“Multivariate Approximation: Theory and Applications”

Dedicated to Leonardo Traversoni for his 60th birthday

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Abstract

This introductory paper describes the main topics of this special issue, dedicated to Leonardo Traversoni, known at international level as the promoter of the conference series “Multivariate Approximation: Theory and Applications”, to celebrate his 60th birthday.

Key words: Interpolation; Multiresolution and subdivision; Image processing; Numerical approximation; Numerical methods related to PDEs.

1. Introduction

The first “Multivariate Approximation: Theory and Applications” conference was held in 1995 at Cancun, Mexico. Each time, a special issue in a journal was completed except for the conference organized in 1999:

- Journal of Computational and Applied Mathematics (Elsevier) vol.73, 1996 (A. Le Méhauté, L.L. Schumaker and L. Traversoni, eds.) for the MATA 1995 conference,

- no special issue for the Second International Conference on Multivariate Scattered Data Fitting, organized by P. Gonzalez- Casanova, A. Le Méhauté, L.L. Schumaker and L. Traversoni in April 1999 at Puerto Vallarta, Mexico,

- Numerical Algorithms (Springer) vol.39, 2005 (D. Apprato, C. Gout, C. Rabut, L. Traversoni, eds.) for the Third Multivariate Approximation: Theory and Applications conference organized in April 2003 at Cancun, Mexico,

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- Numerical Algorithms (Springer) vol.48, 2008 (C. de Boor, A. Kunoth, C. Gout, C. Rabut, eds.) for the Fourth Multivariate Approximation: Theory and Applications conference organized in May 2007 at Cancun, Mexico.

The occasion of this special issue for Journal of Computational and Applied Mathematics in 2012 is also the celebration of the 60th birthday of Leonardo Traversoni. Since from 1995 Leonardo Traversoni has been the main organizer of a conference series entitled "Multivariate Approximation: Theory and Applications", we decided to have a special issue in his honour with the same title.

2. Main themes of the special issue

This special issue of the journal of Computational and Applied Mathematics contains a selected collection of 19 contributions, dealing with recent progress in the field of "Multivariate Approximation: Theory and Applications" (see the References below for the complete list of these papers). All of them have been refereed according to the standard peer review process of the journal. The rate of acceptance is 54% (19/35). The papers in this special issue cover the following topics:

1. *Interpolation* [2, 12, 15, 18]
2. *Multiresolution and subdivision* [6, 7, 14, 19]
3. *Image processing* [3, 8, 13, 16]
4. *Numerical approximation* [4, 5, 11, 17]
5. *Numerical methods related to PDEs* [1, 9, 10].

In the following subsections we provide a short overview of such topics and we describe the corresponding contributions.

2.1. Interpolation

Interpolation is still a hot research topic. A variety of interpolation methods, based either on spline or subdivision approaches, are currently under investigation with the aim of producing smooth shapes passing through a given data set.
In [2], the authors present a general framework for the construction and characterization of piecewise-polynomial local interpolants with given smoothness and approximation order, defined on non-uniform knot partitions. Thanks to its generality, the proposed framework allows to recover uniform local interpolating splines previously proposed in the literature, to generalize them to the non-uniform case, and to complete families of arbitrary support width.

Furthermore it provides new local interpolating polynomial splines with prescribed smoothness and polynomial reproduction properties.
[15] also deals with the development of a framework, but for generating and analyzing non-uniform interpolating subdivision algorithms, closely related to non-uniform spline interpolants. Families of symmetric non-uniform interpolatory $2n$-point schemes of smoothness C^{n-1} are presented for $n = 2, 3, 4$ and even higher orders, as well as a variety of non-uniform 6-point schemes with C^3 continuity.
As an extension of basic interpolation, Hermite interpolation is aimed at producing a curve or surface that not only passes through a given set of points, but also matches the slope at those points.
[12] deals with the problem of interpolating motions of a rigid body in the Hermite sense with geometric continuity conditions. To this end, first rational motions are defined and the concept of geometric continuity is adapted to them. Then, the general G^k continuous interpolation problem for motions is formulated and an approach for rational spline motions with G^1 or G^2 smoothness and interpolating a sequence of assigned positions/orientations is introduced.
In [18], the authors focus on the problem of constructing Hermite interpolants using polynomial splines on T-meshes. In particular, the space of bivariate splines of degree (componentwise) (d_1, d_2) and smoothness (r_1, r_2), with $d_1 \geq 2r_1 + 1, \quad d_2 \geq 2r_2 + 1$ over regular T-meshes without cycles, is considered. Error bounds for Hermite interpolation of smooth functions are provided and compared with similar error bounds for Hermite interpolation by tensor product polynomials.

2.2. Multiresolution and subdivision

The strength and popularity of subdivision schemes is well-established in many fields. Among these, for instance, we find the construction of smooth curves and surfaces and the generation of refinable functions. It is well-known that refinability is a very useful property since it allows to generate a multiresolution analysis and hierarchical structures (or multilevel approaches).

In [6], the authors derive necessary and sufficient conditions for a multivariate subdivision scheme to reproduce polynomials up to a certain degree. These conditions are algebraic and can easily be checked by analysing the symbol of the scheme and its derivatives. They nicely complement the known algebraic conditions for polynomial generation. A key element in the analysis is the derivation of a suitable parameterization for a given subdivision scheme, that is, a sequence of parameter values to which the generated data at each level should be attached.
In [7], the authors present a construction of nested spaces of degree 5, C^2 macro-elements on triangulations of a polygonal domain obtained by uniform refinements of an initial triangulation and a Powell-Sabin 12 split. The

advantage of this construction is in the fact that the degree of the refinable macro-elements is substantially lower than the degree of the currently existing proposals.

[19] illustrates an interesting application of multiresolution representations to the field of digital terrain modelling. The term digital terrain model refers to a representation of the earth's surface in digital form, without including details such as vegetation, bridges, buildings, etc. It is very useful to visualize the terrain in a computer and to study its properties. In [19], the authors propose an algorithm for computing a multiresolution representation of a digital terrain model using level curves. The input curves are sorted by their importance in such a way that the n most relevant curves are contained in the n-th resolution level.

Finally, [14] is devoted to an effective construction of divergence-free wavelets on the square $[0,1]\times[0,1]$, with suitable boundary conditions. To this end it combines the construction of multiresolution analysis of $H_0^1([0,1])$ and the construction of biorthogonal wavelets linked by derivation. As an example, a divergence-free decomposition of a velocity field of lid driven cavity flow is computed and the interpretation of the result is provided.

2.3. Image processing

The development of image processing tools has been of great importance in the last years. Many applications require such tools: segmentation for medical applications, noise removal in Medicine and Geophysics, inpainting and image restoration, registration, classification and so on. For a long time a pure computer science topic, image processing has now become a strong theme in the mathematical community. The contributions of mathematicians were to provide rigorous models of imaging problems. These include variational approaches, the study of existence / uniqueness of solutions and numerical aspects. Parallel implementations (sometimes with GPU) have also been proposed. This is a very active theme, having very strong developments. In this special issue, several imaging problems are studied.

In [13], the authors deal with the restoration of images degraded by a noisy blur kernel. This modelling can be encountered for instance in astronomy or in medical imaging. They empirically show that in several cases classical methods such as the Rudin-Osher model fail to properly clean the degraded image, visual artifacts being still visible, which motivates the introduction of the multiframe approach. The idea is to adopt the multiframe model, which consists in using several degraded frames in the reconstruction of a single restored image. The multiframe model is introduced into two variational minimization problems. They also propose an application of their framework for joint restoration and binary segmentation. Numerical results are given on color images, as well as on real video datasets.

[3] is concerned with an image decomposition problem which consists in separating the geometric component u of an image $f\colon\ \Omega \to \mathcal{R}$, from

the oscillatory part v. The authors introduce auxiliary variables that naturally emerge from the Helmholtz-Hodge decomposition for smooth fields, and which yields to the minimization of the L^∞-norm of the gradients of the new unknowns. This constrained minimization problem is transformed into a series of unconstrained problems by means of Bregman Iteration. The authors prove the existence of minimizers for the involved subproblems. Then a gradient descent method is selected to solve each subproblem, becoming related, in the case of the auxiliary functions, to the infinity Laplacian. Existence/ Uniqueness as well as regularity results of the viscosity solutions of the PDE introduced are proved. Numerical examples are also given.

In [16], the authors present a vector field approximation. In the modelling, they use classical tools such as a D^m-splines approach, leading to the minimization of a functional on a Hilbert space. Several theoretical results are given, and applications on image registration is shown. In image processing, registration is an important task since it allows comparing a subject/time variant template image T with an unbiased reference image R. More precisely, given a reference R and a template T, defined on image domain, the goal is to find a smooth, invertible transformation to map T into an image similar to R. For images of the same modality, a well-registered template has geometric features and intensity distribution matched with the reference.

In [8], the authors are working with specific datasets : geophysical data. Calibrating reservoir models to flow data involves history matching processes. Model parameters are iteratively adjusted to minimize the misfit between the real data and the corresponding simulated responses. The current formulation used to quantify the seismic data mismatch is neither representative of the difference between two images, nor of matching quality. The authors describe an alternative formulation, using methods rooted in image processing (such as Chan-Vese segmentation method, NL means, Hausdorff distance...), to extract relevant information from seismic images and compute their dissimilarity. Numerical results are given.

2.4. Numerical approximation

Splines, orthogonal polynomials and Radial Basis funtions (RBF) are still a very active subject for research in approximation, both from a theoretical point of view and for applications. They usually lead to numerical methods to approximate or interpolate different kinds of datasets.

In [4], the authors present a numerical method for the detection of faults and gradient faults of two dimensional functions when a sample of N uniformly scattered and noisy data is given. The method recognizes the presence of a discontinuity curve by a suitable statistic. The recovering of such curve is discussed and a computational bound of the pointwise error is provided together with a study of the convergence properties of the algorithm as N grows to infinity.

In [5], the authors investigate the performance of RBF-PDE methods to approximate solenoidal vector fields. More precisely, they perform a numerical investigation between RBF global and local methods in order to study the possible advantages of local methods for the approximation of vector fields. The main interest of this approach is to compare the local Hermite interpolation (LHI) method, using inverse multiquadrics, against the non symmetric Kansa's collocation method to approximate solenoidal fields at great scale.
The authors of [11] propose an optimization based Empirical Mode Decomposition (EMD) method (OEMD) for one-dimensional signals. An interesting aspect of the proposed method is that the upper and lower envelopes of the signal are constructed by solving a convex optimization problem instead of using cubic spline interpolation as in the original EMD method. The main advantage of this construction is that it does not create over- or undershoots that have been observed in the original EMD method based on cubic spline interpolation.
[17] discusses Bernstein type inequalities for certain operators on functions defined on the complex sphere that include fractional versions of the Laplace-Beltrami operator. Moreover, it contains a new result relating a bivariate sum involving Jacobi polynomials and Gegenbauer polynomials, which relates the sum of reproducing kernels on spaces of polynomials irreducibly invariant under the unitary group, with the reproducing kernel of the sum of these spaces, which is irreducibly invariant under the action of the orthogonal group.

2.5. Numerical methods related to PDEs

The second half of the twentieth century has witnessed the advent of Partial Differential Equations (PDE) linked to the modelling of many problems like Computational Fluid Dynamics (CFD), a new branch of applied mathematics that deals with numerical simulation of fluid flows, or Level set Methods (Hamilton-Jacobi) or many others famous PDE (Navier-Stokes, Saint-Venant, Schrödinger, Waves, Maxwell and so on...). Many tools have been introduced in this period, including finite differences, finite elements, finite volumes to process the solution of such problems. In this issue, several papers bring novelty in this area.

In [1], the authors quantify the performance and numerical advantages of the operator-based upscaling technique applied to the Laplace problem using a discontinuous Galerkin method. Such approach permits to get the fine scale effects with computations on coarse meshes, leading to reduced computational costs, instead of computing the solution of the problem on a refined mesh in case of small scales.

In [10], the authors present a viscosity method based on wavelets for systems of conservation laws (Euler's equations for gas dynamics). As the viscosity stabilization produces some parasitical oscillations (Gibbs phenomena), the authors discuss different postprocessing techniques known from data and image processing together with a number of numerical comparisons.

In [9], the authors present a new finite elements approach, proposing a construction of $H(div)$ and $H(curl)$ finite element basis functions for triangles and quadrilaterals.

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