''): $V \to V$ denote the canonical projection on V' (resp. on V'') having kernel V'' (resp. V'). Obviously $p' + p'' = \mathsf{id}_V$. Now since the restriction of F_0 to V' is injective and surjective by construction, hence there is a $\mathbb{K}$-linear inverse map $\varphi : M \to V'$, i.e.

$$F_0 \circ \varphi = \mathsf{id}_M, \quad \text{and} \quad \varphi \circ F_0 = p'$$

where we have not written the canonical injection $V' \hookrightarrow V$. Recall that F takes the general form

$$F = \sum_{r=0}^{\infty} F_r \hbar^r$$

where for each nonnegative integer r the symbol F_r denotes a $\mathbb{K}$-linear map $V \to M$, see Lemma 9.1.11, Equation (9.1.10). For $r = 0$ the component F_0 of the formal series of F coincides with the induced map F_0 according to (9.1.3). In the following we shall denote the natural extension of a $\mathbb{K}$-linear map $h : X \to Y$ between $\mathbb{K}$–vector spaces X, Y to an R-linear map $X[[\hbar]] \to Y[[\hbar]]$

(by considering it as a constant formal power series) by the same symbol. Define the R-linear map $H : V[[\hbar]] \to V[[\hbar]]$ by

$$H := p'' + \varphi \circ F = p'' + \varphi \circ F_0 + \hbar \sum_{r=0}^{\infty} \hbar^r (\varphi \circ F_{r+1}) = \mathsf{id}_V + \hbar \sum_{r=0}^{\infty} \hbar^r (\varphi \circ F_{r+1}) =: \mathsf{id}_V + \hbar H^+.$$

Being of the form $\mathsf{id}_V + \hbar H^+$ it is invertible by Lemma 9.1.12 upon choosing $\mathcal{A} = \mathrm{Hom}_{\mathbb{K}}(V, V)$ as the algebra (with composition as multiplication and unit element id_V). Since $F_0 \circ p'' = 0$ and $F_0 \circ \varphi = \mathsf{id}_M$ we get

$$F_0 \circ H = F_0 \circ \big(p'' + \varphi \circ F\big) = 0 + \mathsf{id}_M \circ F = F. \tag{9.2.1}$$

Thanks to the fact that F_0 is surjective and H an isomorphism we get that $F = f \circ \tau$ is surjective, whence f must be surjective.

We can use the R-linear isomorphism H and Equation (9.2.1) to replace the hypotheses of the Theorem by equivalent, but simpler ones: define

$$\tilde{Z} := H(Z) \quad \text{and} \quad \tilde{\tau} := \tau \circ H^{-1}$$

and keep N and f. Since $H : V[[\hbar]] \to V[[\hbar]]$ is a linear isomorphism it is a homeomorphism (see part (*iv*) of Lemma 9.1.4), hence $\tilde{Z}$ is also a closed submodule of $V[[\hbar]]$, and $N = \tilde{\tau}(V[[\hbar]]) \cong V[[\hbar]]/\tilde{Z}$. Moreover we get

$$\tilde{Z} = \ker(\tilde{\tau}) \ \subset \ \ker(f \circ \tilde{\tau}) = \ker(F_0) = V''[[\hbar]].$$

We describe the kernel of f:

$$V''[[\hbar]] = \ker(f \circ \tilde{\tau}) = (f \circ \tilde{\tau})^{-1}\big(\{0\}\big) = \tilde{\tau}^{-1}\big(f^{-1}(\{0\})\big) = \tilde{\tau}^{-1}\big(\ker(f)\big)$$

hence, applying $\tilde{\tau}$ upon using its surjectivity, we find

$$\ker(f) = \tilde{\tau}(V''[[\hbar]]) \cong V''[[\hbar]]/\tilde{Z}.$$

Define the following R-submodules of N

$$N' := \tilde{\tau}\big(V'[[\hbar]]\big) \quad \text{and} \quad N'' := \tilde{\tau}\big(V''[[\hbar]]\big).$$

Obviously, since $E = V'[[\hbar]] \oplus V''[[\hbar]]$ and according to what has already been shown:

$$N' = \ker(f) \quad \text{and} \quad N = N' + N''.$$

In order to show that the sum is direct, let η be an element of $N' \cap N''$. Then there is $\xi' \in V''[[\hbar]]$ such that $\tilde{\tau}(\xi') = \eta$ and $f(\eta) = 0$, hence

$$0 = f\big(\tilde{\tau}(\xi')\big) = F_0(\xi')$$

Hence $\xi' \in \ker(F_0) = V''[[\hbar]]$, but since $\xi' \in V'[[\hbar]]$ it must vanish since the intersection $V'[[\hbar]] \cap V''[[\hbar]] = (V' \cap V'')[[\hbar]]$ vanishes. It follows that

$$N' \cap N'' = \{0\}, \quad \text{hence } N = N' \oplus N''.$$

We shall now pass to the 'classical limit of N', N_0: consider the projection $\pi_0^N : N \to N_0$ whose kernel is given by $N_{(1)} = \hbar N$. Since $N'_{(1)} = \hbar N' \subset \hbar N$ and $N''_{(1)} = \hbar N'' \subset \hbar N$ the projection π_0^N passes to the quotients to define a well-defined R-linear map

$$\phi : N'_0 \oplus N''_0 \to N_0 : \Big(\pi_0^{N'}(\eta'), \pi_0^{N''}(\eta'')\Big) \mapsto \pi_0^N(\eta' + \eta'').$$

Thanks to $N = N' \oplus N''$ the map ϕ is clearly sujective. In order to prove that ϕ is also injective we compute the kernel of ϕ: let $\eta' \in N'$, $\eta'' \in N''$ such that $0 = \phi\left(\pi_0^{N'}(\eta'), \pi_0^{N''}(\eta'')\right)$, hence $\eta' + \eta'' \in \hbar N$. It follows that there are $\check{\eta}' \in N'$ and $\check{\eta}'' \in N''$ such that

$$\eta' + \eta'' = \hbar\check{\eta}' + \hbar\check{\eta}'', \quad \text{hence} \quad \underbrace{\eta' - \hbar\check{\eta}'}_{\in N'} = \underbrace{-\left(\eta'' - \hbar\check{\eta}''\right)}_{\in N''}$$

and thanks to $N = N' \oplus N''$ we have $\eta' - \hbar\check{\eta}' = 0$ whence $\pi_0^{N'}(\eta') = 0$ and $\eta'' - \hbar\check{\eta}'' = 0$ whence $\pi_0^{N''}(\eta'') = 0$ proving that the R-linear map ϕ is an isomorphism.
According to the hypotheses of the Theorem, the induced map $f_0 : N_0 \to M$ is an isomorphism. Since $N'' = \ker(f)$ it follows that $\phi(N_0'')$ is in the kernel of f_0: indeed, let $y'' \in \phi(N_0'')$, then there is $\eta'' \in N''$ such that $y'' = \pi_0^N(0 + \eta'')$ and

$$f_0\left(\pi_0^N(\eta'')\right) \stackrel{(9.1.3)}{=} \pi_0^{M[[\hbar]]}\big(f(\eta'')\big) = 0$$

because $\eta'' \in N''$ is in the kernel of f. Since the kernel of f_0 vanishes by hypothesis it follows that

$$N_0'' = \{0\} \quad \Longleftrightarrow \quad N'' = \hbar N''.$$

This implies that

$$N'' = \bigcap_{n \in \mathbb{N}} N''_{(n)}. \tag{9.2.2}$$

On the other hand, note that the subspace topology of $V''[[\hbar]] \subset V[[\hbar]]$ coincides with the $\hbar$-adic topology of $V''[[\hbar]]$ since the intersection of $\xi + \hbar^n V[[\hbar]]$, $\xi \in V[[\hbar]]$, with $V''[[\hbar]]$ is always of the form $\hat{\xi}'' + \hbar^n V''[[\hbar]]$ with $\hat{\xi}'' \in V''[[\hbar]]$ as can easily be checked for each power of $\hbar$ separately by taking V'- and V''-components in V. It follows that $\tilde{Z}$ is a closed subset in $V''[[\hbar]]$ with respect to the $\hbar$-adic topology, whence the quotient module $V''[[\hbar]]/\tilde{Z} \cong N''$ is Hausdorff according to Lemma 9.1.5, **2.**. But then eqn (9.2.2) implies that

$$\{0\} = N'' = \ker(f) \quad \Longleftrightarrow \quad \tilde{Z} = V''[[\hbar]],$$

and f is an R-linear bijection which proves all the statements of the Theorem. □

Remark 9.2.2. *Note that Enriquez–Etingof's original assumption that f has to be continuous is superfluous since this is automatically the case in view of statement (ii) of Lemma 9.1.4.*

Chapter 10

The universal Drinfeld–Yetter algebra and its combinatorics

In this Chapter we present a description of the universal Drinfeld–Yetter algebra (see Definition 10.3.2) in terms of some combinatorial objects, the Drinfeld–Yetter mosaics (§10.4) and Drinfeld–Yetter looms (§10.5). What follows is based on the forthcoming paper [AR].

10.1 Enriquez's universal algebras

In this Section we present Enriquez's universal algebras (see [Enr01b] [Enr01a] [Enr05]).

The original idea of Enriquez was to define a family of universal algebras $\{\mathcal{U}\mathfrak{S}^n_{univ}\}_{n\geqslant 0}$ with the following properties:

- (Universal property): for any quasi–triangular, finite–dimensional Lie bialgebra $\mathfrak{b}$ there exists a morphism of algebras
$$\rho^n_{\mathfrak{b}} : \mathcal{U}\mathfrak{S}^n_{univ} \to \mathsf{U}(\mathfrak{b})^{\otimes n}. \tag{10.1.1}$$
- There exists a family of insertion-coproduct maps $\mathcal{U}\mathfrak{S}^n_{univ} \to \mathcal{U}\mathfrak{S}^{n+1}_{univ}$ which gives rise to a universal version of the coHochschild differential of $\mathsf{U}(\mathfrak{b})$.

It is well–known that the existence of quantizations of a Lie bialgebra $\mathfrak{b}$ is governed by the Hochschild cohomology of $\mathsf{U}(\mathfrak{b})$, see [Kas12, XVIII]. Hence, the idea of Enriquez was to replicate the Drinfeld's cohomological proof of the existence of quantization of Lie bialgebras, obtaining a cohomological interpretation of the Etingof–Kazhdan quantization. In particular, Enriquez's main result provides, for any Drinfeld associator Φ, a universal twist $J_\Phi \in \mathfrak{S}^2_{univ}$ *killing the associator*. The universal realization maps (10.1.1) allows thus to, for any finite–dimensional, quasi–triangular Lie bialgebra $\mathfrak{b}$, realize the twist on $\mathsf{U}(\mathfrak{b})$, giving rise to a universal quantization.
More specifically, for any $n, N \geqslant 1$ let $\mathcal{A}_N$ be the free algebra in N variables x_i, $i = 1, \ldots, N$ and let $(\mathcal{A}_N^{\otimes n})_{\delta_N}$ be the subspace of $\mathcal{A}_N^{\otimes n}$ generated by elements of degree one in each variable. We have that the symmetric group $\mathfrak{S}_N$ acts diagonally on $(\mathcal{A}_N^{\otimes n})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes n})_{\delta_N}$ by simultaneus permutation of the variables. The n–th **Enriquez's universal algebra** is

$$\mathcal{U}\mathfrak{S}^n_{univ} = \sum_{N\geqslant 0} \left((\mathcal{A}_N^{\otimes n})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes n})_{\delta_N}\right)_{\mathfrak{S}_N}$$

where $\big((\mathcal{A}_N^{\otimes n})_{\delta_N} \otimes (\mathcal{A}_N^{\otimes n})_{\delta_N}\big)_{\mathfrak{S}_N}$ is the space of $\mathfrak{S}_n$–coinvariants. Note that $\mathcal{U}\mathfrak{S}^n_{univ}$ is equipped with a standard basis defined as follows. For any $\underline{N}, \underline{N}' \in \mathbb{N}^n$ with $|\underline{N}| = |\underline{N}'| = N$ and $\sigma \in \mathfrak{S}_N$, consider elements $x_{\underline{N}}$ and $y_{\sigma(\underline{N}')}$ of $(\mathcal{A}_N^{\otimes n})_{\delta_N}$ defined by

$$\begin{aligned} x_{\underline{N}} &= x_1 \cdots x_{N_1} \otimes x_{N_1+1} \cdots x_{N_1+N_2} \otimes \cdots \otimes x_{N_1+\cdots+N_{n-1}+1} \cdots x_N \\ y_{\sigma(\underline{N}')} &= y_{\sigma(1)} \cdots y_{\sigma(N_1')} \otimes \cdots \otimes y_{\sigma(N_1'+\cdots+N_{n-1}'+1} \cdots y_{\sigma(N)} \end{aligned}$$

Then the collection $\{x_{\underline{N}} \otimes y_{\sigma(\underline{N}')}\}$ is a basis of $\mathcal{U}\mathfrak{S}^n_{univ}$. Following [Enr01b], the algebra structure of $\mathcal{U}\mathfrak{S}^n_{univ}$ is provided through a very intricate formula, and is proved to be associative by a lenghty computation. For $n = 1$, one has that $\mathcal{U}\mathfrak{S}^1_{univ}$ is isomorphic – as a vector space – to the direct sum $\bigoplus_{N \geqslant 0} \mathbb{K}[\mathfrak{S}_N]$, and the product is the concatenation of permutations.
Finally, for any finite–dimensional, quasi–triangular Lie bialgebra $\mathfrak{b}$ with r–matrix $r = \sum_i b_i \otimes b^i$, the realization map – we provide for simplicity the case $n = 1$ – $\rho_{\mathfrak{b}}$ is defined as follows:

$$\begin{aligned} \rho_{\mathfrak{b}} : \mathcal{U}\mathfrak{S}^1_{univ} &\to \mathsf{U}(\mathfrak{b}) \\ x_N \otimes y(\sigma(N)) &\mapsto \sum_{i \in I^N} b_{i_1} \cdots b_{i_N} b^{i_{\sigma(1)}} \cdots b^{i_{\sigma(N)}}. \end{aligned}$$

However, it turns out that such a map does not satisfy the desired universal property, as showed in the following

Example 10.1.1. *Consider the quasi–triangular complex Lie bialgebra $\mathfrak{sl}_2$ with standard generators e, f, h and standard r–matrix $r = e \otimes f + \frac{1}{4} h \otimes h$. Take the Poincaré–Birkhoff–Witt basis $\{e^i f^j h^k,\ i, j, k \in \mathbb{N}\}$ of $\mathsf{U}(\mathfrak{sl}_2)$ and consider the elements $\mathsf{id}_1 \in \mathfrak{S}_1$, $\mathsf{id}_2 \in \mathfrak{S}_2$, $(12) \in \mathfrak{S}_2$. Then it is easy to see – through a lenghty but elementary computation – that*

$$\begin{aligned} \rho_{\mathfrak{sl}_2}(\mathsf{id}_1) &= ef + \frac{1}{4} h^2 \\ (\rho_{\mathfrak{sl}_2}(\mathsf{id}_1))^2 &= e^2 f^2 - efh + \frac{efh^2}{2} + \frac{h^4}{16} + 2ef \\ \rho_{\mathfrak{sl}_2}(\mathsf{id}_2) &= e^2 f^2 - efh + \frac{efh^2}{2} + \frac{h^4}{16} + ef \\ \rho_{\mathfrak{sl}_2}((12)) &= e^2 f^2 - efh + \frac{efh^2}{2} + \frac{h^4}{16} \end{aligned}$$

It is clear that $(\rho_{\mathfrak{sl}_2}(\mathsf{id}_1))^2 \neq \rho_{\mathfrak{sl}_2}(\mathsf{id}_2)$, i.e. that the concatenation of permutations does not satisfy the required universal property. On the other hand, we have the following identity

$$(\rho_{\mathfrak{sl}_2}(\mathsf{id}_1))^2 = 2 \cdot \rho_{\mathfrak{sl}_2}(\mathsf{id}_2) - \rho_{\mathfrak{sl}_2}((12)). \tag{10.1.2}$$

In [ATL19] Appel and Toledano Laredo define a family of universal algebra satisfying the desired property (and in particular Equation (10.1.2), see Lemma 10.9.1), see the next Section for more details.

10.2 Drinfeld–Yetter universal algebras

In this Section we present Drinfeld–Yetter universal algebras, defined by Appel and Toledano Laredo – in their attempt to clarify the Enriquez's construction – in [ATL19].

Definition 10.2.1. *Let $n \geqslant 1$. The n–th* ***universal Drinfeld–Yetter algebra*** $\mathfrak{U}^n_{\mathsf{DY}}$ *is*

$$\mathfrak{U}^n_{\mathsf{DY}} \coloneqq \mathsf{End}_{\mathsf{DY}^n}([V_1] \otimes \cdots \otimes [V_n])$$

where DY^n is the n–th Drinfeld Yetter PROP (see §8.6) and the associative multiplication is given by composition of endomorphisms.

The algebra $\mathfrak{U}^n_{\mathsf{DY}}$ has the following vector space structure. For any $N \in \mathbb{N}$ and $\underline{N} = (N_1, \ldots, N_n) \in \mathbb{N}^n$ such that $|\underline{N}| = N$ consider the following morphisms of DY^n

$$\pi^{(\underline{N})} : [N] \otimes \bigotimes_{k=1}^{n} [V_k] \to \bigotimes_{k=1}^{n} [V_k] \qquad \text{and} \qquad \pi^{*(\underline{N})} : \bigotimes_{k=1}^{n} [V_k] \to [N] \otimes \bigotimes_{k=1}^{n} [V_k]$$

which are respectively the ordered composition of N_i actions (resp. coactions) on $[V_i]$. Then we have

Proposition 10.2.2. *(*[ATL19, Prop. 5.12]*) The collection of elements*

$$\left\{ r^{\sigma}_{\underline{N},\underline{N}'} \coloneqq \pi^{(\underline{N})} \circ (\sigma \otimes \mathsf{id}) \circ \pi^{*(\underline{N}')} \right\}$$

where $N \geqslant 0$, $\underline{N}, \underline{N}' \in \mathbb{N}^n$ are such that $|\underline{N}| = |\underline{N}'| = N$, and $\sigma \in \mathfrak{S}_N$ is a basis of $\mathfrak{U}^n_{\mathsf{DY}}$.

We are going to study in detail the case $n = 1$. The following proposition gives the universal property of the algebra $\mathfrak{U}^1_{\mathsf{DY}}$:

Proposition 10.2.3. *For any finite–dimensional, quasi–triangular Lie bialgebra $\mathfrak{b}$ with r–matrix $r = \sum_{i \in I} a_i \otimes b_i$ the map*

$$\begin{aligned} \rho_{\mathfrak{b}} : \mathfrak{U}^1_{\mathsf{DY}} &\to \mathsf{U}(\mathfrak{b}) \\ r^{\sigma}_n &\mapsto \sum_{i_1 \in I} \cdots \sum_{i_n \in I} a_{i_1} \cdots a_{i_n} b_{i_{\sigma^{-1}(n)}} \cdots b_{i_{\sigma^{-1}(1)}} \end{aligned} \tag{10.2.1}$$

is a morphism of algebras, where $r^{\sigma}_n \coloneqq \pi_1^{*(n)} \circ (\sigma \otimes \mathsf{id}_{[V_1]}) \circ \ {\pi_1}^{(n)}$, see next Section for more details.

It is possible to show that for any infinite–dimensional, quasi–triangular Lie bialgebra $\mathfrak{b}$ the map (10.2.1) satisfies – up to completing opportunely the algebra $\mathsf{U}(\mathfrak{b})$ – such a universal property. Next, we give a relationship between the algebras $\mathfrak{U}^n_{\mathsf{DY}}$ and $\mathcal{U}\mathfrak{S}^n_{univ}$ in the following

Proposition 10.2.4. *We have that:*

(i) ([ATL19, p.31]*) There exist algebra homomorphisms $\Delta^n_i : \mathfrak{U}^n_{\mathsf{DY}} \to \mathfrak{U}^{n+1}_{\mathsf{DY}}$ giving to the tower of algebras $\{\mathfrak{U}^n_{\mathsf{DY}}\}_{n \geqslant 0}$ the structure of a cosimplicial complex.*

(ii) ([ATL19, p.37]*) The collection of maps*

$$\begin{aligned} \xi^n : \mathfrak{U}^n_{\mathsf{DY}} &\to \mathcal{U}\mathfrak{S}^n_{univ} \\ r^{\sigma}_{\underline{N},\underline{N}'} &\mapsto x_{\underline{N}} \otimes y_{\tilde{\sigma}(\underline{N}')} \end{aligned}$$

is a collection of isomorphims of vector spaces, where $\tilde{\sigma} \coloneqq \sigma^{-1} \circ \tau$, and τ is the element of $\mathfrak{S}_N$ such that $\tau(i) = N - i$.

(iii) ([ATL19, p.44]*) The isomorphisms ξ^n induce an isomorphism of cosimplicial chains.*

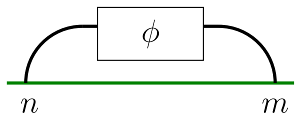
10.3 The universal Drinfeld–Yetter algebra $\mathfrak{U}^1_{\mathsf{DY}}$

In this Section we show that, as a vector space, the algebra $\mathfrak{U}^1_{\mathsf{DY}}$ is isomorphic to the direct sum of all group algebras of all symmetric groups. We then provide an an algorithmic process to calculate the multiplication with respect to the standard basis (i.e. the basis indexed by all permutations of all symmetric groups).

Notation 10.3.1. *From now on we will denote composition between morphisms in PROPs from left to right (as they are read in the respectively pictorial representation).*

Definition 10.3.2. *The universal Drinfeld–Yetter algebra is* $\mathfrak{U}^1_{\mathsf{DY}} := \mathsf{End}_{\mathsf{DY}}([V_1])$.

Note that any element of $\mathfrak{U}^1_{\mathsf{DY}}$ can be represented by a linear combination of oriented diagrams of the form

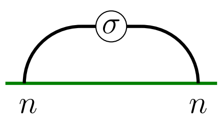


(10.3.1)

i.e. each of them necessarily starting with a certain number $n \geqslant 1$ of coactions, ending with a certain number $m \geqslant 1$ of actions, and containing a morphism $\phi \in \mathsf{Hom}_{\mathsf{DY}^1}([n] \otimes [V_1], [m] \otimes [V_1])$ in the middle. If n is a non–negative integer and $\sigma \in \mathfrak{S}_n$, we denote by r^σ_n the following element of $\mathfrak{U}^1_{\mathsf{DY}}$

$$r^\sigma_n = \pi_1^{*(n)} \circ (\sigma \otimes \mathsf{id}_{[V_1]}) \circ \ {\pi_1}^{(n)},$$

where $\pi_1^{(n)} : [n] \otimes [V_1] \to [V_1]$ denotes the n-th iterated action and $\pi_1^{*(n)} : [V_1] \to [n] \otimes [V_1]$ denotes the n-th iterated coaction. The pictorial representation of r^σ_n is the following



(10.3.2)

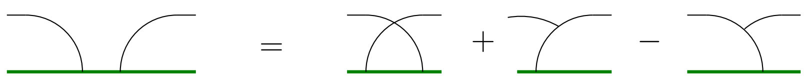
The following result describes the vector space structure of $\mathfrak{U}^1_{\mathsf{DY}}$:

Proposition 10.3.3. *The collection of endomorphisms of* $[V_1]$ *given by*

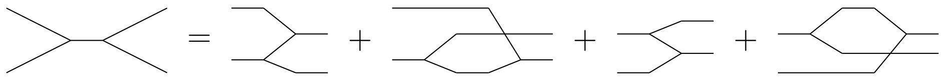
$$\mathcal{B} = \{r^\sigma_n, n \geqslant 0, \sigma \in \mathfrak{S}_n\} \tag{10.3.3}$$

is a basis of $\mathfrak{U}^1_{\mathsf{DY}}$.

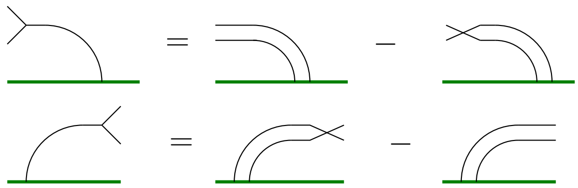
Proof. We proceed as in [ATL19, 5.10]. In order to show that the set $\mathcal{B}$ generates $\mathfrak{U}^1_{\mathsf{DY}}$, let $f \in \mathsf{End}_{\mathsf{DY}}([V_1])$ be represented by a linear combination of oriented diagrams of the form (10.3.1). The Drinfeld–Yetter relation (8.6.4)



allows to reorder π_1 and π_1^*, moving every coaction before any action. The cocycle condition (8.2.4)



allows to reorder brackets and cobrackets in such a way cobrackets horizontally precede brackets. Finally, the relations (8.6.2), (8.6.3)



allow to remove from the graph every μ and every δ involved. It follows that f can be represented as a linear combination of endomorphisms of type (10.3.2), showing that $\mathcal{B}$ is a set of generators for $\mathfrak{U}^1_{\mathsf{DY}}$. In order to show that the r_n^σ's are linearly independent, consider a Lie bialgebra $\mathfrak{b}$ whose underlying Lie algebra is free. Therefore, any non–trivial linear combination $\sum_i c_i r_{n_i}^{\sigma_i} = 0$ would induce, through the universal property of $\mathfrak{U}^1_{\mathsf{DY}}$, a non–trivial relation in $\mathsf{U}(\mathfrak{b})$, contradicting its freeness. ☐

With a similar argument one can construct a basis for $\mathfrak{U}^n_{\mathsf{DY}}$ for $n \geqslant 2$, see [ATL19, 5.12] for more details. As a direct consequence of the result above, we obtain the following

Corollary 10.3.4. *There is a canonical isomorphism of vector spaces*

$$\mathfrak{U}^1_{\mathsf{DY}} \simeq \bigoplus_{n \geqslant 0} \mathbb{K}[\mathfrak{S}_n] \tag{10.3.4}$$

mapping r_n^σ to $\sigma \in \mathfrak{S}_n$.

From now on we shall refer to (10.3.3) as the standard basis of $\mathfrak{U}^1_{\mathsf{DY}}$. In addition to describing the vector space structure of the algebra $\mathfrak{U}^1_{\mathsf{DY}}$, Proposition 10.3.3 gives an algorithmic way to compute the multiplication with respect to the standard basis (10.3.3). Namely, one can proceed in the following way: given $r_n^\sigma, r_m^\tau \in \mathcal{B}$, consider the following algorithm:

(1) Apply repeatedly the Drinfeld–Yetter rule (8.6.4) until there is no action preceding any coaction;

(2) Apply repeatedly the action rule (8.6.2) in order to remove all the brackets from the expansion obtained in (1);

(3) Apply repeatedly the coaction rule (8.6.3) in order to remove all the cobrackets from the expansion obtained in (2).

The result of this process gives the multiplication $r_n^\sigma \circ r_m^\tau$ with respect to the standard basis (10.3.3).

Remark 10.3.5. *Since in every step of the algorithm described above the total number of the strings is preserved, we have that $\mathfrak{U}^1_{\mathsf{DY}}$ is a $\mathbb{N}$–graded algebra: for any $r^\sigma_n, r^\tau_m \in \mathcal{B}$ there exist unique coefficients $c^\gamma_{\sigma,\tau}$ such that*

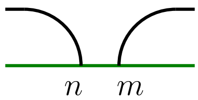
$$r^\sigma_n \circ r^\tau_m = \sum_{\gamma \in \mathfrak{S}_{n+m}} c^\gamma_{\sigma,\tau} \cdot r^\gamma_{n+m}. \tag{10.3.5}$$

Moreover, it follows from the identities involved that the coefficients $c^\gamma_{\sigma,\tau}$ are integers. Therefore, the structure of $\mathfrak{U}^1_{\mathsf{DY}}$ naturally induces a $\mathbb{N}$–graded algebra structure (with integral structure constants) on the vector space $\bigoplus_{n \geqslant 0} \mathbb{K}[\mathfrak{S}_n]$.

A really challenging problem is to find an explicit formula for the structure constants $c^\gamma_{\sigma,\tau}$ in terms of symmetric groups. This problem appears to be highly nontrivial: for instance, the number of summands appearing after the application of the algorithm described above seems to have exponential growth, as conjectured in 10.10.1.
In the next Sections we shall describe the algebra structure of $\mathfrak{U}^1_{\mathsf{DY}}$ through some combinatorial objects, namely the Drinfeld–Yetter mosaics and Drinfeld–Yetter looms.

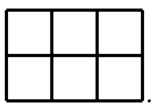
10.4 Drinfeld–Yetter mosaics

In this Section we define the set $\mathfrak{M}_{n,m}$ of $n \times m$ Drinfeld–Yetter mosaics, which will provide, through Proposition 10.6.5, a combinatorial description of the morphism $\pi_1^{(n)} \circ \pi_1^{*(m)}$, which is represented by the picture



More specifically, every element of $\mathfrak{M}_{n,m}$ will represent a morphism appearing in the sum of morphisms generated by the iterated application of the Drinfeld–Yetter rule (8.6.4) to $\pi_1^{(n)} \circ \pi_1^{*(m)}$, hence giving a combinatorial description of the application of step (1) of the algorithm described in the previous Section. We finally present the combinatorial properties of the set $\mathfrak{M}_{n,m}$.

Notation 10.4.1. *Let $n, m \geqslant 1$. We denote by $\mathcal{G}_{n,m}$ the grid with n rows and m columns. For example, $\mathcal{G}_{2,3}$ is*

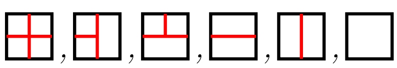


If $\mathcal{T}$ is a given set of tiles, we define a tiling of $\mathcal{G}_{n,m}$ with the elements of $\mathcal{T}$ as a map

$$F : \{1, \dots, n\} \times \{1, \dots, m\} \to \mathcal{T}.$$

We shall denote $F(i, j)$ by $F_{i,j}$. Roughly speaking, a tiling of $\mathcal{G}_{n,m}$ consists of assigning to each position of the empty grid a tile of $\mathcal{T}$.

Definition 10.4.2. *Let $n, m \geqslant 1$ and let $\mathcal{T}_{\mathfrak{M}}$ be the following set of tiles*

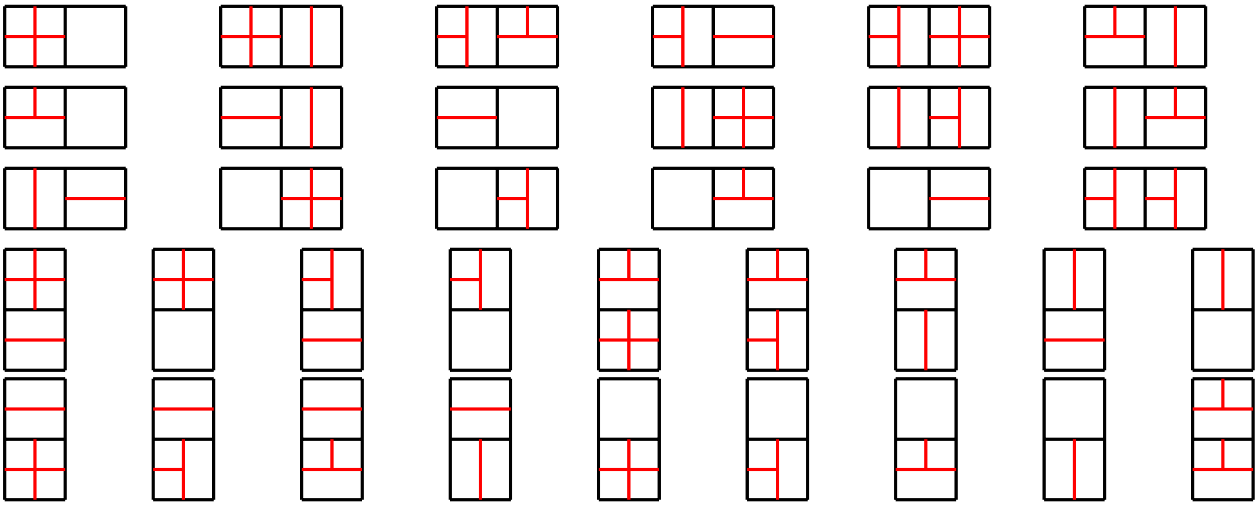
$\mathcal{T}_{\mathfrak{M}} = \Big\{$  $\Big\}$. (10.4.1)

*We define the set of $n \times m$ **Drinfeld–Yetter mosaic** $\mathfrak{M}_{n,m}$ as the set of all possible fillings M of $\mathcal{G}_{n,m}$ with the elements of $\mathcal{T}_{\mathfrak{M}}$ such that the following three conditions are satisfied:*

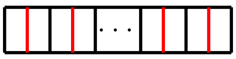
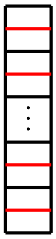
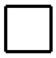
(1): $M_{1,j} \notin \{$  ,  $\}$ *for all* $j \in \{1, \dots, m\}$.

(2): $M_{i,1} \notin \{$  ,  $\}$ *for all* $i \in \{1, \dots, n\}$.

(3): None of the following configurations appear in M:



Roughly speaking, the third condition avoids the existence of Drinfeld–Yetter mosaic in which there is *discontinuity* between the red lines. We set by convention

$\mathfrak{M}_{0,m} = \{$  $\}$ $\quad \mathfrak{M}_{n,0} = \{$  $\}$ $\quad \mathfrak{M}_{0,0} = \{$  $\}$.

We shall respectively call the tiles of (10.4.1) the permutation, bracket, cobracket, action, coaction and empty tile.

Notation 10.4.3. *We will need the following auxiliary functions counting the number of bracket and cobracket tiles in a Drinfeld–Yetter mosaic:*

$$\alpha : \mathfrak{M}_{n,m} \to \mathbb{Z}_{\geqslant 0}$$
$$\beta : \mathfrak{M}_{n,m} \to \mathbb{Z}_{\geqslant 0}$$

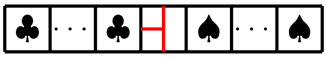
defined by

$\alpha(M) = \#\{M_{i,j} =$  $\}$

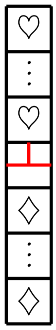
and

$\beta(M) = \#\{M_{i,j} =$  $\}$.

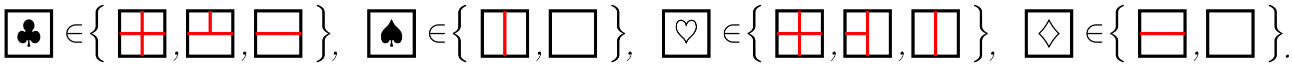
Remark 10.4.4. *Let $n, m \geqslant 1$. It follows by Definition 10.4.2 that the rows and the columns of an element of $\mathfrak{M}_{n,m}$ are respectively of the form*



and



where



We therefore deduce the following facts:

- *Every row of a Drinfeld–Yetter mosaic has at most one bracket tile.*
- *Every column of a Drinfeld–Yetter mosaic has at most one cobracket tile.*
- *The set $\mathfrak{M}_{n,m}$ splits into disjoint union of three subsets*

$$\mathfrak{M}_{n,m} = \mathfrak{M}^{\boxplus}_{n,m} \sqcup \mathfrak{M}^{\boxplus}_{n,m} \sqcup \mathfrak{M}^{\boxplus}_{n,m} \tag{10.4.2}$$

where

$$\mathfrak{M}^{\boxed{\ast}}_{n,m} := \{M \in \mathfrak{M}_{n,m}\ ,\ M_{1,1} = \boxed{\ast}\}.$$

Relying on the previous facts, we obtain the following

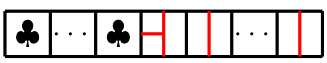
Proposition 10.4.5. *Let $\mathfrak{M}^{(\ell)}_{1,m}$ be the set of all $1 \times m$ Drinfeld–Yetter mosaics satisfying $\alpha(M) = m - \ell$ and $\mathfrak{M}^{[k]}_{n,1}$ be the set of all $n \times 1$ Drinfeld–Yetter mosaics satisfying $\beta(M) = n - k$. Then*

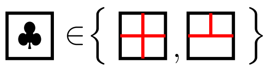
$$\mathfrak{M}_{n,m} = \bigsqcup_{\ell=0}^{m} \mathfrak{M}^{(\ell)}_{1,m} \times \mathfrak{M}_{n-1,\ell} \tag{10.4.3}$$

and

$$\mathfrak{M}_{n,m} = \bigsqcup_{k=0}^{n} \mathfrak{M}^{[k]}_{n,1} \times \mathfrak{M}_{k,m-1}. \tag{10.4.4}$$

Proof. We show (10.4.3), the proof of (10.4.4) is analogous. In order to construct a Drinfeld–Yetter mosaic $M \in \mathfrak{M}_{n,m}$, one can freely assign the first row of M, which is an element y of $\mathfrak{M}_{1,m}$ having the following form

$y =$ 



For any cobracket tile in y, we have that the tiles below are automatically determined. More precisely, according to the defining rules of the Drinfeld–Yetter mosaics, if $M_{1,j}$ is a cobracket tile, then

$M_{i,j} =$  if $M_{i,1} =$ 

 otherwise .

Therefore, if $\alpha(y) = m - \ell$, we have $m - \ell$ columns of M automatically determined. The other ℓ columns can be freely chosen among all the elements of $\mathfrak{M}_{n-1,\ell}$. □

As a consequence of the result above, we obtain two recursive formulas for $|\mathfrak{M}_{n,m}|$:

Corollary 10.4.6. *One has*

$$|\mathfrak{M}_{n,m}| = \sum_{\ell=0}^{m} \binom{m+1}{\ell} |\mathfrak{M}_{n-1,\ell}| \tag{10.4.5}$$

and

$$|\mathfrak{M}_{n,m}| = \sum_{k=0}^{n} \binom{n+1}{k} |\mathfrak{M}_{k,m-1}|. \tag{10.4.6}$$

Proof. We show (10.4.5), the proof of (10.4.6) is analogous. Let $M \in \mathfrak{M}_{1,m}$ and let t be the number of permutation and cobracket tiles of M. Then we have

$$|\mathfrak{M}_{1,m}^{(\ell)}| = \sum_{t=0}^{m} \binom{t}{m-\ell} = \sum_{t=m-\ell}^{m} \binom{t}{m-\ell} = \binom{m+1}{m-\ell+1} = \binom{m+1}{\ell}$$

where the third equality follows by the well–known Hockey–Stick identity. □

We now provide a formula for the cardinality of $\mathfrak{M}_{n,m}$ which involves the Stirling number of the second kind. We set $F_{n,m} := |\mathfrak{M}_{n,m}|$. Then, it is easy to see that $F_{n,m}$ is symmetric (i.e. $F_{n,m} = F_{m,n}$). By iterating equations (10.4.5) and (10.4.6), we obtain

$$F_{n,m} = \sum_{\underline{k} \in I_{n,m}} \prod_{i=0}^{n} \binom{k_i + 1}{k_{i+1}} \tag{10.4.7}$$

where $I_{n,m} := \{(k_0, k_1, \ldots, k_n) \mid k_0 := m,\ k_i \leqslant k_{i-1}, i = 1, \ldots, n\}$.
Recall that the Stirling numbers of the second kind are the non–negative integers $\left\{ {n \atop k} \right\}$ counting the number of ways to partition a set of n labelled objects into k non–empty unlabelled subsets, and they satisfy the recursive relation

$$\left\{ {n+1 \atop k+1} \right\} = (k+1) \cdot \left\{ {n \atop k+1} \right\} + \left\{ {n \atop k} \right\} \tag{10.4.8}$$

with initial conditions

$$\left\{ {0 \atop 0} \right\} = 1 \quad \text{and} \quad \left\{ {n \atop 0} \right\} = \left\{ {0 \atop k} \right\} = 0.$$

One has

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \cdot \binom{k}{i} \cdot (k-i)^n.$$

We shall provide a concise expression of $F_{n,m}$ in terms of Stirling numbers. To this end, we shall use the following

Lemma 10.4.7. *For any* $0 \leqslant k \leqslant n+1$, *we have*

$$k \cdot \left\{ {n+1 \atop k} \right\} = \sum_{\ell=k-1}^{n} (-1)^{n-\ell} \cdot \binom{n+1}{\ell} \cdot \left\{ {\ell+1 \atop k} \right\}. \tag{10.4.9}$$

Proof. We proceed by induction on n. We observe that the cases $k = 0, n+1$ are trivial, while for $k = 1$ it reduces to the identity

$$\sum_{\ell=0}^{n+1} (-1)^{\ell} \binom{n+1}{\ell} = 0.$$

Therefore, the case $n = 1$ is clear.
Assume the result holds for $n - 1 \geqslant 0$. Then,

$$\begin{aligned}
&\sum_{\ell=k-1}^{n} (-1)^{n-\ell} \cdot \binom{n+1}{\ell} \cdot \left\{ {\ell+1 \atop k} \right\} = \\
&\quad = (n+1) \cdot \left\{ {n+1 \atop k} \right\} + \sum_{\ell=k-1}^{n-1} (-1)^{n-\ell} \cdot \binom{n+1}{\ell} \cdot \left\{ {\ell+1 \atop k} \right\} \\
&\quad = (n+1) \cdot \left\{ {n+1 \atop k} \right\} - \sum_{\ell=k-1}^{n-1} (-1)^{n-1-\ell} \cdot \binom{n}{\ell} \cdot \left\{ {\ell+1 \atop k} \right\} \\
&\qquad\qquad - \sum_{\ell=k-1}^{n-1} (-1)^{n-1-\ell} \cdot \binom{n}{\ell-1} \cdot \left\{ {\ell+1 \atop k} \right\} \\
&\quad = (n+1) \cdot \left\{ {n+1 \atop k} \right\} - k \cdot \left\{ {n \atop k} \right\} - k \sum_{\ell=k-1}^{n-1} (-1)^{n-1-\ell} \cdot \binom{n}{\ell-1} \cdot \left\{ {\ell \atop k} \right\} \\
&\qquad\qquad - \sum_{\ell=k-1}^{n-1} (-1)^{n-1-\ell} \cdot \binom{n}{\ell-1} \cdot \left\{ {\ell \atop k-1} \right\} \\
&\quad = (n+1) \cdot \left\{ {n+1 \atop k} \right\} - k \cdot \left\{ {n \atop k} \right\} + k \sum_{j=k-1}^{n-2} (-1)^{n-1-j} \cdot \binom{n}{j} \cdot \left\{ {j+1 \atop k} \right\} \\
&\qquad\qquad + \sum_{j=k-2}^{n-2} (-1)^{n-1-j} \cdot \binom{n}{j} \cdot \left\{ {j+1 \atop k-1} \right\} \\
&\quad = (n+1) \cdot \left\{ {n+1 \atop k} \right\} - k \cdot \left\{ {n \atop k} \right\} + k^2 \cdot \left\{ {n \atop k} \right\} - k \cdot n \cdot \left\{ {n \atop k} \right\} \\
&\qquad\qquad + (k-1) \cdot \left\{ {n \atop k-1} \right\} - n \cdot \left\{ {n \atop k-1} \right\} \\
&\quad = (n+1) \cdot \left\{ {n+1 \atop k} \right\} - (n-k+1) \cdot \left(k \cdot \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} \right) \\
&\quad = k \cdot \left\{ {n+1 \atop k} \right\}
\end{aligned}$$

where the second equality follows from the recursive identity for the binomial coefficient, the third one follows by induction and the recursive identity for the Stirling numbers (10.4.8), the fourth and fifth[1] one by induction, and the last one by (10.4.8). ☐

Relying on the above result, we get the following

Proposition 10.4.8. *For any $n, m \geqslant 0$, we have*

$$F_{n,m} = \sum_{k=1}^{n+1} (-1)^{n-k+1} \cdot k! \cdot k^m \cdot \left\{ {n+1 \atop k} \right\}. \tag{10.4.10}$$

Proof. Set $B_{n,k} \coloneqq (-1)^{n-k} \cdot (k-1)! \cdot \left\{ {n \atop k} \right\}$. Then, the formula (10.4.10) is equivalent to

$$F_{n,m} = \sum_{k=1}^{n+1} B_{n+1,k} \cdot k^{m+1}.$$

Therefore, it is enough to prove that the numbers

$$G_{n,m} \coloneqq \sum_{k=1}^{n+1} B_{n+1,k} \cdot k^{m+1}$$

satisfy the recursive relation (10.4.5). Note that (10.4.9) is equivalent to

$$k \cdot B_{n+1,k} = \sum_{\ell=k-1}^{n} \binom{n+1}{\ell} \cdot B_{\ell+1,k}. \tag{10.4.11}$$

Therefore,

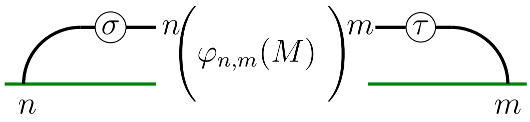
$$\begin{aligned}
\sum_{\ell=0}^{n} \binom{n+1}{\ell} G_{\ell,m-1} &= \sum_{\ell=0}^{n} \binom{n+1}{\ell} \sum_{k=1}^{\ell+1} B_{\ell+1,k} \cdot k^m \\
&= \sum_{\ell=0}^{n} \sum_{k=1}^{\ell+1} \binom{n+1}{\ell} B_{\ell+1,k} \cdot k^m \\
&= \sum_{k=1}^{n+1} \left(\sum_{\ell=k-1}^{n} \binom{n+1}{\ell} B_{\ell+1,k} \right) \cdot k^m \\
&= \sum_{k=1}^{n+1} B_{n+1,k} \cdot k^{m+1} = G_{n,m}
\end{aligned}$$

where the fourth identity follows from (10.4.11). Thus, $G_{n,m} = F_{n,m}$ since they satisfy the same recursion relation. ☐

10.5 Drinfeld–Yetter looms

In this Section, given $n, m \leqslant 1$, we define the set $\mathfrak{L}_{n,m}$ of $n \times m$ Drinfeld–Yetter looms, which will provide, through Lemma 10.6.7, a combinatorial description of the application of steps (2)–(3) of the algorithm of the product of $\mathfrak{U}^1_{\mathsf{DY}}$. More precisely, to any Drinfeld–Yetter mosaic M is associated a morphism pictorially represented by

[1] Note that $\left\{ {k-1 \atop k} \right\} = 0$, thus we can assume the first sum starts at $j = k-1$.

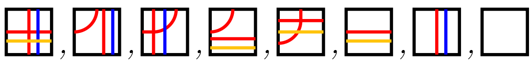


where $\varphi_{n,m}(M)$ is a morphism in $\mathsf{Hom}_{\mathsf{DY}^1}([V] \otimes [n], [V] \otimes [m])$ containing a Lie bracket (resp. a Lie cobracket) if and only if the tile  (resp ) appears in M, see Equation (10.6.1). The process of removing Lie brackets and Lie cobrackets from the morphism

$$\pi_1^{*(n)} \circ (\mathsf{id}_{[1]} \otimes \sigma) \circ \varphi_{n,m}(M) \circ (\mathsf{id}_{[1]} \otimes \tau) \circ \pi_1^{(m)}$$

through formulas (8.6.2) and (8.6.3) will translates in associating to any $M \in \mathfrak{M}_{n,m}$ a set of Drinfeld–Yetter looms $\mathfrak{L}(M)$ (see Equation (10.5.1)). As the set of Drinfeld–Yetter mosaics $\mathfrak{M}_{n,m}$, we shall define the set $\mathfrak{L}_{n,m}$ as a set of tilings of the empty grid. Clearly, we shall use a different set of tiles and of tiling rules, since we need to get rid of Lie bracket and Lie cobracket tiles.

Definition 10.5.1. *Let $n, m \geqslant 1$ and let $\mathcal{T}_{\mathfrak{L}_{n,m}}$ be the following set of tiles*

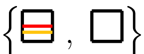
$$\mathcal{T}_{\mathfrak{L}_{n,m}} = \Big\{$$  $$\Big\}$$

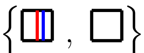
where the yellow line denotes $k \in \{0, \dots, m-1\}$ red horizontal lines and the blue line denotes $\ell \in \{0, \dots, n-1\}$ red vertical lines. To any tile $\mathcal{T}$ of $\mathcal{T}_{\mathfrak{L}_{n,m}}$, we associate a four–tuple of integers (t, b, l, r) that respectively indicates the number of strings occurring on the top, bottom, left and right edge of $\mathcal{T}$. We define the set of $n \times m$ ***Drinfeld–Yetter looms*** *$\mathfrak{L}_{n,m}$ as the set of all possible tilings L of $\mathcal{G}_{n,m}$ with the elements of $\mathcal{T}_{\mathfrak{L}_{n,m}}$ such that the following five conditions are satisfied:*

(1): $b_{i,j} = t_{i+1,j}$ for all $i = 1, \dots, m-1$ and $j = 1, \dots, n$;

(2): $l_{i,j} = r_{i,j+1}$ for all $i = 1, \dots, m$ and $j = 1, \dots, n-1$;

(3): $\sum_{i=1}^{n} l_{i,1} + \sum_{j=1}^{m} b_{n,j} = \sum_{i=1}^{n} r_{i,m} + \sum_{j=1}^{m} t_{1,j} = n + m$;

(4): $R_{1,j} \notin \Big\{$  $\Big\}$ for all $j \in \{1, \dots, m\}$;

(5): $R_{i,1} \notin \Big\{$  $\Big\}$ for all $i \in \{1, \dots, n\}$,

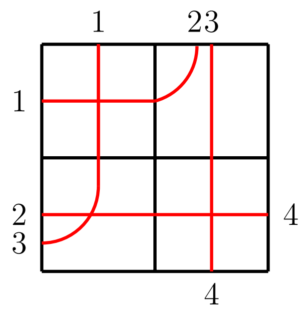
where $(t_{i,j}, b_{i,j}, l_{i,j}, r_{i,j})$ denotes the tuple of weights of $L_{i,j}$. We set by convention $\mathfrak{L}_{0,m} = \mathfrak{M}_{0,m}$, $\mathfrak{L}_{n,0} = \mathfrak{M}_{n,0}$ and $\mathfrak{L}_{0,0} = \mathfrak{M}_{0,0}$.

As in the case of the Drinfeld–Yetter mosaics, the first two conditions avoids the existence of Drinfeld–Yetter looms in which there is discontinuity between the red lines. The fourth and fifth conditions are the analogous of the first two conditions of Definition 10.4.2.

Remark 10.5.2. *Note that, as opposed to the case of Drinfeld–Yetter mosaics, the set of tiles of $\mathfrak{L}_{n,m}$ depends on n and m (more precisely, its cardinality is $nm + 3m + 3n$).*

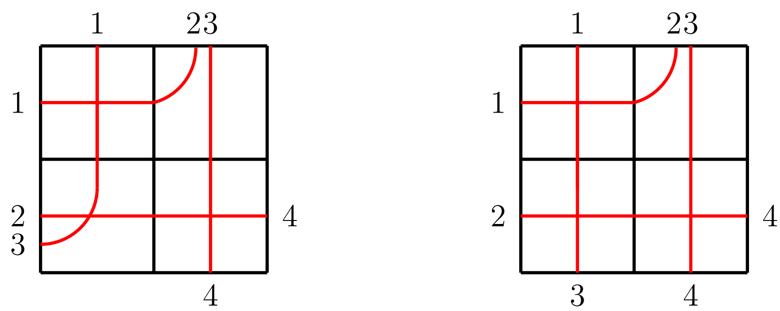
For any $L \in \mathfrak{L}_{n,m}$, we shall interpret the strings occurring in the left edges of the first column of L and the strings occurring in the down edges of the last column of L (resp. the strings occurring in the right edges of the last column of L and the strings occurring in the up edges of the first column of L) as ingoing (resp. ongoing) strings. Note that the path of any ingoing string across the Drinfeld–Yetter loom must end in a outgoing string. We can therefore associate to any Drinfeld–Yetter loom $L \in \mathfrak{L}_{n,m}$ a permutation in $\mathfrak{S}_{n+m}$ in the following way: we assign to any ingoing string of L a number in $\{1, \ldots, n+m\}$, starting from the top left to the bottom left and carrying on from the bottom left to the bottom right. In the same way, we assign to any outgoing string of L a number in $\{1, \ldots, n+m\}$, starting from the top left to the top right and carrying on from the top right to the bottom right.

Example 10.5.3. *The permutation associated to the following Drinfeld–Yetter loom*



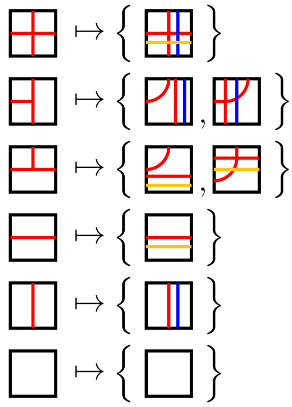
is $(1243) \in \mathfrak{S}_4$.

The procedure described above defines a family of maps $\gamma_{n,m} : \mathfrak{L}_{n,m} \to \mathfrak{S}_{n+m}$. Note that the maps $\gamma_{n,m}$ are in general not injective nor surjective. For example, for the following $L_1, L_2 \in \mathfrak{L}_{2,2}$



we have $\gamma_{2,2}(L_1) = \gamma_{2,2}(L_2)$. On the other hand, one can show through a direct inspection that there is no Drinfeld–Yetter loom $L \in \mathfrak{L}_{1,3}$ such that $\gamma_{1,3}(L) = (12)(34) \in \mathfrak{S}_4$.
Next, we give a connection between the sets $\mathfrak{M}_{n,m}$ and $\mathfrak{L}_{n,m}$. For any $M \in \mathfrak{M}_{n,m}$ with tiles $\{M_{1,1}, \ldots, M_{n,m}\}$ consider the following function associated to M

$$f_M : \{M_{1,1}, \ldots, M_{n,m}\} \to \mathcal{P}(\mathcal{T}_{\mathfrak{L}_{n,m}})$$



where $\mathcal{P}(\mathcal{T}_{\mathfrak{L}_{n,m}})$ denotes the power set of $\mathcal{T}_{\mathfrak{L}_{n,m}}$. Here the numbers of red lines corresponding to the yellow and the blue ones are fixed: the yellow horizontal line in the tiles of $f_M(M_{i,j})$ denotes exactly

$$k = \#\{M_{i,t} = \boxminus, t > j\}$$

red horizontal lines and the blue vertical line in the tiles of $f_M(M_{i,j})$ denotes exactly

$$\ell = \#\{M_{s,j} = \boxminus, s > i\}$$

red vertical lines.

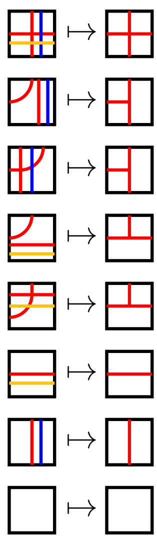
Definition 10.5.4. *The set $\mathfrak{L}(M)$ of all the Drinfeld–Yetter looms related to M is*

$$\mathfrak{L}(M) = \{L \in \mathfrak{L}_{n,m} \mid L_{i,j} \in f_M(M_{i,j})\}. \tag{10.5.1}$$

Proposition 10.5.5. *The collection $\{\mathfrak{L}(M)\}_{M \in \mathfrak{M}_{n,m}}$ defines a partition of $\mathfrak{L}_{n,m}$.*

Proof. It is clear that $\mathfrak{L}(M)$ is non–empty for any $M \in \mathfrak{M}_{n,m}$ and that $M_1 \neq M_2$ implies $\mathfrak{L}(M_1) \cap \mathfrak{L}(M_1) = \emptyset$. It remains to prove that the collection $\{\mathfrak{L}(M)\}_{M \in \mathfrak{M}_{n,m}}$ defines a covering of $\mathfrak{L}_{n,m}$, i.e. that for any $L \in \mathfrak{L}_{n,m}$ there exists $M \in \mathfrak{M}_{n,m}$ such that $L \in \mathfrak{L}(M)$. For any $L \in \mathfrak{L}_{n,m}$, we construct such a M in the following way: consider the following map

$$\chi : \mathcal{T}_{\mathfrak{L}_{n,m}} \to \mathcal{T}_{\mathfrak{M}}$$



We define the Drinfeld–Yetter loom L_M associated to M as the one with $L_{i,j} = \chi(L_{i,j})$. It is easy to see that $L \in \mathfrak{L}(M_M)$, hence the claim is proved. $\square$

Remark 10.5.6. *Note that we have the following formula for the cardinality of* $\mathfrak{L}(M)$*:*

$$|\mathfrak{L}(M)| = 2^{\alpha(M)+\beta(M)}. \tag{10.5.2}$$

Notation 10.5.7. *As in the case of Drinfeld–Yetter mosaics, we shall need some auxiliary functions counting the amount of some tiles occurring in a Drinfeld–Yetter loom: we set*

$$\zeta : \mathfrak{L}_{n,m} \to \mathbb{Z}_{\geqslant 0}$$
$$\xi : \mathfrak{L}_{n,m} \to \mathbb{Z}_{\geqslant 0}$$

defined by

$\zeta(L) = \#\{L_{i,j} =$  , $\}$

$\xi(L) = \#\{L_{i,j} =$  , $\}.$

Proposition 10.5.8. *let* $M \in \mathfrak{M}_{n,m}$ *and let* $L \in \mathfrak{L}(M)$*. Then*

$$(-1)^{\alpha(M)}(-1)^{\zeta(L)} = (-1)^{\xi(L)}.$$

Proof. Set $k = \#\{L_{i,j} =$ $\}$, $\ell = \#\{L_{i,j} =$ $\}$ and $h = \#\{L_{i,j} =$ $\}$. By definition of the auxiliary functions we have $\alpha(M) = k + \ell$, and then $(-1)^{\alpha(M)}(-1)^k = (-1)^\ell$. To end the proof it suffices to multiply both sides for $(-1)^h$. $\square$

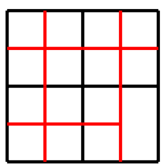
10.6 An explicit formula for the multiplication of $\mathfrak{U}^1_{\mathsf{DY}}$

The aim of this Section is to provide an explicit formula for the multiplication of $\mathfrak{U}^1_{\mathsf{DY}}$ with respect to the standard basis (10.3.3). We shall first present some preliminary results (namely Proposition 10.6.5 and Lemma 10.6.7) and then give the main result in Theorem 10.6.9.

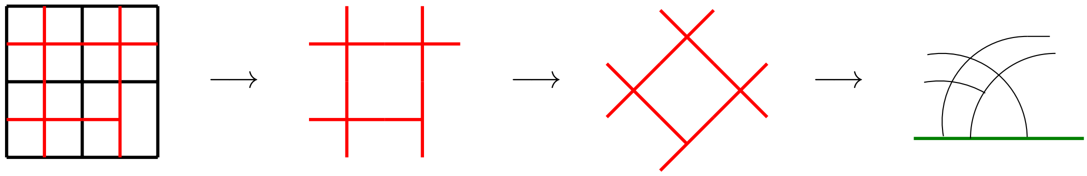
To any Drinfeld–Yetter mosaic $M \in \mathfrak{M}_{n,m}$ we associate a morphism in $\mathsf{Hom}_{\mathsf{DY}^1}([n] \otimes [V_1], [m] \otimes [V_1])$ by considering the picture obtained by removing all borders from the Drinfeld–Yetter mosaic and turning it 45 degrees clockwise. This procedure defines a colletion of maps

$$\varphi_{n,m} : \mathfrak{M}_{n,m} \to \mathsf{Hom}_{\mathsf{DY}^1}([n] \otimes [V_1], [m] \otimes [V_1]). \tag{10.6.1}$$

Example 10.6.1. *Given the following Drinfeld–Yetter mosaic*

$M =$ 

the procedure described above gives the following picture



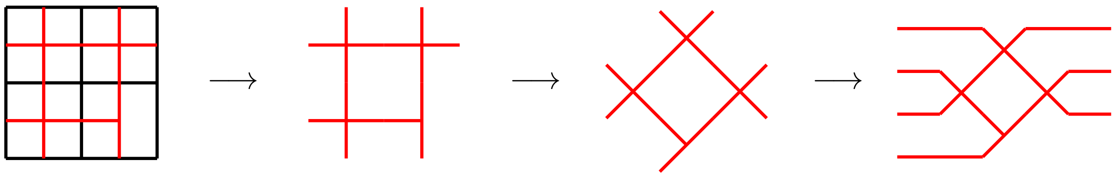
leading to the following morphism of $\mathsf{Hom}_{\mathsf{DY}^1}([2] \otimes [V_1], [2] \otimes [V_1])$*:*

$$\varphi_{2,2}(M) = (\mathsf{id}_{[2]} \otimes \pi_1^{*(2)}) \circ \big(\big((23) \circ \mathsf{id}_{[2]} \otimes \delta \circ (132)\big) \otimes \mathsf{id}_{[V_1]}\big) \circ (\mathsf{id}_{[2]} \otimes \pi_1).$$

Remark 10.6.2. *Note that* $\varphi_{0,m}(M) = \pi_1^{*(m)}$, $\varphi_{n,0}(M) = \pi_1^{(n)}$ *and* $\varphi_{0,0}(M) = \mathsf{id}_{[1]}$.

Similarly, for any $M \in \mathfrak{M}_{n,m}$ we denote by $M^\top$ the morphism of $\mathsf{Hom}_{\mathsf{DY}^1}([n+m-\alpha(M)], [n+m-\beta(M)])$ pictorially represented by removing all borders from M, turning it 45 degrees clockwise and attaching horizontal lines to the end and beginning of any diangonal line.

Example 10.6.3. *For the Drinfeld–Yetter mosaic* M *of the previous example, we obtain the following picture*



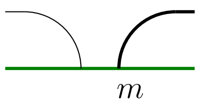
corresponding to the morphism

$$M^\top = (23) \circ (\mathsf{id}_{[2]} \otimes \delta) \circ (132) \in \mathsf{Hom}_{\mathsf{DY}^1}([4], [3]).$$

Note that for any $M \in \mathfrak{M}_{n,m}$ we have

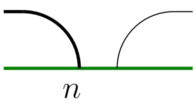
$$\varphi_{n,m}(M) = \big(\mathsf{id}_{[n]} \otimes \pi_1^{*(m-\alpha(M))}\big) \circ (M^\top \otimes \mathsf{id}_{[V_1]}) \circ \big(\mathsf{id}_{[m]} \otimes \pi_1^{(n-\beta(M))}\big). \tag{10.6.2}$$

Lemma 10.6.4. *For any* $n, m \geqslant 0$ *we have*



$$= \sum_{M \in \mathfrak{M}_{1,m}} (-1)^{\alpha(M)} \varphi_{1,m}(M)$$

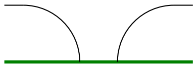
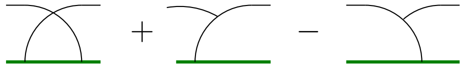
and



$$= \sum_{M \in \mathfrak{M}_{n,1}} (-1)^{\alpha(M)} \varphi_{n,1}(M)$$

.

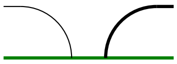
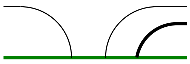
Proof. We prove the first identity by induction on $m \geqslant 0$. For $m = 0$ the claim holds trivially. For $m = 1$ we have

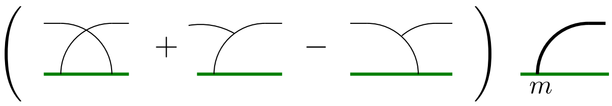
 $=$ 

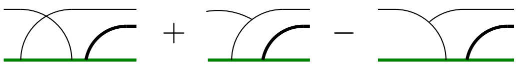
$= \varphi_{1,1}($  $) + \varphi_{1,1}($  $) - \varphi_{1,1}($  $)$

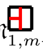
$$= \sum_{M\in\mathfrak{M}_{1,1}} (-1)^{\alpha(M)}\varphi_{1,1}(M)$$

where the first equality follows from Equation (8.6.4). If $m \geqslant 1$, we have

 $m{+}1$ $=$  m

$=$ 

$=$ 

$= \sum_{M\in\mathfrak{M}^{\square}_{1,m+1}} (-1)^{\alpha(M)}\varphi_{1,m+1}(M) + \sum_{M\in\mathfrak{M}^{\square}_{1,m+1}} (-1)^{\alpha(M)}\varphi_{1,m+1}(M)$ (superscript boxes: , )

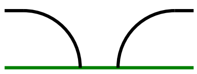
$+ \sum_{M\in\mathfrak{M}^{\square}_{1,m+1}} (-1)^{\alpha(M)}\varphi_{1,m+1}(M)$ (superscript box: )

$$= \sum_{M\in\mathfrak{M}_{1,m+1}} (-1)^{\alpha(M)}\varphi_{1,m+1}(M)$$

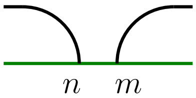
where the second equality follows from Equation (8.6.4) and the fourth equality follows by the inductive hypothesis. Then the first part of the claim is proved. The proof of the second part is analogous. $\square$

We can now prove the main result regarding Drinfeld–Yetter mosaics:

Proposition 10.6.5. *Let $n, m \geqslant 0$. Then*

 $n \quad m$ $= \sum_{M\in\mathfrak{M}_{n,m}} (-1)^{\alpha(M)}\varphi_{n,m}(M)$.

Proof. The cases $n = 0$, $m = 0$ and $n = m = 0$ hold trivially, while the cases $n = 1$ and $m = 1$ hold for Lemma 10.6.4. For $n, m \geqslant 2$ we have



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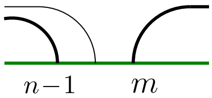

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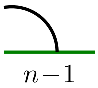
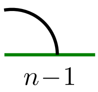
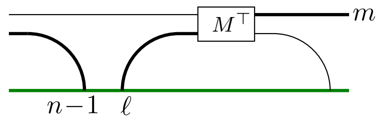


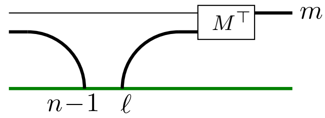
$$\left(\sum_{M\in\mathfrak{M}_{1,m}} (-1)^{\alpha(M)}\varphi_{1,m}(M)\right)$$

$=$


$$\left(\sum_{\ell=0}^{m} \sum_{M\in\mathfrak{M}^{(\ell)}_{1,m}} (-1)^{\alpha(M)}\varphi_{1,m}(M)\right)$$

$$= \sum_{\ell=0}^{m} \sum_{\substack{M\in\mathfrak{M}^{(\ell)}_{1,m}\\ \beta(M)=0}} (-1)^{\alpha(M)} \Bigg($$


$\Bigg)$

$$+ \sum_{\ell=0}^{m} \sum_{\substack{M\in\mathfrak{M}^{(\ell)}_{1,m}\\ \beta(M)=1}} (-1)^{\alpha(M)} \Bigg($$


$\Bigg)$

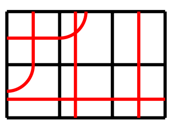
$$= \sum_{\ell=0}^{m} \sum_{M\in\mathfrak{M}^{(\ell)}_{1,m}\times\mathfrak{M}_{n-1,\ell}} (-1)^{\alpha(M)}\varphi_{n,m}(M)$$

$$= \sum_{M\in\mathfrak{M}_{n,m}} (-1)^{\alpha(M)}\varphi_{n,m}(M)$$

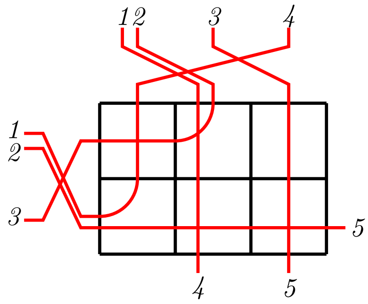
where the second equality follows from Lemma 10.6.4, the third and the fifth equalities follows from Proposition 10.4.5, the fourth equality follows from Equation (10.6.2) and the sixth equality follows from Equation (10.4.3). □

Let $L \in \mathfrak{L}_{n,m}$, $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_m$. We want to associate a permutation in $\mathfrak{S}_{n+m}$ to such data, by *gluing* σ to the left edge of L and τ to the top edge of L. However, since the number of strings occurring in the left (resp. top) edge of L may be greater than n (resp. m), we extend the permutations in such a way they move at the same time multiple strings of a tile. Finally, we get a permutation of $\mathfrak{S}_{n+m}$ by labelling the strings from 1 to $n + m$, following the same argument of Example 10.5.3.

Example 10.6.6. *Consider the following Drinfeld–Yetter loom* $L \in \mathfrak{L}_{2,3}$



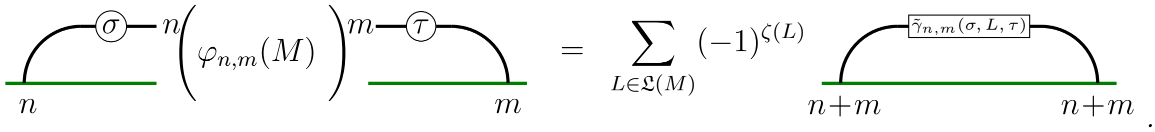
and the permutations $\sigma = (12) \in \mathfrak{S}_2$ *and* $\tau = (132) \in \mathfrak{S}_3$*. By gluing* σ *to the left edge of* L *and* τ *to the top edge of* L*, we obtain the picture*



obtaining the permutation $(14)(253) \in \mathfrak{S}_5$*.*

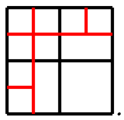
The procedure described above defines a family of maps $\tilde{\gamma}_{n,m} : \mathfrak{S}_n \times \mathfrak{L}_{n,m} \times \mathfrak{S}_m \to \mathfrak{S}_{n+m}$. Note that for all $L \in \mathfrak{L}_{n,m}$ we have $\tilde{\gamma}_{n,m}(\mathsf{id}_n, L, \mathsf{id}_m) = \gamma_{n,m}(L)$.

Lemma 10.6.7. *Let* $M \in \mathfrak{M}_{n,m}$*. Then*

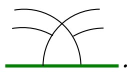


Before proving the Lemma, let us give an

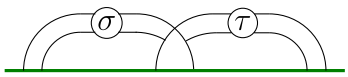
Example 10.6.8. *Let* M *be the following* 2×2 *Drinfeld–Yetter mosaic*



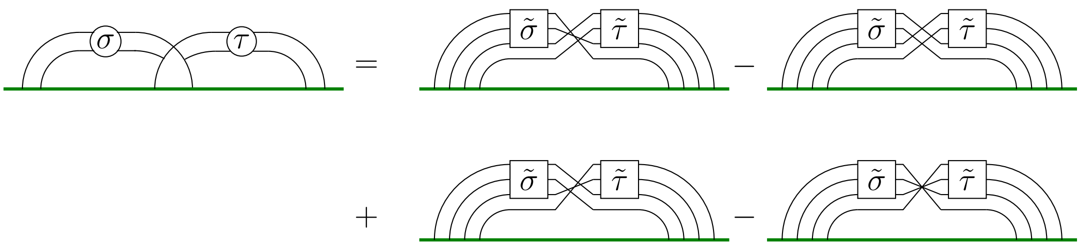
Then the morphism $\varphi_{2,2}(M)$ *is pictorially represented by*



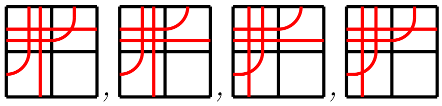
Gluing two permutations $\sigma, \tau \in \mathfrak{S}_2$ *to* $\varphi_{2,2}(M)$ *and composing with* $\pi_1^{*(2)}$ *and* $\pi_1^{(2)}$ *we obtain*



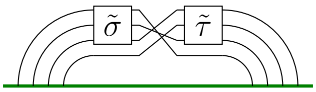
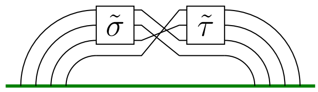
and applying relations (8.6.2) *and* (8.6.3)*, we obtain*

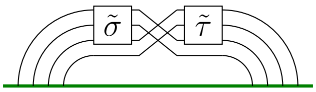
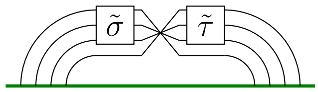


where $\tilde{\sigma}$ and $\tilde{\tau}$ are the permutations of $\mathfrak{S}_3$ moving the first two strings as they were one. On the other hand, we have that

$\mathfrak{L}(M) = \{L_1, L_2, L_3, L_4\} = \Big\{$  $\Big\}$

and it is easy to see that

$r_4^{\tilde{\gamma}_{2,2}(\sigma,L_1,\tau)} =$  $\quad r_4^{\tilde{\gamma}_{2,2}(\sigma,L_3,\tau)} =$ 

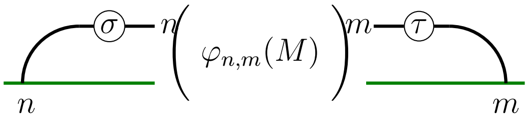
$r_4^{\tilde{\gamma}_{2,2}(\sigma,L_2,\tau)} =$  $\quad r_4^{\tilde{\gamma}_{2,2}(\sigma,L_4,\tau)} =$ 

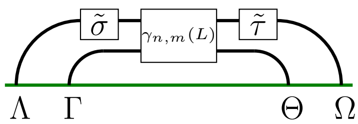
where the permutation in the middle of $r_4^{\tilde{\gamma}_{2,2}(\sigma,L_i,\tau)}$ is exactly $\gamma_{2,2}(L_i)$. We therefore get

$$\pi_1^{*(2)} \circ ((\sigma \otimes \mathsf{id}_{[V_1]}) \circ \varphi_{2,2}(M) \circ (\tau \otimes \mathsf{id}_{[V_1]})) \circ \pi_1^{(2)} = r_4^{\tilde{\gamma}_{2,2}(\sigma,L_1,\tau)} - r_4^{\tilde{\gamma}_{2,2}(\sigma,L_2,\tau)} + r_4^{\tilde{\gamma}_{2,2}(\sigma,L_3,\tau)} - r_4^{\tilde{\gamma}_{2,2}(\sigma,L_4,\tau)}$$

as is claimed in the statement.

Proof. Let $M \in \mathfrak{M}_{n,m}$ and let $\varphi_{n,m}(M)$ be the associated morphism in DY^1. Applying relations (8.6.2), (8.6.3) we get



$= \sum_{L \in \mathfrak{L}(M)} (-1)^{\zeta(L)}$ 

where

$$\Lambda = \sum_{i=1}^{n} l_{i1}, \qquad \Omega = \sum_{j=1}^{m} t_{1j}, \qquad \Gamma = n + m - \Lambda, \qquad \Theta = n + m - \Omega.$$

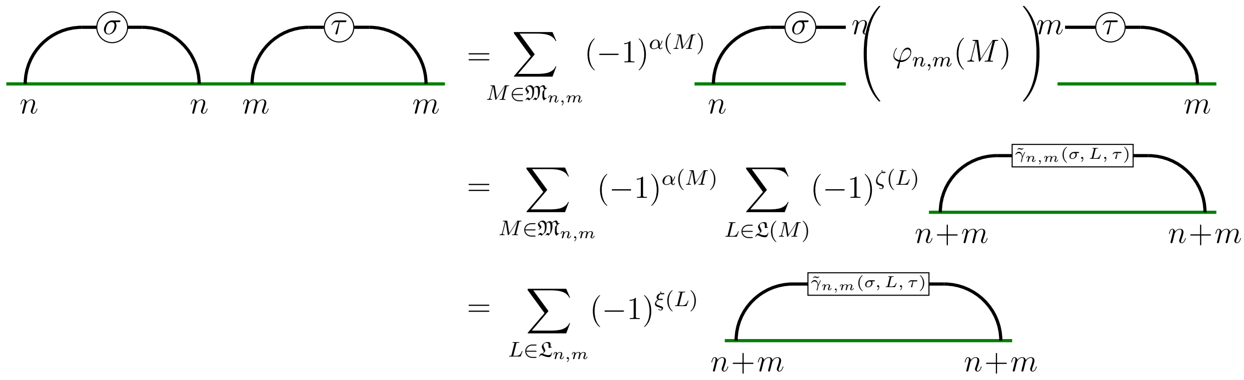
To end the proof it suffices to note that $\tilde{\gamma}_{n,m}(\sigma, L, \tau) = (\tilde{\sigma} \otimes \mathsf{id}_{[\Gamma]}) \circ \gamma_{n,m}(L) \circ (\tilde{\tau} \otimes \mathsf{id}_{[\Theta]})$. □

We now have all the ingredients to give an explicit formula for the multiplication of $\mathfrak{U}^1_{\mathsf{DY}}$.

Theorem 10.6.9. *We have*

$$r_n^{\sigma} \circ r_m^{\tau} = \sum_{L \in \mathfrak{L}_{n,m}} (-1)^{\xi(L)} r_{n+m}^{\tilde{\gamma}_{n,m}(\sigma,L,\tau)} \tag{10.6.3}$$

Proof. The proof is pictorial. We have



where the first equality follows by Proposition 10.6.5, the second equality follows by Lemma 10.6.7 and the third equality follows by Proposition 10.5.5 and Proposition 10.5.8. □

Theorem 10.6.9 gives an explicit formula for the product of $\mathfrak{U}^1_{\mathsf{DY}}$ with respect to the standard basis (10.3.3). However, this does not provide a formula in terms of symmetric groups, i.e. a formula of the following form

$$r_n^\sigma \circ r_m^\tau = \sum_{\pi \in \mathfrak{S}_{n+m}} c^\pi_{\sigma,\tau} r^\pi_{n+m}.$$

In the next Section we propose an approach to find such a formula consisting in determine a subset of $\mathfrak{L}_{n,m}$ depending on σ and τ (namely a set of *(σ–τ)–essential Drinfeld–Yetter looms*) satisfying the property of being a minimal set necessary to describe the product $r_n^\sigma \circ r_m^\tau$ through the formula (10.6.3).

10.7 Essential Drinfeld–Yetter looms

It follows by Theorem 10.6.9 that the number of summands appearing in the multiplication $r_n^\sigma \circ r_m^\tau$ does not depend on σ and τ, but only on n and m, and it is equal to $|\mathfrak{L}_{n,m}|$. However, as conjectured in 10.10.1, we have that $|\mathfrak{L}_{n,m}| << (n+m)!$. This means that there are several Drinfeld–Yetter looms that do not give any contribution to the sum (10.6.3); we shall call such Drinfeld–Yetter looms *negligible*. In order to formalize this definition, let $\sigma \in \mathfrak{S}_n$, $\tau \in \mathfrak{S}_m$ and $\pi \in \mathfrak{S}_{n+m}$. To such permutations we associate the set

$$\Gamma^{\sigma,\tau,\pi}_{n,m} := \{L \in \mathfrak{L}_{n,m} \mid \tilde{\gamma}_{n,m}(\sigma, L, \tau) = \pi\}.$$

Consider also the following non–negative integers

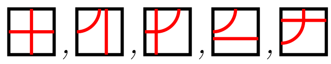
$$P^{\sigma,\tau,\pi}_{n,m} := \#\{L \in \mathfrak{L}_{n,m} \mid \tilde{\gamma}_{n,m}(\sigma, L, \tau) = \pi \ , (-1)^{\xi(L)} = 1\}$$
$$N^{\sigma,\tau,\pi}_{n,m} := \#\{L \in \mathfrak{L}_{n,m} \mid \tilde{\gamma}_{n,m}(\sigma, L, \tau) = \pi \ , (-1)^{\xi(L)} = -1\}$$

which we call the number of positive (resp. negative) Drinfeld–Yetter looms (σ,τ)–associated to the permutation π. It is clear that $\{\Gamma^{\sigma,\tau,\pi}_{n,m}\}_{\pi \in \mathfrak{S}_{n+m}}$ defines a partition of $\mathfrak{S}_{n+m}$ (with eventually some empty blocks). It is also clear that

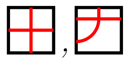
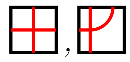
$$|\Gamma^{\sigma,\tau\pi}_{n,m}| = P^{\sigma,\tau\pi}_{n,m} + N^{\sigma,\tau\pi}_{n,m}.$$

Definition 10.7.1. *We say that $L_1, L_2 \in \mathfrak{L}_{n,m}$ is a pair of (σ,τ)–negligible Drinfeld–Yetter looms (and we write $\{L_1, L_2\} \in \mathsf{Neg}^{\sigma,\tau}_{n,m}$) if $\tilde{\gamma}_{n,m}(\sigma, L_1, \tau) = \tilde{\gamma}_{n,m}(\sigma, L_2, \tau)$ and $(-1)^{\xi(L_1)} = -(-1)^{\xi(L_2)}$.*

Example 10.7.2. *Recall that the set of* 1×1 *Drinfeld–Yetter looms is*

$$\mathfrak{L}_{1,1} = \Big\{$$  $$\Big\}.$$

We therefore have the following two (non–disjoint) sets of negligible 1×1 *Drinfeld–Yetter looms:*

$\Big\{$  $\Big\}$ *and* $\Big\{$  $\Big\}$

which both refer to the permutation $\pi = (12) \in \mathfrak{S}_2$.

Fix $n, m \geqslant 1$, $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_m$ and consider the partition $\{\Gamma^{\sigma,\tau,\pi}_{n,m}\}_{\pi \in \mathfrak{S}_{n+m}}$ of $\mathfrak{L}_{n,m}$. For any (non–empty) block $\Gamma^{\sigma,\tau,\pi}_{n,m}$, choose an ordering of its elements

$$\Gamma^{\sigma,\tau,\pi}_{n,m} = \{L_1, \dots, L_{P^{\sigma,\tau,\pi}_{n,m}}, L_{P^{\sigma,\tau,\pi}_{n,m}+1}, \dots, L_{P^{\sigma,\tau,\pi}_{n,m}+N^{\sigma,\tau,\pi}_{n,m}}\} \tag{10.7.1}$$

in such a way the first $P^{\sigma,\tau,\pi}_{n,m}$ elements are positive Drinfeld–Yetter looms and the last $N^{\sigma,\tau,\pi}_{n,m}$ elements are negative Drinfeld–Yetter looms. Next, take $M^{\sigma,\tau,\pi}_{n,m} = \min\{P^{\sigma,\tau,\pi}_{n,m}, N^{\sigma,\tau,\pi}_{n,m}\}$. It is then clear that all the pairs $\{L_i, L_{P^{\sigma,\tau,\pi}_{n,m}+N^{\sigma,\tau,\pi}_{n,m}-i+1}\}_{i=1,\dots,M^{\sigma,\tau,\pi}_{n,m}}$ are disjoint elements of $\mathsf{Neg}^{\sigma,\tau}_{n,m}$. By removing all of these pairs from $\Gamma^{\sigma,\tau,\pi}_{n,m}$, we obtain

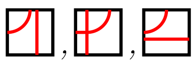
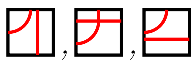
$$\Gamma^{\sigma,\tau,\pi,ess}_{n,m} := \Gamma^{\sigma,\tau,\pi}_{n,m} \setminus \bigcup_{i=1}^{M^{\sigma,\tau,\pi}_{n,m}} \{L_i, L_{P^{\sigma,\tau,\pi}_{n,m}+N^{\sigma,\tau,\pi}_{n,m}-i+1}\}.$$

Finally, we define

$$\mathfrak{L}^{\sigma,\tau,ess}_{n,m} := \bigcup_{\pi \in \mathfrak{S}_{n+m}} \Gamma^{\sigma,\tau,\pi,ess}_{n,m} \subset \mathfrak{L}_{n,m}$$

and we call it a set of (σ,τ)–essential Drinfeld–Yetter looms. Note that the construction of $\mathfrak{L}^{\sigma,\tau,ess}_{n,m}$ strongly depends on the choice of an ordering as in (10.7.1).

Example 10.7.3. *We have the following two sets of essential* 1×1 *Drinfeld–Yetter looms (which are related to Example 10.7.2):*

$\Big\{$  $\Big\}$ *and* $\Big\{$  $\Big\}$.

A set of essential Drinfeld–Yetter looms is a minimal set of Drinfeld–Yetter looms necessary to describe the multiplication of $\mathfrak{U}^1_{\mathsf{DY}}$, as is stated in the following

Proposition 10.7.4. *Let* $n, m \geqslant 1$, $\sigma \in \mathfrak{S}_n$ *and* $\tau \in \mathfrak{S}_m$. *For any* $\pi \in \mathfrak{S}_{n+m}$ *choose an ordering of the elements of* $\Gamma^{\sigma,\tau,\pi}_{n,m}$ *as in 10.7.1 and let* $\mathfrak{L}^{\sigma,\tau,ess}_{n,m}$ *be the associated set of essential Drinfeld–Yetter looms. Then*

(i) For any $\pi \in \mathfrak{S}_{n+m}$, *the set* $\Gamma^{\sigma,\tau,\pi,ess}_{n,m}$ *is made of Drinfeld–Yetter looms having all the same sign (i.e. they are all either positive or negative).*

(ii) The following formula holds

$$r^\sigma_n \circ r^\tau_m = \sum_{L \in \mathfrak{L}^{\sigma,\tau,ess}_{n,m}} (-1)^{\xi(L)} r^{\tilde{\gamma}_{n,m}(\sigma,L,\tau)}_{n+m}.$$

Proof. We observe that (i) follows directly by the construction of $\Gamma_{n,m}^{\sigma,\tau,\pi,ess}$. We have

$$\begin{aligned}
r_n^\sigma \circ r_m^\tau &= \sum_{L\in\mathfrak{L}_{n,m}} (-1)^{\xi(L)} r_{n+m}^{\tilde{\gamma}_{n,m}(\sigma,L,\tau)} \\
&= \sum_{\pi\in\mathfrak{S}_{n+m}} \sum_{L\in\Gamma_{n,m}^{\sigma,\tau,\pi}} (-1)^{\xi(L)} r_{n+m}^{\pi} \\
&= \sum_{\pi\in\mathfrak{S}_{n+m}} \sum_{i=1}^{P_{n,m}^{\sigma,\tau,\pi}+N_{n,m}^{\sigma,\tau,\pi}} (-1)^{\xi(L_i)} r_{n+m}^{\pi} \\
&= \sum_{\pi\in\mathfrak{S}_{n+m}} \sum_{i=M_{n,m}^{\sigma,\tau,\pi}+1}^{P_{n,m}^{\sigma,\tau,\pi}+N_{n,m}^{\sigma,\tau,\pi}-M_{n,m}^{\sigma,\tau,\pi}} (-1)^{\xi(L_i)} r_{n+m}^{\pi} \\
&= \sum_{\pi\in\mathfrak{S}_{n+m}} \sum_{L\in\Gamma_{n,m}^{\sigma,\tau,\pi,ess}} (-1)^{\xi(L)} r_{n+m}^{\pi} \\
&= \sum_{L\in\mathfrak{L}_{n,m}^{\sigma,\tau,ess}} (-1)^{\xi(L)} r_{n+m}^{\tilde{\gamma}_{n,m}(\sigma,L,\tau)}
\end{aligned}$$

where the second and the last equality follow from the fact that the collection $\{\Gamma_{n,m}^{\sigma,\tau,\pi,ess}\}_{\pi\in\mathfrak{S}_{n+m}}$ defines a partition of the set $\mathfrak{L}_{n,m}^{\sigma,\tau,ess}$. ☐

Corollary 10.7.5. *Under the previous notations, we have*

$$r_n^\sigma \circ r_m^\tau = \sum_{\pi\in\mathfrak{S}_{n+m}} (P_{n,m}^{\sigma,\tau,\pi} - N_{n,m}^{\sigma,\tau,\pi}) r_{n+m}^{\pi}.$$

Therefore, finding a complete description of the set $\mathfrak{L}_{n,m}^{\sigma,\tau,ess}$ would directly give an explicit formula for $\circ$ in terms of symmetric groups. However, finding such a description appear to be a very challenging problem, due to the *elusive* nature of the set $\mathfrak{L}_{n,m}$.

In the rest of this Chapter, we shall present some conjectures, combinatorial properties, and explicit computations regarding the set $\mathfrak{L}_{n,m}$ and the multiplication of $\mathfrak{U}_{\mathsf{DY}}^1$.

10.8 Counting Drinfeld–Yetter looms

We set $H_{n,m} = |\mathfrak{L}_{n,m}|$. Then, it is easy to see that $H_{n,m}$ is symmetric (i.e. $H_{n,m} = H_{m,n}$). We have that

$$\begin{aligned}
H_{n,m} &= \sum_{M\in\mathfrak{M}_{n,m}} 2^{\alpha(M)+\beta(M)} \\
&= \sum_{M\in\mathfrak{M}_{n,m}^{\boxbar}} 2^{\alpha(M)+\beta(M)} + \sum_{M\in\mathfrak{M}_{n,m}^{\boxminus}} 2^{\alpha(M)+\beta(M)} + \sum_{M\in\mathfrak{M}_{n,m}^{\boxplus}} 2^{\alpha(M)+\beta(M)} \\
&= 2\sum_{M\in\mathfrak{M}_{n-1,m}} 2^{\alpha(M)+\beta(M)} + 2\sum_{M\in\mathfrak{M}_{n,m-1}} 2^{\alpha(M)+\beta(M)} + \sum_{M\in\mathfrak{M}_{n,m}^{\boxplus}} 2^{\alpha(M)+\beta(M)} \\
&= 2H_{n-1,m} + 2H_{n,m-1} + \sum_{M\in\mathfrak{M}_{n,m}^{\boxplus}} 2^{\alpha(M)+\beta(M)}
\end{aligned}$$

where the first equaility follows from Proposition 10.5.5 and Equation (10.5.2), the second equality follows by Equation (10.4.2), and the third equality follows from the fact any Drinfeld–Yetter mosaic M with $M_{1,1} =$  (resp. $T_{1,1} =$ ) has the first row (resp. the first column) automatically determined, and the rest part of M can be chosen among all the elements of $\mathfrak{M}_{n-1,m}$ (resp. $\mathfrak{M}_{n,m-1}$). Therefore, in order to determine a recursive formula for $H_{n,m}$ it suffices to find a closed formula for the third summand of the right hand side of the Equation above.

Proposition 10.8.1. *Let*

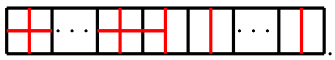
$$C_{n,m} \coloneqq \sum_{M \in \mathfrak{M}^{\boxplus}_{n,m}} 2^{\alpha(M)+\beta(M)}.$$

Then

$$C_{n,m} = (2m-1)H_{n-1,m} + \sum_{k=1}^{m-1} \binom{m-1}{k} 2^k H_{n-1,m-k} + \sum_{k=1}^{m-2} \sum_{\ell=1}^{k} \binom{k}{\ell} 2^{\ell+1} H_{n-1,m-\ell}.$$

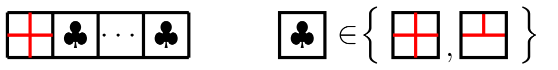
Proof. The proof consist of counting all the possible configurations of the first row y of the elements of $\mathfrak{M}^{\boxplus}_{n,m}$. We divide the discussion in three cases:

- $\alpha(y) = 0$: in this case y must be of the form



  There is exactly one configuration with $\beta(y) = 0$, and $m-1$ configurations with $\beta(y) = 1$. In all of these configurations, we can attach to y any element of $\mathfrak{M}_{n-1,m}$, hence the number of Drinfeld–Yetter looms related to such Drinfeld–Yetter mosaic is $(2m-1)H_{n-1,m}$.

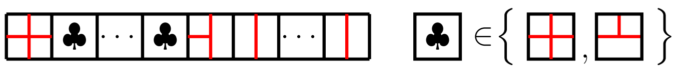
- $\alpha(y) \geqslant 1$ and $\beta(y) = 0$: in this case y must be of the form



  Following the same argument of Proposition 10.4.5, we have that every cobracket tile of y automatically determines the columns below, and to fill the rest part of the Drinfeld–Yetter mosaic we can choose any element of $\mathfrak{M}_{n-1,m-\alpha(y)}$. Letting run $k = \alpha(y)$ from 1 to $m-1$, we obtain that the number of Drinfeld–Yetter looms related to such Drinfeld–Yetter mosaics is

$$\sum_{k=1}^{m-1} \binom{m-1}{k} 2^k H_{n-1,m-k}.$$

- $\alpha(y) \geqslant 1$ and $\beta(y) = 1$: in this case y must be of the form



There are $m-2$ positions in which the unique bracket tile can be (all the positions except the first two), and after this choice the tiles on the right of the bracket tile will be automatically determined. The other tiles (except the first one, which must be of permutation type) of y can be permutation or cobracket tiles; and each occurrence of a cobracket tile automatically determine the whole column of the Drinfeld–Yetter mosaic. As in the previous case, we can fill the rest part of the Drinfeld–Yetter mosaic by choosing any element of $\mathfrak{M}_{n-1,m-\alpha(y)}$. We therefore have that the number of Drinfeld–Yetter looms related to such Drinfeld–Yetter mosaics is

$$\sum_{k=1}^{m-2}\sum_{\ell=1}^{k}\binom{k}{\ell}2^{\ell+1}H_{n-1,m-\ell}.$$

□

Note that the discussion above can be made reasoning on the first column instead on the first row, obtaining

$$C_{n,m}=(2n-1)H_{n,m-1}+\sum_{k=1}^{n-1}\binom{n-1}{k}2^{k}H_{n-k,m-1}+\sum_{k=1}^{n-2}\sum_{\ell=1}^{k}\binom{k}{\ell}2^{\ell+1}H_{n-\ell,m-1}.$$

We therefore obtain the following

Corollary 10.8.2. *We have the following two recursive formulas for $H_{n,m}$:*

$$H_{n,m}=2H_{n-1,m}+(2n+1)H_{n,m-1}+\sum_{k=1}^{n-1}\binom{n-1}{k}2^{k}H_{n-k,m-1}+\sum_{k=1}^{n-2}\sum_{\ell=1}^{k}\binom{k}{\ell}2^{\ell+1}H_{n-\ell,m-1} \quad (10.8.1)$$

and

$$H_{n,m}=2H_{n,m-1}+(2m+1)H_{n-1,m}+\sum_{k=1}^{m-1}\binom{m-1}{k}2^{k}H_{n-1,m-k}+\sum_{k=1}^{m-2}\sum_{\ell=1}^{k}\binom{k}{\ell}2^{\ell+1}H_{n-1,m-\ell}. \quad (10.8.2)$$

10.9 Some explicit computations in $\mathfrak{U}^1_{\mathsf{DY}}$

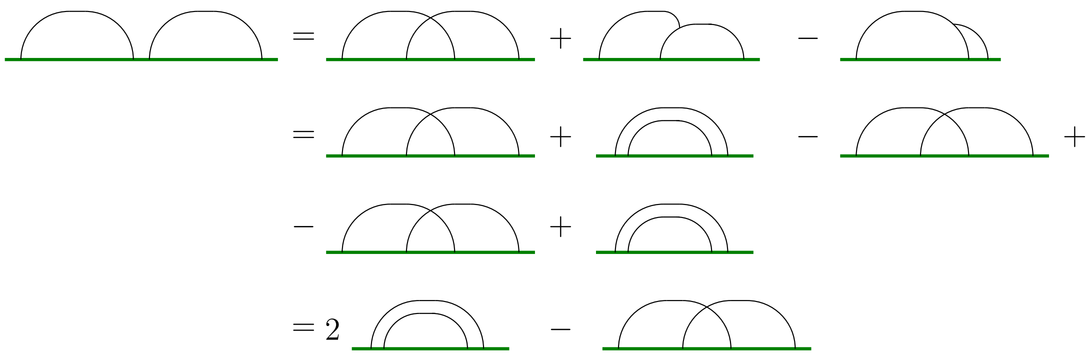
In this Section we exhibit some explicit calculations of the multiplication of $\mathfrak{U}^1_{\mathsf{DY}}$.

The multiplication $r_1^{\mathsf{id}}\circ r_n^{\mathsf{id}}$

Lemma 10.9.1. *We have*

$$r_1^{\mathsf{id}}\circ r_1^{\mathsf{id}}=2\cdot r_2^{\mathsf{id}}-r_2^{(12)}.$$

Proof. We have



where the first equality follows by (8.6.4) and the second equality follows by (8.6.2) and (8.6.3). □

One can generalize the previous result with the following

Proposition 10.9.2. *For any $n \geqslant 1$, we have*

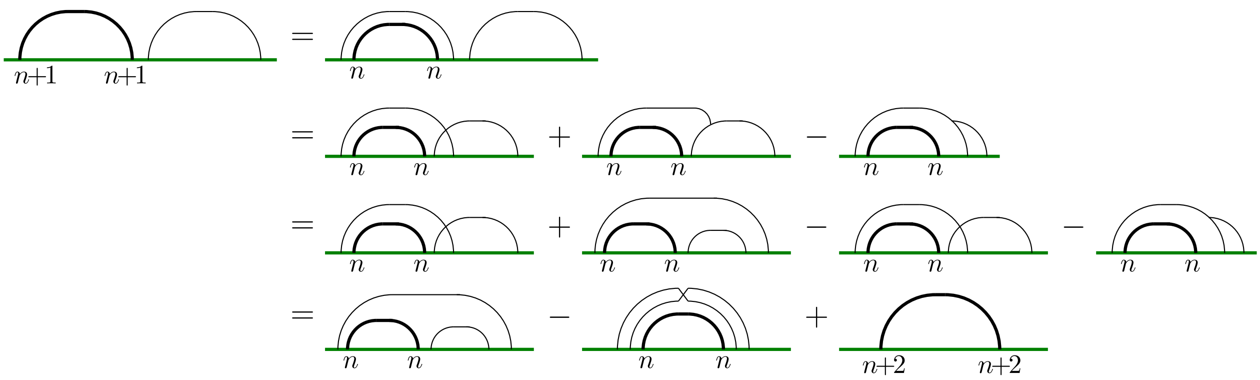
$$r_n^{\mathsf{id}} \circ r_1^{\mathsf{id}} = (n+1) r_{n+1}^{\mathsf{id}} - \sum_{i=1}^{n} r_{n+1}^{(i,i+1)} \tag{10.9.1}$$

and

$$r_1^{\mathsf{id}} \circ r_n^{\mathsf{id}} = (n+1) r_{n+1}^{\mathsf{id}} - \sum_{i=1}^{n} r_{n+1}^{(i,i+1)} \tag{10.9.2}$$

where $(i, i+1)$ denotes the permutation that swaps i and $i+1$ and fixes all the other elements.

Proof. We show (10.9.1) by induction on n, where the base case $n = 1$ is given by the previous Lemma. Let us assume that the statement holds for n. We have



$$= (n+1) r_{n+2}^{\mathsf{id}} - \sum_{i=2}^{n+1} r_{n+2}^{(i,i+1)} - r_{n+2}^{(1,2)} + r_{n+2}^{\mathsf{id}}$$

$$= (n+2) r_{n+2}^{\mathsf{id}} - \sum_{i=1}^{n+1} r_{n+2}^{(i,i+1)}$$

where the second equality follows by (8.6.4), the third follows by (8.6.2), the fourth by (8.6.3) and in the fifth we used the inductive hypothesis. The proof of (10.9.2) is analogous. □

The coefficient of the identity with respect to $r_n^{\mathsf{id}} \circ r_m^{\mathsf{id}}$

By Theorem 10.6.9, it follows that

$$r_n^{\mathsf{id}} \circ r_m^{\mathsf{id}} = \sum_{L \in \mathfrak{L}_{n,m}} (-1)^{\xi(L)} r_{n+m}^{\tilde{\gamma}_{n,m}(\mathsf{id},L,\mathsf{id})} = \sum_{L \in \mathfrak{L}_{n,m}} (-1)^{\xi(L)} r_{n+m}^{\gamma_{n,m}(L)}.$$

Note that for any Drinfeld–Yetter loom L we have that the permutation $\gamma_{n,m}(L)$ is the identity of $n+m$ strings if and only if none of the following tiles appear in L

Hence, denoting $c_{n,m} := c_{\mathsf{id}_n,\mathsf{id}_m}^{\mathsf{id}_{n+m}}$ the coefficient of id_{n+m} with respect to the multiplication $r_n^{\mathsf{id}} \circ r_m^{\tau}$, we have

$$c_{n,m} = \#\{L \in \mathfrak{L}_{n,m}, L_{i,j} \neq \boxplus, \boxplus, \boxplus\} = \#\{M \in \mathfrak{M}_{n,m}, M_{i,j} \neq \boxplus\}$$

where the second equality follows by Proposition 10.5.5. Denoting by $\mathfrak{M}_{n,m}^{\times}$ the number of Drinfeld–Yetter mosaics of $\mathfrak{M}_{n,m}$ without permutation tiles, we have

$$\mathfrak{M}_{n,m}^{\times} = (\mathfrak{M}_{n,m}^{\boxplus})^{\times} \sqcup (\mathfrak{M}_{n,m}^{\boxplus})^{\times}$$

and since

$$|(\mathfrak{M}_{n,m}^{\boxplus})^{\times}| = |\mathfrak{M}_{n-1,m}^{\times}| \quad \text{and} \quad |(\mathfrak{M}_{n,m}^{\boxplus})^{\times}| = |\mathfrak{M}_{n,m-1}^{\times}|$$

we obtain the following recurrence rule for the coefficients $c_{n,m}$:

$$\begin{cases} c_{n,m} = c_{n-1,m} + c_{n,m-1} \\ c_{n,0} = c_{0,m} = 1 \end{cases}$$

It follows that

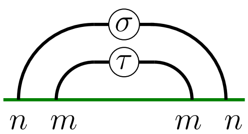
$$c_{n,m} = \frac{(n+m)!}{n!m!}.$$

Furthermore, it seems that the coefficient of r_{n+m}^{id} with respect to the multiplication $r_n^{\mathsf{id}} \circ r_m^{\mathsf{id}}$ is by far the dominant term (i.e. the biggest in absolute value).

The reduction lemma

Definition 10.9.3. *The* ***convolution multiplication*** *in* $\mathfrak{U}_{\mathsf{DY}}^1$ *is* $r_n^{\sigma} \star r_m^{\tau} = r_{n+m}^{\sigma \otimes \tau}$.

Pictorially, $\star$ is represented by the incapsulation of r_m^{τ} inside r_n^{σ}, i.e. by the picture

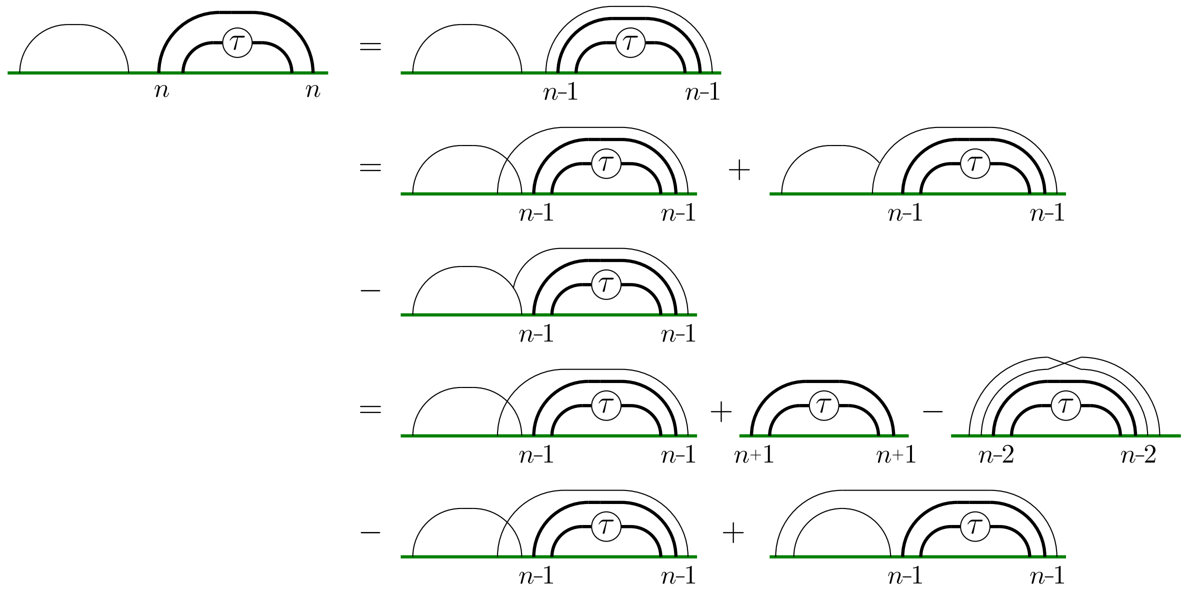


We call the following result the reduction lemma, and it is useful in order to compute the multiplication of $\mathfrak{U}_{\mathsf{DY}}^1$ in some particular cases.

Lemma 10.9.4. *Let $n, m \geqslant 0$ and $\tau \in \mathfrak{S}_m$. Then we have*

$$r_1^{\mathsf{id}} \circ (r_n^{\mathsf{id}} \star r_m^{\tau}) = n \cdot r_{n+1}^{\mathsf{id}} \star r_m^{\tau} + r_n^{\mathsf{id}} \star (r_1^{\mathsf{id}} \circ r_m^{\tau}) - \sum_{k=1}^{n} r_{n+1}^{(k,k+1)} \star r_m^{\tau}.$$

Proof. The proof is by induction on n, where the base case $n = 0$ holds trivially. Let us assume that the statement holds for $n-1$. Using the pictorial representation, we have



$$= r_{n+1}^{\mathsf{id}} \star r_m^{\tau} - r_{n+1}^{(12)} \star r_m^{\tau} + r_1^{\mathsf{id}} \star (r_1^{\mathsf{id}} \circ (r_{n-1}^{\mathsf{id}} \star r_m^{\tau}))$$

where the second equality follows by (8.6.4) and the third by (8.6.2) and (8.6.3). To end the proof it suffices to apply the inductive hypothesis on the last term of the right hand side. □

Remark 10.9.5. *Note that:*

(i) The reduction lemma still holds if one considers any $\phi \in \mathfrak{U}_{\mathsf{DY}}^1$ instead of r_m^{τ}.

(ii) Setting $m = 0$ gives Equation 10.9.2.

(iii) Since r_1^{id} is central (see [ATL19, 9.8] *for a proof), one has also that*

$$(r_n^{\mathsf{id}} \star r_m^{\tau}) \circ r_1^{\mathsf{id}} = n \cdot r_{n+1}^{\mathsf{id}} \star r_m^{\tau} + r_n^{\mathsf{id}} \star (r_m^{\tau} \circ r_1^{\mathsf{id}}) - \sum_{k=1}^{n} r_{n+1}^{(k,k+1)} \star r_m^{\tau}.$$

The multiplication $r_1^{\mathsf{id}} \circ r_n^{(i,j)}$

We are going to provide a closed formula for $r_1^{\mathsf{id}} \circ r_n^{(i,j)}$ for any transposition (i,j) of $\mathfrak{S}_n$. We start the discussion by considering transpositions which fix the first element.

Lemma 10.9.6. *Let $n \geqslant 3$. Then for any k, ℓ such that $0 \leqslant \ell < k \leqslant n-2$ one has*

$$r_n^{(n-k,n-\ell)} \circ r_1^{\mathsf{id}} = (n-k-1) r_{n+1}^{(n-k+1,n-\ell+1)} + r_{n-k-1}^{\mathsf{id}} \star (r_{k+1}^{(1,k-\ell+1)} \circ r_1^{\mathsf{id}}) - \sum_{i=1}^{n-k-1} r_{n+1}^{(i,i+1)(n-k+1,n-\ell+1)}. \quad (10.9.3)$$

Proof. By the reduction Lemma 10.9.4, we can write the second term of the right hand side as

$$\begin{aligned} r^{\mathsf{id}}_{n-k-1} \star (r^{(1,k-\ell+1)}_{k+1} \circ r^{\mathsf{id}}_1) &= (r^{\mathsf{id}}_{n-k-1} \star r^{(1,k-\ell+1)}_{k+1}) \circ r^{\mathsf{id}}_1 - (n-k-1) r^{\mathsf{id}}_{n-k} \star r^{(1,k-\ell+1)}_{k+1} \\ &+ \sum_{i=1}^{n-k-1} r^{(i,i+1)}_{n-k} \star r^{1,k-\ell+1}_{k+1} \\ &= r^{(n-k,n-\ell)}_n \circ r^{\mathsf{id}}_1 - (n-k-1) r^{(n-k+1,n-\ell+1)}_{n+1} + \sum_{i=1}^{n-k-1} r^{(i,i+1)(n-k+1,n-\ell+1)}_{n+1}. \end{aligned}$$

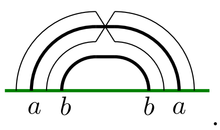
□

Note that the only term of the right hand side of (10.9.3) which is not written with respect to the standard basis is $r^{\mathsf{id}}_{n-k-1} \star (r^{(1,k-\ell+1)}_{k+1} \circ r^{\mathsf{id}}_1)$. Therefore, in order to determine a closed formula for the multiplication $r^{(n-k,n-\ell)}_n \circ r^{\mathsf{id}}_1$ it suffices to determine a closed formula for the multiplication $r^{(1,k-\ell+1)}_{k+1} \circ r^{\mathsf{id}}_1$.

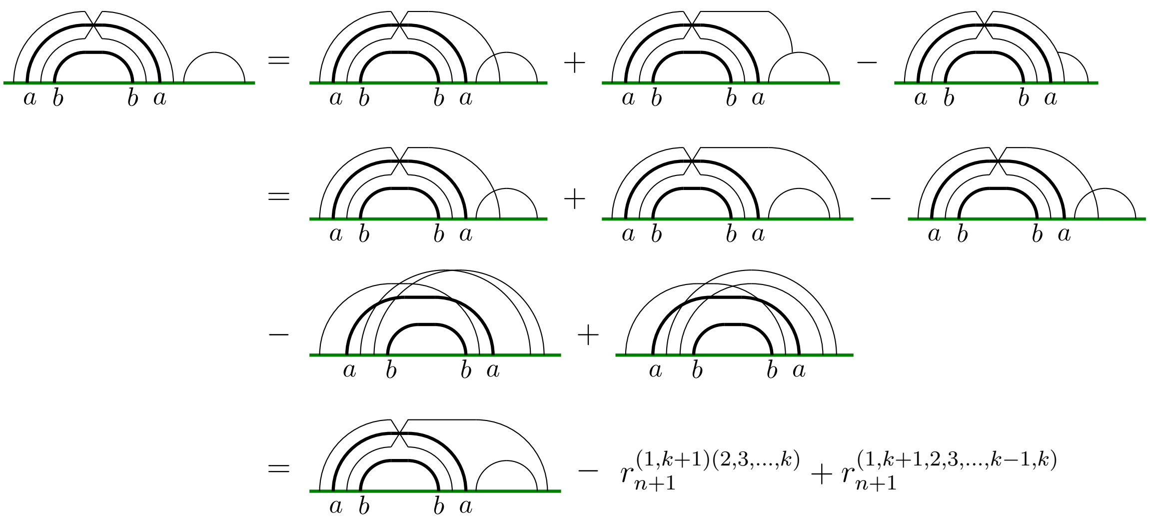
Proposition 10.9.7. *Let $n \geqslant 0$. Then, for any $2 \leqslant k \leqslant n$ one has*

$$\begin{aligned} r^{(1,k)}_n \circ r^{\mathsf{id}}_1 &= (k-2) r^{(1,k+1)}_{n+1} + (n-k+1) r^{(1,k)}_{n+1} - \sum_{i=2}^{k-1} r^{(1,k+1)(i,i+1)}_{n+1} \\ &- \sum_{i=k+1}^{n} r^{(1,k)(i,i+1)}_{n+1} + r^{(1,k+1,2,3,\dots,k-1,k)}_{n+1} - r^{(1,k+1)(2,3,\dots,k)}_{n+1} \\ &+ r^{(1,k,k-1,k-2,\dots,3,2,k+1)}_{n+1} - r^{(1,k+1)(2,k,k-1,\dots,3,2)}_{n+1}. \end{aligned}$$

Proof. The proof is pictorial, where we set $a = k-2$, $b = n-k$. The element $r^{1,k}_n$ is thus represented by the picture

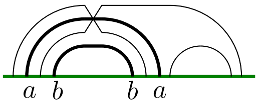


We have

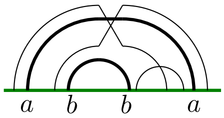


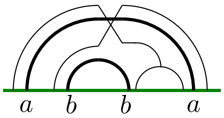
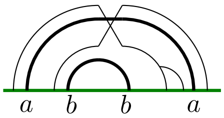
$$- r^{(1,k+1)(2,3,\dots,k)}_{n+1} + r^{(1,k+1,2,3,\dots,k-1,k)}_{n+1}.$$

We can rewrite the first element of the right hand side as

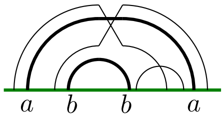


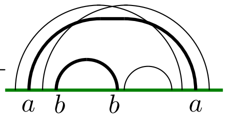
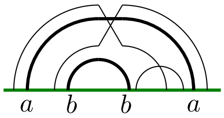
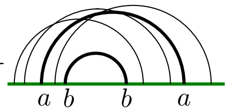
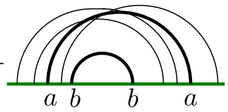
$$= (k-2)r_{n+1}^{(1,k+1)} - \sum_{i=2}^{k-1} r_{n+1}^{(1,k+1)(i,i+1)} +$$



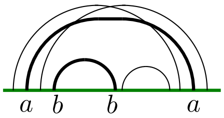
$+$  $-$ 

$$= (k-2)r_{n+1}^{(1,k+1)} - \sum_{i=2}^{k-1} r_{n+1}^{(1,k+1)(i,i+1)} +$$



$+$  $-$  $-$  $+$ 

$$= (k-2)r_{n+1}^{(1,k+1)} - \sum_{i=2}^{k-1} r_{n+1}^{(1,k+1)(i,i+1)} - r_{n+1}^{(1,k+1)(2,k,k-1,\dots,3,2)} + r_{n+1}^{(1,k,k-1,\dots,3,2,k+1)}$$

$+$ 

and therefore to end the proof it suffices to apply the Proposition 10.9.2 to the last term. □

The multiplication for small n, m

Proposition 10.9.8. *Let $n, m \geqslant 0$ such that $n + m \leqslant 4$ and let $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_m$. Then*

$$r_n^\sigma \circ r_m^\tau = r_m^\tau \circ r_n^\sigma.$$

Proof. In view of Proposition 10.10.8 it suffices to show that r_2^{id} commutes with $r_2^{(12)}$. It follows by a direct computation that

$$\begin{aligned} r_2^{\mathsf{id}} \circ r_2^{(12)} &= r_4^{(234)} + r_4^{(243)} - 2r_4^{(24)} + r_4^{(12)} + r_4^{(123)} + r_4^{(132)} \\ &\quad - 2r_4^{(13)} + 2r_4^{(13)(24)} - 2r_4^{(1324)} - 2r_4^{(1423)} + 2r_4^{(14)(23)} \\ &= r_2^{(12)} \circ r_2^{\mathsf{id}}. \end{aligned}$$

□

10.10 Some conjectures related to the algebra $\mathfrak{U}^1_{\mathsf{DY}}$

In this Section we collect some Conjectures related to Drinfeld–Yetter looms and to the algebra $\mathfrak{U}^1_{\mathsf{DY}}$.

A closed formula for the cardinality of $\mathfrak{L}_{n,m}$

Recall that Equations (10.8.1) (10.8.2) give recursive formulas for the cardinality of $\mathfrak{L}_{n,m}$. By a direct computation, we obtain

$$\begin{aligned} H_{n,0} &= 1 \\ H_{n,1} &= 2 \cdot 3^n - 1 \\ H_{n,2} &= 8 \cdot 5^n - 8 \cdot 3^n + 1 \\ H_{n,3} &= 48 \cdot 7^n - 72 \cdot 5^n + 26 \cdot 3^n - 1 \\ H_{n,4} &= 384 \cdot 9^n - 768 \cdot 7^n + 464 \cdot 5^n - 80 \cdot 3^n - 1. \end{aligned}$$

We can therefore conjecture a closed formula for $H_{n,m}$:

Conjecture 10.10.1. *We have*

$$\begin{aligned} H_{n,m} &= \sum_{k=0}^{m} \sum_{i=0}^{k} (-1)^{m-i} \binom{k}{i} (2i+1)^m (2k+1)^n \\ &= \sum_{k=0}^{m} (-1)^{m-k} T_{m,k} (2k+1)^n \end{aligned}$$

where $T_{m,k}$ is defined by the recurrence rule $T_{m,k} = (2k+1)T_{m-1,k} + 2kT_{m-1,k-1}$ with initial conditions $T_{0,0} = 1$ and $T_{0,k} = 0$ for any $k \geqslant 1$.

Knowing a closed formula for the cardinality of $\mathfrak{L}_{n,m}$ would give a better understanding of the computational complexity of an algorithm computing the multiplication.

Drinfeld–Yetter looms and permutation patterns

Recall that, given $\sigma \in \mathfrak{S}_n$ the one–line notation for σ is $\sigma_1 \cdots \sigma_n$, where $\sigma_i = \sigma(i)$ (for example, the identity of $\mathfrak{S}_3$ is represented by 123). Recall the following definition, see [Kit11] and references therein for more details:

Definition 10.10.2. *Let $n \geqslant k \geqslant 1$ be two integers, and let $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$. We say that $\tau = \tau_1 \cdots \tau_k$ occurs in $\sigma = \sigma_1 \cdots \sigma_n$ as a pattern if there exists a subsequence $\sigma_{i_1} \cdots \sigma_{i_k}$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$, such that $\sigma_{i_j} < \sigma_{i_m}$ if and only if $\tau_i < \tau_m$. If not, we say that the permutation σ avoids the pattern τ. If S is a set of permutations, we define the following sets*

$$\mathsf{Av}(S)_n = \{\sigma \in \mathfrak{S}_n \mid \sigma \textit{ avoids every element of } S \textit{ as a pattern}\}$$

and

$$\mathsf{Av}(S) = \bigcup_{n \geqslant 0} \mathsf{Av}(S)_n.$$

If S is given by one element, we omit the curly brackets.

Example 10.10.3. *Let $\sigma = 3241 \in \mathfrak{S}_4$, $\tau = 231 \in \mathfrak{S}_3$ and $\gamma = 123 \in \mathfrak{S}_3$. Then we have that τ appears two times as a pattern in σ (with the subsequences 341 and 241), while σ avoids the pattern γ, so we write $\sigma \in \mathsf{Av}(\gamma)$.*

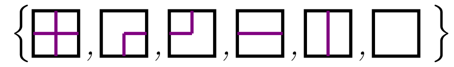
Conjecture 10.10.4. *For any $n, m \geqslant 1$, there is a set of permutations $S_{n,m}$ such that $\gamma_{n,m}(\mathfrak{L}_{n,m}) = \mathsf{Av}(S_{n,m})$.*

It seems indeed reasonable that the defining rules of Drinfeld–Yetter looms do not allow the existence of some specific permutations in $\gamma_{n,m}(\mathfrak{L}_{n,m})$.

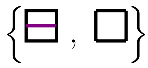
Drinfeld–Yetter looms and bumpless pipedreams

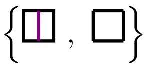
This Subsection will not contain any conjecture, but a connection between Drinfeld–Yetter looms and other combinatorial objects called bumpless pipedreams.

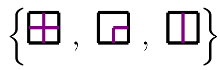
Definition 10.10.5 ([LLS21], [Hec19])**.** *Let $n \geqslant 1$ and let $\mathcal{T}_{\mathcal{BPD}}$ be the following set of tiles*

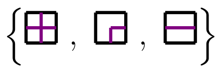
$$\mathcal{T}_{\mathcal{BPD}} =$$


*We define the set of $n \times n$ **bumpless pipedreams** $\mathcal{BPD}_n$ as the set of all possible tilings B of $\mathcal{G}_{n,n}$ with the elements of $\mathcal{T}_{\mathcal{BPD}}$ such that the following three conditions are satisfied:*

(1): $B_{1,j} \notin$  *for all $j \in \{1, \ldots, n\}$.*

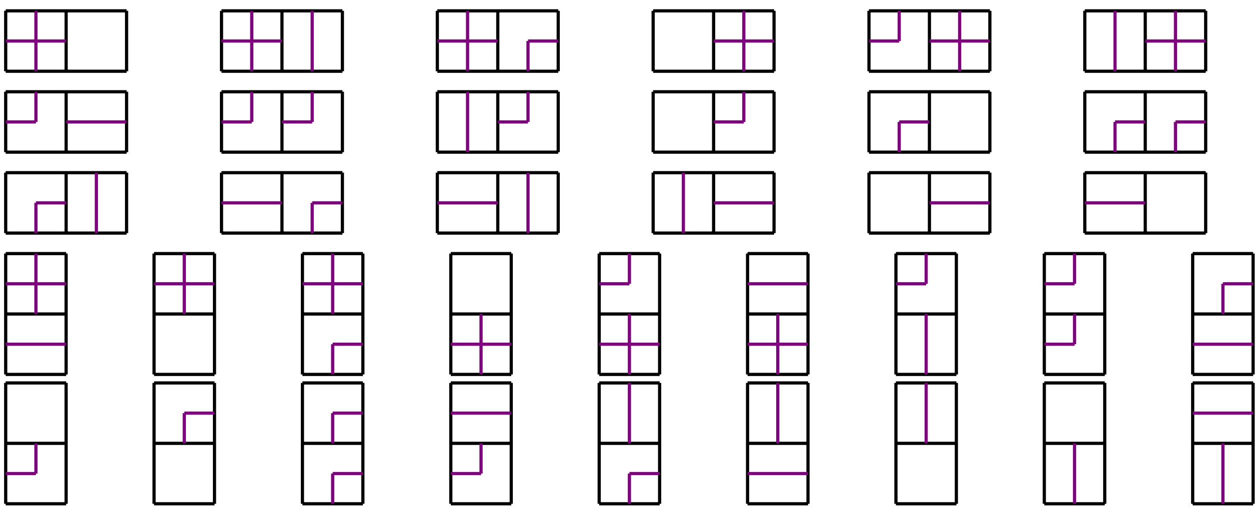
(2): $B_{i,1} \notin$  *for all $i \in \{1, \ldots, n\}$.*

(3): $B_{n,j} \notin$  *for all $j \in \{1, \ldots, n\}$.*

(4): $B_{i,n} \notin$  *for all $i \in \{1, \ldots, n\}$.*

(5): Strings cross pairwise at most once (with respect to the picture obtained by removing all black borders *from B).*

(6): None of the following configurations appear in B:



Note that our definition of bumpless pipedreams differs from the one in [Hec19] in that the direction of the strings goes from the left edge to the top one, and not from the right to the bottom. Note also that there are some similarities between bumpless pipedreams and Drinfeld–Yetter mosaics.
Pipedreams are relevant in may areas of combinatorics, such as calculation of Schubert polynomials (see [MS05]), permutation words (see [Mar13]), and maximal 0-1–fillings of moon polynomials (see [Rub12]).
There is a canonical map $s_n : \mathcal{BPD}_n \to \mathfrak{S}_n$ associating to any $n \times n$ bumpless pipedream a permutation of the symmetric group $\mathfrak{S}_n$; conversely, there is a canonical map $R : \mathfrak{S}_n \to \mathcal{BPD}_n$

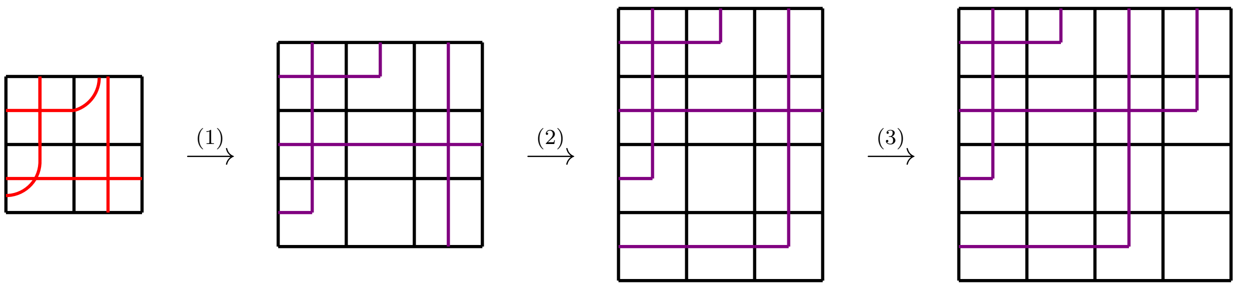
associating to any permutation σ a specific bumpless pipedream, called the **Rothe bumpless pipedream** of σ, see [Hec19, §2] for more details.
We now define a map $f : \mathfrak{L}_{n,m} \to \mathcal{BPD}_{n+m}$ with the property that $s_{n+m}(f(L)) = \gamma_{n,m}(L)$. For any $L \in \mathfrak{L}_{n,m}$, consider the element $f(L) \in \mathcal{BPD}_{n+m}$ constructed according to the following steps:

(1) *Stretch* L horizontally and vertically in such a way every left (resp. top) tile of the first column (resp. row) has only one string. The result of this process gives a tiling of $\mathcal{G}_{\Lambda,\Omega}$ (where $\Lambda = \sum_{i=1}^{n} l_{i1}$ and $\Omega = \sum_{j=1}^{m} t_{1j}$) with elements of $\mathcal{T}_{\mathcal{BPD}}$. However, this in general will not be a bumpless pipedream.

(2) If $\Lambda \neq n+m$, add a row for any string occurring in the bottom edge of the last row. Then, for any tile of the Λ–th row having a string in the bottom edge, attach the tile  to its bottom edge and fill all the left–side tiles of such a row with the tile .

(3) If $\Omega \neq n+m$, add a column for any string occurring in the left edge of the last column. Then, for any tile of the Ω–th column having a string in the left edge, attach the tile  to its left edge and fill all the top–side tiles of such a column with the tile .

Example 10.10.6. *Consider the Drinfeld–Yetter loom $L \in \mathfrak{L}_{2,2}$ of Example 10.5.3. Then the procedure described above gives*



and it is easy to see that $\gamma_{2,2}(L) = s_4(f(L)) = (1243) \in \mathfrak{S}_4$.

Remark 10.10.7. *The reasoning above shows that the set of $\mathfrak{L}_{n,m}$ of $n \times m$ Drinfeld–Yetter looms maps into the set $\mathcal{BPD}_{n+m}$ of $n \times m$ bumpless pipedreams, and so in the symmetric group $\mathfrak{S}_{n+m}$. However, this association is not injective nor surjective. Hence $\mathfrak{L}_{n,m}$ gives a refinement only of the subset of $\mathcal{BPD}_{n+m}$ given by the Rothe bumpless pipedreams.*

The center of $\mathfrak{U}^1_{\mathsf{DY}}$

Recall that the center of an algebra A is defined by $Z(A) = \{a \in A \mid ax = xa \quad \text{for all } x \in A\}$. A very interesting problem is to understand whose elements of $\mathfrak{U}^1_{\mathsf{DY}}$ are central. Indeed, we have that $Z(\mathfrak{U}^1_{\mathsf{DY}})$ is non–empty, as is proved in the following result (see [ATL19, 9.8] for a proof):

Proposition 10.10.8. *The element r_1^{id} of $\mathfrak{U}^1_{\mathsf{DY}}$ is central.*

We state the following

Conjecture 10.10.9. *The center of $\mathfrak{U}^1_{\mathsf{DY}}$ is spanned by r_1^{id}.*

The mirror permutation

We define the n–th mirror permutation as the following element of $\mathfrak{S}_n$:

$$\pi_n := \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (i, n-i+1) \in \mathfrak{S}_n.$$

For example, $\pi_2 = (12) \in \mathfrak{S}_2$, $\pi_3 = (13) \in \mathfrak{S}_3$ and $\pi_4 = (14)(23) \in \mathfrak{S}_4$.

Conjecture 10.10.10. *We have the following*

(i) The following identity holds in $\mathfrak{U}^1_{\mathsf{DY}}$*:*

$$r_1^{\mathsf{id}} \circ r_n^{\pi_n} = -n \cdot r_{n+1}^{\pi_{n+1}} + r_n^{\pi_n} \star r_1^{\mathsf{id}} + \sum_{i=1}^{n} r_{n+1}^{\pi_{n+1} \circ (i,i+1)}.$$

(ii) Let $p_{n,m}$ *be the coefficient of* $r_{n+m}^{\pi_{n+m}}$ *with respect to the multiplication* $r_n^{\mathsf{id}} \circ r_m^{\pi_m}$*. Then* $p_{n,m}$ *is the dominant coefficient (i.e. is the biggest in absolute value). Moreover, the* $p_{n,m}$*'s satisfy the recursive formula* $p_{n,m} = (-1)^n(|p_{n-1,m}| + |p_{n,m-1}|)$ *with initial conditions* $p_{1,1} = -1$*,* $p_{1,m} = -m$ *and* $p_{n,1} = 0$ *for* $n > 1$*.*

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