Descent methods for a class of generalized variational inequalities*

Barbara Panicucci† Massimo Pappalardo† Mauro Passacantando†

Abstract: In this paper we propose a class of differentiable gap functions in order to formulate a generalized variational inequality (GVI) problem, involving a set-valued map with closed and convex graph, as an optimization problem. We also show that under appropriate assumptions on the set-valued map, any stationary point of the equivalent optimization problem is a global optimal solution and solves the GVI. Finally, we describe a descent method, with Armijo-type line search, for solving the optimization problem equivalent to the GVI and we prove its global convergence.

Keywords: generalized variational inequality, set-valued map, gap function, descent method.

1 Introduction

Given a closed and convex set $K \subseteq \mathbb{R}^n$ and a set-valued map $F : K \rightrightarrows \mathbb{R}^n$, the generalized variational inequality (GVI) problem consists in finding $x^* \in K$ and $u^* \in F(x^*)$ such that

$$\langle u^*, y - x^* \rangle \geq 0, \qquad \forall\, y \in K,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product.

This problem is one of the several equilibrium models which represent powerful tools for solving many problems arising in Economics, Engineering, Operations Research and Mathematical Physics (see e.g. [8, 10, 11]).

In the rest of the paper we make the following assumptions.
(A1) *The set $K \subseteq \mathbb{R}^n$ is closed and convex.*
(A2) *The graph of F:*

$$\mathrm{Gr}\,(F) = \{(x, u) : \ x \in K \ \textit{and}\ u \in F(x)\}$$

is closed and convex.
To support assumptions (A1) and (A2), we show an application in the framework of the traffic network equilibrium problem.

Example 1.1. The usual model of a traffic network is given by a graph, consisting of a set of nodes $\mathcal{N}$, a set of links $\mathcal{A}$ and a set of origin/destination pairs $\mathcal{W}$. For each origin/destination pair w there is a known demand $d_w > 0$ distributed among the paths $\mathcal{P}_w$ connecting w. Let $\mathcal{P}$ be the union of $\mathcal{P}_w$ for w ranging over all w in W. Let h_p denote the flow on path p and let $h = (h_p)_{p \in \mathcal{P}}$. Thus the set of feasible

*This work has been supported by the National Research Program PRIN/2005017083 "Innovative Problems and Methods in Nonlinear Optimization". This paper has been published in Computational Optimization and Applications, vol. 45 (2010), pp. 415–425.

†Department of Applied Mathematics - University of Pisa, via Buonarroti 1/c, 56127 Pisa - Italy (panicucc@mail.dm.unipi.it), (pappalardo@dma.unipi.it), (passacantando@dma.unipi.it).

arc flow vectors is given by

$$K = \left\{ x \in \mathcal{R}^{|\mathcal{A}|} : \text{there exists}\, h \geq 0, \text{such that} \sum_{p \in P_w} h_p = d_w, \forall\, w \in W \,\text{and}\, x = \Delta\, h \right\},$$

where Δ is the arc-path incidence matrix, whose entry $\Delta_{a,p} = 1$ if arc a is in path p and $\Delta_{a,p} = 0$, otherwise. It easy to see that K is convex and compact and therefore (A1) holds.

For each arc a there is a cost function $c_a(x)$ representing the delay in traversing the arc a when the arc flow vector is x. In the basic traffic model a standard choice for the travel cost function is the Bureau of Public Road (BPR) function (see, e.g. [3]): $c_a(x) = T_a + \alpha\,(x_a/C_a)^4$, where T_a represents the free flow travel time on the arc a, C_a represents the capacity of the arc a and α is a positive parameter. The traffic network equilibrium problem is to find a feasible flow vector such that path flows are positive only on paths of minimum travel cost. This problem can be reformulated as a classical variational inequality with operator $c = (c_a)_{a \in \mathcal{A}}$ and feasible set K (see, [4]). To generalize the model and to consider the possibility that the delay in traversing arc a may vary over a range, we assume that $c_a(x) = [l_a(x), u_a(x)]$, where $l_a(x)$ is a convex continuous function and $u_a(x)$ is a concave continuous function, e.g. let $l_a(x)$ to be the BPR function and $u_a(x)$ a concave piecewise linear increasing function. The traffic network equilibrium problem with this modified cost function can be reformulated as a GVI with operator $c(x) = \prod_{a \in \mathcal{A}} [l_a(x), u_a(x)]$ and feasible set K. It follows, from the assumptions on l_a and u_a, that the set-valued map c satisfies (A2).

Let us note that assumption (A2) is not new. In particular in [12, 14, 15] the authors introduced the graph-convex polyhedral multifunctions, that is set-valued maps whose graph is a convex polyhedral set.

In this paper we describe a solution approach, for a class of GVI, which consists in reformulating the GVI as a differentiable constrained optimization problem by means of a suitable gap function. We also show that, under suitable assumptions on F, every stationary point of the equivalent optimization problem solves the GVI. Finally, we propose a globally convergent descent method, with an Armijo-type line search, for solving the optimization problem equivalent to GVI.

2 Gap functions for GVI

In this section we reformulate GVI as a differentiable optimization problem by means of a class of gap functions. We say that a function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a gap function for GVI if

- $g(x, u) \geq 0$ for all $(x, u) \in \mathrm{Gr}\,(F)$;
- (x^*, u^*) is a solution of GVI if and only if $(x^*, u^*) \in \mathrm{Gr}\,(F)$ and $g(x^*, u^*) = 0$.

These two properties imply that, if the solution set of GVI is not empty, then it coincides with the set of global minima of the following optimization problem:

$$\left\{ \begin{array}{l} \min\ g(x, u) \\ (x, u) \in \mathrm{Gr}\,(F). \end{array} \right. \tag{1}$$

It is worth noting that, if the optimization problem (1) has a strictly positive optimal value, then the GVI has no solution. On the other hand, when the GVI does not have a solution, problem (1) may have a global minimizer and the corresponding objective value is strictly positive.

Recently in [5] it has been proposed the following gap function for GVI:

$$g(x, u) = \sup_{y \in K} \langle u, x - y \rangle, \qquad\qquad (x, u) \in \mathbb{R}^n \times \mathbb{R}^n.$$

However, this function is, in general, neither finite nor differentiable on $\mathrm{Gr}\,(F)$. Therefore, to construct a more tractable reformulation of the GVI, we consider a regularized gap function defined as:

$$\varphi(x,u) = \sup_{y\in K} \Phi(x,u,y), \qquad (x,u) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{2}$$

where

$$\Phi(x,u,y) = \langle u, x-y\rangle - \frac{1}{2}\,\|x-y\|^2,$$

and $\|\cdot\|$ denotes the Euclidean norm. Specifically, let denote the unique maximizer of (2) by $y(x,u) = \Pi_K(x-u)$, where $\Pi_K(\cdot)$ is projection operator onto the closed convex set K, then the regularized gap function is given by

$$\varphi(x,u) = \langle u, x-y(x,u)\rangle - \frac{1}{2}\,\|x-y(x,u)\|^2. \tag{3}$$

Observe that in the definition of φ, the euclidean norm can be substituted with a norm induced by any symmetric definite positive matrix.

This function has been introduced in a more general context (where K is not fixed but depends on the solution point x) in [5]. Here we consider the set K to be fixed so that the function φ results to be continuously differentiable, as shown in the following theorem.

Theorem 2.1. If the assumption (A1) is fulfilled, then φ is a continuously differentiable gap function for GVI and its gradient is given by

$$\nabla\varphi(x,u) = (u + y(x,u) - x, x - y(x,u)). \tag{4}$$

Proof. Since $y(x,u) = \Pi_K(x-\,u)$, then from the properties of the projection operator, it follows that

$$\langle x-u-y(x,u), x-y(x,u)\rangle \le 0, \qquad \forall\; x\in K. \tag{5}$$

Hence

$$\langle u, x-y(x,u)\rangle \ge \|x-y(x,u)\|^2, \qquad \forall\; x\in K. \tag{6}$$

From (3) and (6) it follows that

$$\begin{aligned}\varphi(x,u) \quad &= \langle u, x-y(x,u)\rangle - \tfrac{1}{2}\,\|x-y(x,u)\|^2 \\ &\ge \|x-y(x,u)\|^2 - \tfrac{1}{2}\,\|x-y(x,u)\|^2 \\ &= \tfrac{1}{2}\|x-y(x,u)\|^2 \\ &\ge 0.\end{aligned} \tag{7}$$

Now, if (x^*,u^*) is a solution of GVI, it is well known that it is equivalent to $y(x^*,u^*) = x^*$, and thus $\varphi(x^*,u^*) = 0$. On the other hand, if $\varphi(x^*,u^*) = 0$, for $(x^*,u^*) \in \mathrm{Gr}\,(F)$, then from (7) it follows that $x^* = y(x^*,u^*)$, namely (x^*,u^*) solves GVI. Therefore φ is a gap function for GVI. Since the function $\Phi(x,u,y)$ is continuously differentiable and, for every fixed x and u, the maximum of $\Phi(x,u,\cdot)$ over K is uniquely attained at $y(x,u)$, it follows that (see e.g. [2]) φ is continuously differentiable and its gradient is given by (4). □

Theorem 2.1 states that φ yields an optimization problem equivalent to the GVI:

$$\begin{cases} \min\;\; \varphi(x,u) \\ (x,u) \in \mathrm{Gr}\,(F), \end{cases} \tag{8}$$

Therefore we can obtain a solution of GVI, if it exists, by finding a global minimum of φ with zero objective value, on the set $\mathrm{Gr}\,(F)$. It is worthwhile to remark that solution methods for a constrained optimization problem of the type (8) does not converge towards a global minimizer, but only to a stationary point (x^*,u^*). Under assumptions (A1)-(A2) such stationary point is characterized by the following property:

$$\langle \nabla\varphi(x^*,u^*), (x,u) - (x^*,u^*)\rangle \ge 0, \qquad \forall\; (x,u) \in \mathrm{Gr}\,(F). \tag{9}$$

Consequently it is important to know conditions under which a stationary point of (8) is actually a global optimal solution. In the following we show that it is possible to consider a class of maps F for which every stationary point of (8) is a solution of GVI.

Let consider the following assumption on the map F:

(A3) $\quad \forall\, x, y \in K,\ x \neq y, \quad \forall\, u \in F(x), \quad \exists\, v \in F(y) \qquad$ *such that* $\qquad \langle u - v, x - y \rangle > 0.$

We remark that in the generalized traffic equilibrium problem introduced in Example 1.1, assumption (A3) holds whenever l_a and u_a are increasing with respect to f_a, for every arc $a \in \mathcal{A}$.
It is worth noting that condition (A3) is satisfied for every strictly monotone map, i.e. such that:

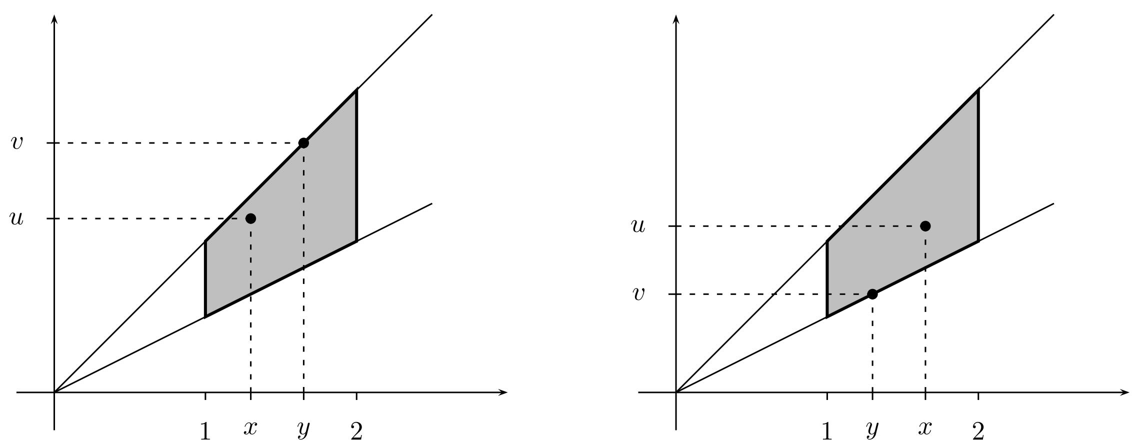
$$\forall\, x, y \in K,\ x \neq y, \quad \forall\, u \in F(x), \quad \forall\, v \in F(y) \qquad \langle u - v, x - y \rangle > 0,$$

instead, it easy to show that (A3) does not necessarily hold for monotone map, i.e. such that:

$$\forall\, x, y \in K,\ x \neq y, \quad \forall\, u \in F(x), \quad \forall\, v \in F(y) \qquad \langle u - v, x - y \rangle \geq 0.$$

Moreover, there are several maps satisfying assumption (A3) which are neither strictly monotone nor monotone, as the following example shows.

Example 2.1. Let $K = [1, 2]$ and $F : K \rightrightarrows \mathbb{R}$ defined by $F(x) = \left[\dfrac{x}{2}\ , x\right]$. This map satisfies assumption (A3) because if $x, y \in K$ and $u \in \left[\dfrac{x}{2}\ , x\right]$, then it is sufficient to choose $v = y$ if $x < y$ and $v = \dfrac{y}{2}$ otherwise (see figure below). However, F is neither strictly monotone nor monotone because for $x = 1$, $y = \dfrac{3}{2}$, $u = 1$, and $v = \dfrac{3}{4}$ one has that $(u - v)\,(x - y) = -\dfrac{1}{8} < 0$.



The following theorem says that, under the additional assumption (A3), each stationary point of problem (8) is a solution of GVI.

Theorem 2.2. Let assumptions $(A1) - (A3)$ be fulfilled. Then $(x^*, u^*) \in \mathrm{Gr}\,(F)$ is a solution of GVI if and only if (9) holds.

Proof. If (x^*, u^*) is a solution of GVI then, by Theorem 2.1, it is a global optimal solution of the problem (8) and thus (9) holds.

Vice versa, let assume that $(x^*, u^*) \in \mathrm{Gr}\,(F)$ satisfies (9). Then, by using (4), we have

$$\langle u^* + y(x^*, u^*) - x^*, x - x^* \rangle + \langle x^* - y(x^*, u^*), u - u^* \rangle \geq 0, \qquad \forall\ (x, u) \in \mathrm{Gr}\,(F).$$

Choosing $x = y(x^*, u^*)$ we obtain that for all $u \in F(y(x^*, u^*))$ one has

$$\begin{aligned} \langle u - u^*, y(x^*, u^*) - x^* \rangle &\leq \langle u^* + y(x^*, u^*) - x^*, y(x^*, u^*) - x^* \rangle \\ &= \langle y(x^*, u^*) - (x^* - u^*), y(x^*, u^*) - x^* \rangle \\ &= \langle x^* - u^* - y(x^*, u^*), x^* - y(x^*, u^*) \rangle. \end{aligned}$$

From (5) it follows that

$$\langle u - u^*, y(x^*, u^*) - x^* \rangle \leq 0 \qquad \forall\, u \in F(y(x^*, u^*)). \tag{10}$$

Since F satisfies assumption (A3), it follows that the inequality (10) holds only if $x^* = y(x^*, u^*)$, i.e. (x^*, u^*) solves GVI. □

3 A descent method

Since the gap function φ yields an optimization problem equivalent to GVI, a natural idea to solve GVI is to develop a descent method for the function φ, that is to move from a point $(x, u) \in \mathrm{Gr}\,(F)$, through the feasible region $\mathrm{Gr}\,(F)$, in a direction d along which the function φ decreases. Observe that if x does not solve GVI, i.e. $x \neq y(x, u)$, then by assumption (A3), there exists $v \in F(y(x, u))$ such that $\langle v - u, y(x, u) - x \rangle > 0$. This fact is the key to derive a descent direction for φ as shown in the following theorem.

Theorem 3.1. Let assumptions $(A1) - (A3)$ hold. Assume that $x \neq y(x, u)$, and consider $v(x, u) \in F(y(x, u))$ such that:

$$\langle v(x, u) - u, y(x, u) - x \rangle > 0.$$

Then the vector

$$d = (y(x, u) - x, v(x, u) - u) \tag{11}$$

satisfies the descent condition $\langle \nabla \varphi(x, u), d \rangle < 0$.

Proof. Since F satisfies assumption (A3), there exists $v(x, u) \in F(y(x, u))$ such that

$$\langle v(x, u) - u, y(x, u) - x \rangle > 0,$$

thus the vector d is well defined. Taking into account (4) and the choice of $v(x, u)$ we have

$$\begin{aligned} \langle \nabla \varphi(x, u), d \rangle \quad & = \langle u + [y(x, u) - x], y(x, u) - x \rangle + \langle x - y(x, u), v(x, u) - u \rangle \\ & = \langle y(x, u) - (x - u), y(x, u) - x \rangle + \langle x - y(x, u), v(x, u) - u \rangle \\ & \leq \langle x - y(x, u), v(x, u) - u \rangle \\ & < 0. \end{aligned} \tag{12}$$

□

A way to generate the descent direction d considered in Theorem 3.1 is to find

$$v(x, u) \in \arg \max_{w \in F(y(x,u))} \langle w - u, y(x, u) - x \rangle. \tag{13}$$

Under assumption (A2), this problem consists of maximizing a linear objective function over a closed and convex set. In particular, when $\mathrm{Gr}\,(F)$ is polyhedral, (13) is a linear programming problem.

The descent method, with an inexact Armijo-type line search procedure, exploiting the descent direction (11) can be summarized as follows.

Algorithm

0. (Initial step)

   Let $\beta, \gamma \in (0, 1)$.

   Choose any vector $(x^0, u^0) \in \mathrm{Gr}\,(F)$ and set $k = 0$.

1. (Stopping criterion)

   If $\varphi(x^k, u^k) = 0$ then STOP.

2. (Line search)

   Compute $y^k = \Pi_K(x^k - u^k)$.

   Set $\nabla\varphi(x^k, u^k) = (u^k + (y^k - x^k), x^k - y^k)$.

   Compute $v^k \in \arg \max\limits_{w \in F(y^k)} \langle w - u^k, y^k - x^k \rangle$.

   Set $d_x^k = y^k - x^k$, $d_u^k = v^k - u^k$, and $d^k = (d_x^k, d_u^k)$.

   Compute the smallest nonnegative integer m such that:

$$\varphi(x^k + \gamma^m d_x^k, u^k + \gamma^m d_u^k) - \varphi(x^k, u^k) \le \beta \gamma^m \langle \nabla\varphi(x^k, u^k), d^k \rangle. \tag{14}$$

   Set $t_k = \gamma^m$.

   Set $x^{k+1} = x^k + t_k d_x^k$, $u^{k+1} = u^k + t_k d_u^k$, $k := k + 1$ and return to step 1.

Additional assumptions that allows us to establish the global convergence of the above algorithm are the following:

(A4) *The set* Gr (F) *is bounded.*

(A5) *The mapping F is lower semicontinuous (l.s.c.) on K, i.e. for each $x^0 \in K$, for any $u^0 \in F(x^0)$ and any neighborhood N of u^0, there exists a neighborhood U of x^0 such that*

$$F(x) \cap N \neq \emptyset \qquad \forall\ x \in U \cap K.$$

It is worth remarking that assumption $(A4)$ implies in particular that the set K is bounded, which is quite natural in applications, as for example in traffic equilibrium problems. It may be shown that in Example 1.1 assumption (A5) is satisfied in view of the continuity of l and u.

We now prove the global convergence of the algorithm.

Theorem 3.2. Let assumptions $(A1) - (A5)$ be fulfilled. Then the algorithm either stops at a solution to GVI after a finite number of iterations, or generates a bounded sequence $\{(x^k, u^k)\}$ such that any of its cluster points solves GVI.

Proof. First, we show that the algorithm is well defined, i.e. the line search procedure is always finite. Assume, by contradiction, that there is an iteration k such that for all $m \in \mathbb{N}$ one has

$$\varphi(x^k + \gamma^m d_x^k, u^k + \gamma^m d_u^k) - \varphi(x^k, u^k) > \beta \gamma^m \langle \nabla\varphi(x^k, u^k), d^k \rangle.$$

Then passing to the limit we have

$$\langle \nabla\varphi(x^k, u^k), d^k \rangle = \lim_{m \to +\infty} \frac{\varphi(x^k + \gamma^m d_x^k, u^k + \gamma^m d_u^k) - \varphi(x^k, u^k)}{\gamma^m} \ge \beta \langle \nabla\varphi(x^k, u^k), d^k \rangle.$$

Hence one has

$$(1 - \beta) \langle \nabla\varphi(x^k, u^k), d^k \rangle \ge 0,$$

which is impossible because $\beta < 1$ and $\langle \nabla\varphi(x^k, u^k), d^k\rangle < 0$ by Theorem 3.1. So the line search procedure is always finite.

The generated sequence $\{(x^k, u^k)\}$ lies in $\mathrm{Gr}\,(F)$, in fact $\mathrm{Gr}\,(F)$ is convex, the starting point $(x^0, u^0) \in \mathrm{Gr}\,(F)$, and $\{(y^k, v^k)\} \in \mathrm{Gr}\,(F)$, $\forall k \in \mathbb{N}$. Now, if the algorithm stops at (x^*, u^*) after a finite number of iterations, then $\varphi(x^*, u^*) = 0$, thus (x^*, u^*) solves GVI by Theorem 2.1.

If the algorithm generates an infinite sequence $\{(x^k, u^k)\}$, then we consider two possible cases: either $\limsup_{k\to\infty} t_k > 0$, or $\limsup_{k\to\infty} t_k = 0$.

Case 1. If $\limsup_{k\to\infty} t_k > 0$, then there are $t^* > 0$ and a subsequence $\{t_{k_s}\}$ such that $t_{k_s} \geq t^* > 0$. Since the sequence $\{(x^k, u^k)\}$ is infinite, we have

$$\begin{aligned} \varphi(x^{k_s}, u^{k_s}) - \varphi(x^{k_s+1}, u^{k_s+1}) &\geq -\beta\, t_{k_s} \langle \nabla\varphi(x^{k_s}, u^{k_s}), d^{k_s}\rangle \\ &\geq -\beta\, t^* \langle \nabla\varphi(x^{k_s}, u^{k_s}), d^{k_s}\rangle \\ &> 0. \end{aligned} \tag{15}$$

The sequence $\{\varphi(x^k, u^k)\}$ is monotone decreasing and bounded below, hence

$$\lim_{k\to\infty} [\varphi(x^k, u^k) - \varphi(x^{k+1}, u^{k+1})] = 0,$$

and in particular we have

$$\lim_{s\to\infty} [\varphi(x^{k_s}, u^{k_s}) - \varphi(x^{k_s+1}, u^{k_s+1})] = 0. \tag{16}$$

Using (15) and (16), we obtain

$$\lim_{s\to\infty} \langle \nabla\varphi(x^{k_s}, u^{k_s}), d^{k_s}\rangle = 0. \tag{17}$$

The subsequences $\{(x^{k_s}, u^{k_s})\}$ and $\{v^{k_s}\}$ are bounded, thus they have cluster points $(\bar{x}, \bar{u})$ and $\bar{v}$, respectively. Then passing to the limit, and taking a subsequence if necessary, we have

$$d_x^{k_s} = (y(x^{k_s}, u^{k_s}) - x^{k_s}) \longrightarrow (y(\bar{x}, \bar{u}) - \bar{x}) = \bar{d}_x.$$

Let now prove that passing to the limit, and taking a subsequence if necessary, we obtain

$$d_u^{k_s} = (v^{k_s} - u^{k_s}) \longrightarrow (\bar{v} - \bar{u}) = \bar{d}_u,$$

where

$$\bar{v} \in \arg \max_{w\in F(y(\bar{x},\bar{u}))} \langle w - \bar{u}, y(\bar{x}, \bar{u}) - \bar{x}\rangle,$$

i.e.

$$\langle \bar{v} - \bar{u}, y(\bar{x}, \bar{u}) - \bar{x}\rangle \geq \langle w - \bar{u}, y(\bar{x}, \bar{u}) - \bar{x}\rangle \qquad \forall\, w \in F(y(\bar{x}, \bar{u})).$$

Let $w \in F(y(\bar{x}, \bar{u}))$. Since F is l.s.c on K, for any neighborhood N of w, there exists a neighborhood V of $y(\bar{x}, \bar{u})$, such that

$$F(x) \cap N \neq \emptyset \qquad \forall\, x \in V \cap K.$$

Then there exists a sequence $\{w^k\}$ converging to w, with $w^k \in F(y(x^k, u^k))$. By definition of v^k we have

$$\langle v^k - u^k, y(x^k, u^k) - x^k\rangle \geq \langle z - u^k, y(x^k, u^k) - x^k\rangle \qquad \forall\, z \in F(y(x^k, u^k)), \quad \forall\, k \in \mathbb{N}.$$

In particular, for $z = w^k$ we have

$$\langle v^k - u^k, y(x^k, u^k) - x^k\rangle \geq \langle w^k - u^k, y(x^k, u^k) - x^k\rangle \qquad \forall\, k \in \mathbb{N}.$$

Passing to the limit we get

$$\langle \bar{v} - \bar{u}, y(\bar{x}, \bar{u}) - \bar{x} \rangle \geq \langle w - \bar{u}, y(\bar{x}, \bar{u}) - \bar{x} \rangle.$$

This prove that

$$d_u^{k_s} = (v^{k_s} - u^{k_s}) \longrightarrow (\bar{v} - \bar{u}) = \bar{d}_u,$$

where

$$\bar{v} \in \arg \max_{w \in F(y(\bar{x}, \bar{u}))} \langle w - \bar{u}, y(\bar{x}, \bar{u}) - \bar{x} \rangle.$$

Therefore passing to the limit, and taking a subsequence if necessary, we have

$$\lim_{s \to \infty} \langle \nabla \varphi(x^{k_s}, u^{k_s}), d^{k_s} \rangle = \langle \nabla \varphi(\bar{x}, \bar{u}), \bar{d} \rangle. \tag{18}$$

From (17) and (18) it follows that

$$\langle \nabla \varphi(\bar{x}, \bar{u}), \bar{d} \rangle = 0,$$

thus, taking into account Theorem 3.1, we deduce that $(\bar{x}, \bar{u})$ is a solution of GVI. Therefore

$$\lim_{s \to \infty} \varphi(x^{k_s}, u^{k_s}) = \varphi(\bar{x}, \bar{u}) = 0.$$

Since $\{\varphi(x^k, u^k)\}$ is a decreasing sequence, then $\lim_{k \to \infty} \varphi(x^k, u^k) = 0$. Finally, let (x^*, u^*) be any cluster point of $\{(x^k, u^k)\}$, then by continuity of φ we obtain $\varphi(x^*, u^*) = 0$, i.e. (x^*, u^*) is a solution of GVI.

Case 2. If $\limsup_{k \to \infty} t_k = 0$, then $\lim_{k \to \infty} t_k = 0$. From the step length rule it follows that for all $k \in \mathbb{N}$ one has

$$\varphi\left(x^k + \frac{t_k}{\gamma}\, d_x^k, u^k + \frac{t_k}{\gamma}\, d_u^k\right) - \varphi(x^k, u^k) > \beta\, \frac{t_k}{\gamma}\, \langle \nabla \varphi(x^k, u^k), d^k \rangle.$$

By the mean value theorem we have

$$\varphi\left(x^k + \frac{t_k}{\gamma}\, d_x^k, u^k + \frac{t_k}{\gamma}\, d_u^k\right) - \varphi(x^k, u^k) = \left\langle \nabla \varphi\left(x^k + \theta_k\, \frac{t_k}{\gamma}\, d_x^k, u^k + \theta_k\, \frac{t_k}{\gamma}\, d_u^k\right), \frac{t_k}{\gamma}\, d^k \right\rangle,$$

for some $\theta_k \in (0, 1)$. Therefore for all $k \in \mathbb{N}$ we have

$$\left\langle \nabla \varphi\left(x^k + \theta_k\, \frac{t_k}{\gamma}\, d_x^k, u^k + \theta_k\, \frac{t_k}{\gamma}\, d_u^k\right), d^k \right\rangle > \beta\, \langle \nabla \varphi(x^k, u^k), d^k \rangle. \tag{19}$$

Let (x^*, u^*) be any cluster point of $\{(x^k, u^k)\}$. Passing to the limit, and taking a subsequence if necessary, we have

$$d^k \longrightarrow d^*,$$

where $d_x^* = y(x^*, u^*) - x^*$ and $d_u^* = v^* - u^*$, with

$$v^* \in \arg \max_{w \in F(y(x^*, u^*))} \langle w - u^*, y(x^*, u^*) - x^* \rangle.$$

Since $\lim_{k \to \infty} t_k = 0$, then passing to the limit in (19), and taking the subsequence if necessary, we have

$$\langle \nabla \varphi(x^*, u^*), d^* \rangle \geq \beta\, \langle \nabla \varphi(x^*, u^*), d^* \rangle.$$

Since $\beta < 1$ we obtain

$$\langle \nabla \varphi(x^*, u^*), d^* \rangle \geq 0.$$

From Theorem 3.1 we deduce that (x^*, u^*) is a solution of GVI.

□

Remark 3.1. *A convergence result similar to Theorem 3.2 can be obtained by replacing the inexact line search in (14) with the exact one of the form*

$$\varphi(x^k + t_k\, d_x^k, u^k + t_k\, d_u^k) = \min_{t\in[0,1]} \varphi(x^k + t\, d_x^k, u^k + t\, d_u^k).$$

In such a case the proof is based on the well-known Zangwill's convergence Theorem [17].

References

[1] Aubin J. P., *L'analyse non linaire et ses motivations economiques*, Masson, 1984.

[2] Auslender A., *Optimisation: mthodes numriques*, Masson, 1976.

[3] Bureau of Public Roads, *Traffic Assignment Manual (U.S. Department of Commerce*, Urban Planning Division, Washington DC) 1964.

[4] Dafermos S., *Traffic equilibria and variational inequalities*, Transportation Science, vol. 14 (1980), pp. 42–54.

[5] Dietrich H., *A smooth dual gap function solution to a class of quasivariational inequalities*, Journal of Mathematical Analysis and Applications, vol. 235 (1999), pp. 380–393.

[6] Fang S. C. and Peterson E. L., *Generalized variational inequalities*, Journal of Optimization Theory and Applications, vol. 38 (1982), pp. 363–383.

[7] Fukushima M., *Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems*, Mathematical Programming, vol. 53 (1992), pp. 99–110.

[8] Harker P. T. and Pang S., *Engineering and Economic Applications of Complementarity Problems*, SIAM Review 39 (1997), pp. 669-713.

[9] Hearn D. W., *The gap function of a convex program*, Operations Research Letters, vol. 1 (1982), pp. 67–71.

[10] Konnov I.,Combined Relaxation Methods for Variational Inequalities, Springer-Verlag: New York, 2001.

[11] Konnov I., Equilibrium Models and Variational Inequalities, Elsevier: Amsterdam, 2007.

[12] Lu, S., *Sensitivity of Static Traffic User Equilibria with Perturbations in Arc Cost Function and Travel Demand*, Transportation Science, vol. 42 (2008), pp. 105–123.

[13] Panicucci B., Pappalardo M., and Passacantando M., *On finite-dimensional generalized variational inequalities*, Journal of Industrial and Management Optimization, vol. 2 (2006), pp. 43–53.

[14] Robinson, S. M., *Solution continuity in monotone affine variational inequalities*, SIAM Journal on Optimization, vol 18 (2007), pp. 1046–1060.

[15] Robinson, S. M. and Lu, S., *Solution continuity in variational conditions*, Journal of Global Optimization, vol 40 (2008), pp. 405–415.

[16] Yao J.-C., *Variational inequalities with generalized monotone operators*, Mathematics of Operations Research, vol. 19 (1994), pp. 691–705.

[17] Zangwill W. I., *Nonlinear programming: a unified approach*, Prentice-Hall, Englewood Cliffs, NJ, 1969.