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Sharp Estimates and Inequalities on Riemannian Manifolds with Euclidean Volume Growth

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Abstract

We obtain sharp estimates for heat kernels and Green's functions on complete noncompact Riemannian manifolds with Euclidean volume growth and nonnegative Ricci curvature. We will then apply these estimates to obtain sharp Moser-Trudinger inequalities on such manifolds.

Keywords Sharp inequalities · Moser-Trudinger · Heat kernels · Green's functions · Euclidean volume growth

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Contents

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1 Introduction

A complete, noncompact, $n-$dimensional Riemannian manifold with nonnegative Ricci curvature is said to have *Euclidean volume growth*, or EVG, if

$$\frac{\mathrm{Vol}\big(B(x,r)\big)}{|B_{\mathbb{R}^n}(r)|} \searrow \sigma_M > 0, \qquad \text{as } r \to \infty \tag{1.1}$$

where $\mathrm{Vol}(B(x,r)$ denotes the volume of the geodesic ball centered at x and radius r, and $|B_{\mathbb{R}^n}(r)|$ the volume of the Euclidean ball of radius r. The volume ratio in (1.1) is decreasing in r by the Bishop comparison theorem, which also guarantees that $\mathrm{Vol}\big(B(x,r)\big) \le |B_{\mathbb{R}^n}(r)|$, and therefore $0 < \sigma_M \le 1$. It's easy to see that the limit in (1.1) is independent of x.

Manifolds with EVG have been extensively studied since the late 80s, see for example [9, 13, 18, 51, 61, 62].

Very recently, several authors found versions of the classical isoperimetric and Sobolev inequalities on manifolds with EVG, whose sharp constants feature explicitly the constant σ_M, known currently as *Asymptotic Volume Ratio*, or AVR. In particular, the sharp isoperimetric inequality holds in the form

$$\mathrm{Per}(\Omega) \ge \sigma_M^{\frac{1}{n}} n B_n^{\frac{1}{n}} \mathrm{Vol}(\Omega)^{\frac{n-1}{n}} \tag{1.2}$$

where $B_n = |B_{\mathbb{R}^n}(1)|$ and where $\mathrm{Per}(\Omega)$ denoted the perimeter of a bounded open set Ω with smooth boundary. This result, along with the characterization of equality, was proved by Brendle [11], while other authors proved the result for more general Ω, and on more general metric measure spaces, with different techniques (see [3, 4, 7]).

In [7], the sharp Sobolev inequality on manifolds with EVG was derived in the form

$$\|u\|_{p^*} \le \sigma_M^{-\frac{1}{n}} C(n,p) \|\nabla u\|_p, \qquad 1 < p < n, \quad p^* = \frac{np}{n-p} \tag{1.3}$$

where $C(n,p)$ denotes the best constant in the Euclidean version of the same inequality [5, 55].

In this paper we will focus on the borderline case of the Sobolev embedding theorem, which in $\mathbb{R}^n$ concerns the Sobolev spaces $W^{\alpha,\frac{n}{\alpha}}$. In particular we will obtain sharp Moser-Trudinger inequalities on manifolds with EVG. In order to state our main results we will need a few more definitions (for some more details see the next section).

Let us define

$$D^\alpha = \begin{cases} (-\Delta)^{\frac{\alpha}{2}} & \text{for } \alpha \text{ even} \\ \nabla(-\Delta)^{\frac{\alpha-1}{2}} & \text{for } \alpha \text{ odd.} \end{cases} \tag{1.4}$$

where Δ is the Laplace-Beltrami operator on M and ∇ is the gradient.

The Bessel potential space is defined for $\alpha > 0$, $p \geq 1$ as

$$H^{\alpha,p}(M) = \{u \in L^p(M) : \ u = (I - \Delta)^{-\frac{\alpha}{2}} f, \ f \in L^p(M)\} \tag{1.5}$$

endowed with the norm $\|(I - \Delta)^{\frac{\alpha}{2}} u\|_p$. These spaces were studied, on general non-compact manifolds, by Strichartz [53], who also showed that $C_c^\infty(M)$ is dense in $H^{\alpha,p}(M)$ and that $\|u\|_p + \|(-\Delta)^{\frac{\alpha}{2}} u\|_p$ is an equivalent norm if $p > 1$. By work of Auscher et al [6], it turns out that in our setting the Riesz transform is continuous in L^p and this implies that $\|u\|_p + \|\nabla(-\Delta)^{\frac{\alpha-1}{2}} u\|_p$ is an equivalent norm in $H^{\alpha,p}$, for $\alpha \geq 1$. See Section 2 for some comments about the relations between $H^{\alpha,p}$ and the classical Sobolev spaces $W^{\alpha,p}$ and $W_0^{\alpha,p}$.

If $\Delta_{\mathbb{R}^n}$ is the standard Laplacian on $\mathbb{R}^n$ then the fundamental solution of $(-\Delta_{\mathbb{R}^n})^{\frac{\alpha}{2}}$ will be denoted by

$$G_\alpha^{\mathbb{R}^n}(\xi, \eta) = N_\alpha(\xi - \eta), \qquad N_\alpha(\xi) = c_\alpha |\xi|^{\alpha - n} \tag{1.6}$$

i.e. the classical Riesz potential, where

$$c_\alpha = \frac{2^{-\alpha}\Gamma\left(\frac{n-\alpha}{2}\right)}{\pi^{n/2}\Gamma(\frac{\alpha}{2})}. \tag{1.7}$$

Throughout this paper the volume of the unit ball in $\mathbb{R}^n$ will also be denoted by B_n:

$$B_n = |B_{\mathbb{R}^n}(1)| = \frac{\omega_{n-1}}{n} \tag{1.8}$$

where $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1} = \partial B_{\mathbb{R}^n}(1)$.

The *Adams constant* is defined as

$$\gamma_{n,\alpha} = \begin{cases} \dfrac{c_\alpha^{-\frac{n}{n-\alpha}}}{B_n} & \text{if } \alpha \text{ even} \\[2ex] \dfrac{\widetilde{c}_\alpha^{\,-\frac{n}{n-\alpha}}}{B_n} & \text{if } \alpha \ \text{ odd,} \end{cases} \tag{1.9}$$

where

$$\widetilde{c}_\alpha = \frac{2^{-\alpha}\Gamma\left(\frac{n-\alpha+1}{2}\right)}{\pi^{n/2}\Gamma(\frac{\alpha+1}{2})} = \begin{cases} (n-\alpha-1)c_{\alpha+1} & \text{if } \alpha+1 < n \\ \dfrac{c_{\alpha-1}}{\alpha-1} & \text{if } 1 < \alpha < n, \end{cases} \tag{1.10}$$

and which gives the sharp exponential constant in the classical Moser-Trudinger inequality on bounded domains of $\mathbb{R}^n$, and in many other related inequalities.

Finally, the *regularized exponential function* is defined as

$$\exp_m(t) = e^t - \sum_{k=0}^{m} \frac{t^k}{k!}, \qquad t \geq 0, \;\; m = 0, 1, 2... \tag{1.11}$$

Theorem 1 *Let (M, g) be a complete, connected, noncompact Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth.*

For α even and $\alpha < n$, or for α odd and $\alpha \leq n/2$, there exists $C > 0$ such that for all $u \in H^{\alpha, \frac{n}{\alpha}}(M)$ with

$$\|D^\alpha u\|_{n/\alpha} \leq 1 \tag{1.12}$$

we have

$$\int_M \frac{\exp_{\lceil \frac{n-\alpha}{\alpha} - 1 \rceil}\left(\sigma_M^{\frac{\alpha}{n-\alpha}} \gamma_{n,\alpha} |u|^{\frac{n}{n-\alpha}} \right)}{1 + |u|^{\frac{n}{n-\alpha}}} d\mu \leq C \|u\|_{n/\alpha}^{n/\alpha}. \tag{1.13}$$

If $\alpha = 1$, $\alpha = 2$, $4 \leq \alpha < \frac{n}{2}$ and α even, $3 \leq \alpha < \frac{n}{2} + 1$ and α odd, then the exponential constant $\gamma_{n,\alpha}$ in (1.13) is sharp, i.e. it cannot be replaced by a larger value. For the same range of α, the power $\frac{n}{n-\alpha}$ in the denominator in (1.13) is sharp, i.e. it cannot be replaced by a smaller number.

Theorem 1 includes $\mathbb{R}^n$ with the standard metric, in which case $\sigma_M = 1$, where the inequalities hold for all $\alpha < n$ (see [31, 40, 42–44, 49]).

On noncompact, simply connected Riemannian manifolds with nonpositive sectional curvature K, i.e. Hadamard manifolds, (1.13) holds (for all $\alpha < n$) with sharp exponential constant $\gamma_{n,\alpha}$, provided $-b^2 \leq K \leq -a^2 < 0$ [45, 46] (see also [45] for a sharp version of (1.13) on the Heisenberg group).

We emphasize that all of the results known so far, on the above Riemannian manifolds, hold with sharp constant $\gamma_{n,\alpha}$, whereas within manifolds with EVG the sharp constant is *lower*, by a factor of $\sigma_M^{\frac{\alpha}{n-\alpha}}$. The main point is that inequalities such as (1.13) under the norm condition (1.12) are very sensitive to the geometry of the ambient space at infinity, in particular volume growth, something that had already been observed in [45] (see for example the exponential constant in the Adams inequality (5.4) below).

In the next theorem, we will show that on manifolds with EVG, and with enough bounded geometry, imposing a norm condition stronger than (1.12) yields sharp Moser-Trudinger inequalities with the usual, classical constant $\gamma_{n,\alpha}$.

We will say that M has k^{th} *order bounded geometry* if there is $B > 0$ such that

$$\mathrm{inj}(M) > 0, \qquad |\nabla^i R| \leq B, \qquad i = 0, 1, ..., k. \tag{1.14}$$

where $\mathrm{inj}(M)$ is the injectivity radius of M, R the curvature tensor, and where ∇^i is the i^{th} covariant derivative.

Theorem 2 *Let (M, g) be a complete, connected, noncompact Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth. For any α integer with $1 \leq \alpha < n$, let M have 1^{st} order bounded geometry for α even, and 2^{nd} order bounded* *geometry for α odd. Then for any $\kappa > 0$ there is C such that for all $u \in H^{\alpha,\frac{n}{\alpha}}(M)$ with*

$$\kappa\|u\|_{n/\alpha}^{n/\alpha} + \|D^\alpha u\|_{n/\alpha}^{n/\alpha} \le 1 \tag{1.15}$$

we have

$$\int_M \frac{\exp_{\lceil \frac{n-\alpha}{\alpha} - 1\rceil}\left(\gamma_{n,\alpha}|u|^{\frac{n}{n-\alpha}}\right)}{1+|u|^{\frac{n}{n-\alpha}}} d\mu \le C\|u\|_{n/\alpha}^{n/\alpha}, \tag{1.16}$$

and

$$\int_M \exp_{\lceil \frac{n-\alpha}{\alpha} - 1\rceil}\left(\gamma_{n,\alpha}|u|^{\frac{n}{n-\alpha}}\right) d\mu \le C. \tag{1.17}$$

The exponential constant $\gamma_{n,\alpha}$ in (1.16) *and* (1.17) *is sharp, and the power $\frac{n}{n-\alpha}$ in the denominator in* (1.16) *is sharp.*

Inequality (1.17) on $\mathbb{R}^n$, was proved in [24, 32, 37, 48]. It is easy to check that on $\mathbb{R}^n$ inequality (1.17) under (1.15) (with $\kappa > 0$) follows from the same result with $\kappa = 1$, by a dilation argument. It is worth mentioning, at this point, that (1.17) is *false* under (1.12), that is under the weaker condition $\|D^\alpha u\|_{n/\alpha} \le 1$. In fact, (1.17) is false even under the full Sobolev norm conditions

$$\max\left\{\|u\|_{n/\alpha}, \|D^\alpha u\|_{n/\alpha}\right\} \le 1 \qquad \left(\|u\|_{n/\alpha}^{qn/\alpha} + \|D^\alpha u\|_{n/\alpha}^{qn/\alpha}\right)^{\frac{\alpha}{qn}} \le 1, \quad q > 1. \tag{1.18}$$

Under such conditions one can only prove (1.17) with $\gamma_{n,\alpha}$ replaced by any $\gamma < \gamma_{n,\alpha}$. In particular, for any $\gamma < \gamma_{n,\alpha}$, there is C (depending on γ) such that under (1.12) there holds

$$\int_{\mathbb{R}^n} \exp_{\lceil \frac{n-\alpha}{\alpha} - 1\rceil}\left(\gamma|u|^{\frac{n}{n-\alpha}}\right) d\mu \le C\|u\|_{n/\alpha}^{n/\alpha}. \tag{1.19}$$

This inequality is often called of Adachi-Tanaka type, since it first appeared when $\alpha = 1$ in a paper by Adachi and Tanaka [AT]; later it was extended to all integer values of α in [24], as a relatively easy consequence of (1.17) under (1.15). In [49] Corollary 1, Qin derived (1.19) as an easy consequence of (1.16) under either (1.12) or (1.15) when u is a general Riesz-like potential. For a short discussion and history of such types of weaker, *subcritical* Moser-Trudinger inequalities, which will not be treated in this paper, see [24] and references therein (in particular, see [24] Corollary 4).

Still on $\mathbb{R}^n$, when $\gamma = \gamma_{n,\alpha}$ inequality (1.19) fails under (1.12), however it becomes true, and sharp, if the integrand is divided by $1 + |u|^{\frac{n}{n-\alpha}}$, which gives (1.16) (and which is the same as the already mentioned (1.13), since $\sigma_M = 1$ on $\mathbb{R}^n$). This was first discovered in [31], when $n = 2$, $\alpha = 1$, followed by the aforementioned papers [40, 42–44, 49], which treated other values of n and α. For a unified discussion of all the known most important Moser-Trudinger inequalities on domains of $\mathbb{R}^n$, see the introduction of [45].

It has been remarked before that on $\mathbb{R}^n$ inequality (1.13) (which is the same as (1.16)) under (1.12) is stronger than (1.17) under (1.15), in the sense that it directly implies it via Hölder's inequality and (1.19) (see [42–44], and also [49], Section 8). With the same arguments one sees, on general manifolds, that if (1.16) holds with exponential constant γ^* under (1.12), then (1.19) also holds for $\gamma < \gamma^*$, and (1.17) holds, under (1.15), with the *same* exponential constant γ^*. With small modifications of the arguments in [49] Section 8, the same is true if one assumes (1.16) with exponential constant γ^* under (1.15), rather than (1.12); in particular, in the context of Theorem 2, estimate (1.16) implies estimate (1.17) (however we do not know if the converse implication is true). One consequence of all this is that on manifolds with EVG and enough bounded geometry, inequality (1.13) under (1.12) is *not* stronger than (1.17) under (1.15), unless $\sigma_M = 1$, i.e. $M = \mathbb{R}^n$, with its standard metric. On such manifolds, therefore, the sharp estimates contained in Theorems 1 and 2 are in a sense independent of one another.

On Hadamard manifolds, (1.17) holds when $\alpha = 1$ under (1.15), provided $-b^2 \leq K \leq 0$ [25] (with the exception $n = 2, 3, 4$, in which cases it holds for $K \leq 0$ [30]). When $-b^2 \leq K \leq -a^2 < 0$, estimate (1.17) holds under the weaker condition (1.12) ([10]). To the best of our knowledge the results in Theorem 2, even when $\alpha = 1$, are not known on other noncompact manifolds with variable Ricci curvature which is only bounded below (or even nonnegative), even assuming positive injectivity radius (see [30, 63] for related results).

The results in Theorem 2, and their proofs, are more local in nature, and somewhat insensitive to the volume growth of large balls. In particular, in [25] the authors introduced the concept of "Local Moser-Trudinger inequality", for the case $\alpha = 1$, and showed that if an inequality like (1.17) holds for functions whose support have volume less than a given number, then the inequality holds globally. In fact, it is possible to prove (1.17) above using this result, for the specific case $\alpha = 1$ (see Remark 13).

There are essentially two methods of proof of basic Moser-Trudinger type inequalities. In the first approach, used mainly in $\mathbb{R}^n$, one tries to replace the original function u with its symmetric decreasing rearrangement $u^\#$, thereby reducing the problem to a sharp one-dimensional estimate. On $\mathbb{R}^n$, and for $\alpha = 1$, this is done via the Pólya-Szegő inequality $\|\nabla u^\#\|_p \leq \|\nabla u\|_p$ (see for example [31, 37, 42, 43, 47, 48]). For $\alpha \geq 2$ one can use Talenti's comparison theorem, to establish more general inequalities for Sobolev functions satisfying Navier boundary conditions see [44, 56].

The second approach is to replace u by a suitable potential, typically after an estimate of type

$$|u(x)| \leq \int_M |K_\alpha(x, y)|\,|D^\alpha u(y)| d\mu(y) \tag{1.20}$$

where K_α is essentially a fundamental solution of D^α. Indeed, one typically takes $K_2(x, y) = G(x, y)$, the Green function of Δ, and $K_1(x, y) = \nabla_y G_(x, y)$. Through sharp asymptotic estimates of K_α along the diagonal, and at infinity for some noncompact manifolds, one is generally able to follow the technique originally developed by Adams on $\mathbb{R}^n$ in [1], and to derive a sharp exponential inequality where u is replaced by the integral operator with kernel K_α. This method has been successfully used in spaces where the symmetrization results mentioned above are not (yet) available, such as compact manifolds, CR manifolds, Hadamard manifolds etc. (see for example [1, 22–24, 45, 49]).

Both methods have shortcomings on general noncompact manifolds satisfying $\operatorname{inj}(M) > 0$ and $\operatorname{Ric} \geq \lambda$. The Pólya-Szegő inequality holds in the form

$$I(\Omega)\|\nabla u^{\#}\|_{L^p(\Omega^{\#})} \leq \|\nabla u\|_{L^p(\Omega)}, \qquad u \in W_0^{1,p}(\Omega) \tag{1.21}$$

where Ω is an open bounded set in M, $\Omega^{\#}$ a ball in $\mathbb{R}^n$ with volume $\operatorname{Vol}(\Omega)$, and where $I(\Omega)$ is the normalized isoperimetric ratio

$$I(\Omega) = \inf_U \frac{\operatorname{Per}(U)}{nB_n^{\frac{1}{n}}\operatorname{Vol}(U)^{\frac{n-1}{n}}} \in [0,1] \tag{1.22}$$

the infimum being taken among smooth open subsets of Ω. Assuming $I(\Omega) > 0$, this result will allow easily to prove a Moser-Trudinger inequality for functions in $W_0^{1,n}(\Omega)$, with exponential constant $I(\Omega)^{\frac{n}{n-1}}\gamma_{n,1}$ (see for ex. [30]). However, proving the sharpness of such constant is not possible without further information on the sharpness of the isoperimetric inequality, and ultimately on the geometry of M.

Secondly, establishing uniform asymptotic estimates for K_α is not always an easy task. For α even, if $K_\alpha(x,y) = G_\alpha(x,y)$ is a fundamental solution of $(-\Delta)^{\frac{\alpha}{2}}$, the "Adams' method" requires at the very least a local asymptotic estimate of type

$$|G_\alpha(x,y) - c_\alpha d(x,y)^{\alpha-n}| \leq Cd(x,y)^{a-n+\epsilon}, \qquad d(x,y) \leq 1 \tag{1.23}$$

with C *independent of* x, y (it is not difficult to show (1.23) for fixed x and with C depending on x). To prove such result one needs to impose further assumptions on the curvature.

Moreover, it is necessary to have some control on the decay of $G_\alpha(x,y)$ at infinity, and for the validity of an inequality such as (1.13) it is also important to establish more precise asymptotic bounds of $G(x,y)$ for large distances.

On manifolds with EVG we are able to prove (1.13) with a technique which is a sort of hybrid between the two methods discussed above. On one hand, there is a sharp Pólya-Szegő inequality

$$\|\nabla u^{\#}\|_{L^p(\Omega^{\#})} \leq \sigma_M^{-\frac{1}{n}}\|\nabla u\|_{L^p(\Omega)}, \qquad u \in W_0^{1,p}(\Omega) \tag{1.24}$$

(see [4, 7]), which readily implies (1.13) for $\alpha = 1$ (but not the sharpness part!); see (5.15) and related comments. On the other hand, to settle the case $\alpha > 1$, in Sect. 4.3 we will obtain sharp Talenti/Weinberger type symmetrization estimates on the fundamental solutions of D^α, by using recent results by Chen-Li [14]. For α even we will obtain the estimate

$$\big(G_\alpha(x,\cdot)\big)^*(t) \leq \sigma_M^{-\frac{\alpha}{n}} N_\alpha^*(t), \qquad t > 0 \tag{1.25}$$

for all $x \in M$, where f^* denotes the decreasing rearrangement of a function f, and where $N_\alpha(\xi)$ is the Riesz kernel on $\mathbb{R}^n$, defined in (1.6). We will also obtain a sharp comparison result for

$$\int_{s_1 < G_\alpha(x,y) \le s_2} |\nabla_y G_\alpha(x,y)|^q \, d\mu(y) \tag{1.26}$$

in terms of N_α, and for $q \le 2$ (see Theorem 3). These results will allow us to get one of the key estimates needed to obtain (1.13), namely

$$\int_{r_1 < d(x,y) \le r_2} |K_\alpha(x,y)|^{\frac{n}{n-\alpha}} \, d\mu(y) \le \sigma_M^{-\frac{\alpha}{n-\alpha}} \gamma_{n,\alpha}^{-1} \log \frac{V_x(r_1)}{V_x(r_2)} + B, \qquad 0 < r_1 < r_2 \tag{1.27}$$

where $K_\alpha = G_\alpha$ for α even and $K_\alpha = \nabla_y G_{\alpha+1}$ for α odd, in which case we require $\frac{n}{n-\alpha} \le 2$, i.e. $\alpha \le n/2$. The inequality in (1.13) will then follow from the representation formula (1.20), combined with an Adams inequality on metric measure spaces, Theorem 4, recently obtained in [45]. At this point we do not know how to handle the case α odd and $n/2 < \alpha < n$.

The proof of (1.16) and (1.17) will instead be based on the uniform local estimate (1.23), and a similar one for $|\nabla_y G_{\alpha+1}|$, and not on the sharp behavior of the fundamental solution at infinity. These estimates will be obtained in Sect. 4.1 as a consequence of local, uniform, small times estimates for the heat kernel, which we will derive in Sect. 3.2. The methods used there are by no means new: we will essentially reload the classical parametrix machinery for the heat kernel, carefully track down the lowest order term and the error term, and estimate the latter uniformly, under suitable global bounds on the curvature tensor.

Establishing the sharpness of the exponential constant in (1.13) is not trivial. Ideally, one would like to fix a point x, and then construct an extremal family of functions in $W^{\alpha,\frac{n}{\alpha}}(M)$ which is equal to $\log(r/d(x,y))$ for large r, and $1 < d(x,y) < r$. This is similar in spirit to the standard construction of the so-called "Moser sequence" for the classical inequalities (which is defined on small annuli around a point x). However, this idea only works for $\alpha = 1$, due to the non-smoothness of the distance function on large balls. To overcome this difficulty, we will replace $d(x,y)$ with the function

$$b(y) = \left(\frac{G_2(x,y)}{\sigma_M^{-1} c_2} \right)^{\frac{1}{2-n}} \sim d(x,y), \qquad \text{as } d(x,y) \to \infty, \tag{1.28}$$

which was featured in several papers by Colding and Colding-Minicozzi (see for ex. [17–19]). One should think of b as a smooth, proper replacement of the distance function, on manifolds with EVG. The extremal family for (1.13) will then be constructed in Sect. 6 either by smoothing $\log(r/b)$ on large annuli (which seems to be a workable method only for $\alpha = 2$), or by taking potentials of suitable compactly supported functions which are essentially equal to $b^{-\alpha}$ on large annuli, when $2 \le \alpha < n/2$ (cf. Remark 15).

In either case it will be necessary to have at hand sharp estimates for the Green function $G_\alpha(x,y)$, for large distances from a given point x. In Sects. 3.3 and 4.2, we will first refine some heat kernel and Green function estimates for large distances from a fixed point, on manifolds with EVG, due to Li-Tam-Wang [39] and Colding-Minicozzi [18]. As a result, we will obtain the sharp asymptotic bounds for α even

$$|G_\alpha(x,y)-\sigma_M^{-1}c_\alpha d(x,y)^{\alpha-n}|\leq C\widetilde{\Lambda}_x\big(d(x,y)\big)d(x,y)^{\alpha-n},\qquad d(x,y)\geq 1 \tag{1.29}$$

$$\bigg(\;\fint_{r<d(x,y)\leq(1+\eta)r}\Big|\nabla_y G_\alpha(x,y)-\sigma_M^{-1}c_\alpha\nabla_y d(x,y)^{\alpha-n}\Big|^2 d\mu(y)\bigg)^{\frac12} \leq C_\eta\widetilde{\Lambda}_x^{\frac12}(r)r^{\alpha-n-1},\qquad \eta>0, r\geq 1 \tag{1.30}$$

where $\widetilde{\Lambda}_x(r)$ is a decreasing function such that $\widetilde{\Lambda}_x(r)\to 0$ as $r\to\infty$. Such function is obtained from the volume ratio remainder

$$\Lambda_x(r)=\frac{V_x(r)}{B_n r^n}-\sigma_M \tag{1.31}$$

via a sort of transform defined in (3.40). In [18] estimates (1.29), (1.30) were obtained for $\alpha=2$, but with an unspecified function $\epsilon(r)$ going to 0 as $r\to\infty$, in place of $\widetilde{\Lambda}_x(r)$. Using an iteration procedure it would be possible to obtain a similar result for all α even, which would also be enough to establish sharpness of (1.13). We believe, however, that our estimates are of independent interest, since they quantify the rate of convergence of the heat kernel/Green function in terms of the rate of convergence of the volume ratio (see Lemma 2 and comments thereafter).

It is noteworthy that in order to establish the sharpness of the power $\frac{n}{n-\alpha}$ in the denominator of (1.13), the bounding radii of the annuli where the extremal functions are built, need to be finely calibrated in terms of the volume ratio remainder (1.31), via its transform $\widetilde{\Lambda}_x(r)$ (see (6.15) and Remark 16).

2 Background Notation

Let (M,g) be a complete, noncompact, $n-$dimensional Riemannian manifold, with metric tensor g. The geodesic distance between two points $x,y\in M$ is denoted by $d(x,y)$, and the open geodesic ball centered at x and with radius r will be denoted as $B(x,r)$. The manifold (M,g) is equipped with the natural Riemannian measure μ which in any chart satisfies $d\mu=\sqrt{|g|}dm$, where $|g|=\det(g_{ij})$, $g=(g_{ij})$ and dm is the Lebesgue measure.

For each given $p\in M$ denote the tangent space at p by T_pM, and the unit sphere in such space as $S_p=\{\xi\in T_pM:|\xi|=1\}$ where

$$\langle X,Y\rangle=g(X,Y),\qquad |X|=\langle X,X\rangle^{1/2},\qquad X,Y\in T_pM. \tag{2.1}$$

Covariant derivatives of order k of a smooth function u are denoted as $\nabla^k u$, and their components in a local chart are denoted as $(\nabla^k u)_{j_1\ldots j_k}$. In particular $\nabla^0 u=u$,

$(\nabla^1 u)_j = \partial_j u$. Define

$$|\nabla^k u|^2 = g^{i_1 j_1} \ldots . g^{i_k j_k} (\nabla^k u)_{i_1 \ldots i_k} (\nabla^k u)_{j_1 \ldots j_k}$$

The gradient and the Laplacian of a smooth function u on M are given as

$$\nabla u = g^{ij} \partial_j u \, \partial_i, \qquad \Delta u = g^{ij} (\nabla^2 u)_{ij} = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} \, g^{ij} \partial_j u \right).$$

and

$$|\nabla u|^2 = g(\nabla u, \nabla u) = g_{ij} (\nabla u)^i (\nabla u)^j = g^{ij} \partial_i u \, \partial_j u = |\nabla^1 u|^2.$$

Given a smooth tensor field $F(x, y)$ on $M \times M$, endowed with the standard product metric, we will use the notation $\nabla^1_x F$ to denote the covariant derivative of F with respect to x. Formally, if Z is any vector field in $T(M \times M)$ identified with $TM \oplus TM$ via the projections $\pi_1(x, y) = x$, $\pi_2(x, y) = y$, then $(\nabla^1_x)_Z F = \nabla^1_X F$, where $X = \pi_{1*} Z$. Similarly one defines $\nabla^1_y F$. Given a smooth function $f(x, y)$, the second mixed covariant derivative of f will be denoted as $\nabla^1_x \nabla^1_y f = \nabla^1_x (\nabla^1_y f)$, i.e. $\nabla^1_x \nabla^1_y f = \mathrm{Hess} f \circ (\pi_{1*}, \pi_{2*})$. In the standard coordinate charts of $M \times M$, inherited from M, one has that $\nabla^1_x \nabla^1_y f = \partial^2_{x_i, y_j} f dx^i dy^j$, and $|\nabla^1_x \nabla^1_y f|^2 = g^{ij} \partial^2_{x_i, y_j} f$.

The curvature tensor in the metric g will be denoted as R, the sectional curvature as K, and the Ricci curvature as Ric.

Remark 1 Assuming k^{th} order bounded geometry, it is possible to guarantee that if $0 < r_0 < \mathrm{inj}(M)$, then there exists C_0, depending on r_0 and the curvature bounds, such that for any $x \in M$, and in any normal coordinate neighborhood of x we have

$$\left| \partial^j_h g_{ij} \big(\exp_x (r\xi) \big) \right| \le C_0, \qquad 0 \le j \le k, \; r \le r_0, \tag{2.2}$$

where h denotes any multi-index with $|h| = j$ (see for example [20]).
We now discuss briefly the Sobolev spaces. For $u \in C^\infty$ define

$$\|u\|_{W^{k,q}} = \sum_{j=0}^{k} \Big(\int_M |\nabla^j u|^q d\mu \Big)^{1/q}, \tag{2.3}$$

and let $C^{k,q}(M) = \big\{ u \in C^\infty(M) : \|u\|_{W^{k,q}} < \infty \big\}$. The Sobolev space $W^{k,q}(M)$ is defined to be the completion of $C^{k,q}(M)$, relative to the norm $\| \cdot \|_{W^{k,q}}$, and it can be viewed as a subspace of $L^q(M)$. Endowed with the obvious norm, still denoted as $\| \cdot \|_{W^{k,q}}$, the space $W^{k,q}$ is a Banach space. The Sobolev space $W^{k,q}_0(M)$ is the closure of $C^\infty_c(M)$ in $W^{k,q}(M)$. We have already defined the Bessel potential spaces $H^{\alpha,m}(M)$ in the introduction. We remark here that on manifolds with $\mathrm{Ric} \ge 0$ the Riesz transform $\nabla(-\Delta)^{\frac{1}{2}}$ is bounded on L^p for $1 < p < \infty$, and $\|\nabla u\|_p$ is equivalent to $\|(-\Delta)^{\frac{1}{2}} u\|_p$, due to results in [6], Thm 1.4, since such M satisfies the doubling property and the heat kernel estimates $H(t, x, y) \le C/V_x(\sqrt{t})$, $|\nabla H(t, x, y)| \le C t^{-1/2}/V_x(\sqrt{t})$ (see Section 3). Hence we can use $\|u\|_p + \|D^\alpha u\|_p$ as a norm in $H^{\alpha,p}$ and we also have $W_0^{\alpha,p} \subseteq W^{\alpha,p} \subseteq H^{\alpha,p}$, for $p > 1$, $\alpha \in \mathbb{N}$. We will not be concerned with the issues related to when equality holds in the previous inclusions. Suffices to say that it's certainly true that for $p \geq 1$ we have $W_0^{\alpha,p} = W^{\alpha,p}$ when $\alpha = 1$, and for $\alpha \geq 2$ provided that $\mathrm{inj}(M) > 0$ and $|\nabla^j \mathrm{Ric}| \leq C$, for $j \leq \alpha - 2$ (see [28], Thms. 2.7, 2.8, and [59] for related counterexamples). Regarding the spaces $H^{\alpha,p}$, it's clear that for $p > 1$ we have $W_0^{\alpha,p} = W^{\alpha,p} = H^{\alpha,p}$ when $\alpha = 1$, and for higher values of α it is believed to be true again when $\mathrm{inj}(M) > 0$ and $|\nabla^j \mathrm{Ric}| \leq C$, $j \leq \alpha - 2$ (see [GP] for results in this direction when $\alpha = 2$), and it's certainly true for all α if $\mathrm{inj}(M) > 0$ and $|\nabla^j R| \leq C$ for all j ([57], p.320).

The cut locus of x can be described as $\mathrm{Cut}_x = \exp_x\{c(\xi)\xi,\ \xi \in S_x\}$, where $\exp_x$ is the exponential map at x, and $c(\xi) < \infty$ is the distance from x to its cut point. Then,

$$\exp_x :\ D_x := \{t\xi : 0 < t < c(\xi),\ \xi \in S_x\} \to M_x := M \setminus (\mathrm{Cut}_x \cup \{x\}) \tag{2.4}$$

is a diffeomorphism. Moreover, Cut_x has zero measure and for any f integrable on M we can write the integral of f in geodesic polar coordinates as

$$\int_M f d\mu = \int_{M_x} f d\mu = \int_{S_x} d\mu_x(\xi) \int_0^{c(\xi)} f(\exp_x t\xi)\sqrt{|g|}(t\xi)t^{n-1}dt, \tag{2.5}$$

where $d\mu_x(\xi)$ is the measure on S_x induced by the Lebesgue measure (see [12], §III.3]).

In this notation we have, for each $x \in M$,

$$V_x(r) = \int_{B(x,r)} d\mu = \int_0^r A_x(s)ds \tag{2.6}$$

where

$$A_x(r) = \int_{\{r<c(\xi)\}} \sqrt{|g|}(r\xi)r^{n-1}d\mu_x(\xi) \tag{2.7}$$

which coincides with the $(n-1)$-dimensional Hausdorff measure of $\partial B(x,r)$, for a.e. $r > 0$.

By the Bishop comparison theorem the functions $A_x(r)/r^{n-1}$ and $V_x(r)/r^n$ are both decreasing for $r > 0$, and

$$\lim_{r\to 0} \frac{V_x(r)}{B_n r^n} = 1. \tag{2.8}$$

Using that $B(x,r) \subseteq B(y, r + d(x,y))$ one sees that

$$\sigma_M := \lim_{r\to+\infty} \frac{V_x(r)}{B_n r^n} \tag{2.9}$$

is independent of x, from which we deduce

$$\sigma_M B_n r^n \leq V_x(r) \leq B_n r^n, \qquad x \in M, \quad r > 0. \tag{2.10}$$

Similarly, since $A_x(r)/r^{n-1}$ is decreasing, (2.9) and (2.10) imply (and are implied by)

$$\sigma_M = \lim_{r\to+\infty} \frac{A_x(r)}{nB_n r^{n-1}}, \qquad \sigma_M n B_n r^{n-1} \le A_x(r) \le nB_n r^{n-1}. \tag{2.11}$$

Clearly, $\sigma_M \le 1$ and it turns out, from the Bishop comparison theorem, that if $\sigma_M = 1$ then (M, g) is isometric to $\mathbb{R}^n$. As a consequence of this, and of the proof of the Bishop comparison, the function $V_x(r)/r^n$ is strictly decreasing, unless (M, g) is isometric to $\mathbb{R}^n$.

3 Uniform Heat Kernel Estimates

In this section (M, g) will denote a complete, $n-$dimensional Riemannian manifold with nonnegative Ricci curvature. The unique positive, symmetric, minimal heat kernel, i.e. the fundamental solution of $\Delta - \partial_t = 0$ on M, will be denoted as $H(x, y, t)$. For the existence of H and its basic properties see for example [34] or [53].

3.1 Uniform Global Bounds.

In this section we recall some known Gaussian bounds for the heat kernel and its space and time derivatives. First, we have the well-known bounds due to Li-Yau [36]: for any $\delta \in (0, 1)$, there exist $C_1, C_2 > 0$, depending only on δ and n, such that for any $x, y \in M$ and any $t > 0$

$$C_1 V_x(\sqrt{t})^{-1} \exp\left(-\frac{d^2(x,y)}{4(1-\delta)t}\right) \le H(x,y,t) \le C_2 V_x(\sqrt{t})^{-1} \exp\left(-\frac{d^2(x,y)}{4(1+\delta)t}\right). \tag{3.1}$$

Regarding the global gradient estimate, there exists $C_3 > 0$ depending only on n such that for all $x, y \in M$, and all $t > 0$

$$|\nabla_y H(x,y,t)| \le C_3 t^{-\frac{1}{2}} \left(1 + \frac{d^2(x,y)}{t}\right) H(x,y,t). \tag{3.2}$$

Estimate (3.2) is due to Souplet-Zhang [52].

The following time derivative estimate was obtained by Grigor'yan [26], Cor. 3.1]: there exists C_4 such that for all $x, y \in M$, and all $t > 0$

$$\left|\frac{\partial H}{\partial t}(x,y,t)\right| \le C_4 t^{-1} V_x(\sqrt{t})^{-1} \left(1 + \frac{d^2(x,y)}{4t}\right)^{2+3n/4} \exp\left(-\frac{d^2(x,y)}{4t}\right). \tag{3.3}$$

Finally, a uniform bound on the mixed second covariant derivative was obtained by Jiayu Li [35]: if $\alpha > 1$, then there exists C_5 depending on n and α such that for all

$x, y \in M$ and all $t > 0$

$$|\nabla_x^1 \nabla_y^1 H(x, y, t)| \leq C_5 t^{-1} H(x, y, t) + 3\alpha \frac{\partial H}{\partial t}(x, y, t) - \alpha \frac{|\nabla_y H(x, y, t)|^2}{H(x, y, t)}. \quad (3.4)$$

Under the assumptions Ric ≥ 0 the Bishop comparison gives $V_x(r) \leq |B_{\mathbb{R}^n}(r)|$ for all $r > 0$. Assuming $\mathrm{inj}(M) > 0$, and sectional curvature $K \leq k_0$, some $k_0 > 0$, (in particular under 1^{st} order bounded geometry), the Günther volume comparison theorem ([33], Thm. 11.14) gives

$$V_x(r) \geq \mathrm{Vol}_{k_0}(r), \qquad \forall x \in M,\ 0 < r < \min\left\{\mathrm{inj}(M), \frac{\pi}{\sqrt{k_0}}\right\}. \quad (3.5)$$

where $\mathrm{Vol}_{k_0}(r) = \omega_{n-1} \int_0^r \left[\frac{1}{\sqrt{k_0}} \sin\left(\sqrt{k_0}\, t\right)\right]^{n-1} dt$ denotes the volume of the geodesic ball of radius r in the space form of constant sectional curvature k_0 (sphere). Therefore, if $r_1 = \min\left\{\mathrm{inj}(M), \pi/\sqrt{k_0}\right\}$, then the Bishop comparison implies that for any $\overline{r} > 0$, $0 < r \leq \overline{r}$ we have

$$\frac{V_x(r)}{|B_{\mathbb{R}^n}(r)|} \geq \frac{V_x(\overline{r})}{|B_{\mathbb{R}^n}(\overline{r})|} \geq \frac{1}{|B_{\mathbb{R}^n}(\overline{r})|} \min\left\{\mathrm{Vol}_{k_0}(\overline{r}), \mathrm{Vol}_{k_0}(r_1)\right\} \quad (3.6)$$

and

$$V_x(r) \geq C_6 r^n, \qquad 0 < r \leq \overline{r} \quad (3.7)$$

some $C_6 > 0$ independent of x (and depending on $\overline{r}$, in general).

Let

$$E(x, y, t) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d^2(x, y)}{4t}\right). \quad (3.8)$$

Taking $\delta = \frac{1}{2}$ in the bounds (3.1), (3.2) and using (3.7) together with the Bishop comparison, we obtain that for all $x, y \in M$ and for $0 < t \leq 1$

$$C_7 E(x, y, t/2) \leq H(x, y, t) \leq C_8 E(x, y, 3t/2) \quad (3.9)$$

$$|\nabla_y H(x, y, t)| \leq C_9 t^{-\frac{1}{2}} E(x, y, 3t) \quad (3.10)$$

for some constants C_7, C_8, C_{10} independent of x, y, t, where the left inequality in (3.9) actually holds for all $t > 0$. Clearly, $|\nabla_x E|$ satisfies the same bound as in (3.10), but within the domain of differentiability of $d(x, y)$ (defined in (2.4)), in particular a.e.:

$$|\nabla_y E(x, y, t)| \leq C_{10} t^{-\frac{1}{2}} E(x, y, 3t), \qquad y \in M_x. \quad (3.11)$$

Remark 2 Note that (3.10), and the right inequality in (3.9), hold for $0 < t \leq T$, any $T > 0$, with C_8 and C_{10} depending on T, and this is true under the assumption Ric ≥ 0, $\mathrm{inj}(M) > 0$, and K bounded. Assuming Euclidean volume growth instead of K bounded, such inequalities hold for all $t > 0$, since $V_x(r) \geq \sigma_M |B_{\mathbb{R}^n}(1)|\, r^n$.

Later we will also need the following Lipschitz estimates for H: assuming K bounded, $T, \eta > 0$, there are C_{11}, C_{12} such that for any $R > (1+\eta)r$, $x' \in B(x, r)$, $y \notin B(x, R)$, and for all $0 < t \leq T$

$$|H(x, y, t) - H(x', y, t)| \leq C_{11} d(x, x') t^{-\frac{1}{2}} V_x(\sqrt{t})^{-1} \exp\left(-\frac{d^2(x, y)}{c_\eta t}\right) \tag{3.12}$$

$$|\nabla_y H(x, y, t) - \nabla_y H(x', y, t)| \leq C_{12} d(x, x') t^{-1} V_x(\sqrt{t})^{-1} \exp\left(-\frac{d^2(x, y)}{c_\eta t}\right) \tag{3.13}$$

for some $c_\eta > 0$ (specifically, we can take $c_\eta = 6(1+\eta)^2/\eta^2$). Estimate (3.12) follows easily from the mean value theorem along a minimizing geodesic joining x and x', and the gradient estimate (3.2), in the x variable, followed by (3.1) (with $\delta = 1/2$). Estimate (3.13) follows in a similar manner, but using instead the mixed gradient estimate (3.4), followed by (3.3) (with $\alpha = 2$), and (3.1).

3.2 Uniform Asymptotic Bounds: Small Times, Small Distances, Bounded Geometry.

It is well known that locally the heat kernel on M, for small times, behaves like the Euclidean heat kernel, i.e. in a suitable sense

$$H(x, y, t) \approx (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d^2(x, y)}{4t}\right), \tag{3.14}$$

for small $d(x, y)$ and small t (cf. Remark 3). In the following proposition we quantify this statement under the assumption of 1^{st} order bounded geometry, and give a corresponding statement for the gradient of H, under 2^{nd} order bounded geometry. While the results seem somewhat intuitive, we were not able to find them in the vast literature, in the particular form stated below. Once again, the main issue is to guarantee uniformity in the error term.

Once and for all let us fix an r_0 such that

$$0 < r_0 < \mathrm{inj}(M). \tag{3.15}$$

Proposition 1 *If M has 1^{st} order bounded geometry, then there exists $C_{13} > 0$ depending only on n, r_0, and the curvature bounds such that for $0 < t \leq 1$ and $x, y \in M$*

$$\left| H(x, y, t) - (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d^2(x, y)}{4t}\right) \right| \leq C_{13} t^{-\frac{n}{2}+\frac{1}{2}} \exp\left(-\frac{d^2(x, y)}{24t}\right). \tag{3.16}$$

If M has 2^{nd} order bounded geometry then there exists $C_{14} > 0$ depending only on n, r_0, and the curvature bounds such that for $0 < t \leq 1$, $x \in M$, and $y \in M_x$

$$\left|\nabla_y H(x, y, t) - \nabla_y (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d^2(x, y)}{4t}\right)\right| \leq C_{14} t^{-\frac{n}{2}} \exp\left(-\frac{d^2(x, y)}{24t}\right). \tag{3.17}$$

Remark 3 The classical statement $H(x, y, t) \sim E(x, y, t)$ as $t \to 0$ and $d(x, y) \to 0$ is made precise by (3.16), which implies that within each region of type $D = \{(x, y, t) : d^2(x, y) \leq Ct\}$, some fixed $C > 0$, we have

$$\lim_{\substack{t \to 0 \\ (x,y,t) \in D}} \frac{H(x, y, t)}{E(x, y, t)} = 1, \tag{3.18}$$

where the convergence is uniform with respect to x, y (within the region D). See also Remark 4.

Proof We note first that (3.16), (3.17) are only meaningful for $d(x, y)$ small. Indeed, assuming $d(x, y) \geq a > 0$ (some fixed $a > 0$), one can easily prove both bounds using the triangle inequality and (3.9), (3.10). Through out the proof we will then assume $d(x, y) \leq r_0$.

Let us start with proving (3.16). We combine the usual construction of the parametrix for heat kernel with the global Gaussian bounds in (3.1), (3.2). In geodesic normal coordinates at x let

$$u_0(x, y) = |g|^{-\frac{1}{4}}(r\xi) \tag{3.19}$$

and let $\phi \in C^\infty(M)$ be so that $\phi = 1$ on $B(x, r_0/2)$, $\phi = 0$ outside $B(x, r_0)$, $0 \leq \phi \leq 1$ and $|\nabla \phi| \leq 2/r_0$.

Letting

$$F(x, y, t) = \phi(y) u_0(x, y) E(x, y, t) \tag{3.20}$$

by standard calculations (see [34], pp. 109-111) we have

$$H(x, y, t) - F(x, y, t) = \int_0^t \int_M H(z, y, t - s)\big(\Delta_z - \partial_s\big) F(x, z, s) dz ds, \tag{3.21}$$

and for $z \in B(x, r_0)$

$$\big(\Delta_z - \partial_t\big) F(x, z, t) = \phi(\Delta_z u_0) E + u_0 E \Delta_z \phi + 2\big\langle \nabla_z \phi, \nabla_z(u_0 E)\big\rangle. \tag{3.22}$$

Inserting (3.22) into (3.21), using Green's identity in the integrals corresponding to the first two terms in (3.22), we obtain

$$\begin{aligned} H(x, y, t) - F(x, y, t) = &- \int_0^t \int_M \big\langle \nabla_z H(z, y, t - s), \nabla_z(\phi u_0)\big\rangle E(x, z, s) dx ds \\ &+ \int_0^t \int_M H(z, y, t - s)\big\langle \nabla_z E(x, z, s), u_0 \nabla_z \phi - \phi \nabla_z u_0\big\rangle dx ds. \end{aligned} \tag{3.23}$$

Using the bounds (3.9), (3.10) we obtain, for $0 < t \leq 1$, $y \in B(x, r_0)$,

$$|H(x, y, t) - F(x, y, t)| \leq C_{15} \int_0^t \left[(t-s)^{-\frac{1}{2}} + s^{-\frac{1}{2}}\right] ds \int_{B(x,r_0)} (u_0 + |\nabla_z u_0|)(x, z) E(z, y, 3(t-s)) E(x, z, 3s) dz \tag{3.24}$$

and using the hypothesis of bounded geometry (1.14) (see Remark 1), the lower bound for H in (3.9) (valid for all $t > 0$, see Remark 2), and the semigroup property

$$\begin{aligned} |H(x, y, t) - F(x, y, t)| &\leq C_{16} \int_0^t s^{-\frac{1}{2}} ds \int_M H(z, y, 6(t-s)) H(x, z, 6s) dz \\ &= C_{16} H(x, y, 6t) \int_0^t s^{-\frac{1}{2}} ds \leq C_{17} t^{\frac{1}{2}-\frac{n}{2}} \exp\left(-\frac{d^2(x, y)}{24t}\right). \end{aligned} \tag{3.25}$$

Finally, note that $u_0(x, x) = \phi(x) = 1$, hence using (1.14) we get $|u_0(x, y)\phi(y) - 1| \leq C_{18} d(x, y)$, for $y \in B(x, r_0)$, and therefore

$$\begin{aligned} |F(x, y, t) - E(x, y, t)| &= |E(x, y, t)||u_0(x, y)\phi(y) - 1| \\ &\leq C_{19} t^{\frac{1}{2}-\frac{n}{2}} \exp\left(-\frac{d^2(x, y)}{8t}\right). \end{aligned} \tag{3.26}$$

To prove (3.17), we use again the representation formula (3.21). This time after inserting (3.22) into (3.21) we take directly the gradient in the y variable, obtaining

$$\begin{aligned} &|\nabla_y H(x, y, t) - \nabla_y F(x, y, t)| \leq \\ &\leq C_{20} \int_0^t \int_M |\nabla_y H(z, y, t-s)| \, |E(x, z, s)| \big(u_0 + |\nabla_z u_0| + |\Delta_z u_0|\big)(x, z) dz ds \\ &\quad + C_{21} \int_0^t \int_M |\nabla_y H(z, y, t-s)| \, |\nabla_z E(x, z, s)| \, u_0(x, z) dz ds. \end{aligned} \tag{3.27}$$

Assuming 2^{nd} order bounded geometry, using the bounds (3.9), (3.10), and arguing just as in (3.24), (3.25) we obtain, for $0 < t \leq 1$, $y \in B(x, r_0)$,

$$\begin{aligned} |\nabla_y H(x, y, t) - \nabla_y F(x, y, t)| &\leq C_{22} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds \\ &\quad \times \int_{B(x,r_0)} E(z, y, 3(t-s)) E(x, z, 3s) dz \\ &\leq C_{23} t^{-\frac{n}{2}} \exp\left(-\frac{d^2(x, y)}{24t}\right). \end{aligned} \tag{3.28}$$

Finally,

$$
\begin{aligned}
|\nabla_y F - \nabla_y E| = \left|\nabla_y\big(E(\phi u_0 - 1)\big)\right| &\leq |\nabla_y u_0| E + |\nabla_y E||\phi u_0 - 1| \\
&\leq C_{24} t^{-\frac{n}{2}} \exp\left(-\frac{d^2(x,y)}{8t}\right)
\end{aligned}
\tag{3.29}
$$

arguing as in (3.26). □

3.3 Uniform Asymptotic Bounds: Large Distances From a Fixed Point

Let M have Euclidean volume growth i.e.

$$
\sigma_x(r) := \frac{V_x(r)}{B_n r^n} \searrow \sigma_M > 0. \tag{3.30}
$$

In [39], Li-Tam-Wang proved the following heat kernel bound: there is $C_{25} > 0$ depending only on n and σ_M, such that for all $x, y \in M$, $t > 0$ and $\delta > 0$

$$
\begin{aligned}
\frac{(4\pi t)^{-\frac{n}{2}}}{\theta_x\big(\delta d(x,y)\big)} \exp\left(-\frac{1+9\delta}{4t} d^2(x,y)\right) &\leq H(x,y,t) \leq \\
&\leq \big(1 + C_{25}(\delta + \beta(x,y,\delta)\big) \frac{(4\pi t)^{-\frac{n}{2}}}{\sigma_M} \exp\left(-\frac{1-\delta}{4t} d^2(x,y)\right)
\end{aligned}
\tag{3.31}
$$

where

$$
\beta(x,y,\delta) = \delta^{-2n} \max_{r \geq (1-\delta)d(x,y)} \left(1 - \frac{\theta_x(r)}{\theta_x(\delta^{2n+1} r)}\right), \tag{3.32}
$$

and where

$$
\theta_x(r) = \frac{A_x(r)}{n B_n r^{n-1}}. \tag{3.33}
$$

(Note that in the notation used in [39], $\theta_x(r) = A_x(r)/(nr^{n-1})$.)

Remark 4 In contrast with (3.18), estimate (3.31) implies that within each region $D = \{(x,y,t) : d^2(x,y) \leq Ct\}$ (some fixed $C > 0$) we have $H(x,y,t) \sim \frac{1}{\sigma_M} E(x,y,t)$, as $d(x,y) \to +\infty$, in the sense that

$$
\lim_{\substack{d(x,y) \to +\infty \\ (x,y,t) \in D}} \frac{H(x,y,t)}{E(x,y,t)} = \frac{1}{\sigma_M}, \tag{3.34}
$$

and the convergence is uniform with respect to t (within D).

To prove (3.34), note that for fixed $\delta > 0$ we have $\beta(x, y, \delta) \to 0$, as $d(x, y) \to +\infty$. From (3.31), fixing $\delta > 0$ and letting $d(x, y) \to +\infty$

$$\frac{1}{\sigma_M} e^{-\frac{9\delta}{4}C} \leq \liminf_{\substack{d(x,y)\to+\infty \\ (x,y,t)\in D}} \frac{H(x, y, t)}{E(x, y, t)} \leq \limsup_{\substack{d(x,y)\to+\infty \\ (x,y,t)\in D}} \frac{H(x, y, t)}{E(x, y, t)} \leq \frac{1}{\sigma_M}(1 + C_{25}\delta)e^{\frac{\delta}{4}C} \tag{3.35}$$

which gives (3.34) after letting $\delta \to 0$.

In the following proposition we work out (3.31) a bit more explicitly, in terms of the *volume ratio remainder* at a given point $x \in M$, defined as

$$\Lambda_x(r) = \sigma_x(r) - \sigma_M = \frac{V_x(r)}{B_n r^n} - \sigma_M. \tag{3.36}$$

Let us first note that using $\theta_x(r)$ in place of $\sigma_x(r)$ in (3.36) gives an equivalent remainder:

Lemma 1 *There is $C_M \in (0, 1]$ such that*

$$C_M\big(\sigma_x(r) - \sigma_M\big) \leq \theta_x(r) - \sigma_M \leq \sigma_x(r) - \sigma_M. \tag{3.37}$$

Proof On one hand we have

$$\sigma_x(r) = \frac{1}{B_n r^n}\int_0^r A_x(t)dt = \frac{1}{nr^n}\int_0^r \theta_x(t)nt^{n-1}dt \geq \frac{\theta_x(r)}{nr^n}\int_0^r nt^{n-1}dt = \theta_x(r). \tag{3.38}$$

On the other hand, assuming $\sigma_M < 1$ (if $\sigma_M = 1$ then $\sigma_x(r) = \theta_x(r) = 1$ for all r), the isoperimetric inequality (1.2) gives

$$\theta_x(r) - \sigma_M \geq \sigma_M^{\frac{1}{n}}\sigma_x(r)^{\frac{n-1}{n}} - \sigma_M = \sigma_M\left(\left(\frac{\sigma_x(r)}{\sigma_M}\right)^{\frac{n-1}{n}} - 1\right) \geq \psi(\sigma_M^{-1})\big(\sigma_x(r) - \sigma_M\big) \tag{3.39}$$

where $\psi(t) = (t^{\frac{n-1}{n}} - 1)/(t - 1)$, a decreasing function on $(1, \infty)$. □

Definition 1 Given $\phi : [0, \infty) \to (0, \infty)$, differentiable, decreasing, and $\phi(r) \to 0$ as $r \to \infty$, let, for $\delta, r > 0$,

$$\widetilde{\phi}(r) = \min_{\delta>0}\big(\delta + \delta^{-2n}\phi(\delta^{2n+1}r)\big). \tag{3.40}$$

It's clear that $\widetilde{\phi}(r)$ is well defined for all $r > 0$, and decreasing. In Lemma 2 we will show that $\widetilde{\phi}(r) \to 0$ as $r \to +\infty$, and in fact we will give an effective, and essentially optimal, bound for $\widetilde{\phi}$ in terms of ϕ.

Proposition 2 *If M has EVG, then there exists C_{26} depending only on n, B, σ_M such that for any $x, y \in M$ with $d(x, y) \geq 1$, and for any $t > 0$*

$$\left|H(x, y, t) - \frac{1}{\sigma_M}E(x, y, t)\right| \leq C_{26}\,\widetilde{\Lambda}_x\big(d(x, y)\big)\,E(x, y, 3t). \tag{3.41}$$

Proof For $0 < \delta \le \frac{1}{2}$ we have

$$\begin{aligned}\beta(x,y,\delta) &\le \delta^{-2n} \max_{r \ge \frac{1}{2}d(x,y)} \left(\frac{\theta_x(\delta^{2n+1}r) - \sigma_M}{\theta_x(\delta^{2n+1}r)} - \frac{\theta_x(r) - \sigma_M}{\theta_x(\delta^{2n+1}r)} \right) \\ &\le \delta^{-2n} \max_{r \ge \frac{1}{2}d(x,y)} \frac{\theta_x(\delta^{2n+1}r) - \sigma_M}{\theta_x(\delta^{2n+1}r)} \le \delta^{-2n} \max_{r \ge \frac{1}{2}d(x,y)} \frac{\theta_x(\delta^{2n+1}r) - \sigma_M}{\sigma_M} \\ &= \frac{\delta^{-2n}}{\sigma_M}\Big(\theta_x\big(\tfrac{1}{2}\delta^{2n+1}d(x,y)\big) - \sigma_M\Big).\end{aligned} \tag{3.42}$$

Also,

$$\frac{1}{\sigma_M} - \frac{1}{\theta_x\big(\delta d(x,y)\big)} \le \frac{\theta_x\big(\delta d(x,y)\big) - \sigma_M}{\sigma_M^2} \le 2\frac{\delta^{-2n}}{\sigma_M^2}\Big(\theta_x\big(\tfrac{1}{2}\delta^{2n+1}d(x,y)\big) - \sigma_M\Big) \tag{3.43}$$

since $\theta_x(\lambda r)/\lambda$ is decreasing in λ.

Since

$$|e^{-\lambda_1 x} - e^{-\lambda_2 x}| \le 3|\lambda_1 - \lambda_2| e^{-x/3}, \qquad \lambda_1, \lambda_2 \ge \tfrac{1}{2},\ x > 0. \tag{3.44}$$

and changing δ into $2^{\frac{1}{2n+1}}\delta$ and using (3.31) we get, for $\delta \le \frac{1}{4}$,

$$\Big|H(x,y,t) - \frac{1}{\sigma_M}E(x,y,t)\Big| \le C_{27}\Big[\delta + \delta^{-2n}\Big(\theta_x\big(\delta^{2n+1}d(x,y)\big) - \sigma_M\Big)\Big]\ E(x,y,3t). \tag{3.45}$$

It's easy to check that the same estimate holds for $\delta > \frac{1}{4}$, so, using Lemma 1, we get (3.41). □

The next Lemma provides more explicit information about the function $\widetilde{\phi}$:

Lemma 2 *Let $\phi, \widetilde{\phi}$ be as in Definition 1, and let $t(r)$ be the unique solution of $t/\phi(t) = r$. Then $t(r)$is differentiable, strictly increasing to $+\infty$, and*

$$\phi(t(r))^{\frac{1}{2n+1}} \le \widetilde{\phi}(r) \le 2\phi(t(r))^{\frac{1}{2n+1}}, \qquad r > 0. \tag{3.46}$$

Proof The statement about $t(r)$ is clear since $t/\phi(t)$ is strictly increasing, and $\phi(t) \to 0$, as $t \to \infty$. Let $\delta_r^{2n+1} = t(r)/r = \phi(t(r))$, which is equivalent to $\delta_r = \delta_r^{-2n}\phi(\delta_r^{2n+1}r)$. For fixed r let $F(\delta) = \delta + \delta^{-2n}\phi(\delta^{2n+1}r)$, so that $F(\delta_r) = 2\delta_r$. If the minimum of F is attained at $\delta_{r,0}$, then an easy calculation shows that $F(\delta_{r,0}) - \delta_{r,0} = \delta_{r,0}^{-2n}\phi(\delta_{r,0}^{2n+1}r) \le \frac{1}{2n}\delta_{r,0}$, and since $F(\delta) - \delta$ is decreasing, this implies that $\delta_r \le \delta_{r,0}$. Hence, we obtain

$$\delta_r \le \delta_{r,0} \le F(\delta_{r,0}) \le F(\delta_r) = 2\delta_r \tag{3.47}$$

which implies (3.46). □

In practical applications, given an estimate $\Lambda_x(r) \leq \phi(r)$, for $r \geq r_0$ with ϕ differentiable, decreasing, and going to 0 at infinity, the above lemma gives the rather explicit upper bound $\widetilde{\Lambda}_x(r) \leq 2\phi(t(r))^{\frac{1}{2n+1}}$, for $r \geq r_0$.

For example, when $\phi(r) = Cr^{-a}(\log r)^{-b}$ with $a \geq 0$, $b \in \mathbb{R}$, and $b > 0$ if $a = 0$, then one can check (using the Lambert function when $b \neq 0$) that for large r

$$t(r) \asymp r^{\frac{1}{1+a}}(\log r)^{-\frac{b}{1+a}} \qquad \widetilde{\phi}(r) \asymp r^{-\frac{a}{(1+a)(2n+1)}}(\log r)^{-\frac{b}{(1+a)(2n+1)}}. \tag{3.48}$$

where $f(r) \asymp g(r)$ means $c_1 g(r) \leq f(r) \leq c_2 g(r)$ for large r.

Asymptotically Locally Euclidean (ALE) manifolds provide an important family of manifolds with EVG such that $\Lambda_x(r) \leq Cr^{-a}$ for some $a > 0$, some $x \in M$, and for large r. For the definitions and properties of ALE manifolds see for example [3, 9].

We point out, however, that it is relatively easy to construct manifolds with EVG, such that $\Lambda_x(r)$ has arbitrarily slow rate of decay at infinity. For example, one can do so by considering rotationally invariant metrics on $\mathbb{R}^n$.

We will now derive sharp asymptotic estimates for the gradient of the heat kernel, and for large distances. In general we do not expect optimal pointwise bounds (see Remark 8) but sharp, aymptotic, L^2 integral bounds on annuli will suffice for our purpose.

For any $r, R > 0$ and $x \in M$ let

$$A_{r,R}(x) = \{y \in M : r \leq d(x, y) \leq R\} \tag{3.49}$$

for simplicity we will often omit the dependence on x in what follows.

The integral average over a Borel measurable set A will be denoted as

$$\fint_A = \frac{1}{\mu(A)} \int_A . \tag{3.50}$$

We will also use the notation

$$E(r, t) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{4t}\right) \tag{3.51}$$

so that $E(x, y, t) = E\big(d(x, y), t\big)$.

Proposition 3 *If M has EVG then there exists C_{28} depending only on n, σ_M such that for any $x \in M$, $\eta > 0$, $t > 0$, and $r \geq 1$ we have*

$$\left(\fint_{A_{r,(1+\eta)r}} \left|\nabla_y H(x, y, t) - \frac{1}{\sigma_M}\nabla_y E(x, y, t)\right|^2 dy\right)^{\frac{1}{2}} \leq C_{28}\Big(1 + \frac{1}{\sqrt{\eta}}\Big)\widetilde{\Lambda}_x(r)^{\frac{1}{2}} r^{-\frac{1}{2}} t^{-\frac{1}{4}} E(r, 3t). \tag{3.52}$$

Remark 5 In (3.52) the vector field $\nabla_y E(x, y, t)$ is intended to be defined for a.e. y, in particular on $M_x = M \setminus \mathrm{Cut}_x \setminus \{x\}$. The same type of remark can be made regarding the integral in (4.18).

Proof Since we are planning on using Stokes' theorem we need to regularize $E(d(x, y), t)$, as a function of y. Using a mollifier it's possible to show that given any compact set $K \subseteq M$ and $x \notin K$ then there exists a sequence of smooth, compactly supported functions $\{d_j\}$ (depending on x and K) such that for any $y \in K$ and any j

i) $|d_j(x, y) - d(x, y)| < \dfrac{1}{j}$
ii) $|\nabla_y d_j(x, y) - \nabla_y d(x, y)| \to 0$ as $j \to \infty$, if $K \subseteq M \setminus \mathrm{Cut}_x$
iii) $|\nabla_y d_j(x, y)| \leq 2$
iv) $\Delta_y d_j(x, y) \leq \dfrac{n-1}{d(x, y)} + \dfrac{1}{j}$

(cf. [16, 39]). The last estimate is basically an approximated form of the Laplacian comparison theorem, i.e. $\Delta_y d(x, y) \leq (n-1)/d(x, y)$, which holds in the sense of distributions and also inside M_x.

Using these estimates, it's straightforward to check that for any $y \in K$ and $t > 0$ and for any $j \geq j_0(x, K)$ large enough

$$|E(d_j(x, y), t) - E(d(x, y), t)| \leq \frac{C_{29}}{j} \operatorname{dist}(x, K)^{-1-n} \tag{3.53}$$

$$\lim_{j \to \infty} |\nabla_y E(d_j(x, y), t) - \nabla_y E(d(x, y), t)| = 0, \quad \text{if } K \subseteq M \setminus \mathrm{Cut}_x \tag{3.54}$$

$$|\nabla_y E(d_j(x, y), t)| \leq C_{30} \operatorname{dist}(x, K)^{-1-n} \tag{3.55}$$

$$\Delta_y E(d_j(x, y), t) \geq -\frac{n}{2} t^{-\frac{n}{2}-1} - \frac{C_{31}}{j} \big(\operatorname{dist}(x, K)^{-1-n} + \operatorname{dist}(x, K)^{-3-n}\big) \tag{3.56}$$

where the constants C_{29}, C_{30}, C_{31} depend only on n (see Appendix A)

Note that the Laplacian comparison (in the distributional sense) gives the estimate

$$\Delta_y E(d(x, y), t) \geq \partial_t E(d(x, y), t) \geq -\frac{n}{2} t^{-\frac{n}{2}-1}. \tag{3.57}$$

For simplicity, in the estimates below we will mute the variables x, y in the integrands, and we will let

$$E_j = E_j(x, y, t) = E(d_j(x, y), t).$$

Since Cut_x has zero measure, then for almost all $r > 0$ the set $\partial B(x, r) \cap \mathrm{Cut}_x$ has zero $(n-1)-$dimensional Hausdorff measure. Following [39] we shall call such r "permissible". If r, R are permissible, with $R > r > 1$, then $A_{r,R}$ is "almost C^1", in the sense that around each point of $\partial A_{r,R} \setminus \mathrm{Cut}_x$ the boundary of $A_{r,R}$ can be represented by a C^1 function. For such domains Stokes' theorem works (see for example [41], Thm. 9.6, for the $\mathbb{R}^n$ version, which can be extended to Riemannian manifolds). Hence, for all permissible r, R with $R > r > 1$ we have

$$\begin{aligned}
\int\limits_{A_{r,R}\setminus \mathrm{Cut}_x} \left|\nabla H - \sigma_M^{-1}\nabla E\right|^2 d\mu &\le \liminf_j \int\limits_{A_{r,R}\setminus \mathrm{Cut}_x} \left|\nabla H - \sigma_M^{-1}\nabla E_j\right|^2 d\mu \\
&= \liminf_j \int_{A_{r,R}} \left|\nabla H - \sigma_M^{-1}\nabla E_j\right|^2 d\mu \\
&= \liminf_j \Big(-\int_{A_{r,R}} (H - \sigma_M^{-1} E_j)\Delta(H - \sigma_M^{-1}E_j) d\mu \\
&\qquad + \int_{\partial A_{r,R}} (H - \sigma_M^{-1}E_j)\nabla(H - \sigma_M^{-1}E_j)\cdot \mathbf{N} d\tilde{\mu} \Big)
\end{aligned} \tag{3.58}$$

where $\mathbf{N}$ is the outward unit normal vector field along $\partial A_{r,R}$ and $\tilde{\mu}$ the induced measure on $\partial A_{r,R}$, and where the first inequality follows from (3.54) and Fatou's lemma. To estimate the first integral note that from (3.41) and (3.53) we have (with $d = d(x, y) \ge r \ge 1$)

$$|H - \sigma_M^{-1} E_j| \le C_{32}\, \widetilde{\Lambda}_x\big(d(x,y)\big)\, E(d, 3t) + \frac{C_{33}}{j} \le C_{32}\, \widetilde{\Lambda}_x(r)\, E(r, 3t) + \frac{C_{33}}{j}. \tag{3.59}$$

Using the heat equation and the time derivative estimate (3.3) we have

$$|\Delta H| = |\partial_t H| \le C_{34}\, t^{-\frac{n}{2}-1}. \tag{3.60}$$

To estimate the integral of $|\Delta E_j|$ over the annulus, we use the same idea as in [39], proof of Thm. 1.1, [16], proof of Lemma 1.4. From the Laplacian comparison (3.56) we have

$$\Delta E_j \ge -\frac{n}{2}\, t^{-\frac{n}{2}-1} - \frac{C_{35}}{j} \qquad \big(\Delta E_j\big)_- \le \frac{n}{2}\, t^{-\frac{n}{2}-1} + \frac{C_{35}}{j} \tag{3.61}$$

where $\big(\Delta E_j\big)_- = \max\big\{-\Delta E_j, 0\big\}$ is the negative part of ΔE_j. Hence, using Stokes' theorem again,

$$\int_{A_{r,R}} |\Delta E_j| = \int_{A_{r,R}} \Delta E_j + 2\int_{A_{r,R}} \big(\Delta E_j\big)_- = \int_{\partial A_{r,R}} \nabla E_j \cdot \mathbf{N} + 2\int_{A_{r,R}} \big(\Delta E_j\big)_- \tag{3.62}$$

From (3.54), (3.55) (and since r, R are permissible)

$$\lim_{j\to\infty} \int_{\partial A_{r,R}} \nabla E_j \cdot \mathbf{N} = \int\limits_{\partial A_{r,R}\setminus \mathrm{Cut}_x} \nabla E \cdot \mathbf{N} = A_x(R)\frac{R}{2t}E(R,t) - A_x(r)\frac{r}{2t}E(r,t) \le C_{36} R^n t^{-\frac{n}{2}-1}. \tag{3.63}$$

Combining the above estimates we obtain

$$\begin{aligned}
&\limsup_j \int_{A_{r,R}} (H-\sigma_M^{-1}E_j)\Delta(H-\sigma_M^{-1}E_j)\\
&\le \limsup_j \Big(C_{32}\,\widetilde{\Lambda}_x(r)\,E(r,3t)+\frac{C_{33}}{j}\Big)\int_{A_{r,R}}\big(|\Delta H|+|\Delta E_j|\big)\\
&\le C_{37}\limsup_j \Big(\widetilde{\Lambda}_x(r)\,E(r,3t)+\frac{1}{j}\Big)\bigg[\int_{A_{r,R}}\Big(t^{-\frac{n}{2}-1}+\frac{1}{j}\Big)+\int_{\partial A_{r,R}}\nabla E_j\cdot\mathbf{N}\bigg]\\
&\le C_{38}R^n t^{-\frac{n}{2}-1}\,\widetilde{\Lambda}_x(r)\,E(r,3t)=C_{39}R^n t^{-n-1}\,\widetilde{\Lambda}_x(r)\exp\Big(-\frac{r^2}{12t}\Big).
\end{aligned}\tag{3.64}$$

To estimate the boundary integral in (3.58), using (3.54), (3.10), (3.11)

$$\begin{aligned}
&\lim_{j\to\infty}\int_{\partial A_{r,R}}(H-\sigma_M^{-1}E_j)\nabla(H-\sigma_M^{-1}E_j)\cdot\mathbf{N}\\
&=\int_{\partial A_{r,R}\setminus \mathrm{Cut}_x}(H-\sigma_M^{-1}E)\nabla(H-\sigma_M^{-1}E)\cdot\mathbf{N}\\
&\le \int_{\partial A_{r,R}\setminus \mathrm{Cut}_x}|H-\sigma_M^{-1}E|\,\big(|\nabla H|+\sigma_M^{-1}|\nabla E|\big)\\
&\le C_{40}\int_{\partial A_{r,R}}\widetilde{\Lambda}_x\big(d(x,y)\big)\,E(d,3t)t^{-\frac{n}{2}-\frac{1}{2}}\\
&\le C_{41}R^{n-1}t^{-\frac{n}{2}-\frac{1}{2}}\widetilde{\Lambda}_x(r)E(r,3t)=C_{42}R^{n-1}t^{-n-\frac{1}{2}}\,\widetilde{\Lambda}_x(r)\exp\Big(-\frac{r^2}{12t}\Big).
\end{aligned}\tag{3.65}$$

Putting together (3.58), (3.64), (3.65) we get, for all permissible r, R with $R > r > 1$,

$$\begin{aligned}
&\int_{A_{r,R}}\big|\nabla H-\sigma_M^{-1}\nabla E|^2\le\\
&\le \limsup_j\int_{A_{r,R}}(H-\sigma_M^{-1}E_j)\Delta(H-\sigma_M^{-1}E_j)d\mu\\
&\qquad+\lim_{j\to\infty}\int_{\partial A_{r,R}}(H-\sigma_M^{-1}E_j)\nabla(H-\sigma_M^{-1}E_j)\cdot\mathbf{N}\\
&\le C_{43}R^{n-1}t^{-n-\frac{1}{2}}\widetilde{\Lambda}_x(r)\big(Rt^{-\frac{1}{2}}+1\big)\exp\Big(-\frac{r^2}{12t}\Big).
\end{aligned}\tag{3.66}$$

Now note that (3.66) is actually true for *all* r, R with $R > r > 1$, since for all such r, R the integral on the left hand side of (3.66) is finite (as it increases as the annulus $A_{r,R}$ gets larger), and therefore it's separately continuous in r and R, as well as the right hand side of (3.66).

Taking $R = (1+\eta)r$ we obtain (3.52). □

4 Uniform Green Functions Estimates

Let us assume once again that M has nonnegative Ricci curvature. Let $G_\alpha(x,y)$ denote the minimal, positive, Green function for $(-\Delta)^{\frac{\alpha}{2}}$ for α even and $0<\alpha<n$, i.e. a positive C^∞ function off the diagonal such that $(-\Delta_y)^{\frac{\alpha}{2}}G_\alpha(x,y)=\delta_x(y)$. In terms of the heat kernel we have

$$G_\alpha(x,y)=\frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty t^{\frac{\alpha}{2}-1}H(x,y,t)dt \tag{4.1}$$

provided the integral on the right hand side converges. Note that with $E(x,y,t)$ defined in (3.8) we have

$$\frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty t^{\frac{\alpha}{2}-1}E(x,y,t)dt=c_\alpha d^{\alpha-n}(x,y), \tag{4.2}$$

where c_α is given in (1.7), and which reduces to the classical Riesz kernel on the standard $\mathbb{R}^n$.

Arguing as in [36], Thm 5.2, using the rough bound (3.1), the integral in (4.1) is convergent if and only if

$$\int_1^\infty \frac{t^{\frac{\alpha}{2}-1}}{V_x(\sqrt{t})}dt<\infty \tag{4.3}$$

(split the integral in (4.1) at $r^2=d^2(x,y)$, make the change $s=r^4/t\geq r^2$ and use the Bishop comparison theorem in the form $V_x(r^2/\sqrt{s})/V_x(\sqrt{s})\geq(r^2/\sqrt{s})^n/(\sqrt{s})^n$).

It's easy to see that if (4.3) holds for one x, then it holds for all x.

When $\alpha=2$ condition (4.3) is equivalent to the non-parabolicity of a complete manifold with non negative ricci curvature M, i.e. the fact that M admits a positive Green's function (see [36, 38, 58])

The argument that leads to (4.3) gives the rough bound

$$C_{44}\int_{d^2(x,y)}^\infty \frac{t^{\frac{\alpha}{2}-1}}{V_x(\sqrt{t})}dt\leq G_\alpha(x,y)\leq C_{45}\int_{d^2(x,y)}^\infty \frac{t^{\frac{\alpha}{2}-1}}{V_x(\sqrt{t})}dt. \tag{4.4}$$

Define, for $0<\alpha<n$,

$$\widetilde{G}_\alpha(x,y)=\frac{1}{\Gamma(\frac{\alpha+1}{2})}\int_0^\infty t^{\frac{\alpha-1}{2}}\nabla_y H(x,y,t)dt \tag{4.5}$$

In view of the gradient bound (3.2) it's easy to see that the above integral converges under (4.3), and

$$|\widetilde{G}_\alpha(x,y)|\leq C_{46}\int_{d^2(x,y)}^\infty \frac{t^{\frac{\alpha}{2}-1}}{V_x(\sqrt{t})}dt. \tag{4.6}$$

If in addition $\alpha+1<n$ then we have $\widetilde{G}_\alpha(x,y)=\nabla_y G_{\alpha+1}(x,y)$.

We have, for $0 < \alpha < n$,

$$\frac{1}{\Gamma(\frac{\alpha+1}{2})}\int_0^\infty t^{\frac{\alpha-1}{2}}\nabla_y E(x,y,t)dt = -\widetilde{c}_\alpha d^{\alpha-n}(x,y)\nabla_y d(x,y), \tag{4.7}$$

where $\widetilde{c}_\alpha$ is given in (1.10).

Assuming that M has Euclidean volume growth then (4.3) is verified, and (4.4), (4.6) give immediately the following global rough bounds:

$$\begin{aligned} C_{47}d(x,y)^{\alpha-n} \le G_\alpha(x,y) \le C_{48}d^{\alpha-n}(x,y),& \\ |\widetilde{G}_\alpha(x,y)| \le C_{49}d^{\alpha-n}(x,y), &\quad \forall x,y\in M \end{aligned} \tag{4.8}$$

In particular, for $\alpha = 1$ (4.8) gives

$$|\nabla_y G_2(x,y)| \le C_{50}d^{1-n}(x,y), \qquad \forall x,y\in M. \tag{4.9}$$

Finally, assuming (4.3), we have the representation formulas for $u\in C_0^\infty(M)$

$$u(x) = \int_M G_\alpha(x,y)(-\Delta)^{\frac{\alpha}{2}}u(y)d\mu(y), \qquad \alpha \text{ even} \tag{4.10}$$

$$u(x) = \int_M \big\langle\widetilde{G}_\alpha(x,y), \nabla(-\Delta)^{\frac{\alpha-1}{2}}u(y)\big\rangle d\mu(y), \qquad \alpha \text{ odd} \tag{4.11}$$

which can be easily obtained using integration by parts, the heat equation, $E(x,\cdot,0) = \delta_x(\cdot)$, and the fact that $t^{\frac{\alpha}{2}-1}V_x(\sqrt{t})^{-1}\to 0$ as $t\to\infty$ (a consequence of (4.3)).

4.1 Uniform Asymptotic Bounds: Small Distances

Proposition 4 *Let M have nonnegative Ricci curvature and*

$$\sup_{x\in M}\int_1^\infty \frac{t^{\frac{\alpha}{2}-1}}{V_x(\sqrt{t})}dt < \infty. \tag{4.12}$$

a) If M has 1^{st} order bounded geometry, then there exists $C_{51} > 0$ such that for $d(x,y)\le 1$

$$\big|G_\alpha(x,y) - c_\alpha\, d^{\alpha-n}(x,y)\big| \le C_{51}\, d^{\alpha-n+1}(x,y). \tag{4.13}$$

b) If M has 2^{nd} order bounded geometry, then there exists $C_{52} > 0$ such that for $d(x,y)\le 1$ and $y\in M_x$

$$\big|\widetilde{G}_\alpha(x,y) + \widetilde{c}_\alpha\, d^{\alpha-n}(x,y)\nabla_y d(x,y)\big| \le C_{52}\, d^{\alpha-n+1}(x,y). \tag{4.14}$$

Remark 6 When M has EVG, then (4.12) holds, for $\alpha < n$.

Proof Assuming 1^{st} order bounded geometry, from (4.1), (4.2), (3.16), we get

$$\begin{aligned}\big|G_\alpha(x,y)-c_\alpha\, d^{\alpha-n}(x,y)\big| &\leq \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^1 t^{\frac{\alpha}{2}-1}\big|H(x,y,t)-E(x,y,t)\big|dt\\ &+\frac{1}{\Gamma(\frac{\alpha}{2})}\int_1^\infty t^{\frac{\alpha}{2}-1}H(x,y,t)dt+\frac{1}{\Gamma(\frac{\alpha}{2})}\int_1^\infty t^{\frac{\alpha}{2}-1}E(x,y,t)dt\\ &\leq \frac{C_{13}}{\Gamma(\frac{\alpha}{2})}\int_0^1 t^{-\frac{n}{2}+\frac{1}{2}+\frac{\alpha}{2}-1}\exp\left(-\frac{d^2(x,y)}{24t}\right)dt\\ &+\frac{C_2}{\Gamma(\frac{\alpha}{2})}\int_1^\infty \frac{t^{\frac{\alpha}{2}-1}}{V_x(\sqrt{t})}dt+\frac{(4\pi)^{-\frac{n}{2}}}{\Gamma(\frac{\alpha}{2})}\int_1^\infty t^{\frac{\alpha}{2}-\frac{n}{2}-1}dt\end{aligned}\tag{4.15}$$

and (4.13) follows from (4.12) and

$$\int_0^\infty t^{-\lambda-1}e^{-\rho/t}dt=\rho^{-\lambda}\Gamma(\lambda)\tag{4.16}$$

The proof of (4.14) under 2^{nd} order bounded geometry is similar, using (4.5), (4.7), and (3.17). □

4.2 Uniform Asymptotic Bounds: Large Distances from a Fixed Point

Proposition 5 *If M has EVG, then there exist C_{53}, C_{54}, depending only on n, σ_M, such that for $x, y \in M$, $d(x,y)\geq 1$*

$$\left|G_\alpha(x,y)-\frac{c_\alpha}{\sigma_M}\,d^{\alpha-n}(x,y)\right|\leq C_{53}\,\widetilde{\Lambda}_x\big(d(x,y)\big)d^{\alpha-n}(x,y),\tag{4.17}$$

and for any $\eta>0$ and any $r\geq 1$

$$\left(\fint_{A_{r,(1+\eta)r}}\left|\widetilde{G}_\alpha(x,y)+\frac{\widetilde{c}_\alpha}{\sigma_M}d^{\alpha-n}(x,y)\nabla_y d(x,y)\right|^2dy\right)^{\frac{1}{2}}\leq C_{54}\Big(1+\frac{1}{\sqrt{\eta}}\Big)\widetilde{\Lambda}_x^{\frac{1}{2}}(r)r^{\alpha-n}.\tag{4.18}$$

Remark 7 Assuming that M has EVG, then the global rough bounds in (4.8) imply that (4.13), (4.14), (4.17), (4.18) hold for all $y\in M$.

Remark 8 When $\alpha=1$ and $n\geq 3$, and under EVG estimate (4.18) becomes

$$\begin{aligned}&\left(\fint_{A_{r,(1+\eta)r}}\left|\nabla_y G_2(x,y)-\frac{(2-n)c_2}{\sigma_M}d^{1-n}(x,y)\nabla_y d(x,y)\right|^2dy\right)^{\frac{1}{2}}\\ &\qquad\leq C_{54}\Big(1+\frac{1}{\sqrt{\eta}}\Big)\widetilde{\Lambda}_x^{\frac{1}{2}}(r)r^{1-n}\end{aligned}\tag{4.19}$$

(in our notation $G_2(x,y)$ is the Green function for $-\Delta$), a version of [18], eq. (4.7), with a more explicit bound in terms of the volume ratio remainder (we note that in

[18] they use a different normalization for the Green function, and also that in (4.7) there should be s^{2-2n}). Compare also (6.9) with [18], (3.38).

Without further restrictions on the geometry one cannot expect a pointwise bound for $\widetilde{G}_\alpha$, in particular a bound of type

$$\Big|\nabla_y G_\alpha(x,y) - \frac{c_\alpha}{\sigma_M}\nabla_y d^{\alpha-n}(x,y)\Big| \le C_{55}\,\widetilde{\Lambda}_x\big(d(x,y)\big)d^{\alpha-n-1}(x,y), \tag{4.20}$$

which would imply (4.18). When $\alpha = 2$, such an estimate was proved in a weaker form in [18], Prop. 4.1, under quadratic curvature decay, and we believe that it should still hold for general α under such assumption.

Proof Estimate (4.17) follows at once from (4.1), (4.2), (3.41), (4.16). Likewise, (4.5), (4.7) imply

$$\begin{aligned}&\bigg(\fint_{A_{r,(1+\eta)r}}\Big|\widetilde{G}_\alpha(x,y)+\frac{\widetilde{c}_\alpha}{\sigma_M}d^{\alpha-n}(x,y)\nabla_y d(x,y)\Big|^2 dy\bigg)^{\frac12}\\ &\le \frac{1}{\Gamma(\frac{\alpha+1}{2})}\int_0^\infty t^{\frac{\alpha-1}{2}}\bigg(\fint_{A_{r,(1+\eta)r}}\Big|\nabla_y H(x,y,t)-\frac{1}{\sigma_M}\nabla_y E(x,y,t)\Big|^2 dy\bigg)^{\frac12}dt\end{aligned} \tag{4.21}$$

and (4.18) follows from (3.52) and (4.16). □

4.3 Uniform Global Rearrangement Bounds: Talenti's Type Estimates

In this section we will derive sharp global estimates for the symmetric decreasing rearrangement of G_α in terms of the ones for the Riesz kernel $N_\alpha(\xi) = c_\alpha|\xi|^{\alpha-n}$ on $\mathbb{R}^n$, as well as similar integral bounds for $|\nabla G_\alpha|^q$. These results are in the spirit of the classical Talenti's comparison estimates [54], but in the context of manifolds with Euclidean volume growth. We will make use of recent results in this direction due to Chen-Li [14].

Let us recall the definition of decreasing rearrangement on a general measure space (M,μ). Given a measurable function $f\;\; M \to [-\infty,\infty]$ its distribution function is defined as

$$\lambda_f(s) = \mu(\{x\in M:\ |f(x)|>s\}),\quad s\ge 0.$$

Assuming that the distribution function of f is finite for all $s > 0$, the decreasing rearrangement of f is the function $f^*: [0,\infty) \to [0,\infty]$ defined as

$$f^*(t) = \inf\{s\ge 0:\ \lambda_f(s)\le t\},\quad t\ge 0, \tag{4.22}$$

where clearly $f^*(t) = 0$ if $\mu(M) \le t < \infty$. The Schwarz symmetric decreasing rearrangement of a measurable function f is the radially symmetric decreasing function $f^\#: \mathbb{R}^n \to \mathbb{R}$ defined as

$$f^\#(\xi) = f^*(B_n|\xi|^n),\qquad \xi\in\mathbb{R}^n. \tag{4.23}$$

Given a measurable function $k(x, y)$ on $M \times M$, we will use the following notation for the rearrangements of the $x-$sections and y-sections of k:

$$k^*(x,t) = \big(k(x,\cdot)\big)^*(t), \qquad k^*(t,y) = \big(k(\cdot,y)\big)^*(t), \qquad x, y \in M,\ t \geq 0. \tag{4.24}$$

Note also that for given $\gamma > 0$

$$\lambda_f(s) \leq A s^{-\frac{1}{\gamma}}, \quad \forall s > 0 \quad \Longleftrightarrow \quad f^*(t) \leq A^{\frac{1}{\gamma}} t^{-\frac{1}{\gamma}}, \quad \forall t > 0 \tag{4.25}$$

and if either inequality holds a.e. then it also holds everywhere.

Theorem 3 *Let M have EVG,* Ric ≥ 0, dim$M = n \geq 3$, *and denote*

$$\lambda_\alpha(x,s) = \mu(\{y : G_\alpha(x,y) > s\}), \qquad x \in M. \tag{4.26}$$

Then, for any α even, $2 \leq \alpha < n$, and all $x \in M$, we have

$$G_\alpha^*(x,t) \leq \sigma_M^{-\frac{\alpha}{n}} N_\alpha^*(t) = \sigma_M^{-\frac{\alpha}{n}} B_n^{\frac{n-\alpha}{n}} c_\alpha\, t^{-\frac{n-\alpha}{n}}, \qquad t > 0, \tag{4.27}$$

and for $0 < q \leq 2$, $0 < s_1 < s_2$, and all $x \in M$, we have

$$\int\limits_{s_1 < G_\alpha(x,\cdot) \leq s_2} |\nabla_y G_\alpha(x,y)|^q dy \leq \sigma_M^{-\frac{q(\alpha-1)}{n}} \int\limits_{\lambda_\alpha(x,s_2) < B_n|\xi|^n \leq \lambda_\alpha(x,s_1)} \big|\nabla N_\alpha(\xi)\big|^q d\xi. \tag{4.28}$$

In particular, for all $x \in M$, $0 < s_1 < s_2$, α even, and $2 \leq \alpha < n$, we have

$$\int\limits_{s_1 < G_\alpha(x,\cdot) \leq s_2} |G_\alpha(x,y)|^{\frac{n}{n-\alpha}} dy \leq \sigma_M^{-\frac{\alpha}{n-\alpha}} \gamma_{n,\alpha}^{-1} \log \frac{\lambda_\alpha(x,s_1)}{\lambda_\alpha(x,s_2)} \tag{4.29}$$

whereas for α odd and $1 \leq \alpha \leq \frac{n}{2}$

$$\int\limits_{s_1 < G_{\alpha+1}(x,\cdot) \leq s_2} |\nabla_y G_{\alpha+1}(x,y)|^{\frac{n}{n-\alpha}} dy \leq \sigma_M^{-\frac{\alpha}{n-\alpha}} \gamma_{n,\alpha}^{-1} \log \frac{\lambda_{\alpha+1}(x,s_1)}{\lambda_{\alpha+1}(x,s_2)}. \tag{4.30}$$

Remark 9 As it will be apparent from the proof below, we will also obtain the sharp bounds (4.27)-(4.30) for the Green functions of the Poisson equation with homogeneous Navier boundary condition on any smooth bounded domain $\Omega \subseteq M$. Specifically, given

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = f \text{ on } \Omega \\ (-\Delta)^j u = 0 \text{ on } \partial\Omega,\ 0 \leq j \leq \lceil(\alpha-1)/2\rceil, \end{cases} \tag{4.31}$$

and the corresponding symmetrized problem

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}} v = f^{\#} \text{ on } \Omega^{\#} \\ (-\Delta)^{j} v = 0 \text{ on } \partial\Omega^{\#}, \ 0 \leq j \leq \lceil(\alpha-1)/2\rceil,\end{cases} \tag{4.32}$$

where $\Omega^{\#}$ denotes the ball of center 0 in $\mathbb{R}^n$ and with volume $\mu(\Omega)$, if $G_\alpha^\Omega(x,y)$ denotes the Green function for (4.31), and $N_\alpha^{\Omega^{\#}}(\xi)$ the Green function for (4.32) with pole 0, then (4.27)-(4.30) hold with G_α^Ω in place of G_α and $N_\alpha^{\Omega^{\#}}$ in place of N_α, where λ_α is of course defined with respect to Ω.

When $M = \mathbb{R}^n$ ($\sigma_M = 1$) bounds (4.27) for the Dirichlet Green function G_2^Ω appear in Bandle [8], whereas for the biharmonic Green function ($\alpha = 4$) they can be deduced from results in [21]. The integral bounds (4.28), (4.29), (4.30) do not seem to appear in the literature, even on $\mathbb{R}^n$. Note that letting $s_2 \to \infty$ and $s_1 \to 0$ in (4.28), allows us to replace the domains of integration with Ω and $\Omega^{\#}$ respectively. Such an estimate on $\mathbb{R}^n$ was given by Weinberger [60] for $\alpha = 2$ and for $q < \frac{n}{n-1}$ (see also [8], Thm 2.5).

Proof We will begin to prove (4.27), (4.28) for the Dirichlet Green function G_2^Ω of a smooth bounded domain Ω in M. Let us recall some classical results first. If $\Omega_1 \subseteq \Omega_2$ then $G_2^{\Omega_1}(x,y) \leq G_2^{\Omega_2}(x,y)$ for $x,y \in \Omega_1$, $x \neq y$, by the maximum principle and the local asymptotic expansion of G_2^Ω (see e.g. [50], p. 82). Secondly, under our assumptions M is non parabolic, and if $\{\Omega_k\}$ is an increasing sequence of smooth, bounded, open sets which exhaust M, then for any fixed $x,y \in M$, ($x \neq y$) we have $x,y \in \Omega_k$ for $k \geq k_0$, and $G_2^{\Omega_k}(x,y) \uparrow G_2(x,y)$, where G_2 is the unique positive, minimal Green function defined earlier.

Recall now a recent result by Chen-Li [14], which gives a version of Talenti's comparison theorem on a noncompact Riemannian manifold M, with EVG and Ric ≥ 0. If Ω is a smooth, bounded open set in M, and $f \in L^2(\Omega)$ then let $u \in W_0^{1,2}(\Omega)$ be the weak solution of

$$\begin{cases}-\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\end{cases} \tag{4.33}$$

and let $v \in W_0^{1,2}(\Omega^{\#})$ be the weak solution of

$$\begin{cases}-\Delta v = f^{\#} & \text{in } \Omega^{\#} \\ v = 0 & \text{on } \partial\Omega^{\#}.\end{cases} \tag{4.34}$$

Then, Chen-Li proved that

$$u^*(t) \leq \sigma_M^{-\frac{2}{n}} v^*(t), \qquad t > 0. \tag{4.35}$$

The proof is based on Talenti's original argument, in the version given by Kesavan [29], combined with the sharp isoperimetric inequality on manifolds with EVG. In particular, if

$$\lambda(s) := \mu(\{|u| > s\}), \qquad F(t) = \int_0^t f^*(\zeta)d\zeta \tag{4.36}$$

then

$$1 \le \sigma_M^{-\frac{2}{n}}(nB_n^{\frac{1}{n}})^{-2}\lambda(s)^{\frac{2}{n}-2}F\big(\lambda(s)\big)(-\lambda'(s)), \qquad \text{a.e. } s > 0. \tag{4.37}$$

Arguing as in [29], p. 51, it's easy to see that for $0 < q \le 2$

$$-\frac{d}{ds}\int_{|u|>s}|\nabla u|^q dx \le F\big(\lambda(s)\big)^{\frac{q}{2}}(-\lambda'(s))^{\frac{2-q}{2}}, \qquad \text{a.e. } s > 0. \tag{4.38}$$

Multiplying the last estimate by (4.37) raised to the power $q/2$, integrating in $s \in [s_1, s_2]$, and changing variables yields

$$\begin{aligned}\int_{s_1<|u|\le s_2}|\nabla u|^q dx &\le \sigma_M^{-\frac{q}{n}}\int_{s_1}^{s_2}\Big((nB_n^{\frac{1}{n}})^{-1}\lambda(s)^{\frac{1}{n}-1}F(\lambda(s))\Big)^q(-\lambda'(s))ds\\ &\le \sigma_M^{-\frac{q}{n}}\int_{\lambda(s_2)}^{\lambda(s_1)}\Big((nB_n^{\frac{1}{n}})^{-1}t^{\frac{1}{n}-1}F(t)\Big)^q dt = \sigma_M^{-\frac{q}{n}}\int_{\lambda(s_2)<B_n|\xi|^n\le\lambda(s_1)}|\nabla v(\xi)|^q d\xi,\end{aligned} \tag{4.39}$$

where for the last identity we used the explicit formula

$$v(\xi) = n^{-2}B_n^{-\frac{2}{n}}\int_{B_n|\xi|^n}^{|\Omega^\#|}\tau^{\frac{2}{n}-2}F(\tau)d\tau. \tag{4.40}$$

With this in mind, we can readily obtain (4.27),(4.28) for the Dirichlet Green function G_2^Ω of a smooth domain Ω inside M.

Fix $x \in \Omega$, and let $f_k \in C_c^\infty(M)$ be such that $f \ge 0$, $\int f_k = 1$ and $\operatorname{supp} f_k \subseteq B(x, \frac{1}{k}) \subseteq \Omega$, for $k \ge k_0$. Then $f_k \to \delta_x$ in the sense of distributions. Let u_k be the solution of the Dirichlet problem (4.33) with $f = f_k$, and let v_k be the solution of the corresponding symmetrized problem (4.34). The function u_k is given by the formula

$$u_k(y) = \int_\Omega G_2^\Omega(y,z)f_k(z)dz, \qquad y \in \Omega \tag{4.41}$$

and it's smooth. Fix $y \in \Omega$, $y \ne x$ and assume $\frac{1}{k_0} < \frac{1}{2}d(y,x)$. If φ is a nonnegative bump function and supported on $B(x, \frac{1}{k_0})$ and equal 1 on $B(x, \frac{1}{k_0+1})$, then for $k > k_0 + 1$

$$u_k(y) = \int_\Omega G_2^\Omega(y,z)\varphi(z)f_k(z)dz \to u(y) := G_2^\Omega(y,x) \tag{4.42}$$

and moreover, since $G_2^\Omega(y,z) \le G_2(y,z) \le C_{48}d(y,z)^{2-n}$ (from (4.8)),

$$0 \le u_k(y) \le \int_{B(x,\frac{1}{k_0})} G_2^\Omega(y,z)\varphi(z)f_k(z)dz$$

$$\leq C_{48}\int_{B(x,\frac{1}{k_0})} d(y,z)^{2-n} f_k(z)dz \leq 2^{n-2}C_{48}d(y,x)^{2-n}. \tag{4.43}$$

Regarding v_k, we have the explicit formula

$$v_k^*(t) = n^{-2}B_n^{-\frac{2}{n}}\int_t^{|\Omega^\#|} \tau^{\frac{2}{n}-2}F_k(\tau)d\tau, \qquad F_k(\tau) = \int_0^\tau f_k^* \tag{4.44}$$

valid for $0 < t \leq |\Omega^\#| = \mu(\Omega)$, where the rearrangement is made with respect to Ω. Since $\int f_k = 1$ we then have $F_k \leq 1$, and in fact for any $t \in (0, |\Omega^\#|]$, we have $F_k(t) = 1$ for k large enough (so that $\mu(\operatorname{supp} f_k) < t$).

Hence,

$$v_k^*(t) \leq \frac{1}{n(n-2)B_n^{\frac{2}{n}}}\big(t^{-\frac{n-2}{n}} - |\Omega^\#|^{-\frac{n-2}{n}}\big) := (N_2^{\Omega^\#})^*(t) \tag{4.45}$$

where, recall, $N_2^{\Omega^\#}$ is the Dirichlet Green function for the ball of volume $|\Omega^\#| = \mu(\Omega)$, and with pole 0.

By (4.35) we have, for $0 < t \leq |\Omega^\#|$

$$u_k^*(t) \leq \sigma_M^{-\frac{2}{n}} v_k^*(t) \leq \sigma_M^{-\frac{2}{n}}(N_2^{\Omega^\#})^*(t) \leq \frac{\sigma_M^{-\frac{2}{n}}}{n(n-2)B_n^{\frac{2}{n}}}\, t^{-\frac{n-2}{n}}, \tag{4.46}$$

and hence

$$\lambda_{u_k}(s) \leq \left(\frac{\sigma_M^{-\frac{2}{n}}}{n(n-2)B_n^{\frac{2}{n}}}\right)^{\frac{n}{n-2}} s^{-\frac{n}{n-2}}, \qquad s > 0. \tag{4.47}$$

Now, since λ_u is discontinuous at countably many points, then $\mu(\{y : |u(y)| = s\}) = 0$ for a.e. $s > 0$. For any such s we have that $\chi_{\{|u_k|>s\}} \to \chi_{\{|u|>s\}}$, μ-a.e., and using the Dominated Convergence Theorem and the fact that $\mu(\Omega) < \infty$, we can conclude that $\lambda_{u_k}(s) \to \lambda_u(s)$. Note that the convergence is actually true for all $s > 0$, in the case of the Green function, since by classical results (see for ex. [15], Thm. 2.2) we have $\mu(\{z \in \Omega : G_2^\Omega(y,z) = s\}) = 0$ for all $s > 0$ (see also Remark 14). Thus we can conclude that

$$\lambda_{u_k}(s) \leq \left(\frac{\sigma_M^{-\frac{2}{n}}}{n(n-2)B_n^{\frac{2}{n}}}\right)^{\frac{n}{n-2}} s^{-\frac{n}{n-2}} \tag{4.48}$$

for a.e. $s > 0$ and hence for all $s > 0$. The above inequality implies then (4.27), when $\alpha = 2.$, and for the Green function G_2^Ω.

The proof of (4.28) for $\alpha = 2$ follows similarly from (4.39) applied to u_k, v_k. Indeed, using (4.41) we can see easily that $|\nabla_y u_k(y)|^2 \to |\nabla_y u(y)|^2 = |\nabla_y G_2^\Omega(y,x)|^2$ for $y \in \Omega$ and Fatou's lemma combined with (4.39) gives (4.28).

The proof of (4.27), (4.28) for general α even, is done by iteration. using the relations

$$G_\alpha^\Omega(x,y) = \int_M G_2^\Omega(x,z)G_{\alpha-2}^\Omega(z,y)dz, \qquad -\Delta_y G_\alpha^\Omega(x,\cdot) = G_{\alpha-2}^\Omega(x,\cdot) \tag{4.49}$$

where G^{Ω}_{α} denotes the Green function for the problem $(-\Delta)^{\frac{\alpha}{2}}u = f$ on Ω, with homogeneous Navier boundary condition $(-\Delta)^{j}u = 0$ on $\partial\Omega$, for $0 \le j \le \lceil(\alpha-1)/2\rceil$.

Indeed, fix $x \in M$ and let $u_{k,\alpha}$, $v_{k,\alpha}$ be the solutions to the problems

$$\begin{cases} -\Delta u_{k,\alpha} = u_{k,\alpha-2} & \text{in } \Omega \\ u_{k,\alpha} = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta v_{k,\alpha} = u^{\#}_{k,\alpha-2} & \text{in } \Omega^{\#} \\ v_{k,\alpha} = 0 & \text{on } \partial\Omega^{\#} \end{cases} \tag{4.50}$$

where $u_{k,2} = f_k$.

Assuming that for each $y \in \Omega$

$$u_{k,\alpha-2}(y) \to G^{\Omega}_{\alpha-2}(y,x), \qquad \text{as } k \to \infty \tag{4.51}$$

$$0 \le u_{k,\alpha-2}(y) \le C_\alpha d(x,y)^{\alpha-2-n}, \qquad \text{for } k > 2k_0 \tag{4.52}$$

and

$$u^{*}_{k,\alpha-2}(t) \le \sigma_M^{-\frac{\alpha-2}{n}} B_n^{\frac{n-\alpha-2}{n}} c_{\alpha-2} t^{-\frac{n-\alpha-2}{n}}, \qquad t > 0, \tag{4.53}$$

where C_α is a constant independent of x, y, Ω, by induction we can show that see that (4.51), (4.52), (4.53), and hence (4.27), also hold for $u_{k,\alpha}$.

Indeed for $\alpha = 4$, estimate (4.52) is is just (4.43), whereas for $\alpha > 4$ we have

$$\begin{aligned} 0 \le u_{k,\alpha}(y) &= \int_\Omega G^{\Omega}_2(y,z)u_{k,\alpha-2}(z)dz \le C_\alpha \int_\Omega G^{\Omega}_2(y,z)d(z,x)^{\alpha-2-n}dz \\ &\le C_\alpha C_{47}\int_M G_2(y,z)G_{\alpha-2}(z,x)dz = C_\alpha C_{47} G_\alpha(y,x) \le C_{\alpha+2} d(y,x)^{\alpha-n}. \end{aligned} \tag{4.54}$$

where we used that $G^{\Omega}_2 \le G_2$, and the global bounds (4.8). The proof of (4.51) for $u_{k,\alpha}$ follows by induction using the Dominated Convergence Theorem. The proof of (4.53) for $u^{*}_{k,\alpha}$ follows also by induction, since by Talenti's estimate and

$$\begin{aligned} u^{*}_{k,\alpha}(t) \le \sigma_M^{-\frac{2}{n}} v^{*}_{k,\alpha}(t) &= \sigma_M^{-\frac{2}{n}} n^{-2} B_n^{-\frac{2}{n}} \int_t^{|\Omega^{\#}|} \tau^{\frac{2}{n}-2} \int_0^{\tau} u^{*}_{k,\alpha-2}(\zeta)d\zeta \\ &\le \sigma_M^{-\frac{2}{n}} n^{-2} B_n^{-\frac{2}{n}} \int_t^{|\Omega^{\#}|} \tau^{\frac{2}{n}-2} \int_0^{\tau} \sigma_M^{-\frac{\alpha-2}{n}} B_n^{\frac{n-\alpha+2}{n}} c_{\alpha-2} \zeta^{-\frac{n-\alpha+2}{n}} d\zeta \\ &= \sigma_M^{-\frac{\alpha}{n}} B_n^{\frac{n-\alpha}{n}} \frac{c_{\alpha-2}}{(\alpha-2)(n-\alpha)} \big(t^{-\frac{n-\alpha}{n}} - |\Omega^{\#}|^{-\frac{n-\alpha}{n}}\big) \end{aligned} \tag{4.55}$$

and the result follows from the identity $c_\alpha(\alpha-2)(n-\alpha) = c_{\alpha-2}$.

To prove (4.28) for $2 < \alpha \le \frac{n}{2}$, use the relation

$$u_{k,\alpha}(y) = \int_\Omega G^{\Omega}_2(y,z)u_{k,\alpha-2}(z)dz. \tag{4.56}$$

Note that from the classical gradient bound for positive harmonic functions (see e.g. [34], Theorem 6.1) we get that for all $y, z \in \Omega$, $y \neq z$

$$
\begin{aligned}
|\nabla_y G_2^\Omega(y,z)| &\leq C(n) G_2^\Omega(y,z) d(y,z)^{-1} \\
&\leq C(n) G_2(y,z) d(y,z)^{-1} \leq C(n) C_{48} d(y,z)^{1-n},
\end{aligned} \tag{4.57}
$$

for some $C(n)$ depending only on n. From here, using the Dominated Convergence Theorem one sees easily that $|\nabla_y u_{k,\alpha}(y)|^2 \to |\nabla_y G_\alpha^\Omega(y,x)|^2$ for all $y \in \Omega$, and Fatou's lemma combined with (4.39) applied with $u = G_\alpha^\Omega(x,y)$ gives (4.28).

The proof of (4.29) and(4.30), for the Dirichlet Green functions, is a simple computation based on (4.27), (4.28); we leave the details to the reader.

To prove the theorem for the global Green functions, consider an increasing sequence of smooth bounded sets $\{\Omega_k\}$ such that $\Omega_k \uparrow M$, so that the corresponding Dirichlet Green functions satisfy $G_2^{\Omega_k} \uparrow G_2$, and as a consequence, for any α even $G_\alpha^{\Omega_k} \uparrow G_\alpha$. From this relation it's clear that for any $x \in M$ and $t > 0$ we have $(G_\alpha^{\Omega_k})^*(x,t) \uparrow (G_\alpha)^*(x,t)$ so that (4.27) holds.

Next, note that if $\lambda_\alpha^k(x,s) = \mu(\{y \in \Omega_k : G_\alpha^{\Omega_k}(x,y) > s\})$ then $\lambda_\alpha^k(x,s) \uparrow \lambda_\alpha(x,s)$. Also, Using the gradient bound applied to the positive harmonic function $G_2(x,\cdot) - G_2^{\Omega^k}(x,\cdot)$ we obtain, for fixed $x, y \in M$, $x \neq y$,

$$
\left|\nabla_y\big(G_2(x,y) - G_2^{\Omega^k}(x,y)\big)\right| \leq C(n)\big(G_2(x,y) - G_2^{\Omega^k}(x,y)\big) d(x,y)^{-1} \to 0, \tag{4.58}
$$

as $k \to \infty$, which implies $|\nabla_y G_2^{\Omega_k}(x,y)| \to |\nabla_y G_2(x,y)|$, as $k \to +\infty$. Using that

$$
G_\alpha^{\Omega_k}(x,y) = \int_{\Omega_k} G_{\alpha-2}^{\Omega_k}(x,z) G_2^{\Omega_k}(z,y) dz \tag{4.59}
$$

the bounds $G_{\alpha-2}^{\Omega_k}(x,z) \leq C_{48} d(x,z)^{\alpha-2-n}$, estimate (4.57), and the Dominated Convergence Theorem, gives (4.28), (4.29), (4.30). □

Remark 10 As observed in Remark 9, the previous yields (4.27)-(4.30) for the Green functions of smooth bounded domains. One just has to be a bit more careful and assume, in the induction process, that (4.53) is valid in the stronger form $u_{k,\alpha-2}^*(t) \leq \sigma_M^{-\frac{\alpha-2}{n}} (N_{\alpha-2}^{\Omega^\#})^*(t)$, and using that

$$
\big(N_\alpha^{\Omega^\#}\big)^*(t) = n^2 B_n^{-\frac{2}{n}} \int_t^{|\Omega^\#|} \tau^{\frac{2}{n}-2} \int_0^\tau \big(N_{\alpha-2}^{\Omega^\#}\big)^*(\zeta) d\zeta. \tag{4.60}
$$

5 Proofs of Theorems 1 and Theorem 2: Inequalities

Let us recall the Adams inequalities on metric measure spaces proved recently in [45].

Let (M, d, μ) be a metric measure space, that is a set M endowed with a distance function d and a Borel measure μ. Define

$$B(x, r) = \{y : \ d(x, y) \le r\}, \qquad V_x(r) = \mu\big(B(x, r)\big).$$

and assume that that for all $x \in M$

$$i) \ \ \forall r > 0, V_x(r) < \infty, \qquad ii) \ r \to V_x(r) \text{ continuous}. \tag{5.1}$$

In particular, the boundary of any ball has zero measure, and μ is nonatomic.

In [45] we defined a measurable $k : M \times M \to \mathbb{R}$ to be *Riesz-like kernel of order* $\beta > 1$ *and normalization constants* $A_0 > 0$ *and* $A_\infty \ge 0$ if there exists $B \ge 0$ such that for all $x \in M$

$$\int\limits_{r_1 \le d(x,y) \le r_2} |k(x, y)|^\beta d\mu(y) \le A_0 \log \frac{V_x(r_2)}{V_x(r_1)} + B, \quad 0 < V_x(r_1) < V_x(r_2) \le 1, \tag{K1}$$

$$\int\limits_{r_1 \le d(x,y) \le r_2} |k(x, y)|^\beta d\mu(y) \le A_\infty \log \frac{V_x(r_2)}{V_x(r_1)} + B, \quad 1 \le V_x(r_1) < V_x(r_2); \tag{K2}$$

for all $x, y \in M$

$$|k(x, y)| \le B V_x\big(d(x, y)\big)^{-1/\beta} \qquad |k(x, y)| \le B V_y\big(d(x, y)\big)^{-1/\beta}; \tag{K3}$$

for each $\delta > 0$ there is $B_\delta > 0$ such that for all $x \in M$

$$\int\limits_{d(x,y) > R} |k(x, y) - k(x', y)|^\beta d\mu(y) \le B_\delta, \quad V_x(R) \ge (1+\delta)V_x(r), \quad \forall x' \in B(x, r). \tag{K4}$$

A Riesz-like potential is an integral operator

$$T f(x) = \int_M k(x, y) f(y) d\mu(y). \tag{5.2}$$

where $k(x, y)$ is Riesz-like. Such operator is well defined for $f \in L^{\beta'}$ and with compact support. Here and from now on β' will denote the exponent conjugate to β:

$$\frac{1}{\beta} + \frac{1}{\beta'} = 1.$$

Theorem 4 ([45]) *If $k(x, y)$ is a Riesz-like kernel of order $\beta > 1$ and normalization constants A_0, A_∞, and if $p \geq 1$, then there is a constant C such that for any measurable function f supported in a ball with*

$$\|f\|_{\beta'} \leq 1 \tag{5.3}$$

we have

$$\int_M \frac{\exp_{\lceil p/\beta - 1\rceil}\left(\dfrac{1}{\max\{A_0, A_\infty\}}|Tf|^\beta\right)}{1 + |Tf|^{p\beta/\beta'}}\, d\mu \leq C\|Tf\|_p^p. \tag{5.4}$$

Moreover, if $A_\infty > 0$ and given any $\kappa > 0$ there exists C (depending on κ) such that for any measurable f supported in a ball with

$$\|f\|_{\beta'}^{\beta'} + \kappa\|Tf\|_{\beta'}^{\beta'} \leq 1 \tag{5.5}$$

we have

$$\int_M \frac{\exp_{\lceil p/\beta - 1\rceil}\left(\dfrac{1}{A_0}|Tf|^\beta\right)}{1 + |Tf|^{p\beta/\beta'}}\, d\mu \leq C\|Tf\|_p^p, \tag{5.6}$$

and

$$\int_M \exp_{\lceil \beta' - 2\rceil}\left(\frac{1}{A_0}|Tf|^\beta\right) d\mu \leq C. \tag{5.7}$$

Remark 11 The conclusions of Theorem 4 hold if k and f are vector-valued. Specifically, if $k = (k_1, ..., k_m)$ is defined on $M \times M$ and measurable, and if $f = (f_1, ..., f_m)$ is measurable on M, then let $Tf(x) = \int_M k(x, y) \cdot f(y) d\mu$ where the "$\cdot$" denotes the standard Euclidean inner product on $\mathbb{R}^m$. If k satisfies (K1)-(K4), with $|k| = (k \cdot k)^{1/2}$, $|f| = (f \cdot f)^{1/2}$, $\|f\|_{\beta'} = \left(\int_M |f|^{\beta'}\right)^{1/\beta'}$, then the conclusions of Theorem 4 hold.

Clearly, we would like to apply the above theorem when (M, μ, d) is a Riemannian manifold satisfying the assumptions of Theorems 1,2 (it is known that the conditions in (5.1) are true). Given the representation formulas (4.10), (4.11), we want to take $T = T_\alpha$ to be the integral operator with kernel $G_\alpha(x, y)$ for α even, and $\widetilde{G}_\alpha(x, y)$ for α odd. What we will verify is that without bounded geometry G_α and $\widetilde{G}_\alpha$ satisfy conditions (K1)-(K4), with $\beta = \frac{n}{n-\alpha}$, $A_0 = A_\infty = \sigma_M^{-\frac{\alpha}{(n-\alpha)}}\gamma_{n,\alpha}^{-1}$, and if bounded geometry is assumed then $A_0 = \gamma_{n,\alpha}^{-1} < A_\infty$. This will imply the result for $u \in C_c^\infty(M)$, and by density for $u \in H^{\alpha, \frac{n}{\alpha}}(M)$.

Remark 12 When α is odd the operator defined by (4.11) is not one of the types described in Remark 11, i.e. involving a Euclidean inner product. However, we can argue as in the proof of Theorem 7 in [45], to reduce matters to Euclidean inner products. Namely, let us pick any point $p \in M$ and a chart in geodesic polar coordinates at p, and in the notation of Sect. 2, $\exp_p$ is a diffeomorphism from D_p onto $M \setminus (\mathrm{Cut}_p \cup$

$\{p\}$), where we can identify D_p with $\{r\xi : 0 < r < c(\xi), \xi \in \mathbb{S}^{n-1}\} \subseteq \mathbb{R}^n$. For any $\overline{y} \in D_p$ we have $(g_{ij}(\overline{y})) = R_{\overline{y}}^T R_{\overline{y}}$ for some invertible matrix $R_{\overline{y}}$, and

$$u(x) = T_\alpha f(\overline{x}) = \int_{D_p} K_\alpha(\overline{x}, \overline{y}) \cdot f(\overline{y})\sqrt{|g(\overline{y})|}\, dm(\overline{y}), \qquad x = \exp_p \overline{x}, \tag{5.8}$$

where

$$K_\alpha(\overline{x}, \overline{y}) = R_{\overline{y}} \widetilde{G}_\alpha(x, y), \qquad f(\overline{y}) = R_{\overline{y}} \nabla D^{\alpha-1} u(y) = R_{\overline{y}} D^\alpha u(y). \tag{5.9}$$

Since $K_\alpha \cdot K_\alpha = |\widetilde{G}_\alpha|^2$, $f \cdot f = |\nabla D^{\alpha-1} u|^2$, and Cut_p has zero measure, we can check conditions (K1)-(K4) using the norm in the metric g, and then apply Theorem 4 and Remark 11 to the operator T_α defined on the space D_p, endowed with metric $d(\overline{x}, \overline{y}) = d(x, y)$ and measure $d\mu(\overline{y}) = \sqrt{|g(\overline{y})|}\, dm(\overline{y})$.

We will now derive (K1)-(K4) without any bounded geometry assumptions, based on the bounds of Theorem 3. Assume $n \geq 3$. The global bounds (4.8) imply that for any fixed $x \in M$, and any $r_2 > r_1 > 0$

$$\{y : r_1 \leq d(x, y) < r_2\} \subseteq \{y : C_{47} r_2^{\alpha-n} < G_\alpha(x, y) \leq C_{48} r_1^{\alpha-n}\} \tag{5.10}$$

and for the distribution function defined in (4.26)

$$V_x\big(C_{56} s^{-\frac{1}{n-\alpha}}\big) \leq \lambda_\alpha(x, s) \leq V_x\big(C_{57} s^{-\frac{1}{n-\alpha}}\big), \qquad s > 0. \tag{5.11}$$

The above estimates combined with (4.27), (4.28) and (2.10), instantly imply that there is $B > 0$ such that for any $x \in M$

$$\int\limits_{r_1 \leq d(x,y) \leq r_2} G_\alpha(x, y)^{\frac{n}{n-\alpha}} dy \leq \sigma_M^{-\frac{\alpha}{n-\alpha}} \gamma_{n,\alpha}^{-1} \log \frac{V_x(r_2)}{V_x(r_1)} + B, \qquad 0 < r_1 < r_2 \tag{5.12}$$

when α is an even integer, and the same inequality holds for $\widetilde{G}_\alpha(x, y)$ when α is an odd integer smaller or equal $n/2$. This shows that (K1), (K2) hold with $A_0 = A_\infty = \sigma_M^{-\frac{\alpha}{n-\alpha}} \gamma_{n,\alpha}^{-1}$.

Regarding (K3), such estimate follows from (2.10) and the global bounds (4.8).

To prove (K4), assume $V_x(R) \geq (1+\delta) V_x(r)$, for some $\delta > 0$. Since $V_x(R)/|B_{\mathbb{R}}^n(r)|$ is decreasing and smaller than 1, then $V_x(R) - V_x(r) \leq B_n(R^n - r^n)$, from which we deduce $R^n - r^n \geq \sigma_M \delta r^n$, and therefore $R \geq (1 + \eta_\delta) r$, with $\eta_\delta = (\sigma_M \delta)^{\frac{1}{n}}$. By integrating in t (3.12) and (3.13) we then get the following Lipschitz estimates, for $d(x, x') \leq r,\ d(x, y) \geq R$:

$$|G_\alpha(x, y) - G_\alpha(x', y)| \leq C_\delta\, d(x, x') d^{\alpha-n-1}(x, y) \tag{5.13}$$

$$|\widetilde{G}_\alpha(x, y) - \widetilde{G}_\alpha(x', y)| \leq \widetilde{C}_\delta\, d(x, x') d^{\alpha-n-1}(x, y), \tag{5.14}$$

from which (K4) follows easily. This proves the inequality part of Theorem 1, in case $n \geq 3$.

When $n = 2$ we only need to consider $\alpha = 1$. In this case however we can use directly the Pólya-Szegő inequality (1.21) to prove (1.13) for $\alpha = 1$ and any $n \geq 2$. Indeed, assuming $\|\nabla u\|_n \leq 1$ we have $\|\sigma_M^{\frac{1}{n}} \nabla u^\#\|_n \leq 1$, so using the fact that (1.13) is true on $\mathbb{R}^n$,

$$\begin{aligned}
&\int_M \frac{\exp_{\lceil n-2 \rceil}\left(\sigma_M^{\frac{1}{n-1}} \gamma_{n,1} |u|^{\frac{n}{n-1}}\right)}{1 + |u|^{\frac{n}{n-1}}} d\mu \\
&= \int_{\mathbb{R}^n} \frac{\exp_{\lceil n-2 \rceil}\left(\gamma_{n,1} |\sigma_M^{\frac{1}{n}} u^\#|^{\frac{n}{n-1}}\right)}{1 + |u^\#|^{\frac{n}{n-1}}} dm \leq C\|u^\#\|_n^n = C\|u\|_n^n,
\end{aligned} \tag{5.15}$$

where dm denotes the Lebesgue measure on $\mathbb{R}^n$. This concludes the proof the inequalities in Theorem 1. Now let us prove (1.16) and (1.17).

Assuming that α even, and that M has 1^{st} order bounded geometry, let us show that (K1) holds with $A_0 = \gamma_{n,\alpha}^{-1}$. Note that if $V_x(r) \leq 1$ then $r \leq \left(\sigma_M B_n\right)^{-\frac{1}{n}}$. Using the inequality $|A^p - B^p| \leq p|A - B| \max\{A^{p-1}, B^{p-1}\}$, $(A, B \geq 0,\ p > 1)$, and using (4.13), (4.8), and Remark 7, we obtain, for $d(x, y) \leq \left(\sigma_M B_n\right)^{-\frac{1}{n}}$,

$$\begin{aligned}
\left| |G_\alpha(x, y)|^{\frac{n}{n-\alpha}} - \left(c_\alpha d^{\alpha-n}(x, y)\right)^{\frac{n}{n-\alpha}} \right| &\leq C_{58} d^{\alpha-n+1}(x, y) \left(d^{\alpha-n}(x, y)\right)^{\frac{\alpha}{n-\alpha}} \\
&= C_{58} d^{1-n}(x, y)
\end{aligned} \tag{5.16}$$

hence, for $0 < r_1 < r_2 \leq \left(\sigma_M B_n\right)^{-\frac{1}{n}}$

$$\left| \int_{r_1 \leq d(x,y) \leq r_2} |G_\alpha(x, y)|^{\frac{n}{n-\alpha}} dy - c_\alpha^{\frac{n}{n-\alpha}} \int_{r_1 \leq d(x,y) \leq r_2} d^{-n}(x, y) dy \right| \leq C_{59}. \tag{5.17}$$

From the Bishop comparison (2.10)

$$\begin{aligned}
\int_{r_1 \leq d(x,y) \leq r_2} d^{-n}(x, y) dy &\leq \int_{r_1 \leq d(x,y) \leq r_2} \frac{B_n}{V_x\left(d(x, y)\right)} dy \\
&= B_n \int_{V_x(r_1)}^{V_x(r_2)} \frac{1}{t} dt = B_n \log \frac{V_x(r_2)}{V_x(r_1)}
\end{aligned} \tag{5.18}$$

which implies (K1) with $A_0 = c^{\frac{n}{n-\alpha}} B_n$. Note that the first identity in (5.18) follows from the fact that

$$\left(\frac{1}{V_x\left(d(x, \cdot)\right)}\right)^*(t) = \frac{1}{t} \tag{5.19}$$

and the fact that balls are level sets of $V_x(d(x, y))^{-1}$ (up to sets of zero measure on general metric measure spaces satisfying (5.1)).

To show (K1) in the case α odd, under 2[nd] order bounded geometry, we proceed in exactly the same manner, using (4.14), and the fact that $|\nabla_y d(x, y)| = 1$.

Since we have already checked (K2),(K3),(K4), and $A_0 \leq A_\infty$, the inequality part of Theorem 2 is proved, for $n \geq 3$.

Remark 13 When $\alpha = 1$ the proof of (1.17) can be achieved using the results in [25]. Indeed,(4.9) and (4.14) imply that

$$|\nabla_y G_2(x, \cdot)|^*(t) \leq \gamma_n^{-\frac{n-1}{n}} t^{-\frac{n-1}{n}} (1 + C t^{\frac{1}{n}}), \qquad 0 < t \leq 1 \tag{5.20}$$

and

$$|\nabla_y G_2(x, \cdot)|^*(t) \leq C t^{-\frac{n-1}{n}}, \qquad t > 0, \tag{5.21}$$

and the constant C in both inequalities is independent of x. According to [25], Observation 3, we can conclude that the Moser-Trudinger inequality holds locally, i.e. for functions supported on sets with measure less than 1, and hence globally (see [25], Theorem 4).

6 Proofs of Theorem 1 and Theorem 2: Sharpness

Fix a point $x \in M$. For simplicity, for the rest of this section we will use the notation

$$d = d(x, \cdot), \qquad V(r) = V_x(r) = \mu(B(x, r)), \qquad \Lambda(r) = \Lambda_x(r), \tag{6.1}$$

and we will assume that (M, g) is not isometric to $\mathbb{R}^n$, in which case Λ is strictly decreasing to 0.

Later we will need the following estimate

$$\Lambda(r) \leq \widetilde{\Lambda}(r) = \min_{\delta > 0} \big(\delta + \delta^{-2n} \Lambda(\delta^{2n+1} r)\big). \tag{6.2}$$

This holds since Λ is decreasing, $0 < \Lambda \leq 1 - \sigma_M$ and

$$\delta + \delta^{-2n} \Lambda(\delta^{2n+1} r) \geq \begin{cases} \Lambda(r) & \text{if } 0 < \delta \leq 1 \\ 1 & \text{if } \delta > 1. \end{cases} \tag{6.3}$$

The Euclidean volume growth condition can be written as

$$\frac{\sigma_M B_n}{V(r)} \leq \frac{1}{r^n} = \frac{\sigma_M B_n}{V(r)} + \frac{B_n\ \Lambda(r)}{V(r)} \leq \frac{\sigma_M B_n}{V(r)} + B_n \frac{\Lambda\big(B_n^{-\frac{1}{n}} V^{\frac{1}{n}}(r)\big)}{V(r)}. \tag{6.4}$$

We will also repeatedly use that for $a, r > 0$

$$a^n \sigma_M V(r) \leq V(ar) \leq \frac{a^n}{\sigma_M} V(r), \tag{6.5}$$

and

$$a^{-\frac{1}{2n+1}} \min\{a,1\}\widetilde{\Lambda}(r) \le \widetilde{\Lambda}(ar) \le a^{-\frac{1}{2n+1}} \max\{a,1\}\widetilde{\Lambda}(r). \tag{6.6}$$

Assume first $n \ge 3$. Following the notation used in [18, 19], we consider the function

$$b(y) = \left(\frac{\sigma_M}{c_2}\, G_2(x,y)\right)^{\frac{1}{2-n}}, \qquad y \in M \setminus \{x\}. \tag{6.7}$$

which should be thought of as a good, smooth replacement of the distance function, for large distances.

In particular, under our hypothesis the main estimates (4.17) and (4.18) for $\alpha = 2$ can be written in terms of b as follows:

$$|b-d| \le C_{60}\, d\, \widetilde{\Lambda}(d), \qquad d = d(x,y) \ge 1 \tag{6.8}$$

$$\left(\fint_{A_{r,(1+\eta)r}} \left|\nabla b - \nabla d\right|^2\right)^{\frac{1}{2}} \le C_{61}(1+\eta)^{n-1}\left(1+\frac{1}{\sqrt{\eta}}\right)\widetilde{\Lambda}^{\frac{1}{2}}(r), \qquad r \ge r_1 \tag{6.9}$$

for a suitable $r_1 > 1$ and for any fixed $\eta > 0$.

Note that, as pointed out in [19] and [17], by the Laplacian comparison and the maximum principle, one can show that $G_2(x,y) \ge c_2 d^{2-n}(x,y)$, which implies $b \le \sigma_M^{\frac{1}{2-n}} d$. In [17] Colding proved that we also have $|\nabla b| \le \sigma_M^{\frac{1}{2-n}}$, with equality at one point if and only if M is the flat $\mathbb{R}^n$.

Remark 14 In [19], Remark 2.11, it was remarked that the critical sets $\{b = r\} \cap \{\nabla b = 0\}$ are closed sets of codimension 2, by results of Cheng in [15], Thm. 2.2, which also show that $\{b = r\} \setminus \{\nabla b = 0\}$ is an $(n-1)-$dim. smooth submanifold of M. Hence each set $\{b = r\}$ has zero measure in M.

Inequality (6.8) follows from (4.17), with $\alpha = 2$, combined with the rough global bound for G_2, given in (4.8), which can be written in terms of b as

$$C_{62}\, d \le b \le C_{63}\, d, \qquad d = d(x,y) > 0. \tag{6.10}$$

To prove (6.9), we have

$$\nabla b = -\frac{\sigma_M}{(n-2)c_2} b^{n-1} \nabla G_2 \tag{6.11}$$

and from (4.8), (4.9) we get $|\nabla b| \le C_{64}$. Also, using (6.11) estimate (4.19) can be rewritten as

$$\left(\fint_{A_{r,(1+\eta)r}} \left|b^{1-n}\nabla b - d^{1-n}\nabla d\right|^2\right)^{\frac{1}{2}} \le C_{65}\left(1+\frac{1}{\sqrt{\eta}}\right)\widetilde{\Lambda}^{\frac{1}{2}}(r) r^{1-n}. \tag{6.12}$$

From (6.8) we find $C_{66}, C_{67} > 0$ and $r_1 > 1$ such that

$$d^{-n}\left(1 - C_{66}\widetilde{\Lambda}(d)\right) \le b^{-n} \le d^{-n}\left(1 + C_{67}\widetilde{\Lambda}(d)\right), \qquad d = d(x,y) \ge r_1 \tag{6.13}$$

and using $|b^{1-n} - d^{1-n}| \le C_{68} d^{1-n} \widetilde{\Lambda}(d)$ for $d \ge r_1$ large enough, we obtain (6.9).

Additionally, from (6.13) and (6.4), (6.2), and the fact that $\Lambda(r) \le 1 - \sigma_M$, we find $a_1, a_2 > 0$ such that

$$\frac{B_n \sigma_M}{V(d)} \Big(1 - a_1 \widetilde{\Lambda}\big(B_n^{-\frac{1}{n}} V^{\frac{1}{n}}(d)\big)\Big) \le b^{-n} \le \frac{B_n \sigma_M}{V(d)} \Big(1 + a_2 \widetilde{\Lambda}\big(B_n^{-\frac{1}{n}} V^{\frac{1}{n}}(d)\big)\Big), \qquad d \ge r_1. \tag{6.14}$$

For any $r > 0$ let ρ be uniquely related to r as follows

$$V(r) = V(\rho)\ \exp\Big(\widetilde{\Lambda}^{-\frac{1}{2}}\big(V^{\frac{1}{n}}(\rho)\big)\Big) \tag{6.15}$$

(note that $V(r)$ is strictly increasing in r). Clearly $r > \rho$ and $\rho \to +\infty$ if $r \to +\infty$, and from (6.5), (6.6) we also have

$$\widetilde{\Lambda}^{\frac{1}{2}}(\rho) \log \frac{r}{\rho} \le C_{69} \widetilde{\Lambda}^{\frac{1}{2}}\big(V^{\frac{1}{n}}(\rho)\big) \log \frac{V(r)}{V(\rho)} + C_{70} = C_{69} + C_{70}. \tag{6.16}$$

For $t \in \mathbb{R}$, $r > 0$, $\alpha > 0$ define, with ρ satisfying (6.15),

$$h_{\alpha,r}(t) = \begin{cases} \rho^{-\alpha} - r^{-\alpha} & \text{if } t \le \rho \\ t^{-\alpha} - r^{-\alpha} & \text{if } \rho < t \le r \\ 0 & \text{if } t > r \end{cases} \qquad h_{0,r}(t) = \begin{cases} \log \frac{r}{\rho} & \text{if } t \le \rho \\ \log \frac{r}{t} & \text{if } \rho < t \le r \\ 0 & \text{if } t > r \end{cases} \tag{6.17}$$

Consider the family of functions

$$u_{\alpha,r}(y) = T_\alpha\big(h_{\alpha,r}(b)\big)(y) = \int_M G_\alpha(y,z) h_{\alpha,r}(b(z)) dz \tag{6.18}$$

for α even, and for α odd

$$\tilde{u}_{\alpha,r} = \begin{cases} \dfrac{u_{\alpha-1,r}}{\alpha - 1} & \text{if } \alpha \ge 3 \\ h_{0,r}(b) & \text{if } \alpha = 1 \end{cases} \tag{6.19}$$

which clearly satisfy

$$\begin{cases} D^\alpha u_{\alpha,r} = h_{\alpha,r}(b) & \text{if } \alpha \text{ even} \\ D^\alpha \tilde{u}_{\alpha,r} = \dfrac{\nabla h_{\alpha-1,r}(b)}{\alpha - 1} & \text{if } \alpha \text{ odd, } \alpha \ge 3 \\ D^\alpha \tilde{u}_{1,r} = \nabla h_{0,r}(b) & \text{if } \alpha = 1. \end{cases} \tag{6.20}$$

We will prove that for r large enough

$$\|D^\alpha u_{\alpha,r}\|_{n/\alpha}^{n/\alpha} = B_n \sigma_M \log \frac{V(r)}{V(\rho)} + O(1) \tag{6.21}$$

$$\|D^\alpha \tilde{u}_{\alpha,r}\|_{n/\alpha}^{n/\alpha} = B_n\sigma_M \log\frac{V(r)}{V(\rho)} + O(1) \tag{6.22}$$

$$u_{\alpha,r}(y) \geq c_\alpha B_n \log\frac{V(r)}{V(\rho)} - C, \qquad y \in B(x,\rho) \tag{6.23}$$

$$\tilde{u}_{\alpha,r}(y) \geq \tilde{c}_\alpha B_n \log\frac{V(r)}{V(\rho)} - C, \qquad y \in B(x,\rho) \tag{6.24}$$

$$\|u_{\alpha,r}\|_{n/\alpha}^{n/\alpha} \leq CV(r), \qquad \alpha < \frac{n}{2}, \tag{6.25}$$

$$\|\tilde{u}_{\alpha,r}\|_{n/\alpha}^{n/\alpha} \leq CV(r), \qquad \alpha < \frac{n}{2}+1, \tag{6.26}$$

where "$O(1)$" denotes a function of r (depending on the fixed x) which is bounded as $r \to \infty$.

From (6.10) we find $a_3, a_4 > 0,\ a_3 < 1 < a_4$, such that

$$B(x, a_3 r) \subseteq \{b \leq r\} \subseteq B(x, a_4 r), \qquad r > 0. \tag{6.27}$$

***Proof of* (6.21), (6.22)** Let α be even. If r so large so that $a_3\rho \geq r_1$, using (6.27) we have

$$\begin{aligned}
&\int_M |D^\alpha u_{\alpha,r}|^{\frac{n}{\alpha}} = \int_M (h_{\alpha,r}(b))^{\frac{n}{\alpha}} = \int_{b\leq\rho} (\rho^{-\alpha} - r^{-\alpha})^{\frac{n}{\alpha}} + \int_{\rho<b\leq r} (b^{-\alpha} - r^{-\alpha})^{\frac{n}{\alpha}} \\
&\leq V(a_4) + \int_{a_3\rho<d\leq a_4 r} b^{-n} \leq V(a_4) + \int_{a_3\rho<d\leq a_4 r} \frac{B_n\sigma_M}{V(d)}\Big(1 + a_2\widetilde{\Lambda}\big(B_n^{-\frac{1}{n}} V^{\frac{1}{n}}(d)\big)\Big) \\
&\leq V(a_4) + B_n\sigma_M \int_{V(a_3\rho)}^{V(a_4 r)} \frac{1}{t}\Big(1 + a_2\widetilde{\Lambda}\big(B_n^{-\frac{1}{n}} t^{\frac{1}{n}}\big)\Big)dt \\
&= V(a_4) + B_n\sigma_M \log\frac{V(a_4 r)}{V(a_3\rho)} + B_n\sigma_M a_2\widetilde{\Lambda}\big(B_n^{-\frac{1}{n}} V^{\frac{1}{n}}(a_3\rho)\big)\log\frac{V(a_4 r)}{V(a_3\rho)} \\
&\leq B_n\sigma_M \log\frac{V(r)}{V(\rho)} + C_{71}\widetilde{\Lambda}\big(V^{\frac{1}{n}}(\rho)\big)\log\frac{V(r)}{V(\rho)} + C_{72} \leq B_n\sigma_M \log\frac{V(r)}{V(\rho)} + C_{73}
\end{aligned} \tag{6.28}$$

where we used (6.16).

In the other direction, for r large enough, and arguing as above,

$$\begin{aligned}
\int_M (h_{\alpha,r}(b))^{\frac{n}{\alpha}} &\geq \int_{\rho<b\leq r} b^{-n}\bigg(1 - C_{74}\Big(\frac{b}{r}\Big)^{\alpha}\bigg) \\
&\geq \int_{a_4\rho<d\leq a_3 r} \frac{B_n\sigma_M}{V(d)}\Big(1 - a_1\widetilde{\Lambda}\big(B_n^{-\frac{1}{n}} V^{\frac{1}{n}}(d)\big)\Big) - C_{75} r^{-\alpha}\int_{d\leq a_3 r} d^{\alpha-n} \\
&\geq B_n\sigma_M \log\frac{V(r)}{V(\rho)} - C_{77} - C_{76} r^{-\alpha}\int_{d\leq a_3 r} V^{\frac{\alpha-n}{n}}(d) \\
&= B_n\sigma_M \log\frac{V(r)}{V(\rho)} - C_{77} - C_{78} r^{-\alpha}\int_0^{V(a_3 r)} t^{\frac{\alpha-n}{n}}\,dt
\end{aligned} \tag{6.29}$$

which gives (6.21).

Now let α be odd. If $\alpha = 1$ we get

$$\int_M |D^\alpha \tilde{u}_{1,r}|^{\frac{n}{\alpha}} = \int_M |\nabla h_{0,r}(b)|^n = \int_{\rho<b\le r} b^{-n} = B_n \sigma_M \log \frac{V(r)}{V(\rho)} + O(1). \tag{6.30}$$

If $\alpha \ge 3$,

$$\begin{aligned}\int_M |D^\alpha \tilde{u}_{\alpha,r}|^{\frac{n}{\alpha}} &= \int_M \left|\frac{\nabla h_{\alpha-1,r}(b)}{\alpha-1}\right|^{\frac{n}{\alpha}} \\ &= \int_{\rho<b\le r} |\nabla b^{-\alpha+1}|^{\frac{n}{\alpha}} \\ &= \int_{\rho<b\le r} b^{-n}|\nabla b|^{\frac{n}{\alpha}} = \int_{\rho<b\le r} b^{-n} + \int_{\rho<b\le r} b^{-n}\big(|\nabla b|^{\frac{n}{\alpha}} - |\nabla d|^{\frac{n}{\alpha}}\big).\end{aligned} \tag{6.31}$$

To estimate the last integral, let $J_r \in \mathbb{N}$ be such that $\rho 2^{J_r-1} \le r < \rho 2^{J_r}$, and using (6.9) (with $\eta = 1$ and $r = \rho 2^j$) we get

$$\begin{aligned}&\int_{\rho<b\le r} b^{-n}\big||\nabla b|^{\frac{n}{\alpha}} - |\nabla d|^{\frac{n}{\alpha}}\big| \le C_{79}\int_{\rho<b\le r} b^{-n}|\nabla b - \nabla d| \le C_{79}\sum_{j=0}^{J_r-1}\int_{\rho 2^j\le d\le \rho 2^{j+1}} b^{-n}|\nabla b - \nabla d| \\ &\le C_{79}\sum_{j=0}^{J_r-1}\Big(\int_{\rho 2^j\le d\le\rho 2^{j+1}} |\nabla b-\nabla d|^2\Big)^{1/2}\Big(\int_{\rho 2^j\le d\le\rho 2^{j+1}} b^{-2n}\Big)^{1/2} \\ &\le C_{80}\sum_{j=0}^{J_r-1}\widetilde{\Lambda}^{\frac12}(\rho 2^j)V^{\frac12}(\rho 2^{j+1})\Big(\int_{\rho 2^j\le d\le \rho 2^{j+1}} V^{-2}(d)\Big)^{1/2} \le C_{80}\sum_{j=0}^{J_r-1}\widetilde{\Lambda}^{\frac12}(\rho 2^j)\Big(\frac{V(\rho 2^{j+1})}{V(\rho 2^j)}\Big)^{1/2} \\ &\le C_{81}\sum_{j=0}^{J_r-1}\widetilde{\Lambda}^{\frac12}(\rho 2^j) = C_{81}\widetilde{\Lambda}^{\frac12}(\rho) + \frac{C_{81}}{\log 2}\sum_{j=1}^{J_r-1}\int_{\rho 2^{j-1}}^{\rho 2^j}\widetilde{\Lambda}^{\frac12}(\rho 2^j)\,\frac{dt}{t} \le C_{82} + C_{83}\sum_{j=1}^{J_r-1}\int_{\rho 2^{j-1}}^{\rho 2^j}\widetilde{\Lambda}^{\frac12}(t)\,\frac{dt}{t} \\ &\le C_{84} + C_{85}\int_\rho^r \widetilde{\Lambda}^{\frac12}(t)\,\frac{dt}{t} \le C_{84} + C_{85}\,\widetilde{\Lambda}^{\frac12}(\rho)\log\frac{r}{\rho} \le C_{86}.\end{aligned} \tag{6.32}$$

which, combined with (6.31), gives (6.22). □

***Proof of* (6.23), (6.24)** Let α be even. For r large enough, and with $d = d(x,z)$, using (4.17) and (6.13) we obtain

$$\begin{aligned}u_{\alpha,r}(y) &\ge \int_{\rho<b\le r} G_\alpha(y,z)(b^{-\alpha} - r^{-\alpha})dz \ge \int_{\rho<b\le r} G_\alpha(x,z)b^{-\alpha}dz \\ &\quad + \int_{\rho<b\le r}\big(G_\alpha(y,z) - G_\alpha(x,z)\big)b^{-\alpha}dz - \int_{b\le r} G_\alpha(y,z)r^{-\alpha}dz \\ &\ge \int_{a_4\rho\le d\le a_3 r} c_\alpha\sigma_M^{-1}b^{-n}\big(1 - C_{87}\widetilde{\Lambda}(d)\big)dz - C_{88}\int_{d\le a_3 r} d^{\alpha-n}(y,z)r^{-\alpha}dz \\ &\quad - C_{89}\int_{a_4\rho\le d\le a_3 r}|G_\alpha(y,z) - G_\alpha(x,z)|d^{-\alpha}(x,z)dz.\end{aligned} \tag{6.33}$$

Since $a_4 > 1$, using the Lipschitz estimate (5.13) and arguing as in (6.29), using (6.14), we get that for $y \in B(x,\rho)$

$$\begin{aligned} u_{\alpha,r}(y) &\geq c_\alpha B_n \log\frac{V(r)}{V(\rho)} - C_{90} - C_{91}\int_{d\geq a_4\rho} d(x,y)d^{-n-1}(x,z)dz \\ &\geq c_\alpha B_n \log\frac{V(r)}{V(\rho)} - C_{92}. \end{aligned} \tag{6.34}$$

If α odd and $\alpha \geq 3$ then (6.24) follows since $\tilde{c}_\alpha = c_\alpha/(\alpha-1)$, whereas if $\alpha = 1$ then for $y \in B(x,\rho) \subseteq \{b \leq \rho/a_3\}$, then

$$\tilde{u}_{0,r}(y) = \begin{cases} \log(r/\rho) & \text{if } b \leq \rho \\ \log(r/b) & \text{if } \rho < b \leq r \end{cases} \geq \frac{1}{n}\log\frac{V(r)}{V(\rho)} - C_{93} = \tilde{c}_1 B_n \log\frac{V(r)}{V(\rho)} - C_{93}. \tag{6.35}$$

□

***Proof of* (6.25), (6.26)** Let α be even. If $d(x,y) \geq 2a_4 r$ then

$$\begin{aligned} |u_{\alpha,r}(y)| &= \int_{b\leq r} G_\alpha(y,z)h_{\alpha,r}(b)(z)dz \leq \int_{b\leq r} G_\alpha(y,z)b^{-\alpha}(z)dz \\ &\leq C_{94}\int_{d(x,z)\leq a_4 r} d^{\alpha-n}(y,z)d^{-\alpha}(x,z)dz \leq C_{95}d(x,y)^{\alpha-n}\int_{d(x,z)\leq a_4 r} d^{-\alpha}(x,z) \\ &\leq C_{96}d^{\alpha-n}(x,y)r^{n-\alpha}. \end{aligned} \tag{6.36}$$

If $\alpha < n/2$ then $d^{\alpha-n}(x,\cdot) \in L^{\frac{n}{\alpha}}(B(x,R)^c)$ for any $R > 0$, and

$$\begin{aligned} \int_{d(x,y)\geq 2a_4 r} |u_{\alpha,r}(y)|^{\frac{n}{\alpha}}dy &\leq C_{97}r^{(n-a)\frac{n}{\alpha}}\int_{d(x,y)\geq 2a_4 r} d(x,y)^{(\alpha-n)\frac{n}{\alpha}}dx \\ &\leq C_{98}r^n \leq C_{99}V(r). \end{aligned} \tag{6.37}$$

Now, due to (4.8), the operator T_α in (6.18) is bounded from $L^p(M)$ to $L^q(M)$ with $1 < p < \frac{n}{\alpha}$ and $q^{-1} = p^{-1} - \frac{\alpha}{n}$ (this is the Hardy-Littlewood -Sobolev inequality for generalized, Riesz-like potentials, see [2]). So if we pick any p such that $\frac{n}{2\alpha} < p < \frac{n}{\alpha}$ then $q > \frac{n}{\alpha}$, and for $R = 2a_4 r$

$$\begin{aligned} &\left(\int_{d(x,y)\leq R} |u_{\alpha,r}(y)|^{\frac{n}{\alpha}}dy\right)^{\frac{\alpha}{n}} \leq C_{100}V(R)^{\frac{\alpha}{n}-\frac{1}{q}}\left(\int_{d(x,y)\leq R} |T_\alpha\big(h_{\alpha,r}(b)\big)|^q dy\right)^{\frac{1}{q}} \\ &\leq C_{101}V(R)^{\frac{\alpha}{n}-\frac{1}{q}}\left(\int_{d(x,y)\leq R} \big|h_{\alpha,r}(b)\big|^p\right)^{\frac{1}{p}} \\ &\leq C_{102}V(R)^{\frac{\alpha}{n}-\frac{1}{q}}R^{\frac{n}{p}-\alpha} \leq C_{103}V(R)^{\frac{\alpha}{n}} \leq C_{104}V(r)^{\frac{\alpha}{n}}, \end{aligned} \tag{6.38}$$

which concludes the proof of (6.25) for α even, $\alpha < n/2$, and therefore also for (6.26) for α odd, $3 \leq \alpha < \frac{n}{2}+1$. When $\alpha = 1$ one checks easily that (6.26) also holds.

Let us now prove the sharpness statement. Suppose that α is even, $\alpha < \frac{n}{2}$, $n \geq 3$, and consider $v_{\alpha,r} = u_{\alpha,r}/\|D^\alpha u_{\alpha,r}\|_{n/\alpha}$. From (6.21), (6.23), (6.25), we get, for r large enough,

$$|v_{\alpha,r}(y)|^{\frac{n}{n-\alpha}} \geq c_\alpha^{\frac{n}{n-\alpha}} B_n \sigma_M^{-\frac{\alpha}{n-\alpha}} \log \frac{V(r)}{V(\rho)} - C, \qquad y \in B(x,\rho) \tag{6.39}$$

$$\|D^\alpha v_{\alpha,r}\|_{n/\alpha} = 1, \qquad \|v_{\alpha,r}\|_{n/\alpha}^{n/\alpha} \leq C \frac{V(r)}{\log \frac{V(r)}{V(\rho)}}, \tag{6.40}$$

and recall that from (6.15)

$$\log \frac{V(r)}{V(\rho)} = \frac{1}{\widetilde{\Lambda}^{\frac{1}{2}}\big(V^{\frac{1}{n}}(\rho)\big)} \to +\infty, \qquad r \to \infty. \tag{6.41}$$

Hence, for $\theta > 1$ and r large enough, recalling that $\gamma_{n,\alpha}^{-1} = c_\alpha^{\frac{n}{n-\alpha}} B_n$, and from estimate (3.2) in [49],

$$\begin{aligned}
&\frac{1}{\|v_{\alpha,r}\|_{n/\alpha}^{n/\alpha}} \int_M \frac{\exp_{\lceil \frac{n-\alpha}{\alpha}-1\rceil}\left(\theta \sigma_M^{\frac{\alpha}{n-\alpha}} \gamma_{n,\alpha} |v_{\alpha,r}|^{\frac{n}{n-\alpha}}\right)}{1+|v_{\alpha,r}|^{\frac{n}{n-\alpha}}} dy \\
&\geq \frac{1}{\|v_{\alpha,r}\|_{n/\alpha}^{n/\alpha}} \int_{|v_{\alpha,r}|\geq 1} \frac{\exp\left(\theta \sigma_M^{\frac{\alpha}{n-\alpha}} \gamma_{n,\alpha} |v_{\alpha,r}|^{\frac{n}{n-\alpha}}\right)}{1+|v_{\alpha,r}|^{\frac{n}{n-\alpha}}} dy - C \\
&\geq \frac{1}{\|v_{\alpha,r}\|_{n/\alpha}^{n/\alpha}} \int_{B(x,\rho)} \frac{\exp\left(\theta \sigma_M^{\frac{\alpha}{n-\alpha}} \gamma_{n,\alpha} |v_{\alpha,r}|^{\frac{n}{n-\alpha}}\right)}{1+|v_{\alpha,r}|^{\frac{n}{n-\alpha}}} dy - C \\
&\geq C \frac{V(\rho)}{V(r)} \log \frac{V(r)}{V(\rho)} \cdot \frac{\exp\left(\theta \log \frac{V(r)}{V(\rho)} - C\right)}{\log \frac{V(r)}{V(\rho)}} - C \\
&= C\left(\frac{V(r)}{V(\rho)}\right)^{\theta-1} - C \to +\infty, \qquad r \to +\infty
\end{aligned} \tag{6.42}$$

which gives the sharpness of the exponential constant in (1.13), for α even. Regarding the sharpness of the power in the denominator of (1.13), using the same estimates as in (6.42) we get for $\theta < 1$

$$
\begin{aligned}
&\frac{1}{\|v_{\alpha,r}\|_{n/\alpha}^{n/\alpha}}\int_M \frac{\exp_{\lceil\frac{n-\alpha}{\alpha}-1\rceil}\left(\sigma_M^{\frac{\alpha}{n-\alpha}}\gamma_{n,\alpha}|v_{\alpha,r}|^{\frac{n}{n-\alpha}}\right)}{1+|v_{\alpha,r}|^{\theta\frac{n}{n-\alpha}}}dy\\
&\qquad\geq \frac{1}{\|v_{\alpha,r}\|_{n/\alpha}^{n/\alpha}}\int_{|v_{\alpha,r}|\geq 1} \frac{\exp\left(\sigma_M^{\frac{\alpha}{n-\alpha}}\gamma_{n,\alpha}|v_{\alpha,r}|^{\frac{n}{n-\alpha}}\right)}{1+|v_{\alpha,r}|^{\theta\frac{n}{n-\alpha}}}dy-C\\
&\qquad\geq \frac{1}{\|v_{\alpha,r}\|_{n/\alpha}^{n/\alpha}}\int_{B(x,\rho)} \frac{\exp\left(\sigma_M^{\frac{\alpha}{n-\alpha}}\gamma_{n,\alpha}|v_{\alpha,r}|^{\frac{n}{n-\alpha}}\right)}{1+|v_{\alpha,r}|^{\theta\frac{n}{n-\alpha}}}dy-C\\
&\qquad\geq C\frac{V(\rho)}{V(r)}\log\frac{V(r)}{V(\rho)}\cdot\frac{\exp\left(\log\frac{V(r)}{V(\rho)}-C\right)}{\left(\log\frac{V(r)}{V(\rho)}\right)^{\theta}}-C\\
&\qquad= C\left(\log\frac{V(r)}{V(\rho)}\right)^{1-\theta}-C\to+\infty,\qquad r\to+\infty.
\end{aligned}
\tag{6.43}
$$

The sharpness for α odd, $\alpha<\frac{n}{2}+1$, follows in the same exact manner, using $v_{\alpha,r}=\tilde{u}_{\alpha,r}/\|D^\alpha\tilde{u}_{\alpha,r}\|_{n/\alpha}$, and (6.22), (6.24), (6.26).

It remains to consider the cases $\alpha=1$, $n=2$, and $\alpha=2$, $n=3,4$.. For $\alpha=1$ we note that when $n\geq 3$ we could have just as well considered the functions $\tilde{u}_{1,r}(y)=h_{0,r}\big(d(x,y)\big)$, which are defined also for $n=2$, since they are Lipschitz, compactly supported, hence in $W^{1,n}(M)$, and satisfy the same estimates as in (6.22), (6.24), (6.26), and therefore (6.42), for the corresponding $v_{1,r}$.

When $\alpha=2$ the proof of sharpness for arbitrary $n\geq 3$ can be effected as follows. Let h_r be smooth on $[0,\infty)$, decreasing, such that $|h_r'|\leq C$, $|h_r''|\leq C$ with

$$
h_r(t)=\begin{cases}\log\dfrac{r}{\rho} & \text{if } 0<t<\rho\\ \log\dfrac{r}{t} & \text{if } \rho+1<t\leq r-1\\ 0 & \text{if } t>r,\end{cases}
\tag{6.44}
$$

and $h_r(t)\leq\log(r/t)$ on $[r-1,r]$. We then let $u_r=h_r(b)$, which belongs to $W_0^{2,n/2}(M)$. One checks that $\Delta b=(n-1)b^{-1}|\nabla b|^2$, and therefore

$$
\Delta u_r=\Delta h_r(b)=\big(h_r''(b)+h_r'(b)(n-1)b^{-1}\big)|\nabla b|^2
\tag{6.45}
$$

and in particular

$$
\Delta u_r=(2-n)b^{-2}|\nabla b|^2,\qquad \rho+1<b<r-1.
\tag{6.46}
$$

Using the same estimates as in (6.31), (6.32) we get for r large enough

$$\begin{aligned}\int_M |\Delta u_r|^{\frac{n}{2}} &= (n-2)^{\frac{n}{2}} \int_{\rho<b\le r} b^{-n}|\nabla b|^n + O(1)\\ &= (n-2)^{\frac{n}{2}} \int_{\rho<b\le r} b^{-n} + (n-2)^{\frac{n}{2}}\\ &\quad \times \int_{\rho<b\le r} b^{-n}\big(|\nabla b|^n - |\nabla d|^n\big) + O(1)\\ &= (n-2)^{\frac{n}{2}} B_n \sigma_M \log\frac{V(r)}{V(\rho)} + O(1)\big),\end{aligned} \tag{6.47}$$

' we used $|\nabla b| \le C$ and

$$\int_{r-1<b\le r} b^{-n} \le C, \tag{6.48}$$

and the same estimate with ρ rather than r.

Hence,

$$\|\Delta u_r\|_{n/2} = \frac{\sigma_M^{2/n}}{nc_2 B_n^{\frac{n-2}{n}}}\left(\log\frac{V(r)}{V(\rho)}\right)^{\frac{2}{n}} + O(1) \tag{6.49}$$

and furthermore, if $d(x,y) \le \rho$ then $b \le \rho/a_3 < r-1$, for r large enough, so for $y \in B(x,\rho)$

$$u_r(y) \ge \begin{cases}\log(r/\rho) & \text{if } b<\rho\\ \log(r/b) & \text{if } \rho \le b \le \rho/a_3\end{cases} \ge \log\frac{r}{\rho} - C \ge \tilde{c}_1 B_n \log\frac{V(r)}{V(\rho)} - C. \tag{6.50}$$

$$\|u_r\|_{n/2}^{n/2} \le C\log r + \int_{2<b\le r}\left(\log\frac{r}{b}\right)^{n/2} \le CV(r). \tag{6.51}$$

Then, the function $v_r = u_r/\|\Delta u_r\|_{n/2}$ satisfies the same estimates as $v_{2,r}$ in (6.39), (6.40), (6.42), which gives the sharpness statements for (1.13), when $\alpha = 2$ and for any $n \ge 3$. □

Remark 15 Except for the case $\alpha = 2$, $n = 3, 4$, the proof of sharpness goes through using $d(x,y)$ instead of $b(y)$. We decided to use b throughout, since the estimates needed for $\alpha = 2$, using the family $\{u_r\}$, are basically the same as those used for $\alpha > 2$ using the family $\{u_{\alpha,r}\}$ (e.g. (6.31), (6.32)). We were not able to use the family $\{u_r\}$ above, to settle the sharpness for $\alpha > 2$.

Remark 16 The fine calibration between ρ and r given in (6.15) is only needed to prove sharpness of the power in the denominator of (1.13). Indeed, the proof of the sharpness of the exponential constant in (1.13) goes through using the same families of functions as above, but with $\rho = 1$.

To prove the sharpness statement for (1.16) and (1.17) consider the function h_r satisfying (6.44), and such that $|h_r^{(j)}| \le C$, $j = 1, ..., \alpha$. For a fixed r_0 s.t. $0 < r_0 <$

inj(M) let

$$w_\epsilon(y)=h_{1/\epsilon}\Big(\frac{d(x,y)}{\epsilon r_0}\Big) \tag{6.52}$$

which is nothing but the manifold version of the standard Adams-Moser family, i.e. a smoothing of the function $\log(r_0/d(x,y))$, as defined on $2\epsilon r_0<d(x,y)\le r_0-\epsilon r_0$, for ϵ small enough. With calculations as in [22] one finds that for α even and ϵ small enough

$$\|D^\alpha w_\epsilon\|_{n/\alpha}=\frac{1}{nc_\alpha B_n^{\frac{n-\alpha}{n}}}\Big(\log\frac{1}{V(\epsilon)}\Big)^{\frac{\alpha}{n}}+O(1) \tag{6.53}$$

$$|w_\epsilon(y)|\ge\frac{1}{n}\log\frac{1}{V(\epsilon)}-C,\qquad y\in B(x,\epsilon r_0) \tag{6.54}$$

$$\|w_\epsilon\|_{n/\alpha}\le C \tag{6.55}$$

and the same estimates hold for α odd, with $\widetilde{c}_\alpha$ in place of c_α. The sharpness of the exponential constants in (1.16), (1.17), and of the power in the denominator of (1.16), follows easily from the above estimates, considering the family $v_\epsilon=w_\epsilon/(\kappa\|w_\epsilon\|_{n/\alpha}^{n/\alpha}+\|D^\alpha w_\epsilon\|_{n/\alpha}^{n/\alpha})^{\alpha/n}$. □

Appendix A Proof of (3.53)-(3.56)

For a fixed t let $f(r)=E(r,t)$ so that

$$f'(r)=-\frac{r}{2t}E(r,t),\qquad f''(r)=\Big(-\frac{1}{2t}+\frac{r^2}{4t^2}\Big)E(r,t).$$

Note that

$$|f'(r)|=\frac{r}{2(4\pi)^{n/2}}\,t^{-n/2-1}\exp\Big(-\frac{r^2}{4t}\Big)\le C_{105}r^{-n-1}.$$

Hence using the MVT, for each y there is $\tilde d_j$ with $|\tilde d_j-d|\le|d_j-d|<1/j$ s.t. for $j>\frac{1}{2}\mathrm{dist}(x,K)$

$$|E(d_j,t)-E(d,t)|=|d_j-d|f'(\tilde d_j)\le\frac{C_{106}}{j}\Big(d-\frac{1}{j}\Big)^{-n-1}\le\frac{C_{107}}{j}\mathrm{dist}(x,K)^{-n-1}.$$

We have

$$\nabla E(d_j,t)=-\nabla d_j\frac{d_j}{2t}E(d_j,t)\to-\nabla d\frac{d}{2t}E(d,t)=\nabla E(d,t),$$

and

$$|\nabla E(d_j,t)|\le\frac{d_j}{t}E(d_j,t)\le C_{108}\mathrm{dist}(x,K)^{-n-1}.$$

Using that $\Delta(f\circ g)=f'(g)\Delta g+f''(g)|\nabla g|^2$, and using iv), we get

$$\begin{aligned}\Delta E(d_j,t)&=\left(-\frac{d_j}{2t}\Delta d_j-\frac{1}{2t}+\frac{d_j^2}{4t^2}\right)E(d_j,t)\\&\geq\left(-\frac{n-1}{2t}\frac{d_j}{d}-\frac{d_j}{2jt}-\frac{1}{2t}+\frac{d_j^2}{4t^2}\right)E(d_j,t)\\&\geq\left(-\frac{n-1}{2t}\left|\frac{d_j}{d}-1\right|-\frac{d_j}{2jt}-\frac{n}{2t}+\frac{d_j^2}{4t^2}\right)E(d_j,t)\\&\geq-\left(\frac{n-1}{2jd}+\frac{d_j}{2j}\right)t^{-n/2-1}\exp\left(-\frac{d_j^2}{4t}\right)-\frac{n}{2}t^{-n/2-1}\\&\geq-\frac{C_{109}}{j}\left(\mathrm{dist}(x,K)^{-n-3}+\mathrm{dist}(x,K)^{-n-1}\right)-\frac{n}{2}t^{-n/2-1}.\end{aligned}\tag{A.1}$$

Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Competing interests The authors declare no conflicts of interest, whether direct or indirect, related to this work.

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