Hamiltonian approach to 2-layer dispersive stratified fluids

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Acknowledgement

During the time studying in the Joint PhD Program in Mathematics (Milano Bicocca,Pavia,INdAM), I have received a great deal of support and assistance from the professors and colleagues there. I would like to express my special thanks and gratitude for being a member of the INFN group based in Bicocca, and being a member of the research group including Prof. R.Camassa, Prof. G.Falqui, Prof. G.Ortenzi and Prof. M.Pedroni. We have had a great chance to collaborate and co-author 3 papers so far. I especially want to thank my thesis supervisor, Prof. Gregorio Falqui, for his invaluable advice and support throughout my writing of PhD thesis. I also want to thank Prof. Roberto Camassa for the opportunity to do an exchange program at the Department of Mathematics at University of North Carolina at Chapel Hill, under his support and guidance. This thesis project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant no 778010 *IPaDEGAN*. I also want to show gratitude to my parents for giving me time and emotional support during the completion of the PhD program.

Finally, I want to thank my friends at the VNU-HCM, University of Science, and VIASM, and my expat friends in Milan for their advice and encouragement at hard times.

Best.

Thao Thuan Vu Ho

CHAPTER 1

Introduction

1. From classical mechanics to Hamiltonian mechanics

As it is well-known from classical mechanics, to find the trajectory $q(t) \in \mathbb{R}^n$ of a point particle of mass m under the force $\overrightarrow{F}(q)$, we use Newton's equation:

$$m\ddot{q} = \overrightarrow{F}(q) \tag{1}$$

This ordinary differential equation can be solved locally, either analytically by Cauchy-Lipschitz theorem or numerically, if the force field $\overrightarrow{F}(q)$ satisfies sufficient smoothness condition.
However, the formulation of the Newton's equation of motion depends heavily on the choice of coordinates. To overcome this difficulty, we can formulate the equation of motion in a coordinate-free presentation by casting the problem into Lagrangian mechanics, in which the equation of motion is the Euler-Lagrange equation defined in a generalized coordinate system. The derivation of such an Euler-Lagrange equation comes from the variational principle (a.k.a least action principle), i.e finding the critical points of the action $S = \int L dt$ in which L is the Lagrangian functional.
This Lagrangian mechanics point of view is not limited to motion of rigid bodies in finite-dimensional configuration space, but can be extended to infinite-dimensional ones, such as motion of a continuum mechanical system. In the case of a continuum system like fluids, it is possible to see the equations of fluid motions as the Euler-Lagrange equations corresponding to a specific Lagrangian functional, in an infinite-dimensional configuration space (this is the motivation behind chapter 2 concerning the variational Euler-Poincaré formulation for the motion of a 2-layered stratified fluid).

On the other hand, the Hamiltonian point of view provides a canonical way to trade the n second order Euler-Lagrange equations for a Dynamical System of $2n$ first order equations. For example, consider a system of mass m at position q, under the action of a conservative force field $\overrightarrow{F}(q)$, thus there exists a potential function V such that:

$$\overrightarrow{F}(q) = -\nabla V(q) \tag{2}$$

Then, the Hamiltonian functional of the system is

$$H(q,p) = \frac{p^2}{2m} + V(q) \tag{3}$$

and the new variable p is the momentum, corresponds to $m\dot{q}$ in Newtons's equation. Then second-order Newton's equation becomes equivalent to the following first-order Hamiltonian system:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned} \tag{4}$$

This Hamiltonian approach to dynamics provides some advantages. Hamiltonian structure of evolutionay equations provides a useful setting to find symmetry groups associated with the equations (via Noether's theorem), from which conservation laws of the system can be deduced. These conservation laws help reduce the degree of freedom of the configuration space where the motion trajectory of the dynamics system lies. The number of conserved quantities of a dynamical system can be used, for example, to check its integrability. Moreover, the Hamiltonian structure of the evolutionary equations is of great help in studying the stability of the resulting solutions (see, e.g. [**9**] for a collection of these ideas in fluid dynamics)

2. Introduction to fluids and stratified fluids

Fluid waves are ubiquitous phenomena in marine and atmosphere science. One important cause that creates internal waves in fluid is density stratification. Displacement of fluid parcels from their neutral buoyancy position within a density stratified flow can result in internal wave motion. Dynamics of these internal waves has been of great interest and resulted in numerous investigations, from [**2**], [**4**] to the more recent papers [**10**], [**11**] [**1**], [**12**] and others.

In this thesis, we focus on the dynamics of two-layered stratified fluids, starting from their dispersionless quasi-linear limits, with a view towards including dispersion from an Hamiltonian standpoint.

The Euler's equation for the motion of an incompressible, irrotational and non-viscous fluid are

$$\begin{aligned} \rho\frac{D\mathbf{u}}{Dt} &= -\nabla p - \mathbf{F} \\ \nabla.\mathbf{u} &= 0 \text{ (incompressibility condition)} \\ \frac{D\rho}{Dt} &= 0 \text{ (advection of the density)} \end{aligned}$$

where **u** denotes the velocity field, p the pressure field, **F** the external forces field and ρ denotes the density of the fluid, which can be constant or not.
The Euler equation is the basic equation, and is used to describe many phenomena in fluid dynamics. Based on the Euler equation, various models have been developed to provide a simplified description of such complex phenomena. For example, in the case of stratified fluids, we present in chapter 2 and 3 two classical models for sharply stratified 2-layer fluids, namely the Camassa-Choi model and the Wu model. These models both base on the Euler's equation as the starting point and then use long-wave small parameter series expansion to derive the 1D effective dispersive equations of motion for multi-layer flows. As we shall see, a crucial hypothesis to derive these models is to regard the fluid domain as being confined in a vertical channel of fixed total height. That is, besides the presence of an impermeable bottom surface, one considers the presence of an impermeable top lid. This hypothesis is empirically justified by the fact that surface and internal oscillations are decoupled, and the ocean surface acts as an effective lid.

A general Hamiltonian setting for 2D heterogeneous incompressible fluids was derived by T.-B. Benjamin in 1986 in [**3**] and later generalized to 3D in [**8**]. In such multi-dimensional cases, the Poisson tensor is a Lie-theoretic one, while the Hamiltonian is the sum of the kinetic energy and the gravitational potential energy. The basic idea of this thesis is to derive effective 1D equations of motion by means of a Hamiltonian reduction process starting from Benjamin's in the spirit of the Marsden-Ratiu general Hamiltonian reduction scheme [**7**]. As we shall see, the reduced Poisson structure, besides becoming a constant one, has the fundamental property of being independent of the densities ρ_j of the individual layers. Thus the construction of the models boils down to the reduction of the Hamiltonian (the natural energy) to an effective one. This has been performed in various cases starting from the long-wave dispersionless approximation (with an extension to the 3-layer case) to dispersive case, namely the classical the weakly non linear approximation and a further case, termed *Mildly non-linear* approximation, where the relative scaling $\alpha \simeq \epsilon^2$ typical of the weakly non-linear regime between the dispersion parameter ϵ and the non linear parameter α (see, e.g., [**17**]) is replaced by the relative scaling $\epsilon^2 < \alpha < \epsilon$.

3. Outline of the thesis

We first consider the Lagrangian mechanics setting proposed by Darryl Holm and his collaborators, in which the evolutionary equations of internal waves can be obtained via the Euler-Poincaré variational reduction technique, under the so-called columnar motion ansatz assumption. Then, we go on explaining the classical model proposed by T. Wu and Camassa-Choi to formulate evolutionary equations of the internal waves, by means of Taylor-expansion in the vertical direction z from the velocites at the boundaries. Finally, based on the classical Hamiltonian formalism for 2D wave motions in heterogeneous fluids by Benjamin in [**3**], we discuss the Hamiltonian structure of 2-layer dispersionless stratified fluids. The Hamiltonian approach to dispersive stratified fluids, on the other hand, was inspired by the work of Dubrovin and his collaborators, but detailed study in the case when dispersion is included has not been thoroughly studied and related results have not been published anywhere yet, as far as the author knows. Therefore, the thesis aims to delve deep into the issue of Hamiltonian approach to dispersive stratified fluids. The Hamiltonian matrix, a.k.a the Poisson tensor, in the case of dispersive stratified fluids is the same as the one in the non-dispersive case. However, the Hamiltonian functional is a lot more complicated than the one in the dispersionless case, and has not been explicitly computed anywhere yet. Therefore, in this thesis, we focus on computing the Hamiltonian functional of dispersive 2-layer stratified fluids, firstly by using the Weakly Nonlinear assumption and Boussinesq approximation to simplify the Hamiltonian functional. We then try to relax these assumptions, by computing the Hamiltonian functional in the general case of Mildly Nonlinear assumption and without Boussinesq approximation. The main technique used during the computation process involves the usual geometric series expansion, long-wave asymptoptics, but requires technical computational manipulation to obtain the final simple Hamiltonian functional.

*Chapter 1:*Introduction to the thesis.

Chapter 2:

In this chapter, we apply the Euler-Poincaré reduction technique discussed in [**19**] to derive equations for internal waves in 2-layer fluid system, under the additional condition that the columnar ansatz is needed. Detailed computations and explanations are written explicitly for the reader's convenience. The technique can be used in the general case of an n-layer flow as well.

Chapter 3

Here we consider the classical study of internal waves in a two-fluid system derived by Camassa and Choi, in which the variables are layer-averaged horizontal velocities of each fluid with the use of long-wave asymptotics as described in [**4**], [**2**].

Chapter 4

Here, we consider the classical models for 2-layer stratified fluids via long-wave small parameter series expansion by Wu in 4. In his study, the motion of the interfaces in stratified fluids is described via Taylor-expansion in the vertical direction, in which the variables of the evolutionary equations are interfacial horizontal velocities.

Chapter 5
We first summarize the classical result of the study of internal waves in heterogeneous fluids proposed by Benjamin in [**3**], in which the Poisson tensor for the motion of internal waves was formulated in the coordinates density ρ and weighted vorticity σ. Following this approach, we present Hamiltonian reduction technique to derive the Poisson tensor in terms of layer-averaged variables of ρ and σ. After some changes of coordinates, the Poisson tensor is proved to be canonical. Notice that during the computation, we ignore dispersive effect in the 2-fluid system. These results are already published in the paper [**12**] .
Chapter 6
The Hamiltonian setup for 2-layer dispersionless fluids described in the previous chapter can be extended to the 3-layer case. In this chapter, we explicitly show the Hamiltonian structure for 3-layer dispersionless stratified fluids. These results are already published in the paper [**1**].
Chapter 7
The Hamiltonian approach presented in this chapter is the same as the Hamiltonian approach in the previous chapter. However, here we focus on the Hamiltonian structure of 2-layer fluids when dispersive effect is take into account. The inclusion of dispersive effect does not alter the Poisson tensor of the system, but changes the relation between basic variables and complicate the corresponding Hamiltonian functional. Thus, we employ Weakly-Nonlinear assumption and Boussinesq approximation to reduce the complexity of the Hamiltonian functional.
Chapter 8
This chapter deals with a more general assumption than the Weakly nonlinear assumption presented in the previous chapter.
Specifically, we focus on deriving the Hamiltonian structure for 2-layer dispersive stratified fluids, under the Mildly nonlinear assumption, in both the Boussinesq and non-Boussinesq case. Finally, we consider solutions of special form, namely traveling wave solutions and unidirectional waves, as well as computing the dispersion relation. Some of these results were included in the submitted paper [**22**].
Chapter 9
The conclusion first summarizes the main results obtained during the completion of this thesis. We first make a comparison between the different models discussed in previous chapters. We then go on making some comments and predictions concerning the number of conserved quantities, existence of bi-Hamiltonian structure and integrability, etc. of the obtained system. We conclude the chapter with open problems for future study.
Chapter 10
In this appendix chapter, we explain the geometric setting to study motion of a

continuum system from the Lagrangian mechanics point of view, based on [**23**]. Motion of incompressible fluid at time t is described by a diffeomorphism $g(t)$ in the infinite-dimensional configuration space $G = Diff_{vol}(D)$ (in which $Diff_{vol}(D)$ is group of volume-preserving diffeomorphisms on flow domain D). We show that the Lagrangian functional of a fluid flow is invariant under the group action $G = Diff_{vol}(D)$. Thus, the Euler-Lagrange equation is reduced to the Euler-Poincaré equation of motion. Finally, we show that the classical Euler's equation for an incompressible fluid can be seen as an Euler-Poincaré equation.

CHAPTER 2

Variational Euler-Poincaré formulation

A detailed explanation of the Euler-Poincaré formulation for the motion of a rigid body and the motion of a continuum system like fluids can be found in the appendix. The following variational Euler-Poincaré formulation for 2-layered stratified fluids (can be extended to the n-layer case) under columnar ansatz assumption is based on the approach in [**19**].

1. Problem setting

Consider a system of two homogeneous fluid layers, which are under the gravitational force defined by a constant acceleration g. Each layer has a constant density ρ_i such that $\rho_1 < \rho_2$. We choose to number the layers downwards from the top (rigid lid) to bottom. Denote h_i the depth of the layer interface with respect to the rigid lid at which $z = 0$. Define $b = -h_3$ the fixed bottom topography to which the 2nd-layer is attached to. In the following computation, we assume b is a fixed constant, independent from the horizontal x direction (i.e even bottom). Denote $\gamma_1(\gamma_2)$ the initial undisturbed thickness of the upper (lower) layer, respectively. γ_1 and γ_2 are known constants. Let (u_i, w_i) denote the horizontal and vertical components of the fluid velocity in the ith-layer. In the general case, u_i and w_i are functions of the spatial variables (x, z) and time t.
Note: These notations are different from the notations in the Camassa-Choi model [**4**]. In [**4**], $h_1(h_2)$ is the undisturbed thickness of the upper (lower) layer, and $z = 0$ denotes the interface when undisturbed, while in [**12**], $z = 0$ denotes the bottom of the fluid system.

2. The Lagrangian functional

Mass conservation and incompressibility condition within each layer gives:

$$div(u_i, w_i) = 0, i = 1, 2 \tag{5}$$

or

$$u_{ix} + w_{iz} = 0, i = 1, 2 \tag{6}$$

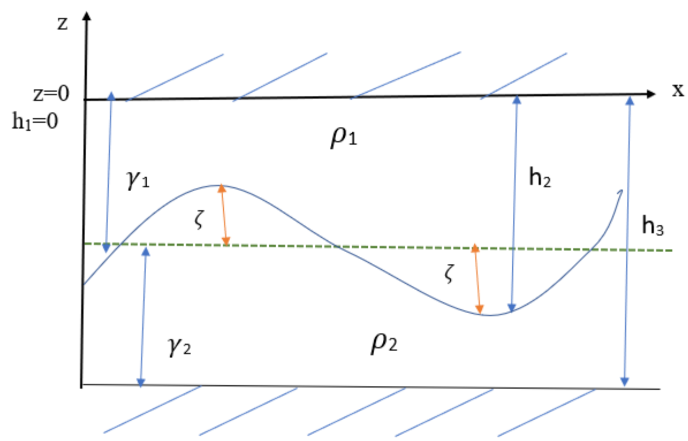

FIGURE 1. Two-layer fluid setup with displacement vector of the interface ζ

Also, as in [**19**], we assume the conservation of fluid in each layer, which gives a transport equation for layer thicknesses $\eta_i = h_i - h_{i+1}$:

$$\frac{\partial \eta_i}{\partial t} + (\eta_i u_i)_x = 0 \tag{7}$$

By the columnar condition, i.e

$$\frac{\partial u_i}{\partial z} = 0, \qquad \text{for } i = 1, 2 \tag{8}$$

we also have u_{ix} is also independent from z. We have that u_i and u_{ix} are functions of (z, t) only.

Combine this columnar condition with the previous incompressibility condition, we conclude that the vertical components of velocity in the ith-layer can be written as a function of z as follows:

$$w_i(z) = -z u_{ix} + c_i \tag{9}$$

in which c_i is a constant to be defined.

Similarly, we have $w_{i+1}(z) = -z u_{i+1,x} + c_{i+1}$. Moreover, we have the boundary condition at the interface, which is the continuity of the vertical velocity, i.e

$$w_i(h_{i+1}) = h_{i+1,t} + u_i h_{i+1,x} \tag{10}$$

$$w_{i+1}(h_{i+1}) = h_{i+1,t} + u_{i+1} h_{i+1,x} \tag{11}$$

i.e we have the following relation, from which we can find the constants c_i.

$$h_{i+1,t} + u_i h_{i+1,x} = w_i(h_{i+1}) = -h_{i+1} u_{ix} + c_i \tag{12}$$

$$h_{i+1,t} + u_{i+1} h_{i+1,x} w_{i+1}(h_{i+1}) = -h_{i+1} u_{i,x} + c_{i+1} \tag{13}$$

Thus, we have

$$c_i = h_{i+1,t} + u_i.h_{i+1,x} + h_{i+1}u_{ix} \tag{14}$$

in which $h_{i+1,t} = -h_{i+1}u_{i+1,x} + c_{i+1} - u_{i+1}h_{i+1,x}$.
Thus, we have

$$c_i = c_{i+1} - [h_{i+1}(u_{i+1} - u_i)]_x \tag{15}$$

For the boundary condition at the bottom $z = h_3 = -b$, we have that the $(u_2, w_2).\mathbf{n} = 0$, where $\mathbf{n} = (b_x, 1)$ (thus in case of the even bottom, $\mathbf{n} = (0, 1)$) is the outward normal vector at the bottom $z = h_3 = -b < 0$, i.e

$$u_2 b_x + w_2(-b) = 0 \tag{16}$$

Thus, we have $w_2(-b) = bu_{2,x} + c_2 = -u_2 b_x$

$$c_2 = -u_2 b_x - bu_{2,x} = -(bu_2)_x = [h_3 u_2]_x \tag{17}$$

From $c_i = c_{i+1} - [h_{i+1}(u_{i+1} - u_i)]_x$, and $c_2 = [h_3 u_2]_x$, we have

$$c_1 = c_2 - [h_2(u_2 - u_1)]_x = -[bu_1]_x - [\eta_2(u_2 - u_1)]_x \tag{18}$$

where $\eta_2 = h_2 - h_3$ is the thickness of the lower layer.
Therefore, the vertical velocity of each layer is

$$w_1(z) = -zu_{1,x} - [bu_1]_x - [\eta_2(u_2 - u_1)]_x \tag{19}$$

$$w_2(z) = -zu_{2,x} - (bu_2)_x \tag{20}$$

We can therefore write the vertical velocity of each layer as

$$w_i(z) = -zu_{i,x} - (bu_i)_x - \Sigma_{j=i+1}^{2}[\eta_j(u_j - u_i)]_x \forall i = 1, 2 \tag{21}$$

Denote W_i the vertical velocity at the bottom of each layer, which are equal to

$$W_i = w_i(h_{i+1}) = u_i h_{i+1,x} - \Sigma_{j=i+1}^{2}[\eta_j u_j]_x$$

Thus, we have

$$W_1 = w_1(h_2) = u_1 h_{2x} - [\eta_2 u_2]_x \tag{22}$$

$$W_2 = w_2(h_3) = 0 \tag{23}$$

since there is no fluid penetrating through the bottom of the fluid system, so $w_2(h_3) = 0$.
The kinetic energy of the system is

$$\int K dx := \Sigma_{i=1}^{2} \int \frac{\rho_i}{2} \int_{h_{i+1}}^{h_i} u_i^2 + w_i^2 dz dx \tag{24}$$

The gravitational potential energy is

$$\int V dx := g\Sigma_{i=1}^{2}\rho_i \int_{h_{i+1}}^{h_i} z dx dz = g \int \Sigma_{i=1}^{2}\frac{\rho_i}{2}(h_i^2 - h_{i+1}^2)dx \tag{25}$$

Let's compute the kinetic energy term.

(26)

$$\begin{aligned}
K &= \sum_{i=1}^{2}\int_{h_{i+1}}^{h_i} u_i^2 + w_i^2 dz \\
&= \sum_{i=1}^{2}\int_{h_{i+1}}^{h_i} u_i^2 + (zu_{i,x})^2 + 2zu_{i,x}[(bu_i)_x + \Sigma_{j=i+1}^{2}[\eta_j(u_j - u_i)]_x] + [(bu_i)_x + \Sigma_{j=i+1}^{2}[\eta_j(u_j - u_i)]_x]^2 dz \\
&= \sum_{i=1}^{2} z[u_i^2 + [(bu_i)_x + \Sigma_{j=i+1}^{2}[\eta_j(u_j - u_i)]_x]^2] + \frac{z^3}{3}u_{i,x}^2 \\
&+ z^2 u_{i,x}[(b.u_i)_x + \Sigma_{j=i+1}^{2}[\eta_j(u_j - u_i)]_x] \text{ for } z \in (h_{i+1}, h_i) \\
&= \sum_{i=1}^{2}\eta_i[u_i^2 + [(b.u_i)_x + \Sigma_{j=i+1}^{2}[\eta_j(u_j - u_i)]_x]^2 \\
&+ \frac{\eta_i^3 + 3h_i h_{i+1}\eta_i}{3}(u_{i,x})^2 \\
&+ \eta_i(h_i + h_{i+1})u_{i,x}[(bu_i)_x + \Sigma_{j=i+1}^{2}[\eta_j(u_j - u_i)]_x] \\
&= \sum_{i=1}^{2}\eta_i u_i^2 + \eta_i[W_i^2 - W_i u_{i,x}\eta_i]
\end{aligned}$$

in which we use the relation $\eta_i = h_i - h_{i+1}$, and $w_i(z) = -zu_{i,x} - (bu_i)_x - \Sigma_{j=i+1}^{2}[\eta_j(u_j - u_i)]_x$, so $W_i = w_i(h_{i+1}) = u_i h_{i+1,x} - \Sigma_{j=i+1}^{2}(\eta_j u_j)_x$, thus

$$[(bu_i)_x + \Sigma_{j=i+1}^{2}(\eta_j(u_j - u_i))_x]^2 = [-W_i - h_{i+1}u_{i,x}]^2$$

Then, the Lagrangian function is a function of horizontal velocities u_i and layer thicknesses $\eta_i = h_i - h_{i+1}$.
For the 2-layer fluid system with a rigid lid constraint, the Lagrangian function becomes:

(27)

$$\boxed{l = \sum_{i=1}^{2}\frac{\rho_i}{2}\left(\eta_i u_i^2 + \eta_i\left[\frac{\eta_i^2}{3}(u_{ix})^2 - \eta_i u_{ix} W_i + W_i^2\right] - g(h_i^2 - h_{i+1}^2)\right) - \phi.[\Sigma_{j=1}^{2}\eta_j - b]}$$

where $\phi(x)$ is the Lagrange multiplier to impose rigid lid constraint, so that the sum of the thicknesses η_j equal the bottom topography $b = -h_3$.
We have:

. $\eta_1 = h_1 - h_2 = -h_2$, or $\eta_1^2 = h_2^2$ in rigid lid constraint case ($h_1 \neq 0$ only in the free upper boundary case). Thus,

$$h_1^2 - h_2^2 = -\eta_1^2 \tag{28}$$

. $\eta_2 = h_2 - h_3$. Thus,

$$h_2^2 - h_3^2 = (h_2 - h_3)(h_2 + h_3) = \eta_2(h_2 - h_3 + 2h_3) = \eta_2(\eta_2 + 2h_3) = \eta_2^2 + 2\eta_2 h_3 \tag{29}$$

. Vertical velocity at the bottom of the upper layer is: $W_1 = w_1(h_2) = u_1.h_{2x} - (\eta_2 u_2)_x$

. Rigid lid constraint means vertical velocity on the top surface vanishes, i.e

$$\begin{aligned}
0 &= w_1(z=0) = -0 \cdot u_{1x} - (bu_1)_x - [\eta_2(u_2 - u_1)]_x \\
&= (h_3 u_1)_x + ((h_2 - h_3)u_1)_x - (\eta_2)u_2)_x \quad (\text{ since } -b = h_3; \eta_2 = h_2 - h_3) \\
&= u_1 h_{2x} + h_2 u_{1x} - (\eta_2 u_2)_x = -\eta_1 u_{1x} + W_1
\end{aligned} \tag{30}$$

Thus,

$$0 = -\eta_1 u_{1x} + W_1 \tag{31}$$

or

$$W_1 = \eta_1 u_{1x} \tag{32}$$

so we have

$$-\eta_1 u_{1x} W_1 + W_1^2 = 0 \tag{33}$$

. Similarly, there is no vertical velocity at the bottom topography $z = h_3$, thus we have $w_2(h_3) = 0$ or

$$W_2 = w_2(h_3) = 0 \tag{34}$$

From the above reasoning and $W_2 = 0$, the Lagrangian is thus reduced to

$$\boxed{\begin{aligned}
l = &\frac{\rho_1}{2}\left(\eta_1(u_1)^2 + \left[\frac{\eta_1^3}{3}(u_{1x})^2\right] + g\eta_1^2\right) \\
&+ \frac{\rho_2}{2}\left(\eta_2(u_2)^2 + \left[\frac{\eta_2^3}{3}(u_{2x})^2\right] - g\eta_2^2 - 2g\eta_2 h_3\right) \\
&- \phi[\eta_1 + \eta_2 - b]
\end{aligned}} \tag{35}$$

and Hamilton's principle is

$$\delta S = \delta \int \int l dxdt = 0$$

This Lagrangian function for the 2-layer fluid system here consists of the kinetic energy, potential energy and rigid lid constraint, whereas the Lagrangian function in the derivation of Euler's equation consists of kinetic energy, potential energy and incompressibility constraint. The incompressibility constraint for the 2-layer system is derived separately via the common mass conservation of fluid within each layer.

3. Euler-Poincaré equations for the 2-layer case under rigid lid constraint, columnar ansatz

We can see that $Mdxdz$ is the volume element (mass density) in 2D-space (x,z). Take out the x-integral, integrate in the z-direction in each layer of element Mdz gives a function in the layer-thickness η_i, so we can replace M by layer thickness η_i in this case of two-layer fluid system. Thus, from 373 in the appendix chapter, we can deduce the Euler-Poincaré equation in this case of two-layer fluid system as:

$$\frac{\partial}{\partial t}\left(\frac{m_i}{\eta_i}\right) + u_i\left(\frac{m_i}{\eta_i}\right)_x + \frac{m_i}{\eta_i}u_{ix} = \left(\frac{\delta l}{\delta \eta_i}\right)_x \tag{36}$$

where $m_i = \dfrac{\delta l}{\delta u_i}$ is the momentum of the $i-th$ layer. η_i is the thickness of $i-th$ layer.

We thus have

$$\frac{\delta l}{\delta u_1} = \frac{\rho_1}{2}\left(\eta_1 2u_1 + \frac{\eta_1^3}{3}(-2u_{1xx})\right) = \rho_1\left[\eta_1 u_1 - \frac{\eta_1^3}{3}u_{1xx}\right] \tag{37}$$

in which $\dfrac{\delta\int(u_{1x})^2dx}{\delta u_1} = -2u_{1xx}$, and

$$\frac{\delta l}{\delta \eta_1} = \frac{\rho_1}{2}\left[u_1^2 + (\eta_1 u_{1x})^2 + 2g\eta_1\right] - \phi \tag{38}$$

Divide both sides by ρ_1, Euler-Poincaré equation for the first layer $i = 1$ becomes

$$\begin{aligned}&\frac{\partial}{\partial t}\left(u_1 - \frac{\eta_1^2}{3}u_{1xx}\right) + u_1\left[u_{1x} - \left(\frac{\eta_1^2}{3}u_{1xx}\right)_x\right] + u_{1x}\left(u_1 - \frac{\eta_1^2}{3}u_{1xx}\right)\\ &= \frac{1}{2}\left[2u_1u_{1x} + 2(\eta_1u_{1x})(\eta_1u_{1x})_x + 2g\eta_{1x}\right] - \frac{\phi_x}{\rho_1}\end{aligned} \tag{39}$$

Rearranging all terms, we get

$$\boxed{\frac{\partial}{\partial t}u_1+u_1u_{1x}-g\eta_{1x} = \frac{-\phi_x}{\rho_1}+\left[(\eta_1 u_{1x})(\eta_1 u_{1x})_x + \left(\frac{\eta_1^2}{3}u_{1xx}\right)_t + \left(u_1\frac{\eta_1^2}{3}u_{1xx}\right)_x\right]} \tag{40}$$

Similarly, for the second layer $i = 2$, we have

$$\frac{\delta l}{\delta u_2} = \frac{\rho_2}{2}\left(\eta_2 2u_2 + \frac{\eta_2^3}{3}(-2u_{2xx})\right) = \rho_2\left[\eta_2 u_2 - \frac{\eta_2^3}{3}u_{2xx}\right] \tag{41}$$

where we use $\dfrac{\delta\int(u_{2x})^2dx}{\delta u_2} = -2u_{2xx}$
and

$$\frac{\delta l}{\delta \eta_2} = \frac{\rho_2}{2}\left[u_2^2 + (\eta_2 u_{2x})^2 - 2g\eta_2 - 2gh_3\right] - \phi \tag{42}$$

Thus, the Euler-Poincaré equation for the second layer gives

$$\boxed{\frac{\partial}{\partial t}u_2+u_2u_{2x}+g(\eta_2+h_3)_x = \frac{-\phi_x}{\rho_2}+\left[(\eta_2 u_{2x})(\eta_2 u_{2x})_x + \left(\frac{\eta_2^2}{3}u_{2xx}\right)_t + \left(u_2\frac{\eta_2^2}{3}u_{2xx}\right)_x\right]} \tag{43}$$

In the case of the even bottom, we have h_3 is independent of the horizontal direction x, so $h_{3x} = 0$, thus the Euler-Poincaré equation for the second layer is

$$\boxed{\frac{\partial}{\partial t}u_2+u_2u_{2x}+g\eta_{2x} = \frac{-\phi_x}{\rho_2}+\left[(\eta_2 u_{2x})(\eta_2 u_{2x})_x + \left(\frac{\eta_2^2}{3}u_{2xx}\right)_t + \left(u_2\frac{\eta_2^2}{3}u_{2xx}\right)_x\right]} \tag{44}$$

Recall the notation as in the picture. $\gamma_1$and γ_2 are the undisturbed thickness of the upper and lower layer, respectively, which are constants. Make a change of variable from $(u_1, u_2, \eta_1, \eta_2)$ to (u_1, u_2, ζ). The layer thicknesses η_1, η_2 are related to the displacement at the interface ζ as:

$$\eta_1 = h_1 - h_2 = \gamma_1 - \zeta \tag{45}$$

$$\eta_2 = h_2 - h_3 = \gamma_2 + \zeta \tag{46}$$

Since $\gamma_1(\gamma_2)$ are the undisturbed thicknesses, which are constants, so

$$\eta_{1x} = -\zeta_x \tag{47}$$

$$\eta_{2x} = \zeta_x \tag{48}$$

Thus, the above Euler-Poincaré equation gives the evolution equation of the fluid in each layer, under the rigid lid constraint and gravitational force, as follows:

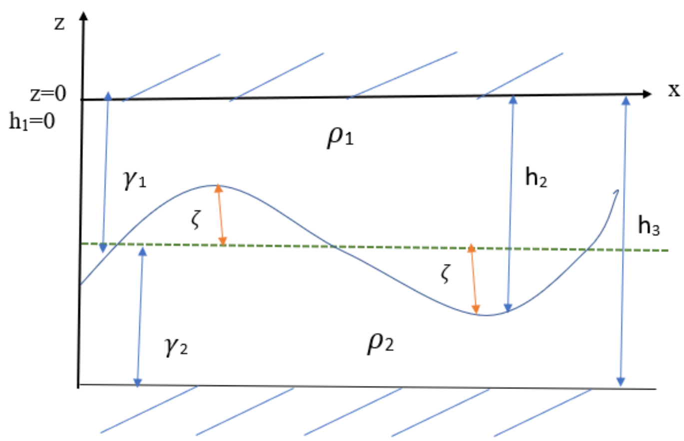

FIGURE 2. Two-layer fluid setup with displacement vector of the interface ζ

(49)

$$\frac{\partial}{\partial t}u_1 + u_1u_{1x} + g\zeta_x = \frac{-\phi_x}{\rho_1} + \left[(\eta_1 u_{1x})(\eta_1 u_{1x})_x + \left(\frac{\eta_1^2}{3}u_{1xx}\right)_t + \left(u_1\frac{\eta_1^2}{3}u_{1xx}\right)_x\right]$$

$$\frac{\partial}{\partial t}u_2 + u_1u_{2x} + g\zeta_x = \frac{-\phi_x}{\rho_2} + \left[(\eta_2 u_{2x})(\eta_2 u_{2x})_x + \left(\frac{\eta_2^2}{3}u_{2xx}\right)_t + \left(u_2\frac{\eta_2^2}{3}u_{2xx}\right)_x\right]$$

where the terms in brackets [ ] are dispersion terms.

Moreover, we have the transport equation for the layer thicknesses $\eta_1 = \gamma_1 - \zeta$ and $\eta_2 = \gamma_2 + \zeta$, based on the conservation of fluid within each layer in 2D (x,z).

$$\frac{\partial \eta_i}{\partial t} + \nabla.(\eta_i u_i) = \frac{\partial \eta_i}{\partial t} + (\eta_i u_i)_x = 0 \tag{50}$$

Thus, we obtain a closed system of equations for u_1, u_2, ζ.

CHAPTER 3

Camassa-Choi formulation

Below is the detailed computation to derive Camassa-Choi equation in paper [**2**].
$i = 1 (i = 2)$ stands for the upper (lower) fluid (see figure 1) so $\rho_1 < \rho_2$ is assumed. Denote $h_1(h_2)$ be the initial undisturbed thickness of the upper (lower) layer, respectively. ζ is the displacement at the interface between the 2 fluids.
Let (u_i, w_i) denote the horizontal and vertical components of the fluid velocity in the ith-layer. In the general case, u_i, w_i are functions of the spatial variables (x, z) and time t.

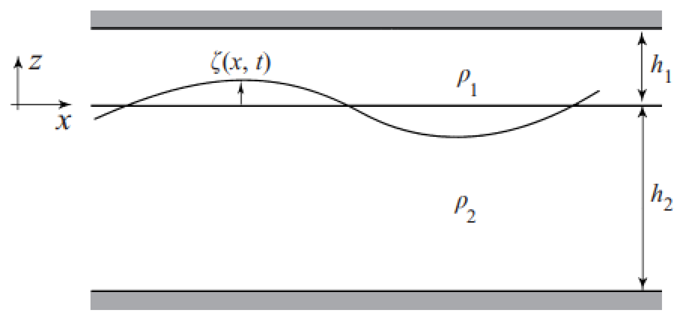

FIGURE 1. Two-layer fluid setup with displacement vector of the interface ζ

For an inviscid and incompressible fluid of density ρ_i, the velocity components in Cartesian coordinates (u_i, w_i) and the pressure satisfy the continuity equation and the Euler equations

$$
\begin{aligned}
&u_{ix} + w_{iz} = 0\\
&u_{it} + u_i u_{ix} + w_i u_{iz} = -p_{ix}/\rho_i\\
&w_{it} + u_i w_{ix} + w_i w_{iz} = -p_{iz}/\rho_i - g
\end{aligned}
\tag{51}
$$

The boundary conditions at the interface are the continuity of normal velocity and pressure:

$$\zeta_t + u_1\zeta_x = w_1,\ \ \zeta_t + u_2\zeta_x = w_2,\ \ p_1 = p_2 \quad \text{at } z = \zeta(x, t) \tag{52}$$

At the upper and lower rigid surfaces, the kinematic boundary conditions are

$$w_1(x, h_1, t) = 0,\ w_2(x, -h_2, t) = 0 \tag{53}$$

where $h_1(h_2)$ is the undisturbed thickness of the upper(lower) fluid layer.
Based on the scaling relation,

$$\begin{aligned} w_i/u_i &= O(h_i/L) = O(\epsilon) \ll 1 \\ u_i/U_0 &= O(\zeta/h_i) = O(1) \end{aligned} \tag{54}$$

(where L is a typical wavelength, U_0 is a characteristic speed), we then non-dimensionalize all physical variables as

$$\begin{gathered} x = Lx^\star, \qquad z = h_1 z^\star, \qquad t = (L/U_0)t^\star \\ \zeta = h_1\zeta^\star, \qquad p_i = (\rho_1 U_0^2)p_i^\star, \qquad u_i = U_0 u_i^\star, \qquad w_i = \epsilon U_0 w_i^\star \end{gathered} \tag{55}$$

By integrating (51) for i=1 across the upper layer ($\zeta \leq z \leq 1$) and imposing boundary conditions (52), we obtain the layer-mean equations for the upper fluid

$$\begin{gathered} \eta_{1t} + (\eta_1\overline{u_1})_x = 0,\ \eta_1 = 1 - \zeta \\ (\eta_1\overline{u_1})_t + (\eta_1\overline{u_1u_1})_x = -\eta_1\overline{p_{1x}} \end{gathered} \tag{56}$$

where the layer-mean quantity $\overline{f}$ of any function is defined as

$$\overline{f}(x,t) = \frac{1}{\eta_1}\int_\zeta^1 f(x,z,t)dz$$

and drop the asterisks for dimensionless variables.
The quantities $\overline{u_1u_1}$ and $\overline{p_{1x}}$ prevent closure of the system of layer-mean equations (51). The following analysis will therefore focus on expressing these quantities in terms of the two unknowns ζ and $\overline{u}_1$.
From the scalings, then the dimensionless form of the vertical momentum equation (51) for upper layer is written as

$$p_{1z} = -1 - \epsilon^2[w_{1t} + u_1w_{1x} + w_1w_{1z}] \tag{57}$$

Now, we seek an asymptotic expansion of $f = (u_1, w_1, p_1)$ in powers of ϵ^2.

$$f(x,z,t) = f^{(0)} + \epsilon^2 f^{(1)} + O(\epsilon^4) \tag{58}$$

From

$$p_{1z} = -1 - \epsilon^2[w_{1t} + u_1w_{1x} + w_1w_{1z}]$$

we then have $p_{1x}^{(0)} = -1$, so $p_1(x,z,t)^{(0)} = -z + c(x,t)$.
Pressure continuity across the interface means

$$p_1(x,\zeta,t)^{(0)} = -\zeta + c(x,t) = P(x,t) \tag{59}$$

where $P(x,t) = p_2(x,\zeta,t)$ is the pressure at the interface.
Thus, $c(x,t) = P(x,t) + \zeta$, or the leading-order pressure $p_1^{(0)}$ is

$$p_1^{(0)} = -(z-\zeta) + P(x,t) \tag{60}$$

Substitute (58),(60) in (51), one obtains

$$\boxed{u_1^{(0)} = u_1^{(0)}(x,t) \ \text{ if } u_{1z}^{(0)} = 0 \text{ at } t = 0} \tag{61}$$

meaning that the leading-order of u_1 does not depend on z if initially it does not depend on z. This condition is automatically satisfied if we assume the flow is initially irrotational.
From (51), then $w_{1z}^{(0)} = -u_{1x}^{(0)}$, or $w_1^{(0)} = -u_{1x}^{(0)}z + c(x,t)$.
Combining with the boundary condition at the interface, we then have

$$w_1^{(0)}(x,\zeta,t) = \zeta_t + u_1(x,\zeta,t)\zeta_x = -u_{1x}^{(0)}\zeta + c(x,t) \tag{62}$$

Thus, $c(x,t) = u_{1x}\zeta + (\partial_t + u_1^{(0)}\partial_x)\zeta$, or

$$w_1^{(0)} = -(u_{1x}^{(0)})(z-\zeta) + D_1\zeta \tag{63}$$

where D_1 stands for the material derivative $D_1 = \partial_t + u_1^{(0)}\partial_x$.
Since $u_1(x,z,t) = u_1^{(0)} + O(\epsilon^2)$ and $u_1^{(0)}$ does not depend on z, therefore we compute the layer-average horizontal velocity of the upper layer $\overline{u_1}$ as follows

(64)

$$\overline{u_1} = \frac{1}{\eta_1}\int_\zeta^1 u_1^{(0)}(x,t)dz + O(\epsilon^2) = \frac{1}{\eta_1}u_1^{(0)}(1-\zeta) + O(\epsilon^2) = u_1^{(0)} + O(\epsilon^2) \ \text{ since } \eta_1 = 1-\zeta$$

We have

(65)

$$\overline{u_1u_1} = \frac{1}{\eta_1}\int_\zeta^1 u_1^{(0)}(x,t)u_1^{(0)}(x,t)dz + O(\epsilon^4) = u_1^{(0)}(x,z,t)u_1^{(0)}(x,z,t) + O(\epsilon^4) = \bar{u}_1\bar{u}_1$$

Thus, we have

$$\eta_1\overline{u_1u_1} = \eta_1\bar{u}_1\bar{u}_1 + O(\epsilon^4) \tag{66}$$

Thus, the dimensionless form of the layer-mean horizontal momentum equation (56) becomes

$$\overline{u_1}_t + \bar{u}_1\overline{u_1}_x = -\overline{p_{1x}} + O(\epsilon^4) \tag{67}$$

Remark 3.1. Notice that in general $\overline{u_1}_x \neq \overline{u_{1x}}$. Here, we consider the system of layer-averaged variables $\bar{u}_1, \bar{u}_2$ only. △

From (57) (58), at order $O(\epsilon^2)$, we have

$$p^{(1)}_{1z} = -[w^{(0)}_{1t} + u^{(0)}_1 w^{(0)}_{1x} + w^{(0)}_1 w^{(0)}_{1z}] \tag{68}$$

Now, we compute $p^{(1)}_{1z}$. Starting from (63), and $\overline{u_1} = u^{(0)}_1 + O(\epsilon^2)$, we have:

$$\begin{aligned}
-w_{1t} &= \overline{u_1}_{xt}(z-\zeta) + \overline{u_1}_x(-\zeta_t) - (D_1\zeta)_t \\
-w_{1x} &= \overline{u_1}_{xx}(z-\zeta) + \overline{u_1}_x(-\zeta_x) - (D_1\zeta)_x \\
-u_1 w_{1x} &= \overline{u_1}.\overline{u_1}_{xx}(z-\zeta) - \bar{u}_1\overline{u_1}_x\zeta_x - \overline{u_1}(D_1\zeta)_x \\
-w_{1z} &= \overline{u_1}_x \text{ (since } (D_1\zeta)_z = 0 \text{ since } \zeta \text{ does not depend on z)} \\
-w_1 w_{1z} &= -\overline{u_1}^2_x(z-\zeta) + (D_1\zeta)\overline{u_1}_x
\end{aligned} \tag{69}$$

Thus, we have

$$p^{(1)}_{1z} = (z-\zeta)(\overline{u_1}_{xt}+\bar{u}_1\overline{u_1}_{xx}-\overline{u_1}^2_x)-\overline{u_1}_x\zeta_t-(D_1\zeta)_t-\bar{u}_1\overline{u_1}_x\zeta_x-\overline{u_1}(D_1\zeta)_x+\overline{u_1}_x(D_1\zeta) \tag{70}$$

Denote

$$G_1(x,t) = \overline{u_1}_{xt} + \bar{u}_1\overline{u_1}_{xx} - \overline{u_1}^2_x \tag{71}$$

Thus,

$$p^{(1)}_{1z} = G_1(z-\zeta) - \overline{u_1}_x\zeta_t - (D_1\zeta)_t - \overline{u_1 u_1}_x\zeta_x - \overline{u_1}(D_1\zeta)_x + \overline{u_1}_x(D_1\zeta) \tag{72}$$

Moreover, by definition, we have $D_1\zeta = -D_1\eta_1$ and

$$D_1\eta_1 = \eta_{1t} + u^{(0)}_1\eta_{1x} = \eta_{1t} + \overline{u_1}\eta_{1x} + O(\epsilon^2) = -\eta_1\overline{u_1}_x + O(\epsilon^2) \tag{73}$$

where we use $\overline{u_1} = u^{(0)}_1 + O(\epsilon^2)$ and mass-conservation of fluid within each layer $\eta_{1t} + (\eta_1\overline{u_1})_x = 0$.
We thus have

$$\begin{aligned}
D_1\zeta &= -D_1\eta_1 = \eta_1\overline{u_1}_x + O(\epsilon^2) \\
D^2_1\zeta &= D_1(D_1\zeta) = D_1(\eta_1\overline{u_1}_x + O(\epsilon^2)) \\
&= \partial_t(\eta_1\overline{u_1}_x) + \overline{u_1}\partial_x(\eta_1\overline{u_1}_x) + O(\epsilon^4) \\
&= \eta_{1t}\overline{u_1}_x + \eta_1\overline{u_1}_{xt} + \overline{u_1}\eta_{1x}\overline{u_1}_x + \overline{u_1}\eta_1\overline{u_1}_{xx} \\
&= \overline{u_1}_x(\eta_{1t} + \overline{u_1}\eta_{1x}) + \eta_1(\overline{u_1}_{xt} + \overline{u_1}_{xx}\overline{u_1}) \\
&= \overline{u_1}_x D_1\eta_1 + \eta_1 D_1\overline{u_1}_x
\end{aligned} \tag{74}$$

Thus, replace $D_1\zeta$ by $\eta_1\overline{u_1}_x + O(\epsilon^2)$, we then have

(75)

$$\begin{aligned}
p^{(1)}_{1z} &= G_1(z-\zeta) - \overline{u_1}_x\zeta_t - \eta_1\overline{u_1}_{xt} - \eta_{1t}\overline{u_1}_x - \bar{u}_1\overline{u_1}_x\zeta_x - \eta_{1x}\bar{u}_1\overline{u_1}_x - \eta_1\bar{u}_1\overline{u_1}_{xx} + \eta_1\overline{u_1}^2_x \\
&= G_1(z-\zeta) - \eta_1(\overline{u_1}_{xt} + \bar{u}_1\overline{u_1}_{xx} - \overline{u_1}^2_x) - \overline{u_1}_x\zeta_t - \eta_{1t}\overline{u_1}_x - \bar{u}_1\overline{u_1}_x\zeta_x - \bar{u}_1\overline{u_1}_x\eta_{1x} \\
&= G_1(z-\zeta) - \eta_1 G_1
\end{aligned}$$

(since $-\zeta_t = \eta_{1t}, \ -\zeta_x = \eta_{1x}$)

Also, from $D_1(D_1\zeta) = \overline{u_1}_x D_1\eta_1 + \eta_1 D_1\overline{u_1}_x$, we have

$$
\begin{aligned}
D_1(D_1\zeta) &= \overline{u_1}_x D_1\eta_1 + \eta_1 D_1\overline{u_1}_x \\
&= \overline{u_1}_x(\eta_{1t} + \overline{u_1}\eta_{1x}) + \eta_1(\overline{u_1}_{xt} + \overline{u}_1\overline{u}_{1xx}) \\
&= \overline{u_1}_x[(\eta_{1t} + (\overline{u}_1\eta_1)_x) - \overline{u_1}_x\eta_1] + \eta_1(\overline{u_1}_{xt} + \overline{u}_1\overline{u_1}_{xx}) \\
&= \overline{u_1}_x(-\overline{u_1}_x\eta_1) + \eta_1(\overline{u_1}_{xt} + \overline{u}_1\overline{u_1}_{xx}) \\
&\quad \text{since } (\eta_{1t} + (\overline{u}_1\eta_1)_x) = 0 \\
&= \eta_1 G_1
\end{aligned}
\tag{76}
$$

Integrating $G_1(x,t)$ from ζ to z, we obtain the second-order term for pressure of the upper layer at a specific height z, $p_1^{(1)}(x,z,t)$ as

$$
\begin{aligned}
p_1^{(1)} &= G_1(x,t)\int_\zeta^z (z-\zeta)dz - \eta_1(x,t)G_1(x,t)\int_\zeta^z dz \\
&= \frac{1}{2}G_1(x,t)(z-\zeta)^2 - \eta_1 G_1(x,t)(z-\zeta)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
p_1(x,z,t) &= p_1^{(0)}(x,z,t) + \epsilon^2 p_1^{(1)}(x,z,t) + O(\epsilon^4) \\
&= -(z-\zeta) + P(x,t) + \frac{1}{2}\epsilon^2 G_1(x,t)(z-\zeta)^2 - \epsilon^2\eta_1 G_1(x,t)(z-\zeta) + O(\epsilon^4)
\end{aligned}
\tag{77}
$$

Therefore,

$$
\begin{aligned}
\overline{p_{1x}(x,z,t)} &= \overline{\zeta_x + P_x(x,t) + \frac{1}{2}\epsilon^2 G_{1x}(z-\zeta)^2 + \frac{1}{2}\epsilon^2 G_1 2(z-\zeta)(-\zeta_x) - \epsilon^2[\eta_1 G_1]_x(z-\zeta) - \epsilon^2\eta_1 G_1(-\zeta_x) + O(\epsilon^4)} \\
&= \overline{\zeta_x + P_x + \frac{1}{2}\epsilon^2 G_{1x}(z-\zeta)^2 - \epsilon^2 G_1(z-\zeta)\zeta_x - \epsilon^2[\eta_1 G_1]_x(z-\zeta) + \epsilon^2\eta_1 G_1\zeta_x} + O(\epsilon^4) \\
&= \zeta_x + P_x + \overline{\frac{1}{2}\epsilon^2 G_{1x}(z-\zeta)^2 - \epsilon^2 G_1(z-\zeta)\zeta_x - \epsilon^2[\eta_1 G_1]_x(z-\zeta) + \epsilon^2\eta_1 G_1\zeta_x} + O(\epsilon^4)
\end{aligned}
\tag{78}
$$

Note that $\eta_1 = 1-\zeta$, we thus obtain

$$
\begin{aligned}
&\bullet\ \overline{\frac{1}{2}\epsilon^2 G_{1x}(z-\zeta)^2} = \frac{1}{2}\epsilon^2 G_{1x}\frac{1}{\eta_1}\int_\zeta^1 (z-\zeta)^2 dz = \frac{1}{2}\frac{\epsilon^2}{\eta_1}G_{1x}\frac{\eta_1^3}{3} \\
&\bullet\ \overline{-\epsilon^2 G_1(z-\zeta)\zeta_x} = -\frac{1}{2}\frac{\epsilon^2}{\eta_1}G_1\zeta_x\eta_1^2 \\
&\bullet\ \overline{-\epsilon^2[\eta_1 G_1]_x(z-\zeta)} = -\frac{1}{2}\frac{\epsilon^2}{\eta_1}[\eta_1 G_1]_x\eta_1^2 \\
&\bullet\ \overline{\epsilon^2\eta_1 G_1\zeta_x} = \epsilon^2\eta_1 G_1\zeta_x
\end{aligned}
\tag{79}
$$

Sum up the above 4 terms, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\epsilon^2}{\eta_1}G_{1x}\frac{\eta_1^3}{3} - \frac{1}{2}\frac{\epsilon^2}{\eta_1}G_1\zeta_x\eta_1^2 - \frac{1}{2}\frac{\epsilon^2}{\eta_1}[\eta_1 G_1]_x\eta_1^2 + \epsilon^2\eta_1 G_1\zeta_x \\
&= \frac{-\epsilon^2}{\eta_1}\left[-\frac{1}{2}G_{1x}\frac{\eta_1^3}{3} + \frac{1}{2}G_1\zeta_x\eta_1^2 + (\eta_1 G_1)_x\frac{\eta_1^2}{2} - \eta_1^2 G_1\zeta_x\right] \\
&= \frac{-\epsilon^2}{\eta_1}\left[-\frac{1}{2}G_{1x}\frac{\eta_1^3}{3} - \frac{1}{2}G_1\eta_{1x}\eta_1^2 + \eta_{1x}G_1\frac{\eta_1^2}{2} + \eta_1 G_{1x}\frac{\eta_1^2}{2} + \eta_1^2 G_1\eta_{1x}\right] \qquad (80)\\
&= \frac{-\epsilon^2}{\eta_1}\left[\frac{\eta_1^3}{3}G_{1x} + G_1\eta_1^2\eta_{1x}\right] \\
&= \frac{-\epsilon^2}{\eta_1}\left(\frac{1}{3}\eta_1^3 G_1\right)_x
\end{aligned}
$$

(where we replace $\zeta_x = -\eta_{1x}$)
Thus, from formula for $\overline{p_{1x}}$, we obtain

$$\overline{p_{1x}} = \zeta_x + P_x - \frac{\epsilon^2}{\eta_1}\left(\frac{1}{3}\eta_1^3 G_1\right)_x + O(\epsilon^4) \tag{81}$$

Thus, the layer-mean horizontal momentum equation $\overline{u_1}_t + \overline{u_1}.\overline{u_1}_x = -\overline{p_{1x}} + O(\epsilon^4)$ becomes

$$\overline{u_1}_t + \overline{u}_1\overline{u_1}_x + g\zeta_x = -\frac{P_x}{\rho_1} + \frac{1}{\eta_1}(\frac{1}{3}\eta_1^3 G_1)_x + O(\epsilon^4) \tag{82}$$

Similarly, we have the horizontal momentum equation for the lower layer

$$\overline{u_2}_t + \overline{u}_2\overline{u_2}_x + g\zeta_x = -\frac{P_x}{\rho_2} + \frac{1}{\eta_2}(\frac{1}{3}\eta_2^3 G_2)_x + O(\epsilon^4) \tag{83}$$

where $G_2(x,t) = \overline{u_2}_{xt} + \overline{u}_2\overline{u_2}_{xx} - \overline{u_2}_x^2 = -\dfrac{D_2^2\zeta}{\eta_2}$

CHAPTER 4

Wu's model

The model explained below is based on Wu's approach [**5**], with some slight differences in notations from the original work of Wu, in order to be consistent with the rest of this thesis. In particular,

Remark 1: Concerning the density numbering.

In this chapter, we denote $\rho_1 < \rho_2$, so the heavier fluid will have density ρ_2. The interface displaced from its rest position at $z = 0$ to elevation $z = \zeta(x, t)$ as a function of horizontal position x and time t. Thickness of the upper layer is $h_1 - \zeta$, and the lower layer's thickness is $\zeta + h_2$, in which $h_1(h_2)$ is the unperturbed layer-thickness.

Remark 2: Concerning the η notation.

In Wu's approach, he denoted η_2 the 2nd-layer thickness, whereas η_1 equals minus the 1st-layer thickness for technical reason concerning the Taylor expansion. To avoid confusion, the $\eta' s$ notations in Wu's original approach will be changed to $\beta' s$ in this thesis, and the $\eta' s$ will denote the layer-thickness as in other models' approach.

The effects of nonlinearity and dispersion can be closely estimated by two key parameters, namely

$$\alpha = a/h; \quad \epsilon = h/\lambda \tag{84}$$

for characterizing waves of amplitude a and typical wavelength λ in water of rest depth h.

We adopt the Euler's equation for describing two-dimensional inviscid wave motion in two layers of stratified fluid under the action of gravity and surface tension. The two fluid layers have rest thickness h_j and constant density ρ_j, with $j = 2$ for the lower and $j = 1$ for the upper layer, and $\rho_2 > \rho_1$. The two fluid layers are bounded below by a rigid horizontal bottom at $z = -h_2$ and on top by another horizontal rigid lid of infinite extent at $z = h_1$, so the system has only the interface free to move. The fluids move with velocity (u_j, w_j) in each layer, with the interface displaced from its rest position at $z = 0$ to elevation $z = \zeta(x, t)$ as a function of horizontal position x and time t. We have the mass conservation/ continuity equation (assuming incompressible fluids) and Euler's momentum conservation for the

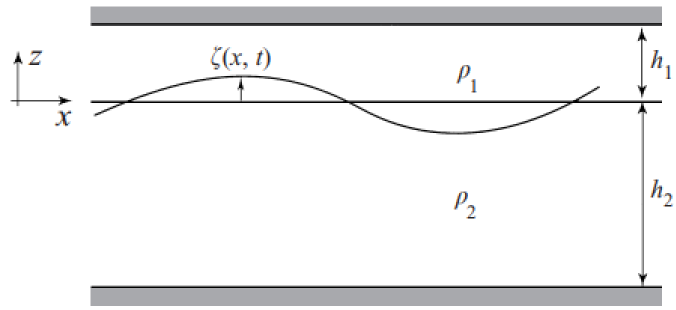

FIGURE 1. Two-layer fluid setup with displacement vector of the interface ζ

inviscid fluids as:

$$u_{jx} + w_{jz} = 0 \tag{85}$$
$$\frac{d(u_j, w_j)}{dt} = \frac{\partial}{\partial t}(u_j, w_j) + (u_j, w_j).\nabla(u_j, w_j) = -\frac{1}{\rho_j}(p_{jx}, p_{jz}) - (0, g)$$

The momentum equation is thus written in horizontal and vertical direction as:

$$\begin{aligned} \frac{du_j}{dt} &= \frac{\partial}{\partial t}u_j + u_j.u_{jx} + w_j.u_{jz} = -\frac{1}{\rho_j}p_{jx} \\ \frac{dw_j}{dt} &= \frac{\partial}{\partial t}w_j + u_j.w_{jx} + w_j.w_{jz} = -\frac{1}{\rho_j}p_{jz} - g \end{aligned} \tag{86}$$

where $p_j(x, z)$ is the pressure and g is the gravitational acceleration.
Boundary conditions (at the bottom and upper rigid lid) are:

$$\begin{aligned} &\tilde{w}_j = \tilde{D}_j\zeta \text{ on } z = \zeta(x, t), \text{ where } \tilde{D}_j = \partial_t + \tilde{u}_j.\partial_x \\ &p_1 = p_2 \text{ on } z = \zeta(x, t) \\ &w_1 = 0 \text{ on } z = h_1 \\ &w_2 = 0 \text{ on } z = -h_2 \end{aligned} \tag{87}$$

where $\tilde{u}_j(x, t) = u_j(x, \zeta, t)$ is the value of u_j at the interface, likewise for w_j. To include surface tension at the interface, we use $p_1 = p_2 + \rho_1\gamma.n_x$, where $\rho_1\gamma$ is the uniform surface tension of the interface and n is the upward unit vector normal to the interface. Here, we just use the continuity of pressure at the interface (87).
We adopt the interfacial variables $\tilde{u}_j, \tilde{w}_j$ and ζ as the unknowns in formulating the theory for the unsteady nonlinear interfacial waves in two-layered fluids.
Thus, to proceed, we project momentum equations under boundary conditions (87) onto the free interface to obtain the projected equations in terms of the unknowns. For an arbitrary flow variable $f_j(x, z, t)$, it approaches, as $z \to \zeta(x, t)$ from within

the j-side fluid domain, the interfacial value $\tilde{f}_j(x,t)$: $\tilde{f}_j(x,\zeta(x,t),t) = \tilde{f}_j(x,t)$.
By the chain rule, their time derivatives are related by

$$\partial_t \tilde{f}_j(x,t) = [\partial_t f(x,z,t) + \zeta_t \partial_z f(x,z,t)]_{z=\zeta(x,t)} \tag{88}$$

and similarly for derivative with respect to x.
The Lagrangian time derivative and Eulerian time derivative for a variable f_j carried along the flow with velocity $(\tilde{u}_j, \tilde{w}_j)$, evaluated at $z = \zeta(x,t)$ are related by

$$\frac{df_j}{dt}\Big|_{z=\zeta(x,t)} = \partial_t f_j|_{z=\zeta} + \tilde{u}_j.f_{jx}|_{z=\zeta} + \tilde{w}_j f_{jz}|_{z=\zeta} \tag{89}$$

$$= \partial_t f_j\Big|_{z=\zeta} + \tilde{u}_j.f_{jx}|_{z=\zeta} + (\zeta_t + \tilde{u}_j\zeta_x) f_{jz}|_{z=\zeta} \tag{90}$$

Moreover, we have that

$$\begin{aligned} \tilde{D}_j\tilde{f}_j &:= (\partial_t + \tilde{u}_j.\partial_x)\tilde{f}_j \\ &= \partial_t\tilde{f}_j + \tilde{u}_j\tilde{f}_{jx} \\ &= \partial_t f_j|_{z=\zeta} + \zeta_t\partial_z f_j|_{z=\zeta} + \tilde{u}_j.[f_{jx} + \zeta_x.f_{jz}]_{z=\zeta} \\ &= \partial_t f|_{z=\zeta} + \tilde{u}_j f_x|_{z=\zeta} + (\zeta_t + \tilde{u}_j.\zeta_x).f_z]_{z=\zeta} \end{aligned} \tag{91}$$

Thus, we have

$$\frac{df_j}{dt}\Big|_{z=\zeta(x,t)} = \tilde{D}_j\tilde{f}_j \tag{92}$$

Therefore, the horizontal momentum equation, projected on $z = \zeta(x,t)$ becomes

$$\frac{du_j}{dt}\Big|_{z=\zeta} = \tilde{D}_j\tilde{u}_j = -\frac{1}{\rho_j}\tilde{p}_{j\,x} \tag{93}$$

where

$$-\frac{1}{\rho_j}\tilde{p}_{j\,x} = -\frac{1}{\rho_j}[\partial_x p_j + \zeta_x\partial_z p_j]_{z=\zeta} \tag{94}$$

in which, from the projected vertical momentum equation, we have

$$-\frac{1}{\rho_j}\partial_z p_j\Big|_{z=\zeta} = \frac{dw_j}{dt}\Big|_{z=\zeta} + g = \tilde{D}_j^2\zeta + g \tag{95}$$

since

$$\frac{dw_j}{dt}\Big|_{z=\zeta} = \tilde{D}_j\tilde{w}_j = \tilde{D}_j(\tilde{D}_j\zeta) = \tilde{D}_j^2\zeta$$

Therefore, we arrive at

$$-\frac{1}{\rho_j}\tilde{p_{jx}} = -\frac{1}{\rho_j}p_{jx} + \zeta_x.(\tilde{D}_j^2 + g) \tag{96}$$

Thus, the horizontal momentum equation (86), projected on $z = \zeta(x,t)$ becomes

$$\tilde{D}_j\tilde{u}_j + [g + \tilde{D}_j^2\zeta]\zeta_x = -\frac{1}{\rho_j}\tilde{p_{jx}} \tag{97}$$

The projected horizontal momentum equation for each layer becomes:

$$\tilde{D}_1\tilde{u}_1 + [g + \tilde{D}_1^2\zeta]\zeta_x = -(1/\rho_1)\tilde{p}_{1x} \tag{98}$$
$$\tilde{D}_2\tilde{u}_2 + [g + \tilde{D}_1^2\zeta]\zeta_x = -(1/\rho_2)\tilde{p}_{2x}$$

Combining with the dynamic boundary condition ($p_1 = p_2$ on $(z = \zeta(x,t))$, we get

$$\tilde{D}_1\tilde{u}_1 - \mu\tilde{D}_2\tilde{u}_2 + [g_e + \tilde{D}_1^2\zeta - \mu\tilde{D}_2^2\zeta]\zeta_x = 0 \tag{99}$$

where

$$\mu = \rho_2/\rho_1; \quad g_e = (1-\mu)g$$

1. (A) The (u_{oj}, ζ) system- the boundary velocity basis

From now on, we assume the flow to be continuously differentiable, and irrotational in the flow domain of the two fluids, except for the interface, across which any discontinuities of the tangential velocity will cause a vortex sheet.
Since the flow is irrotational in both fluids, the velocity fields are represented by a scalar potentials $\phi_j(x,z,t)$, s.t

$$u_j = \phi_{jx}; \quad w_j = \phi_{jz} \quad (-h_2 \le z \le \zeta) \text{ for } j = 2; \quad (\zeta \le z \le h_1) \text{ for } j = 1; \tag{100}$$

The incompressibility of the two fluids gives $u_{jx} + w_{jz} = 0$, and from definition of ϕ_j, we thus have ϕ_j satisfies the Laplacian equation

$$\phi_{jxx} + \phi_{jzz} = \Delta\phi_j = 0 \tag{101}$$

To pursue analysis for small ϵ, we let the vertical lengths scaled by h_1, and horizontal lengths scaled by a typical wavelength λ.

$$z = z^\star h_1; \quad x = x^\star \lambda$$

Thus,

$$\frac{d\phi_j}{dx^\star} = \phi_{jx}.\frac{\partial x}{\partial x^\star} = \lambda\phi_{jx}$$
$$\phi_{j,x^\star x^\star} = \frac{d^2\phi_j}{d^2x^\star} = \lambda^2\phi_{jxx}$$
$$\phi_{j,z^\star z^\star} = \frac{d^2\phi_j}{d^2z^\star} = h_1^2\phi_{jzz}$$

Thus,

$$\phi_{jxx} = \frac{1}{\lambda^2}\phi_{j,x^\star x^\star} \tag{102}$$
$$\phi_{jzz} = \frac{1}{h_1^2}\phi_{j,z^\star z^\star}$$

So the Laplacian equation (101), written in the coordinates of scaled independent variables $(x^\star, z^\star)$ becomes

$$\frac{1}{\lambda^2}\phi_{jx^\star x^\star} + \frac{1}{h_1^2}\phi_{jz^\star z^\star} = 0$$

or equivalently

$$\epsilon^2\phi_{jx^\star x^\star} + \phi_{jz^\star z^\star} = 0 \tag{103}$$

where $\epsilon = h_1/\lambda$.
From now on, we use the coordinates of scaled variables $(x^\star, z^\star)$, and omit the $\star$.
Then, scale ϕ_j by $c\lambda$ where $c = \sqrt{g_o h_1}$ is a linear wave speed (g_o is a reference value of g_e), i.e $\phi_j = \phi_j^\star c\lambda$.
ϕ_j may assume a series expansion depending on ϵ as follows: take a standard approach for small deviations and represent ϕ as a power series in the z-direction

$$\begin{aligned} &\phi_j(x^\star, z^\star, t; \epsilon) = \Sigma_{n=0}^{\infty}[\epsilon H_j(z)]^n \phi_{jn}(x, t, \epsilon) \\ &H_1(z) = z - h_1 = z^\star - 1; \quad H_2(z) = h_2 + z = \frac{h_2}{h_1} + z^\star \end{aligned} \tag{104}$$

in which $[\epsilon H_j(z)]$ represents the scaled deviation from the boundaries (rigid lid or bottom), and ϕ_{j0} is the velocity potential at the boundaries. The components coefficients ϕ_{jn} are independent from vertical direction z.

REMARK 4.1. As explicitly shown later, the coefficients $\phi_{jn}(x, t; \epsilon)$ can be computed via the potential at the boundary $\phi_{j0}(x, t; \epsilon)$. In the end, ϕ_{j0}, which is the only unknown involved in ϕ_j, may depend on the parameter ϵ as a result of taking an appropriate regrouping of the complementary solutions of the higher-order equations for ϕ_{jn} such that $\nabla\phi_{j0}(x, t; \epsilon) = O(\alpha)$ as $\epsilon \to 0$. This regrouping is admissible provided the medium is uniform (h_j = const, $j = 1, 2$) and horizontally unbounded in the absence of any boundary effects of specific orders in magnitude. △

Then, from (104), we have

$$\begin{aligned} \phi_{jz}(x, z, t, \epsilon) &= \Sigma_{n=1}^{\infty}\epsilon^n n H_j^{n-1}(z)\phi_{jn}(x, t, \epsilon) \\ &= \epsilon^1 1 H_j^0(z)\phi_{j1}(x, t, \epsilon) + \epsilon^2 2 H_j^1 \phi_{j2}(x, t, \epsilon) + \text{higher order terms} \end{aligned} \tag{105}$$

At the bottom (rigid lid), $H_j^{n-1}(z) = 0$ for $n > 1$. Since there is no-flux condition at the boundary $w_j = \phi_{jz} = 0$ at $z = -h_2$ (or $z = h_1$), we arrive at

$$0 = w_j(z = -h_2/z = h_1) = (1/\epsilon)\phi_{jz}\Big|_{z=-h_2/z=h_1} = H_j^0\phi_{j1} \tag{106}$$

Thus $\phi_{j1} = 0$, i.e the 1st-z-derivative of ϕ_j evaluated at the boundary is 0.
We also have, from (105)

$$\begin{aligned}\phi_{j,zz}(x,z,t,\epsilon) &= \Sigma_{n=2}^{\infty}\epsilon^n n(n-1)H_j^{n-2}(z)\phi_{jn}(x,t,\epsilon)\\ &= \Sigma_{n=0}^{\infty}\epsilon^n(n+2)(n+1)H_j^n(z)\phi_{jn}(x,t,\epsilon)\end{aligned}\tag{107}$$

Insert this expansion of ϕ_j into the Laplace equation $\phi_{j,xx} + \phi_{j,zz} = \Delta\phi_j = 0$, we get

$$\Sigma_{n=0}^{\infty}M_j^n[\phi_{jn,xx} + (n+2)(n+1)\phi_{j,n+2}] = 0$$

where $M_j^n(z) = [\epsilon H_j(z)]^n$. Thus,

$$\phi_{jn,xx} + (n+2)(n+1)\phi_{j(n+2)} = 0$$

Since $\phi_{j1}(x,t) = 0$ (so $\phi_{j1,xx} = 0$), from the above recurrence relation among the coefficients, we have $\phi_{j3} = 0$, and similarly all the odd components of ϕ_j vanish. On the other hand, the even components $\phi_{j(n+2)}$ will be computed via the 2nd-x derivative of ϕ_{jn}.
For example,

$$\begin{aligned}\phi_{j2} &= \frac{-\phi_{j0,xx}}{2!}\\ \phi_{j4} &= \frac{-\phi_{j2,xx}}{(3.4)} = \frac{-1}{3.4}\left(\frac{-\phi_{j0,xx}}{2}\right)_{xx}\\ &= \frac{(-1)^2}{4!}\partial_x^{(2.2)}\phi_{j0}\end{aligned}\tag{108}$$

In general, $\phi_{j(2n+1)} = 0$ and

$$\phi_{j(2n)} = \frac{(-1)^n}{(2n)!}\partial_x^{(2n)}\phi_{j0}\tag{109}$$

In brief, we have

(110)

$$\begin{aligned}\phi_j(x,z,t;\epsilon) &= \Sigma_{n=0}^{\infty}[\epsilon H_j(z)]^n\phi_{jn}(x,t,\epsilon)\\ &= \Sigma_{n=0}^{\infty}[\epsilon H_j(z)]^{2n}\phi_{j2n}(x,t,\epsilon) \quad \text{(since the odd components vanish)}\\ &= \Sigma_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}[\epsilon H_j(z)]^{2n}\partial_x^{(2n)}\phi_{jo}(x,t,\epsilon)\end{aligned}$$

$$H_1(z) = z - h_1 = z^\star - 1; \qquad H_2(z) = h_2 + z = \frac{h_2}{h_1} + z^\star$$

From this, we deduce the horizontal and the vertical velocity components u_j, w_j are both scaled by $c = \sqrt{g_o h_1}$, from $u_j = \phi_{jx}; w_j = \epsilon^{-1}\phi_{jz}$.
From (110), we have

$$u_j(x,z,t,\epsilon) = \phi_{jx}(x,z,t,\epsilon) = \Sigma_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}[\epsilon H_j(z)]^{2n}\nabla^{2n}(\nabla\phi_{jo}(x,t,\epsilon))\tag{111}$$

in which

$$\nabla\phi_{j0} = \partial_x\phi_{j0} = u_{j0}$$

is the horizontal velocity at the boundary (at the lid or bottom) of the j-th fluid. Thus, the interfacial horizontal velocity $\tilde{u_j}$ at $z = \zeta(x, t)$ is

$$\tilde{u_j} = \phi_{jx}(x, \zeta, t, \epsilon) = A_j[u_{jo}(x, t)] = \Sigma_{n=0}^{\infty}\epsilon^{2n}A_{jn}u_{jo}(x, t) \tag{112}$$

$$\text{where } A_{jn} = \frac{(-1)^n}{(2n)!}\beta_j^{2n}\nabla^{2n} \tag{113}$$

where $\beta_j = H_j(\zeta)$, i.e

$$\beta_1 = \zeta - h_1 < 0; \;\; \beta_2 = \zeta + h_2 > 0 \tag{114}$$

so $\beta_1^2(\beta_2^2)$ is the layer-thickness squared.
Denote $\eta_1(\eta_2)$ the 1st and 2nd-layer thickness, respectively. Thus,

$$\eta_1 = -\beta_1; \qquad \eta_2 = \beta_2 \tag{115}$$

Combined with the Laplacian equation for ϕ_j, (103) , we can similarly derive the expansion for vertical velocity at $z = \zeta$,

$$\begin{aligned}
&\phi_{j,zz}(x, z, t) = -\epsilon^2\phi_{jxx} = -\epsilon^2\Sigma_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}[\epsilon H_j(z)]^{2n}\nabla^{2n+1}u_{jo}(x, t, \epsilon)\\
&w_j(x, z, t) = \epsilon^{-1}\phi_{jz} = -\epsilon\Sigma_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\epsilon^{2n}\int[H_j(z)]^{2n}dz\nabla^{2n+1}u_{jo}(x, t, \epsilon)\\
&w_j(x, z, t) = \Sigma_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!}\epsilon^{2n+1}\frac{H_j^{2n+1}}{2n+1}\nabla^{2n+1}u_{jo}(x, t, \epsilon)
\end{aligned} \tag{116}$$

Thus, at $z = \zeta$,

(117)

$$\tilde{w_j}(x, t) = w_j(x, \zeta, t) = \Sigma_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}\epsilon^{2n+1}\beta_j^{2n+1}\nabla^{2n+1}u_{jo}(x, t, \epsilon) = \Sigma_{n=0}^{\infty}\epsilon^{2n+1}B_{jn}u_{jo}(x, t)$$

where

$$B_{jn} = \frac{(-1)^{n+1}}{(2n+1)!}\beta_j^{2n+1}\nabla^{2n+1} = \frac{(-1)^{n+1}}{(2n+1)!}\beta_j^{2n+1}\partial_x^{2n+1} \tag{118}$$

For the (u_{jo}, ζ) system, the basic equations consist of kinematic interfacial boundary condition, the interfacial projected momentum and series relation

$$\begin{aligned}
&\zeta_t = (1/\epsilon)\tilde{w}_j - \tilde{u}_j.\zeta_x, (j = 1, 2)\\
&\tilde{D_1}\tilde{u}_1 + [g + \epsilon^2\tilde{D_1^2}\zeta]\zeta_x = -\frac{1}{\rho_1}\tilde{p}_{1,x}\\
&\tilde{D_2}\tilde{u}_2 + [g + \epsilon^2\tilde{D_2^2}\zeta]\zeta_x = -\frac{1}{\rho_2}\tilde{p}_{2,x}\\
&\tilde{u_j}(x, t) = A_j[u_{jo}(x, t)]; \quad \tilde{w}_j = B_j[u_{jo}(x, t)] \quad (j = 1, 2)
\end{aligned} \tag{119}$$

2. (B) The $(\tilde{u_j}, \zeta)$ system- the interface velocity basis

The above model (A)-(u_{jo}, ζ) system is quite complicated for applications due to the appearance of the series and the series products in the equations. To overcome such complication, $(\tilde{u_j}, \zeta)$ are chosen as the basic variables.
To do this, we obtain the inverse of equation. (112) as

$$u_{jo}(x,t) = J_j[\tilde{u_j}(x,t)] = \sum_{n=0}^{\infty} \epsilon^{2n} J_{jn}\tilde{u_j}(x,t) \tag{120}$$

The inversion is identified via the relation $A_j J_j = 1 = J_j A_j$. Since
(121)
$A_j = \Sigma_{n=0}^{\infty}\epsilon^{2n}A_{jn} = A_0+A_1\epsilon^2+A_2\epsilon^4+\cdots;$ $\qquad J_j = \Sigma_{n=0}^{\infty}\epsilon^{2n}J_{jn} = J_0+A_1\epsilon^2+J_2\epsilon^4+\cdots$
Thus

$$\begin{aligned} A_j J_j &= A_{j0}J_{j0} + A_{j0}J_{j1}\epsilon^2 + A_{j0}J_{j2}\epsilon^4 + ... \\ &+ A_{j1}J_{j0}\epsilon^2 + A_{j1}J_{j1}\epsilon^4 + A_{j1}J_{j2}\epsilon^6 + ... \\ &+ A_{j2}J_{j0}\epsilon^4 + ... \\ &= A_{j0}J_{j0} + (A_{j0}J_{j1} + A_{j1}J_{j0})\epsilon^2 + (A_{j0}J_2 + A_{j1}J_{j1})\epsilon^4 + ... \end{aligned} \tag{122}$$

To have $A_j J_j = 1$, we should have $A_{j0}J_{j0} = 1$, and the coefficients of the ϵ terms must be 0, i.e J_j must satisfy

$$J_{j0} = 1(\text{ since } A_{j0} = 1); \quad J_{jn} = -\Sigma_{m=0}^{n-1}A_{j(n-m)}J_{jm}, \tag{123}$$

Thus,
(124)
$\tilde{w}_j(x,t) = B_j u_{jo} = B_j J_j \tilde{u}_j = K_j[\tilde{u}_j] = \Sigma_{n=0}^{\infty}\epsilon^{2n+1}K_{jn}\tilde{u}_j(x,t) = \epsilon\Sigma_{n=0}^{\infty}\epsilon^{2n}K_{jn}\tilde{u}_j(x,t)$
where $K_{jn} = \Sigma_{m=0}^{n}B_{jm}J_{j(n-m)} \quad (n = 0,1,2,...)$
We have the expression for $\tilde{w}_j$ in terms of $\tilde{u}_j$. From $\zeta_t = (1/\epsilon)\tilde{w}_j - \tilde{u}_j.\zeta_x$ (i.e $\tilde{w}_j = \epsilon\tilde{D}_j\zeta$)in model (A) , we are ready to establish another model called:
(B) : $(\tilde{u}_j, \zeta)$ system-the interfacial velocity basis, which consists of the equations:

$$\begin{aligned} &\zeta_t + \tilde{u}_j.\zeta_x = \Sigma_{n=0}^{\infty}\epsilon^{2n}K_{jn}\tilde{u}_j, \quad (j = 1,2) \\ &\tilde{D}_1\tilde{u}_1 + [g + \epsilon^2\tilde{D}_1^2\zeta]\zeta_x = -\frac{1}{\rho_1}\tilde{p}_{1,x} \\ &\tilde{D}_2\tilde{u}_2 + [g + \epsilon^2\tilde{D}_2^2\zeta]\zeta_x = -\frac{1}{\rho_2}\tilde{p}_{2,x} \end{aligned} \tag{125}$$

This system (B)-interfacial velocity basis, is more simple and elegant than system (A), as (B) involves only a single series expansion.

CHAPTER 5

Hamiltonian reduction: 2 layers in the dispersionless case

In this chapter, we detail the Hamiltonian structure for 2-layer dispersionless fluids as published in paper [**12**], in which we denote $z = 0$ the bottom of the fluid system, $\rho_1 < \rho_2$.

1. 2D Benjamin model for heterogeneous fluids in a channel

Consider two-dimensional fluid in a vertical channel of height h. The incompressible Euler equations for the velocity field $\mathbf{u} = (u, w)$ and non-constant density $\rho(x, z, t)$ in the presence of gravity $-g\mathbf{k}$ are

$$\begin{aligned} &\frac{D\rho}{Dt} = 0, \nabla \cdot \mathbf{u} = 0 \\ &\frac{D(\rho\mathbf{u})}{Dt} + \nabla p + \rho g\mathbf{k} = 0 \end{aligned} \tag{126}$$

with boundary conditions

(127)

$$\mathbf{u}(x = \boldsymbol{\infty}) = \mathbf{0}, \qquad w(x, -h_2, t) = w(x, h_1, t) = 0, \qquad x \in \mathbb{R}, z \in (-h_2, h_1), t \in \mathbb{R}^+$$

where as usual $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the material derivative.
The above Euler system admits a Hamiltonian structure in [**3**], with the basic variables : density ρ and weighted vorticity ς defined by $\varsigma = \nabla \times (\rho\mathbf{u}) = (\rho w)_x - (\rho u)_z$. The equations of motion for these two fields, ensuing from the above Euler equations, are

$$\begin{aligned} &\rho_t + u\rho_x + w\rho_z = 0 \\ &\varsigma_t + u\varsigma_x + w\varsigma_z + \rho_x\left(gz - \frac{1}{2}(u^2 + w^2)\right)_z + \frac{1}{2}\rho_z(u^2 + w^2)_x = 0 \end{aligned} \tag{128}$$

These can be written in the form

$$\rho_t = -\left[\rho, \frac{\delta H}{\delta\varsigma}\right], \qquad \varsigma_t = -\left[\rho, \frac{\delta H}{\delta\rho}\right] - \left[\varsigma, \frac{\delta H}{\delta\varsigma}\right] \tag{129}$$

where by definition $[A, B] = A_x B_z - A_z B_x$, and the functional

$$H = \int_D \rho\left(\frac{1}{2}|u|^2 + gz\right) dxdz \tag{130}$$

is simply the sum of the kinetic and potential energy, D being the fluid domain in $\mathbb{R}\times[-h_2,h_1]$. As shown by Benjamin in [**3**], the equations 129 admit a Hamiltonian structure, for the Poisson tensor defined by

$$J_B = -\begin{pmatrix} 0 & \rho_x\partial_z - \rho_z\partial_x \\ \rho_x\partial_z - \rho_z\partial_x & \varsigma_x\partial_z - \varsigma_z\partial_x \end{pmatrix} \tag{131}$$

1.1. Euler's system in layer-averaged variables. A simplification of the system of Euler equations which retains the essential properties of stratification can be obtained by considering a system of two fluids of homogeneous densities $\rho_2 > \rho_1$ in the channel $\mathbb{R}\times[0,h]$. The interface between the two homogeneous fluids is described by a smooth function $\zeta = \zeta(x,t)$. In this case the density and velocity fields can be described as

$$\begin{aligned} &\rho(x,z,t) = \rho_2 + (\rho_1 - \rho_2)\theta(z - \zeta(x,t)) \\ &u(x,z,t) = u_2(x,z,t) + (u_1(x,z,t) - u_2(x,z,t))\theta(z - \zeta(x,t)) \\ &w(x,z,t) = w_2(x,z,t) + (w_1(x,z,t) - w_2(x,z,t))\theta(z - \zeta(x,t)), \end{aligned} \tag{132}$$

where θ is the Heaviside function.

A nowadays standard way to reduce the dimensionality of the model is to introduce the layer-averaged velocities as set forth by T. Wu [**6**], since in the case of fluids stratified by gravity the vertical direction plays a distinguished role. Let us denote by

$$\overline{u}_1(x,t) := \frac{1}{\eta_1(x,t)}\int_\eta^h u_1(x,z,t)\,\mathrm{d}z, \quad \overline{u}_2(x,t) = \frac{1}{\eta_2(x,t)}\int_0^\eta u_2(x,z,t)\,\mathrm{d}z,$$

the layer-averaged velocities, where $\eta_1 = h_1 - \eta$, and η_2 are the thicknesses of the layers. Letting $P(x,t)$ denote the interfacial pressure, the non-homogeneous incompressible Euler equations result in the (non-closed) system

$$\begin{aligned} &\eta_{i_t} + (\eta_i\overline{u}_i)_x = 0, \qquad i = 1,2, \\ &\overline{u}_{1t} + \overline{u}_1\overline{u}_{1x} - g\eta_{1x} + \frac{P_x}{\rho_1} + D_1 = 0, \\ &\overline{u}_{2t} + \overline{u}_2\overline{u}_{2x} + g\eta_{2x} + \frac{P_x}{\rho_2} + D_2 = 0. \end{aligned} \tag{133}$$

The terms D_1, D_2 at the right hand side of system (133) are

$$D_i = \frac{1}{3\eta_i}\partial_x[\eta_i^3(\overline{u}_{ixt} + \overline{u}_i\overline{u}_{ixx} - (\overline{u}_{ix})^2)] + \cdots, \qquad i = 1,2, \tag{134}$$

where the unwritten part (denoted by dots) represent terms with nonlocal dependence on the averaged velocities. These terms collect the non-hydrostatic correction to the pressure field, and make the evolution of system (133) dispersive. When an asymptotic expansion based on the long-wave assumption $\epsilon \equiv \max[\eta_i/L] \ll$

1, $i = 1, 2$, is carried out (where L is a typical wavelength), expressions (134) explicitly define the leading order dispersive terms in the small parameter ϵ; truncating at this order makes equations local in the layer averaged velocities, resulting in the strongly nonlinear system studied in, e.g., [**2**].
Equations (133) come equipped with two constraints. Namely, we have the obvious geometrical constraint $\eta_1 + \eta_2 = h$ and its consequence obtained by summing the equations in the first line of (133) $(\eta_1 + \eta_2)_t = h_t = 0$, we then have

$$(\eta_1 \overline{u}_1 + \eta_2 \overline{u}_2)_x = 0\,. \tag{135}$$

We remark that under suitable far-field boundary conditions (such as vanishing velocities for $x \to \pm\infty$) this relation translates into the *dynamical* constraint

$$\eta_1 \overline{u}_1 + \eta_2 \overline{u}_2 = 0\,. \tag{136}$$

2. Hamiltonian reduction: Two-layer dispersionless case

The Benjamin's Poisson tensor 131 is written in the ρ, ς (density, weight-vorticity) coordinates. Details of this Hamiltonian reduction can be found in [**12**]. By means of the Dirac δ and Heavisde θ functions, and $\theta'() = \delta()$, we have The two momentum components are

$$\begin{aligned} \rho u &= \rho_2 u_2(x,z) + (\rho_1 u_1(x,z) - \rho_2 u_2(x,z))\theta(z - \zeta(x))\,, \\ \rho w &= \rho_2 w_2(x,z) + (\rho_1 w_1(x,z) - \rho_2 w_2(x,z))\theta(z - \zeta(x))\,, \end{aligned} \tag{137}$$

so that the weighted vorticity $\varsigma = (\rho w)_x - (\rho u)_z$ is

$$\begin{aligned} \varsigma =& \rho_2(w_{2\,x} - u_{2\,z}) + (\rho_1(w_{1\,x} - u_{1\,z}) - \rho_2(w_{2\,x} - u_{2\,z})\theta(z - \zeta(x)) \\ &- (\rho_1 u_1(x,z) - \rho_2 u_2(x,z) + \eta_x(\rho_1 w_1(x,z) - \rho_2 w_2(x,z)))\delta(z - \zeta(x))\,, \end{aligned} \tag{138}$$

where $\delta(\cdot)$ is the Dirac delta function.

We assume that the motion in each layer is irrotational, so that we are left with a "momentum vortex line" along the interface, that is,

$$\varsigma = (\rho_2 u_2(x,z) - \rho_1 u_1(x,z) + \eta_x(\rho_2 w_2(x,z) - \rho_1 w_1(x,z)))\delta(z - \zeta(x)). \tag{139}$$

where $\eta = \eta_2$ is the layer thickness of the lower fluid.
We define a projection map 2D $\to$ 1D as

$$\eta(x) = \frac{1}{\rho_{\scriptscriptstyle\Delta}} \int_0^h (\rho(x,z) - \rho_2)\,\mathrm{d}z \,-h_2, \qquad \bar{\sigma}(x) = \frac{1}{h}\int_0^h \varsigma(x,z)\,\mathrm{d}z\,, \tag{140}$$

where $\rho_{\scriptscriptstyle\Delta} = \rho_2 - \rho_1$. Since

$$\begin{aligned} h\bar{\sigma} = \int_0^h \varsigma(x,z)dz &= \rho_2 u_2(x,\eta) - \rho_1 u_1(x,\eta) + \eta_x(\rho_2 w_2(x,\eta) - \rho_1 w_1(x,\eta)) \\ &= \rho_2 \overline{u}_2(x) - \rho_1 \overline{u}_1(x) + O(\epsilon) \end{aligned} \tag{141}$$

where $w_i = O(\epsilon)$ can be neglected in this long-wave dynamics and dispersionless case.
Denote M_1 the manifold of averaged quantities $(\bar{\sigma}, \eta)$ and M_2 the manifold of 2D quantities $(\rho(x,z), \varsigma(x,z))$.
Let $(\alpha_\eta(x), \alpha_{\bar{\sigma}}(x))$ be a 1-form on the manifold M_1. We lift this 1-form, under the projection map to obtain

$$\left(\frac{1}{\rho_\Delta}\alpha_\eta(x), \frac{1}{h}\alpha_{\bar{\sigma}}(x)\right) \tag{142}$$

Applying the Benjamin Poisson's tensor 131, we then get

$$\begin{pmatrix} \dot{\rho}(x,z) \\ \dot{\varsigma}(x,z) \end{pmatrix} = -\begin{pmatrix} 0 & \rho_\Delta \delta(z-\eta(x))\partial_x \\ \rho_\Delta \delta(z-\eta(x))\partial_x & h\bar{\sigma}\delta_z(z-\eta(x))\partial_x \end{pmatrix} \begin{pmatrix} \alpha_\eta(x)/\rho_\Delta \\ \alpha_{\bar{\sigma}}(x)/h \end{pmatrix} \tag{143}$$

This gives

$$\begin{pmatrix} \dot{\rho}(x,z) \\ \dot{\varsigma}(x,z) \end{pmatrix} = -\begin{pmatrix} \rho_\Delta \delta(z-\eta(x))\,(\alpha_{\bar{\sigma}}(x)/h)_x \\ \rho_\Delta \delta(z-\eta(x))\left(\frac{1}{\rho_\Delta}\alpha_\eta(x)\right)_x + \frac{1}{h}\bar{\sigma(x)}\delta_z(z-\eta(x))(\alpha_{\bar{\sigma}}(x))_x \end{pmatrix}$$

Pushing this vector to M_1 via 140 to get

$$\dot{\eta} = \frac{1}{\rho_\Delta}\int_0^h \dot{\rho}(x,z)dz, \quad \dot{\bar{\sigma}} = \frac{1}{h}\int_0^h \dot{\varsigma}(x,z)dz \tag{144}$$

which yields

$$(\dot{\eta}, \dot{\bar{\sigma}}) = \left(-\frac{1}{h}\partial_x(\alpha_{\bar{\sigma}}); -\frac{1}{h}\partial_x(\alpha_\eta)\right) \tag{145}$$

owing to the fact that $\int_0^h \delta'(z-\eta(x))dz = 0$. We have shown that the reduction of the Benjamin Poisson tensor 131 in the $(\eta, \bar{\sigma})$ coordinates is given by

$$J_{red} = -\frac{1}{h}\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}$$

and the Hamiltonian functional is

$$H_{red}(\eta, \bar{\sigma}) = \int_{\mathbb{R}} \left(\frac{h^2}{2}\frac{\eta(h-\eta)}{\rho_2\eta_1 + \rho_1\eta_2}\bar{\sigma}^2 + \frac{g}{2}(\rho_2 - \rho_1)\eta^2\right)\mathrm{d}x \tag{146}$$

Remark 5.1. According to the terminology favored by the Russian school, for Hamiltonian quasi-linear systems of PDEs the coordinates (ξ_l, τ_l) and, *a fortiori*, the coordinates (ζ_l, σ_l), are "flat" coordinates for the system. In view of the particularly simple form of the Poisson tensor, the latter set could be called a system of flat *Darboux* coordinates. △

REMARK 5.2. In [**1**] we conjectured that in the n-layered case, with a stratification given by densities $\rho_1 < \rho_2 < \cdots < \rho_n$ and interfaces $\zeta_1 > \zeta_2 > \cdots > \zeta_{n-1}$ a good procedure yielding a natural Hamiltonian formulation for the averaged problem was to consider intervals

$$I_1 = [0, h],\ I_2 = \left[0, \frac{\zeta_1 + \zeta_2}{2}\right], I_3 = \left[0, \frac{\zeta_2 + \zeta_3}{2}\right], \ldots, I_n = \left[0, \frac{\zeta_{n-2} + \zeta_{n-1}}{2}\right], \tag{147}$$

In the next chapter, we will show in details the Hamiltonian structure for the $n = 3$ layer case.
Concerning the $n = 4$ layer case, in [**16**], we explicitly proved that the quantities

$$(\zeta_1, \zeta_2, \zeta_3, \sigma_1, \sigma_2, \sigma_3), \tag{148}$$

where $\sigma_k = \rho_{k+1}\overline{u}_{k+1} - \rho_k\overline{u}_k$ are flat Darboux coordinates for the reduced Poisson structure. $\triangle$

CHAPTER 6

Hamiltonian reduction: 3 layers in dispersionless case

In this chapter, we detail the Hamiltonian structure for 3-layer dispersionless fluids, whose result is also published in [**1**].

1. Hamiltonian reduction technique

By means of the Heaviside θ and Dirac δ generalized functions, a three-layer fluid configuration can be introduced with constant densities ρ_i and velocity components $u_i(x,z)$, $w_i(x,z)$, $i = 1,2,3$ (for the upper $i = 1$, middle $i = 2$, and lower layer $i = 3$, respectively), with interfaces ζ_1 and ζ_2, where $z = 0$ denotes the bottom. The global density and velocity variables can be written as

$$
\begin{aligned}
\rho(x,z) &= \rho_3 + (\rho_2 - \rho_3)\theta(z-\zeta_2) + (\rho_1 - \rho_2)\theta(z-\zeta_1) \\
u(x,z) &= u_3 + (u_2 - u_3)\theta(z-\zeta_2) + (u_1 - u_2)\theta(z-\zeta_1) \\
w(x,z) &= w_3 + (w_2 - w_3)\theta(z-\zeta_2) + (w_1 - w_2)\theta(z-\zeta_1)\,.
\end{aligned}
\tag{149}
$$

Thus, the density-weighted vorticity $\Sigma = (\rho w)_x - (\rho u)_z$ is

$$
\begin{aligned}
\Sigma =&\rho_3(w_{3x} - u_{3z}) + \theta(z-\zeta_2)\left(\rho_2 w_{2x} - \rho_2 u_{2z} + \rho_3 u_{3z} - \rho_3 w_{3x}\right) \\
&+ \theta(z-\zeta_1)\left(\rho_1 w_{1x} - \rho_1 u_{1z} + \rho_2 u_{2z} - \rho_2 w_{2x}\right) \\
&+ \delta(z-\zeta_2)\left((\rho_3 w_3 - \rho_2 w_2)\zeta_{2x} + (\rho_3 u_3 - \rho_2 u_2)\right) \\
&+ \delta(z-\zeta_1)\left((\rho_2 w_2 - \rho_1 w_1)\zeta_{1x} + (\rho_2 u_2 - \rho_1 u_1)\right) \\
=&\rho_3\Omega_3 + \theta(z-\zeta_2)(\rho_2\Omega_2 - \rho_3\Omega_3) + \theta(z-\zeta_1)(\rho_1\Omega_1 - \rho_2\Omega_2) \\
&+ \left((\rho_3 u_3 - \rho_2 u_2) + (\rho_3 w_3 - \rho_2 w_2)\zeta_{2x}\right)\delta(z-\zeta_2) \\
&+ \left((\rho_2 u_2 - \rho_1 u_1) + (\rho_2 w_2 - \rho_1 w_1)\zeta_{1x}\right)\delta(z-\zeta_1)\,,
\end{aligned}
$$

where $\Omega_i = w_{ix} - u_{iz}$ for $i = 1,2,3$. Next, we assume the motion in each layer to be irrotational, so that $\Omega_i = 0$ for all $i = 1,2,3$. Therefore the density weighted vorticity acquires the form

$$
\begin{aligned}
\Sigma = &\left((\rho_3 u_3 - \rho_2 u_2) + (\rho_3 w_3 - \rho_2 w_2)\zeta_{2x}\right)\delta(z-\zeta_2) \\
&+ \left((\rho_2 u_2 - \rho_1 u_1) + (\rho_2 w_2 - \rho_1 w_1)\zeta_{1x}\right)\delta(z-\zeta_1)\,.
\end{aligned}
\tag{150}
$$

In the long wave asymptotics and dispersionless case, we only take the leading-order terms, and neglect the vertical velocities w_i and trade the horizontal velocities

u_i with their layer-averaged counterparts. Thus, we obtain

$$\begin{aligned}\rho(x,z) &= \rho_3 + (\rho_2-\rho_3)\theta(z-\zeta_2) + (\rho_1-\rho_2)\theta(z-\zeta_1)\\ \Sigma(x,z) &= (\rho_3\overline{u}_3 - \rho_2\overline{u}_2)\,\delta(z-\zeta_2) + (\rho_2\overline{u}_2 - \rho_1\overline{u}_1)\,\delta(z-\zeta_1)\,.\end{aligned} \tag{151}$$

The x and z-derivative of the Benjamin's variables given by equations (151) are generalized functions supported at the surfaces $\{z=\zeta_1\}\cup\{z=\zeta_2\}$, and are computed as

$$\begin{aligned}\rho_x =& -(\rho_2-\rho_3)\zeta_{2x}\delta(z-\zeta_2) - (\rho_1-\rho_2)\zeta_{1x}\delta(z-\zeta_1)\\ \rho_z =&(\rho_2-\rho_3)\delta(z-\zeta_2) + (\rho_1-\rho_2)\delta(z-\zeta_1)\,,\end{aligned} \tag{152}$$

and

$$\begin{aligned}\Sigma_z =&(\rho_3\overline{u}_3-\rho_2\overline{u}_2)\delta'(z-\zeta_2) + (\rho_2\overline{u}_2-\rho_1\overline{u}_1)\delta'(z-\zeta_1)\\ \Sigma_x =& -(\rho_3\overline{u}_3-\rho_2\overline{u}_2)\delta'(z-\zeta_2)\zeta_{2x} - (\rho_2\overline{u}_2-\rho_1\overline{u}_1)\delta'(z-\zeta_1)\zeta_{1x}\\ &+(\rho_3\overline{u}_{3x}-\rho_2\overline{u}_{2x})\,\delta(z-\zeta_2) + (\rho_2\overline{u}_{2x}-\rho_1\overline{u}_{1x})\,\delta(z-\zeta_1)\,.\end{aligned} \tag{153}$$

Contrary to 2-layer case described in chapter 2, now we need to integrate on two vertical slices of the channel in order to obtain four x-dependent fields.

Once equipped with the isopycnal $z=f(x)$, we can define a projection $\pi:\widetilde{\mathcal{M}}\equiv\mathcal{M}\times\mathcal{F}\to(C^\infty(\mathbb{R}))^4$ by means of

(154)
$$\begin{aligned}&\pi\,(\rho(x,z),\Sigma(x,z),f) = (\xi_1,\xi_2,\tau_1,\tau_2)\\ &=\left(\int_0^h(\rho(x,z)-\rho_{max})\,\mathrm{d}z, \int_0^f(\rho(x,z)-\rho_{max})\,\mathrm{d}z, \int_0^h\Sigma(x,z)\mathrm{d}z, \int_0^f\Sigma(x,z)\mathrm{d}z\right)\,.\end{aligned}$$

where

$$f=\frac{\zeta_1+\zeta_2}{2}\,, \tag{155}$$

hereafter denoted by $\overline{\zeta}$.

(156)
$$\pi\left(\rho(x,z),\Sigma(x,z),\overline{\zeta}\right) = \left(\int_0^h(\rho(x,z)-\rho_3)\,\mathrm{d}z, \int_0^{\overline{\zeta}}(\rho(x,z)-\rho_3)\,\mathrm{d}z, \int_0^h\Sigma(x,z)\mathrm{d}z, \int_0^{\overline{\zeta}}\Sigma(x,z)\mathrm{d}z\right)\,.$$

to project from Benjamin's manifold of $2D$ fluid configurations to the space of effective $1D$ fields $\mathcal{S}$, parametrized by the four quantities (ξ_k,τ_k)

To obtain a Hamiltonian structure on the manifold $\mathcal{S}$ (defined by averaged variables)by reducing Benjamin's parent structure (131), we perform the following steps:

(i) Starting from a 1-form on the manifold $\mathcal{S}$, represented by the 4-tuple $(\alpha_S^{(1)}, \alpha_S^{(2)}, \alpha_S^{(3)}, \alpha_S^{(4)})$, we construct its pull-back to $\mathcal{I}$, that is, the 1-form $\boldsymbol{\alpha}_M = (\alpha_M^{(1)}, \alpha_M^{(2)}, 0)$ at $T^*\widetilde{M}|_{\mathcal{I}}$ satisfying the relation

$$\int_{-\infty}^{+\infty}\int_0^h (\alpha_M^{(1)}\dot{\rho} + \alpha_M^{(2)}\dot{\Sigma})\,\mathrm{d}x\,\mathrm{d}z = \int_{-\infty}^{+\infty}\sum_{k=1}^{4}\alpha_S^{(k)}\cdot\Big(\pi_*(\dot{\rho},\dot{\Sigma})\Big)^k\,\mathrm{d}x\ , \tag{157}$$

where π_* is the tangent map to ((156)).

(ii) We apply Benjamin's operator (131) to the lifted one form $\boldsymbol{\alpha}_M$ to get the vector field

$$\mathbf{Y} \equiv \begin{pmatrix} Y_M^{(1)} \\ Y_M^{(2)} \\ 0 \end{pmatrix} = J_B \cdot \begin{pmatrix} \alpha_M^{(1)} \\ \alpha_M^{(2)} \\ 0 \end{pmatrix}. \tag{158}$$

(iii) Thanks to the form of Benjamin's Poisson structure, it turns out that $\mathbf{Y}$ is still supported on the locus $\{z=\zeta_1\}\cup\{z=\zeta_2\}$, and can be easily projected with π_* to obtain the vector field (X_1, X_2, X_3, X_4) on $\mathcal{S}$. The latter depends linearly on $\{\alpha_S^{(i)}\}_{i=1,\ldots,4}$, and defines the reduced Poisson operator $\mathcal{P}$ on $\mathcal{S}$.

This construction essentially works as in the two-layer case considered in [**12**], provided one point is taken into account, namely that the integrals in the second and fourth component of π have a variable upper bound. We have, for $(\dot{\rho},\dot{\Sigma}) \in TM\big|_{\mathcal{I}}$,

$$\pi_*\begin{pmatrix} \dot{\rho} \\ \dot{\Sigma} \\ \dot{\overline{\zeta}} \end{pmatrix} = \begin{pmatrix} \int_0^h \dot{\rho}\,\mathrm{d}z \\ \int_0^{\overline{\zeta}} \dot{\rho}\,\mathrm{d}z + \dot{\overline{\zeta}}\,(\rho(x,\overline{\zeta}) - \rho_3) \\ \int_0^h \dot{\Sigma}\,\mathrm{d}z \\ \int_0^{\overline{\zeta}} \dot{\Sigma}\,\mathrm{d}z + \dot{\overline{\zeta}}\,\Sigma(x,\overline{\zeta})\,. \end{pmatrix}. \tag{159}$$

We remark that, since the inequalities $\zeta_2 < \overline{\zeta} \equiv \dfrac{\zeta_1+\zeta_2}{2} < \zeta_1$ hold in the strict sense, the second term of this vector's fourth component vanishes. The same cannot be said for the analogous term in the vector's second component, since $\rho(x,\overline{\zeta}) - \rho_3 = \rho_2 - \rho_3 \neq 0$. As anticipated above, on $\mathcal{I}$ the mean of the tangent vector component coming from the ζ's, $\dot{\overline{\zeta}} = (\dot{\zeta}_1 + \dot{\zeta}_2)/2$, can be expressed in terms of the tangent vector component $\dot{\rho}$. To this end, we can use the analogue of relations (152), which generically give

$$\dot{\rho} = (\rho_3 - \rho_2)\dot{\zeta}_2\delta(z-\zeta_2) + (\rho_2 - \rho_1)\dot{\zeta}_1\delta(z-\zeta_1)\,. \tag{160}$$

Integrating this with respect to z in $[0,h]$ yields

$$\int_0^h \dot{\rho}\,\mathrm{d}z = (\rho_3-\rho_2)\dot{\zeta}_2 + (\rho_2-\rho_1)\dot{\zeta}_1\,, \tag{161}$$

while by integrating over $[0, \overline{\zeta}]$ we obtain

$$\int_0^{\overline{\zeta}} \dot{\rho}\, \mathrm{d}z = (\rho_3 - \rho_2)\dot{\zeta}_2\,. \tag{162}$$

Solving the linear system given by (161) and (162) gives

$$(\rho_2 - \rho_3)\dot{\overline{\zeta}} = \Delta_\rho \int_0^h \dot{\rho}\, \mathrm{d}z - (\frac{1}{2} + \Delta_\rho) \int_0^{\overline{\zeta}} \dot{\rho}\, \mathrm{d}z\,, \tag{163}$$

where

$$\Delta_\rho := \frac{\rho_2 - \rho_3}{2(\rho_2 - \rho_1)}\,. \tag{164}$$

Thus, formula ((159)) for the components of the push forward π_* does not explicitly depend on the last component $\dot{\overline{\zeta}}$, and can be written as

$$\pi_* \begin{pmatrix} \dot{\rho} \\ \dot{\Sigma} \\ \dot{\overline{\zeta}} \end{pmatrix} = \begin{pmatrix} \int_0^h \dot{\rho}\, \mathrm{d}z \\ \Delta_\rho \int_0^h \dot{\rho}\, \mathrm{d}z + (\frac{1}{2} - \Delta_\rho) \int_0^{\overline{\zeta}} \dot{\rho}\, \mathrm{d}z \\ \int_0^h \dot{\Sigma}\, \mathrm{d}z \\ \int_0^{\overline{\zeta}} \dot{\Sigma}\, \mathrm{d}z \end{pmatrix}. \tag{165}$$

By substituting this result in relation ((157)) we find

$$\begin{aligned} &\int_{-\infty}^{+\infty} \int_0^h (\dot{\rho}\, \alpha_M^{(1)} + \dot{\Sigma}\, \alpha_M^{(2)})\, \mathrm{d}x\, \mathrm{d}z \\ &= \int_{-\infty}^{+\infty} \left(\int_0^h \dot{\rho}(\alpha_S^{(1)} + \Delta_\rho \alpha_S^{(2)})\, \mathrm{d}z + \int_0^{\overline{\zeta}} \dot{\rho}\, (\frac{1}{2} - \Delta_\rho)\alpha_S^{(2)})\, \mathrm{d}z + \int_0^h \dot{\Sigma}\alpha_S^{(3)}\, \mathrm{d}z + \int_0^{\overline{\zeta}} \dot{\Sigma}\alpha_S^{(4)}\, \mathrm{d}z \right) \mathrm{d}x\,. \end{aligned} \tag{166}$$

Hence, the non-vanishing components of the 1-form $\boldsymbol{\alpha}_M$ pulled back to $\mathcal{I}$ are

$$\alpha_M^{(1)} = \alpha_S^{(1)} + \Delta_\rho \alpha_S^{(2)} \theta(\overline{\zeta} - z) + (\frac{1}{2} - \Delta_\rho)\alpha_S^{(2)}\,, \qquad \alpha_M^{(2)} = \alpha_S^{(3)} + \alpha_S^{(4)} \theta(\overline{\zeta} - z)\,. \tag{167}$$

Applying the Poisson tensor (131) to this 1-form yields, after some manipulations that crucially use the fact that products of Dirac's δ supported at different locations can be consistently set to zero, we obtain the vector field

$$\begin{aligned} Y_M^{(1)} =& ((\rho_2 - \rho_3)\delta(z - \zeta_2) + (\rho_1 - \rho_2)\delta(z - \zeta_1))\, \alpha_{S,x}^{(3)} + (\rho_2 - \rho_3)\delta(z - \zeta_2)\, \alpha_{S,x}^{(4)} \\ Y_M^{(2)} =& ((\rho_2 - \rho_3)\delta(z - \zeta_2) + (\rho_1 - \rho_2)\delta(z - \zeta_1))\alpha_{S,x}^{(1)} \\ &+ (\frac{1}{2}(\rho_2 - \rho_3)\delta(z - \zeta_2) + (\rho_1 - \rho_2)\Delta_\rho \delta(z - \zeta_1))\alpha_{S,x}^{(2)} \\ &+ ((\tau_1 - \tau_2)\delta'(z - \zeta_1)\zeta_{1x} + \tau_2 \delta'(z - \zeta_2)\zeta_{2x})\alpha_{S,x}^{(3)}\,. \end{aligned} \tag{168}$$

The push-forward under the map π_* of this vector field gives the following four components:

(169)

$$
\begin{aligned}
&\int_0^h Y_M^{(1)}\,\mathrm{d}z = (\rho_1-\rho_3)\alpha^{(3)}_{S,x} + (\rho_2-\rho_3)\alpha^{(4)}_{S,x} \\
&\Delta_\rho \int_0^h Y_M^{(1)}\,\mathrm{d}z + (\frac{1}{2}-\Delta_\rho)\int_0^{\overline{\zeta}} Y_1\,\mathrm{d}z \\
&\qquad = \Delta_\rho((\rho_1-\rho_3)\alpha^{(3)}_{S,x} + (\rho_2-\rho_3)\alpha^{(4)}_{S,x}) + (\frac{1}{2}-\Delta_\rho)(\rho_2-\rho_3)(\alpha^{(3)}_{S,x}+\alpha^{(4)}_{S,x}) \\
&\qquad = (\Delta_\rho(\rho_1-\rho_3) + (\frac{1}{2}-\Delta_\rho)(\rho_2-\rho_3))\alpha^{(3)}_{S,x} + \frac{1}{2}(\rho_2-\rho_3)\alpha^{(4)}_{S,x} \\
&\qquad = \frac{1}{2}(\rho_2-\rho_3)\alpha^{(4)}_{S,x} \\
&\int_0^h Y_M^{(2)}\,\mathrm{d}z = (\rho_1-\rho_2)\alpha^{(1)}_{S,x} + (\frac{\rho_2-\rho_3}{2} + \Delta_\rho\,(\rho_1-\rho_2))\alpha^{(2)}_{S,x} = (\rho_1-\rho_3)\alpha^{(1)}_{S,x} \\
&\int_0^{\overline{\zeta}} Y_M^{(2)}\,\mathrm{d}z = (\rho_2-\rho_3)\alpha^{(1)}_{S,x} + \frac{\rho_2-\rho_3}{2}\alpha^{(2)}_{S,x}\,,
\end{aligned}
$$

where we used definition (164) of Δ_ρ and the fact that the terms with the z-derivatives of the Dirac's δ give a vanishing contribution since they are integrated against functions of x only.

From (169) we obtain the expression of the reduced Poisson tensor $\mathcal{P}$ on $\mathcal{S}$, in the coordinates $(\xi_1, \xi_2, \tau_1, \tau_2)$, as

$$
\mathcal{P} = \begin{pmatrix} 0 & 0 & \rho_1-\rho_3 & \rho_2-\rho_3 \\ 0 & 0 & 0 & \frac{1}{2}(\rho_2-\rho_3) \\ \rho_1-\rho_3 & 0 & 0 & 0 \\ \rho_2-\rho_3 & \frac{1}{2}(\rho_2-\rho_3) & 0 & 0 \end{pmatrix} \partial_x\,. \tag{170}
$$

The variables $(\xi_1, \xi_2, \tau_1, \tau_2)$ are related to $(\zeta_1, \zeta_2, \sigma_1 = \rho_2\overline{u}_2 - \rho_1\overline{u}_1, \sigma_2 = \rho_3\overline{u}_3 - \rho_2\overline{u}_2)$ by

(171)

$$
\begin{aligned}
&\xi_1 = (h-\zeta_2)(\rho_2-\rho_3) + (h-\zeta_1)(\rho_1-\rho_2)\,, \qquad \xi_2 = \frac{1}{2}\,(\rho_2-\rho_3)(\zeta_1-\zeta_2)\,, \\
&\tau_1 = \sigma_1+\sigma_2\,, \qquad \tau_2 = \sigma_2\,.
\end{aligned}
$$

Solving these relations for ζ_i's and the σ_i's gives:

(172)

$$
\begin{aligned}
&\zeta_1 = -\frac{\xi_1}{\rho_1-\rho_3} + \frac{2\xi_2}{\rho_1-\rho_3} + h\,, \qquad \zeta_2 = -\frac{\xi_1}{\rho_1-\rho_3} - \frac{2\,(\rho_1-\rho_2)\,\xi_2}{(\rho_2-\rho_3)\,(\rho_1-\rho_3)} + h\,, \\
&\sigma_1 = \tau_1-\tau_2\,, \qquad \sigma_2 = \tau_2\,.
\end{aligned}
$$

A straightforward computation shows that in these coordinates the Poisson operator (170) acquires the particularly simple form

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & -\partial_x & 0 \\ 0 & 0 & 0 & -\partial_x \\ -\partial_x & 0 & 0 & 0 \\ 0 & -\partial_x & 0 & 0 \end{pmatrix} . \tag{173}$$

2. The Hamiltonian functional

The full energy (per unit length) of the $2D$ fluid in the channel is just the sum of the kinetic and potential energy,

$$H = \int_{-\infty}^{+\infty} \int_0^h \frac{1}{2} \rho \left(u^2 + w^2\right) \mathrm{d}x \, \mathrm{d}z + \int_{-\infty}^{+\infty} \int_0^h g\rho z \, \mathrm{d}x \, \mathrm{d}z . \tag{174}$$

The potential energy is readily reduced, using the first of (149), to

$$U = \int_{-\infty}^{+\infty} \frac{1}{2} \left(g\left(\rho_2 - \rho_1\right) {\zeta_1}^2 + g\left(\rho_3 - \rho_2\right) {\zeta_2}^2 \right) \mathrm{d}x . \tag{175}$$

When the layer thicknesses are not asymptotically zero, both energies can be appropriately renormalized subtracting the far field contributions of ζ_i. To obtain the reduced kinetic energy density, we use the fact that at order $O(\epsilon^2)$ we can disregard the vertical velocity w, and trade the horizontal velocities with their layer-averaged means. Thus the x-density is computed as

(176)

$$\mathcal{T} = \frac{1}{2} \left(\int_0^{\zeta_2} \rho_3 \overline{u}_3^2 \, \mathrm{d}z + \int_{\zeta_2}^{\zeta_1} \rho_2 \overline{u}_2^2 \, \mathrm{d}z + \int_{\zeta_1}^{h} \rho_1 \overline{u}_1^2 \, \mathrm{d}z \right) = \frac{1}{2} \left(\rho_3 \zeta_2 \overline{u}_3^2 + \rho_2 (\zeta_1 - \zeta_2) \overline{u}_2^2 + \rho_1 (h - \zeta_1) \overline{u}_1^2 \right) ,$$

so that the reduced kinetic energy is

$$T = \int_{-\infty}^{+\infty} \frac{1}{2} \left(\rho_3 \zeta_2 \overline{u}_3^2 + \rho_2 (\zeta_1 - \zeta_2) \overline{u}_2^2 + \rho_1 (h - \zeta_1) \overline{u}_1^2 \right) \mathrm{d}x . \tag{177}$$

We now use the dynamical constraint for localized solutions, whereby velocities vanish at infinty,

$$(h - \zeta_1) \overline{u}_1 + (\zeta_1 - \zeta_2) \overline{u}_2 + \zeta_2 \overline{u}_3 = 0 , \tag{178}$$

to obtain

$$\overline{u}_1 = \frac{(\zeta_1 - \zeta_2) \overline{u}_2 + \zeta_2 \overline{u}_3}{\zeta_1 - h} . \tag{179}$$

Next, we express $\overline{u}_2, \overline{u}_3$ in terms of σ_1, σ_2 as

$$\begin{aligned} \overline{u}_2 &= \frac{\rho_3 \left(h - \zeta_1\right) \sigma_1}{\Psi} - \frac{\zeta_2 \rho_1 \sigma_2}{\Psi} \\ \overline{u}_3 &= \frac{\rho_2 (h - \zeta_1) \sigma_1}{\Psi} + \frac{\left(h\rho_2 + (\rho_1 - \rho_2)\zeta_1 - \zeta_2 \rho_1\right) \sigma_2}{\Psi} , \end{aligned} \tag{180}$$

where

$$\Psi = h\rho_2\rho_3 - \rho_3(\rho_2 - \rho_1)\zeta_1 - \rho_1(\rho_3 - \rho_2)\zeta_2\,. \tag{181}$$

The kinetic energy density turns out to be, in the new set of variables,

$$\begin{aligned}\mathcal{T} =& \frac{1}{2\Psi}\Big((h-\zeta_1)(\rho_3\zeta_1 + (\rho_2-\rho_3)\zeta_2)\,\sigma_1^2 \\ &+ 2(h-\zeta_1)\rho_2\,\zeta_2\,\sigma_1\sigma_2 + \left((\rho_1-\rho_2)\zeta_2\,\zeta_1 + \rho_2\,\zeta_2\,h - \rho_1\,\zeta_2^2\right)\sigma_2^2\Big)\,,\end{aligned} \tag{182}$$

so that the Hamiltonian functional is

$$H = \int_{-\infty}^{+\infty}\left(\mathcal{T} + \frac{1}{2}\,g\left((\rho_2-\rho_1)\zeta_1^2 + (\rho_3-\rho_2)\zeta_2^2\right)\right)\mathrm{d}x\,. \tag{183}$$

Explicitly, the equations of motion can be written as conservation laws,

$$\begin{pmatrix} \zeta_{1t} \\ \zeta_{2t} \\ \sigma_{1t} \\ \sigma_{2t} \end{pmatrix} = -\begin{pmatrix} (\delta_{\sigma_1}H)_x \\ (\delta_{\sigma_2}H)_x \\ \left(\delta_{\zeta_1}H\right)_x \\ \left(\delta_{\zeta_2}H\right)_x \end{pmatrix}, \tag{184}$$

where the gradient of the Hamiltonian is, explicitly,

(185)

$$\begin{aligned}\delta_{\zeta_1}H =& \frac{1}{2\Psi}\left((h\rho_3 - 2\,\zeta_1\rho_3 + \zeta_2(\rho_3-\rho_2))\,{\sigma_1}^2 - 2\,\zeta_2\rho_2\sigma_2\sigma_1 - \zeta_2\,(\rho_2-\rho_1)\,{\sigma_2}^2\right) \\ &- \frac{\rho_3(\rho_1-\rho_2)\mathcal{T}}{\Psi} + g\,(\rho_2-\rho_1)\,\zeta_1\,, \\ \delta_{\zeta_2}H =& \frac{1}{2\Psi}\left((h-\zeta_1)\,(\rho_2-\rho_3)\,{\sigma_1}^2 + 2\,(h-\zeta_1)\,\rho_2\sigma_2\sigma_1 + (\rho_2 h + \zeta_1\rho_1 - \zeta_1\rho_2 - 2\,\zeta_2\rho_1)\,{\sigma_2}^2\right) \\ &- \frac{\rho_1\,(\rho_2-\rho_3\mathcal{T})}{\Psi} + g\,(\rho_3-\rho_2)\,\zeta_2\,, \\ \delta_{\sigma_1}H =& \frac{1}{\Psi}\left((\zeta_1\rho_3 + \zeta_2\rho_2 - \zeta_2\rho_3)\,(h-\zeta_1)\,\sigma_1 + \rho_2\zeta_2\,(h-\zeta_1)\,\sigma_2\right)\,, \\ \delta_{\sigma_2}H =& \frac{1}{\Psi}\left(\rho_2\zeta_2\,(h-\zeta_1)\,\sigma_1 + \zeta_2\,(h\rho_2 + \zeta_1(\rho_1-\rho_2) - \zeta_2\rho_1)\,\sigma_2\right)\,.\end{aligned}$$

The two-dimensional fluid system above has a conserved identity T_i if

$$0 = \frac{dT_i}{dt} = \frac{\partial T_i}{\partial t} + \{T, H\}$$

in which $H(\zeta_1, \zeta_2, \sigma_1, \sigma_2)$ is the reduced Hamiltonian functional as defined above, and σ_j is the momentum shear: $\sigma_1 = \rho_2\overline{u}_2 - \rho_1\overline{u}_1$ and $\sigma_2 = \rho_3\overline{u}_3 - \rho_2\overline{u}_2$.
$\{.,.\}$ is the Poisson bracket between two functionals as defined in [**21**]. For example, given two functionals

$$F = \int_{\mathbb{R}} f(u, v, u_x, v_x, ...)dx, \qquad G = \int_{\mathbb{R}} g(u, v, u_x, v_x, ...)dx \tag{186}$$

then the Poisson bracket is again a functional given by

$$\{F, G\} = \int_{\mathbb{R}} \frac{\delta F}{\delta u(x)} \partial_x \frac{\delta G}{\delta v(x)} + \frac{\delta F}{\delta v(x)} \partial_x \frac{\delta G}{\delta u(x)} dx \tag{187}$$

where

$$\frac{\delta F}{\delta u(x)} = E_u f = \left(\frac{\partial}{\partial u} - \partial_x \frac{\partial}{\partial u_x} + \partial_x^2 \frac{\partial}{\partial u_{xx}} \right) f \tag{188}$$

As can be seen from the above formulae, even with the simple, constant Hamiltonian operator (173), this Hamiltonian gradient leads to rather lengthy (albeit explicit) expressions for the evolution equations, which are not particularly illuminating and hence are omitted here. Suffices to say that the conciseness of the Hamiltonian formalism allows to show quickly the existence of at least six conserved quantities,

$$\begin{aligned} &Z_j = \int_{-\infty}^{+\infty} \zeta_j \, \mathrm{d}x, \qquad S_j = \int_{-\infty}^{+\infty} \sigma_j \, \mathrm{d}x, \quad j = 1, 2 \\ &H\,, \quad K = \int_{-\infty}^{+\infty} (\zeta_1 \sigma_1 + \zeta_2 \sigma_2) \, \mathrm{d}x\,. \end{aligned} \tag{189}$$

3. The Boussinesq approximation

A dramatic simplification of the problem is provided by the so-called *Boussinesq approximation*, that is, the double scaling limit

(190)

$$\rho_i \to \overline{\rho} = \frac{\rho_1 + \rho_2 + \rho_3}{3}, \quad i = 1, 2, 3, \quad \text{with } g(\rho_1 - \rho_2) \text{ and } g(\rho_2 - \rho_3) \text{ both finite.}$$

Since the Poisson tensor (173) is independent of the densities, this double scaling limit can be implemented most simply within the Hamiltonian formulation. While the potential energy is unchanged, from (182) the kinetic energy acquires the form

$$\frac{(h - \zeta_1)\, \zeta_1 {\sigma_1}^2}{2h\overline{\rho}} + \frac{\zeta_2\, (h - \zeta_1)\, \sigma_2 \sigma_1}{h\overline{\rho}} + \frac{\zeta_2\, (h - \zeta_2)\, {\sigma_2}^2}{2h\overline{\rho}}\,, \tag{191}$$

so that the Hamiltonian energy functional in this Boussinesq limit is

$$\begin{aligned} H_B = \int_{-\infty}^{+\infty} &\frac{1}{2\, h\overline{\rho}} \left(\zeta_1 (h - \zeta_1) {\sigma_1}^2 + 2\zeta_2\, (h - \zeta_1)\, \sigma_1 \sigma_2 + \zeta_2\, (h - \zeta_2)\, {\sigma_2}^2 \right) \\ &+ g((\rho_2 - \rho_1)\, {\zeta_1}^2 + (\rho_3 - \rho_2)\, {\zeta_2}^2)\, \mathrm{d}x\,. \end{aligned} \tag{192}$$

The ensuing equations of motion are

$$\begin{pmatrix} \zeta_{1t} \\ \zeta_{2t} \\ \sigma_{1t} \\ \sigma_{2t} \end{pmatrix} = \mathcal{P} \begin{pmatrix} \delta_{\zeta_1} H_B \\ \delta_{\zeta_2} H_B \\ \delta_{\sigma_1} H_B \\ \delta_{\sigma_2} H_B \end{pmatrix}, \tag{193}$$

with $\mathcal{P}$ the "canonical" Poisson tensor ((173)). As a system of quasi-linear equations, they can be cast in the form

$$\begin{pmatrix} \zeta_{1t} \\ \zeta_{2t} \\ \sigma_{1t} \\ \sigma_{2t} \end{pmatrix} + \mathbf{A} \begin{pmatrix} \zeta_{1x} \\ \zeta_{2x} \\ \sigma_{1x} \\ \sigma_{2x} \end{pmatrix} = 0\,, \tag{194}$$

where the charateristic matrix reads
(195)

$$\mathbf{A} = \frac{1}{h\overline{\rho}} \begin{pmatrix} \begin{array}{l}(2\,\zeta_1 - h)\,\sigma_1 \\ +\sigma_2\zeta_2\end{array} & (\zeta_1 - h)\,\sigma_2 & \zeta_1\,(\zeta_1 - h) & \zeta_2\,(\zeta_1 - h) \\ \zeta_2\sigma_1 & \begin{array}{l}(\zeta_1 - h)\,\sigma_1 \\ +(2\,\zeta_2 - h)\,\sigma_2\end{array} & \zeta_2\,(\zeta_1 - h) & \zeta_2\,(\zeta_2 - h) \\ \sigma_1^2 + \tilde{g}\,(\rho_1 - \rho_2) & \sigma_1\sigma_2 & \begin{array}{l}(2\,\zeta_1 - h)\,\sigma_1 \\ +\sigma_2\zeta_2\end{array} & \zeta_2\sigma_1 \\ \sigma_1\sigma_2 & \sigma_2^2 + \tilde{g}\,(\rho_2 - \rho_3) & (\zeta_1 - h)\,\sigma_2 & \begin{array}{l}(\zeta_1 - h)\,\sigma_1 \\ +(2\,\zeta_2 - h)\,\sigma_2\end{array} \end{pmatrix}$$

and $\tilde{g} = g\,h\,\overline{\rho}$ is the reduced gravity.
As shown in paper [**1**], the system does not admit Riemmann invariants.

3.1. Symmetric solutions. In the recent paper [**20**] the authors have focused on the symmetric solutions defined by the equality of the upper ($i = 1$) and lower ($i = 3$) layer thicknesses, i.e., $\zeta_2 = h - \zeta_1$, and the averaged horizontal velocities, $\overline{u}_1 = \overline{u}_3$. In the Boussinesq approximation, our variables (σ_1, σ_2) are actually proportional to the velocity shears,

$$\sigma_1 = \overline{\rho}(\overline{u}_2 - \overline{u}_1), \quad \sigma_2 = \overline{\rho}(\overline{u}_3 - \overline{u}_2)\,, \tag{196}$$

so that the symmetric solutions found in [**20**] are given by the relations

$$\zeta_2 = h - \zeta_1, \quad \sigma_2 = -\sigma_1\,. \tag{197}$$

A straightforward computation confirms that the submanifold defined by these relations is invariant under the flow (194) if and only if the relation

$$\rho_3 - \rho_2 = \rho_2 - \rho_1 \tag{198}$$

is fulfilled among the density differences. In this case system ((194)) reduces to a system with 2 "degrees of freedom," parametrized, e.g., by the pair ($\zeta_2 \equiv \zeta, \sigma_2 \equiv \sigma$). The reduced "symmetric" equations of the motion are

$$\begin{pmatrix} \zeta_t \\ \sigma_t \end{pmatrix} = \frac{1}{\overline{\rho}h} \begin{pmatrix} (4\,\zeta - h)\,\sigma & \zeta\,(2\,\zeta - h) \\ 2\,\sigma^2 - g\overline{\rho}\rho_{_\Delta}h & (4\,\zeta - h)\,\sigma \end{pmatrix} \begin{pmatrix} \zeta_x \\ \sigma_x \end{pmatrix}, \tag{199}$$

where we have defined $\rho_{\Delta} = \rho_3 - \rho_2$. These equations follow from the Hamiltonian functional

$$H_{B,S} = \int_{-\infty}^{+\infty} \left(\frac{\zeta\,(h - 2\,\zeta)\,\sigma^2}{2\overline{\rho} h} + \frac{1}{2}\, g\rho_{\Delta} \left(\zeta - \frac{h}{2} \right)^2 \right) \mathrm{d}x\,, \tag{200}$$

with the “standard” Poisson tensor

$$\mathcal{P}_{(2)} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}, \tag{201}$$

where the reference level for the potential energy in the Hamiltonian density is chosen at the midpoint of the channel.

CHAPTER 7

Dispersion in the 2-layer Boussinesq case under the Weakly non linear assumption

1. Hamiltonian reduction process for 2-layer dispersive stratified fluids

The reduction process for dispersive stratified fluids is the same as the Hamiltonian reduction process for non-dispersive stratified fluids as presented in previous chapters and papers [**12**], [**1**]. Only the Hamiltonian functional for dispersive fluids is different from the Hamiltonian functional for non-dispersive fluids. Moreover, unlike the setup for 2-layer dispersionless case in [**12**], in this section, for convenience purpose, the coordinates for the reduced Poisson structure are chosen to be the interfacial displacement ζ and tangential momentum shear at the interface σ. For the sake of completeness, we detail the Hamiltonian reduction process in the ζ, σ coordinates as follows.
By means of the Heaviside θ and Dirac δ generalized functions, a 2-layer fluid configuration can be described within Benjamin's setting as in [**12**].
The two momentum components are

$$\begin{aligned} \rho u &= \rho_2 u_2(x,z) + (\rho_1 u_1(x,z) - \rho_2 u_2(x,z))\theta(z-\zeta(x))\,, \\ \rho w &= \rho_2 w_2(x,z) + (\rho_1 w_1(x,z) - \rho_2 w_2(x,z))\theta(z-\zeta(x))\,, \end{aligned} \tag{202}$$

so that the weighted vorticity

is

$$\begin{aligned} \varsigma =& \rho_2(w_{2\,x} - u_{2\,z}) + (\rho_1(w_{1\,x} - u_{1\,z}) - \rho_2(w_{2\,x} - u_{2\,z})\theta(z-\zeta(x)) \\ &- (\rho_1 u_1(x,z) - \rho_2 u_2(x,z) + \zeta_x(\rho_1 w_1(x,z) - \rho_2 w_2(x,z)))\delta(z-\zeta(x))\,, \end{aligned} \tag{203}$$

where $\delta(\cdot)$ is the Dirac delta function.

We assume that the motion in each layer is irrotational, so that we are left with a "momentum vortex line" along the interface, that is,

$$\varsigma = (\rho_2 u_2(x,z) - \rho_1 u_1(x,z) + \zeta_x(\rho_2 w_2(x,z) - \rho_1 w_1(x,z)))\delta(z-\zeta(x)). \tag{204}$$

We define a projection map 2D $\to$ 1D as

$$\zeta(x) = \frac{1}{\rho_\Delta}\int_{-h_2}^{h_1} (\rho(x,z) - \rho_1)\,\mathrm{d}z\, - h_2, \qquad \sigma(x) = \int_{-h_2}^{h_1} \varsigma(x,z)\,\mathrm{d}z\,, \tag{205}$$

where $\rho_{\Delta} = \rho_2 - \rho_1$. When applied to 2-layer configurations, the first of these relations is obtained from the first of equations ((132)). Moreover, in the 2-layer bulk irrotational case,

$$\sigma(x) = \rho_2 u_2(x,\zeta) - \rho_2 u_1(x,\zeta) + \zeta_x(\rho_2 w_2(x,\zeta) - \rho_1 w_1(x,\zeta)), \tag{206}$$

i.e., the averaged weighted vorticity σ is the tangential momentum shear at the interface.

1) We have the relation from the basic variables $(\rho(x,z), \varsigma(x,z))$ with the variables $\zeta(x), \sigma(x)$.

$$\zeta(x) = \frac{1}{\rho_\Delta}\int_{-h_2}^{h_1}(\rho(x,z)-\rho_1)dz - h_2, \qquad \sigma(x) = \int_{-h_2}^{h_1}\varsigma(x,z)dz \tag{207}$$

The first relation concerning the interface displacement ζ and the non-constant density $\rho(x,z)$ can be easily checked by computation. Concerning the second relation,the averaged weighted vorticity reduces to the tangential momentum shear at the interface $\sigma(x)$.

2) Define manifold M given by the 2-layer configuration space

$$\{\rho(x,z) = \rho_2 - \rho_\Delta\theta(z-\zeta(x)), \varsigma(x,z) = \sigma(x)\delta(z-\zeta(x))\} \tag{208}$$

Since on M we have

$$\begin{aligned} &\rho_x = \rho_\Delta\zeta_x\delta(z-\zeta), \qquad \rho_z = -\rho_\Delta\delta(z-\zeta), \\ &\zeta_x = \sigma_x\delta(z-\zeta) - \sigma\zeta_x\delta'(z-\zeta), \qquad \zeta_z = \sigma\delta'(z-\zeta) \end{aligned} \tag{209}$$

The restriction of the Poisson tensor (131) on M has the form

$$J_{B|M} = -\begin{pmatrix} 0 & \delta(z-\zeta)\rho_\Delta(\zeta_x\partial_z + \partial_x) \\ \delta(z-\zeta)\rho_\Delta(\zeta_x\partial_z+\partial_x) & \delta(z-\zeta)\sigma_x\partial_z - \delta'(z-\zeta)\sigma(\zeta_x\partial_z+\partial_x) \end{pmatrix} \tag{210}$$

to obtain the vector field $\dot\rho(x,z); \dot\varsigma(x,z)$ on M.
Apply the map (207) to $(\dot\rho(x,z); \dot\varsigma(x,z))$,

$$\dot\zeta = \frac{1}{\rho_\Delta}\int_{-h_2}^{h_1}\dot\rho(x,z)dz, \qquad \dot\sigma = \int_{-h_2}^{h_1}\dot\varsigma(x,z)dz \tag{211}$$

we obtain the vector fields $(\dot\zeta, \dot\sigma)$ as follows

$$(\dot{\zeta,\varsigma}) = (-\partial_x\mu_\sigma, -\partial_x\mu_\zeta) \tag{212}$$

where μ_σ, μ_ζ is a generic 1-form on manifold M^1 parametrized by $(\zeta(x), \sigma(x))$. Thus, the expression of the Benjamin Poisson tensor J_B on manifold M^1 given in the coordinates $(\zeta(x), \sigma(x))$ by the constant tensor

$$J_{red} = -\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{213}$$

2. The perturbed Hamiltonian functional (in terms of $\tilde{u}_1; \tilde{u}_2, \eta_1, \eta_2$)

For the case of 2-layer dispersive stratified fluids, the Poisson tensor remains the same as in the non-dispersive case, whereas the Hamiltonian functional is different.
The dispersive effect causes a perturbation added to the Hamiltonian H_0 of the dispersionless system, so that the resulting Hamiltonian functional of the dispersive stratified fluid system is a perturbed Hamiltonian of the following form

$$H = H_0 + \epsilon^2 H_2 + O(\epsilon^4) \tag{214}$$

in which

$$H_0 = \int k_0(\mathbf{u})dx$$

is the Hamiltonian of the system in the dispersionless case, which is a functional that involves only the unknown **u** and no derivatives of **u** are involved;
and

$$H_2 = \int k_2(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx})dx$$

which includes both the unknown **u** and its first and second-order derivatives.
u is the vector of the unknowns, which in the dispersionless case in [**12**] is the layer thickness η_2 and tangential momentum shear $\tilde{\sigma}$ at the interface, i.e

$$\mathbf{u} = (\eta_2, \tilde{\sigma})$$

Then, the perturbed Hamiltonianian functional of the dispersive stratified fluid can be derived, following the Dubrovin's approach in [**13**] and [**14**].

3. Weakly Nonlinear asymptotic

To do so, first we make more simplifications by

(i) Taking Boussinesq approximation, i.e $\rho_1 = \bar{\rho} = \rho_2$ in the kinetic term T, preserving the density difference in the potential energy term V.
(ii) Starting first to derive the Hamiltonian formalism under **Small-Amplitude (Weakly Nonlinear)** assumption, i.e

$$\eta_j = h_j\left(1 + (-1)^j \frac{a}{h_j}\zeta\right)$$

with

$$h_1 \approx h_2 \approx h$$

and

$$\frac{a}{h} = \alpha$$

and most importantly $\alpha \approx \epsilon^2$ so that α^2 is at the same order of magnitude with ϵ^4 .Thus,

$$\eta_{j,x} = \frac{a}{h}(-1)^j \zeta_x \tag{215}$$

scales as $\frac{a}{h}\frac{h}{L} = \alpha.\epsilon$

Recall that in the dispersionless case, i.e when $\epsilon = 0$, the horizontal velocities at the interface $\tilde{u}_j$ are equal to the layer-averaged $\bar{u}_j$ and boundary horizontal velocities u_{j0}, i.e

$$\tilde{u}_j = u_{j0} = \bar{u}_j \tag{216}$$

Thus, the resulting equations of motion are all the same in the dispersionless case, no matter whether the basic variables are interfacial horizontal velocities, boundary velocities, or layer-averaged ones.
However, when dispersion is included in the system, this is no more true. See chapter 3 and 4 to check the difference when dispersion is included.
For the sake of convenience and simplicity, hereafter, we will use interfacial horizontal velocities as the basic variables in the Hamiltonian formalism of dispersive stratified fluids.
Starting from the series expansion in Wu's model, we have, up to ϵ^2, the interfacial $(\tilde{u_j}, \tilde{w_j})$ are related to the boundary velocities (i.e velocities at the bottom or rigid lid) (u_{j0}, w_{j0}) by

$$\begin{aligned} u_{j0} &= \Sigma_{n=0}^{\infty} \epsilon^{2n} J_{jn} \tilde{u}_j(x,t) = J_{j0}\tilde{u}_j + \epsilon^2 J_{j1}\tilde{u}_j + O(\epsilon^4) \\ &= \tilde{u}_j + \frac{\epsilon^2}{2}\beta_j^2 \partial_{xx}\tilde{u}_j + O(\epsilon^4) \\ \tilde{w_j} &= \epsilon K_{j0}\tilde{u_j} + \epsilon^3 K_{j1}\tilde{u_j} \\ &= -\epsilon\beta_j \tilde{u_j}_x + O(\epsilon^3) \end{aligned} \tag{217}$$

Based on formula (111) for the horizontal velocity $u_j(x,z,t\epsilon)$, we have, up to ϵ^2

$$u_j(x,z,t,\epsilon) = u_{j0}(x,t,\epsilon) - \frac{1}{2}\epsilon^2 H_j(z)^2 \partial_{xx} u_{j0}(x,t,\epsilon) \tag{218}$$

Then, from (217), we can rewrite $u_j(x,z,t,\epsilon)$ in terms of $\tilde{u_j}$

$$\begin{aligned} u_j(x,z,t,\epsilon) &= (\tilde{u}_j + \frac{\epsilon^2}{2}\beta_j^2\partial_{xx}\tilde{u}_j) - \frac{1}{2}\epsilon^2 H_j(z)^2 \left(\tilde{u}_j + \frac{\epsilon^2}{2}\beta_j^2\partial_{xx}\tilde{u}_j\right)_{xx} \\ &= \tilde{u}_j + \frac{\epsilon^2}{2}\tilde{u}_{j,xx}(\beta_j^2 - H_j(z)^2) + O(\epsilon^4) \end{aligned} \tag{219}$$

Moreover, from (116), we have

$$w_j(x,z,t,\epsilon) = \frac{(-1)}{1}\epsilon H_j(z)u_{j0,x} + \frac{1}{3!}\epsilon^3 H_j^3(z)u_{j0,xxx} = -\epsilon H_j(z)u_{j0,x} + O(\epsilon^3) \tag{220}$$

written in terms of $\tilde{u}_j$ as

$$w_j(x,z,t,\epsilon) = -\epsilon H_j(z)(\tilde{u}_j - \frac{\epsilon^2}{2}\beta_j^2\partial_{xx}\tilde{u}_j)_x = -\epsilon H_j(z)\tilde{u}_{j,x} + O(\epsilon^3) \tag{221}$$

so the interfacial vertical velocities $\tilde{w}_j$ are

$$\begin{aligned} \tilde{w}_1 &= \epsilon\eta_1\tilde{u}_1 + O(\epsilon^3) \\ \tilde{w}_2 &= -\epsilon\eta_2\tilde{u}_2 + O(\epsilon^3) \end{aligned} \tag{222}$$

Thus, the kinetic energy density of the system, written in terms of the interfacial horizontal velocities $\tilde{u}_j$

$$\begin{aligned} T &= \frac{\rho_2}{2}\int_{-h_2}^{\zeta}\left(\left[\tilde{u}_2 + \frac{\epsilon^2}{2}\tilde{u}_{2,xx}(\beta_2^2 - H_2(z)^2)\right]^2 + (-\epsilon H_2(z)\tilde{u}_{2,x})^2\right)dz \\ &+ \frac{\rho_1}{2}\int_{\zeta}^{h_1}\left(\left[\tilde{u}_1 + \frac{\epsilon^2}{2}\tilde{u}_{1,xx}(\beta_1^2 - H_1(z)^2)\right]^2 + (-\epsilon H_1(z)\tilde{u}_{1,x})^2\right)dz \\ &= \frac{\rho_2}{2}\left[\int_{-h_2}^{\zeta}(\tilde{u}_2^2 + \epsilon^2\tilde{u}_2\tilde{u}_{2,xx}\beta_2^2)dz + \epsilon^2(\tilde{u}_{2,x}^2 - \tilde{u}_2\tilde{u}_{2,xx})\int_{-h_2}^{\zeta}H_2(z)^2dz\right] \\ &+ \frac{\rho_1}{2}\left[\int_{\zeta}^{h_1}(\tilde{u}_1^2 + \epsilon^2\tilde{u}_1\tilde{u}_{1,xx}\beta_1^2)dz + \epsilon^2(\tilde{u}_{1,x}^2 - \tilde{u}_1\tilde{u}_{1,xx})\int_{\zeta}^{h_1}H_1(z)^2dz\right] \end{aligned} \tag{223}$$

Since

$$H_2(z) = z + h_2; \qquad H_1(z) = z - h_1$$

and

$$\beta_2 = H_2(\zeta) = \zeta + h_2 > 0; \qquad \beta_1 = H_1(\zeta) = \zeta - h_1 = -(h_1 - \zeta) < 0$$

$$\begin{aligned} \int_{-h_2}^{\zeta}H_2(z)^2dz &= \frac{H_2^3}{3}\Big|_{-h_2}^{\zeta} = \frac{1}{3}\beta_2^3 \\ \int_{\zeta}^{h_1}H_1(z)^2dz &= \frac{H_1^3}{3}\Big|_{\zeta}^{h_1} = -\frac{1}{3}\beta_1^3 \end{aligned} \tag{224}$$

and neglect the ϵ^4 terms in the computation, thus the kinetic energy density becomes

$$\begin{aligned} T &= \frac{\rho_2}{2}\left[(\tilde{u}_2^2 + \epsilon^2\tilde{u}_2\tilde{u}_{2,xx}\beta_2^2)\beta_2 + \frac{\epsilon^2}{3}\beta_2^3(\tilde{u}_{2,x}^2 - \tilde{u}_2\tilde{u}_{2,xx})\right] \\ &+ \frac{\rho_1}{2}\left[-(\tilde{u}_1^2 - \epsilon^2\tilde{u}_1\tilde{u}_{1,xx}\beta_1^2)\beta_1 - \frac{\epsilon^2}{3}(\beta_1^3)(\tilde{u}_{1,x}^2 - \tilde{u}_1\tilde{u}_{1,xx})\right] \end{aligned} \tag{225}$$

written in terms of the layer-thickness $\eta_1 = -\beta_1, \eta_2 = \beta_2$ and interfacial velocity $\tilde{u}_1; \tilde{u}_2$ as

$$
\begin{aligned}
T &= \frac{\rho_2}{2}\left[(\tilde{u}_2^2 + \epsilon^2\tilde{u}_2\tilde{u}_{2,xx}\eta_2^2)\eta_2 + \frac{\epsilon^2}{3}\eta_2^3(\tilde{u}_{2,x}^2 - \tilde{u}_2\tilde{u}_{2,xx})\right] \\
&\quad + \frac{\rho_1}{2}\left[(\tilde{u}_1^2 + \epsilon^2\tilde{u}_1\tilde{u}_{1,xx}\eta_1^2)\eta_1 + \frac{\epsilon^2}{3}\eta_1^3(\tilde{u}_{1,x}^2 - \tilde{u}_1\tilde{u}_{1,xx})\right] \\
&= \frac{\rho_2}{2}\left[\tilde{u}_2^2\eta_2 + \frac{\epsilon^2}{3}\eta_2^3(\tilde{u}_{2,x}^2 + 2\tilde{u}_2\tilde{u}_{2,xx})\right] \\
&\quad + \frac{\rho_1}{2}\left[\tilde{u}_1^2\eta_1 + \frac{\epsilon^2}{3}\eta_1^3(\tilde{u}_{1,x}^2 + 2\tilde{u}_1\tilde{u}_{1,xx})\right]
\end{aligned}
\tag{226}
$$

The potential energy density of the system is

(227)

$$
\begin{aligned}
V &= \int_{-h_2}^{\zeta} \rho_2 gz dz + \int_{\zeta}^{h_1} \rho_1 gz dz \\
&= \frac{g}{2}[\rho_2(\zeta^2 + h_2^2) + \rho_1(h_1^2 - \zeta^2)] \\
&= \frac{g}{2}[\rho_2 - \rho_1]\zeta^2
\end{aligned}
$$

up to constant terms which do not influence variational derivatives in the Hamiltonian structure

in which $h_1(h_2)$ is the unperturbed layer thickness.

The Hamiltonian functional becomes

(228)

$$
\begin{aligned}
H &= \int_{\mathbb{R}} (T + V) dx \\
&= \int_{\mathbb{R}} \frac{\rho_2}{2}\left[\tilde{u}_2^2\eta_2 + \frac{\epsilon^2}{3}\eta_2^3(\tilde{u}_{2,x}^2 + 2\tilde{u}_2\tilde{u}_{2,xx})\right] + \frac{\rho_1}{2}\left[\tilde{u}_1^2\eta_1 + \frac{\epsilon^2}{3}\eta_1^3(\tilde{u}_{1,x}^2 + 2\tilde{u}_1\tilde{u}_{1,xx})\right] \\
&\qquad + \frac{g}{2}[\rho_2 - \rho_1]\zeta^2 dx
\end{aligned}
$$

However, the Hamiltonian variables in the dispersionless case (in the previously-done paper [**12**]) is the thickness of the lower layer η_2, and the tangential momentum shear at the interface $\tilde{\sigma}$. Thus, we need to rewrite the above Hamiltonian in terms of $\eta_2, \tilde{\sigma}$. To do so, beside the geometric constraint $\eta_1 = h - \eta_2$, we need to find the relation between the interfacial velocities $\tilde{u}_1$ and $\tilde{u}_2$; and the relation between $\tilde{u}_2$ and $\tilde{\sigma}$.

These relations in general are complicated. Thus, to simplify the problem further, we start first with the Boussinesq approximation, and use Small-Amplitude(Weakly nonlinear) assumption (so that $\tilde{\sigma} = \bar{\rho}(\tilde{u}_2 - \tilde{u}_1)$).

3.1. Mass conservation at $O(\epsilon^2)$-Weak dynamical constraint. From the mass-conservation constraint in Wu's model (125), we have up to ϵ^2:

$$\begin{aligned}\zeta_t + \tilde{u}_j.\zeta_x &= [K_{j0}\tilde{u}_j] + [\epsilon^2 K_{j1}\tilde{u}_j] + O(\epsilon^3)\\ &= -\beta_j \tilde{u}_{jx} - \epsilon^2\beta_j^2\beta_{jx}\tilde{u}_{jxx} - \frac{1}{3}\epsilon^2\beta_j^3\tilde{u}_{jxxx} + O(\epsilon^3)\end{aligned}$$

or equivalently

$$\zeta_t + \tilde{u}_j.\zeta_x + \beta_j\tilde{u}_{jx} + \epsilon^2\left[\frac{1}{3}\beta_j^3\tilde{u}_{jxx}\right]_x + O(\epsilon^3) = 0 \tag{229}$$

in which $\beta_1 = \zeta - h_1$ and $\beta_2 = \zeta + h_2$, so

$$\beta_{1,x} = \zeta_x = \eta_{2,x}; \qquad \beta_{1,t} = \zeta_t = \beta_{2,t}$$

Recall the notation η for layer thickness, thus

$$\beta_1 = -\eta_1; \qquad \beta_2 = \eta_2 \tag{230}$$

so

$$-\eta_{1,t} = \zeta_t = \eta_{2,t}; \qquad -\eta_{1,x} = \zeta_x = \eta_{2,x} \tag{231}$$

Then, the above mass- conservation (229) equations becomes

$$\begin{aligned}\eta_{2,t} + \tilde{u}_2.\eta_{2,x} + \eta_2\tilde{u}_{2x} + \epsilon^2\left[\frac{1}{3}\eta_2^3\tilde{u}_{2xx}\right]_x + O(\epsilon^3) = 0\\ -\eta_{1,t} - \tilde{u}_1.\eta_{1,x} - \eta_1\tilde{u}_{1x} - \epsilon^2\left[\frac{1}{3}\eta_1^3\tilde{u}_{1xx}\right]_x + O(\epsilon^3) = 0\end{aligned} \tag{232}$$

or equivalently,

$$\begin{aligned}\eta_{2,t} + [\tilde{u}_2.\eta_2]_x + \epsilon^2\left[\frac{1}{3}\eta_2^3\tilde{u}_{2xx}\right]_x + O(\epsilon^3) = 0\\ \eta_{1,t} + [\tilde{u}_1.\eta_1]_x + \epsilon^2\left[\frac{1}{3}\eta_1^3\tilde{u}_{1xx}\right]_x + O(\epsilon^3) = 0\end{aligned} \tag{233}$$

Thus, summing (233) over $j = \overline{1;2}$ we have

$$(\eta_1 + \eta_2)_t + (\eta_1\tilde{u}_1 + \eta_2\tilde{u}_2)_x + \epsilon^2\left[\frac{1}{3}\eta_1^3\tilde{u}_{1,xx} + \frac{1}{3}\eta_2^3\tilde{u}_{2,xx}\right]_x = 0 \tag{234}$$

and since the total height h is constant, thus $(\eta_1 + \eta_2)_t = h_t = 0$

$$\left[\eta_1\tilde{u}_1 + \eta_2\tilde{u}_2 + \epsilon^2\left(\frac{1}{3}\eta_1^3\tilde{u}_{1,xx} + \frac{1}{3}\eta_2^3\tilde{u}_{2,xx}\right)\right]_x = 0 \tag{235}$$

This means $\eta_1\tilde{u}_1 + \eta_2\tilde{u}_2 + \epsilon^2\left(\frac{1}{3}\eta_1^3\tilde{u}_{1,xx} + \frac{1}{3}\eta_2^3\tilde{u}_{2,xx}\right)$ is independent from x, and since the velocities vanish at the infinity $lim_{x\to\infty}\tilde{u}_j = 0$, thus

(236)

$$\begin{aligned}&\eta_1\tilde{u}_1(x) + \eta_2\tilde{u}_2(x) + \epsilon^2\left(\frac{1}{3}\eta_1^3\tilde{u}_{1,xx}(x) + \frac{1}{3}\eta_2^3\tilde{u}_{2,xx}(x)\right)\\ &= \eta_1\tilde{u}_1(x=\infty) + \eta_2\tilde{u}_2(x=\infty) + \epsilon^2\left(\frac{1}{3}\eta_1^3\tilde{u}_{1,xx}(x=\infty) + \frac{1}{3}\eta_2^3\tilde{u}_{2,xx}(x=\infty)\right) = 0\end{aligned}$$

which gives the dynamical constraint

$$\eta_1\tilde{u}_1 + \eta_2\tilde{u}_2 + \epsilon^2\left(\frac{1}{3}\eta_1^3\tilde{u}_{1,xx} + \frac{1}{3}\eta_2^3\tilde{u}_{2,xx}\right) = O(\epsilon^4) \tag{237}$$

Boussinesq approximation means $\rho_1 = \rho_2 = \bar{\rho}$ in the kinetic energy and preserving the density difference in the potential energy.
As said before, Weakly nonlinear, or Small Interface amplitude assumption introduces another small parameter α, defined as

$$\alpha = \frac{a}{h} \text{ with } h_1 \approx h_2 \approx h \tag{238}$$

in which a is a characteristic wave amplitude at the interface , so

$$\zeta/h_1 = O(\zeta/h_2) = O(\alpha) \tag{239}$$

Here, we just consider $\alpha \approx \epsilon^2$ and neglect terms of order $\alpha^2 \approx \epsilon^4$. Then,

$$\eta_j = h_j\left(1 + (-1)^j\frac{a}{h_j}\zeta\right) \tag{240}$$

which gives

$$\begin{aligned}\eta_j^3 = h_j^3\left(1 + (-1)^j\frac{a}{h_j}\zeta\right)^3 &= h_j^3\left(1 + 3\frac{a^2}{h_j^2}\zeta^2 + 3(-1)^j\frac{a}{h_j}\zeta + (-1)^j\left(\frac{a}{h_j}\zeta\right)^3\right)\\ &= h_j^3(1 + O(\alpha^2) + O(\alpha) + O(\alpha^3))\\ &\approx h_j^3\end{aligned} \tag{241}$$

The α terms in η_j^3 is omitted since later we multiply η_j^3 with ϵ^2 terms which give $O(\epsilon^4)$ and higher-order terms that needs omitting (with $\alpha \approx \epsilon^2 \ll 1$).
For example, the dynamical constraint derived above (237), under this weakly nonlinear assumption, give the **Weak dynamical constraint**

$$\tilde{u}_1\eta_1 + \tilde{u}_2\eta_2 + \frac{\epsilon^2}{3}\left(h_1^3\tilde{u}_{1,xx} + h_2^3\tilde{u}_{2,xx}\right) = 0 \tag{242}$$

3.2. Kinetic energy under Weakly Non-linear assumption. Moreover, the kinetic density is also simplified, which, under the Boussinesq approximation, gives

$$T_{WNL} = \frac{\bar{\rho}}{2}\left(\eta_1\tilde{u}_1^2 + \eta_2\tilde{u}_2^2 + \epsilon^2\left[h_1^3\left(\frac{2}{3}\tilde{u}_1\tilde{u}_{1,xx} + \frac{1}{3}(\tilde{u}_{1,x})^2\right) + h_2^3\left(\frac{2}{3}\tilde{u}_2\tilde{u}_{2,xx} + \frac{1}{3}(\tilde{u}_{2,x})^2\right)\right]\right) \tag{243}$$

Notice that

$$\begin{aligned}\frac{2}{3}\tilde{u}_j\tilde{u}_{j,xx} + \frac{1}{3}(\tilde{u}_{j,x})^2 &= \frac{1}{3}\left(\tilde{u}_j\tilde{u}_{j,xx} + (\tilde{u}_{j,x})^2\right) + \frac{1}{3}\tilde{u}_j\tilde{u}_{j,xx} \\ &= \frac{d}{dx}\left(\frac{1}{3}\tilde{u}_j\tilde{u}_{j,x}\right) + \frac{1}{3}\tilde{u}_j\tilde{u}_{j,xx} \\ &= \frac{1}{3}\tilde{u}_j\tilde{u}_{j,xx}\end{aligned} \tag{244}$$

since $\frac{d}{dx}\left(\frac{1}{3}\tilde{u}_j\tilde{u}_{j,x}\right)$ will be canceled after we perform integration of the kinetic density $\int_{\mathbb{R}} T_{WNL}dx$ and vanishing velocity at the infinity. Thus, the kinetic density becomes

$$\begin{aligned}T_{WNL} &= \frac{\bar{\rho}}{2}\left(\eta_1\tilde{u}_1^2 + \eta_2\tilde{u}_2^2 + \frac{\epsilon^2}{3}\left[h_1^3\tilde{u}_1\tilde{u}_{1,xx} + h_2^3\tilde{u}_2\tilde{u}_{2,xx}\right]\right) \\ V &= \frac{1}{2}g(\rho_2 - \rho_1)\eta_2^2 \quad \text{(up to constant terms)}\end{aligned} \tag{245}$$

3.3. Momentum shear at the interface $\tilde{\sigma}_B$ under WNL and Boussinesq approximation. Since

$$\begin{aligned}\tilde{w}_1 &= \epsilon\eta_1\tilde{u}_{1,x} + O(\epsilon^3) \\ \tilde{w}_2 &= -\epsilon\eta_2\tilde{u}_{2,x} + O(\epsilon^3)\end{aligned} \tag{246}$$

thus the momentum shear at the interface, under Boussinesq approximation, becomes

$$\begin{aligned}\tilde{\sigma}_B &= \bar{\rho}[(\tilde{u}_2 - \tilde{u}_1) + \eta_{2,x}(\tilde{w}_2 - \tilde{w}_1)] \\ &= \bar{\rho}[(\tilde{u}_2 - \tilde{u}_1) + \eta_{2,x}(-\epsilon\eta_2\tilde{u}_{2,x} - \epsilon\eta_1\tilde{u}_{1,x} + O(\epsilon^3))] \\ &= \bar{\rho}(\tilde{u}_2 - \tilde{u}_1) - \bar{\rho}\epsilon\eta_{2,x}(\eta_2\tilde{u}_{2,x} + \eta_1\tilde{u}_{1,x})\end{aligned} \tag{247}$$

Since $\eta_j = h_j\left(1 + (-1)^j \dfrac{a}{h_j}\zeta\right)$, so $\eta_{j,x} = (-1)^j \dfrac{a}{h_j}.h_j\zeta_x = O\left(\dfrac{a}{h_j}.\dfrac{h_j}{L}\right) = O(\alpha.\epsilon)$, thus $\Sigma_{j=1,2}\eta_{j,x}.\tilde{u}_j = O(\epsilon.\alpha)$, in which L is the characteristic wavelength.
Take the term $\Sigma_{j=1,2}\eta_{j,x}.\tilde{u}_j = O(\epsilon.\alpha)$, then multiply it by $\epsilon.\eta_{2,x}$ in $\tilde{\sigma}_B$ will give $O(\epsilon^3.\alpha)$ which is fine to be omitted. Thus,

$$
\begin{aligned}
\tilde{\sigma}_B &= \bar{\rho}[\tilde{u}_2 - \tilde{u}_1 - \epsilon\eta_{2,x}(\eta_2\tilde{u}_{2,x} + \eta_1\tilde{u}_{1,x})] \\
&\approx \bar{\rho}[\tilde{u}_2 - \tilde{u}_1 - \epsilon\eta_{2,x}(\eta_2\tilde{u}_{2,x} + \eta_1\tilde{u}_{1,x} + \eta_{2,x}\tilde{u}_2 + \eta_{1,x}\tilde{u}_1)] \\
&= \bar{\rho}[\tilde{u}_2 - \tilde{u}_1 - \epsilon\eta_{2,x}(\eta_1\tilde{u}_1 + \eta_2\tilde{u}_2)_x] \\
&\approx \bar{\rho}[\tilde{u}_2 - \tilde{u}_1]
\end{aligned} \tag{248}
$$

since $\Sigma_j(\eta_j\tilde{u}_j)_x = O(\epsilon^2)$ from the dynamical constraint.
Thus, we have the momentum shear $\tilde{\sigma}_B$ which is the same as the momentum shear in the dispersionless case

$$
\boxed{\tilde{\sigma}_B = \bar{\rho}[\tilde{u}_2 - \tilde{u}_1]} \tag{249}
$$

3.4. Relation between $\tilde{u}_1$ and $\tilde{u}_2$. The relation between $\tilde{u}_1$ and $\tilde{u}_2$ can be found via the dynamical constraint at $O(\epsilon^2)$, which gives

$$
\begin{aligned}
&\left(\eta_1 + \frac{\epsilon^2}{3}\eta_1^3\partial_x^2\right)\tilde{u}_1 = -(\eta_2 + \frac{\epsilon^2}{3}\eta_2^3\partial_x^2)\tilde{u}_2 \\
&\Rightarrow \eta_1\left(\mathbb{I} + \frac{\epsilon^2}{3}h_1^2\partial_x^2\right)\tilde{u}_1 = -\eta_2\left(\mathbb{I} + \frac{\epsilon^2}{3}h_2^2\partial_x^2\right)\tilde{u}_2 \\
&\text{since } \eta_j^2 = h_j^2 + O(\alpha) \approx h_j^2 \text{ after multiplied by } \frac{\epsilon^2}{3} \\
&\Rightarrow \left(\mathbb{I} + \frac{\epsilon^2}{3}h_1^2\partial_x^2\right)\tilde{u}_1 = -\frac{\eta_2}{h - \eta_2}\left(\mathbb{I} + \frac{\epsilon^2}{3}h_2^2\partial_x^2\right)\tilde{u}_2 \\
&\Rightarrow \tilde{u}_1 = \left(\mathbb{I} + \frac{\epsilon^2}{3}h_1^2\partial_x^2\right)^{-1}.\left[\frac{-\eta_2}{h - \eta_2}\left(\mathbb{I} + \frac{\epsilon^2}{3}h_2^2\partial_x^2\right)\right]\tilde{u}_2 \\
&\Rightarrow \tilde{u}_1 = \left(\mathbb{I} - \frac{\epsilon^2}{3}h_1^2\partial_x^2 + O(\epsilon^4)\right).\left[\frac{-\eta_2}{h - \eta_2}\left(\mathbb{I} + \frac{\epsilon^2}{3}h_2^2\partial_x^2\right)\right]\tilde{u}_2
\end{aligned} \tag{250}
$$

Let's consider $\dfrac{\eta_2}{h-\eta_2}$

$$
\begin{aligned}
\frac{\eta_2}{h-\eta_2} &= \frac{h_2\left(1+\dfrac{\zeta}{h_2}\right)}{h_1\left(1-\dfrac{\zeta}{h_1}\right)} \\
&= \frac{h_2\left(1+\dfrac{\zeta}{h_2}\right)}{h_1}.\left(\frac{1}{1-\dfrac{\zeta}{h_1}}\right) \\
&= \frac{h_2}{h_1}+\frac{\zeta}{h_1}.\left(1+\frac{\zeta}{h_1}+\frac{\zeta^2}{h_1^2}\right) \\
&= \frac{h_2}{h_1}+O(\alpha)
\end{aligned} \tag{251}
$$

Thus, replace $\dfrac{\eta_2}{h-\eta_2} \approx \dfrac{h_2}{h_1}$ after multiplied by $\dfrac{\epsilon^2}{3}$ terms, so $\tilde{u}_1$ is computed as

$$
\begin{aligned}
\tilde{u}_1 &= -\frac{\eta_2}{h-\eta_2}\tilde{u}_2 - \frac{\epsilon^2}{3}h_1^2\left(-\frac{h_2}{h_1}\tilde{u}_{2,xx}\right) - \frac{h_2}{h_1}\frac{\epsilon^2}{3}h_2^3\tilde{u}_{2,xx} \\
&= -\frac{\eta_2}{h-\eta_2}\tilde{u}_2 + \frac{\epsilon^2}{3}\left(h_1h_2-\frac{h_2^3}{h_1}\right)\tilde{u}_{2,xx} \\
&= -\frac{\eta_2}{h-\eta_2}\tilde{u}_2 + \frac{\epsilon^2}{3}\frac{h_2}{h_1}.[h_1^2-h_2^2]\tilde{u}_{2,xx}
\end{aligned} \tag{252}
$$

$$
\boxed{\tilde{u}_1 = -\frac{\eta_2}{h-\eta_2}\tilde{u}_2 + \frac{\epsilon^2}{3}\frac{h_2}{h_1}.[h_1^2-h_2^2]\tilde{u}_{2,xx}} \tag{253}
$$

3.5. Relation between $\tilde{u}_2$ and $\tilde{\sigma}_B$. From the above computation (249), we have $\tilde{\sigma}_B = \bar{\rho}(\tilde{u}_2 - \tilde{u}_1)$, and the relation between velocities $\tilde{u}_1$ and $\tilde{u}_2$ (253), we thus have

$$
\begin{aligned}
\frac{\tilde{\sigma}_B}{\bar{\rho}} &= \tilde{u}_2 - \tilde{u}_1 = \tilde{u}_2\left(1+\frac{\eta_2}{h-\eta_2}\right) - \frac{\epsilon^2}{3}\frac{h_2}{h_1}(h_1^2-h_2^2)\tilde{u}_{2,xx} \\
&= \tilde{u}_2\left(\frac{h}{h-\eta_2}\right) - \frac{\epsilon^2}{3}\frac{h_2}{h_1}(h_1^2-h_2^2)\tilde{u}_{2,xx} \\
&= \frac{h}{h-\eta_2}\left(\mathbb{I} - \frac{\epsilon^2}{3}\frac{h_1}{h_1+h_2}\frac{h_2}{h_1}(h_1^2-h_2^2)\partial_x^2\right)\tilde{u}_2
\end{aligned} \tag{254}
$$

since
(255)

$$\begin{aligned}\frac{h-\eta_2}{h} &= \frac{h_1\left(1-\dfrac{\zeta}{h_1}\right)}{h_1+h_2}\\ &= \frac{h_1}{h_1+h_2}.\frac{1}{1+\dfrac{\zeta}{h_1}+\dfrac{\zeta^2}{h_1^2}},\ \left(\text{from Taylor's series for } \left(1-\frac{\zeta}{h_1}\right) \text{ for } \frac{\zeta}{h_1} \ll 1\right)\\ &= \frac{h_1}{h_1+h_2} + O(\alpha)\end{aligned}$$

Thus, $\tilde{u}_2$ is computed as

$$\begin{aligned}&\left(\mathbb{I}-\frac{\epsilon^2}{3}\frac{h_2}{h_1+h_2}(h_1^2-h_2^2)\partial_x^2\right)\tilde{u}_2 = \frac{h-\eta_2}{h}.\frac{\tilde{\sigma}_B}{\bar{\rho}}\\ &\Rightarrow \tilde{u}_2 = \left(\mathbb{I}-\frac{\epsilon^2}{3}\frac{h_2}{h_1+h_2}(h_1^2-h_2^2)\partial_x^2\right)^{-1}.\frac{h-\eta_2}{h}.\frac{\tilde{\sigma}_B}{\bar{\rho}}\\ &\Rightarrow \tilde{u}_2 = \frac{h-\eta_2}{h}\left[\frac{\tilde{\sigma}_B}{\bar{\rho}}+\frac{\epsilon^2}{3}\frac{h_2}{h_1+h_2}(h_1^2-h_2^2)\frac{\tilde{\sigma}_{B,xx}}{\bar{\rho}}\right]\end{aligned} \tag{256}$$

$$\boxed{\tilde{u}_2 = \frac{h-\eta_2}{h}\left[\frac{\tilde{\sigma}_B}{\bar{\rho}}+\frac{\epsilon^2}{3}\frac{h_2}{h_1+h_2}(h_1^2-h_2^2)\frac{\tilde{\sigma}_{B,xx}}{\bar{\rho}}\right]} \tag{257}$$

From the relation (253), thus

$$\tilde{u}_{1,xx} = \frac{-h_2}{h_1}\tilde{u}_{2,xx} + O(\epsilon^2) \tag{258}$$

in which the $O(\epsilon^2)$ term in $\tilde{u}_{1,xx}$ will later can be omitted when multiplied with the ϵ^2 in the kinetic T_{WNL}.
Thus,

$$\tilde{\sigma}_{B,xx} = \bar{\rho}.\left(\tilde{u}_{2,xx}+\frac{h_2}{h_1}\tilde{u}_{2,xx}\right) = \bar{\rho}\tilde{u}_{2,xx}.\left(\frac{h_1+h_2}{h_1}\right) \tag{259}$$

which gives

$$\tilde{u}_{2,xx} = \frac{1}{\bar{\rho}}\frac{h_1}{h_1+h_2}\tilde{\sigma}_{B,xx} \tag{260}$$

3.6. The Hamiltonian under weakly nonlinear assumption in Boussinesq case in terms of $\eta_2;\tilde{\sigma}_B$. From (245),

$$T_{WNL} = \frac{\bar{\rho}}{2}\left(\eta_1\tilde{u}_1^2+\eta_2\tilde{u}_2^2+\frac{\epsilon^2}{3}\left[h_1^3\tilde{u}_1\tilde{u}_{1,xx}+h_2^3\tilde{u}_2\tilde{u}_{2,xx}\right]\right) \tag{261}$$

Compute the ϵ^2 term in T_{WNL}

$$\tilde{u}_2\tilde{u}_{2,xx} = \left(\frac{h-\eta_2}{h}\right)\left[\frac{\tilde{\sigma}_B}{\bar{\rho}} + O(\epsilon^2)\right].\left(\frac{1}{\bar{\rho}}\frac{h_1}{h}\tilde{\sigma}_{xx}\right)$$
$$(262) \qquad = \frac{1}{\rho^2}\frac{h_1(h-\eta_2)}{h^2}\tilde{\sigma}_B\tilde{\sigma}_{B,xx} + O(\epsilon^2)$$

$$\tilde{u}_1\tilde{u}_{1,xx} = \left(-\frac{\eta_2}{h-\eta_2}\tilde{u}_2 + \frac{\epsilon^2}{3}\frac{h_2}{h_1}.[h_1^2 - h_2^2]\tilde{u}_{2,xx}\right).\left(\frac{-h_2}{h_1}\tilde{u}_{2,xx}\right)$$
$$(263) \qquad = \frac{\eta_2 h_2}{h_1(h-\eta_2)}\tilde{u}_2\tilde{u}_{2,xx} + O(\epsilon^2)$$
$$= \frac{1}{\rho^2}\frac{h_2\eta_2}{h^2}\tilde{\sigma}_B.\tilde{\sigma}_{B,xx}$$

Thus

$$\frac{\epsilon^2}{3}[h_1^3\tilde{u}_1\tilde{u}_{1,xx} + h_2^3\tilde{u}_2\tilde{u}_{2,xx}]$$
$$= \frac{\epsilon^2}{3}\frac{1}{\rho^2}\tilde{\sigma}_B.\tilde{\sigma}_{B,xx}.\left[\frac{h_1^3h_2\eta_2}{h^2} + \frac{h_2^3h_1(h-\eta_2)}{h^2}\right]$$
$$(264) \qquad = \frac{\epsilon^2}{3}\frac{1}{\rho^2}\tilde{\sigma}_B.\tilde{\sigma}_{B,xx}\left(\frac{\eta_2h_1h_2(h_1^2-h_2^2)}{h^2} + \frac{h_2^3h_1h}{h^2}\right)$$
$$= \frac{\epsilon^2}{3}\frac{1}{\rho^2}\tilde{\sigma}_B.\tilde{\sigma}_{B,xx}\left(\frac{h_1h_2[\eta_2(h_1-h_2)+h_2^2]}{h}\right)$$
$$= \frac{\epsilon^2}{3}\frac{1}{\rho^2}\tilde{\sigma}_B.\tilde{\sigma}_{B,xx}\frac{h_1^2h_2^2}{h}$$

since we replace $\eta_2 = h_2$ in ϵ^2 term.
Now let's compute the other other in T_{WNL}

$$\eta_1\tilde{u}_1^2 = (h-\eta_2)\left(\left(\frac{\eta_2}{h-\eta_2}\right)^2\tilde{u}_2^2 - 2\frac{\epsilon^2}{3}\left(\frac{\eta_2}{h-\eta_2}\right)^2(h_1^2-h_2^2)\tilde{u}_2\tilde{u}_{2,xx} + O(\epsilon^4)\right)$$
$$(265) \qquad = \frac{\eta_2^2}{h-\eta_2}\tilde{u}_2^2 - 2\frac{\epsilon^2}{3}\frac{\eta_2^2}{h-\eta_2}(h_1^2-h_2^2)\tilde{u}_2\tilde{u}_{2,xx}$$

thus

(266)

$$\eta_1\tilde{u}_1^2 + \eta_2\tilde{u}_2^2 = \frac{h\eta_2}{h-\eta_2}\tilde{u}_2^2 - 2\frac{\epsilon^2}{3}\frac{\eta_2^2}{h-\eta_2}(h_1^2-h_2^2)\tilde{u}_2\tilde{u}_{2,xx}$$
$$= \frac{(h-\eta_2)\eta_2\tilde{\sigma}_B^2}{h\bar{\rho}^2} + \frac{2}{\bar{\rho}^2}\frac{\epsilon^2}{3}\frac{h_2(h_1-h_2)(h-\eta_2)\eta_2}{h}\tilde{\sigma}_B\tilde{\sigma}_{B,xx} - \frac{2}{\bar{\rho}^2}\frac{\epsilon^2}{3}\frac{\eta_2^2h_1(h_1-h_2)}{h}\tilde{\sigma}_B\tilde{\sigma}_{B,xx}$$
$$= \frac{(h-\eta_2)\eta_2\tilde{\sigma}_B^2}{h\bar{\rho}^2}$$

since the ϵ^2 term vanishes since we replace $\eta_2 \approx h_2$.
Thus, $T_{WNL} + V$ is obtained as

$$T_{WNL} + V = \frac{1}{2}\frac{(h-\eta_2)\eta_2\tilde{\sigma}_B^2}{\bar{\rho}h} + \frac{\epsilon^2}{6\bar{\rho}}\frac{h_1^2 h_2^2}{h}\tilde{\sigma}_B\tilde{\sigma}_{B,xx} + \frac{1}{2}g(\rho_2-\rho_1)\eta_2^2 \tag{267}$$

The above ϵ^2 term can be reduced when we take integration in x, so

$$\int_{\mathbb{R}} \tilde{\sigma}_B\tilde{\sigma}_{B,xx}dx = \tilde{\sigma}_B\tilde{\sigma}_{B,x}|_{-\infty}^{\infty} - \int_{\mathbb{R}} \tilde{\sigma}_{B,x}^2 dx = -\int_{\mathbb{R}} \tilde{\sigma}_{B,x}^2 dx \tag{268}$$

since $\tilde{\sigma}_B(x = \pm\infty) = \bar{\rho}(\tilde{u}_2(x = \pm\infty) - \tilde{u}_1(x = \pm\infty)) = 0$.
Thus, the Hamiltonian density is

$$\boxed{H_B = \frac{1}{2}\frac{(h-\eta_2)\eta_2}{\bar{\rho}h}\tilde{\sigma}_B^2 - \frac{\epsilon^2}{6\bar{\rho}}\frac{h_1^2 h_2^2}{h}\tilde{\sigma}_{B,x}^2 + \frac{1}{2}g(\rho_2-\rho_1)\eta_2^2} \tag{269}$$

Similar approach can be done in the non-Boussinesq case, however the relation between σ and $\tilde{u}_2$ will be longer, and the resulting Hamiltonian functional in the non-Boussinesq Weakly Nonlinear case is too.
In the next chapter, we consider a more general case than this Weakly Nonlinear non-Boussinesq case. We will first start with the non-Boussinesq Mildly Nonlinear case, then derive the Hamiltonian functional under Boussinesq approximation.

CHAPTER 8

Adding dispersion to the 2-layer case in the Mildly non linear assumption

In this chapter, we study the Hamiltonian structure for 2-layer dispersive stratified fluids under the Mildly Nonlinear assumption, in both the Boussinesq and non-Boussinesq case, whose results were submitted for peer-review recently.

1. Hamiltonian variables $\tilde{\sigma}$ and ζ

Following the previously mentioned model of Wu in chapter 4, we use the assumed bulk irrotationality of the fluid flow to introduce the bulk velocity potentials $\varphi_j(x,z)$, which we Taylor expand with respect to the vertical variable z. By the vanishing of the vertical velocity at the physical boundaries $z = h_1$, and $z = -h_2$ we obtain the Taylor expansions

$$\varphi_j(x,z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} H_j(z)^{2n} \partial_x^{2n} \varphi_{0\,j}(x) \tag{270}$$

where

$$H_1(z) = z - h_1, \quad H_2(z) = z + h_2\,, \tag{271}$$

and $\varphi_{0\,1}(x) = \varphi_1(x,h_1)$, $\varphi_{0\,2} = \varphi_2(x,-h_2)$ are the values of the potential at the rigid lids.

The horizontal velocities are then given by

$$u_j = \partial_x \varphi_j(x,z) = \sum_{j=0}^{\infty} \frac{(-1)^n}{(2n)!} H_j(z)^{2n} \partial_x^{2n} \partial_x \varphi_{0\,j}(x) = \sum_{j=0}^{\infty} \frac{(-1)^n}{(2n)!} H_j(z)^{2n} \partial_x^{2n} u_{0\,j}(x)\,, \tag{272}$$

$u_{0\,j}(x)$ being the horizontal velocities at $z = h_1$ (for $j = 1$) and at $z = -h_2$ (for $j = 2$).

Likewise, the vertical velocities are given by

$$w_j(x,z) = \partial_z \varphi_j(x,z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} H_j(z)^{2n+1} \partial_x^{2n+1} u_{0\,j}(x) \tag{273}$$

Notice that the boundary conditions $w_1(x,h_1) = w_2(x,-h_2) = 0$ are satisfied.

Since

$$H_1(\zeta) = -\eta_1, \quad H_2(\zeta) = \eta_2, \quad \text{i.e., } H_j(\zeta) = (-1)^j \eta_j,\ j = 1, 2\,, \tag{274}$$

where $\eta_1(x) = h_1 - \zeta(x)$ (resp. $\eta_2(x) = h_2 + \zeta(x)$) is the thickness of the upper (resp. lower) layer, the interface velocities can be directly obtained by formulas (272) and (273) as

$$\tilde{u}_j = \sum_{j=0}^{\infty} \frac{(-1)^n}{(2n)!} \eta_j^{2n} \partial_x^{2n} u_{0,j}(x)\,, \qquad \tilde{w}_j = (-1)^{j-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \eta_j^{2n+1} \partial_x^{2n+1} u_{0\,j}(x)\,. \tag{275}$$

For later use when needed, we also express (from the same formulas) the layer-mean horizontal velocities in terms of the fluid thicknesses and the (respective) boundary velocities as

$$\begin{aligned} \overline{u}_1(x) &\equiv \frac{1}{\eta_1} \int_{\zeta}^{h_1} u_1(x,z)\,\mathrm{d}z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \eta_1^{2n} \partial_x^{2n} u_{0\,1}(x) \\ \overline{u}_2(x) &\equiv \frac{1}{\eta_2} \int_{-h_2}^{\zeta} u_2(x,z)\,\mathrm{d}z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \eta_2^{2n} \partial_x^{2n} u_{0\,2}(x)\,. \end{aligned} \tag{276}$$

2. Mildly Nonlinear and Weakly Nonlinear Asymptotics assumption

In this section, we consider a mildly nonlinear assumption, which is a more general case than the Weakly Nonlinear asymptotic presented before.
Here, in the Mildly Nonlinear asymptotics, we assume

1. The interface displacement ζ will be understood to be scaled by its maximum value a to yield the amplitude nondimensional small parameters $\alpha = \dfrac{a}{h} \ll 1$. Thus, the non-dimensional fluid thickness η_j will be written as

$$\eta_j = h_j + (-1)^j \alpha\zeta \tag{277}$$

2. We shall make an asymptotic expansion in the small parameters α and ϵ and mainly consider the "Mildly Non-linear" (MNL) case, defined by the relative scalings $\epsilon^2 \ll \alpha \ll \epsilon$. we thus discard terms of order $\alpha\epsilon^2, \epsilon^3$ and higher, but retains terms of order α^2.

The usual Weakly Non-linear (WNL) case where $\alpha = O(\epsilon^2)$ in previous sections can be seen as a special case of the MNL case. The consequences of such MNL scaling limits are the following:

i) The slope ζ_x of the normalized interface is small and scales as $\frac{a}{h}\frac{h}{L} = O(\alpha\epsilon)$

ii) $\tilde{w}_j$ scales as ϵ and ζ_x scales as $\alpha\epsilon$, the Hamiltonian variable $\sigma = \rho_2\tilde{u}_2 - \rho_1\tilde{u}_1 + \zeta_x(\rho_2\tilde{w}_2 - \rho_1\tilde{w}_1)$ becomes

$$\sigma = \rho_2\tilde{u}_2 - \rho_1\tilde{u}_1 \tag{278}$$

iii) Approximate dynamical constraint gets simplified and reads

$$\eta_1\tilde{u}_1 + \eta_2\tilde{u}_2 + \frac{\epsilon^2}{3}(h_1^3\tilde{u}_{1xx} + h_2^3\tilde{u}_{2xx}) = O(\epsilon^4) \tag{279}$$

iv) The total kinetic energy density becomes

$$T = T_1 + T_2 = \frac{h}{2}\left(\rho_1\left(\eta_1\tilde{u}_1^2 + \frac{\epsilon^2}{3}h_1^3\tilde{u}_1\tilde{u}_{1xx}\right) + \rho_2\left(\eta_2\tilde{u}_2^2 + \frac{\epsilon^2}{3}h_2^3\tilde{u}_2\tilde{u}_{2xx}\right)\right) \tag{280}$$

3. Hamiltonian functional in Mildly Nonlinear and Non-Boussinesq case- (ABC) system

Kinetic energy density

$$T = \frac{h}{2}\left(\rho_1\left(\eta_1\tilde{u}_1^2 + \frac{\epsilon^2}{3}h_1^3\tilde{u}_1\tilde{u}_{1xx}\right) + \rho_2\left(\eta_2\tilde{u}_2^2 + \frac{\epsilon^2}{3}h_2^3\tilde{u}_2\tilde{u}_{2xx}\right)\right) \tag{281}$$

Recall the relations that we have

(282)

$$\tilde{u}_1 = -\frac{\eta_2}{\eta_1}\tilde{u}_2 + \frac{\epsilon^2}{3}\frac{h_2}{h_1}(h_1^2 - h_2^2)\tilde{u}_{2xx}$$

$$\tilde{u}_{1xx} = -\frac{h_2}{h_1}\tilde{u}_{2xx}$$

$$\tilde{u}_1\eta_1 + \tilde{u}_2\eta_2 + \frac{1}{3}\epsilon^2\tilde{u}_{2xx}h_2(h_2^2 - h_1^2) = 0 \text{ (as in the above weak dynamical constraint)}$$

and

(283)

$$\tilde{\sigma} = \rho_2\tilde{u}_2 - \rho_1\tilde{u}_1 + \zeta_x(\rho_2\tilde{w}_2 - \rho_1\tilde{w}_1) = \rho_2\tilde{u}_2 - \rho_1\tilde{u}_1$$

(within this mildly nonlinear asymptotics, and the same as in dispersionless case)

$$\Rightarrow \tilde{u}_1 = -\frac{\sigma\eta_2}{\eta_1\rho_2 + \eta_2\rho_1} + \frac{\epsilon^2}{3}\frac{\tilde{u}_{2xx}h_2\rho_2(h_1^2 - h_2^2)}{\eta_1\rho_2 + \eta_2\rho_1}$$

$$\Rightarrow \tilde{u}_2 = \frac{\sigma\eta_1}{\eta_1\rho_2 + \eta_2\rho_1} + \frac{\epsilon^2}{3}\frac{\tilde{u}_{2xx}h_2\rho_1(h_1^2 - h_2^2)}{\eta_1\rho_2 + \eta_2\rho_1}$$

$$\sigma_{xx} = \rho_2\tilde{u}_{2xx} - \rho_1\tilde{u}_{1xx}$$

$$\Rightarrow \tilde{u}_{2xx} = \frac{\sigma_{xx}h_1}{h_1\rho_2 + h_2\rho_1}$$

$$\Rightarrow \tilde{u}_{1xx} = \frac{1}{\rho_1}\left(\rho_2\frac{\sigma_{xx}h_1}{h_1\rho_2 + h_2\rho_1} - \sigma_{xx}\right)$$

From the relation between $\tilde{u}_1$ and $\tilde{u}_2$, $\tilde{u}_2$ and σ, we arrive at the relation

(284)
$$\begin{aligned}\tilde{u}_1 &= -\frac{\sigma\eta_2}{\eta_1\rho_2 + \eta_2\rho_1} + \rho_2\frac{\epsilon^2}{3}\frac{h_1h_2(h_1^2 - h_2^2)}{(h_1\rho_2 + h_2\rho_1)(\eta_1\rho_2 + \eta_2\rho_1)}\sigma_{xx}\\ \tilde{u}_2 &= \frac{\sigma\eta_1}{\eta_1\rho_2 + \eta_2\rho_1} + \rho_1\frac{\epsilon^2}{3}\frac{h_1h_2(h_1^2 - h_2^2)}{(h_1\rho_2 + h_2\rho_1)(\eta_1\rho_2 + \eta_2\rho_1)}\sigma_{xx}\end{aligned}$$

We have the relations

(285)
$$\begin{aligned}\frac{\epsilon^2}{3}h_1^3\tilde{u}_1\tilde{u}_{1xx} &= \frac{\epsilon^2}{3}h_1^3\left(-\frac{\sigma\eta_2}{\eta_1\rho_2 + \eta_2\rho_1} + O(\epsilon^2)\right)\left(\frac{\rho_2\sigma_{xx}h_1}{h_1\rho_2 + h_2\rho_1}\right)\\ &= \frac{\epsilon^2}{3}h_1^3\left(-\frac{h_1\rho_2}{h_1\rho_2 + h_2\rho_1}.\frac{\eta_2}{\eta_1\rho_2 + \eta_2\rho_1} + \frac{\eta_2}{\eta_1\rho_2 + \eta_2\rho_1}\right)\sigma\sigma_{xx}\end{aligned}$$

Recall the relation

(286)
$$\frac{\eta_2}{\eta_1\rho_2 + \eta_2\rho_1} = \frac{h_2}{h_1\rho_2 + h_2\rho_1} + O(\alpha)$$

Thus,

(287)
$$\begin{aligned}\frac{\epsilon^2}{3}h_1^3\tilde{u}_1\tilde{u}_{1xx} &= \frac{\epsilon^2}{3}h_1^3\left(-\frac{h_1\rho_2}{h_1\rho_2 + h_2\rho_1}.\frac{h_2}{h_1\rho_2 + h_2\rho_1} + \frac{h_2}{h_1\rho_2 + h_2\rho_1}\right)\sigma\sigma_{xx}\\ &= \frac{\epsilon^2}{3}\frac{h_1^3h_2^2\rho_1}{(h_1\rho_2 + h_2\rho_1)^2}\end{aligned}$$

Similarly, we have

(288)
$$\frac{\epsilon^2}{3}h_2^3\tilde{u}_2\tilde{u}_{2xx} = \frac{\epsilon^2}{3}\frac{h_2^3h_1^2\rho_2}{(h_1\rho_2 + h_2\rho_1)^2}$$

Thus, we obtain the ϵ^2 term in the kinetic energy density

$$\frac{\epsilon^2}{3}h_1^3\tilde{u}_1\tilde{u}_{1xx} + \frac{\epsilon^2}{3}h_2^3\tilde{u}_2\tilde{u}_{2xx} = \frac{h_1^2h_2^2(h_1\rho_1 + h_2\rho_2)}{(h_1\rho_2 + h_2\rho_1)^2} \tag{289}$$

We have

(290)

$$\begin{aligned}
\rho_1\eta_1\tilde{u}_1^2 + \rho_2\eta_2\tilde{u}_2^2 &= \rho_1\eta_1\left(\frac{\sigma^2\eta_2^2}{(\eta_1\rho_2 + \eta_2\rho_1)^2} - 2\rho_2\frac{\epsilon^2}{3}\frac{\eta_2h_1h_2(h_1^2 - h_2^2)}{(h_1\rho_2 + h_2\rho_1)(\eta_1\rho_2 + \eta_2\rho_1)^2}\sigma\sigma_{xx} + O(\epsilon^4)\right)\\
&\quad + \rho_2\eta_2\left(\frac{\sigma^2\eta_1^2}{(\eta_1\rho_2 + \eta_2\rho_1)^2} + +2\rho_1\frac{\epsilon^2}{3}\frac{\eta_1h_1h_2(h_1^2 - h_2^2)}{(h_1\rho_2 + h_2\rho_1)(\eta_1\rho_2 + \eta_2\rho_1)^2}\sigma\sigma_{xx} + O(\epsilon^4)\right)\\
&= \frac{\sigma^2\eta_1\eta_2(\rho_1\eta_2 + \rho_2\eta_1)}{(\eta_1\rho_2 + \eta_2\rho_1)^2} + O(\epsilon^4)\\
&= \frac{\eta_1\eta_2\sigma^2}{\rho_2\eta_1 + \rho_1\eta_2} + O(\epsilon^4)
\end{aligned}$$

Substituting these relations to the kinetic energy density above leads to the following expression

$$T = \frac{h}{2}\left(\frac{\eta_1\eta_2\sigma^2}{\rho_2\eta_1 + \rho_1\eta_2} + \frac{\epsilon^2}{3}\frac{h_1^2h_2^2(h_1\rho_1 + h_2\rho_2)}{(h_1\rho_2 + h_2\rho_1)^2}\sigma\sigma_{xx}\right) \tag{291}$$

Next, the first term σ^2 of the kinetic must be expanded in powers of α. We have

(292)

$$\begin{aligned}
\frac{\eta_1\eta_2}{\rho_2\eta_1 + \rho_1\eta_2} &= \frac{(h_1 - \alpha\zeta)(h_2 + \alpha\zeta)}{\rho_1h_2 + \rho_2h_1 - \rho_\Delta\alpha\zeta}\\
&= \frac{(h_1 - \alpha\zeta)(h_2 + \alpha\zeta)}{\phi\left(1 - \frac{\rho_\Delta\alpha\zeta}{\phi}\right)}, \qquad \text{for } \phi = \rho_1h_2 + \rho_2h_1 \qquad \rho_\Delta = \rho_2 - \rho_1\\
&= \frac{h_1h_2 + \alpha\zeta(h_1 - h_2) - \alpha^2\zeta^2}{\phi}\left[1 + \frac{\rho_\Delta\alpha\zeta}{\phi} + \frac{\rho_\Delta^2\alpha^2\zeta^2}{\phi^2} + \frac{\rho_\Delta^3\alpha^3\zeta^3}{\phi^3}\right]\\
&= \frac{h_1h_2}{\phi} + \frac{\alpha\zeta}{\phi^2}\left(\phi(h_1 - h_2) + h_1h_2\rho_\Delta\right) - \frac{\alpha^2\zeta^2}{\phi^3}\left(\phi^2 - \rho_\Delta(h_1 - h_2)\phi - h_1h_2\rho_\Delta^2\right)\\
&= \frac{h_1h_2}{\phi} + \frac{\alpha\zeta}{\phi^2}(\rho_2h_1^2 - \rho_1h_2^2) - \frac{\alpha^2\zeta^2}{\phi^3}(\rho_1\rho_2h^2)
\end{aligned}$$

Thus, the kinetic energy density is written in powers of α as

$$\begin{aligned}
T &= h\left(\frac{1}{2}\frac{h_1h_2}{h_1\rho_2 + h_2\rho_1}\sigma^2 + \frac{\alpha}{2}\frac{h_1^2\rho_2 - h_2^2\rho_1}{(h_1\rho_2 + h_2\rho_1)^2}\zeta\sigma^2 - \frac{\alpha^2}{2}\frac{h^2\rho_1\rho_2}{(h_1\rho_2 + h_2\rho_1)^3}\zeta^2\sigma^2\right)\\
&\quad + h\left(\frac{\epsilon^2}{6}\frac{h_1^2h_2^2(h_1\rho_1 + h_2\rho_2)}{(h_1\rho_2 + h_2\rho_1)^2}\sigma\sigma_{xx}\right)
\end{aligned} \tag{293}$$

and the total energy density at this order is

$$\begin{aligned} E = h\left(\frac{1}{2}\frac{h_1h_2}{h_1\rho_2+h_2\rho_1}\sigma^2 + \frac{\alpha}{2}\frac{h_1^2\rho_2-h_2^2\rho_1}{(h_1\rho_2+h_2\rho_1)^2}\zeta\sigma^2 - \frac{\alpha^2}{2}\frac{h^2\rho_1\rho_2}{(h_1\rho_2+h_2\rho_1)^3}\zeta^2\sigma^2\right) \\ + h\left(\frac{\epsilon^2}{6}\frac{h_1^2h_2^2(h_1\rho_1+h_2\rho_2)}{(h_1\rho_2+h_2\rho_1)^2}\sigma\sigma_{xx}\right) + \frac{1}{2}h^2g(\rho_2-\rho_1)\zeta^2 \end{aligned} \tag{294}$$

It is convenient to introduce the non-dimensional momentum shear $\sigma^\star$ by

$$\sigma = \sqrt{hg(\rho_2-\rho_1)(h_1\rho_2+h_2\rho_1)}\sigma^\star \tag{295}$$

so that the non-dimensional form of the total energy is (dropping asterisks for ease of notation)

$$\begin{aligned} E &= \frac{1}{2}h_1h_2\sigma^2 + \frac{\alpha}{2}\frac{h_1^2\rho_2-h_2^2\rho_1}{(h_1\rho_2+h_2\rho_1)}\zeta\sigma^2 - \frac{\alpha^2}{2}\frac{h^2\rho_1\rho_2}{(h_1\rho_2+h_2\rho_1)^2}\zeta^2\sigma^2 \\ &\quad + \left(\frac{\epsilon^2}{6}\frac{h_1^2h_2^2(h_1\rho_1+h_2\rho_2)}{(h_1\rho_2+h_2\rho_1)}\sigma\sigma_{xx}\right) + \frac{1}{2}\zeta^2 \\ &= \frac{1}{2}(A\sigma^2 + \alpha B\zeta\sigma^2 - \alpha^2 C\zeta^2\sigma^2 + \zeta^2 + \epsilon^2\kappa\sigma\sigma_{xx}) \end{aligned} \tag{296}$$

where we denote

$$A = h_1h_2, \qquad B = \frac{h_1^2\rho_2-h_2^2\rho_1}{h_1\rho_2+h_2\rho_1}, \qquad C = \frac{h^2\rho_1\rho_2}{(h_1\rho_2+h_2\rho_1)^2}, \qquad \kappa = \frac{1}{3}\frac{h_1^2h_2^2(h_1\rho_1+h_2\rho_2)}{h_1\rho_2+h_2\rho_1} \tag{297}$$

where total height h in the constant C is scaled to equal 1, so that it can be dropped in the formula.

3.1. The ensuing equations of motion (ABC-system). Applying the Poisson tensor to the variational differential of the energy $\mathcal{E} = \int E dx$ yields the equations of motion as

$$\begin{pmatrix} \zeta \\ \sigma \end{pmatrix}_t = -\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta\mathcal{E}}{\delta\zeta} \\ \frac{\delta\mathcal{E}}{\delta\sigma} \end{pmatrix}$$

where t is the non-dimensional time, related with the physical time by

$$t \to \epsilon\sqrt{\frac{g(\rho_2-\rho_1)}{h(h_1\rho_2+h_2\rho_1)}}t \tag{298}$$

The resulting system in conservation form is

$$\begin{aligned} \zeta_t + (A\sigma + \alpha B\zeta\sigma - \alpha^2 C\zeta^2\sigma + \epsilon^2\kappa\sigma_{xx})_x &= 0 \\ \sigma_t + (\zeta + \alpha\frac{B\sigma^2}{2} - \alpha^2 C\zeta\sigma^2)_x &= 0 \end{aligned} \tag{299}$$

or carrying out the spatial differentiations explicitly,

$$\zeta_t + A\sigma_x + \alpha B(\zeta\sigma)_x - \alpha^2 C(\zeta^2\sigma)_x + \epsilon^2\kappa\sigma_{xxx} = 0 \tag{300}$$
$$\sigma_t + \zeta_x + \alpha B\sigma\sigma_x - \alpha^2 C(\zeta\sigma^2)_x = 0$$

which from now on will be referred to as the ABC-system.

Remark 8.1. A few comments on the parameters A,B,C and their relations with the physical parameters ρ_1, ρ_2, h_1, h_2. First, the parameter A is just the square of the linear wave velocity , in nondimensional form it ranges from 0 to $\frac{1}{4}$. Next, note that the parameter κ is non-negative, and vanishes only when $h_1 \to 0$ or $h_2 \to 0$. Similarly, the parameter C is non-negative, and vanishes only in the air-water limit $\rho_1 \to 0$. The most interesting parameter is B, which is non sign definite and appears in front of the cubic term $\sigma^2\zeta$ of the Hamiltonian. It vanishes at the critical ratio

$$\frac{\rho_1}{\rho_2} = \frac{h_1^2}{h_2^2} \tag{301}$$

△

3.2. The associated conserved quantities. From theorem 4.1 in [**18**], we can quickly show that the Hamiltonian formulation of the 300 system provides three additional constants of the motions besides the energy E. They are the two Casimir functionals

$$K_1 = \int_{\mathbb{R}} \zeta dx, \qquad K_2 = \int_{\mathbb{R}} \sigma dx \tag{302}$$

and the generator of the x-translation

$$\Pi = \int_{\mathbb{R}} \zeta\sigma dx \tag{303}$$

They are conserved quantities for any choice of the parameters A,B and C.

4. Hamiltonian functional under Boussinesq approximation and Midly-Nonlinear assumption

Starting from the above total energy density 296, now we add Boussinesq approximation, i.e retaining density differences in the potential energy while neglecting the associated inertial differences in the kinetic density of 296 by setting $\rho_1 = \rho_2 = \bar{\rho}$.
This Boussinesq approximation simplifies significantly the weakly or mildly nonlinear asymptotics; indeed variable for weighted shear reduces to $\sigma = \bar{\rho}(\tilde{u}_2 - \tilde{u}_1)$,

and the non-dimensional energy density of the system becomes

$$
\begin{aligned}
E_B &= \frac{1}{2}h_1h_2\sigma^2 + \frac{\alpha}{2}(h_1-h_2)\zeta\sigma^2 - \frac{\alpha^2}{2}\zeta^2\sigma^2 + \frac{\epsilon^2}{4}h_1^2h_2^2\sigma\sigma_{xx} + \frac{1}{2}\zeta^2 \\
&= \frac{1}{2}(A_B\sigma^2 + \alpha B_B\zeta\sigma^2 - \alpha^2 C_B\zeta^2\sigma^2 + \zeta^2 + \epsilon^2\kappa_B\sigma\sigma_{xx})
\end{aligned} \tag{304}
$$

where

$$
A_B = h_1h_2, \qquad B_B = h_1 - h_2, \qquad C_B = 1, \qquad \kappa_B = \frac{1}{3}h_1^2h_2^2 \tag{305}
$$

5. Linearization and dispersion relation of the ABC-system

Recall the (ABC-system) (300)

$$
\begin{aligned}
&\zeta_t + A\sigma_x + \alpha B(\zeta\sigma)_x - \alpha^2 C(\zeta^2\sigma)_x + \epsilon^2\kappa\sigma_{xxx} = 0 \\
&\sigma_t + \zeta_x + \alpha B\sigma\sigma_x - \alpha^2 C(\zeta\sigma^2)_x = 0
\end{aligned} \tag{306}
$$

Linearize the ABC-system of evolutionary equations (300) around the constant solution $\zeta = Z + z(x,t)$ and $\sigma = S + s(x,t)$ yields

$$
\begin{aligned}
&z_t + As_x + \alpha B(Zs_x + Sz_x) - \alpha^2 C(Z^2 s_x + 2SZz_x) + \epsilon^2\kappa s_{xxx} = 0 \\
&s_t + z_x + \alpha BS s_x - \alpha^2 C(S^2 z_x + 2SZs_x) = 0
\end{aligned} \tag{307}
$$

(Notice that we omit small terms like ss_x and zs_x).
written in matrix form

$$
\begin{pmatrix} z \\ s \end{pmatrix}_t + \begin{pmatrix} \alpha BS - 2\alpha^2 CSZ & A + \alpha BZ - \alpha^2 CZ^2 + \epsilon^2\kappa\partial_{xx} \\ 1 - \alpha^2 CS^2 & \alpha BS - 2\alpha^2 CSZ \end{pmatrix} \begin{pmatrix} z \\ s \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{308}
$$

Looking for the wave solutions of the form $(z,s) = (a_z, a_s)e^{i(kx-\omega t)}$, we thus have

$$
(-i\omega)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_z \\ a_s \end{pmatrix} e^{i(kx-\omega t)} + (ik)\begin{pmatrix} \alpha BS - 2\alpha^2 CSZ & A + \alpha BZ - \alpha^2 CZ^2 - \epsilon^2\kappa k^2 \\ 1 - \alpha^2 CS^2 & \alpha BS - 2\alpha^2 CZS \end{pmatrix}\begin{pmatrix} a_z \\ a_s \end{pmatrix} e^{i(kx-\omega t)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{309}
$$

and denoting $c_p = \omega/k$, we thus have

$$
\begin{pmatrix} \alpha BS - 2\alpha^2 CSZ - c_p & A + \alpha BZ - \alpha^2 CZ^2 - \epsilon^2\kappa k^2 \\ 1 - \alpha^2 CS^2 & \alpha BS - 2\alpha^2 CZS - c_p \end{pmatrix}\begin{pmatrix} a_z \\ a_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \forall \begin{pmatrix} a_z \\ a_s \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{310}
$$

Therefore,

$$
det\begin{pmatrix} \alpha BS - 2\alpha^2 CSZ - c_p & A + \alpha BZ - \alpha^2 CZ^2 - \epsilon^2\kappa k^2 \\ 1 - \alpha^2 CS^2 & \alpha BS - 2\alpha^2 CZS - c_p \end{pmatrix} = 0 \tag{311}
$$

i.e

$$
c_p^2 - 2(\alpha BS - 2\alpha^2 CSZ)c_p + (\alpha BS - 2\alpha^2 CSZ)^2 - (1 - \alpha^2 CS^2)(A + \alpha BZ - \alpha^2 CZ^2 - \epsilon^2\kappa k^2) = 0 \tag{312}
$$

Solve for $c_p = \omega/k$ gives the dispersion relation

$$c_p = \alpha(BS - 2\alpha CSZ) \pm \sqrt{(1-\alpha^2 CS^2)(A + \alpha BZ - \alpha^2 CZ^2 - \epsilon^2\kappa k^2)} \tag{313}$$

whereby the critical threshold wavenumber

$$k_c^2 = (A + \alpha BZ - \alpha^2 CZ^2)/(\epsilon^2\kappa) \tag{314}$$

is identified.
Notice that when k is sufficiently large ($k > k_c$), c_P becomes a complex solution, which makes the system ill-posed.
Note that the factor $1 - \alpha^2 CS^2$ needs to be positive.

5.1. Shift of the dependent variable σ. In this section, we consider making a shift of the dependent variable σ, which will result in the positive wavenumber c_P in the end.
We have

$$\begin{aligned} &\bar{\sigma} = \sigma + \epsilon^2\frac{\kappa}{A}\bar{\sigma}_{xxx} \\ &\Rightarrow \sigma = \bar{\sigma} - \epsilon^2\frac{\kappa}{A}\bar{\sigma}_{xx} \\ &\Rightarrow A\sigma_x = A\bar{\sigma}_x - \epsilon^2\kappa\bar{\sigma}_{xxx} \\ &\text{and } \sigma_t = \bar{\sigma}_t - \epsilon^2\bar{\kappa}\bar{\sigma}_{xxt} \end{aligned} \tag{315}$$

where

$$\bar{\kappa} = \kappa/A \tag{316}$$

As hinted by the notation used, this step is equivalent to that of using layer averaged velocities, in defining the density weighted vorticity, instead of the velocities at the interface between layers.
Thus the above ABC-system becomes

$$\begin{aligned} &\zeta_t + A\bar{\sigma}_x + \alpha B(\zeta\sigma)_x - \alpha^2 C(\zeta^2\sigma)_x = 0 \\ &\bar{\sigma}_t + \zeta_x + \alpha B\bar{\sigma}\bar{\sigma}_x - \alpha^2 C(\zeta\bar{\sigma}^2)_x - \epsilon^2\bar{\kappa}\bar{\sigma}_{xxt} = 0 \end{aligned} \tag{317}$$

Note that $-\bar{\sigma}_{xxt} = \bar{\sigma}_{xxx} + O(\alpha)$, so

$$\begin{aligned} &\zeta_t + A\bar{\sigma}_x + \alpha B(\zeta\sigma)_x - \alpha^2 C(\zeta^2\sigma)_x = 0 \\ &\bar{\sigma}_t + \zeta_x + \alpha B\bar{\sigma}\bar{\sigma}_x - \alpha^2 C(\zeta\bar{\sigma}^2)_x + \epsilon^2\bar{\kappa}\bar{\sigma}_{xxx} = 0 \end{aligned} \tag{318}$$

From now, we can replace $\bar{\kappa}$ by κ for convenience.
The dispersion relation for system (318) linearized around constant states $\sigma = S$

and $\zeta = Z$ is obtained from modifying (310)

(319)
$$\begin{pmatrix} \alpha BS - 2\alpha^2 CSZ - c_p & A + \alpha BZ - \alpha^2 CZ^2 \\ 1 - \alpha^2 CS^2 & \alpha BS - 2\alpha^2 CZS - c_p(1 + \epsilon^2 \kappa k^2) \end{pmatrix} \begin{pmatrix} a_z \\ a_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \forall \begin{pmatrix} a_z \\ a_s \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The $det = 0$ gives the relation

$$(1 + \epsilon^2 \kappa k^2) c_P^2 - q_1 (2 + \epsilon^2 \kappa k^2) c_P - q_2 + q_1^2 = 0 \tag{320}$$

where we have introduced the shorthand notation

$$q_1 = \alpha BS - 2\alpha^2 CSZ; \qquad q_2 = (A + \alpha BZ - \alpha^2 CZ^2)(1 - \alpha^2 CS^2) \tag{321}$$

The discriminant of (320) is

$$\Delta = q_1^2 (2 + \epsilon^2 \kappa k^2)^2 + 4(1 + \epsilon^2 \kappa k^2)(q_2 - q_1^2) \tag{322}$$

For the case of zero constant states $Z = S = 0$,

- In the ABC-system (300), the dispersion relation c_P is

$$c_P^2 = A - \epsilon^2 \kappa k^2 \tag{323}$$

- In the ABC-system with a shift of σ (318), the dispersion relation c_P is

$$c_P^2 = \frac{A}{1 + \epsilon^2 \kappa k^2} \tag{324}$$

Unlike the dispersion relation (313) in which c_P could be a complex number when the wavenumber k is sufficiently large, here by a shift of the dependent variable σ, we obtain the dispersion relation c_P which is always positive.

6. Travelling wave solutions of ABC system

The ABC-system rewritten, setting $\alpha = \epsilon = 1$ are

$$\begin{aligned} &\zeta_t + (A\sigma + B\zeta\sigma - C\zeta^2\sigma + \kappa\sigma_{xx})_x = 0 \\ &\sigma_t + \left(\zeta + \frac{B}{2}\sigma^2 - C\zeta\sigma^2\right)_x = 0 \end{aligned} \tag{325}$$

Via the traveling wave ansatz $\zeta(x, t) = \zeta(x - ct); \sigma(x, t) = \sigma(x - ct)$ become

$$\begin{aligned} &[-c\zeta + (A\sigma + B\zeta\sigma - C\zeta^2\sigma + \kappa\sigma_{xx})]' = 0 \\ &[-c\sigma + \left(\zeta + \frac{B}{2}\sigma^2 - C\zeta\sigma^2\right)]' = 0 \end{aligned} \tag{326}$$

Thus,

$$\begin{aligned} &- c\zeta + (A\sigma + B\zeta\sigma - C\zeta^2\sigma + \kappa\sigma_{xx})] = K_1 const \\ &- c\sigma + \left(\zeta + \frac{B}{2}\sigma^2 - C\zeta\sigma^2\right)] = K_2 const \end{aligned} \tag{327}$$

We seek solitary wave solutions propagating into a quiescent state , i.e $\zeta \to 0; \sigma \to 0$ as $x \to \pm\infty$, thus the constants $K_1 = 0 = K_2$. The second equation above yields the relation between ζ and σ,

$$\zeta = \frac{\sigma(B\sigma/2 - c)}{C\sigma^2 - 1} \tag{328}$$

the first equation of (327) gives the Newton-like equation

$$\begin{aligned} \kappa\sigma_{xx} &= c\zeta - A\sigma - B\zeta\sigma + C\zeta^2\sigma = F(\sigma) = -\frac{dU}{d\sigma} \\ &= -A\sigma - \frac{\sigma(B\sigma - 2c)\left(B\sigma\left(C\sigma^2 - 2\right) + 2c\right)}{4\left(C\sigma^2 - 1\right)^2} \end{aligned} \tag{329}$$

for $\zeta(\sigma) = \frac{\sigma(B\sigma/2-c)}{C\sigma^2-1}$.
Thus, potential U is

$$U(\sigma) = -\int F(\sigma)d\sigma = -\frac{4AC^3\sigma^4 - 4AC^2\sigma^2 + B^2C^2\sigma^4 - B^2C\sigma^2 + B^2 - 4BcC^2\sigma^3 + 4c^2C}{8C^2\left(C\sigma^2 - 1\right)} \tag{330}$$

$$U(0) = -\frac{B^2 + 4c^2C}{8C^2} \tag{331}$$

Normalize the potential as U(0)=0 (by just define $U = -\int F(\sigma)d\sigma - U(0)$, this way, potential U becomes more simple

$$U(\sigma) = \sigma^2 \frac{\left(4A\left(C\sigma^2 - 1\right) + (B\sigma - 2c)^2\right)}{8(C\sigma^2 - 1)} \tag{332}$$

Below is the plot of potential $U(\sigma)$ in the case of different values of parameters A,B,C. When parameters $A = 0.154, B = 0.623, C = 0.987$(from [**1**]) , the critical value of the speed c (the value for having kinks) is $c_k \simeq \pm 0.5023049708$ in the region $\sigma \in [-0.7, 0.05]$. When the wave speed $c < 0$, we obtain waves traveling to the left, and $c > 0$ when waves travel to the right.

For values of $|c|$ slightly less than c_k,for example when $c = -0.502$, we obtain a soliton

When $|c| = |c_k|$, U has a second double zero (one zero is at $\sigma = 0$) and obtain a kink solution.

When $|c| > c_k$, the second maximum value of U at $\sigma \neq 0$ is below 0, meaning that the solutions stemming from 0 are no more bounded (from the corresponding phase portrait). For example, when $c = -0.503$, we have

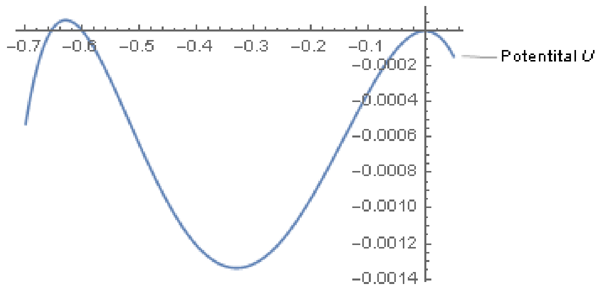

FIGURE 1. Plot of potential U(σ) when $A = 0.154, B = 0.623, C = 0.987, |c| = |-0.502| < c_k$

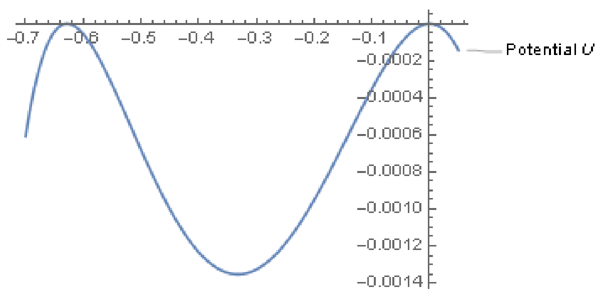

FIGURE 2. Plot of potential U(σ) when $A = 0.154, B = 0.623, C = 0.987, c = c_k = -0.5023049708$

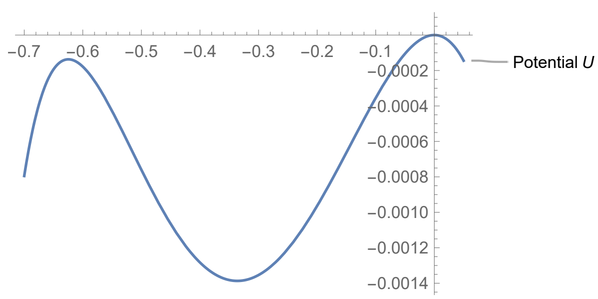

FIGURE 3. Plot of potential U(σ) when $A = 0.154, B = 0.623, C = 0.987, |c| = |-0.503| > c_k$

7. Unidirectional waves

To obtain unidirectional waves equations for our model, we at first observe that the rescaling $\sigma \to \sqrt{A}\sigma$ simplifies the Hamiltonian density (296) to

$$\tilde{H} = \frac{1}{2}(\sigma^2 + \zeta^2 + \alpha\tilde{B}\zeta\sigma^2 - \alpha^2\tilde{C}\zeta^2\sigma^2 + \epsilon^2\tilde{\kappa}\sigma\sigma_{xx}) \tag{333}$$

with $\tilde{B} = \dfrac{B}{A}$ and etc.
The resulting Hamiltonian equations are

$$\begin{aligned} \zeta_t + \sigma_x + \alpha B(\zeta\sigma)_x - C\alpha^2(\zeta^2\sigma)_x + \epsilon^2\kappa\sigma_{xxx} &= 0 \\ \sigma_t + \zeta_x + \alpha B\left(\frac{\sigma^2}{2}\right)_x - C\alpha^2(\zeta\sigma^2)_x &= 0 \end{aligned} \tag{334}$$

For unidirectional waves in the right direction, we can decompose (following the classical step of [**17**] chapter 13)

$$\zeta = \sigma + \alpha F + \alpha^2 G + \epsilon^2 M \tag{335}$$

Substituting this ζ into the above system of equations, we get

(336)
$$\begin{aligned} \sigma_t + \sigma_x + \alpha F_t + \alpha^2 G_t + \epsilon^2 M_t + \alpha B[(\sigma + \alpha F)\sigma]_x - C\alpha^2(\sigma^3)_x + \epsilon^2\kappa\sigma_{xxx} &= 0 \\ \sigma_t + F_x + \alpha B\left(\frac{\sigma^2}{2}\right)_x - C\alpha^2(\sigma^3)_x + \alpha F_x + \alpha^2 G_x + \epsilon^2 M_x &= 0 \end{aligned}$$

7.1. At order 1, O(1). At order 1, O(1),(not taking α, ϵ terms into consideration), we then have

$$\sigma_t + \sigma_x = 0 \tag{337}$$

7.2. At order α. At order α, we have

$$\begin{aligned} \sigma_t + \sigma_x + \alpha F_t + 2\alpha B\sigma\sigma_x &= 0 \\ \sigma_t + \sigma_x + \alpha B\sigma\sigma_x + \alpha F_x &= 0 \end{aligned} \tag{338}$$

Since $\eta_t = -\eta_x + O(\alpha)$, we thus have

$$\begin{aligned} \sigma_t + \sigma_x - \alpha F_x + 2\alpha B\sigma\sigma_x &= 0 \\ \sigma_t + \sigma_x + \alpha(B\sigma\sigma_x + F_x) &= 0 \\ \Rightarrow -F_x + 2B\sigma\sigma_x &= B\sigma\sigma_x + F_x \\ \Rightarrow F_x &= \frac{B\sigma\sigma_x}{2} \end{aligned} \tag{339}$$

$$\Rightarrow F = \frac{1}{4}B\sigma^2 + K \tag{340}$$

Therefore,

$$\begin{aligned} \sigma_t + \sigma_x - \frac{\alpha}{2}B\sigma\sigma_x + 2\alpha B\sigma\sigma_x &= 0 \\ \Rightarrow \sigma_t + \sigma_x + \frac{3\alpha}{2}B\sigma\sigma_x &= 0 \end{aligned} \tag{341}$$

$$\Rightarrow \partial_t = -\partial_x - \frac{3}{2}\alpha B\sigma\partial_x \tag{342}$$

7.3. At order α^2.

(343)

$$\sigma_t + \sigma_x + \alpha F_t + \alpha^2 G_t + \epsilon^2 M_t + \alpha(B\sigma^2)_x + \alpha^2 B(F\sigma)_x - C\alpha^2(\sigma^3)_x + \epsilon^2\kappa\sigma_{xxx} = 0$$

$$\sigma_t + \sigma_x + \alpha F_x + \alpha\left(\frac{B\sigma^2}{2}\right)_x + \alpha^2(G_x - C(\sigma^3)_x) + \epsilon^2 M_x = 0$$

Substituting (340) and (342) ,

(344)

$$\sigma_t + \sigma_x - \frac{\alpha}{2}B\sigma\sigma_x - \frac{3}{4}\alpha^2 B^2\sigma^2\sigma_x - \alpha^2 G_x - \epsilon^2 M_x + 2\alpha B\sigma\sigma_x + \alpha^2 B\left[\sigma^2\frac{B}{2}\sigma_x + \frac{1}{4}B\sigma^2\sigma_x + K\sigma_x\right]$$

$$- C\alpha^2(\sigma^3)_x + \epsilon^2 M\sigma_{xxx} = 0$$

$$\sigma_t + \sigma_x + \frac{\alpha}{2}B\sigma\sigma_x + \alpha B\sigma\sigma_x + \alpha^2 G_x - \alpha^2 C(\sigma^3)_x + \epsilon^2 M_x = 0$$

- α term: $\frac{3}{2}\alpha B\sigma\sigma_x = \frac{3}{2}\alpha B\sigma\sigma_x$
- α^2 term: $-G_x + BK\sigma_x - C(\sigma^3)_x = G_x - C(\sigma^3)_x$

$$\Rightarrow 2G_x = BK\sigma_x \Rightarrow G = \frac{BK\sigma}{2} \tag{345}$$

- ϵ^2 term: $-M_x + \kappa\sigma_{xxx} = M_x$

$$\Rightarrow 2M_x = \kappa\sigma_{xxx} \Rightarrow M = \kappa\sigma_{xx}/2 \tag{346}$$

Thus, we obtain the link between ζ and σ as

$$\zeta(\sigma) = \sigma + \alpha F + \alpha^2 G + \epsilon^2 M = \sigma + \alpha\frac{1}{4}B\sigma^2 + \alpha^2\frac{B\sigma}{2} + \frac{\epsilon^2}{2}\tilde{\kappa}\zeta_{xx} \tag{347}$$

We can perform the other way around,i.e seek for a relation $\sigma = \sigma(\zeta)$ of the form

$$\sigma = \zeta + \alpha F(\zeta) + \alpha^2 G(\zeta) + \epsilon^2 K(\zeta) \tag{348}$$

with F,G,K being differential polynomials in ζ such that at $O(\alpha^2, \epsilon^2)$, the resulting equations obtained substituting (348) in (334) coincide. This procedure can be carried out the same as the derivation of the KdV equation in [**17**], the only difference is that at $O(\alpha)$, one has to use the relation

$$\partial_t = -\partial_x - \frac{3}{2}\tilde{B}\alpha\zeta\partial_x \tag{349}$$

The outcome is

- The link between ζ and σ of (348) is given by

$$\sigma = \zeta - \frac{1}{4}\alpha\tilde{B}\zeta^2 + \frac{1}{8}\alpha^2\tilde{B}^2\zeta^3 - \frac{1}{2}\epsilon^2\tilde{\kappa}\zeta_{xx} \tag{350}$$

- The resulting unidirectional equation of motion is the (defocusing)Gardner (or KdV-mKdV) equation

$$\zeta_t = -\zeta_x - \frac{3}{2}\alpha\tilde{B}\zeta\zeta_x + \left(3\alpha^2\tilde{C} + \frac{3}{8}\alpha^2\tilde{B}^2\right)\zeta^2\zeta_x - \frac{1}{2}\epsilon^2\tilde{\kappa}\zeta_{xxx} \tag{351}$$

We first notice that, in the Weakly Non Linear (WNL) approximation, i.e at order $\alpha = O(\epsilon^2)$, the (351) equation becomes the KdV equation, and relation (348) reduces to the one of [**17**] §13.

CHAPTER 9

Conclusion

1. Summary

We have followed the technique of Hamiltonian reduction, in the presence of constraints, to derive model equations that inherit their structure from the parent Benjamin Hamiltonian formulation of a density stratified ideal fluid, under the asymptotic scalings of small amplitudes of fluid parcel displacements from their equilibrium positions and under slow variations in their horizontal positions, i.e., long wave approximation and small dispersion.

2. Comparison between Wu's model-Camassa-Choi model and Euler-Poincaré reduction model

The Camassa-Choi and the Wu models have to be equivalent in the dispersionless case. To show this, we do direct computations of the asymptotic formulas connecting averaged and interface velocities as follows. Specifically, starting from the series expansion in Wu's model, for the velocity fields we have

$$
\begin{aligned}
u_j(x,z) &= \sum_{j=0}^{\infty} \frac{(-1)^n}{(2n)!} \epsilon^{2n} H_j(z)^{2n} \partial_x^{2n} u_{0j}(x) \\
w_j(x,z) &= (-1)^{j-1} \epsilon \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \epsilon^{2n} H_j(z)^{2n+1} \partial_x^{2n+1} u_{0j}(x)
\end{aligned}
\tag{352}
$$

and

$$
\begin{aligned}
\tilde{u}_j &= \sum_{j=0}^{\infty} \frac{(-1)^n}{(2n)!} \epsilon^{2n} \eta_j^{2n} \partial_x^{2n} u_{0j} = u_{0j} - \frac{\epsilon^2}{2} \eta_j^2 u_{0jxx} + O(\epsilon^4) \\
\tilde{w}_j &= (-1)^{j-1} \epsilon \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \epsilon^{2n} \eta_j^{2n+1} \partial_x^{2n+1} u_{0j}(x) \\
&= \epsilon (-1)^{j-1} \left(\eta_j u_{0jx} - \frac{\epsilon^2}{6} \eta_j^3 u_{0jxxx} + O(\epsilon^4) \right) \\
\bar{u}_j &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \epsilon^{2n} \eta_j^{2n} \partial_x^{2n} u_{0j}(x) = u_{0j}(x) - \frac{\epsilon^2}{6} \eta_j^2 u_{0jxx} + O(\epsilon^4)
\end{aligned}
\tag{353}
$$

At leading order in the expansion with respect to the small dispersion parameter ϵ, we have

$$\bar{u}_j = \tilde{u}_j, \bar{w}_j = \tilde{w}_j \approx 0 \text{ with } \sigma = \rho_2\tilde{u}_2 - \rho_1\tilde{u}_1 = \rho_2\bar{u}_2 - \rho_1\bar{u}_1 \tag{354}$$

that is, σ reduces to the *horizontal* momentum shear. At this order one can view the motion as satisfying the so-called columnar motion *ansatz* (in , [**19**]). Thus, at the leading order (i.e when dispersion term is neglected), the evolutionary equations in Wu, Camassa-Choi formulation, and the Euler-Poicanré variational formulation coincide.

On the other hand, at higher orders, leading to the case that dispersive terms are included in the equations, this fails, since we have

$$\sigma = \rho_2\tilde{u}_2 - \rho_1\tilde{u}_1 + \epsilon\zeta_x(\rho_2\tilde{w}_2 - \rho_1\tilde{w}_1) \tag{355}$$

and columnar motion in [**19**] is no longer consistent with (352). Thus, in the dispersive case, the three models do not coincide. The interfacial pressure P (in Wu and Camassa-Choi models) is expected not to equal the pressure Φ in the Holm and Cotter Euler-Poincaré model in [**19**], since Φ represents the Lagrange multiplier for the rigid lid constraint.

3. Open problems and future development on the Hamiltonian approach to stratified fluids

Open problem in this direction might involve the study of the existence of Riemann invariants, conserved quantities and integrability issue of the obtained system. However, to our prediction, it is very unlikely that the systems (184) be linearly degenerate. Hence, since they do not admit Riemann invariants, according to the results of the paper [**24**] and [**25**], the only conserved quantities are the one listed chapter 6. Thus, we do not expect them to possess other conserved quantities, and any second Hamiltonian structure. However, detailed study is welcomed to justify this prediction.

Future development on the Hamiltonian approach to stratified fluids might involve considering the Hamiltonian structure of other hydrodynamics equations, and related issues concerning the symmetry groups, conserved quantities and integrability, solutions of special forms for example, traveling wave solutions, unidirectional wave solutions and symmetric solutions.

CHAPTER 10

Appendix

1. Brief overview of Euler-Poincaré reduction theory for a continuum system

In this chapter, we provide a summary of the Euler-Poincaré reduction technique, which is used to systematically derive many equations in geophysical fluid dynamics, as discussed in [**23**]. The Euler-Poincaré framework we will describe below is useful in formulating mathematical models for numerical simulations of many geophysical fluid dynamics (GFD) problems. When doing numerical simulations of these GFD problems, one may come across different GFD (approximate or exact) equations and ask " Is this equation good or better than other equations?". The issue of what is good about a certain GFD equation/ model relies on whether it satisfies a Kelvin circulation theorem and other conservation laws or symmetries. Within this Euler-Poincaré framework, we are in the proper context to study and check these symmetries, via the Kelvin-Noether circulation theorem.
Motivation of this Euler-Poincaré framework: To study the dynamics of a certain physical phenomenon, one may start from the Hamilton's variational principle, which gives the corresponding Euler-Lagrange equation. In the special case, when the action in Hamilton's principle or its associated Lagrangian function is invariant under certain groups actions, the resulting Euler-Lagrange equation will be reduced to a different equation called Euler-Poincaré equation. This method is known as Euler-Poincaré reduction (or reduction of Hamilton's Principle by symmetry). In this section, we will discuss this Euler-Poincaré reduction method, according to the approach presented in [**15**].

2. Geometric setting to study fluid motion

• **Configuration space for the motion of an incompressible fluid is the group $G = Diff(D)$ of diffeomorphisms which preserve volume on a region** $D \in \mathbb{R}^n$ occupied by the fluid. Fluid motion is described by g_t for some $g_t \in Diff(D)$.

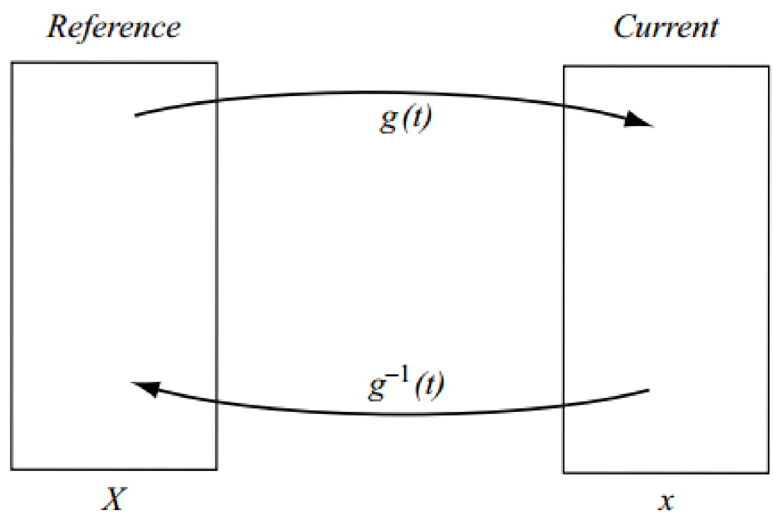

FIGURE 1. The map from Lagrange reference coordinates X in the fluid to the current Eulerian spatial position x is performed by the time-dependent diffeomorphisms, so that $x(t, X) = g(t) \cdot X$

Indeed, a fluid flow determines for every time moment t the map $g(t)$ of the flow domain to itself (the initial position of every fluid particle is taken to its terminal position at the moment t). All the terminal positions, i.e., configurations of the system (or permutations of particles), form the infinite-dimensional manifold $Diff(D)$.

• **Lagrangian velocity U** along motion g_t (keeping fluid particle label X fixed):

$$U(X, t) := \frac{\partial}{\partial t} g_t \cdot X = \frac{\partial}{\partial t} x(X, t)$$

• **Eulerian velocity u** along motion g_t (keeping the Eulerian spatial point x fixed), i.e if $x = x(X, t) = g_t X$ then

$$u(x, t) := U(X, t) = U(g_t^{-1}(x), t) \tag{356}$$

• Thus, we have the relation between Eulerian velocity and Lagrangian velocity

$$\boldsymbol{u_t = U_t g_t^{-1} = \dot{g}_t g_t^{-1}}, \text{ or } \boldsymbol{U_t = u_t \circ g_t} \tag{357}$$

• Lie algebra of $Diff(D)$ (tangent space to $Diff(D)$ at the identity) is

$$\mathfrak{g}(Diff(D)) := T_e Diff(D) = \mathfrak{X}(D) \tag{358}$$

where $\mathfrak{X}(D)$ is the set of smooth vector fields on D.

• The Lie algebra of Diff(D) is $\mathfrak{X}(D)$ endowed with the **Lie bracket** defined by

$$[u, v]_L := -[u, v] \forall u, v \in \mathfrak{g}(Diff(D)) \tag{359}$$

where $[\cdot, \cdot]$ on RHS is standard **Jacobi-Lie bracket**
• Proposition:

$$ad_u v = -[u, v] \forall u, v \in \mathfrak{g}(Diff(D)) \tag{360}$$

where $ad_u v$ denotes the adjoint action of the right Lie algebra of Diff(D) on itself.
•Dual Lie algebra. Definition of diamond operation $\diamond$:
The **diamond operation $\diamond$** between elements $A \in V$ and $a \in V^*$ produces an element of the dual Lie algebra $\mathfrak{g}(Diff(D))^*$ and is defined as

$$\langle A \diamond a, w\rangle_{\mathfrak{g}^*,\mathfrak{g}} = \langle A, -\mathfrak{L}_w a\rangle_{V\times V^*} = -\int_D A \cdot \mathfrak{L}_w a \tag{361}$$

where $A \in V; a \in V^*; w \in \mathfrak{g}(Diff(D)) = \mathfrak{X}(D)$ is a smooth vector field on D; $\mathfrak{L}_w a$ is Lie-derivative of a along vector field w.
i.e diamond operation $\diamond$ represents the dual of the Lie algebra action $\mathfrak{g}$ on the tensor a.
• **Advected parameter $a \in V^* = (\mathfrak{T} \otimes dVol)(D)$ (tensor field density on D)**
Physical advected quantity a is a tensor field (e.g buoyancy, heat, mass density, magnetic flux, etc.), which is Lie transported by Eulerian velocity field u.

$$\dot{a} := -\mathfrak{L}_u a \tag{362}$$

and define right action of any $u \in \mathfrak{g}(D) = \mathfrak{X}(D)$ on advected quantity a to be

$$au := \mathfrak{L}_u a \tag{363}$$

Advected quantity a must satisfy advection equation

$$\dot{a} = -\mathfrak{L}_u a = -au \tag{364}$$

whose solution is

$$a(t) = g_{t^*} a_0 = a_0 g^{-1}(t)$$

where g_t is the flow of $u = \dot{g}g^{-1}(t)$, and a_0 initial condition.

• The **Lagrangian of a continuum mechanical system** is a function

$$L : TDiff(D) \times V^* \to \mathbb{R}, \tag{365}$$

which is

- right invariant relative to the tangent lift of right translation of $Diff(D)$ on itself
- invariant w.r.t pull-back on the tensor field a

Explanation: The kinetic energy appeared in the Lagrangian function is right invariant under the action of the group $G = Diff(D)$. This is because the action of an arbitrary group element $h \in G = Diff_{Vol}(D)$ from the right is regarded as a (volume preserving) renumeration of fluid particles. The velocity of the particle occupying a certain position at a given moment does not change under the renumeration, and therefore the kinetic energy is preserved. Also, the potential energy due to advected quantities a present in the Lagrangian function is invariant under the action of group element h on the tensor field densities V^* (property of advection). Thus, the Lagrangian function L of a continuum mechanical system is right invariant relative to the tangent lift of right translation of $Diff(D)$ on itself and pull-back on the tensor field densities.
Invariance of the Lagrangian function L (defined on tangent bundle TDiff(D)) induces a function

$$l : \mathfrak{X}(D) \times V^* \to \mathbb{R}$$

(defined on Lie algebra $\mathfrak{g}(Diff(D)) = \mathfrak{X}(D)$ only), given by

$$l(u, a) := L(u \circ g_t, g_t^* a),$$

where $g_t^* a$ is invariant w.r.t pull-back of the advected quantity a by diffeomorphism $g_t \in Diff$.

3. Euler-Poincaré reduction Theorem for continua with advected quantities:

Consider a path g_t in $Diff(D)$ with Lagrangian velocity vector field U and Eulerian velocity u (thus $\boldsymbol{U_t = u_t \circ g_t}$). Then, the followings are equivalent:

1) Hamilton's variational principle

$$\boldsymbol{\delta \int_{t_1}^{t_2} L(X, U_t(X), a_0(X)) dt = 0} \tag{366}$$

holds, for variations δg_t vanishing at the endpoints in time.
2) g_t satisfies the Euler-Lagrange equations for L_{a_0} on $Diff(D)$
3) The reduced variational principle in Eulerian coordinates

$$\boldsymbol{\delta \int_{t_1}^{t_2} l(u, a) dt = 0} \tag{367}$$

holds on $\boldsymbol{\mathfrak{g}(Diff(D)) \times V^*}$, for variations of the form

$$\boldsymbol{\delta u = \frac{\partial w}{\partial t} + [u, w] = \frac{\partial w}{\partial t} - ad_u w, \quad \delta a = -\mathfrak{L}_w a,} \tag{368}$$

where $w_t = \delta g_t \circ g_t^{-1}$ vanishes at the endpoints.

4) The Euler-Poincaré equations for continua

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta u} = -ad^*_u\frac{\delta l}{\delta u} + \frac{\delta l}{\delta a}\diamond a = -\mathfrak{L}_u\frac{\delta l}{\delta u} + \frac{\delta l}{\delta a}\diamond a, \tag{369}$$

holds, with auxiliary advection equations $(\partial_t+\mathfrak{L}_u)a = 0$ for each advected quantity a(t).

Proof: • The equivalence 1) and 2) holds for any configuration space, so it holds in this case.
• Also, the equivalence between 3) and 4) can be proved in the following way, using integration by parts, constrained variation of v and a, and adjoint operation.

$$\begin{aligned} 0 = \delta\int_{t_1}^{t_2} l(u,a)dt &= \int_{t_1}^{t_2}\left(\frac{\delta l}{\delta u}\delta u + \frac{\delta l}{\delta a}\delta a\right)dt \\ &= \int_{t_1}^{t_2}\left[\frac{\delta l}{\delta u}.\left(\frac{\partial w}{\partial t} - ad_u w\right) - \frac{\delta l}{\delta a}.\mathfrak{L}_w a\right]dt \\ &= \int_{t_1}^{t_2} w\cdot\left[-\frac{\partial}{\partial t}\frac{\delta l}{\delta u} - ad^*_u\frac{\delta l}{\delta u} + \frac{\delta l}{\delta a}\diamond a\right]dt \quad \text{for any w} \end{aligned} \tag{370}$$

This implies the equivalence between 3) and 4).
• To prove the equivalence between 1) and 3), first note that the two integrands in 1) and 3) are equal, thanks to the invariance of the Lagrangian function.
Next, we need to check if all variations $\delta g(t)$ vanishing at the endpoints induce and are induced by variations $\delta u = \dot{w} + [u, w]$ for $w_t = \delta g_t \circ g_t^{-1}$ vanishes at endpoints, and $\delta a = -\mathfrak{L}_w a$.
Since $U_t = u_t \circ g_t$, then $u_t = U_t \circ g_t^{-1} = \dot{g}_t \circ g_t^{-1}$.
Note that variation of diffeomorphism g is defined as $\delta g := \frac{d}{d\epsilon}|_{\epsilon=0} g_\epsilon$.
Take variation of u_t w.r.t variable g and use definition $w_t = \delta g_t \circ g_t^{-1}$, then

(371)

$$\begin{aligned} \delta u = \delta(\dot{g}.g^{-1}) &= \frac{\delta\dot{g}}{\delta g}g^{-1} + \dot{g}\frac{\delta g^{-1}}{\delta g} \\ &= \frac{\delta}{\delta g}\left(\frac{dg}{dt}\right)g^{-1} + \dot{g}(-1)g^{-1}\delta g g^{-1} \\ &= \frac{d}{d\epsilon}|_{\epsilon=0}\left(\frac{dg}{dt}\right)g^{-1} - \dot{g}g^{-1}\delta g g^{-1} \\ &= \frac{d^2 g}{dt d\epsilon}g^{-1} - \dot{g}g^{-1}\delta g g^{-1} \quad \text{(since time derivative and variational derivative commute)} \end{aligned}$$

Now, compute derivative of w_t.

$$\frac{\partial w}{\partial t} = \frac{\partial(\delta g_t)}{\partial t}g_t^{-1} + \delta g_t\frac{\partial g_t^{-1}}{\partial t} = \frac{d^2}{dt d\epsilon}|_{\epsilon=0} g^{-1} - \delta g g^{-1}\dot{g}g^{-1}$$

Thus

(372) $$\delta u - \frac{\partial w}{\partial t} = -(\dot{g}g^{-1})\delta g g^{-1} + (\delta g g^{-1})\dot{g}g^{-1} = -uw + wu = -ad_u w = [u, w]$$

Moreover, since $a = a_0 g^{-1}$, then

$$\delta a = a_0 \delta(g^{-1}) = a_0(-1)g^{-1}\delta g g^{-1} = -(a_0 g^{-1})w = -aw = -\mathfrak{L}_w a$$

Thus, 1) is equivalent to 3).

4. Derivation of Euler's equation from Euler-Poincaré reduction technique

Consider the Eulerian motion equation for an ideal incompressible fluid with a constant mass density ρ in a two-dimensional space. The reduced Lagrangian $l(u, M)$ and reduced action are as follows:

$$\mathfrak{s}_{red} = \int dt l(u, M) = \int dt \int d^2x \left[\frac{1}{2}\rho M|u|^2 - p(M-1)\right]$$

i.e the advected tensor field a here consists of volume element Md^2x only. Thus, the term $\frac{1}{2}\rho M|u|^2$ represents kinetic energy, and p is the Lagrange multiplier for the incompressibility constraint $M = 1$. The constraint function is $\phi = M - 1$, zero level set of ϕ is $\phi^{-1}(0)$ is equivalent to constraint $M = 1$

The, the Euler-Poincaré equation for continua expressed in Kelvin-Noether form in this case (from 369) is

(373) $$\left(\frac{\partial}{\partial t} + \mathfrak{L}_u\right)\left(\frac{1}{M}\frac{\delta l}{\delta u}.dx\right) - \nabla\left(\frac{\delta l}{\delta M}\right).dx = 0$$

Then, the variations of each variable are as follows:

$$\frac{1}{M}\frac{\delta l}{\delta u} = \rho u; \qquad \frac{\delta l}{\delta M} = \frac{1}{2}\rho|u|^2 - p; \qquad \frac{\delta l}{\delta p} = -(M-1)$$

Thus, the above Euler-Poincaré equation becomes

(374) $$\rho\left(\frac{\partial}{\partial t} + \mathfrak{L}_u\right)(u.dx) - \nabla\left(\frac{1}{2}\rho|u|^2 - p\right).dx = 0$$

We have:

$$\mathfrak{L}_u(u.dx) = [(u.\nabla)u + u^j\nabla u^j].dx = [-u \times (\nabla \times u) + \nabla(u.u)].dx$$

Also, from the formula

$$\nabla(u.v) = u \times (\nabla \times v) + (u.\nabla)v + v \times (\nabla \times u) + (v.\nabla)u$$

Thus

$$\nabla(u.u) = 2[u \times (\nabla \times u) + (u.\nabla)u]$$

Combining all of the above computation, we obtain:

$$\begin{aligned}
&\rho\mathfrak{L}_u(u.dx) - \frac{1}{2}\rho\nabla|u|^2.dx \\
&= \rho[-u\times(\nabla\times u) + \nabla(u.u)].dx + \rho\left[\frac{-1}{2}\nabla(u.u)\right].dx \\
&= \rho\left[-u\times(\nabla\times u) + \frac{1}{2}\nabla(u.u)\right].dx \\
&= \rho\left[-u\times(\nabla\times u) + \frac{1}{2}.2[u\times(\nabla\times u) + (u.\nabla)u]\right].dx \\
&= \rho\left[(u.\nabla)u\right].dx
\end{aligned}$$

Thus, the Euler-Poincaré eqn becomes

$$\rho\left[\frac{\partial u}{\partial t} + (u.\nabla)u + \frac{1}{\rho}\nabla p\right].dx = 0$$

which gives

$$\frac{\partial u}{\partial t} + (u.\nabla)u + \frac{1}{\rho}\nabla p = 0$$

which is the Euler's equation for motion of an ideal fluid.

Thus, we have proved that the Euler's equation can be derived from the Euler-Poincaré equation with the Lagrangian function specified above. **If we want to include gravitational force in the Euler's equation, we just need to add potential energy term V =** $\int$ **$\mathbf{d^2x(\rho Mgz)}$ into the above Lagrangian function.** Then,

$$\mathfrak{s}_{red} = \int dtl(u,M) = \int dt\int d^2x\left[\frac{1}{2}\rho M|u|^2 - p(M-1) - \rho Mgz\right]$$

The term $\dfrac{\delta l}{\delta M}$ in the Euler-Poincaré equation will be different, which will equal

$$\frac{\delta l}{\delta M} = \frac{1}{2}\rho|u|^2 - p - \rho gz$$

The Euler-Poincaré equation then gives

$$\rho\left[\frac{\partial u}{\partial t} + (u.\nabla)u + \frac{1}{\rho}\nabla p + g\nabla z\right].dx = 0$$

which gives

$$\frac{\partial u}{\partial t} + (u.\nabla)u + \frac{1}{\rho}\nabla p + g\hat{z} = 0$$

(where $\hat{z}$ is the unit vector in the vertical direction). We finally obtain the common Euler's equation.

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