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On quadratically defined Lie algebras and their subalgebras

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ON QUADRATICALLY DEFINED LIE ALGEBRAS AND THEIR SUBALGEBRAS

Abstract. This thesis focuses on characterizing a specific class of Lie algebras that share a significant cohomological trait with absolute Galois groups of fields.

The affirmative solution of the longstanding Bloch-Kato conjecture reveals that closed subgroups of maximal pro-p quotients $G_k(p)$ of absolute Galois groups of fields k containing primitive pth roots of 1 have quadratic $\mathbb{F}_p$-cohomology — i.e., their $\mathbb{F}_p$-cohomology rings are generated by elements of degree 1, subjected to relations of degree 2—, leading to the designation of such groups as Bloch-Kato pro-p groups. It has also been conjectured by I. Efrat ([10], [11]) that, if $G_k(p)$ is finitely generated, then it has a specific structure, namely, it is of elementary type, meaning that it can be obtained by iteratively performing certain semidirect products with the infinite cyclic pro-p group $\mathbb{Z}_p$ and free pro-p products by starting with Demuškin groups and the trivial group.

Quadratic algebras hold significant importance in the cohomology theory of graded algebras, as they form the diagonal part of the bigraded cohomology ring ([31]). Notably, within the realm of quadratic algebras, there exists a well-behaved subset known as Koszul algebras, distinguished by their ability to admit linear free resolutions of the trivial module. In turn, this is equivalent to the fact that the cohomology ring is concentrated on its diagonal part ([35], [23]). Computations of the cohomology of Koszul algebras are conveniently achieved through the so-called Koszul dual construction, involving straightforward linear-algebra computations. This advantageous feature makes Koszul algebras particularly amenable to cohomological analyses. Although not all quadratic algebras are Koszul, intriguingly, all currently known instances of Bloch-Kato groups do exhibit Koszul cohomology. This observation prompted T. Weigel [58] and L. Positselski [34] to propose a conjecture asserting that the cohomology of $G_k(p)$, as well as the restricted Lie algebra associated with the Zassenhaus filtration, are Koszul algebras. Additionally, these algebras are expected to be Koszul dual to each other, representing a captivating avenue of exploration in this research area. All groups of elementary type have been proved to satisfy this conjecture (see [27]).

In this thesis, we delve into the study of Bloch-Kato (BK) Lie algebras, a class of quadratic Lie algebras where subalgebras generated by elements of degree 1 are also quadratic. These Lie algebras, introduced in [1], are demonstrated to be Koszul, and their cohomology rings are *universally Koszul*. The thesis introduces novel tools for investigating quadratic Lie algebras through HNN-extensions and presents comprehensive characterizations of BK Lie algebras within specific classes of Lie algebras. In fact, the BK property is explored within the context of holonomy Lie algebras of hyperplane arrangements. To accomplish this, a new class of graded Lie algebras associated with simplicial complexes is introduced, enabling a more generalised treatment of holonomies and right-angled Artin Lie algebras. It turns out that, for this generalised class of holonomy Lie algebras, a Lie-version of the Elementary Type Conjecture holds true. However, we prove that all quadratic 2-relator Lie algebras are BK, so that one cannot expect that the above conjecture holds true for all BK Lie algebras.

Such a construction also allows one to prove that several non-supersolvable arrangements have Koszul holonomy Lie algebra, which answers to a question asked in [44].

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If p is an odd prime, one can easily translate phenomena of quadratic $\mathbb{F}_p$-Lie algebras into the context of restricted p-Lie algebras, and vice versa.

Overall, this thesis contributes to a deeper understanding of quadratic Lie algebras and provides insights into the nature of Bloch-Kato Lie algebras and their cohomology.

The research sheds light on the intriguing connections between these Lie algebras and absolute Galois groups of fields, offering promising avenues for further exploration and analysis.

Sommario. Questa tesi si concentra sulla caratterizzazione di una particolare classe di algebre di Lie che condividono una significativa proprietà coomologica con i gruppi assoluti di Galois di campi.

La soluzione affermativa della congettura di Bloch-Kato rivela che i sottogruppi chiusi dei quozienti pro-p massimali dei gruppi assoluti di Galois $G_k(p)$ dei campi k che contengono radici primitive p-esime dell'unità hanno coomologia quadratica su $\mathbb{F}_p$ — ovvero, i loro anelli di coomologia a coefficienti in $\mathbb{F}_p$ sono generati da elementi di grado 1, soggetti a relazioni di grado 2 —, portando alla designazione di tali gruppi come gruppi pro-p di Bloch-Kato. È stato anche congetturato da I. Efrat ([10], [11]) che, se $G_k(p)$ è finitamente generato, allora ha una struttura specifica, ovvero è di tipo elementare: ciò significa che può essere ottenuto eseguendo iterativamente certi prodotti semidiretti con il gruppo pro-p ciclico infinito $\mathbb{Z}_p$ e prodotti liberi pro-p partendo dai gruppi Demuškin e dal gruppo banale.

Le algebre quadratiche rivestono un'importanza significativa nella teoria delle coomologie delle algebre graduate, in quanto formano la parte diagonale dell'anello di coomologia bigraduato ([31]). In particolare, nel contesto delle algebre quadratiche, esiste una sottoclasse con un buon comportamento coomologico, note come algebre di Koszul, distinte dal fatto di ammettere risoluzioni lineari libere del modulo banale. A loro volta, ciò è equivalente al fatto che l'anello di coomologia è concentrato sulla sua parte diagonale ([35], [23]). I calcoli della coomologia delle algebre di Koszul sono convenientemente realizzati attraverso la cosiddetta costruzione duale di Koszul, che coinvolge semplici calcoli di algebra lineare. Questa caratteristica vantaggiosa rende le algebre di Koszul particolarmente adatte alle analisi coomologiche. Sebbene non tutte le algebre quadratiche siano di Koszul, sorprendentemente, tutte le istanze attualmente conosciute dei gruppi di Bloch-Kato mostrano una coomologia con tale proprietà. Questa osservazione ha spinto T. Weigel [58] e L. Positselski [34] a proporre una congettura che afferma che la coomologia di $G_k(p)$, così come l'algebra di Lie ristretta associata alla filtrazione di Zassenhaus, siano algebre di Koszul. Inoltre, si prevede che queste algebre siano una la duale di Koszul dell'altra, rappresentando un'affascinante via di esplorazione in questo campo di ricerca. È stato dimostrato che tutti i gruppi di tipo elementare soddisfano questa congettura (vedi [27]).

In questa tesi, approfondiamo lo studio delle algebre di Lie di Bloch-Kato (BK), una classe di algebre di Lie quadratiche in cui le sottoalgebre generate da elementi di grado 1 sono anch'esse quadratiche. Queste algebre di Lie, introdotte in [1], sono algebre di Koszul, e i loro anelli di coomologia soddisfano una certa proprietà di essere *universalmente Koszul*. La tesi introduce nuovi strumenti per investigare le algebre di Lie quadratiche attraverso le estensioni HNN e presenta caratterizzazioni complete delle algebre di Lie di BK all'interno di specifiche classi di algebre di Lie. Infatti, la proprietà BK è esplorata nel contesto delle algebre di Lie di olonomia di arrangiamenti di iperpiani. Per lo studio di tali algebre, viene introdotta una nuova classe di algebre di Lie graduate associate a complessi simpliciali, consentendo un trattamento più generalizzato di olonomie e delle cosiddette algebre di Lie RAAG. Risulta che, per questa classe generalizzata di algebre di Lie di olonomia, una versione per algebre di Lie della Congettura di Tipo Elementare è vera. Tuttavia, dimostriamo che tutte le algebre di Lie quadratiche con solo 2 relatori sono di BK, quindi non ci si può aspettare che la suddetta congettura sia vera per tutte le algebre di Lie di BK.

Tale costruzione consente anche di dimostrare che diversi arrangiamenti non supersolubili hanno una omonimia Lie di Koszul, rispondendo a una domanda posta in [44].

Se p è un numero primo dispari, è possibile tradurre facilmente i fenomeni delle algebre di Lie quadratiche $\mathbb{F}_p$ nel contesto delle algebre di Lie p-ristrette, e viceversa.

In generale, questa tesi contribuisce a una comprensione più approfondita delle algebre di Lie quadratiche e fornisce intuizioni sulla natura delle algebre di Lie di Bloch-Kato e sulla

loro coomologia. La ricerca getta luce sulle connessioni intriganti tra queste algebre di Lie e i gruppi di Galois assoluti dei campi, offrendo prospettive promettenti per ulteriori esplorazioni e analisi.

Resumen. Esta tesis se centra en caracterizar una clase específica de álgebras de Lie que comparten un rasgo cohomológico significativo con los grupos absolutos de Galois de los campos.

La solución afirmativa de la longeva conjetura de Bloch-Kato revela que los subgrupos cerrados de los cocientes pro-p maximales $G_k(p)$ de los grupos de Galois absolutos de los cuerpos k que contienen raíces primitivas p-ésimas de 1 tienen cohomología cuadrática $\mathbb{F}_p$; es decir, sus anillos de cohomología $\mathbb{F}_p$ están generados por elementos de grado 1, sometidos a relaciones de grado 2, lo que lleva a designar a tales grupos como grupos pro-p de Bloch-Kato. También se ha conjecturado por I. Efrat ([10], [11]) que, si $G_k(p)$ es finitamente generado, entonces tiene una estructura específica, es decir, es de tipo elemental, lo que significa que se puede obtener realizando iterativamente ciertos productos semidirectos con el grupo pro-p cíclico infinito $\mathbb{Z}_p$ y productos libres pro-p comenzando con grupos Demuškin y el grupo trivial.

Las álgebras cuadráticas tienen una importancia significativa en la teoría de cohomología de álgebras graduadas, ya que forman la parte diagonal del anillo de cohomología bigraduado ([31]). Notablemente, dentro del ámbito de las álgebras cuadráticas, existe un subconjunto con bueno comportamiento cohomológico conocido como el de las álgebras de Koszul, distinguido por su capacidad de admitir resoluciones libres lineales del módulo trivial. A su vez, esto es equivalente al hecho de que el anillo de cohomología esté concentrado en su parte diagonal ([35], [23]). Los cálculos de la cohomología de las álgebras de Koszul se logran de manera conveniente mediante la llamada construcción dual de Koszul, que los reduce a cálculos sencillos de álgebra lineal. Esta característica ventajosa hace que las álgebras de Koszul sean particularmente adecuadas para análisis cohomológicos. Aunque no todas las álgebras cuadráticas son Koszul, curiosamente, todas las instancias conocidas de grupos Bloch-Kato exhiben cohomología de Koszul. Esta observación llevó a T. Weigel [58] y L. Positselski [34] a proponer una conjetura que afirma que la cohomología de $G_k(p)$, así como el álgebra de Lie restringida asociada con la filtración de Zassenhaus, son álgebras de Koszul. Además, se espera que estas álgebras sean duales entre sí, representando un camino fascinante de exploración en esta área de investigación. Se ha demostrado que todos los grupos de tipo elemental satisfacen esta conjetura (ver [27]).

En esta tesis, profundizamos en el estudio de las álgebras de Lie de Bloch-Kato (BK), una clase de álgebras de Lie cuadráticas donde las subálgebras generadas por elementos de grado 1 también son cuadráticas. Estas álgebras de Lie, introducidas en [1], son de Koszul, y sus anillos de cohomología son *universalmente de Koszul*. La tesis presenta nuevas herramientas para investigar álgebras de Lie cuadráticas a través de extensiones HNN y ofrece caracterizaciones exhaustivas de las álgebras de Lie BK dentro de clases específicas de álgebras de Lie. De hecho, la propiedad BK se explora en el contexto de las álgebras de Lie de holonomía de arreglos de hiperplanos. Para lograr esto, se introduce una nueva clase de álgebras de Lie graduadas asociadas con complejos simpliciales, lo que permite un tratamiento más generalizado de las holonomías y álgebras de Lie de Artin de ángulo recto (RAAG). Resulta que, para esta clase generalizada de álgebras de Lie de holonomía, se cumple una versión de Lie de la Conjetura de Tipo Elemental. Sin embargo, demostramos que todas las álgebras de Lie cuadráticas con 2 relatores son BK, por lo que no se puede esperar que la conjetura anterior sea cierta para todas las álgebras de Lie BK.

Esta construcción también permite demostrar que varios arreglos no supersolubles tienen álgebras de Lie de holonomía de Koszul, lo que responde a una pregunta formulada en [44].

Si p es un primo impar, se pueden traducir fácilmente fenómenos de álgebras de Lie cuadráticas $\mathbb{F}_p$ en el contexto de álgebras de Lie restringidas p, y viceversa.

En resumen, esta tesis contribuye a una comprensión más profunda de las álgebras de Lie cuadráticas y proporciona información interesante sobre la naturaleza de las álgebras de Lie de Bloch-Kato y su cohomología.

La investigación arroja luz sobre las conexiones intrigantes entre estas álgebras de Lie y los grupos de Galois absolutos de los cuerpos, ofreciendo perspectivas prometedoras para una mayor exploración y análisis.

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Contents

Introduction

Lie algebras were introduced in the context of Lie groups to study the concept of infinitesimal transformations by Marius Sophus Lie in the 1870s, and were independently discovered by Wilhelm Killing in the 1880s. Mathematicians were interested in understanding the local behaviour of Lie groups by some kind of linearization of their elements. Since then, Lie algebras have gained more and more emancipation from the Lie group-theoretic context and are now extensively studied by physicists as well for exploring the symmetries of our Universe.

During the early 20^{th} century, Cartan discovered a connection between the De Rham cohomology groups of a Lie group – which are related with its holes as a manifold – and the cohomology groups of the associated Lie algebra. In particular, he proved that for a compact, simply connected Lie group, its cohomology groups are isomorphic to those of the associated Lie algebra.

In the study of discrete groups, Magnus theory allows one to associate to any group a graded Lie algebra whose homogeneous components are the factors of the lower-central series of the group. In this case, one may expect that the group cohomology of the given group is related to the cohomology of the associated Lie algebra. In fact, under some mild hypotheses, there is a spectral sequence due to J.P. May [24] involving the graded cohomology groups of the Lie algebra and converging to the cohomology of the given group.

Our interest in the cohomology theory of Lie algebras mainly comes from the realm of Galois theory. A primitive version of cohomology theory of Galois theory made its first appearance in the 1920s thanks to Emmy Noether, who first noticed that both the normal basis theorem and Hilbert's Theorem 90 can be stated in a unified way: The first cohomology group of the Galois group of a Galois extension $\mathbb{L}/\mathbb{F}$ with coefficients in a module M vanishes when M is either the additive or the multiplicative group of the field $\mathbb{L}$. Later, many more Galois theoretic results were translated into the cohomological language.

The main problem in modern Galois theory is that of recognising those profinite groups that occur as absolute Galois groups of some field. A big step forward was accomplished by the positive solution of the longstanding Bloch-Kato conjecture, now known as Norm-residue isomorphism theorem ([54], [57]), which predicts that the $\mathbb{F}_p$-cohomology of the absolute Galois group of a field $\mathbb{F}$ containing a primitive pth root of 1 is isomorphic with the Milnor K-theory mod p of $\mathbb{F}$. In particular, for such fields $\mathbb{F}$, the $\mathbb{F}_p$-cohomology ring of any closed subgroup of the absolute Galois group of $\mathbb{F}$ is a quadratic algebra.

By using the Jennings-Zassenhaus filtration of a pro-p group – a variation of the lower-central series –, one can associated to any such a group a graded (restricted) Lie algebra which encodes information about the group itself. For instance, a version of May's spectral sequence relates relates the cohomology ring of the restricted $\mathbb{N}$-graded Lie algebra associated to a pro-p group G with the cohomology ring of the pro-p group G.

Since the absolute Galois groups of those special fields as above satisfy the hereditary cohomological property predicted by the Bloch-Kato conjecture – that is hence called the Bloch-Kato property–, we are led to consider graded Lie algebras with an analogous feature. In fact, all known examples of absolute Galois groups of fields containing primitive pth roots of 1 have associated Lie algebras satisfying such a property.

Another important – yet far from being proven – conjecture is the so-called Elementary Type Conjecture due to Ido Efrat [10] which would provide the structure of all finitely generated absolute Galois pro-p groups in terms of some elementary constructions. More precisely, the conjecture predicts that such groups can be obtained by iteratively performing certain semidirect products with the infinite cyclic pro-p group $\mathbb{Z}_p$ and free pro-p products by starting with Demuškin groups and locally uniform pro-p groups.

So far, all known examples of pro-p groups with the Bloch-Kato property have this shape, though it is not known whether this is always the case. A Lie algebraic version of such a problem might either give evidence in support of the conjecture or give a hint on how to disprove it.

Now we provide an outline of the present thesis, which consists of three main parts.

In the first part we define the main characters and tools that are used in the rest of the thesis. We start with the definitions of graded algebras and Lie algebras. In particular, we focus on Lie algebras $\mathscr{L}$ defined over an arbitrary field k, having vector space decompositions $\mathscr{L} = \bigoplus_{i=1}^{\infty} \mathscr{L}_i$ which are compatible with the Lie brackets, i.e., $[\mathscr{L}_i, \mathscr{L}_j] \subseteq \mathscr{L}_{i+j}$.

Next, we define the cohomology rings of (Lie) algebras and show some elementary facts about their behaviour in the graded case. Each cohomology group is a graded vector space, with respect to the so-called internal grading, and the degree of the elements of the first and second cohomology groups are related with the degrees of minimal generators and relations of the (Lie) algebra, respectively. There, we start by restricting our focus to quadratic Lie algebras, that are quotients of free Lie algebras with respect to some ideal generated by linear combinations of simple Lie brackets $[x, y]$ between the generators, i.e., the nth cohomology group is concentrated in degree n for $n = 1$ (the generators have degree 1) and $n = 2$ (the relations have degree 2). Among **quadratic Lie algebras**, there is the special subclass of Koszul Lie algebras, defined in terms of their bigraded cohomology: a Lie algebra $\mathscr{L}$ is **Koszul** if its cohomology is concentrated on the diagonal, i.e., the internal grading coincides with the homological one. Equivalently, the cohomology ring is generated as an algebra by the degree 1 elements of the first cohomology group. By a result of Löfwall [23], for Koszul Lie algebras presented by generators and relations, computing their cohomology rings is just linear algebra on the presentation.

A conjecture by Positselski predicts that the Lie algebra associated with an absolute pro-p Galois group is always Koszul [32]. Though, Koszulity of a Lie algebra must not be related with the structure of its subalgebras, e.g., they must not be even quadratic. We are thus led to consider the more special class of **Bloch-Kato Lie algebras**, which share the hereditary property of absolute Galois groups given by the solution of the Bloch-Kato conjecture, whence the name. We say that a graded Lie algebra is Bloch-Kato if it is quadratic, as well as all of its subalgebras that are generated by elements of degree 1 (that are called standard subalgebras). The aim of this work is to study such a class of Lie algebras, guided by a translation into the Lie context of the Elementary Type Conjecture.

In order to approach the study of Bloch-Kato Lie algebras (and of quadratic Lie algebras as well), we make use of the so-called HNN-extension, which is a construction coming from geometric group theory that was translated into the realm of Lie algebras by Lichtman and Shirvani in [22]. We show that all quadratic Lie algebras split as HNN-extensions over some subalgebras generated by elements of degree 1. The consequence is that Bloch-Kato Lie algebras are Koszul, as well as all of their standard subalgebras. Also, By using results of [18], HNN-extensions allow us to show that all finitely presented, positively-graded Lie algebras embed into some quadratic Lie algebra

We end the first part with some transversal problems concerning Lie algebras and their cohomology: We state a version of Kurosh subalgebra theorem, proven by the candidate in [1], and we prove a Stallings-type decomposition theorem for some Lie algebras with "more than one end". Moreover, we deal with restricted Lie algebras and their cohomology, and we prove that in characteristic $\neq 2$, the theories of quadratic restricted Lie algebras and of quadratic Lie algebras are equivalent.

In the second part we consider the class of Poincaré duality Lie algebras, that are defined cohomologically in terms of the non-degeneracy of the maps induced by the multiplication of the cohomology. We prove by hand that all quadratic Lie algebras that are Poincaré duality of dimension 2 are Bloch-Kato, and an explicit presentation of them is given. Interestingly, these are precisely the associated Lie algebras of Demuškin groups that appear in the Elementary Type Conjecture. Such a classification allows us to prove that all quadratic 2-relator Lie algebras are Bloch-Kato, suggesting a possible example of a pro-p group that is Bloch-Kato but not of elementary type.

Motivated by a phenomenon that occurs for Poincaré duality Lie algebras, we also deal with the homogeneity of the cohomological dimensions of the subalgebras of a given Lie algebra. We prove that if $\mathscr{L}$ is Bloch-Kato of cohomological dimension n and $m \leq n$ is a positive integer, then $\mathscr{L}$ contains a subalgebra of cohomological dimension m.

The third and last part is devoted to the study of the Bloch-Kato property for Lie algebras within very specific classes defined by some combinatorial structures.

The first class under concern is that of **graph products** of Lie algebras, which generalises the Lie algebra analogue of right-angled Artin (pro-p) groups. For defining such objects one fixes a graph whose vertices are labelled by some given Lie algebras (the local-algebras), and then defines its graph product as the Lie algebra generated by all these Lie algebras where the elements of different local-algebras commute if their corresponding vertices are adjacent. We provide a complete classification of Bloch-Kato graph products in terms of the local-algebras and of the shape of the defining graph.

As a second test bench we explore the **holonomy Lie algebras** of hyperplane arrangements. The holonomy Lie algebra of a finite collection of linear hyperplanes in some finite-dimensional complex space is related with the **Orlik-Solomon algebra** of the arrangement, i.e., the cohomology of the fundamental group of its complement, and it only depends on its combinatorics. In particular, one can give an explicit presentation of the holonomy in terms of the intersection lattice of the arrangement. For instance, when the arrangement satisfies a specific combinatorial property called supersolvability, the holonomy Lie algebra is Koszul, as well as the Orlik-Solomon algebra. Although it is not known whether there exist non-supersolvable arrangements with Koszul Orlik-Solomon algebra, we show that there do exist such arrangements with Koszul holonomy, answering to a question of Shen and Suciu [44].

Finally, we consider the Bloch-Kato property for holonomy Lie algebras and we provide a complete classification of them in terms of the low-rank elements of the intersection lattice of the arrangement. In order to do so, we concocted a wider new class of Lie algebras defined by simplicial complexes. In the case when the simplicial complex is a graph, the corresponding Lie algebra is a right-angled Artin Lie algebra.

We conclude our work with the study of a distinguished class of Lie algebras defined as the kernels of some linear characters of Lie algebras associated to graphs (or, more generally, to simplicial complexes), namely the **Bestvina-Brady Lie algebras**. The group theoretic counterparts of these objects were introduced by Bestvina and Brady, generalising an example of Stallings, as instances of groups where homological and geometric finiteness properties do not agree. Although in the case of graded Lie algebras these concepts coincide, Bestvina-Brady Lie algebras still give a rich class of examples where the homological finiteness properties can be checked by looking at the defining graph. We provide necessary conditions on a graph for the associated Bestvina-Brady Lie algebra to be Bloch-Kato.

§ I. Definitions

1. Graded modules

If G is an abelian group and R is any ring with 1, by a G-graded R-module we mean an R-module V with a specified decomposition

$$V = \bigoplus_{g\in G} V_g$$

into R-submodules V_g. If $0 \neq v \in V_g$, then we say that v is homogeneous of degree $|v| := g$. If all the components V_g are finitely generated over R, then we say that the graded module V is **locally finite**, or of finite type. If X is a subset of G and $V_g = 0$ for all $g \in G \setminus X$, we may also call V an X-graded R-module.

One can then define an R-morphism $f : V \to W$ between G-graded R-modules V and W to be **graded** (or homogeneous) of degree $d \in G$ if $f(V_g) \subseteq W_{g+d}$ ($g \in G$). For instance, the identity map $V \to V$ is a graded morphism of degree 0_G. We will denote by $\mathrm{Hom}_R^{-d}(V, W)$ the set of all R-linear graded maps of degree d.

If $d \in G$, one can define the d-shift of V to be the G-graded R-module $V[d]$ whose degree g submodule is V_{g+d}. With this notation, one has

$$\mathrm{Hom}_R^{-d}(V, W) = \mathrm{Hom}_R^0(V[-d], W) = \mathrm{Hom}_R^0(V, W[d]).$$

One can also define a G-graded structure on the tensor product $V \otimes_R W$ by setting

$$(V \otimes_R W)_g = \bigoplus_{x\in G} V_x \otimes_R W_{g-x}.$$

If A is an associative algebra over a commutative ring k, such that its underlying k-module is G-graded, then we say that A is a graded k-algebra if the multiplication mapping $A \otimes A \to A$ is graded of degree 0. Moreover, if the underlying k-module of an A-module M is G-graded we say that M is a **graded A-module** if the natural map $A \otimes M \to M$ is graded of degree 0.

Henceforth, we will only consider the case in which $G = \mathbb{Z}$ is the group of integers, and k is a field of characteristic $\neq 2$.

Example 1.1. *Let $\mathfrak{X}$ be a set of indeterminates and let $S = k[\mathfrak{X}]$ be the corresponding polynomial ring on a commutative ring k. Then, S is a graded k-algebra with respect to the grading given by the polynomial-degree of its elements, i.e.,*

$$S_n = \mathrm{Span}_k \left\{ x_1 x_2 \cdots x_n \;\middle|\; x_i \in \mathfrak{X} \right\}.$$

More generally, if we allow the degree d_x of a generator $x \in \mathfrak{X}$ to be any natural number, the induced grading on S is given by

$$S_n = \mathrm{Span}_k \left\{ x_1 x_2 \cdots x_m \;\middle|\; d_{x_1} + \cdots + d_{x_m} = n,\ x_i \in \mathfrak{X} \right\}.$$

2. Lie algebras

Let k be an arbitrary field. By a **Lie algebra**, we mean a k-vector space $\mathscr{L}$ endowed with a k-bilinear map

$$[\_,\_] : \mathscr{L} \times \mathscr{L} \to \mathscr{L},$$

called Lie bracket, that satisfies the following two properties:

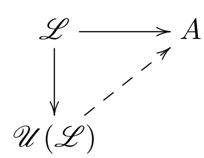
(1) Skew-symmetry: for all $x \in \mathscr{L}$, $[x, x] = 0$.
(2) Jacobi identity: for all $x, y, z \in \mathscr{L}$, one has

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Notice that for any bilinear map $[\_,\_]$, condition (1) is equivalent to requiring that, for all $x, y \in \mathscr{L}$, $[x, y] + [y, x] = 0$, whenever k has characteristic different from 2.

An easy example of a Lie algebra is given by considering the commutator of any associative (non-necessarily unital) k-algebra. Namely, if A is such an algebra, then A is naturally a Lie algebra by defining its Lie brackets by $[a, b] := ab - ba$ $(a, b \in A)$. We thus get a functor $(\_)_{[\cdot,\cdot]} : \mathfrak{Alg}_k \to \mathfrak{Lie}_k$ from the category of associative k-algebras to that of k-Lie algebras.

To any Lie algebra $\mathscr{L}$ one usually attaches an associative algebra $\mathscr{U}(\mathscr{L})$, the **universal enveloping algebra**, with a map $\mathscr{L} \to \mathscr{U}(\mathscr{L})$ which is universal with respect to morphisms of Lie algebras $\mathscr{L} \to A_{[\cdot,\cdot]}$, where A is any unital associative algebra. Universality means that every such Lie map yields a unique morphism of algebras $\mathscr{U}(\mathscr{L}) \to A$ making the following diagram commutative



It turns out that $\mathscr{U}(\mathscr{L})$ exists and is unique up to a unique isomorphism. Moreover, the functor $\mathscr{U}(\_) : \mathfrak{Lie}_k \to \mathfrak{Alg}_k$ is left-adjoint to $(\_)_{[\cdot,\cdot]}$, i.e., there are natural isomorphisms

$$\mathrm{Hom}_{\mathfrak{Lie}}(\mathscr{L}, A_{[\cdot,\cdot]}) \simeq \mathrm{Hom}_{\mathfrak{Alg}}(\mathscr{U}(\mathscr{L}), A).$$

One can explicitly construct $\mathscr{U}(\mathscr{L})$ as the quotient of the tensor algebra $T(\mathscr{L})$ over $\mathscr{L}$ by the ideal generated by elements $[a, b] - a \otimes b + b \otimes a$, where $a, b \in \mathscr{L}$. Recall that the **tensor algebra** $T_\bullet(V)$ over a vector space V is the graded k-algebra with degree n component $T_n(V) = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$ for $n \geq 0$, $T_n(V) = 0$ for $n < 0$, and multiplication given by concatenation of tensors.

The degree filtration

$$F^p T(\mathscr{L}) = \sum_{n \geq p} T_n(\mathscr{L})$$

of the tensor algebra gives rise to a descending filtration of $\mathscr{U}(\mathscr{L})$. One can thus define a graded object

$$\operatorname{gr}\mathscr{U}(\mathscr{L}) = \bigoplus_{p\geq 0} F^p\mathscr{U}(\mathscr{L})/F^{p+1}\mathscr{U}(\mathscr{L})$$

which is completely described by the following well-known theorem. Before stating it, recall that for a vector space V, the **symmetric algebra** on V is the quotient of the tensor algebra $T(V)$ by the ideal generated by $v \otimes w - w \otimes v$, for $v, w \in V$ (see Example 1.1).

Theorem 2.1 (Poincaré-Birkhoff-Witt [16]). *Let $\mathscr{L}$ be a Lie algebra over a field k.*

(1) *If $\mathcal{B}$ is an ordered basis of $\mathscr{L}$, then the elements $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, for $x_i \in \mathcal{B}$, $\alpha_i \geq 1$ and $x_1 < x_2 < \cdots < x_n$, form a basis for $\mathscr{U}(\mathscr{L})$.*
(2) *The natural map $\iota : \mathscr{L} \to \mathscr{U}(\mathscr{L})$ is injective.*
(3) *The graded universal enveloping algebra $\operatorname{gr}\mathscr{U}(\mathscr{L})$ is isomorphic to the symmetric algebra $S(\mathscr{L})$ on $\mathscr{L}$.*

In particular, $\mathscr{U}(\mathscr{L})$ has no non-trivial zero-divisors.

2.1. **Graded Lie algebras.** Let G be a (pro-p) group. One can thus consider the lower-central series (resp. Zassenhaus filtration) G_i ($i \geq 1$) of G and form the graded abelian group (resp. graded $\mathbb{F}_p$-vector space)

$$\operatorname{gr} G := \bigoplus_{i\geq 1} G_i/G_{i+1}.$$

It follows by Hall-Witt commutator formulas that $\operatorname{gr} G$ naturally inherits a structure of a Lie ring (resp. $\mathbb{F}_p$-Lie algebra). This object can be used to study properties of the group G (e.g. see [62] and [63]). The underlying module of such Lie algebras is graded and the Lie brackets respect such a grading. If G is an abstract group, by tensoring $\operatorname{gr} G$ with any field k, we get a graded k-Lie algebra generated by degree 1 elements. In this section we will abstractly define graded Lie algebras.

If the underlying vector space of a Lie algebra $\mathscr{L}$ is $\mathbb{Z}$-graded, we say that $\mathscr{L}$ is an $\mathbb{N}$-graded Lie algebra (or just a **graded Lie algebra**) if

(1) $\mathscr{L}_i = 0$ for $i \leq 0$, and
(2) $[\_,\_]$ is a graded linear map of degree 0 on the tensor product $\mathscr{L} \otimes \mathscr{L}$, i.e., $[\mathscr{L}_i, \mathscr{L}_j] \subseteq \mathscr{L}_{i+j}$.

Notice that in the literature, by a graded Lie algebra one often means a graded vector space satisfying a graded version of skew-symmetry and of Jacobi identity. They are also called Lie super-algebras and, forgetting the grading, they need not be ordinary Lie algebras. Although we will refer to such a generalization just once, it is worth observing that Lie algebras which are graded with respect to our definition are graded Lie superalgebras concentrated in even degrees.

If $\mathscr{L}$ is a graded Lie algebra, then its universal enveloping algebra inherits the structure of a graded associative algebra, which is different from the one inducing the degree filtration. We can either define it via the PBW Theorem or by inducing it from a different grading of the tensor algebra.

(1) Let $\mathcal{B}$ be a homogeneous basis of $\mathscr{L}$. The degree n component of $\mathscr{U}(\mathscr{L})$ is the space generated by the elements $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, where $x_i \in \mathcal{B}$, $x_i < x_j$ $(i < j)$ and $\sum_i \alpha_i |x_i| = n$. Since $\mathscr{U}(\mathscr{L})$ is generated by elements of that form with $n \geq 0$, we get a graduation of $\mathscr{U}(\mathscr{L})$.

(2) Define the induced graduation on $T(\mathscr{L})$ as we did in (1). Now, for homogeneous elements $a, b \in \mathscr{L}$, the element $[a, b] - a \otimes b + b \otimes a$ lies in the degree $|a| + |b|$ component of $T(\mathscr{L})$, i.e., it generates a graded ideal J of $T(\mathscr{L})$. It follows that the quotient $\mathscr{U}(\mathscr{L}) = T(\mathscr{L})/J$ is a graded algebra.

If V is a graded vector space, we can define the **free Lie algebra** $\mathscr{F}(V)$ over V as the Lie subalgebra of $T(V)_{[\cdot,\cdot]}$ generated by V. It satisfies the following universal property which shows that it is a free object in the category of graded Lie algebras: Any graded morphism $V \to \mathscr{L}$ of V to a graded Lie algebra $\mathscr{L}$ can be uniquely lifted to a Lie map $\mathscr{F}(V) \to \mathscr{L}$. If $\mathcal{B}$ is a homogeneous basis of the vector space V, we will also say that $\mathscr{F}(V)$ is free over $\mathcal{B}$.

By a **standard (Lie) algebra** we will mean a 1-generated (Lie) algebra, i.e., a graded (Lie) algebra that is generated by elements of degree 1. Equivalently, it is a quotient of the free (Lie) algebra $T(V)$ (resp. $\mathscr{F}(V)$), where $V = V_1$, by a homogeneous ideal.

2.2. **Cohomology of graded algebras.** Like most algebraic structures, cohomology theory applies to (Lie) algebras. In fact, in the 1930s, Cartan and De Rham realised that the De Rham cohomology of a compact Lie group may be computed using left-invariant differentials, and it was gradually noticed that the Lie algebra of left invariant vector fields of the group determines its cohomology. In their seminal paper in 1948, Chevalley and Eilenberg defined the cohomology of any Lie algebra, and they were also able to translate Whitehead Lemmas into the homological language. E.g., the first Whitehead Lemma, which states that there are no outer derivations from a finite dimensional Lie algebra $\mathfrak{g}$ of characteristic 0 into a simple $\mathfrak{g}$-module V, can now be translated into the more concise statement "$H^1(\mathfrak{g}, V) = 0$".

Since then, cohomology of Lie algebras has been intensively studied and it was proved that many – though not all – group-theoretic cohomological results can be translated into the context of Lie algebras. One of the aims of this thesis walks toward this direction, as we will try to show evidence on more similarities between the cohomological behaviours of (pro-p) groups and of graded Lie algebras.

If $\mathscr{L}$ is a k-Lie algebra and M is an $\mathscr{L}$-module, i.e., a $\mathscr{U}(\mathscr{L})$-module one defines the nth cohomology group of $\mathscr{L}$ with coefficients in M as

$$H^n(\mathscr{L}, M) := \mathrm{Ext}^n_{\mathscr{U}(\mathscr{L})}(k, M).$$

We now treat the graded case in a more detailed way.

Fix a $\mathbb{N}_0$-graded **connected** associative algebra A. This amounts to saying that A is a non-negatively graded k-algebra with an **augmentation map** $\varepsilon : A \to k$ that is a homomorphism of graded algebras – where k is concentrated in degree 0 –, and an isomorphism in degree 0. The augmentation ideal $A_+ = \ker \varepsilon$ is the sum of the positive degree components of A. Recall that by a graded (left) A-module M we mean a $\mathbb{Z}$-graded vector space endowed with a degree 0 map $A \otimes M \to M$ which makes M into an A-module. *Henceforth we will refer to such algebras and modules simply as graded k-algebras and graded A-modules, respectively.* The augmentation induces on k the structure of an A-module, which is called the **trivial module**.

It is easy to see that the category of locally finite A-modules has enough projectives, for any free module being projective.

If M and N are two A-modules, we denote by $\mathrm{Ext}^{\bullet j}_A(M, N)$ the right-derived functor of

$$\mathrm{Hom}^j_A(M, N) = \left\{ f : M \to N \text{ morphism of } A\text{-modules} \;\middle|\; f(M_k) \subseteq N_{k-j} \right\}.$$

It is straightforward to check that

$$\mathrm{Ext}^{ij}_A(M[m], N[n]) = \mathrm{Ext}^{i,j+m-n}_A(M, N) \tag{2.1}$$

for graded A-modules M and N and integers m, n. Indeed, by definition of right-derived functors, it is enough to check it holds for $i = 0$, that is, $\mathrm{Hom}^j_A(M[m], N[n]) = \mathrm{Hom}^{j+m-n}_A(M, N)$.

For a module M over a k-algebra A we call $H^\bullet(A) := \mathrm{Ext}^\bullet_A(k, k)$ and $H^\bullet(A, M) = \mathrm{Ext}^\bullet_A(k, M)$ the *cohomology* of A and of M, respectively.

The Ext-groups can be computed as follows: Let $P_\bullet \to M$ be a graded projective A-resolution and apply the functor $\mathrm{Hom}^j_A(\_, N)$ to get the cochain complex

$$\cdots \to \mathrm{Hom}^j_A(P_{i-1}, N) \to \mathrm{Hom}^j_A(P_i, N) \to \mathrm{Hom}^j_A(P_{i+1}, N) \to \ldots.$$

Then

$$\mathrm{Ext}^{ij}_A(M, N) \simeq H^i(\mathrm{Hom}^j_A(P_\bullet, N)) = \frac{\ker(\mathrm{Hom}^j_A(P_i, N) \to \mathrm{Hom}^j_A(P_{i+1}, N))}{\mathrm{Im}(\mathrm{Hom}^j_A(P_{i-1}, N) \to \mathrm{Hom}^j_A(P_i, N))}$$

There is a distinguished projective resolution that one can always use for computing cohomology, that is the so-called *Bar resolution*

$$\cdots \to A \otimes A_+^{\otimes n} \otimes M \to \cdots \to A \otimes A_+ \otimes A_+ \otimes M \to A \otimes A_+ \otimes M \to A \otimes M \to 0$$

with differential given by

$$\partial(a_0\otimes\cdots\otimes a_i\otimes m)=\sum_{s=1}^{i}(-1)^s a_0\otimes\cdots\otimes a_{s-1}a_s\otimes\cdots\otimes a_i\otimes m+(-1)^{i+1}a_0\otimes\cdots\otimes a_i m.$$

Although the Bar complex is a natural construction, it is almost useless to actually compute cohomology groups (of high degrees). This because it has infinite length and all the terms $A\otimes A_+^{\otimes n}\otimes M$ are often too big. However, one can always find a (non-natural) "small" resolution.

Definition 2.2. *Let A be a graded algebra. A bounded above complex of graded free modules*

$$\cdots\to P_2\to P_1\to P_0$$

is called ***minimal*** *if all the induced maps $k\otimes_A P_{i+1}\to k\otimes_A P_i$ vanish.*

The following result is classical and relies on a version of Nakayama Lemma. Here we present the proof given in [31] (see also [58]).

Lemma 2.3. *Let A be a graded algebra and let M be a non-negatively graded A-module. Then, there is a free A-resolution $P_\bullet\twoheadrightarrow M$ of M such that $P_\bullet$ is a minimal complex.*

Proof. We start by showing that a homogeneous subspace X of M generates M as an A-module iff the composite $\bar f: X\to M\to k\otimes_A M$ is surjective. Indeed, X generates M iff the natural map $f: A\otimes X\to M$ is surjective. If f is surjective, then, by applying the right exact functor $k\otimes_A \_$ to f we get the surjectivity of $X\simeq k\otimes_A A\otimes M\to k\otimes_A M$. Conversely, if $X\to M\to k\otimes_A M$ is surjective, we may argue by induction on n for proving that f_n is surjective. Since M is non-negatively graded, for $n<0$, it is clearly true. Assume f_i is surjective for $i<n$. If $m\in M_n$, since $\bar f$ is surjective, there exists some $x\in X_n$ such that $1\otimes x=\bar f(x)=1\otimes m$. Notice that, since A acts on k via the augmentation ε, $k\otimes_A M\simeq M/A_+M$. Hence, $m-x\in A_+M$, i.e., $m-x$ belongs to the image of f, and thus the same is true for m.

We can now construct a minimal resolution by iterating this process. Let $P_0=A\otimes X_0$, where X_0 is a generating space for M that is isomorphic with $k\otimes_A M_n$. Assume P_i has been defined for $i<n$ and let M_n be the kernel of $P_n\to P_{n-1}$. Choose a generating system X_n for M_n that is isomorphic with $k\otimes_A M$ and put $P_n=A\otimes X_n$. ☐

Free resolutions as the one constructed in Lemma 2.3 are called **minimal**.

Even though minimal resolutions are not unique (see [31]), the involved free modules are uniquely determined by M: If $A\otimes X_\bullet\to M$ is such a resolution, then, $X_i^*\simeq \mathrm{Ext}^i_A(M,k)$.

Definition 2.4. *The **cohomological dimension*** $\operatorname{cd} A$ *of a graded k-algebra A is defined as*

$$\operatorname{cd} A = \min\left(\left\{n \in \mathbb{N} \;\middle|\; \operatorname{Ext}^n_A(k, M) = 0 \text{ for all } A\text{-modules } M\right\} \cup \{\infty\}\right)$$

Corollary 2.5 ([58]). *A graded algebra A has cohomological dimension $n < \infty$ if, and only if, $H^n(A, k) \neq 0 = H^{n+1}(A, k)$.*

The following results give a description of the low degree Ext-groups (see [31]).

Proposition 2.6. *Let A be a graded k-algebra, and M a graded A-module with $M_i = 0$ for $i < 0$. Then the following hold:*

(1) $\operatorname{Ext}^{ij}_A(M, k) = 0$ *for all $i > j$.*

(2) *There exist graded vector spaces X and Y such that M is presented as*

$$A \otimes Y \to A \otimes X \to M \to 0$$

and

$$X^*_j \simeq \operatorname{Ext}^{0,j}_A(M, k)$$
$$Y^*_j \simeq \operatorname{Ext}^{1,j}_A(M, k)$$

(3) *There exist graded vector spaces V and W such that A is presented as an algebra as*

$$T_\bullet(V) \otimes W \otimes T_\bullet(V) \to T_\bullet(V) \to A \to 0$$

and

$$V^*_j \simeq \operatorname{Ext}^{1,j}_A(k, k)$$
$$W^*_j \simeq \operatorname{Ext}^{2,j}_A(k, k)$$

From (2) one can easily deduce:

Corollary 2.7. *Let M be a non-negatively graded A-module.*

(1) If $\operatorname{Ext}^{0,\ell}_A(M, k) = 0$, then M has no minimal generator of degree ℓ, i.e., if $m_\ell \in M_\ell$ then there are elements $a_i \in A_i$ and $m_i \in M_i$ such that

$$m_\ell = \sum_{i=1}^{\ell} a_i m_{\ell - i}.$$

(2) If $\operatorname{Ext}^{1,\ell}_A(M, k) = 0$, then M has no minimal relation of degree ℓ.

The vector space $H^\bullet(A) = \bigoplus_{i,j} H^{i,j}(A)$ can also be endowed with a product, the so-called **cup-product** $\smile$, that makes it into a bigraded (or $\mathbb{N}_0 \times \mathbb{N}_0-$) algebra, i.e.,

$$\smile : H^{ij}(A, k) \otimes H^{pq}(A, k) \to H^{i+p, j+q}(A, k).$$

Since $\operatorname{Ext}^{\bullet,j}_A(\_, k)$ is the right derived functor of $\operatorname{Hom}^j_A(\_, k)$, one recovers a bigraded version of the long exact sequence theorem.

Proposition 2.8. *Let A be a graded k-algebra. If $0 \to L \to M \to N \to 0$ is an exact sequence of graded A-modules, then for all j there is an induced long exact sequence*

$$0 \to \mathrm{Ext}_A^{0,j}(N,k) \to \mathrm{Ext}_A^{0,j}(M,k) \to \mathrm{Ext}_A^{0,j}(L,k) \to \mathrm{Ext}_A^{1,j}(N,k) \to \dots$$
$$\dots \to \mathrm{Ext}_A^{i,j}(N,k) \to \mathrm{Ext}_A^{i,j}(M,k) \to \mathrm{Ext}_A^{i,j}(L,k) \to \mathrm{Ext}_A^{i+1,j}(N,k) \to \dots$$

Also a version of the Eckmann-Shapiro Lemma involving the internal degree of cohomology holds. In order to state such result it is necessary to endow the tensor product with a grading. Let A and B be two $\mathbb{N}_0$-graded algebras, M a graded (A,B)-bimodule and N a graded left B-module. For M being a graded (A,B)-bimodule means that $M = \bigoplus M_i$ is a graded vector space and both the action of A and B preserve the grading, i.e., $A_j M_i + M_i B_j \subseteq M_{i+j}$. Then, the tensor product $M \otimes_B N$ is an A-module whose grading is given by

$$(M \otimes_B N)_n = \left(\bigoplus_{i+j=n} M_i \otimes_k N_j \right) / S_n$$

where S_n is the vector subspace generated by elements of the form $xb \otimes y - x \otimes by$ for $x \in M_i$, $y \in N_j$ and $b \in B_{n-i-j}$.

In turn, this construction allows one to define the **Tor-functor**: If A is a graded k-algebra, for a right A-module M and left A-module N, let $\mathrm{Tor}^A_{\bullet j}(M,N)$ be the left-derived functor to the right-exact functor $M \mapsto (M \otimes_A N)_j$. By using the Bar-complex of N, one can see that there is a natural duality $\mathrm{Ext}^i_A(N, M^*) = \mathrm{Tor}^A_i(M,N)^*$.

Theorem 2.9 (Eckmann-Shapiro Lemma)**.** *Let A be a graded k-algebra and let B be a homogeneous subalgebra of A. Suppose that A is flat as a right B-module. Let M be a left B-module and let N be a left A-module. Then,*

$$\mathrm{Ext}^{ij}_A(A \otimes_B M, N) \simeq \mathrm{Ext}^{ij}_B(M,N).$$

Proof. Let $P_\bullet \to M$ be a projective resolution of M over B. Since A is a flat right B-module, the sequence

$$A \otimes_B P_\bullet \to A \otimes_B M$$

is a projective resolution of $A \otimes_B M$ over A.

Now, the functors $\mathrm{Hom}^j_A(A \otimes_B P_i, \_)$ and $\mathrm{Hom}^j_B(P_i, \_)$ are naturally isomorphic, and hence $H^i(\mathrm{Hom}^j_A(A \otimes_B P_\bullet, N)) \simeq H^i(\mathrm{Hom}^j_B(M,N))$. □

2.3. **Cohomology of Lie algebras.** Let $\mathscr{L}$ be a graded Lie algebra. If M is an $\mathscr{L}$-module (i.e., a graded $\mathscr{U}(\mathscr{L})$-module) one can define the cohomology of $\mathscr{L}$ with coefficients in M as

$$H^\bullet(\mathscr{L}, M) := H^\bullet(\mathscr{U}(\mathscr{L}), M) = \mathrm{Ext}^\bullet_{\mathscr{U}(\mathscr{L})}(k, M).$$

It follows that $H^\bullet(\mathscr{L}, M)$ is bigraded.

We start by showing an easy example which is a complete analogue to Stallings-Swan theorem on groups of cohomological dimension 1 (see [51]).

Proposition 2.10. *Let $\mathscr{L}$ be a graded Lie algebra. Then $\mathscr{L}$ is free if, and only if, $H^2(\mathscr{L}, k) = 0$. In particular, subalgebras of free Lie algebras are free, too.*

Proof. If $\mathscr{L}$ is free, then its universal enveloping algebra is a tensor algebra $T(V)$. We can write down a free resolution of the trivial $T(V)$-module k

$$0 \to T(V) \otimes V \to T(V) \to k \to 0,$$

whence we get $H^2(\mathscr{L}, k) = H^2(T(V), k) = 0$.

Conversely, from Proposition 2.6, we get that the universal enveloping algebra $\mathscr{U}(\mathscr{L})$ is presented by the sequence

$$T(V) \otimes H^2(\mathscr{L}) \otimes T(V) \to T(V) \to \mathscr{U}(\mathscr{L}) \to 0,$$

which yields an isomorphism $\mathscr{U}(\mathscr{L}) \simeq T(V)$.

Now, if $\mathscr{M}$ is a subalgebra of a free Lie algebra $\mathscr{L}$, then its cohomological dimension cannot exceed that of $\mathscr{L}$, and hence it is free. □

As proved in [25], in the ungraded context, there exist Lie algebras over finite fields (of characteristic > 2) that are not free but have cohomological dimension 1. Also, notice that the fact that subalgebras of free Lie algebras are free also holds in the ungraded context, as it was proven in the 1950's by Shirshov [46] and Witt [60].

In order to compute the Lie algebra cohomology one can use the Chevalley-Eilenberg complex $\mathrm{Hom}_k(\Lambda_\bullet(\mathscr{L}), M)$ (cf. [56]), namely,

$$H^\bullet(\mathscr{L}, M) = H^\bullet(\mathrm{Hom}_k(\Lambda_\bullet(\mathscr{L}), M)).$$

With respect to such a description, the cup product on $H^\bullet(\mathscr{L}, k)$ is induced by the natural map $\wedge : \Lambda^n(\mathscr{L}) \otimes \Lambda^m(\mathscr{L}) \to \Lambda^{n+m}(\mathscr{L})$. It is then clear that the cup product makes $H^\bullet(\mathscr{L}, k)$ into a graded-commutative algebra, i.e., $ab = (-1)^{nm} ba$ for $a \in H^n(\mathscr{L}, k)$ and $b \in H^m(\mathscr{L}, k)$.

This description of cohomology allows one to treat elements of $H^i(\mathscr{L}, M)$ as equivalence classes of multilinear alternating maps $\underbrace{\mathscr{L} \times \cdots \times \mathscr{L}}_{i \text{ times}} \to M$.

Also, one can see that the second cohomology groups describe all equivalence classes of abelian extensions. We state such a result for the graded case, as the ordinary case is well-known.

Proposition 2.11. *Let $\mathscr{L}$ be a graded Lie algebra, and M a graded $\mathscr{L}$-module. Then, $H^{2,n}(\mathscr{L}, M)$ classifies all the equivalence classes of extensions of graded algebras*

$$0 \to M \to \mathscr{E} \to \mathscr{L} \to 0$$

where M embeds into $\mathscr{E}$ via a graded map of degree $-n$.

In particular, if $[c] \in H^{2,n}(\mathscr{L}, M)$, then $\mathscr{E}$ may be presented as a vector space as $\mathscr{E} = M \times \mathscr{L}$, with product given by

$$[(m,x),(m',x')] = (xm' - x'm + c(x,x'), [x,x']),$$

and $\mathscr{E}$ has the following grading:

$$\mathscr{E}_i = M_{i+n} \times \mathscr{L}_i.$$

We now want to explicitly describe the cup product of pairs of elements of degree 1 in the cohomology $H^\bullet(\mathscr{L})$. At the end, we will recover an analogous result on the low-degree cup-product of the Galois cohomology of pro-p groups [29, Prop. 3.9.13].

Recall that, if $\mathscr{L}$ is a Lie algebra and M and N are $\mathscr{L}$-modules, then there is a natural induced $\mathscr{L}$-module structure on $\mathrm{Hom}(N, M)$. Indeed, for a linear map $c : N \to M$ and an element $x \in \mathscr{L}$, define the linear map $x \cdot c : N \to M$ by

$$(x \cdot c)(y) := x \cdot c(y) - c(x \cdot y), \quad y \in \mathscr{L}.$$

Proposition 2.12. *Let $\mathscr{G}$ be a Lie algebra with an ideal $\mathscr{L}$, and let M be a $\mathscr{G}$-module. Then the natural action of $\mathscr{G}$ on $\mathrm{Hom}(\mathscr{U}(\mathscr{L}), M)$ induces an action on $H^\bullet(\mathscr{L}, M)$ that is trivial when restricted to $\mathscr{L}$.*

We are only interested in the action of $\mathscr{G}$ on $H^1(\mathscr{L}, M)$, and we will give a proof for this case. The general case follows similarly, though with more elaborated computations.

Proof of Proposition 2.12. Let $c : \mathscr{L} \to M$ be a linear map and assume that $\partial c = 0$, i.e., $\forall u, v \in \mathscr{L}$,

$$c([u,v]) = uc(v) - vc(u).$$

We now show that $\partial(x \cdot c) = 0$ for any $x \in \mathscr{G}$. Indeed, for $y, z \in \mathscr{L}$,

$$\begin{aligned}\partial(x \cdot c)(y,z) &= y(x \cdot c)(z) - z(x \cdot c)(y) - (x \cdot c)([y,z]) = \\ &= yxc(z) - yc([x,z]) - zxc(y) + zc([x,y]) - xc([y,z]) + c([x,[y,z]])\end{aligned}$$

Now, recall that for $a \in M$, $u, v \in \mathscr{L}$ one has

$$[u,v]a = uva - vua.$$

It follows that

$$\begin{aligned}&\partial(x \cdot c)(y,z) = \\ &= ([y,x]c(z) + xyc(z)) - yc([x,z]) - ([z,x]c(y) + xzc(y)) + \\ &\quad + zc([x,y]) - xc([y,z]) + c([x,[y,z]]).\end{aligned}$$

Since $\partial c = 0$, one has

$$0 = x\partial c(y,z) = x(yc(z) - zc(y) - c([y,z])),$$

and, by the Jacobi identity, one has

$$c([x,[y,z]]) = -c([[y,x],z]) - c([[x,z],y]).$$

All together, $\partial(x \cdot c)(y,z)$ is the sum of the following two elements

$$[y,x]c(z) - zc([y,x]) - c([[y,x],z]) = \partial c([y,x],z) = 0$$
$$[x,z]c(y) - yc([x,z]) - c([[x,z],y]) = \partial c([x,z],y) = 0$$

and hence it vanishes.

Now, if $x \in \mathscr{L}$, we prove that $x \cdot c$ is a 1-coboundary, i.e., there is some element $a \in M$ such that, $\forall y \in \mathscr{L}$

$$x \cdot c(y) = \partial a(y) = ya.$$

Since $\partial c = 0$, we have $c([x,y]) = xc(y) - yc(x)$, and hence

$$x \cdot c(y) = xc(y) - c([x,y]) = yc(x).$$

By setting $a = c(x)$, we have the desired identity. □

As for group cohomology, one can define the transgression map defined on the fixed points of the first cohomology group of an ideal to the second cohomology group of the quotient Lie algebra.

Lemma 2.13. *Let $\mathscr{G}$ be a k-Lie algebra and let $\mathscr{L}$ be an ideal of $\mathscr{G}$. Let $[f] \in H^1(\mathscr{L})^{\mathscr{G}/\mathscr{L}}$ be the class of an invariant 1-cocycle of $\mathscr{L}$. If $s : \mathscr{G}/\mathscr{L} \to \mathscr{G}$ is a linear section of the quotient map $\mathscr{G} \to \mathscr{G}/\mathscr{L}$, define $h : \mathscr{G} \to \mathscr{L}$ via*

$$h(x) = x - s(x + \mathscr{L}), \quad x \in \mathscr{G}.$$

Then, the transgression $\operatorname{tg}[f] \in H^2(\mathscr{G}/\mathscr{L})$ is represented by the element t defined on the cartesian square of $\mathscr{G}/\mathscr{L}$ by

$$t(\bar{x},\bar{y}) = -f([h(\bar{x}), h(\bar{y})])$$

Proof. Since f represents an element of $H^1(\mathscr{L})^{\mathscr{G}/\mathscr{L}}$, for every $x \in \mathscr{G}$, there exists a scalar g_x such that $(x + \mathscr{L}) \cdot f = \partial g_x$. However, $\partial g_x = 0$. On the other hand, for $y \in \mathscr{L}$,

$$0 = (x + \mathscr{L}) \cdot f(y) = xf(y) - f([x,y]),$$

proving that $f : \mathscr{L} \to k$ is a homomorphism of Lie algebras satisfying $f|_{[\mathscr{G},\mathscr{L}]} = 0$.

Now, define the map $F : \mathscr{G} \to k$ by extending f via $h(\cdot)$, i.e.,

$$F(x) = f(h(x)), \quad x \in \mathscr{G}.$$

By taking the differential of F, one gets

$$\partial F(x,y) = -F([x,y]).$$

If x or y belong to $\mathscr{L}$, then $[x,y] \in \mathscr{L}$, i.e., $h([x,y]) = [x,y] \in [\mathscr{G},\mathscr{L}]$, and hence

$$\partial F(x,y) = -F([x,y]) = -f([x,y]) = 0.$$

In particular, ∂F defines a 2-cocycle of $\mathscr{G}/\mathscr{L}$, that we denote by t. □

Corollary 2.14. *If $\mathscr{G}$ is free and $\mathscr{H} \leq \mathscr{G}' := [\mathscr{G},\mathscr{G}]$, then* $\operatorname{tg} : H^1(\mathscr{H})^{\mathscr{G}/\mathscr{H}} \to H^2(\mathscr{G}/\mathscr{H})$ *is an isomorphism.*

Proof. Let $[t] \in H^2(\mathscr{G}/\mathscr{H})$. Then $t : \mathscr{G}/\mathscr{H} \times \mathscr{G}/\mathscr{H} \to k$ is a linear map satisfying $\partial t = 0$. We can lift t to a map $T : \mathscr{G} \times \mathscr{G} \to k$ by composing t with the pair of projections $\mathscr{G} \times \mathscr{G} \to \mathscr{G}/\mathscr{H} \times \mathscr{G}/\mathscr{H}$.

Now, if $x, y, z \in \mathscr{G}$, one has

$$\partial T(x,y,z) = -T([x,y],z) + T([x,z],y) - T([y,z],x) = \partial t(\bar{x},\bar{y},\bar{z}) = 0,$$

where we denote by $\bar{x},\bar{y},\bar{z}$ the projections of $x,y,z \in \mathscr{G}$, respectively, to $\mathscr{G}/\mathscr{H}$.

This means that T is a 2-cocycle of $\mathscr{G}$. Since $\mathscr{G}$ is free, 2-cocycles have primitive integrals, i.e., there exists a linear map $F : \mathscr{G} \to k$ such that $\partial F = T$. Denote by $f : \mathscr{H} \to k$ the restriction of F to $\mathscr{H}$.

If x or y belong to $\mathscr{H}$, then $[x,y] \in \mathscr{H}$ and

$$-f([x,y]) = \partial f(x,y) = T(x,y) = t(\bar{x},\bar{y}) = 0,$$

proving that f is a 1-cocycle of $\mathscr{H}$ that vanishes on $[\mathscr{G},\mathscr{H}]$.

If $F' : \mathscr{G} \to k$ is a linear map satisfying $\partial F' = T = \partial F$, then $\partial(F - F') = 0$, i.e., $u = F - F'$ is a 1-cocycle of $\mathscr{G}$, and hence $u : \mathscr{G} \to k$ is a Lie homomorphism. In particular, $u|_{\mathscr{H}} = 0$, for $\mathscr{H} \subseteq \mathscr{G}'$.

It follows that F and F' have the same restriction to $\mathscr{H}$. Similarly, if $t = \partial p$, $p : \mathscr{G}/\mathscr{H} \to k$, then the above procedure gives $f = 0$, proving that $t \mapsto f$ defines a linear map $\beta : H^2(\mathscr{G}/\mathscr{H}) \to H^1(\mathscr{H})^{\mathscr{G}/\mathscr{H}}$.

Of course, $\operatorname{tg}[f] = [t]$ and also the composition $H^1(\mathscr{H})^{\mathscr{G}/\mathscr{H}} \xrightarrow{\operatorname{tg}} H^2(\mathscr{G}/\mathscr{H}) \xrightarrow{\beta} H^1(\mathscr{H})^{\mathscr{G}/\mathscr{H}}$ is the identity map. □

The latter also follows from the 5-term cohomology sequence induced by the Lyndon-Hochschild-Serre spectral sequence associated to the extension $\mathscr{H} \hookrightarrow \mathscr{G} \twoheadrightarrow \mathscr{G}/\mathscr{H}$.

If $0 \to R \to \mathscr{F} \to \mathscr{L} \to 0$ is an exact sequence of Lie algebras, for $\rho \in R$, define the **trace map**

$$\operatorname{tr}_\rho : H^2(\mathscr{L}) \to k$$

by $\operatorname{tr}_\rho(\phi) = (\operatorname{tg}^{-1}\phi)(\rho)$

Proposition 2.15. *Let $0 \to R \to \mathscr{F} \to \mathscr{L} \to 0$ be a minimal free presentation of the (possibly graded) finitely generated Lie algebra $\mathscr{L}$, i.e., $\mathscr{F}$ is free and $R \subseteq \mathscr{F}'$. Let $x_1, \ldots, x_n$ be a basis of the free Lie algebra $\mathscr{F}$ and let $\chi_1, \ldots, \chi_n \in$*

$H^1(\mathscr{F}) = H^1(\mathscr{L}) = \mathrm{Hom}(\mathscr{L}, k)$ be the dual basis. If $\mathcal{R} = \{\rho_i \mid i \in I\}$ is a minimal system of relators of $\mathscr{L}$, and

$$\rho_i = \sum_{k,l} \alpha_i^{kl}[x_k, x_l] + \rho_i', \quad \rho_i' \in \mathscr{F}''$$

then the bilinear form

$$H^1(\mathscr{L}) \times H^1(\mathscr{L}) \overset{\smile}{\to} H^2(\mathscr{L}) \overset{\mathrm{tr}_{\rho_i}}{\to} k$$

is given by the matrix $B_i = (b_i^{kl})$, with respect to the chosen basis $\{\chi_1, \dots, \chi_n\}$, where

$$b_i^{kl} = \mathrm{tr}_{\rho_i}(\chi_k \smile \chi_l) = \begin{cases} -a_i^{kl}, & k < l \\ a_i^{lk}, & \textit{otherwise} \end{cases}$$

More explicitly, one has

$$\chi_k \smile \chi_l = \sum b_i^{kl} \rho_i^*$$

where we identify $H^2(\mathscr{L})$ with the dual space of $\mathrm{Span}(\rho_i : i \in I)$*.*

Proof. The proof is essentially identical to that of the pro-p group case (see [29, Prop. 3.9.13]). Let $c_0 : \mathscr{L} \times \mathscr{L} \to k$ be the cocycle representing the cohomology class $\chi_k \smile \chi_l \in H^2(\mathscr{L})$, i.e.,

$$c_0(\sigma, \tau) = \chi_k(\sigma)\chi_l(\tau), \quad \sigma, \tau \in \mathscr{L}$$

Let $c = \inf_{\mathscr{F}}^{\mathscr{L}} c_0$. Since $\inf_{\mathscr{F}}^{\mathscr{L}} : H^2(\mathscr{L}) \to H^2(\mathscr{F}) = 0$, there exists a cochain $u = u^{kl} : \mathscr{F} \to k$ satisfying $c = \partial u$, i.e.,

$$c(x, y) = -u([x, y]), \quad x, y \in \mathscr{F}.$$

The Lie homomorphism $h : \mathscr{F} \to k$ defined by $h(x_j) = u(x_j)$ is of course a cocycle, and hence $\partial(u - h) = \partial u = c$ and $(u - h)(x_j) = 0$. We may thus assume $u(x_j) = 0$.

Since $u([x, y]) = -\chi_k(x)\chi_l(y)$, one has $u([x, y]) = 0$ whenever x or y belong to $\mathscr{F}'$. In particular, $u|_{\mathscr{F}'} : \mathscr{F}' \to k$ is a Lie homomorphism that vanishes on $\mathscr{F}''$.

Set $v := u|_R : R \to k$. Then, $[v] \in \mathrm{Hom}(R, k) = H^1(R)$. Now, $[v] \in H^1(R)^{\mathscr{L}}$, as $\forall x \in \mathscr{L}$, $\forall y \in R$,

$$(x \cdot v)(y) = v([x, y]) = 0.$$

By definition of the transgression map,

$$\mathrm{tg}(v) = [\partial u] = \chi_k \smile \chi_l,$$

and hence

$$b_i^{kl} := \mathrm{tr}_{\rho_i}(\chi_k \smile \chi_k) = \mathrm{tr}_{\rho_i}(\mathrm{tg}(v)) = v(\rho_i) = u^{kl}(\rho_i).$$

We compute $u^{kl}([x_\nu, x_\mu])$ for $k < l$. We have

$$u^{kl}([x_\nu, x_\mu]) = - \chi_k(x_\nu)\chi_l(x_\mu) = \\ = \delta_{k\nu}\delta_{l\mu} = \begin{cases} -1, & \text{if } k = \nu \text{ and } l = \mu \\ 0, & \text{otherwise} \end{cases}$$

Hence,

$$\begin{aligned} u^{kl}(\rho_i) =& u^{kl}\left(\sum \alpha_i^{\nu\mu}([x_\nu, x_\mu]) + \rho_i'\right) = \\ =& \sum \alpha_i^{\nu\mu} u^{kl}([x_\nu, x_\mu]) + 0 = \qquad (\rho_i' \in \mathscr{F}'' \subseteq \ker u^{kl}|_{\mathscr{F}'}) \\ =& - \alpha_i^{kl}. \end{aligned}$$

□

3. Quadratic algebras

Along most of this thesis, we are interested in quadratic (Lie) algebras. The motivating example comes from Galois Theory. Let G be the **maximal pro-p Galois group** of a field $\mathbb{F}$ containing a primitive pth root of 1, i.e., the maximal pro-p quotient of the absolute Galois group of $\mathbb{F}$, or, equivalently, the absolute Galois group of the maximal p-extension of $\mathbb{F}$. Then, by the celebrated affirmative answer to the Bloch-Kato conjecture, mainly due to Rost, Voevodski and Weibel, it turns out that the $\mathbb{F}_p$-cohomology ring of G is quadratic. Moreover, since closed subgroups of G are again maximal pro-p Galois groups of fields containing primitive pth roots of 1, G is hereditarily of quadratic cohomology.

Let A be a graded connected algebra. Recall that A is **1-generated**, or standard, if it is generated by A_1, i.e., the smallest connected subalgebra of A containing A_1 is the whole algebra A. This amounts to saying that the first cohomology group $H^1(A, k)$ is concentrated in degree 1 (see Corollary 2.7), i.e., $H^{1,j}(A, k) = 0$ for $j \neq 1$. Equivalently, one can present A as a quotient of the tensor algebra $T(A_1)$ by a homogeneous ideal I.

Let $\pi : T(A_1) \to A$ be the natural epimorphism that presents A as a 1-generated algebra, i.e., π is induced by the inclusion $A_1 \to A$. If $\ker \pi$ is generated by elements of degree 2, we call A a **quadratic algebra** and denote it by $Q(A_1, R)$, where $R = \ker \pi_2$ and π_2 is the restriction of π to $A_1 \otimes A_1$. This amounts to saying that the second cohomology group $H^2(A, k)$ is concentrated in degree 2.

To every graded algebra A, one can attach the quadratic algebra $\mathrm{q}A = Q(A_1, \ker \pi_2)$. It is straightforward to see that the quotient map $p : \mathrm{q}A \to A$ is surjective if, and only if, A is 1-generated. In this case, p is an isomorphism in degree 0, 1 and 2 and it is called the **quadratic cover** of A. In general, p is an isomorphism if, and only if, A is quadratic.

It is worth noticing that if B is a 1-generated subalgebra of a graded algebra A with $\mathrm{q}A = Q(V,R)$, then $\mathrm{q}B = Q(B_1, R')$, where $R' = R \cap (B_1 \otimes B_1)$.

Let $\mathfrak{alg}_k$ be the category of graded k-algebras of finite type and let $\mathfrak{salg}_k$ and $\mathfrak{qalg}_k$ be the full subcategories of standard and quadratic algebras, respectively. We will drop the subscript k, if the field is well-understood.

Lemma 3.1. *$A \mapsto \mathrm{q}A$ defines a functor $\mathfrak{alg} \to \mathfrak{qalg}$.*

Moreover, it is right adjoint to the forgetful functor $\mathfrak{qalg} \to \mathfrak{salg}$.

Proof. Let A and B be two algebras of finite type. Denote by $\alpha : T(A_1) \to A$ and $\beta : T(B_1) \to B$ the natural maps.

If $\phi : A \to B$ is a homomorphism of graded algebras, then we can lift ϕ to a homomorphism $\bar{\phi} : T(A_1) \to T(B_1)$, so that $\phi\alpha = \beta\bar{\phi}$.

Since $\beta\bar{\phi}(\ker\alpha) = \phi\alpha(\ker\alpha) = 0$, one has $\bar{\phi}(\ker\alpha_2) \leq \ker\beta_2$, whence we recover a well-defined morphism $\psi = \mathrm{q}\phi : \mathrm{q}A = T(A_1)/(\ker\alpha_2) \to T(B_1)/(\ker\beta_2) = \mathrm{q}B$. This implies that q is a functor.

For 1-generated algebras A and B, denote by $\pi_A : \mathrm{q}A \to A$ and $\pi_B : \mathrm{q}B \to B$ the quadratic covers.

Let A be quadratic. Then, $\mathrm{Hom}_{\mathfrak{qalg}}(A, \mathrm{q}B) \to \mathrm{Hom}_{\mathfrak{salg}}(A, B) : \ \phi \mapsto \pi_B\phi$ is an isomorphism. Indeed, if $\pi_B\phi = 0$, then ϕ_1 is zero because π_B is an isomorphism in degree 1, whence $\phi = 0$. Conversely, if $\psi : A \to B$ is a homomorphism, then we have an induced homomorphism $\phi = \mathrm{q}\psi : A = \mathrm{q}A \to \mathrm{q}B$ such that $\pi_B\phi = \psi$. □

Quadratic algebras play a deep role in the theory of cohomology of graded algebras: we will see that the cohomology ring always contains a canonical quadratic algebra which is related with the quadratic part of the algebra. In order to state this key result we need to define a natural construction involving quadratic algebras of finite type.

Definition 3.2. *Let A be a finitely generated quadratic k-algebra with minimal presentation $T(V)/(R)$, R being a degree 2 subspace of $T(V)$. Since A is finitely generated, it is of finite type, and one can identify the dual space $(V \otimes V)^*$ with $V^* \otimes V^*$. Let $R^\perp$ be the orthogonal complement of R in $V^* \otimes V^*$, i.e., the subspace fitting in the exact sequence*

$$0 \to R^\perp \to (V \otimes V)^* \to R^* \to 0.$$

*The **Koszul dual** (or quadratic dual) of A is the quadratic k-algebra*

$$A^! := T(V^*)/(R^\perp).$$

Notice that, since the dimension of $A_1 = V$ is finite, the Koszul dual construction is an involution on the category $\mathfrak{qalg}$.

Now, let A be a standard algebra with minimal presentation $\langle V \,|\, W \rangle$, i.e., $A = T(V)/(W)$, $V = V_1$ and $W \leq T(V)_+$ is a homogeneous subspace. By

Proposition 2.6, there are graded isomorphisms $H^0(A,k) = k = H^{0,0}(A,k)$, $H^1(A,k) = V^* = H^{1,1}(A,k)$ and $H^{2,j}(A,k) \leq W_j^*$. In particular, if we look at the diagonal part of $H^\bullet(A,k)$ in degrees 0, 1 and 2, we see that $H^{1,1}(A,k) = V^*$ and $H^{2,2}(A,k) = W_2^*$, i.e., there are isomorphisms of vector spaces $H^{i,i}(A,k) = (\mathrm{q}A)^!_i$ for $i \leq 2$. The same phenomenon extends to higher degrees, as it was proved by Löfwall [23].

Theorem 3.3 (Löfwall)**.** *Let A be a graded algebra. Then, the diagonal part*

$$dH^\bullet(A,k) := \bigoplus_{i\geq 0} \mathrm{Ext}_A^{i,i}(k,k)$$

of the cohomology of A is a quadratic algebra and there is an isomorphism of quadratic algebras

$$dH^\bullet(A,k) = (\mathrm{q}A)^!.$$

It follows that, for a standard algebra A, the cohomology $H^\bullet(A,k)$ is 1-generated if, and only if, it is quadratic. This phenomenon led S. Priddy [35] to define a special class of quadratic algebras, namely, Koszul algebras.

Definition 3.4. *A* 1*-generated algebra is said to be* ***Koszul*** *if its cohomology ring is* 1*-generated.*

In particular, Koszul algebras are quadratic, for their second cohomology groups being concentrated in degree 2. One can thus see Koszul algebras as quadratic algebras which are more "homogeneous" in cohomology. If $H^i(A)$ is concentrated in degree i for every $i \leq n$, then one says that A is an n**-Koszul algebra**.

We are interested in such algebras because of the following conjecture due to T.S. Weigel [59] and L. Positselski [34].

Conjecture 3.5. *Let G be the maximal pro-p Galois group of a field $\mathbb{F}$ containing a primitive pth root of* 1*. Suppose that G is finitely generated, i.e., $H^1(G,\mathbb{F}_p)$ is finite. Then,*

1. *$H^\bullet(G,\mathbb{F}_p)$ is Koszul;*
2. *The graded p-Lie algebra $\mathrm{gr}\, G$ associated with the Zassenhaus filtration of G is a Koszul Lie algebra (see Sections 3.2 and 5);*
3. *$H^\bullet(G,\mathbb{F}_p)$ and $\mathrm{gr}\,\mathbb{F}_pG = \underline{u}(\mathrm{gr}\, G)$ are Koszul dual to each other. (See Section 5)*

Such a conjecture has been proven to hold for several classes of fields, like those satisfying the elementary type conjecture (see [27], [28]).

By Lemma 2.3, we see that having diagonal cohomology, i.e., $H^{\bullet,\bullet}(A,k) = dH^\bullet(A,k)$, is equivalent to the existence of a minimal free resolution $P_\bullet$ of the trivial A-module that is **linear**, i.e., P_i is a free graded A-module that is generated by elements of degree i.

In particular, we get several equivalent conditions for a graded algebra being Koszul.

Proposition 3.6 ([31]). *Let A be a finitely generated graded k-algebra. Then, the following are equivalent.*

(1) *A is Koszul;*
(2) *The cohomology ring $H^{\bullet,\bullet}(A,k)$ is diagonal, i.e., $H^{i,j}(A,k)=0$ for $i\neq j$.*
(3) *The cohomology ring $H^{\bullet}(A,k)$ is generated by $H^{1,1}(A,k)$;*
(4) *A and $H^{\bullet}(A,k)$ are both 1-generated algebras;*
(5) *A is 1-generated and the cohomology ring $H^{\bullet}(A,k)$ is a quadratic k-algebra;*
(6) *The trivial A-module k admits a free linear A-resolution.*

Remark 3.7. *We notice that a standard algebra A of cohomological dimension 2, i.e., $H^3(A,k)=0$, is Koszul precisely when it is quadratic. Indeed, by Proposition 2.6, the only possible non-diagonal, non-trivial component of $H^{\bullet,\bullet}(A,k)$ appears in the second cohomology group.*

Definition 3.8. *Given a graded A-module M, where A is quadratic, one calls M an **(n-generated) Koszul module** if $\mathrm{Ext}_A^{ij}(M,k)=0$ for $i+n\neq j$, which amounts to saying that M admits a free linear A-resolution.*

For more details on Koszul modules, see [31].

Koszul algebras admit a canonical resolution for their trivial module, the so-called Koszul (or Priddy) complex.

3.1. **Koszul complex.** If A is a standard algebra minimally generated by elements $x_1,\dots,x_d\in A_1$, one defines the **Koszul complex**

$$\cdots\to K_2\to K_1\to K_0\to k$$

where $K_i=A\otimes(\mathrm{q}A)_i^{!*}\simeq \mathrm{Hom}_k(\mathrm{q}A_i^!,A)$ and the differential $d_i:K_i\to K_{i-1}$ is defined by

$$d_if(a)=\sum_j f(x_ja)x_j,\quad a\in \mathrm{q}A_{i-1}^!.$$

It can be proved that the algebra A is Koszul precisely when its Koszul complex is acyclic (see [31]). By symmetry of the Koszul complex, it follows that a quadratic algebra A is Koszul if, and only if, so is $A^!$.

Moreover, one has the following numerical formula, that was first noticed by Fröberg [13] in full generality, easily provable by using the Koszul complex.

Lemma 3.9 (Fröberg's formula). *Let A be a Koszul algebra and let $H_A(t)$ be the Hilbert polynomial of A, i.e.,*

$$H_A(t)=\sum_{i\geq 0}\dim A_it^i.$$

Then,

$$H_A(t)H_{A^!}(-t)=1.$$

3.2. **Quadratic and Koszul Lie algebras.** If $\mathscr{L}$ is a graded Lie algebra, then we say that $\mathscr{L}$ is standard (quadratic) if $\mathscr{U}(\mathscr{L})$ is standard (resp. quadratic). This is equivalent to requiring that the natural morphism $\mathscr{F}(\mathscr{L}_1) \to \mathscr{L}$ is surjective (resp. it is surjective and its kernel is generated as an ideal by elements of degree 2). Again, one can interpret such properties in cohomological terms: $\mathscr{L}$ is 1-generated iff $H^{1,j}(\mathscr{L}, k) = 0$ for $j \neq 1$, and it is quadratic if $H^{ij}(\mathscr{L}, k) = 0$ for $i \neq j$, $i \leq 2$. For Lie algebras there are functors as in Lemma 3.1.

We also say that a Lie algebra $\mathscr{L}$ is Koszul if its universal envelope $\mathscr{U}(\mathscr{L})$ is. In [31], it is proven that Koszul Lie algebras are either abelian or have exponential growth, i.e., $\dim \mathscr{L}_i$ grows exponentially.

Quadratic Lie algebras have special properties coming either from the fact that their cohomology rings are graded-commutative or from the simple shape their defining relations can have.

Lemma 3.10. (1) *If A is a quadratic graded-commutative algebra, then $A^!$ is the universal envelope of a quadratic Lie algebra.*
(2) *Conversely, if $\mathscr{L}$ is a quadratic Lie algebra, then $\mathscr{U}(\mathscr{L})^!$ is a (quadratic) graded-commutative algebra.*

Proof. (1) Let $A = \Lambda_\bullet(V^*)/(\Omega)$, where $\Omega \leq \Lambda_2(V^*)$. Choose a basis (ω_i) for Ω, and elements $t_i \in V \otimes V$ such that $\omega_i(t_j) = \delta_{ij}$.

Define the Lie algebra given by the following presentation:

$$\mathscr{L} = \left\langle V \,\middle|\, [v, v'] - \sum_i \omega_i(v, v') c(t_i) : \; v, v' \in V \right\rangle,$$

where $c(x \otimes y) = x \otimes y - y \otimes x$. We want to show that $\mathscr{U}(\mathscr{L})$ is isomorphic to $A^!$.

To this end, we present A as a quadratic algebra $A = Q(V^*, R)$, where $R = \tilde{\Omega} \oplus \mathrm{Span}_k(\alpha \otimes \alpha : \alpha \in V) \leq V^* \otimes V^*$, and $\tilde{\Omega}$ is a lifting of Ω to a subspace of $V^* \otimes V^*$ with respect to the canonical projection $V^* \otimes V^* \to V^* \wedge V^*$.

It is easy to see that $\ker(T_\bullet(V) \to \mathscr{U}(\mathscr{L})) \subseteq \ker(T_\bullet(V) \to A^!)$, as

$$\begin{aligned} \omega_j \left((v \otimes v' - v' \otimes v) - \sum_i \omega_i(v, v') c(t_i) \right) &= \\ = 2w_j(v, v') - \sum_i \omega_i(v, v') \omega_j(c(t_i)) &= \\ = 2\omega_j(v, v') + \sum_i \omega_i(v, v') 2\delta_{ij} &= 0 \end{aligned}$$

Also the opposite inclusion holds true.

(2) Since $\mathscr{U}(\mathscr{L})$ is quadratic, by Theorem 3.3, its dual is isomorphic to the diagonal subalgebra $dH^\bullet(\mathscr{U}(\mathscr{L}), k)$ of the cohomology algebra $H^\bullet(\mathscr{L}, k)$, which is well-known to be graded-commutative. □

3.3. **HNN-extensions and free products.** Let $\mathscr{M}$ be a standard Lie algebra and let $\mathscr{A}$ be a standard subalgebra of $\mathscr{M}$. Fix a degree d derivation $\phi : \mathscr{A} \to \mathscr{M}$, i.e., ϕ is a linear map satisfying $\phi([x,y]) = [\phi(x), y] + [x, \phi(y)]$, for $x, y \in \mathscr{A}$, and $\phi(\mathscr{A}_i) \subseteq \mathscr{M}_{i+d}$ $(i \geq 1)$. Then, one defines the **(differential) HNN-extension**

$$\mathrm{HNN}_\phi(\mathscr{M}, t) := \left\langle \mathscr{M}, t \,\middle|\, \text{relations of } \mathscr{M}, [t,a] = \phi(a) : \ a \in \mathscr{A} \right\rangle,$$

where the generator t has degree d and is called the *stable letter*. By [22] and [55], $\mathscr{M}$ naturally embeds into the HNN-extension.

Quadratic Lie algebras satisfy the following fundamental Lemma.

Lemma 3.11. *Let $\mathscr{L}$ be a quadratic Lie algebra. Then, for every maximal standard subalgebra $\mathscr{M}$, there is a decomposition of $\mathscr{L}$ as the HNN-extension of $\mathscr{M}$ with respect to a derivation $\phi : \mathscr{A} \to \mathscr{M}$ of degree 1 on a standard subalgebra $\mathscr{A}$ of $\mathscr{M}$. Explicitly, if x is belongs to $\mathscr{L}_1$ but not to $\mathscr{M}$, then $\mathscr{A}$ can be chosen to be generated by the elements $m \in \mathscr{M}_1$ such that $[x,m] \in \mathscr{M}$.*

Proof. Pick $x \in \mathscr{L}_1 \setminus \mathscr{M}$. Fix a free Lie algebra $\mathscr{F}$ on the space $\mathscr{L}_1$ and identify its degree 1 component with $\mathscr{L}_1$. Let $\mathscr{G}$ be the subalgebra of $\mathscr{F}$ generated by $\mathscr{M}_1$. Let $r_1, \ldots, r_m$ be minimal (quadratic) relations of $\mathscr{L}$, i.e., $\mathscr{L} = \mathscr{F}/(r_1, \ldots, r_m)$. Since $\mathscr{F}_2 = [\mathscr{L}_1, \mathscr{L}_1] = [\mathscr{G}_1, \mathscr{G}_1] \oplus [\mathscr{G}_1, x]$, we may assume that there is some $1 \leq s \leq m$ such that r_i belongs to $\mathscr{G}$ precisely when $1 \leq i \leq s$. Now, for every $s < i \leq m$, there are elements $a_i \in \mathscr{M}_1$ and $m_i \in \mathscr{G}_2$ such that $r_i = [x, a_i] + m_i$.

Let $\mathscr{A}$ be the subalgebra of $\mathscr{M}$ generated by the images of the elements a_i. Since $\mathscr{A}$ is a subalgebra of $\mathscr{L}$, the adjoint map $\mathrm{ad}(x) : \mathscr{L} \to \mathscr{L} : y \mapsto [x,y]$ defines a derivation $\phi = \mathrm{ad}(x)|_{\mathscr{A}} : \mathscr{A} \to \mathscr{M}$. Notice that $\phi(a_i) = -m_i$.

It is then clear that $\mathscr{L}$ is generated by $\mathscr{M}$ and x with only additional relations given by $[x,a] = \phi(a)$, $a \in \mathscr{A}_1$. □

Notice that the quadraticity of $\mathscr{L}$ is necessary for having a partition of the relation set that tells apart those relations in which x appears in a single Lie bracket.

This construction will be useful for proving some results on Bloch-Kato Lie algebras, which are the main characters of this thesis.

Let $\mathscr{L}$ be a graded Lie algebra. If $\mathscr{H} \subseteq \mathscr{L}$ is graded-Lie subalgebra, then by the Poincaré-Birkhoff-Witt Theorem, $\mathscr{U}(\mathscr{L})$ is a $\mathbb{N}_0$-graded free right $\mathscr{U}(\mathscr{H})$-module. Let M be a $\mathbb{N}_0$-graded left $\mathscr{H}$-module, and define

$$\mathrm{ind}_{\mathscr{H}}^{\mathscr{L}}(M) := \mathscr{U}(\mathscr{L}) \otimes_{\mathscr{U}(\mathscr{H})} M.$$

The functor $\mathrm{ind}_{\mathscr{H}}^{\mathscr{L}}(\_)$ is called the **induction functor**. It is a covariant additive exact functor which is left adjoint to the restriction functor $\mathrm{res}_{\mathscr{H}}^{\mathscr{L}}(\_)$, i.e. one has natural isomorphisms of bifunctors

$$\mathrm{Hom}_{\mathscr{L}}(\mathrm{ind}_{\mathscr{H}}^{\mathscr{L}}(M), Q) \simeq \mathrm{Hom}_{\mathscr{H}}(M, \mathrm{res}_{\mathscr{H}}^{\mathscr{L}}(Q)). \tag{3.1}$$

The natural isomorphisms (3.1) are also called Nakayama relations. It follows that the identity map $k \to k$ of the trivial $\mathscr{H}$-module k induces a homomorphism of graded left $\mathscr{L}$-modules

$$\varepsilon^{\mathscr{H}} : \mathrm{ind}_{\mathscr{H}}^{\mathscr{L}}(k) \longrightarrow k. \tag{3.2}$$

In [18], the authors translated some cohomological phenomena of groups acting on trees to the context of Lie algebras. In particular, they prove the following result.

Theorem 3.12. *(1) Let $\mathscr{A}$ and $\mathscr{B}$ be two graded Lie algebras with a common graded subalgebra $\mathscr{H}$. Let $\mathscr{L}$ be the* ***free product*** *of $\mathscr{A}$ and $\mathscr{B}$* ***with amalgamated subalgebra*** *$\mathscr{H}$, i.e., $\mathscr{L}$ is the quotient of the free product $\mathscr{A} \amalg \mathscr{B}$ with respect to the ideal generated by the elements $\iota_{\mathscr{A}}(h) - \iota_{\mathscr{B}}(h)$ $(h \in \mathscr{H})$, where $\iota_{\mathscr{A}} : \mathscr{H} \to \mathscr{L}$ is the composition of the inclusion $\mathscr{H} \leq \mathscr{A}$ with the natural inclusion $\mathscr{A} \leq \mathscr{A} \amalg \mathscr{B}$ (and $\iota_{\mathscr{B}}$ is defined analogously) . Then, there is an exact sequence of $\mathscr{L}$-modules*

$$0 \to \mathrm{ind}_{\mathscr{H}}^{\mathscr{L}}(k) \to \mathrm{ind}_{\mathscr{A}}^{\mathscr{L}}(k) \oplus \mathrm{ind}_{\mathscr{B}}^{\mathscr{L}}(k) \to k \to 0.$$

(2) Let $\phi : \mathscr{A} \to \mathscr{M}$ be a degree 1 derivation of a subalgebra $\mathscr{A}$ of $\mathscr{M}$. Let $\mathscr{L} = \mathrm{HNN}_{\phi}(\mathscr{M}, t)$. Then, there is an exact sequence of $\mathscr{L}$-modules

$$0 \to \mathrm{ind}_{\mathscr{A}}^{\mathscr{L}}(k)[-1] \to \mathrm{ind}_{\mathscr{M}}^{\mathscr{L}}(k) \to k \to 0.$$

In fact, in [18] the above result is stated with much more generality, allowing the Lie algebra to be the so-called fundamental Lie algebra of a graph of Lie algebras. However, the two decoupled results fit better to the aim of the present work.

Lemma 3.13. *Let $\mathscr{A}$ and $\mathscr{B}$ be two Koszul Lie algebras with a common Koszul subalgebra $\mathscr{H}$. Then $\mathscr{A} \amalg_{\mathscr{H}} \mathscr{B}$ is Koszul.*

Proof. Set $\mathscr{L} = \mathscr{A} \amalg_{\mathscr{H}} \mathscr{B}$. By Theorem 3.12-(1), there is an exact sequence of $\mathscr{L}$-modules

$$0 \to \mathrm{ind}_{\mathscr{H}}^{\mathscr{L}}(k) \to \mathrm{ind}_{\mathscr{A}}^{\mathscr{L}}(k) \oplus \mathrm{ind}_{\mathscr{B}}^{\mathscr{L}}(k) \to k \to 0. \tag{3.3}$$

By applying the functor $\mathrm{Ext}_{\mathscr{U}(\mathscr{L})}^{\bullet,j}(\_, k)$, one recovers a long exact sequence

$$\cdots \to \mathrm{Ext}_{\mathscr{U}(\mathscr{L})}^{ij}(\mathrm{ind}_{\mathscr{H}}^{\mathscr{L}}(k), k) \to \mathrm{Ext}_{\mathscr{U}(\mathscr{L})}^{i+1,j}(k,k) \to \mathrm{Ext}_{\mathscr{U}(\mathscr{L})}^{i+1,j}(\mathrm{ind}_{\mathscr{A}}^{\mathscr{L}}(k) \oplus \mathrm{ind}_{\mathscr{B}}^{\mathscr{L}}(k), k) \to \dots$$

which reads, by the graded version of Eckmann-Shapiro Lemma (Theorem 2.9),

$$\cdots \to H^{ij}(\mathscr{H}) \to H^{i+1,j}(\mathscr{L}) \to H^{i+1,j}(\mathscr{A}) \oplus H^{i+1,j}(\mathscr{B}) \to \dots$$

If $j > i+1$, the latter shows that $H^{i+1,j}(\mathscr{L}) = 0$, proving that $\mathscr{L}$ is Koszul. □

Remark 3.14. *Notice that one may replace the Koszulity hypothesis on the amalgamated subalgebra $\mathscr{H}$ with the following weaker assumption:*

$$H^{ij}(\mathscr{H}) = 0, \quad j > i+1.$$

This means that $\mathscr{H}$ has bidiagonal cohomology (i.e. it is concentrated on the first two diagonals).

An analogous result holds for HNN-extensions.

Lemma 3.15. *Let $\mathscr{M}$ be a graded Lie algebra and let $\phi : \mathscr{A} \to \mathscr{M}$ be a derivation of degree 1 on a homogeneous Lie subalgebra $\mathscr{A}$. Consider the HNN-extension*

$$\mathscr{L} = \mathrm{HNN}_\phi(\mathscr{M}, z)$$

If $\mathscr{M}$ and $\mathscr{A}$ are Koszul, then $\mathscr{L}$ is Koszul, too.

Proof. The proof is identical to that of Proposition 3.13, except for the usage of the exact sequence given by Theorem 3.12-(2)

$$0 \to \mathrm{ind}_{\mathscr{A}}^{\mathscr{L}}(k)[-1] \to \mathrm{ind}_{\mathscr{M}}^{\mathscr{L}}(k) \to k \to 0,$$

instead of (3.3) The map $\mathrm{ind}_{\mathscr{A}}^{\mathscr{L}}(k) \to \mathrm{ind}_{\mathscr{M}}^{\mathscr{L}}(k)$ has degree 1, and hence one gets the long exact sequence

$$\cdots \to H^{i-1,j-1}(\mathscr{A}) \to H^{ij}(\mathscr{L}) \to H^{ij}(\mathscr{M}) \to H^{i,j-1}(\mathscr{A}) \to H^{i+1,j}(\mathscr{L}),$$

whence, we can deduce that the Koszulity of $\mathscr{A}$ and $\mathscr{M}$ implies that of $\mathscr{L}$. □

It is easy to see that the two above results have partial converses.

Corollary 3.16. *Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{H}$ and $\mathscr{M}$ be graded Lie subalgebras of a $\mathbb{N}$-graded Lie algebra $\mathscr{L}$.*

(1) *Assume $\mathscr{L} = \mathscr{A} \coprod_{\mathscr{H}} \mathscr{B}$.*
   (a) *If $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{H}$ are Koszul, then so is $\mathscr{L}$.*
   (b) *If $\mathscr{L}$ and $\mathscr{H}$ are Koszul, then so are $\mathscr{A}$ and $\mathscr{B}$.*
   Moreover, if this is the case, the Betti numbers satisfy the following formula:
$$b_i(\mathscr{L}) + b_i(\mathscr{H}) = b_i(\mathscr{A}) + b_i(\mathscr{B}).$$
(2) *Assume $\mathscr{L} = \mathrm{HNN}_\phi(\mathscr{M}, z)$ where $\phi : \mathscr{H} \to \mathscr{M}$ is a derivation of degree 1.*
   (a) *If $\mathscr{H}$ and $\mathscr{M}$ are Koszul, then so is $\mathscr{L}$.*
   (b) *If $\mathscr{L}$ and $\mathscr{M}$ are Koszul, then so is $\mathscr{H}$.*
   Moreover, if this is the case, then the kernel of the restriction $\mathrm{res}_{\mathscr{M}}^{\mathscr{L}} : H^\bullet(\mathscr{L}) \to H^\bullet(\mathscr{M})$ is the $H^\bullet(\mathscr{L})$-module $H^\bullet(\mathscr{H})[-1]$ and hence the Betti numbers satisfy the following formula:
$$b_{i+1}(\mathscr{L}) = b_i(\mathscr{H}) + b_{i+1}(\mathscr{M}).$$
   In particular, $\mathrm{cd}\,\mathscr{H} < \mathrm{cd}\,\mathscr{L} \leq \mathrm{cd}\,\mathscr{M} + 1$.

Proof. (1) Consider the exact sequence

$$\cdots \to H^{i-1,j}(\mathscr{H}) \to H^{ij}(\mathscr{L}) \to H^{ij}(\mathscr{A}) \oplus H^{ij}(\mathscr{B}) \to H^{ij}(\mathscr{H}) \to \ldots$$

Implication (a) has already been proved.

(b) If $j > i$, then $H^{ij}(\mathscr{L}) = H^{ij}(\mathscr{H}) = 0$, and hence $H^{ij}(\mathscr{A}) \oplus H^{ij}(\mathscr{B}) = 0$.

If all the Lie algebras are Koszul, then

$$H^{i-1,i}(\mathscr{H}) = 0 \to H^{ii}(\mathscr{L}) \to H^{ii}(\mathscr{A}) \oplus H^{ii}(\mathscr{B}) \to H^{ii}(\mathscr{H}) \to H^{i+1,i}(\mathscr{L}) = 0$$

is exact, so that

$$b_i(\mathscr{L}) - (b_i(\mathscr{A}) + b_i(\mathscr{B})) + b_i(\mathscr{H}) = 0.$$

(2) One has the following exact sequence:

$$\cdots \to H^{i,j+1}(\mathscr{M}) \to H^{ij}(\mathscr{H}) \to H^{i+1,j+1}(\mathscr{L}) \to H^{i+1,j+1}(\mathscr{M}) \to \dots$$

(a) has already been proved.

(b) If $j > i$, then $H^{i,j+1}(\mathscr{M}) = H^{i+1,j+1}(\mathscr{L}) = 0$, whence $H^{ij}(\mathscr{H}) = 0$.

If all the Lie algebras are Koszul, then

$$H^{i,i+1}(\mathscr{M}) = 0 \to H^{ii}(\mathscr{H}) \to H^{i+1,i+1}(\mathscr{L}) \to H^{i+1,i+1}(\mathscr{M}) \to H^{i,i+1}(\mathscr{H}) = 0$$

is exact, so that

$$b_i(\mathscr{H}) - b_{i+1}(\mathscr{L}) + b_{i+1}(\mathscr{M}) = 0.$$

Notice that the morphisms $H^i(\mathscr{L}) \to H^i(\mathscr{M})$ are the restriction homomorphisms, so that $\ker(\mathrm{res}^{\mathscr{L}}_{\mathscr{M}}) = H^{\bullet-1}(\mathscr{H}) = H^{\bullet}(\mathscr{H})[-1]$. □

We conclude this section by giving a characterization for the decomposability of a Lie algebra into the free product of subalgebras. Plus, we compute the cohomology of free products.

Proposition 3.17. *Let $\mathscr{L}$ be a graded standard k-Lie algebra containing homogeneous subalgebras $\mathscr{A}$ and $\mathscr{B}$ satisfying $\mathscr{L} = \langle \mathscr{A}, \mathscr{B} \rangle$. Then the following are equivalent:*

(i) $\mathscr{L} \simeq \mathscr{A} \amalg \mathscr{B}$,

(ii) $\ker\left(\varepsilon^{\mathscr{A}} - \varepsilon^{\mathscr{B}} : \mathrm{ind}^{\mathscr{L}}_{\mathscr{A}}(k) \oplus \mathrm{ind}^{\mathscr{L}}_{\mathscr{B}}(k) \to k\right) \simeq \mathscr{U}(\mathscr{L})$.

Proof. The implication (i)⇒(ii) is Theorem 3.12.

Suppose now that (ii) holds. Note that the canonical homomorphism $\phi : \mathscr{F} \to \mathscr{L}$ is surjective, where $\mathscr{F} = \mathscr{A} \amalg \mathscr{B}$. By (ii), the sequence

$$0 \to \mathscr{U}(\mathscr{L}) \to \mathrm{ind}^{\mathscr{L}}_{\mathscr{A}}(k) \oplus \mathrm{ind}^{\mathscr{L}}_{\mathscr{B}}(k) \to k \to 0 \tag{3.4}$$

is exact. The same short exact sequence holds for $\mathscr{F}$, by Bass-Serre theory for graded Lie algebras. By the Eckmann-Shapiro Lemma 2.9,

$$\mathrm{Ext}^{\bullet}_{\mathscr{U}(\mathscr{L})}(\mathscr{U}(\mathscr{L}) \otimes_{\mathscr{U}(\mathscr{A})} k, k) \simeq \mathrm{Ext}^{\bullet}_{\mathscr{U}(\mathscr{A})}(k,k),$$
$$\mathrm{Ext}^{\bullet}_{\mathscr{U}(\mathscr{L})}(\mathscr{U}(\mathscr{L}) \otimes_{\mathscr{U}(\mathscr{B})} k, k) \simeq \mathrm{Ext}^{\bullet}_{\mathscr{U}(\mathscr{B})}(k,k).$$

Thus we get a long exact sequences from Theorem 2.8 induced by (3.4)

$$\mathrm{Ext}^1_{\mathscr{U}(\mathscr{L})}(\mathscr{U}(\mathscr{L}),k) \to \mathrm{Ext}^2_{\mathscr{U}(\mathscr{A})}(k,k) \oplus \mathrm{Ext}^2_{\mathscr{U}(\mathscr{B})}(k,k) \to \\ \to \mathrm{Ext}^2_{\mathscr{U}(\mathscr{L})}(k,k) \to \mathrm{Ext}^2_{\mathscr{U}(\mathscr{L})}(\mathscr{U}(\mathscr{L}),k) \to \dots$$

But $\mathrm{Ext}^i_{\mathscr{U}(\mathscr{L})}(\mathscr{U}(\mathscr{L}),k) = 0$ for $i > 0$, and thus the inflation $\inf : H^2(\mathscr{L},k) \to H^2(\mathscr{F},k)$ is an isomorphism $H^2(\mathscr{L},k) \simeq H^2(\mathscr{A},k) \oplus H^2(\mathscr{B},k) \simeq H^2(\mathscr{U}(\mathscr{F}),k)$.

Now, consider the short exact sequence associated with the surjection $\phi : \mathscr{F} \to \mathscr{L}$,

$$0 \to I \to \mathscr{F} \xrightarrow{\phi} \mathscr{L} \to 0.$$

This yields the 5-term exact sequence in cohomology

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\mathscr{L},k) & \xrightarrow{H^1(\phi)} & H^1(\mathscr{F},k) & \xrightarrow{\alpha} & H^1(I,k)^{\mathscr{L}} \\ & & & & & & \downarrow \\ & & & & H^2(\mathscr{F},k) & \longleftarrow & H^2(\mathscr{L},k) \end{array}$$

Since $H^i(\mathscr{L},k) \to H^i(\mathscr{F},k)$, for $i = 1,2$, coincide with the isomorphisms $H^1(\phi)$ and inf, one has $H^1(I,k)^{\mathscr{L}} = 0$. As I is an $\mathbb{N}$-graded ideal of $\mathscr{F}$, one has $H^1(I,k) = \mathrm{Hom}_{\mathfrak{lie}}(I,k) = \mathrm{Hom}_k(I/[I,I],k)$. Thus, $H^1(I,k)^{\mathscr{L}} = 0$ implies $I/[\mathscr{F},I] = 0$, and therefore $I = [\mathscr{F},I]$. Now, suppose $n = \min\left\{m \geq 0 \;\middle|\; I_m \neq 0\right\}$ is finite. One has $I_n = [I,\mathscr{F}]_n = \sum_{1\leq j<n}[I_j,\mathscr{F}_{n-j}] = 0$, since I is a graded ideal, whence $I = 0$, proving that $\ker(\phi) = I = 0$. □

Lemma 3.18. *Let $\mathscr{A}$ and $\mathscr{B}$ be two graded finitely generated k-Lie algebras. Then there is an isomorphism of graded algebras*

$$H^\bullet(\mathscr{A} \amalg \mathscr{B},k) \simeq H^\bullet(\mathscr{A},k) \sqcap H^\bullet(\mathscr{B},k),$$

where $\sqcap$ denotes the product in the category of graded connected algebras.

Proof. Let $\mathscr{L} = \mathscr{A} \amalg B$. Then, by Theorem 3.12, there is a short exact sequence of $\mathscr{U}(\mathscr{L})$-modules

$$0 \to \mathscr{U}(\mathscr{L}) \to \mathrm{ind}^{\mathscr{L}}_{\mathscr{A}}(k) \oplus \mathrm{ind}^{\mathscr{L}}_{\mathscr{B}}(k) \to k \to 0.$$

The latter and the Eckmann-Shapiro Lemma induce a long exact sequence

$$\dots \to \mathrm{Ext}^i_{\mathscr{U}(\mathscr{L})}(\mathscr{U}(\mathscr{L}),k) \to \mathrm{Ext}^{i+1}_{\mathscr{U}(\mathscr{A})}(k,k) \oplus \mathrm{Ext}^{i+1}_{\mathscr{U}(\mathscr{B})}(k,k) \to \\ \to \mathrm{Ext}^{i+1}_{\mathscr{U}(\mathscr{L})}(k,k) \to \mathrm{Ext}^{i+1}_{\mathscr{U}(\mathscr{L})}(\mathscr{U}(\mathscr{L}),k) \to \dots$$

and it follows that for $i \geq 1$,

$$\mathrm{Ext}^i_{\mathscr{U}(\mathscr{L})}(k,k) \simeq \mathrm{Ext}^i_{\mathscr{U}(\mathscr{A})}(k,k) \oplus \mathrm{Ext}^i_{\mathscr{U}(\mathscr{B})}(k,k), \tag{3.5}$$

or $H^i(\mathscr{L},k) \simeq H^i(\mathscr{A},k) \oplus H^i(\mathscr{B},k)$.

Now, the natural inclusions of $\mathscr{A}$ and $\mathscr{B}$ into $\mathscr{A} \amalg \mathscr{B}$ induce the algebra homomorphisms $H^\bullet(\mathscr{A} \amalg \mathscr{B}) \to H^\bullet(\mathscr{A})$ and $H^\bullet(\mathscr{A} \amalg \mathscr{B}) \to H^\bullet(\mathscr{B})$. By the universal property of the direct product, we get an algebra homomorphism $H^\bullet(\mathscr{A} \amalg \mathscr{B}) \to H^\bullet(\mathscr{A}) \sqcap H^\bullet(\mathscr{B})$, that is an isomorphism, in the light of (3.5). □

Lemma 3.19. *Let $\mathscr{L}$ be a graded Lie algebra. Then the direct sum[1] $\mathscr{L} \sqcap k$ has cohomology*

$$H^\bullet(\mathscr{L} \sqcap k) = H^\bullet(\mathscr{L}, k) \wedge \Lambda(t^*).$$

3.4. **Cocyclic ideals of Koszul Lie algebras.** If $\mathscr{M}$ is a homogeneous cocyclic ideal of a standard Lie algebra $\mathscr{L}$, i.e., $\mathscr{L}/\mathscr{M}$ is a 1-dimensional Lie algebra, one easily sees that $\mathscr{L}$ can be decomposed into the semidirect product $\mathscr{L} = \mathscr{M} \rtimes k$, where $k \simeq \mathscr{L}/\mathscr{M}$ is the Lie subalgebra of $\mathscr{L}$ generated by any $t \in \mathscr{L}_1 \setminus \mathscr{M}$. For the purpose of this work, it is useful to notice that such a semidirect product is equivalent to the HNN-extension $\mathrm{HNN}_\phi(\mathscr{M}, t)$, where $\phi : \mathscr{M} \to \mathscr{M}$ is the adjoint map of t.

Corollary 3.20. *Let $\mathscr{L}$ be a Koszul Lie algebra and let $\mathscr{M}$ be a homogeneous cocyclic ideal of $\mathscr{L}$. If $\mathscr{M}$ is a* Lie algebra of type FP*, i.e., all cohomology groups $H^i(\mathscr{M})$ are finite dimensional spaces, then $\mathscr{M}$ is Koszul. In particular, $\mathscr{M}$ is a standard subalgebra of $\mathscr{L}$.*

Proof. As noticed above, since $\mathscr{M}$ is cocyclic, one has a decomposition of $\mathscr{L}$ into the HNN-extension $\mathrm{HNN}_\phi(\mathscr{M}, t)$ for $\phi : x \in \mathscr{M} \mapsto [t, x]$. By Theorem 3.12, we recover the exact sequences

$$H^{ij}(\mathscr{L}) \to H^{ij}(\mathscr{M}) \to H^{i,j-1}(\mathscr{M}) \to H^{i+1,j}(\mathscr{L}),$$

for every $j \geq i \geq 1$. If $j > i + 1$, as $\mathscr{L}$ is Koszul, the sequence gives rise to an isomorphism $H^{ij}(\mathscr{M}) \simeq H^{i,j-1}(\mathscr{M})$. Now, if $\mathscr{M}$ was not Koszul, then there would exist some indices $q > p \geq 1$ such that $H^{pq}(\mathscr{M}) \neq 0$ and hence, the above isomorphism would imply $H^{p,q}(\mathscr{M}) = H^{p,q+s}(\mathscr{M}) \neq 0$ for every $s \geq 0$, which contradicts the fact that $H^p(\mathscr{M})$ is finite dimensional. □

In the same way, one can prove a stronger statement involving the weaker property of n-Koszulity.

Proposition 3.21. *Let $\mathscr{L}$ be an n-Koszul Lie algebra, $n \geq 0$ and let $\mathscr{M}$ is a cocyclic ideal of $\mathscr{L}$. Then, $\mathscr{M}$ is $(n-1)$-Koszul if, and only if, $\mathscr{M}$ is of type FP_{n-1}.*

Recall that, for a Lie algebra $\mathscr{L}$ of type FP, the **Euler characteristic** of $\mathscr{L}$ is the integer

$$\chi(\mathscr{L}) := P_{\mathscr{L}}(-1) = \sum_{i=0}^{\operatorname{cd}\mathscr{L}} (-1)^i \dim_k H^i(\mathscr{L}),$$

[1]By abuse of notation, when the field k is treated as a 1-dimensional Lie algebra, we will always mean it to be generated in degree 1, i.e., as graded Lie algebras, $k := k[-1]$.

where $P_{\mathscr{L}} \in \mathbb{Z}[t]$ denotes the **Poincaré series** of $\mathscr{L}$, i.e., the Hilbert series of $H^\bullet(\mathscr{L})$. The positive integers $b_{ij}(\mathscr{L}) = \dim H^{ij}(\mathscr{L})$ are called the **(bigraded) Betti numbers** of the Lie algebra. The ith Betti number of $\mathscr{L}$ is $b_i(\mathscr{L}) = \sum_j b_{ij}(\mathscr{L})$.

Corollary 3.22. *Let $\mathscr{L}$ be a Lie algebra of type FP with a cocyclic ideal $\mathscr{M}$ of type FP. Then, the Euler characteristic of $\mathscr{L}$ is zero.*

Proof. The natural exact sequence $0 \to \mathscr{M} \to \mathscr{L} \to k \to 0$ splits, and hence it gives rise to a decomposition $\mathscr{L} = \mathrm{HNN}_\phi(\mathscr{M}, t)$, where $t \in \mathscr{L} \setminus \mathscr{M}$. By Theorem 3.12, we get a long exact sequence of finite length

$$0 \to H^0(\mathscr{M}) \to H^0(\mathscr{M}) \to H^1(\mathscr{L}) \to \cdots \to H^d(\mathscr{L}) \to 0,$$

where $d = \operatorname{cd} \mathscr{L}$. By hypothesis, all the spaces $H^i(\mathscr{L})$ and $H^i(\mathscr{M})$ are finite dimensional, and hence one has

$$0 = b_0(\mathscr{M}) - b_0(\mathscr{M}) + b_0(\mathscr{L}) - b_1(\mathscr{M}) + \cdots + (-1)^d b_d(\mathscr{L}) = \chi(\mathscr{L}).$$

□

Remark 3.23. *Assume now $\mathscr{L}$ to be Koszul and let $\mathscr{M}$ be a graded cocyclic ideal of type FP. Notice that $\mathscr{M}$ is a Koszul algebra by Corollary 3.20. By Corollary 3.16.(2), one deduces that the cohomological dimension of $\mathscr{M}$ is $\operatorname{cd} \mathscr{L} - 1$ and the Betti numbers satisfy*

$$b_{i+1}(\mathscr{L}) = b_{i+1}(\mathscr{M}) + b_i(\mathscr{M}). \tag{3.6}$$

It follows that $b_n(\mathscr{M}) = \sum_{i=0}^n (-1)^{n-i} b_i(\mathscr{L})$, for all $n \geq 0$.

Eventually, for sufficiently big d,

$$0 = b_d(\mathscr{M}) = \sum_{i=0}^{d} (-1)^{d-i} b_i(\mathscr{L}) = \pm\chi(\mathscr{L}).$$

One may also observe that Equation 3.6 implies $P_{\mathscr{L}}(t) = (1+t)P_{\mathscr{M}}(t)$, whence $\chi(\mathscr{L}) = P_{\mathscr{L}}(-1) = 0$. One can also compute the Euler characteristic of $\mathscr{M}$ in terms of the Betti numbers of $\mathscr{L}$:

$$\chi(\mathscr{M}) = \sum_{n=0}^{d} (-1)^{n+1} n b_n(\mathscr{L})$$

where $d = \operatorname{cd} \mathscr{L}$.

Indeed, since $P_{\mathscr{L}}(t) = (1+t)P_{\mathscr{M}}$, by taking formal derivatives, it follows that

$$P'_{\mathscr{L}}(t) = (1+t)P'_{\mathscr{M}}(t) + P_{\mathscr{M}}(t)$$

and hence $\chi(\mathscr{M}) = P_{\mathscr{M}}(-1) = P'_{\mathscr{L}}(-1)$.

Recall from Lemma 3.9 that, for any Koszul Lie algebra $\mathscr{L}$, Fröberg's formula holds, i.e.,

$$H_{\mathscr{U}(\mathscr{L})}(t) H_{\mathscr{L}^!}(-t) = 1,$$

where $\mathscr{L}^! := \mathscr{U}(\mathscr{L})^!$. In 1995, two works by Roos [42] and Positselski [32] proved that algebras satisfying Fröberg's formula must not be Koszul. However, the following result shows that in some specific cases, Fröberg formula implies Koszulity of the Lie algebra.

Corollary 3.24. *Let $\mathscr{L}$ be a Koszul Lie algebra with a cocyclic ideal $\mathscr{M}$. Then, $\mathscr{M}$ is Koszul if, and only if, Fröberg's formula holds for $\mathscr{M}$.*

Proof. Recall that from the PBW Theorem it follows that the graded object $\operatorname{gr}\mathscr{U}(\mathfrak{g})$ associated with the canonical filtration of a Lie algebra $\mathfrak{g}$ is the symmetric algebra on the vector space $\mathfrak{g}$. Plus, if $\mathscr{L}=\mathfrak{g}$ is $\mathbb{N}$-graded, then the Hilbert series of $\mathscr{U}(\mathscr{L})$ equals that of $\operatorname{gr}\mathscr{U}(\mathscr{L})$ endowed with the grading induced by that of $\mathscr{L}$. In particular,

$$H_{\mathscr{U}(\mathscr{L})}(t)=\prod_{i\geq 1}\frac{1}{(1-t^i)^{\ell_i}},$$

where $\ell_i=\dim\mathscr{L}_i$ (see [31, Ch. 2.2, Example 2]).

Now, if $\mathscr{M}$ is a cocyclic ideal of $\mathscr{L}$, then $\dim\mathscr{L}_i=\dim\mathscr{M}_i+\delta_{1i}$, for all $i\geq 1$, where δ_{ij} is the Kronecker delta. Therefore,

$$H_{\mathscr{U}(\mathscr{L})}(t)=\frac{1}{(1-t)}H_{\mathscr{U}(\mathscr{M})}(t).$$

By Theorem 3.12, there are long exact sequences

$$H^{i-1,i}(\mathscr{L})\to H^{i-1,i}(\mathscr{M})\to H^{i-1,i-1}(\mathscr{M})\to H^{i,i}(\mathscr{L})\to \\ \to H^{i,i}(\mathscr{M})\to H^{i,i-1}(\mathscr{M})=0$$

for all $i\geq 0$. Since $\mathscr{L}$ is Koszul, and since $H^{ii}(\mathscr{M})=(\mathrm{q}\mathscr{M})^!_i$ is finite dimensional, we recover the following formulae involving the bigraded Betti numbers:

$$b_{i,i}(\mathscr{L})=b_{i-1,i-1}(\mathscr{M})+b_{i,i}(\mathscr{M})-b_{i-1,i}(\mathscr{M}).$$

In particular, the Hilbert polynomial of $\mathscr{L}^!$ (i.e., the Poincaré polynomial of $\mathscr{L}$) is given by

$$H_{\mathscr{L}^!}(t)=\sum_i b_{ii}(\mathscr{L})t^i=\sum_i(b_{i-1,i-1}(\mathscr{M})+b_{i,i}(\mathscr{M})-b_{i-1,i}(\mathscr{M}))t^i= \\ =(1+t)H_{(\mathrm{q}\mathscr{M})^!}(t)+tQ(t),$$

where we have put $Q=\sum_i b_{i,i+1}(\mathscr{M})t^i$.

Now, by Fröberg's formula for $\mathscr{L}$, we get

$$1=H_{\mathscr{U}(\mathscr{L})}(t)H_{\mathscr{L}^!}(-t)=\frac{1}{1-t}H_{\mathscr{U}(\mathscr{M})}(t)\cdot\Big((1-t)H_{(\mathrm{q}\mathscr{M})^!}(-t)-tQ(-t)\Big)= \\ =H_{\mathscr{U}(\mathscr{M})}(t)H_{(\mathrm{q}\mathscr{M})^!}(-t)-\frac{t}{1-t}H_{\mathscr{U}(\mathscr{M})}Q(-t).$$

It follows that, if the Fröberg formula holds for $\mathscr{M}$, then $Q(t)=0$, which in turn implies that $H^{i,i+1}(\mathscr{M})=0$ for all $i\geq 0$. Eventually, from the proof of

Corollary 3.20, it follows that $H^{\bullet,\bullet}(\mathscr{M})$ is concentrated on the diagonal, proving that $\mathscr{M}$ is Koszul. □

4. Bloch-Kato Lie algebras

As we have seen above, from the Bloch-Kato conjecture it follows that maximal pro-p Galois groups of fields containing primitive pth roots of 1 have quadratic cohomology as well as any of its closed subgroups. This motivates the study of Lie algebras with an analogous behavior with respect to the cohomology of standard subalgebras.

The following may not be the regular definition of having "hereditarily" a property for a graded Lie algebra, yet we will adopt it for the rest of the work.

Definition 4.1. *Let $\mathcal{P}$ be a property of standard Lie algebras. A standard Lie algebra $\mathscr{L}$ is said to be* ***hereditarily*** *$\mathcal{P}$ if all of its standard subalgebras satisfy $\mathcal{P}$.*

We can now define the main characters.

Definition 4.2. *A standard Lie algebra $\mathscr{L}$ is said to be* ***Bloch-Kato*** *if it is hereditarily Koszul.*

Recall that, for the cohomology of a standard Lie algebra, quadraticity is equivalent to being 1-generated. Moreover, if $\mathscr{M}$ is a standard subalgebra of $\mathscr{L}$, then the restriction homomorphism in degree 1 is the natural surjection $\mathscr{L}_1^* \to \mathscr{M}_1^*$. Notice that such a map is the restriction homomorphism $H^1(\mathscr{L}, k) \to H^1(\mathscr{M}, k)$. Hence, for a Koszul Lie algebra $\mathscr{L}$, being Bloch-Kato is equivalent to the restriction morphism $H^\bullet(\mathscr{L}, k) \to H^\bullet(\mathscr{M}, k)$ being surjective for all $\mathscr{M}$.

We start by showing some examples of Bloch-Kato Lie algebras.

Examples 4.3. (1) Let $\mathscr{L}$ be a free standard Lie algebra. By Proposition 2.10 and Remark 3.7, $H^i(\mathscr{L}, k) = 0$ for $i \geq 2$ and $\mathscr{L}$ is Koszul. Now, if $\mathscr{M}$ is a standard subalgebra of $\mathscr{L}$, then $H^2(\mathscr{M}, k) = 0$ and hence it is free and Koszul. It follows that free standard Lie algebras are Bloch-Kato.

(2) On the opposite side, abelian Lie algebras are Bloch-Kato. Indeed, if $\mathscr{A}$ is a standard Lie algebra with trivial brackets, then its universal envelope is the whole polynomial ring on $\dim \mathscr{A}_1$ generators which is Koszul. Since any subalgebra of $\mathscr{A}$ is abelian and standard, the same argument proves that $\mathscr{A}$ is Bloch-Kato.

(3) Consider the quadratic Lie algebra

$$\mathscr{L} = \left\langle x, y, z, w \,\middle|\, [x, y] - [z, w] = [x, z] = [x, w] = 0 \right\rangle.$$

In $\mathscr{L}$, the elements x and y do not commute, though $[x, [x, y]] = [x, [z, w]] = [[x, z], w] + [z, [x, w]] = 0$. It follows that the subalgebra $\mathscr{M}$ generated by x and y is not quadratic and hence $\mathscr{L}$ is not Bloch-Kato.

(4) Consider the quadratic Lie algebra

$$\mathscr{L} = \left\langle x_1, x_2, y_1, y_2 \,\middle|\, [x_i, y_j] : 1 \le i, j \le 2 \right\rangle .$$

We see that $[x_1 + y_1, x_2] = [x_1, x_2]$ and hence

$$[[x_1 + y_1, x_2], y_2] = 0.$$

Let $\mathscr{M}$ be the subalgebra generated by $t = x_1 + y_1$, $x = x_2$ and $y = y_2$. It is a (proper) quotient of the Lie algebra $\tilde{\mathscr{M}} = \left\langle t, x, y \,\middle|\, [x, y], [[t, x], y] \right\rangle$. Since the dimensions of $\mathscr{M}$ and $\tilde{\mathscr{M}}$ agree up to degree 2, it follows that $\mathscr{M}$ is not quadratic.

By using HNN-decomposition and Lemma 3.15, one sees that the Lie algebras of (3) and (4) are Koszul.

Also, notice that the quadratic Lie algebra $\mathscr{L}$ of (4) is isomorphic to the direct sum of two free non-abelian Lie algebras

$$\mathscr{L} \simeq \left\langle x_1, x_2 \,\middle|\, \emptyset \right\rangle \sqcap \left\langle y_1, y_2 \,\middle|\, \emptyset \right\rangle .$$

Hence, that example suggests the following general phenomenon:

Proposition 4.4. *Let $\mathscr{A}$ and $\mathscr{B}$ be non-abelian standard Lie algebras. Then $\mathscr{A} \sqcap \mathscr{B}$ is not Bloch-Kato.*

Proof. It is enough to prove the result when $\mathscr{A}$ and $\mathscr{B}$ are Bloch-Kato. Since $\mathscr{A}$ is not abelian, it contains degree 1 elements x_1, x_2 such that $[x_1, x_2] \neq 0$, and for $\mathscr{A}$ being Bloch-Kato, the subalgebra $\mathscr{F}$ generated by the x_i's must be free non-abelian. Analogously, one can find a free non-abelian standard subalgebra $\mathscr{G}$ inside $\mathscr{B}$.

It follows that the subalgebra of $\mathscr{A} \sqcap \mathscr{B}$ generated by $\mathscr{F}$ and $\mathscr{G}$ contains a copy of the Lie algebra of Example 4.3(4), and hence it is not Bloch-Kato. □

We will provide many more examples of Bloch-Kato Lie algebra in later sections. However, so far we notice that in (3) and (4) the fact of not being BK is due to the existence not only of a non-Koszul subalgebra, but also of a non-quadratic subalgebra. This is a general fact which can be easily proven by arguing inductively on the number of minimal generators and by using Fundamental Lemma 3.11 on the splitting into an HNN extension over arbitrary maximal standard Lie subalgebras.

Theorem 4.5. *Hereditarily quadratic Lie algebras are Bloch-Kato.*

Proof. Let n be the dimension of $\mathscr{L}_1$. The case $n = 1$ is trivial.

Let $n > 1$. Assume that all hereditarily quadratic Lie algebras generated by less than n elements are Bloch-Kato.

If $\mathscr{L}$ is hereditarily quadratic generated by n elements and $\mathscr{M}$ is an arbitrary standard proper subalgebra, then $\mathscr{M}$ is hereditarily quadratic and hence Bloch-Kato. So it is enough to prove that $\mathscr{L}$ is Koszul. But this easily follows from the combination of Lemma 3.11 and Lemma 3.15. Indeed, if $\mathscr{M}$ is a maximal standard subalgebra of $\mathscr{L}$, there exists a standard subalgebra $\mathscr{A}$ of $\mathscr{M}$ and a derivation $\phi : \mathscr{A} \to \mathscr{M}$ such that $\mathscr{L} = \mathrm{HNN}_\phi(\mathscr{M}, t)$. Since both $\mathscr{M}$ and $\mathscr{A}$ are Koszul, so is $\mathscr{L}$. □

It can also be proved that all quadratic Lie algebras generated by at most 3 generators are automatically Bloch-Kato. This follows from the fact that quadratic Lie algebras with at most 2 relations are BK (see Theorem 9.10).

Bloch-Kato Lie algebras are also determined by the following embedding property: a standard Lie algebra $\mathscr{L}$ is BK if, and only if, for all homomorphisms $f : \mathscr{M} \to \mathscr{L}$ from a quadratic Lie algebra $\mathscr{M}$ into $\mathscr{L}$, then f is injective iff it is injective in degrees 1 and 2. The proof is straightforward and relies on the following.

Fact 4.6. *Let $f : \mathscr{M} \to \mathscr{L}$ be a homomorphism of quadratic (Lie) algebras. If f is an isomorphism in degrees* 1 *and* 2*, then f is an isomorphism.*

Example 4.7. *Consider the Lie algebra $\mathscr{L}$ of Example 4.3–(4). The quadratic Lie algebra $\mathscr{M} = \langle t, x, y \,|\, [x, y] \rangle$ admits a homomorphism $f : \mathscr{M} \to \mathscr{L}$ sending $t \mapsto x_1 + y_1$, $x \mapsto x_2$ and $y \mapsto y_2$. One can show that f is injective in degree* 1 *and* 2 *but not in all degrees.*

4.1. **Universally Koszul algebras.** If $\mathscr{L}$ is a Koszul Lie algebra, then $\mathscr{L}$ is determined by its cohomology algebra. Indeed, one has $\mathscr{U}(\mathscr{L})^! = H^\bullet(\mathscr{L})$, and the Koszul dual functor is an involution, so that $\mathscr{L}$ can be recovered by taking the Lie subalgebra of $H^\bullet(\mathscr{L}, k)^!$ generated by $H^1(\mathscr{L}, k)^*$.

One might thus be led to guess that for a standard Lie algebra $\mathscr{L}$, being BK is related with the fact that any quotient of the cohomology $H^\bullet(\mathscr{L}, k)$ with respect to ideals generated by degree 1 elements is Koszul. However, this request is too weak for $\mathscr{L}$ being BK, as Examples 4.3(3)-(4) show: both Lie algebras are generated by 4 elements and are Koszul; in particular, $H^\bullet(\mathscr{L}, k)$ is Koszul and any proper quotient by 1-generated ideals is quadratic and has less than 4 generators, hence Koszul.

Definition 4.8. *Let A and B be two graded algebras. A morphism $f : A \to B$ is called a* ***left-Koszul homomorphism*** *if B is a Koszul left A-module with respect to the action induced by f, i.e., if there is a free A-resolution of graded modules*

$$\cdots \to P_1 \to P_0 \to B \to 0,$$

such that $P_i = A \cdot (P_i)_i$.

In [31], the authors prove the following results:

Proposition 4.9. *Let $A \to B$ be a left-Koszul homomorphism of graded algebras. Then A is Koszul if, and only if, B is Koszul.*

In fact, A is Koszul if, and only if, the augmentation $A \to k$ is a Koszul homomorphism.

Lemma 4.10 (See [31])**.** *Let $f : A \to B$ be a homomorphism of Koszul algebras. Then f is left-Koszul if, and only if, $A^!$ is a free right $B^!$-module. If this is the case, then the dual morphism $B^! \to A^!$ is injective, and B and $A^!/A^!B^!_+$ are dual Koszul modules over A and $A^!$, respectively.*

Notice that for a graded-commutative algebra (e.g. the cohomology of a Lie algebra), there is no distinction between left and right modules. We may thus drop the word "left", and consider Koszul homomorphism.

Following [8], we give the introduce the following definition:

Definition 4.11. *A graded-commutative algebra A is said to be **universally Koszul** if $A \to A/I$ is a Koszul homomorphism for any ideal I that is generated by elements of degree 1.*

Consider the following set of ideals of A:

$$\mathcal{L}(A) = \left\{ I \lhd A \;\middle|\; I \text{ is generated by elements of degree } 1 \right\}.$$

We also introduce a notation for pairs of ideals I and J of A; we write $(I : J)$ for the ideal of elements $a \in A$ such that $a \cdot J \subseteq I$.

In [8], the following equivalent conditions are given.

Proposition 4.12. *Let A be a graded-commutative algebra. The following conditions are equivalent:*

(1) *A is universally Koszul*
(2) *For every $I \in \mathcal{L}(A)$, one has $\operatorname{Tor}^A_{2,j}(A/I, \mathbb{F}) = 0$ for every $j > 2$.*
(3) *For every $I \in \mathcal{L}(A)$ and $x \in A_1 \setminus I$, one has $I : (x) \in \mathcal{L}(A)$.*
(4) *$\forall I \in \mathcal{L}(A) \setminus \{0\}$ $\exists J \in \mathcal{L}(A)$, such that $J \subset I$, I/J is cyclic and $(J : I) \in \mathcal{L}(A)$.*
(5) *$\forall I \in \mathcal{L}(A)$, A/I is a Koszul A-module.*

Corollary 4.13. *Let A be a graded-commutative algebra, and assume that for every $z \in A_1 \setminus \{0\}$, one has $(0 : z) \in \mathcal{L}(A)$ and $A/(z)$ is universally Koszul. Then A is universally Koszul.*

Proof. We show that $I : x \in \mathcal{L}(A)$ for every $I \in \mathcal{L}(A)$ and $x \in A_1 \setminus I$. If $I = 0$, then this is clearly true. If $I \neq 0$, then take a non-zero linear form $x \in I$. By assumption, $A/(z)$ is universally Koszul, and hence the ideal $(I : x)/(z) = (I/(z) : (x + I))$ is generated in degree 1. It follows that $(I : x) \in \mathcal{L}(A)$. □

4.2. **Cohomology of Bloch-Kato Lie algebras.** By the PBW Theorem, it follows that the universal enveloping algebra of a graded Lie algebra $\mathscr{L}$ is free over $\mathscr{U}(\mathscr{M})$, for every graded subalgebra $\mathscr{M}$ of $\mathscr{L}$.

Hence, we can specialise Lemma 4.10.

Corollary 4.14. *Let $\mathscr{M} \to \mathscr{L}$ be a homomorphism of Koszul Lie algebras. Then, the restriction $H^\bullet(\mathscr{L}, k) \to H^\bullet(\mathscr{M}, k)$ is a Koszul homomorphism if, and only if, $\mathscr{M} \to \mathscr{L}$ is injective.*

Proof. If $\mathscr{M} \to \mathscr{L}$ is injective, then $\mathscr{U}(\mathscr{L})$ is a free $\mathscr{U}(\mathscr{M})$-module, and hence $H^\bullet(\mathscr{L}, k) \to H^\bullet(\mathscr{M}, k)$ is Koszul, by Lemma 4.10.

Conversely, the dual map of the restriction map is $\mathscr{U}(\mathscr{M}) \to \mathscr{U}(\mathscr{L})$ and, by Lemma 4.10, it is injective, whence the induced map $\mathscr{M} \to \mathscr{L}$ is injective. □

We can now prove the following main result, which first appeared in the author's paper [1].

Theorem 4.15. *Let $\mathscr{L}$ be a graded Lie algebra with cohomology algebra $A = H^\bullet(\mathscr{L}, k)$. Then $\mathscr{L}$ is Bloch-Kato if, and only if, A is universally Koszul.*

Proof. (1) Let $\mathscr{L}$ be a Bloch-Kato Lie algebra. If $I_1 \leq A_1 = \mathscr{L}_1^*$, consider the standard subalgebra

$$\mathscr{M} = \langle m \in \mathscr{L}_1 \,|\, i(m) = 0,\ \forall i \in I_1 \rangle = \langle I_1^\perp \rangle.$$

Since $\mathscr{L}$ is Bloch-Kato, the Lie algebra $\mathscr{M}$ is Koszul, and $H^\bullet(\mathscr{M}, k) \simeq \mathscr{U}(\mathscr{M})^!$.

By definition, res : $H^\bullet(\mathscr{L}, k) \to H^\bullet(\mathscr{M}, k)$ is surjective and

$$\ker \mathrm{res}_1 = \left\{ \alpha \in \mathscr{L}_1^* \;\middle|\; \alpha|_{\mathscr{M}_1} = 0 \right\} = I_1.$$

Therefore, there is an ideal $J = I(I_1) \lhd A$ such that

$$H^\bullet(\mathscr{L}, k)/I(I_1) \simeq B = H^\bullet(\mathscr{M}, k)$$

is Koszul and $I(I_1)_1 = I_1$. We want to show that J is generated by I_1 as an ideal of A.

Now, the projection $A \to B$ is Koszul, by Proposition 4.14, since $A^! = \mathscr{U}(\mathscr{L})$ is a free right $B^!$-module, for $B^! = \mathscr{U}(\mathscr{M})$ and the PBW Theorem. Thus, B is a Koszul A-module.

The exact sequence of A-modules $0 \to J \to A \to B \to 0$ induces a long exact sequence involving the Ext functor, that is,

$$\mathrm{Ext}_A^{0,j}(A, k) \to \mathrm{Ext}_A^{0,j}(J, k) \to \mathrm{Ext}_A^{1,j}(B, k) \to \mathrm{Ext}_A^{1,j}(A, k)$$

Notice that $J_0 = 0$, for $B_0 \simeq A_0$.

For B being a Koszul A-module it means that $\mathrm{Ext}_A^{\bullet,\bullet}(B, k)$ is concentrated on the diagonal, and hence the long exact sequence shows that J is generated in degree 1, namely $J = AI_1$. (More precisely $\mathrm{Ext}_A^{i,j}(J, k) = 0$ for $j \neq i + 1$, and J is a Koszul A-module.)

(2) Suppose that A is a universally Koszul graded-commutative algebra. Then A is Koszul and $A^! = \mathscr{U}(\mathscr{A})$ is the universal envelope of a Koszul Lie algebra $\mathscr{A}$ generated by A_1^*, by Lemma 3.10. Let $V \leq A_1^*$, and $\mathscr{B} = \langle V \rangle$ be the Lie subalgebra of $\mathscr{A}$ generated in degree 1 by V. Let I be the ideal of A generated by $V^\perp \leq A_1$ in degree 1.

Thus A/I is Koszul since A is universally Koszul, and $(A/I)^!$ is a quadratic subalgebra of $A^!$ by Lemma 4.10. It follows that $(A/I)^! = \mathscr{U}(\mathscr{N})$ for some (quadratic) Lie algebra $\mathscr{N}$ generated by $(A_1/I_1)^* = V$. But, by Lemma 4.14, $\mathscr{N}$ is a Lie subalgebra of $\mathscr{A}$, and hence, $\mathscr{N} = \mathscr{B}$, as they have the same generating space. Thus, $\mathscr{B}$ is Koszul and

$$H^\bullet(\mathscr{B}, k) \simeq A/I.$$

Eventually, notice that since $A = H^\bullet(\mathscr{L}, k)$ is 1-generated, $\mathscr{U}(\mathscr{L})$ is a Koszul algebra, and hence $A^! = \mathscr{U}(\mathscr{L})$, namely $\mathscr{A} = \mathscr{L}$. □

Let A and B be graded-commutative universally Koszul algebras. Then, by [27, Prop. 30] , the direct sum $A \sqcap B$ is again universally Koszul. Plus, the wedge product $A \wedge \Lambda(x)$ of A with the exterior algebra on one generator is universally Koszul. Since the cohomology of a free product of Lie algebras is the direct sum of the cohomologies of the factors, it follows that the free product of BK Lie algebras is again BK. The same is true for the direct sum $\mathscr{A} \sqcap k$. Recall that, by Proposition 4.4, for the direct sum of BK Lie algebras to be BK it is necessary that at least one is abelian.

Theorem 4.16. *Let $\mathscr{A}$ and $\mathscr{B}$ be Bloch-Kato Lie algebra. Then, both the free product $\mathscr{A} \amalg \mathscr{B}$ and the direct sum $\mathscr{A} \sqcap k$ are Bloch-Kato Lie algebras.*

4.3. **Kurosh theorem.** Kurosh's subgroup theorem [21] provides the structure of any subgroup of a free product of groups and its more elegant proof relies on Bass-Serre theory of groups acting on trees. In the case of Lie algebras, such a theory does not exist and the analogue of Kurosh theorem is false in general, as it was first noticed by A.I. Shirshov in [45].

Theorem 4.17 (Kurosh subgroup theorem)**.** *Let A and B be two (pro-p) groups. Let H be any subgroup of the free (pro-p) product $G = A \amalg B$. Then, there exist subgroups A_i of A and B_j of B and elements $g_i, g'_j \in G$ such that H is isomorphic to the following free (pro-p) product*

$$F \amalg \coprod_i A_i^{g_i} \amalg \coprod_j B_j^{g'_j},$$

where F is a free (pro-p) group.

For a proof of this result, see [21] for the abstract case and [41] for the pro-p case.

In this section we will state the main theorem proved by the author in [1], which is so far the best known analogue in the realm of Lie algebras. We will anyway refer to this as the Kurosh subalgebra theorem.

Theorem 4.18. *Let $\mathscr{A}$ and $\mathscr{B}$ be two standard Lie algebras, and let $\mathscr{H} \subseteq \mathscr{A} \amalg \mathscr{B}$ be a standard subalgebra of their free product. If $\mathscr{H}$ is Bloch-Kato, then*

$$\mathscr{H} \simeq \langle \mathscr{H}_1 \cap \mathscr{A} \rangle \amalg \langle \mathscr{H}_1 \cap \mathscr{B} \rangle \amalg \mathscr{F} \tag{4.1}$$

where $\mathscr{F}$ is a free Lie algebra generated by any distinguished subspace $W \subseteq \mathscr{A}_1 \oplus \mathscr{B}_1$ such that $\mathscr{H}_1 = W \oplus (\mathscr{H}_1 \cap \mathscr{A}) \oplus (\mathscr{H}_1 \cap \mathscr{B})$.

In particular, if $\mathscr{A}$ and $\mathscr{B}$ are Bloch-Kato, then a form of Kurosh subalgebra theorem holds for standard subalgebras.

4.4. **Lie algebras of elementary type.** Let $\mathcal{C}$ be a class of BK Lie algebras. Since free products of Lie algebras from $\mathcal{C}$ are BK, and so are direct sums with standard abelian Lie algebras, we can consider the smallest class $ET(\mathcal{C})$ containing $\mathcal{C}$, that is obtained by applying such constructions to the elements of $\mathcal{C}$. We will call it the **class of $\mathcal{C}$-elementary type** (or $\mathcal{C}$-ET) Lie algebras. Plus, $\mathcal{C}$ is **closed** if the following condition holds for every Lie algebra $\mathscr{L}$ in $\mathcal{C}$:

If $\mathscr{M}$ is a standard subalgebra of $\mathscr{L}$, then $\mathscr{M}$ belongs to $\mathrm{ET}(\mathcal{C})$.

We say that $\mathcal{C}$ is the **trivial class** if it only contains the zero Lie algebra.

Proposition 4.19. *Let $\mathscr{E}$ be a $\mathcal{C}$-ET Lie algebra, where $\mathcal{C}$ is a closed class of BK Lie algebras. Then every standard subalgebra of $\mathscr{E}$ belongs to ET($\mathcal{C}$).*

Proof. The iterated definition of $\mathrm{ET}(\mathcal{C})$ allows one to argue by induction on the number of constructions that one has to perform in order to get $\mathscr{E}$ from elements of $\mathcal{C}$.

If $\mathscr{E}$ belongs to $\mathcal{C}$, the claim follows from the fact that $\mathcal{C}$ is closed.

Suppose that $\mathscr{E} = \mathscr{A} \amalg \mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are non-trivial $\mathcal{C}$-ET Lie algebras, and let $\mathscr{M}$ be a standard subalgebra of $\mathscr{E}$. Then, by the Kurosh Subalgebra theorem, $\mathscr{M} = \mathscr{A}' \amalg \mathscr{B}' \amalg \mathscr{F}$, where $\mathscr{F}$ is a free Lie algebra, and $\mathscr{A}'$ and $\mathscr{B}'$ are 1-generated subalgebras of $\mathscr{A}$ and $\mathscr{B}$ respectively. By induction, such subalgebras belong to $\mathrm{ET}(\mathcal{C})$, and thus the same is true for $\mathscr{M}$.

Suppose that $\mathscr{E} = \mathscr{A} \sqcap k$, and denote by t the generator of the abelian Lie algebra k. If $\mathscr{M}$ is a standard subalgebra of $\mathscr{E}$, we may suppose that it is generated by a standard subalgebra $\mathscr{B}$ of $\mathscr{A}$ and an element $a+t$, where $a \in \mathscr{A}_1$.

If $\bar{\mathscr{A}} = \langle \mathscr{B}, a\rangle \neq \mathscr{A}$, then $\mathscr{M} \leq \langle \mathscr{B}, a, t\rangle = \bar{\mathscr{A}} \sqcap k$ is of elementary type since $\bar{\mathscr{A}}$ is. Now suppose $\langle \mathscr{B}, a\rangle = \mathscr{A}$. Then, $\mathscr{M}$ is isomorphic to $\mathscr{A}$, since the mappings $\mathscr{B} \hookrightarrow \mathscr{A}$ and $a+t \mapsto a$ induce a well defined Lie algebra homomorphism $\mathscr{B} \to \mathscr{A}$ that is an isomorphism in degree 1, and hence in all the degrees. □

4.5. **Bogomolov's property.** Motivated by a question of Bogomolov [4], Positselski conjectured in [33, Conj. 2] that, whenever a field contains an algebraically closed field, both the Sylow pro-p subgroup of the absolute Galois group of $\mathbb{F}$ and the maximal pro-p Galois group of $\mathbb{F}$ have free commutator subgroups. Although the condition about algebraically closed fields cannot be dropped, it appears that the truth of this conjecture should rely on the fact that the field $\mathbb{F}$ contains all the roots of unity of degrees equal to powers of p.

It has been proved in [39] that the Elementary Type Conjecture implies the above conjecture.

If $\mathcal{C}$ is a class of BK Lie algebras, we thus call $\mathcal{C}$ a Bogomolov class if the commutator subalgebra of every Lie algebra in $\mathcal{C}$ is a free Lie algebra. Henceforth Lie algebras with free derived subalgebras will be called **Bogomolov Lie algebras**.

Proposition 4.20. *Let $\mathscr{L}$ be a Lie algebra of $\mathcal{C}$-ET, where $\mathcal{C}$ is a Bogomolov class. Then $[\mathscr{L}, \mathscr{L}]$ is a free Lie algebra.*

Proof. Notice that if $\mathscr{L}$ is a Bogomolov Lie algebra, then all standard subalgebras of $\mathscr{L}$ are Bogomolov, for subalgebras of free Lie algebras being free. In particular, the closure of $\mathcal{C}$ is a Bogomolov class, and we can thus assume $\mathcal{C}$ to be closed.

We now argue by induction. As in Proposition 4.19, the base case is trivially true. Suppose that $\mathscr{L} = \mathscr{A} \amalg \mathscr{B}$ and that $\mathscr{A}$ and $\mathscr{B}$ have free derived algebras. Then $[\mathscr{A}, \mathscr{A}] = \mathscr{A} \cap [\mathscr{L}, \mathscr{L}]$ and $[\mathscr{B}, \mathscr{B}] = \mathscr{B} \cap [\mathscr{L}, \mathscr{L}]$, and hence, by [18, Prop. 2.10], $[\mathscr{L}, \mathscr{L}]$ is free.

Assume now that $\mathscr{L} = \mathscr{A} \sqcap k$. Then $[\mathscr{L}, \mathscr{L}] = \mathscr{L}_{\geq 2} = \mathscr{A}_{\geq 2} = [\mathscr{A}, \mathscr{A}]$ is free. □

From the point of view of Lie algebras, by [18, Prop. 5.8] Bogomolov property implies that the Lie algebra is coherent, i.e., it is locally finitely presentable.

Proposition 4.21. *Let $\mathscr{L}$ be a Bogomolov Lie algebra. Then $\mathscr{L}$ is locally FP_∞. In particular, it is coherent.*

In similarity with the Bogomolov-Positselski conjecture, we suspect that the following Lie algebra-theoretic phenomenon occurs.

Conjecture 4.22. *All BK Lie algebras are Bogomolov.*

Notice that Positselski writes in [33] that "apparently, no one of the two conjectures (Bloch-Kato's and Bogomolov's) implies directly the other one; rather, they are somewhat complementary".

5. Restricted Lie algebras

We have seen above how graded Lie algebras naturally come out in the study of lower-central series of abstract groups. However, if G is a pro-p group, one can perform a similar construction by considering the graded object associated with the Zassenhaus filtration of G.

Recall that this is the fastest descending central series (G_i) of G satisfying $G_1 = G$ and $G_i^p \leq G_{ip}$. Since (G_i) is central, one can endow the graded object

$$\operatorname{gr} G = \bigoplus_{i \geq 1} G_i / G_{i+1}$$

with a Lie algebra structure. It turns out that the pth power on G yields a set-theoretic map $\operatorname{gr} G \to \operatorname{gr} G$ that makes it into a restricted p-Lie algebra over the field $\mathbb{F}_p$.

Theorem 5.1. *Let G be a pro-p group, p an odd prime. The Zassenhaus filtration of G can be defined inductively as follows*

(1) $G_1 = G$
(2) $G_i = G_{i^*}^p[G_{i-1}, G]$ *for $i > 1$, where i^* is the least integer such that $i \leq pi^*$.*

Following Jacobson [15], we define the concept of **restricted p-Lie algebras** (or simply p-Lie algebras). A set-theoretic map $[p] : \mathscr{L} \to \mathscr{L}$ on a graded $\mathbb{F}_p$-Lie algebra is called a **p-operation** if

(1) $\mathrm{ad}(x^{[p]}) = \mathrm{ad}(x)^p$ $(x \in \mathscr{L})$; in particular, $\mathscr{L}_i^{[p]} \leq \mathscr{L}_{ip}$ for $i \geq 1$;
(2) $(tx)^{[p]} = tx^{[p]}$ $(t \in \mathbb{F}_p,\ x \in \mathscr{L})$;
(3) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(x,y)}{i}$, where $s_i(x,y)$ is the coefficient of u^{i-1} in the formal expression $\mathrm{ad}(ux+y)^{p-1}(x)$.

A graded restricted p-Lie algebra is a graded Lie algebra with a p-operation. It is clear that all $\mathbb{N}_0$-graded $\mathbb{F}_p$-algebras are naturally restricted p-Lie algebras with respect to the pth power map $x \mapsto x^p$ and the commutator as Lie bracket. This construction defines a functor $(\_)_{[\cdot,\cdot]_p}$ from the category of algebras to that of restricted Lie algebras.

In complete analogy with the ordinary graded cases, for any restricted Lie algebra, one can define the universal envelope which gives rise to a left-adjoint functor to $(\_)_{[\cdot,\cdot]_p}$. Explicitly, if $\mathscr{L}$ is a graded restricted p-Lie algebra, the **restricted universal envelope** of $\mathscr{L}$, denoted by $\underline{u}(\mathscr{L})$, is the quotient of $\mathscr{U}(\mathscr{L})$ with respect to the two-sided ideal generated by the elements $x^p - x^{[p]}$ $(x \in \mathscr{L})$.

Theorem 5.2. *Let G be a pro-p group. The augmentation ideal ω of the completed group algebra $R = \mathbb{F}_p[\![G]\!]$ gives rise to a filtration (ω^i) of R with associated graded object* $\mathrm{gr}\, R$.

The restricted universal enveloping algebra of $\mathrm{gr}\, G$ *is naturally isomorphic with* $\mathrm{gr}\, R$ *(see [40]).*

5.1. **Restrictification.** Let $\mathscr{L}$ be any Lie algebra defined over the field $\mathbb{F}_p$. Consider the universal enveloping algebra $\mathscr{U}(\mathscr{L})$ of $\mathscr{L}$ and the set of **primitive elements**

$$\mathscr{P}\mathscr{U}(\mathscr{L}) := \left\{x \in \mathscr{U}(\mathscr{L}) \;\middle|\; \Delta(x) = x \otimes 1 + 1 \otimes x\right\}$$

where $\Delta : \mathscr{U}(\mathscr{L}) \to \mathscr{U}(\mathscr{L}) \otimes \mathscr{U}(\mathscr{L})$ is the Hopf-comultiplication of $\mathscr{U}(\mathscr{L})$ induced by setting $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathscr{L}$.

Denote $\mathscr{P}\mathscr{U}(\mathscr{L})$ by $\hat{\mathscr{L}}$. We claim that $\hat{\mathscr{L}}$ is a restricted Lie algebra.

Let $x, y \in \hat{\mathscr{L}}$. Then

$$\Delta([x,y]) = \Delta(x)\Delta(y) - \Delta(y)\Delta(x) = 1 \otimes [x,y] + [x,y] \otimes 1$$

and

$$\Delta(x^p) = \sum_{i=0}^{p} \binom{p}{i} x^i \otimes x^{p-i} = 1 \otimes x^p + x^p \otimes 1$$

as $p \cdot 1 = 0$ in $\mathbb{F}_p$.

Now, consider the restricted universal envelope $\underline{u}(\hat{\mathscr{L}})$ of $\hat{\mathscr{L}}$. It solves uniquely the following extension problems

$$\begin{array}{ccc} \hat{\mathscr{L}} & \longrightarrow & A \\ \big\downarrow & \nearrow & \\ \underline{u}(\hat{\mathscr{L}}) & & \end{array}$$

where A is any associative algebra and $\hat{\mathscr{L}} \to A$ is a morphism of p-Lie algebras.

Since the natural inclusion $\hat{\mathscr{L}} \to \mathscr{U}(\mathscr{L})$ is a morphism of p-Lie algebras, we recover an algebra homomorphism $f : \underline{u}(\hat{\mathscr{L}}) \to \mathscr{U}(\mathscr{L})$.

Now, by definition of the comultiplication mapping Δ, one sees that $\mathscr{L}$ is a Lie subalgebra of the non-restricted Lie algebra $\hat{\mathscr{L}}$. Thus, the extension problem

$$\begin{array}{ccccc} \mathscr{L} & \longrightarrow & \hat{\mathscr{L}} & \longrightarrow & \underline{u}(\hat{\mathscr{L}}) \\ \big\downarrow & & & \nearrow & \\ \mathscr{U}(\mathscr{L}) & & & & \end{array}$$

has a unique solution $g : \mathscr{U}(\mathscr{L}) \to \underline{u}(\hat{\mathscr{L}})$. Eventually, notice that f and g are inverse to each other. We call $\hat{\mathscr{L}}$ the **p-restrictification** of $\mathscr{L}$.

Lemma 5.3. *If $\mathscr{L}$ is a graded Lie algebra, then there is a natural isomomorphism $\mathscr{U}(\mathscr{L}) \simeq \underline{u}(\hat{\mathscr{L}})$.*

If $\mathscr{L}$ is a standard Lie algebra, one has a natural isomorphism between $\mathscr{L}$ and the standard Lie subalgebra of $\hat{\mathscr{L}}$ generated by $\hat{\mathscr{L}}_1$.

5.2. **Translation to restricted context.** We say that a p-Lie algebra $\mathscr{L}$ is quadratic if the kernel of the natural surjection $\mathscr{F}(\mathscr{L}_1) \to \mathscr{L}$, where $\mathscr{F}(\mathscr{L}_1)$ denotes the free p-Lie algebra generated by the space $\mathscr{L}_1$, is generated in degree 2. Notice that it is equivalent to requiring that its restricted universal envelope $\underline{u}(\mathscr{L})$ is quadratic as an associative algebra.

If p is odd, then any quadratic p-Lie algebra $\mathscr{L}$ can be presented as the quotient of the free p-Lie algebra $\mathscr{F}$ on $\mathscr{L}_1$ with respect to an ideal generated by linear combinations of commutators of elements of $\mathscr{F}_1$. In particular, the equation $x^{[p]} = 0$ has a unique solution $x = 0$. From this observation we get

Fact 5.4. *Let $\mathscr{L}$ be a p-Lie algebra, where p is an odd prime. If $\mathscr{L}$ is quadratic, then there is a quadratic Lie algebra $\mathfrak{g}$ such that $\mathscr{L} = \hat{\mathfrak{g}}$.*

Henceforth, we will assume p to be a fixed odd prime number.

Definition 5.5. *We say that a standard p-Lie algebra $\mathscr{L}$ is **BK** if any of its standard p-subalgebras are quadratic.*

Cohomology theory also applies to the context of restricted p-Lie algebras. We define the **restricted cohomology ring** of a restricted p-Lie algebra $\mathscr{L}$ as $H_r^{\bullet,\bullet}(\mathscr{L},k)=\mathrm{Ext}^{\bullet,\bullet}_{\underline{u}(\mathscr{L})}(k,k)$.

Lemma 5.6. *Let $\mathscr{L}$ be a standard $\mathbb{F}_p$-Lie algebra. Then, $\mathscr{L}$ is BK if, and only if, $\hat{\mathscr{L}}$ is BK.*

Proof. First of all, notice that $\underline{u}(\hat{\mathscr{L}})=\mathscr{U}(\mathscr{L})$ and hence

$$H_r^{\bullet,\bullet}(\hat{\mathscr{L}},k)=\mathrm{Ext}^{\bullet,\bullet}_{\underline{u}(\hat{\mathscr{L}})}(k,k)=\mathrm{Ext}^{\bullet,\bullet}_{\mathscr{U}(\mathscr{L})}(k,k)=H^{\bullet,\bullet}(\mathscr{L},k).$$

In particular, $\mathscr{L}$ is Koszul if, and only if, $\hat{\mathscr{L}}$ is.

Assume $\hat{\mathscr{L}}$ is BK and consider a standard subalgebra $\mathscr{M}$ of $\mathscr{L}$. Define $\hat{\mathscr{M}}$ as the p-subalgebra of $\hat{\mathscr{L}}$ generated by $\mathscr{M}_1$. Since $\hat{\mathscr{L}}$ is BK, $\hat{\mathscr{M}}$ is quadratic and hence it is the restrictification of the Lie algebra $\mathscr{M}$, which is thus quadratic. Indeed, it is clear that the associative subalgebra $\mathscr{U}(\mathscr{M})$ of $\mathscr{U}(\mathscr{L})=\underline{u}(\hat{\mathscr{L}})$ coincides with $\underline{u}(\hat{\mathscr{M}})$, as they are both generated by $\hat{\mathscr{M}}_1=\mathscr{M}_1$.

For the converse, assume $\mathscr{L}$ to be Bloch-Kato and let $\bar{\mathscr{M}}$ be a standard p-subalgebra of $\hat{\mathscr{L}}$. Then $\underline{u}(\mathscr{M})$ is the subalgebra of $\mathscr{U}(\mathscr{L})$ generated by $\mathscr{M}_1$ and hence it coincides with $\mathscr{U}(\mathscr{M})$, where $\mathscr{M}$ is the Lie subalgebra of $\mathscr{L}$ generated by $\mathscr{M}_1$, i.e., $\bar{\mathscr{M}}=\hat{\mathscr{M}}$. As $\mathscr{M}$ is quadratic, the same holds true for $\bar{\mathscr{M}}$. □

Corollary 5.7. *Let $\mathscr{L}$ be a standard p-Lie algebra. Then, $\mathscr{L}$ is Bloch-Kato if, and only if, its restricted cohomology algebra $H_r^{\bullet}(\mathscr{L},k)$ is universally Koszul.*

Proof. Since $\mathscr{L}$ is 1-generated, if $H_r^{\bullet}(\mathscr{L},k)$ is 1-generated, then $\mathscr{L}$ is Koszul and hence quadratic. It follows that if either $H_r^{\bullet}(\mathscr{L},k)$ is Koszul or $\mathscr{L}$ is BK, then $\mathscr{L}$ is a quadratic Lie algebra. By Fact 5.4, there is a standard (ordinary) Lie algebra $\mathscr{M}$ such that $\underline{u}(\mathscr{L})=\mathscr{U}(\mathscr{M})$, from which follows $H_r^{\bullet}(\mathscr{L},k)\simeq H^{\bullet}(\mathscr{M},k)$.

Now, $\mathscr{L}$ is BK if, and only if, $\mathscr{M}$ is BK, which is true precisely when $H_r^{\bullet}(\mathscr{L},k)\simeq H^{\bullet}(\mathscr{M},k)$ is universally Koszul. □

We say that a p-Lie algebra is **torsion-free** if the p-map only vanishes on the zero element.

Corollary 5.8. *Quadratic p-Lie algebras are torsion-free for p an odd prime.*

Proof. Let $\mathscr{L}$ be a quadratic p-Lie algebra and let $\mathfrak{g}$ be the Lie algebra such that $\hat{\mathfrak{g}}=\mathscr{L}$. Hence $\mathscr{U}(\mathfrak{g})=\underline{u}(\mathscr{L})$ has no zero divisor, and, in particular, it has no non-zero element whose pth power vanishes. □

Theorem 5.9. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two $\mathbb{N}$-graded Lie algebras. Then $\widehat{\mathfrak{a}\amalg\mathfrak{b}}\simeq\hat{\mathfrak{a}}\amalg_p\hat{\mathfrak{b}}$, where $\amalg_p$ denotes the free product in the category of p-Lie algebras.*

Proof. The free product $\mathfrak{a} \amalg \mathfrak{b}$ can be given by a presentation

$$\big\langle X_1, X_2 \,\big|\, R_1, R_2 \big\rangle$$

where $\mathfrak{a} = \big\langle X_1 \,\big|\, R_1 \big\rangle$ and $\mathfrak{b} = \big\langle X_2 \,\big|\, R_2 \big\rangle$. Then, $\hat{\mathfrak{a}} = \big\langle X_1 \,\big|\, R_1 \big\rangle_p$ and $\hat{\mathfrak{b}} = \big\langle X_2 \,\big|\, R_2 \big\rangle_p$, whence $\widehat{\mathfrak{a} \amalg \mathfrak{b}} = \big\langle X_1, X_2 \,\big|\, R_1, R_2 \big\rangle_p = \hat{\mathfrak{a}} \amalg_p \hat{\mathfrak{b}}$. □

A version of Kurosh theorem holds true for p-Lie algebras as well.

Corollary 5.10. *Let $\mathscr{A}$ and $\mathscr{B}$ be two Bloch-Kato p-Lie algebras and $\mathscr{M}$ be a 1-generated p-subalgebra of $\mathscr{A} \amalg_p \mathscr{B}$. Then there is a free p-Lie algebra $\mathscr{F}$ such that*

$$\mathscr{M} = \langle \mathscr{M}_1 \cap \mathscr{A} \rangle \amalg_p \langle \mathscr{M}_1 \cap \mathscr{B} \rangle \amalg_p \mathscr{F}$$

Proof. Let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{m}$ be the Lie algebras whose restrictifications are $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{M}$, respectively. Then, $\mathfrak{a}$ and $\mathfrak{b}$ are Bloch-Kato and $\mathscr{A} \amalg_p \mathscr{B} = \widehat{\mathfrak{a} \amalg \mathfrak{b}}$. Now, $\mathfrak{m}$ is a 1-generated subalgebra of $\mathfrak{a} \amalg \mathfrak{b}$, and hence there is a free Lie algebra $\mathfrak{f}$ such that

$$\mathfrak{m} = \langle \mathfrak{m}_1 \cap \mathfrak{a} \rangle \amalg \langle \mathfrak{m}_1 \cap \mathfrak{b} \rangle \amalg \mathfrak{f}.$$

Since the restrictification of free Lie algebras are free p-Lie algebras, we get the above description of $\mathscr{M}$. □

6. On the center of Koszul Lie algebras

Let $\mathscr{L}$ be a standard Lie algebra and assume that z is a central element of $\mathscr{L}$. Since $\mathscr{L}$ is positively graded, an element $z \in \mathscr{L}$ is central if, and only if, the subalgebra generated by z is an ideal of $\mathscr{L}$. If z has degree 1, then clearly $\mathscr{L}$ splits as the direct sum of standard algebras $\mathscr{L}/(z)$ and $\langle z \rangle$. Also, $\mathscr{L}$ is quadratic (resp. Koszul) precisely when $\mathscr{L}/(z)$ is so. Moreover, notice that the dimension of the center cannot be greater than the cohomological dimension of the Lie algebra, since the cohomological dimension of abelian Lie algebras is equal to their dimension and it cannot increase while passing to subalgebras.

Let $\mathcal{C}$ be a closed BK-class of Lie algebras. Assume that, if $\mathscr{L}$ belongs to $\mathcal{C}$, then the center $Z(\mathscr{L})$ is concentrated in degree 1, so that $\mathscr{L} \simeq Z(\mathscr{L}) \sqcap \mathscr{L}/Z(\mathscr{L})$. Now, let $\mathscr{L}$ be a Lie algebra of $\mathcal{C}$-ET and let $\mathscr{A}$ be a maximal abelian standard subalgebra of $\mathscr{L}$, such that there is a decomposition $\mathscr{L} = \mathscr{A} \sqcap \mathscr{B}$. Since $\mathscr{B}$ is of $\mathcal{C}$-ET by Proposition 4.19, there follows by the maximality of $\mathscr{A}$ either that $\mathscr{B}$ is a free product of $\mathcal{C}$-ET Lie algebras or $\mathscr{B}$ belongs to $\mathcal{C}$. In both cases, the center of $\mathscr{B}$ is trivial.

We have thus proven

Proposition 6.1. *Let $\mathcal{C}$ be a closed BK-class of Lie algebras with center concentrated in degree 1. Then the center of every $\mathcal{C}$-ET Lie algebra is concentrated in degree 1.*

We suspect that the centers of all BK Lie algebras are concentrated in degree 1. However, all known Koszul Lie algebras have center concentrated in degree 1, which suggests us to state the following

Conjecture 6.2. *The center of Koszul Lie algebras is concentrated in degree* 1*.*

The following example shows that one cannot expect the conjecture to be true for quadratic Lie algebras.

Example 6.3. *Consider the Lie algebra $\mathscr{H}$ generated by $2n > 2$ elements $x_1, y_1, \ldots, x_n, y_n$ subject to the quadratic relations*

$$\begin{cases} [x_i, x_j] = 0 \\ [y_i, y_j] = 0 \\ [x_i, y_j] = \delta_{ij}[x_1, y_1] \end{cases}$$

*Then, the degree-*2 *component $\mathscr{H}_2 \neq 0$ is central. However, $\mathscr{H}$ is not Koszul because it is not abelian and does not have exponential growth, as $\mathscr{H}_i = 0$ for $i > 2$. In [31], such Lie algebra is called the Heisenberg Lie algebra, where the name likely comes from the fact that, for $n = 2$, $\mathscr{H}$ is the smallest quadratic Lie algebra containing the* classical *Heisenberg Lie algebra given by the presentation*

$$\left\langle x, y \,\middle|\, [x,[x,y]], [y,[x,y]] \right\rangle .$$

In Section 7, we will deal with the problem of embedding graded Lie algebras into quadratic ones.

In the following, we will give a partial solution to this conjecture, by showing that the center of such Lie algebras must be concentrated in "small" odd degrees. In turn, this shows that the Lie algebra of Example 6.3 is not Koszul, independently from the growth argument used above.

Bøgvad [3] proves the following.

Lemma 6.4. *Let $\mathscr{L}$ be a graded* super-*Lie algebra of type FP. If $Z(\mathscr{L})_n \neq 0$, then $1/H_{\mathscr{U}(\mathscr{L})}(t)$ is a polynomial divisible by $1 - t^n$.*

This can also be applied to graded Lie algebras: If $\mathscr{L}$ is a graded Lie algebra, then it is also a super-Lie algebra $\mathfrak{g}$ with homogeneous components $\mathfrak{g}_{2n} = \mathscr{L}_n$. Plus, $H_{\mathscr{U}(\mathfrak{g})}(t) = H_{\mathscr{U}(\mathscr{L})}(t^2)$. If $Z(\mathscr{L})_n \neq 0$, then $Z(\mathfrak{g})_{2n} \neq 0$, and by Lemma 6.4, $1 - t^{2n}$ divides $1/H_{\mathscr{U}(\mathscr{L})}(t^2)$.

Corollary 6.5. *Let $\mathscr{L}$ be a Koszul Lie algebra. Then, $Z(\mathscr{L})$ is concentrated in degrees $< \operatorname{cd}\mathscr{L}/2 + 1$.*

Proof. Consider the universal envelope $\mathscr{U}(\mathscr{L})$. Since $\mathscr{L}$ is Koszul, it is of type FP and Fröberg's formula gives $H_{\mathscr{U}(\mathscr{L})}(t)H_{H^\bullet(\mathscr{L})}(-t) = 1$. Put $p(t) = H_{H^\bullet(\mathscr{L})}(t)$ and notice that $p(t)$ is a polynomial of degree $n = \operatorname{cd}\mathscr{L}$ with positive coefficients. Explicitly, if $b_j = \dim H^j(\mathscr{L}, k)$ denotes the jth Betti number of $\mathscr{L}$, then

$$p(t) = \sum_{j=0}^{n} b_j t^j.$$

Let $z \in \mathscr{L}$ be a degree i central element. By contradiction, assume $z \neq 0$, so that, by Lemma 6.4, $1 - t^i$ divides $p(-t)$, i.e., there are integers a_i such that

$$p(-t) = (1 - t^i)(a_0 + a_1 t + \cdots + a_{n-i} t^{n-i}).$$

By expanding the right-hand side

$$p(-t) = a_0 + a_1 t + \cdots + a_{n-i} t^{n-i} - a_0 t^i - a_1 t^{i+1} + \cdots - a_{n-i} t^n, \tag{6.1}$$

we see that, if $n - i \leq i - 2$, then the polynomial is written as a sum of monomials of increasing degree, and the coefficient of t^{n-i+1} in $p(t)$ is zero. However, such a coefficient cannot vanish, since it equals $(-1)^{i+1} b_{i+1}$. □

It follows from the above proof that the polynomial $p(t)$ with positive coefficients is divisible by $1 - (-t)^i$, whenever $\mathscr{L}$ contains a non-trivial central element of degree i. Notice that one can recognise the parity of i by looking at the roots of the polynomial $1 - (-t)^i$: The number i is even iff -1 is a root of that polynomial.

We are thus led to study the roots of the Poincaré polynomial of Koszul Lie algebras. This was one of the aims of T. Weigel's work [58]. We will recall here some of the results we are interested in.

Let $\mathscr{L}$ be a Lie algebra of type FP. Since the constant term of $P_{\mathscr{L}}(t)$ is non-zero, there are complex numbers $\lambda_1, \ldots, \lambda_n$ such that

$$P_{\mathscr{L}}(-t) = \prod_{i=1}^{n} (1 - \lambda_i t).$$

These complex numbers are called the **eigenvalues** of $\mathscr{L}$.

As noticed by Weigel in [58], there is a constraint on the real eigenvalues.

Lemma 6.6. *The real eigenvalues of a Lie algebra of type FP are positive.*

Proof. It follows from Descartes criterion on the sign of the roots of a real polynomial. □

From this, it follows:

Corollary 6.7. *Let $\mathscr{L}$ be a Koszul Lie algebra. Then, the center of $\mathscr{L}$ is concentrated in odd degrees.*

Proof. Let z be a non-trivial central element of $\mathscr{L}$ of degree i. Hence, as above, $1 - (-t)^i$ divides the Poincaré polynomial $P_{\mathscr{L}}(t)$. Now, by Lemma 6.6, all real eigenvalues of $\mathscr{L}$ are positive. It follows that $1 - (-t)^i$ has no negative root, and hence i must be odd. □

Putting all together, we have:

Theorem 6.8. *The center of any Koszul Lie algebra $\mathscr{L}$ is concentrated in odd degrees $< \operatorname{cd}\mathscr{L}/2+1$.*

For instance, the center of a Koszul Lie algebra of cohomological dimension ≤ 4 is concentrated in degree 1.

Notice that the definition of the eigenvalues still applies to arbitrary (not necessarily graded) algebras of type FP. We can thus give a partial answer to Question 2 of [58] (see also [59]).

Proposition 6.9. *Let A be an algebra of type FP and cohomological dimension $n \geq 1$. If all the eigenvalues of A are real, then*

$$b_2(A) \leq \frac{n-1}{2n} b_1(A)^2.$$

Notice that the formula relates the maximal number of minimal relations of A in terms of a minimal generating system and of its cohomological dimension.

In order to prove the result, notice that, if $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are the eigenvalues of A, then

$$b_1(A) = \sum_{i=1}^{n} \lambda_i,$$
$$b_2(A) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j.$$

In particular, Proposition 6.9 may be reduced to a numerical inequality of real numbers.

Lemma 6.10. *Let $n \geq 2$ be an integer and let d_i be real numbers, $1 \leq i \leq n$. Then,*

$$\sum_{i=2}^{n} \sum_{j=1}^{i-1} d_i d_j \leq \frac{n-1}{2n} \left(\sum_{i=1}^{n} d_i \right)^2$$

Proof. Denote by A_n the left-hand side of the inequality and let $B_n = \left(\sum_{i=1}^{n} d_i\right)^2$, so that we need to prove $2nA_n \leq (n-1)B_n$. By setting $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{R}^n$ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$, the Cauchy-Schwarz Inequality gives

$$\sum_{i=1}^{n} 1 \sum_{i=1}^{n} d_i^2 = ||\mathbf{1}||^2 \cdot ||d||^2 \geq (\mathbf{1} \cdot \mathbf{d})^2 = \left(\sum_{i=1}^{n} d_i \right)^2.$$

The latter can be rewritten as

$$n(B_n - 2A_n) \geq B_n$$

which provides the claim. □

Since A_n and $\sqrt{B_n}$ are elementary symmetric polynomials, the latter result may also be deduced from Newton's inequalities with $k = 1$.

Theorem 6.11 (Newton's inequalities)**.** *Let e_i be the ith elementary symmetric polynomial in some positive real numbers $a_1, \ldots, a_n$. Set $\binom{n}{i} S_i = e_i$. Then,*

$$S_{k-1}S_{k+1} \leq S_k^2,$$

that is

$$e_{k-1}e_{k+1} \leq \frac{k(n-k)}{(k+1)(n-k+1)} e_k^2.$$

As noticed in [58], there exist Koszul Lie algebras with some non-real eigenvalues. We now provide an example of a Koszul Lie algebra having complex eigenvalues with negative real parts.

Example 6.12. *Let Γ be the graph obtained as the disjoint union of a complete graph on 7 vertices and 8 pairwise non-adjacent vertices. Then, the RAAG Lie algebra $\mathscr{L}_\Gamma$ (see Section 13) is Koszul with Poincaré polynomial (equal to the clique polynomial of Γ)*

$$P_{\mathscr{L}_\Gamma}(t) = (1+t)^7 + 8t.$$

A numerical computation shows that some of the roots of $P_{\mathscr{L}_\Gamma}(-t)$ are approximately $1/\lambda_\pm \approx -0.02463 \pm 0.80986\, i$, and hence the eigenvalues $\lambda_\pm$ have negative real parts.

In fact, the set of complex roots of all clique polynomials is dense in $\mathbb{C}$, as it was proved in [6], which provides a negative answer to Question 2 of [59].

However, by Turán's Theorem [53], if a graph Γ on n vertices does not contain any $(r+1)$-clique, then, the number of edges of Γ does not exceed

$$\mathbf{e}_{\max} = \left(1 - \frac{1}{r}\right)\frac{n^2 - s^2}{2} + \binom{s}{2},$$

where $0 \leq s < r$ and $s \equiv n \mod r$. The number $\mathbf{e}_{\max}$ is the number of vedges of the Turán graph $T(n,r)$. In particular, as noticed in [58], if $r \geq 2$, then $\mathbf{e}_{\max} \leq \frac{r-1}{2r} n^2$, proving that for RAAG Lie algebras, the formula of Proposition 6.9 holds, despite their eigenvalues may be non-real.

7. Embedding into quadratic Lie algebras

So far, we have used HNN-extensions of Lie algebras several times as they apply well to the study of quadratic Lie algebras. Classically, group theoretic HNN-extensions have prolific applications in embedding results, like Hall's theorem, which asserts that every countable group embeds into a finitely generated simple group (see [14]). In the same fashion, and with the same tools, a work of Lichtman and Shirvani [22] shows that the same phenomenon happens in the Lie algebra case. However, in the positively-graded case, one cannot expect to achieve a similar result, as all non-abelian $\mathbb{N}$-graded Lie algebras have proper ideals (e.g., the commutator subalgebra), so that there is no simple Lie algebra of that kind.

However, motivated by Example 6.3, we could prove that graded Lie algebras embed into quadratic Lie algebras, under suitable assumptions.

Theorem 7.1. *Every finitely presented positively-graded Lie algebra can be embedded into some quadratic Lie algebra.*

First of all, we show that one can get rid of the high-degree generators of a finitely generated, non-standard Lie algebra, and embed it into a standard one.

Lemma 7.2. *Let $\mathscr{L}$ be a finitely generated $\mathbb{N}$-graded Lie algebra. Then, $\mathscr{L}$ is a homogeneous subalgebra of a (finitely generated) standard Lie algebra.*

Proof. Up to taking the direct product of $\mathscr{L}$ with any standard Lie algebra, we may suppose that $\mathscr{L}_1 \neq 0$. Let $\{x_i^n : (i,n) \in I\}$ be a minimal homogeneous generating system of $\mathscr{L}$, where $x_i^n \in \mathscr{L}_n$. We argue by induction on the maximal N for which there is some generator x_i^N of degree N.

If $N = 1$, then there is nothing to prove.

Assume that $N > 1$. Since $\mathscr{L}_1 \neq 0$, pick $x \in \mathscr{L}_1 \setminus \{0\}$ and consider, for all $(i,N) \in I$, the derivations $\phi_i : \langle x\rangle \to \mathscr{L}$ sending x to x_i^N; such maps are homogeneous of degree $N-1$. Then, the iterated HNN-extension $\mathscr{H}$ of $\mathscr{L}$ with respect to all the derivations ϕ_i, $(i,N) \in I$, is a Lie algebra containing $\mathscr{L}$ and generated by the elements x_i^n, for $(i,n) \in I$ and $i < N$, and by the stable letters t_i of degree $N-1$. By induction, $\mathscr{H}$ embeds into a standard Lie algebra, and hence $\mathscr{L}$ does. □

Notice that the standard Lie algebra $\mathscr{S}$ containing $\mathscr{L}$, as constructed in Lemma 7.2, satisfies $\dim H^{2,j}(\mathscr{S}) = \dim H^{2,j}(\mathscr{L})$ for all $j \geq 3$, provided that $\mathscr{L}_1 \neq 0$. Indeed, if one puts $\mathscr{L}^1 = \mathrm{HNN}_{\phi_1}(\mathscr{L}, t_1)$ and $\mathscr{L}^{i+1} = \mathrm{HNN}_{\phi_{i+1}}(\mathscr{L}^i, t_{i+1})$, then

$$\mathscr{S} = \bigcup_{i:(i,N)\in I} \mathscr{L}^i$$

and one has exact sequences

$$H^{1,j-N+1}(\langle x\rangle) \to H^{2,j}(\mathscr{L}^{i+1}) \to H^{2,j}(\mathscr{L}^i) \to H^{2,j-N+1}(\langle x\rangle) = 0$$

for every j. If $j \geq N+1$, then $H^{1,j-1}(\langle x\rangle) = 0$ and hence $H^{2,j}(\mathscr{L}^i) \simeq H^{2,j}(\mathscr{L}^{i+1})$. In particular, $\mathscr{S}$ has the same number of relations of all degrees $\geq N+1$ as $\mathscr{L}$, but more relations of degree N, where N is the degree of the generator we want to get rid of. Eventually, if $\mathscr{L}$ is finitely presented, then the same holds for $\mathscr{S}$.

Proof of Theorem 7.1. By Lemma 7.2, we may assume $\mathscr{L}$ to be a standard finitely presented Lie algebra. Let d be the maximal degree of the minimal relations of $\mathscr{L}$ and assume that $d \geq 3$.

Let $r = \sum_i [x_i, a_i]$ be a relation of degree d of $\mathscr{L}$, where $(x_i)_{1\leq i\leq n}$ is a minimal generating system and the a_i's are homogeneous elements of degree $d-1$.

Consider the direct sum $\mathscr{Q}^1 = \mathscr{L} \sqcap k$, where the abelian Lie algebra k is generated by the degree-1 element t_1.

One can define, for $2 \leq i \leq n$, the degree-$(d-1)$ derivations $\phi_i : t_1 \mapsto a_i$ and put

$$\mathscr{Q}^2 = \mathrm{HNN}_{\phi_i}(\mathscr{Q}^1, s_i)$$

for the multiple HNN-extension of $\mathscr{Q}^1$ with respect to the derivations ϕ_i, $2 \leq i \leq n$.

Finally, let $\psi : \mathrm{Span}\{t_1, x_1\} \to \mathscr{Q}^2$ be the linear map defined by $t_1 \mapsto a_1$ and $x_1 \mapsto \sum_{i=2}^n [x_i, s_i]$. Such map is a derivation of the abelian Lie algebra $\langle t_1, x_1\rangle$ into $\mathscr{Q}^2$, as

$$\begin{aligned}
[x_1, \psi(t_1)] + [\psi(x_1), t_1] &= [x_1, a_1] + \sum_{2\leq i\leq n} [[x_i, s_i], t_1] = \\
&= [x_1, a_1] + \sum_{2\leq i\leq n} ([[x_i, t_1], s_i] + [x_i, [s_i, t_1]]) = \\
&= [x_1, a_1] + 0 + \sum_{2\leq i\leq n} [x_i, a_i] = \sum_{1\leq i\leq n} [x_i, a_i] = 0.
\end{aligned}$$

Hence, the HNN-extension $\mathscr{Q} = \mathrm{HNN}_{\psi}(\mathscr{Q}^2, t_2)$ has one relation in degree d less than $\mathscr{L}$. Plus, $\mathscr{Q}$ has no relations of degree $> d$, but it is not 1-generated as the elements s_i are minimal generators of degree $d-2$. Notice that $\mathscr{Q}$ contains $\mathscr{L}$. By Lemma 7.2, we get a standard Lie algebra $\mathscr{L}^{(r)}$ containing $\mathscr{Q}$, and hence $\mathscr{L}$, with no relations of degree $> d$ and with no more relations of degree d than $\mathscr{Q}$. Indeed, as noticed above, $H^{2,j}(\mathscr{L}^{(r)}) \simeq H^{2,j}(\mathscr{Q})$ for every $j \geq d$.

Since $\mathscr{L}$ is a standard finitely presented Lie algebra with relations of degree $\leq d$ only, one can thus proceed by induction. □

Notice that the so obtained quadratic Lie algebra is far from being the minimal quadratic Lie algebra containing $\mathscr{L}$.

For an explicit example of an embedding of a non-quadratic Lie algebra into a quadratic one, we refer to Example 6.3.

Question 7.3. *If $\mathscr{L}$ is a quadratic algebra, does there exist a Koszul Lie algebra containing it?*

8. Stallings decomposition theorem

John R. Stallings proved in [49] a structure theorem for finitely generated groups with infinitely many ends. The number of ends $e(G)$ of a finitely generated group G is either 0, 1, 2 or ∞. In particular, it is 0 iff G is finite. If $e(G) \neq 0$, then the number of ends has a cohomological interpretation, namely $e(G) = 1 + \mathrm{rk}_{\mathbb{Z}} H^1(G, \mathbb{Z}G)$.

Theorem 8.1 (Stallings decomposition theorem)**.** *Let G be an infinite finitely-generated group. Then G has infinitely many ends, i.e., $H^1(G, \mathbb{Z}G)$ has infinite rank, iff either*

(1) *G is the HNN-extension of a proper subgroup with respect to finite subgroups of index ≥ 2, or*
(2) *G is the free product of two proper subgroups amalgamated over a common finite subgroup of index > 2 in at least one of the groups.*

In particular, if G is torsion-free, then it has infinitely many ends iff it splits as the free product of two proper subgroups.

As a consequence, Stallings proved the celebrated Stallings-Swan theorem on groups of cohomological dimension 1.

One may wonder whether an analogue of Theorem 8.1 holds for Lie algebras. Notice that the natural G-module $\mathbb{Z}G$ would be replaced with the universal envelope of the Lie algebra, and that finite groups have no Lie algebra counterpart, since universal envelopes are (almost) always infinite dimensional, even if the Lie algebra is finite dimensional. Hence, one may try to translate the result for torsion-free groups to the Lie algebra world.

Though, in [18], it is proved that, even in the graded case, the Lie theoretic Stallings theorem is not true: There exists a finitely generated graded Lie algebra with infinite dimensional $H^1(\mathscr{L}, \mathscr{U}(\mathscr{L}))$ and that is not a free product. Here $\mathscr{U}(\mathscr{L})$ is meant to be endowed with the $\mathscr{L}$-action induced by the regular left-representation of $\mathscr{U}(\mathscr{L})$.

We start by showing that all free products of Lie algebras have "infinitely many ends".

Proposition 8.2. *Let $\mathscr{L} = \mathscr{A} \amalg \mathscr{B}$ be a free product of non-trivial standard Lie algebras. Then, $H^1(\mathscr{L}, \mathscr{U}(\mathscr{L}))$ has infinite dimension.*

Proof. Consider the exact sequence of Theorem 3.12-(1)

$$0 \to \mathscr{U}(\mathscr{L}) \to \operatorname{ind}_{\mathscr{A}}^{\mathscr{L}} k \oplus \operatorname{ind}_{\mathscr{B}}^{\mathscr{L}} k \to k \to 0.$$

By applying the functor $\operatorname{Ext}^{\bullet}_{\mathscr{U}(\mathscr{L})}(\_, \mathscr{U}(\mathscr{L}))$ to the above sequence, one gets an exact sequence

$$\operatorname{Hom}_{\mathscr{L}}(\operatorname{ind}_{\mathscr{A}}^{\mathscr{L}} k \oplus \operatorname{ind}_{\mathscr{B}}^{\mathscr{L}} k, \mathscr{U}(\mathscr{L})) \to \operatorname{Hom}_{\mathscr{L}}(\mathscr{U}(\mathscr{L}), \mathscr{U}(\mathscr{L})) \to {\operatorname{Ext}^1}_{\mathscr{U}(\mathscr{L})}(k, \mathscr{U}(\mathscr{L}))$$

By Eckmann-Shapiro Lemma 2.9, it reads

$$\operatorname{Hom}_{\mathscr{A}}(k, \mathscr{U}(\mathscr{L})) \oplus \operatorname{Hom}_{\mathscr{B}}(k, \mathscr{U}(\mathscr{L})) \to \operatorname{Hom}_{\mathscr{L}}(\mathscr{U}(\mathscr{L}), \mathscr{U}(\mathscr{L})) \to H^1(\mathscr{L}, \mathscr{U}(\mathscr{L}))$$

Now, by the PBW theorem, $\mathscr{U}(\mathscr{L})$ has no zero divisor, so that $\operatorname{Hom}_{\mathscr{U}(\mathscr{L})}(k, \mathscr{U}(\mathscr{L}))$ is zero, as well as $\operatorname{Hom}_{\mathscr{U}(\mathscr{A})}(k, \mathscr{U}(\mathscr{L}))$ and $\operatorname{Hom}_{\mathscr{U}(\mathscr{B})}(k, \mathscr{U}(\mathscr{L}))$ (cf. Lemma 10.2). Thus, $H^1(\mathscr{L}, \mathscr{U}(\mathscr{L}))$ contains $\operatorname{Hom}_{\mathscr{U}(\mathscr{L})}(\mathscr{U}(\mathscr{L}), \mathscr{U}(\mathscr{L})) \simeq \mathscr{U}(\mathscr{L})$ and hence it has infinite dimension. □

Notice that, since $\mathscr{L}$ is 1-generated, there exists a $\mathscr{U}(\mathscr{L})$-resolution of k given by

$$\cdots \to P_2 \xrightarrow{\partial_2} \mathscr{U}(\mathscr{L}) \otimes \mathscr{L}_1 \xrightarrow{\partial_1} \mathscr{U}(\mathscr{L}) \xrightarrow{\varepsilon} k \to 0.$$

Since, under the hypotheses of Proposition 8.2, the sequence

$$0 \to \mathscr{U}(\mathscr{L}) \to \operatorname{ind}_{\mathscr{A}}^{\mathscr{L}} k \oplus \operatorname{ind}_{\mathscr{B}}^{\mathscr{L}} k \to k \to 0$$

is exact, by the comparison theorem there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \mathscr{U}(\mathscr{L}) \otimes \mathscr{L}_1 & \xrightarrow{\partial_1} & \mathscr{U}(\mathscr{L}) & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow \xi & & \downarrow \mu & & \| & & \\
0 & \longrightarrow & \mathscr{U}(\mathscr{L}) & \xrightarrow{\pi} & M_\xi & \longrightarrow & k & \longrightarrow & 0
\end{array}$$

where $M_\xi = \operatorname{ind}_{\mathscr{A}}^{\mathscr{L}} k \oplus \operatorname{ind}_{\mathscr{B}}^{\mathscr{L}} k := A \oplus B$, and the two $\mathscr{U}(\mathscr{L})$-maps $\mu : \mathscr{U}(\mathscr{L}) \to M_\xi$ sending $1 \mapsto (1 \otimes 1, 0)$, and $\xi : \mathscr{U}(\mathscr{L}) \otimes \mathscr{L}_1 \to \mathscr{U}(\mathscr{L})$ sending $1 \otimes x \mapsto x^{\mathscr{B}}$, where $x^{\mathscr{B}}$ denotes the projection onto $\mathscr{B}_1$. Finally, $\pi(1) = (1 \otimes 1, 1 \otimes 1) \in A \oplus B$.

Theorem 8.3 (Decomposition theorem). *Let $\mathscr{L}$ be a standard Lie algebra acting on its universal enveloping algebra via the regular representation of $\mathscr{U}(\mathscr{L})$ itself. Suppose that $\mathscr{L}$ admits a* non-trivial *derivation $d : \mathscr{L} \to \mathscr{U}(\mathscr{L})$ of degree 0 satisfying:*

1. *$d(d(x)) = d(x)$, for every $x \in \mathscr{L}_1$, and*
2. *$\ker d \cap \mathscr{L}_1 \neq 0$.*

Then $\mathscr{L}$ is the free product of two proper 1-generated Lie subalgebras. More precisely, if $\mathscr{A} = \langle \ker d \cap \mathscr{L}_1 \rangle$ and $\mathscr{B} = \langle \operatorname{Im} d \cap \mathscr{L}_1 \rangle$, then $\mathscr{L} = \mathscr{A} \amalg \mathscr{B}$.

Proof. By classical computations, we see that the space of outer derivation of $\mathscr{L}$, namely the quotient $\operatorname{Der}(\mathscr{L}, \mathscr{U}(\mathscr{L}))/\operatorname{Ider}(\mathscr{L}, \mathscr{U}(\mathscr{L}))$, is isomorphic with the cohomology group $\operatorname{Ext}^1_{\mathscr{U}(\mathscr{L})}(k, \mathscr{U}(\mathscr{L}))$, which is a quotient of $\operatorname{Hom}_{\mathscr{U}(\mathscr{L})}(\mathscr{U}(\mathscr{L}) \otimes \mathscr{L}_1, \mathscr{U}(\mathscr{L}))$, where the isomorphism is induced by sending any derivation d to $\xi : 1 \otimes x \mapsto d(x)$.

Now, let d be as in the hypothesis and denote by ξ the associated cocycle $1 \otimes x \mapsto d(x)$. By hypothesis, d is an outer derivation, i.e., $d \notin \operatorname{Ider}(\mathscr{L}, \mathscr{U}(\mathscr{L}))$, and ξ is not a coboundary, i.e., its cohomology class is non-trivial.

Consider a resolution of the trivial $\mathscr{U}(\mathscr{L})$-module of the following shape:

$$\cdots \to P_2 \to P_1 = \mathscr{U}(\mathscr{L}) \otimes \mathscr{L}_1 \to P_0 = \mathscr{U}(\mathscr{L}) \to k \to 0.$$

The map $\xi : \mathscr{U}(\mathscr{L}) \otimes \mathscr{L}_1 \to \mathscr{U}(\mathscr{L})$ yields the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \mathscr{U}(\mathscr{L}) \otimes \mathscr{L}_1 & \xrightarrow{\partial_1} & \mathscr{U}(\mathscr{L}) & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow \xi & & \downarrow \mu & & \| & & \\
0 & \longrightarrow & \mathscr{U}(\mathscr{L}) & \xrightarrow{\pi} & M_\xi & \longrightarrow & k & \longrightarrow & 0
\end{array}$$

where ∂_1 denotes the multiplication mapping of $\mathscr{U}(\mathscr{L})$, M_ξ is the pushout of the left-hand square, i.e.,

$$M_\xi = \frac{\mathscr{U}(\mathscr{L}) \oplus \mathscr{U}(\mathscr{L})}{R},$$

R being the submodule of $\mathscr{U}(\mathscr{L}) \oplus \mathscr{U}(\mathscr{L})$ generated by

$$(\xi(v), -\partial_1(v)), \quad v \in \mathscr{U}(\mathscr{L}) \otimes \mathscr{L}_1,$$

and $\pi(x) = (x, 0) + R$, $\mu(x) = (0, x) + R$.

Let $f = d|_{\mathscr{L}_1} : \mathscr{L}_1 \to \mathscr{L}_1$ be the degree 1 component of d. As f is idempotent, if $\mathscr{A}_1 = \ker f$ and $\mathscr{B}_1 = \operatorname{Im} f$, then

$$\mathscr{L}_1 = \mathscr{A}_1 \oplus \mathscr{B}_1.$$

By setting $v = 1 \otimes x$ and $x = a + b$, with $a \in \mathscr{A}_1$ and $b \in \mathscr{B}_1$, one can see that R is generated by elements $(f(a+b), -a-b) = (f(b), -a-b)$. As $\mathscr{B}_1 = \operatorname{Im} f$, if $b = f(y)$, $y \in \mathscr{L}_1$, one has that $b = f(y) = f(f(y)) = f(b)$, and hence R is the submodule

$$R = \left\langle (b, -a-b) \,\middle|\, a \in \mathscr{A}_1,\, b \in \mathscr{B}_1 \right\rangle.$$

In particular, $R_0 = 0$, whence $(M_\xi)_0 \simeq k \oplus k$.

Now, for $a \in \mathscr{A}_1$, $b \in \mathscr{B}_1$,

$$(\pi - \mu)(a + b) = (a + b, -a - b) + R = (a, 0) + R$$

and

$$\mu(a + b) = (0, a + b) + R = (-b, 0) + R.$$

Consider the $\mathscr{U}(\mathscr{L})$-submodules $A = \operatorname{Im} \mu$ and $B = \operatorname{Im}(\pi - \mu)$ of M_ξ and notice that $A_1 = \mu(\mathscr{B}_1)$ and $B_1 = (\pi - \mu)\mathscr{A}_1$.

We claim that $M_\xi = A \oplus B$.

First of all, $M_\xi = A + B$, since the generators of M_ξ can be obtained as follows

$$(1, 0) + R = (\pi - \mu)(1) + \mu(1),$$

and

$$(0, 1) + R = \mu(1).$$

Now, let $x, y \in \mathscr{U}(\mathscr{L})$ be such that $(\pi - \mu)(x) = \mu(y)$ in M_ξ.

Since π and μ are degree 0 linear maps, x and y must have the same degree. If $|x| = |y| = 0$, then $x = y = 0$, as $R_0 = 0$.

Assume $|x| = |y| > 0$. Using PBW theorem, decompose x and y into products of elements of degree 1 and denote by $v^{\mathscr{A}}$ (resp. $v^{\mathscr{B}}$) the sum of monomials of v ending with an element of $\mathscr{A}_1$ (resp. $\mathscr{B}_1$), for $v \in \{x, y\}$. We can assume that $x^{\mathscr{A}}$ does not belong to the ideal generated by $\mathscr{B}_1$ in case it is non-zero, i.e., $x^{\mathscr{A}} = 0$ or $x^{\mathscr{A}} \in \mathscr{U}(\mathscr{L})\mathscr{A}_1 \setminus \mathscr{U}(\mathscr{L})\mathscr{B}_1$.

Then,

$$(\pi - \mu)(x) - \mu(y) = (x^{\mathscr{A}} + x^{\mathscr{B}}, -x^{\mathscr{A}} - x^{\mathscr{B}} - y^{\mathscr{A}} - y^{\mathscr{B}}) \in R$$

which implies

$$(x^{\mathscr{A}}, -y^{\mathscr{B}}) \in R.$$

Since $x^{\mathscr{A}}$ belongs to $\mathscr{U}(\mathscr{L})\mathscr{B}_1$, it must be zero. Finally, $(y^{\mathscr{B}}, 0) \in R$ implies that there is some $v \in \mathscr{U}(\mathscr{L}) \otimes \mathscr{L}_1$ such that $\xi(v) = y^{\mathscr{B}}$ and $\partial_1(v) = 0$. Then there is some $w \in P_2$ such that $v = \partial_2(w)$, and hence $y^{\mathscr{B}} = \xi(v) = \xi\partial_2(w) = 0$, since ξ is a cocycle. It follows that $\mu(y) = (0, y^{\mathscr{A}}) + R = 0$.

We recover the short exact sequence

$$0 \to \mathscr{U}(\mathscr{L}) \to A \oplus B \to k \to 0$$

where the first map is defined by sending an element $v \in \mathscr{U}(\mathscr{L})$ to the pair consisting of the images of v into the quotients A and B of $\mathscr{U}(\mathscr{L})$, and the second map is defined by

$$(1, 0) \mapsto 1, \text{ and } (0, 1) \mapsto -1$$

We want to show that $A = \operatorname{ind}_{\mathscr{A}}^{\mathscr{L}} k$ and $B = \operatorname{ind}_{\mathscr{B}}^{\mathscr{L}} k$, so that, by Proposition 3.17, $\mathscr{L} = \mathscr{A} \amalg \mathscr{B}$. In order to do this, it is enough to prove that the sequence

$$\mathscr{U}(\mathscr{L}) \otimes \mathscr{A}_1 \xrightarrow{\partial_1} \mathscr{U}(\mathscr{L}) \xrightarrow{\mu} A \to 0$$

is exact (as well as the analogue sequence for B). Indeed, this proves that $A = \mathscr{U}(\mathscr{L})/(\mathscr{U}(\mathscr{L})\mathscr{A}_1) = \mathscr{U}(\mathscr{L})/(\mathscr{U}(\mathscr{L})\mathscr{U}(\mathscr{A})_+) = \operatorname{ind}_{\mathscr{A}}^{\mathscr{L}} k$, where $\mathscr{A} = \langle \mathscr{A}_1 \rangle$.

Let us compute $\ker \mu$. Let $x \in \mathscr{U}(\mathscr{L})$ and suppose that $\mu(x) = 0$. By the PBW theorem, since $\mathscr{L}$ is 1-generated, one can write x as the sum of two elements $x^{\mathscr{A}}$ and $x^{\mathscr{B}}$ that are linear combinations of products of 1-degree elements ending with an element of $\mathscr{A}_1$ and $\mathscr{B}_1$ respectively. We can also assume that $x^{\mathscr{A}} \notin \mathscr{U}(\mathscr{L})\mathscr{B}_1$ whenever $x^{\mathscr{A}} \neq 0$. Then $R = \mu(x) = (0, x) + R = (0, x^{\mathscr{A}} + x^{\mathscr{B}}) + R = (x^{\mathscr{B}}, 0) + R$. As before, this implies that $x^{\mathscr{B}} = 0$, and hence $x = x^{\mathscr{A}} \in \partial_1(\mathscr{U}(\mathscr{L}) \otimes \mathscr{A}_1)$.

Also, $0 = (\pi - \mu)(x) = (x^{\mathscr{A}} + x^{\mathscr{B}}, -x^{\mathscr{A}} - x^{\mathscr{B}}) + R = (x^{\mathscr{A}}, 0) + R$ implies $x = x^{\mathscr{B}} \in \partial_1(\mathscr{U}(\mathscr{L}) \otimes \mathscr{B}_1)$. $\square$

§ II. Poincaré duality Lie algebras

A form of Poincaré duality was first stated by Henri Poincaré in 1893. It was expressed in terms of Betti numbers, i.e., the dimensions of the (co)homology groups: the kth and the $(n-k)$th real Betti numbers of a closed orientable n-manifold are equal.

In this chapter we will consider Poincaré duality for graded Lie algebras, and we will provide some results concerning BK Lie algebras which satisfy such a homological property. Notice that the abelian Lie algebra of dimension 1 is BK and its 0th and 1st cohomology groups have the same dimension. It is a first example, though trivial, of a Poincaré duality Lie algebra.

Example 8.4. *Let A be a non-negatively graded connected algebra and let n be a natural number. If $D = \mathrm{Ext}^i_A(k, A) = 0$ for every $i \neq n$, we say that A is a **duality algebra** of dimension n. If D is 1-dimensional, A is a **Poincaré duality algebra**. A graded Lie algebra $\mathscr{L}$ is a Poincaré duality Lie algebra if $\mathscr{U}(\mathscr{L})$ is so.*

We will simply say that $\mathscr{L}$ is PD^n if the Lie algebra is of Poincaré duality of dimension n. PD^n Lie algebras are also called Gorenstein (see [47]). Notice that, unlike the group-theoretic case, in the graded framework, there is no non-trivial action of an algebra on a 1-dimensional vector space, and hence the notion of non-orientable Poincaré duality lacks of meaning.

We will say that a standard algebra A **satisfies PD^n** if A_n is 1-dimensional and the multiplication of A induces perfect (or non-degenerate) pairings

$$A_i \otimes A_{n-i} \to A_n$$

for all $0 \leq i \leq n$.

9. Surface Lie algebra

Denote by $\mathscr{G}_{2d}$ the Lie algebra with presentation

$$\mathscr{G}_{2d} = \langle x_1, y_1, \dots, x_d, y_d \mid \sum_{1\leq i\leq d} [x_i, y_i]\rangle_{Lie}.$$

We call it a **surface Lie algebra**. It can be shown that $\mathscr{G}_{2d}$ is the graded k-Lie algebra associated to the lower central series of the surface groups, whence its name. If $k = \mathbb{F}_p$, p odd, the p-restrictification of $\mathscr{G}_{2d}$ is the restricted Lie algebra associated with the Zassenhaus p-filtration of any $2d$-generated Demuškin group (see [38]). The latter groups play an interesting role in Galois Theory, since they appear as maximal pro-p Galois groups of local number fields containing a primitive pth root of unity. So far, Demuškin groups are the only known groups that appear as "bulding blocks" (in the point of view Efrat's Elementary Type Conjecture) of maximal pro-p Galois groups and that are not uniform groups.

The universal envelope of $\mathscr{G}_{2d}$ is given by the same presentation in the category of (associative) k-algebras, i.e.

$$A = \mathscr{U}(\mathscr{G}_{2d}) = \langle x_1, y_1, \dots, x_d, y_d \mid \sum_{1\le i\le d} [x_i, y_i]\rangle_{k-alg}$$

We want to show that $\mathscr{G}_{2d}$ is a Bloch-Kato Lie algebra and its cohomology satisfies a version of Poincaré duality. In doing so, we will show several general results for (Poincaré) duality algebras. Notice that for $d = 1$, $\mathscr{G}_2$ is the abelian Lie algebra of dimension 2.

The following is clearly a *free resolution* of k over A:

$$[F_\bullet \to k] = \cdots \to 0 \to A \xrightarrow{d_1} A^{2d} \xrightarrow{d_0} A \xrightarrow{\varepsilon} k \to 0$$

where

$$d_0 : (a_1, b_1, \dots, a_d, b_d) \mapsto \sum a_i x_i + b_i y_i,$$
$$d_1 : a \mapsto a(y_1, -x_1, \dots, y_d, -x_d)$$

and

$$\varepsilon : x_i, y_i \mapsto 0,\ 1 \mapsto 1.$$

That resolution is linear, whence, $\mathscr{L}$ (and A) is Koszul of cohomological dimension $\operatorname{cd}\mathscr{L} = 2$.
In order to compute the cohomology $\operatorname{Ext}^\bullet_A(k, A)$ we apply $\operatorname{Hom}_A(\_, A)$ to $F_\bullet$. Since $\operatorname{Hom}_A(A^r, A) \simeq A^r$ as a (right) A-module, we get the associated cochain complex

$$A \xrightarrow{d_0^*} A^{2d} \xrightarrow{d_1^*} A \to 0 \to \dots,$$

with

$$d^0 = d_0^* : a \mapsto (x_1, y_1, \dots, x_d, y_d)a,$$
$$d^1 = d_1^* : (a_1, b_1, \dots, a_d, b_d) \mapsto \sum (y_i a_i - x_i b_i).$$

This shows that

$$\operatorname{Ext}^2_A(k, A) \simeq k$$

with the right action of A induced by the augmentation map $\varepsilon : A \to k$. Moreover, $\operatorname{Ext}^i_A(k, A) = 0$ for $i \neq 2$, whence A, and thus $\mathscr{G}_{2d}$, satisfies Poincaré-duality in dimension 2.

Proposition 9.1 (Duality in cohomology algebras)**.** *Let A be a duality algebra of dimension n and of type FP.*

Let D be the right A-module $\operatorname{Ext}^n_A(k, A) = H^n(A, A)$. Then for every left A-module M, the following homological duality holds

$$H^i(A, M) \simeq \operatorname{Tor}^A_{n-i}(D, M).$$

Proof. Notice that one has $\operatorname{cd} A = n$ since A is of type FP. Let $P_\bullet = (P_i)_{0 \le i \le n}$ be a finite projective resolution of k over A, i.e. the sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to k \to 0$$

is exact and $P_\bullet$ is a finitely generated projective A-module. Let $\bar{P}^{-i} = \operatorname{Hom}_A(P_i, A)$ be the dual complex. Thus, $\bar{P}^\bullet$ is projective. The sequence $\bar{P}^0 \to \cdots \to \bar{P}^{-n}$ is exact, as $H_{-i}(\bar{P}^\bullet) = H^i(A, A) = 0$, for $i \neq n$. Moreover, $\operatorname{Ext}^n_A(k, A) = H_{-n}(\bar{P}^\bullet) = \bar{P}^{-n}/im(\bar{P}^{-n+1} \to \bar{P}^{-n})$, and thus we have a projective A-resolution for $D = H^n(A, A) = \operatorname{Ext}^n_A(k, A)$

$$\cdots \to \bar{P}^{-(n-1)} \to \bar{P}^{-n} \to D;$$

more precisely, $\bar{P}^{n+\bullet}$ is the projective resolution. For every i and every A-module M, $\operatorname{Hom}_A(P_i, M) \simeq \bar{P}^{-i} \otimes_A M$, for $P_\bullet$ is finitely generated and projective. Finally,

$$\begin{aligned} H^i(A, M) &= \operatorname{Ext}^i_A(k, M) = H_{-i}(\operatorname{Hom}_A(P_\bullet, M)) \\ &\simeq H_{-i}(\bar{P}^\bullet \otimes_A M) = H_{n-i}(\bar{P}^{n+\bullet} \otimes_A M) \\ &= \operatorname{Tor}^A_{n-i}(D, M) \end{aligned}$$

i.e.

$$H^i(A, M) = \operatorname{Ext}^i_A(k, M) \simeq \operatorname{Tor}^A_{n-i}(D, M)$$

□

It follows that, for the surface associative algebra $A = \mathscr{U}(\mathscr{G}_{2d})$, it holds

$$\begin{aligned} \operatorname{Hom}_A(k, \_) = \operatorname{Ext}^0_A(k, \_) &\simeq \operatorname{Tor}^A_2(k, \_) \\ \operatorname{Ext}^1_A(k, \_) &\simeq \operatorname{Tor}^A_1(k, \_) \\ \operatorname{Ext}^2_A(k, \_) &\simeq \operatorname{Tor}^A_0(k, \_) = k \otimes_A \_ \end{aligned}$$

But since the vector space dual functor $\_^* = \operatorname{Hom}_k(\_, k)$ is exact on finite dimensional vector spaces, one has

$$\operatorname{Tor}^A_2(k, k) \simeq \operatorname{Ext}^2_A(k, k)^*. \tag{9.1}$$

Let now $\mathscr{M}$ be a proper Lie subalgebra of $\mathscr{L}$; we want to show that $\mathscr{M}$ has cohomological dimension at most 1, and thus it is free. This finally proves $\mathscr{L}$ to be BK.

In order to do that, we compute $H^2(\mathscr{M}, k) \simeq H_2(\mathscr{M}, k)$ (see also Lemma 10.2). By applying Eckmann-Shapiro Lemma to the subalgebra $B = \mathscr{U}(\mathscr{M})$ of A, we get

$$\operatorname{Tor}^B_2(k, k) \simeq \operatorname{Tor}^A_2(k, A \otimes_B k).$$

Therefore, for the duality relations of A,

$$\operatorname{Tor}^B_2(k, k) \simeq \operatorname{Ext}^0_A(k, A \otimes_B k) = \operatorname{Hom}_A(k, A \otimes_B k).$$

It follows from PBW theorem that the latter space is trivial, namely, the A-module $A \otimes_B k$ has no A-fixed points. Indeed, let $\alpha : k \to A \otimes_B k$ be a A-linear map. As k is a simple A-module, the map α is determined by $\alpha(1)$. Complete a k-basis $\{y_i,\ i \in I\}$ of $\mathscr{M}$ to a k-basis $\{y_i, z_j,\ i \in I,\ j \in J\}$ for $\mathscr{L}$. By PBW theorem, B has k-basis $\{y_{i_1} \cdots y_{i_n} |\ i_1, \ldots, i_n \in I,\ n \in \mathbb{N}_0\}$ and it is infinite-codimensional in A, whose k-basis can be suitably chosen to be $\{z_{j_1} \cdots z_{j_m} y_{i_1} \cdots y_{i_n} |\ i_1, \ldots, i_n \in I,\ j_1 \ldots, j_m \in J,\ m, n \in \mathbb{N}_0, m \neq 0\}$. Note now that the induced module can be written as the following quotient of A:

$$A \otimes_B k \simeq A/AB_+ =: A_B,$$

via the map $a \otimes x \mapsto ax = xa$. With respect to the above isomorphism, if $\alpha(1) = u + AB_+$, one can choose $u \in A$ to be of the form

$$u = \sum r_{j_1,\ldots,j_r} z_{j_1} \cdots z_{j_r}.$$

Finally,

$$0 = \alpha(z_j \cdot 1) = z_j \alpha(1),\ j \in J$$

implies $\sum r_{j_1,\ldots,j_r} z_j z_{j_1} \cdots z_{j_r} \in AB_+$, i.e., $u = 0$.

We conclude that $\operatorname{cd} \mathscr{M} < 2 = \operatorname{cd} \mathscr{L}$, and thus the Lie algebra $\mathscr{L} = \mathscr{G}_{2d}$ is Bloch-Kato, for graded Lie algebras of cohomological dimension ≤ 1 are free by Proposition 2.10. In particular, surface Lie algebras satisfy a version of the Freiheitssatz. One can see that a converse statement holds (cf. Proposition. 9.6). So we have

Theorem 9.2. *The surface Lie algebra $\mathscr{G}_{2d}$ is Bloch-Kato.*

From the fact that the proper Lie subalgebras of $\mathscr{G}_{2d}$ are free we deduce the following.

Corollary 9.3. *If $d > 1$ then the graded Lie algebra $\mathscr{G}_{2d}$ is centerless and* $\amalg$-*indecomposable.*

Proof. Suppose $\mathscr{G}_{2d} = \mathscr{L} \amalg \mathscr{M}$ is a proper free product. We proved that both $\mathscr{L}$ and $\mathscr{M}$ are free Lie algebras and thus $\mathscr{G}_{2d}$ should be free. Alternatively, it follows from Proposition 8.2 and the fact that $H^1(\mathscr{G}_{2d}, \mathscr{U}(\mathscr{G}_{2d})) = 0$.

Suppose now there exists a central element $x \neq 0$ in $\mathscr{G}_{2d}$. Pick an element $y \in \mathscr{G}_{2d}$ that is linear independent with x. The Lie subalgebra of $\mathscr{G}_{2d}$ generated by x and y is 2-dimensional, and hence it is not free. $\square$

From the fact that $\mathscr{G}_{2d}$ is a Koszul Lie algebra, one can easily compute its cohomology algebra by taking its Koszul dual.

$$H^\bullet(\mathscr{G}_{2d}) = \mathscr{U}(\mathscr{G}_{2d})^! = \left\langle \xi_i, \eta_i :\ i = 1, \ldots, d \,\middle|\, \xi_i \wedge \xi_j,\ \eta_i \wedge \eta_j,\ \xi_i \wedge \eta_j - \delta_{ij} \xi_1 \wedge \eta_1 \right\rangle$$

that is, the only non-trivial products are the $\xi_i \eta_i$'s and the $\eta_j \xi_j$'s, and they are all the same up to a sign.

9.1. **Quadratic one-relator Lie algebras.** Let k be a field of characteristic different from 2. Let A be a standard graded-commutative k-algebra. Suppose $\dim A_2 = 1$ and $A_r = 0$, $r \geq 3$. The multiplication map restricts to a bilinear form

$$\beta : A_1 \times A_1 \to A_2 = k.$$

Consider the **radical** of β,

$$R = \operatorname{rad}\beta = \left\{ a \in A_1 \;\middle|\; \beta(a,b) = 0,\ \forall b \in A_1 \right\}$$

Let $C_1 \subseteq A_1$ be a complement of the k-subspace R in A_1, i.e. $A_1 = R \oplus C_1$. In particular, the restriction of β to C_1 defines a symplectic form, i.e. it is a non-degenerate (skew-symmetric) pairing, and the subalgebra C generated by C_1 satisfies PD^2. Consider also the subalgebra B of A generated by R. Then,

$$\begin{aligned} B &= k \cdot 1 + R, \\ C &= k \cdot 1 + C_1 + A_2. \end{aligned}$$

Since $R \cdot C_1 = R \cdot A_2 = 0$, it follows that $A_\bullet = B_\bullet \sqcap C_\bullet$. Also, as the restriction of β to C_1 defines a symplectic form, one can find a Darboux basis $x_1, y_1, \ldots, x_d, y_d$ for C_1, i.e. a basis such that, for the symplectic form $\beta_C : C_1 \otimes C_1 \to A_2 = k$, it holds

$$\begin{aligned} &\beta_C(x_i \otimes y_j) = \delta_{ij}, \\ &\beta_C(x_i \otimes x_j) = \beta(y_i \otimes y_j) = 0. \end{aligned}$$

Now, both B and C are quadratic algebras, given by presentations

$$\begin{aligned} C &= \frac{T(C_1)}{(x_i \otimes y_j - \delta_{ij} x_1 \otimes y_1, x_i \otimes x_j, y_i \otimes y_j, v \otimes v)} \\ B &= \frac{T(R)}{T^2(R)} \end{aligned}$$

It follows that $B^! = T_\bullet(R)$ and $C^! \simeq \mathscr{U}(\mathscr{G}_{2d})$. Indeed, if $f \in T^2(C_1^*)$ vanishes on every relation of C, by some linear algebra computations, $f = \alpha \sum_i (x_i^* \otimes y_i^* - y_i^* \otimes x_i^*)$, for $\alpha \in k$. It follows that the Koszul dual of every standard algebra A with $\dim_{\mathbb{F}} A_2 = 1$, $A_3 = 0$, decomposes as

$$A^! = F \amalg \mathscr{U}(\mathscr{G}_{2d}),$$

where F is a tensor algebra.

We can thus prove

Proposition 9.4. *Let $\mathscr{L}$ be a quadratic 1-relator k-Lie algebra. Then, $\mathscr{L} \simeq \mathscr{G}_{2d} \amalg \mathscr{F}$, for some positive integer d, and some free Lie algebra $\mathscr{F}$.*

In particular, quadratic 1-relator Lie algebras are BK.

Proof. Since $\mathscr{L}$ is defined by only one relation, its second cohomology group is 1-dimensional, and the quadraticity of $\mathscr{L}$ implies that it is concentrated in degree 2. By Theorem 3.3, the cohomology ring of $\mathscr{L}$ is a standard algebra A with $A_2 \simeq k$ and $A_3 = 0$.

It follows that $\mathscr{L}$ is Koszul and $\mathscr{U}(\mathscr{L}) = A^! = F \amalg \mathscr{U}(\mathscr{G}_{2d})$. Since $\mathscr{L}$ is the maximal standard Lie subalgebra of $\mathscr{U}(\mathscr{L})_{[\cdot,\cdot]}$, it follows that $\mathscr{L} = \mathscr{F} \amalg \mathscr{G}_{2d}$. □

Since the minimal number of defining relations is the dimension of the second cohomology group of a Lie algebra, we deduce the following.

Corollary 9.5. *All quadratic PD^2 Lie algebras are surface Lie algebras.*

One can use the theory of HNN extensions to prove that surface Lie algebras are characterised from the fact of having only free proper subalgebras.

Proposition 9.6. *Let $\mathscr{L}$ be a quadratic Lie algebra. If $\mathscr{L}$ is not free and all proper standard subalgebras of $\mathscr{L}$ are free, then $\mathscr{L}$ is a surface Lie algebra.*

Proof. Assume $\mathscr{L}$ has more than 1 minimal relation. Then, there is a quadratic Lie algebra $\tilde{\mathscr{L}}$ with 2 relations, and an epimorphism $\pi : \tilde{\mathscr{L}} \to \mathscr{L}$ that is an isomorphism in degree 1. If $\mathscr{M}$ is a 1-generated proper subalgebra of $\tilde{\mathscr{L}}$, then its image under π is a proper standard subalgebra of $\mathscr{L}$, which needs to be free by hypothesis. In particular, $\mathscr{M}$ is free. Indeed, if $\mathscr{F}$ is the free Lie algebra on $\mathscr{M}_1$, then the composition $\mathscr{F} \to \mathscr{M} \to \pi(\mathscr{M})$ is an isomorphism.

Hence, $\tilde{\mathscr{L}}$ satisfies the hypothesis of the theorem. If such a Lie algebra cannot exist, then the theorem is proved. Thus, assume $\mathscr{L}$ to be defined by two (non-necessarily independent) quadratic relations $s \neq 0$ and r. Let $\mathscr{G}$ be the 1-relator Lie algebra defined by s that covers $\mathscr{L}$.

Since $\mathscr{G}$ is 1-generated, there is a non-zero element $x \in \mathscr{G}_1$, a maximal standard subalgebra $\mathscr{M}$ of $\mathscr{G}$ not containing x, and elements $z \in \mathscr{M}_1$ and $c \in \mathscr{M}_2$ such that

$$r = [x, z] + c.$$

By setting $\phi(z) = -c$, one has the well defined derivation $\phi : \langle z \rangle \to \mathscr{M}$, and the HNN decomposition

$$\mathscr{L} \simeq \mathrm{HNN}_\phi(\mathscr{M}, t)$$

where t corresponds to x in the isomorphism. Since $\mathscr{M}$ embeds into $\mathscr{L}$, it is free, whence the Lie algebra $\mathscr{L}$ has just one relation corresponding to $[t, z] = -c$.

It follows from Proposition 9.4 that $\mathscr{L}$ is a free product of a free Lie algebra $\mathscr{F}$ and a surface Lie algebra. If $\mathscr{F} \neq 0$, then $\mathscr{L}$ contains a proper subalgebra isomorphic with a surface Lie algebra, contradicting the hypotheses. □

Corollary 9.7. *Let $\mathscr{L}$ be a BK Lie algebra. If $\mathscr{L}$ is not free, then $\mathscr{L}$ contains $\mathscr{G}_{2d}$ as a standard subalgebra for some $d \geq 1$ (where $\mathscr{G}_2 = k^2$).*

Proof. Let $\mathfrak{X}$ be the class of all r-relator BK Lie algebras with $r \geq 2$ that do not contain 1-relator standard subalgebras. There is a natural partition

$$\mathfrak{X} = \bigcup_{r \geq 2} \mathfrak{X}_r$$

where $\mathfrak{X}_r$ contains the Lie algebras in $\mathfrak{X}$ with exactly r minimal defining relations.

By contradiction, assume $\mathfrak{X}$ is non-empty and let $r \geq 2$ be the minimal integer with $\mathfrak{X}_r \neq \emptyset$. Let $\mathscr{L}$ be a Lie algebra in $\mathfrak{X}_r$ such that $\dim \mathscr{L}_1$ is minimal among the Lie algebras of $\mathfrak{X}_r$.

Since $\mathscr{L}$ is not free nor 1-relator, by Proposition 9.6, there exists a proper standard subalgebra $\mathscr{M}$ of $\mathscr{L}$ which is not free. Notice that $\mathscr{M}$ is BK, as well, and does not contain any 1-relator standard subalgebra. In particular, $\mathscr{M}$ belongs to $\mathfrak{X}_{r'}$ with $1 < r' \leq r$.

Since $\mathfrak{X}_i$ is empty for $i < r$, we deduce that $r' = r$ which contradicts the minimality of $\dim \mathscr{L}_1$.

It follows that $\mathscr{L}$ contains a 1-relator standard subalgebra, which always contains $\mathscr{G}_{2d}$ for some $d \geq 1$, by Proposition 9.4. □

It is unknown whether a pro-p group version of the latter result holds, and it would provide a step forward in the proof of the elementary type conjecture.

9.2. **2-relator Lie algebras.** We have seen that all quadratic 1-relator Lie algebras are BK. In this subsection we prove that the same holds for 2-relator Lie algebras. Again, we will use Lemma 3.11 on HNN-extensions for quadratic Lie algebras.

The study of quadratic 2-relator Lie algebras has been motivated by a work of C. Quadrelli's [37], in which he proves that for the cohomology of a 2-relator group being quadratic is equivalent to being universally Koszul. However, in order to do so, he uses arguments which cannot be easily translated (if they even can) in terms of Lie algebras, like the mildness-property of groups.

The above-mentioned property of mildness for groups is only used to prove that the cohomological dimension of a 2-relator group with quadratic cohomology is 2. We prove a Lie analogue of that, by using HNN-extensions.

Lemma 9.8. *Let $\mathscr{L}$ be a quadratic 2-relator Lie algebra. Then $\operatorname{cd} \mathscr{L} \leq 2$, and hence it is Koszul.*

Proof. Let $\mathscr{F}$ be the free Lie algebra on $\mathscr{L}_1$ and let $r, s \in \mathscr{F}_2$ be the degree 2 relators of $\mathscr{L}$, i.e., $\mathscr{L} = \mathscr{F}/(r, s)$.

Since $\mathscr{F}/(r)$ is a quadratic 1-relator Lie algebra, it is isomorphic to the free product $\mathscr{G}_{2d} \amalg \mathscr{A}$ of the surface Lie algebra on $2d$ generators ($d \geq 0$) and a free Lie algebra $\mathscr{A}$, by Proposition 9.4.

Let $x_1, y_1, \ldots, x_d, y_d$ be the canonical set of generators of $\mathscr{G}_{2d}$ and let $z_1, \ldots, z_\ell$ be a homogeneous generator system of $\mathscr{A}$.

Denote again by s the image of the relator s into the quotient Lie algebra $\mathscr{F}/(r) = \mathscr{G}_{2d} \amalg \mathscr{A}$. Since $s \neq 0$, there are scalars α and β, not both zero, such that

$$s = \alpha s_1 + \beta s_2,$$

where $s_1 \in [\mathscr{G}_{2d}, \mathscr{F}/(r)]$ and $s_2 \in [\mathscr{A}, \mathscr{A}]$ are (non-uniquely determined) non-zero elements of degree 2 of $\mathscr{F}/(r)$.

If $\alpha = 0$, then s is an element of $\mathscr{A}'$, and hence

$$\mathscr{L} = \mathscr{G}_{2d} \amalg \mathscr{A}/(s)$$

is a free product of surface Lie algebras and a free Lie algebra — a free factor of the 1-relator Lie algebra $\mathscr{A}/(s)$—, proving that $\operatorname{cd} \mathscr{L} = 2$.

If $\alpha \neq 0$, then we can suppose s to be the sum of a Lie bracket $[x_1, z]$, $z \neq 0$, and a linear combination c of elements of degree 2 such that x_1 does not appear in the expression of c. Let V and W be the vector subspaces generated respectively by $\{y_1, x_2, y_2, \dots, x_d, y_d, z_1, \dots, z_\ell\}$ and by $\{z, y_1\}$. Let $\mathscr{M} = \langle V \rangle$ and $\mathscr{N} = \langle W \rangle$. Since z appears in the Lie bracket with x_1, we may assume that it is an element of $\mathscr{M}$, and hence $\mathscr{N} \leq \mathscr{M}$. By the Kurosh Subalgebra Theorem 4.18, and since proper subalgebras of $\mathscr{G}_{2d}$ are free, both $\mathscr{M}$ and $\mathscr{N}$ are free Lie algebras. It follows that the map $\phi : \mathscr{N} \to \mathscr{M}$ defined by $\phi(z) = -c$ and $\phi(y_1) = -\sum_{i \geq 2}[x_i, y_i]$ extends to a derivation and $\mathscr{L} \simeq \mathrm{HNN}_\phi(\mathscr{M}, t)$, where the stable letter t corresponds to the generator x_1 of $\mathscr{L}$.

By Bass-Serre theory for Lie algebras, we see that $\mathscr{L}$ has cohomological dimension 2.

Indeed, one has the short exact sequence of $\mathscr{U}(\mathscr{L})$-modules

$$0 \to \operatorname{ind}_{\mathscr{N}}^{\mathscr{L}}(k) \to \operatorname{ind}_{\mathscr{M}}^{\mathscr{L}}(k) \to k \to 0,$$

which yields the exact sequence

$$\operatorname{Ext}^2_{\mathscr{U}(\mathscr{L})}(\operatorname{ind}_{\mathscr{N}}^{\mathscr{L}}(k), k) \to \operatorname{Ext}^3_{\mathscr{U}(\mathscr{L})}(k, k) \to \operatorname{Ext}^3_{\mathscr{U}(\mathscr{L})}(\operatorname{ind}_{\mathscr{M}}^{\mathscr{L}}(k), k).$$

Now, by Eckmann-Shapiro Lemma, since the natural maps $\mathscr{M}, \mathscr{N} \to \mathscr{L}$ are injective, the above exact sequence reads

$$H^2(\mathscr{N}) \to H^3(\mathscr{L}) \to H^3(\mathscr{M}).$$

Since $\mathscr{M}$ and $\mathscr{N}$ are free, their cohomological dimension is 1, and hence $H^3(\mathscr{L}) = 0$. □

Lemma 9.9. *Let A be a graded-commutative standard k-algebra. If $A_3 = 0$ and $\dim A_2 = 2$, then A is universally Koszul. In particular, it is Koszul.*

Proof. The proof is precisely that of Theorem 3.7 in [37].

Let $x_1, \dots, x_n$ be a k-basis for A and let A_2 be spanned by the monomials $x_1 x_2$ and $x_s x_t$, for some $1 \leq s, t \leq n$ with $\{s, t\} \neq \{1, 2\}$. In order to prove that A is universally Koszul, we use the equivalent condition of Proposition 4.12-(3).

Let $I \neq A_+$ be a 1-generated ideal of A and pick $b \in A_1 \setminus I$. We will show that the colon ideal

$$J := (I : b) = \{x \in A \mid xb \in I\}$$

is generated by elements of degree 1.

Since $A_{\geq 3} = 0$, one has $bA_2 = 0$, and hence $A_2 = J_2$. Plus, as $b^2 = 0$, one has $b \in J_1$. Thus, it is enough to prove that $bA_1 = A_2$.

We may distinguish three cases depending on the dimension of bA_1.

Case 1: $\dim(bA_1) = 0$. From $bA_1 = 0$ it follows that $A_1 = J_1$ which is a contradiction (however, one has that $J = A_+$ is a 1-generated ideal).

Case 2: $\dim(bA_1) = 1$. There exists a non-zero pair of scalars $p, q \in k$ such that $px_1x_2 + qx_sx_t$ is a basis of bA_1. Now, for $i = 1, 2$, there are scalars $\lambda_i \in k$ such that

$$x_ib = \lambda_i(px_1x_2 + qx_sx_)).$$

If one of the λ_i's is zero, say λ_1, then $x_1b = 0$, from which follows that $x_1 \in J_1$, and hence $x_1x_2 \in J_1A_1$. On the other hand, if $\lambda_1 \neq 0 \neq \lambda_2$, then

$$\left(x_1 - \frac{\lambda_1}{\lambda_2}x_2\right) b = 0,$$

which implies that $x_1 - \frac{\lambda_1}{\lambda_2}x_2 \in J_1$, and hence

$$x_1x_2 = \left(\frac{\lambda_1}{\lambda_2}x_2\right) x_2 \in J_1A_1.$$

A similar argument shows that $a_sa_t \in J_1A_1$, and hence $J_1A_1 = A_2 = J_2$.

Case 3: $\dim(bA_1) = 2$. Since $\dim A_2 = 2$, one trivially has $bA_1 = A_2 = J_2$. □

With these ingredients we may prove:

Theorem 9.10. *Let $\mathscr{L}$ be a quadratic 2-relator Lie algebra. Then $\mathscr{L}$ is BK.*

Proof. By Lemma 9.16, $\mathscr{L}$ has cohomological dimension 2 and it is Koszul. Then, $H^\bullet(\mathscr{L}, k)$ satisfies the hypotheses of Lemma 9.9, and hence it is universally Koszul. From Theorem 4.15, it follows that $\mathscr{L}$ is BK. □

Corollary 9.11. *Let $\mathscr{L}$ be a quadratic 2-relator Lie algebra. Then the derived subalgebra $\mathscr{L}'$ is free, i.e. $\mathscr{L}$ is a Bogomolov Lie algebra.*

Proof. Assume $\mathscr{L}$ is a quadratic 2-relator Lie algebra such that all proper 2-relator standard subalgebras of $\mathscr{L}$ are Bogomolov. By Proposition 9.6, there is a standard subalgebra $\mathscr{M}$ of $\mathscr{L}$ that is not free, and clearly we can take $\mathscr{M}$ to be maximal. Hence, we can decompose $\mathscr{L}$ as $\mathrm{HNN}_\phi(\mathscr{M}, t)$, for some degree 1 derivation $\phi : \mathscr{A} \to \mathscr{M}$, where $\mathscr{A}$ is a standard subalgebra of $\mathscr{M}$. Since $\mathscr{L}$ is BK, using Corollary 3.16–2b, we get the following exact sequence

$$0 \to H^1(\mathscr{A}) \to H^2(\mathscr{L}) \to H^2(\mathscr{M}) \to 0.$$

As $\mathscr{M}$ is not free, its second cohomology group is not zero, and hence the dimension of $H^1(\mathscr{A})$ is at most 1, i.e., either $\mathscr{A} = 0$ or it is 1-dimensional, so that $\mathscr{A} \subseteq \mathscr{L}_1$.

Now, $\mathscr{L}' \cap \mathscr{A} = 0$ and $\mathscr{L}' \cap \mathscr{M} = \mathscr{M}'$ is free. Indeed, $\mathscr{M}$ is either 1-relator, whence Bogomolov, or 2-relator, in which case, by assumption, $\mathscr{M}'$ is free. By invoking [22, Theorem 3], one concludes the claim. □

By Proposition 4.21, it follows that quadratic 2-relator Lie algebras are coherent.

If $\mathscr{L}$ is a quadratic Lie algebra generated by at most 3 elements, then it is Bloch-Kato, since either the number of defining relations is less than 3 or the Lie algebra is abelian. This phenomenon leads to the following

Example 9.12. *Let $\mathscr{L}$ be a quadratic non-Koszul Lie algebra generated by 4 elements. Then all maximal standard subalgebras of $\mathscr{L}$ are not quadratic.*

Indeed, if there is some quadratic subalgebra $\mathscr{M}$ generated by 3 elements, then $\mathscr{M}$ is Bloch-Kato and hence $\mathscr{L}$ is Koszul, by Lemma 3.15.

For instance, the Lie algebra $\mathscr{H}$ introduced in Example 6.3 with $n = 2$ has no quadratic subalgebra minimally generated by 3 elements.

Notice that we have also proved that if A is a quadratic graded-commutative algebra with $\dim A_2 = 2$, then $A_3 = 0$. In particular, one can weaken the hypotheses of Lemma 9.9.

Corollary 9.13. *Let A be a graded-commutative standard algebra with* $\dim A_2 = 2$*. Then $A_3 = 0$ and A is universally Koszul.*

Proof. We may assume A quadratic. Indeed, the quadratic cover $\mathrm{q}A$ satisfies the hypotheses if A does, and if $\mathrm{q}A_3 = 0$, then $A_3 = 0$ and hence $A = qA$.

The quadratic dual of a graded-commutative algebra is the universal enveloping algebra of a quadratic Lie algebra. So, let $A^! = \mathscr{U}(\mathscr{L})$. As $\mathscr{L}$ is quadratic, $H^2(\mathscr{L}) = H^{2,2}(\mathscr{L}) = A_2$ has dimension 2, i.e., $\mathscr{L}$ is a quadratic 2-relator Lie algebra and hence it is Bloch-Kato of cohomological dimension 2, proving that $A = H^\bullet(\mathscr{L})$ is zero in degree ≥ 3. □

Remark 9.14. *Notice that one can prove directly that $A_3 = 0$ without passing to the associated quadratic Lie algebra.*

To do that, let A be a graded-commutative standard algebra with $\dim A_2 = 2$*. Let x be any non-zero element of degree 1. We will prove that $xA_2 = 0$.*

If $xA_1 = 0$ or $xA_1 = A_2$, then $xA_2 = 0$. If xA_1 is 1-dimensional, say generated by an element $a \neq 0$, then there is a basis $z_1 = x,\ z_2 = y,\ z_3, \dots, z_n$ for A_1 such that $xy = a$ and $xz_i = 0$ for $i \geq 3$. Let $b \in A_2$ so that $A_2 = \mathrm{Span}\{a, b\}$. Let $\alpha_{ij} \in k$ be the coefficient of $z_i z_j$ in some expression of b. Since $xz_j = 0$ for $j > 2$, we can take $\alpha_{1j} = 0$ for $j > 2$, and, up to replacing b with $b - \alpha_{12}a$, one can suppose that $\alpha_{12} = 0$. In particular, b is in the subalgebra generated by z_i, $i \geq 2$, and hence $bx = 0$.

In turn, it follows that a quadratic 2-relator Lie algebra $\mathscr{L}$ has quadratic dual $\mathscr{L}^!$ which is 2-dimensional in degree 2 and zero in degree 3. Hence, $\mathscr{L}^!$ is Koszul by Lemma 9.9, proving that $\mathscr{L}$ is Koszul.

Interestingly, from the above corollary we get the following linear algebra fact.

Proposition 9.15. *Let X be a codimension 2 subspace of the wedge square $V \wedge V$ of a finite dimensional vector space V. Then,*

$$V \wedge X := \operatorname{Span}_k \left\{ v \wedge x \;\middle|\; v \in V, x \in X \right\} = \Lambda^3(V).$$

Proof. Set $A = \Lambda(V)/(X)$. Then, A is a graded-commutative quadratic algebra of dimension 2 in degree 2, and hence, by the previous corollary, $A_3 = 0$, proving that $(X)_3 = \Lambda^1(V) \wedge X = \Lambda^3(V)$. □

Lemma 9.16. *Let $\mathscr{G}$ be the surface Lie algebra on $2d$ generators and let $r \in \mathscr{G}_2$ be a non trivial element. Let $R = (r)$ be the ideal of $\mathscr{G}$ generated by r, and $\mathscr{L} = \mathscr{G}/R$. If $H^1(\mathscr{L}, H^1(R)) = 0$, then $\operatorname{cd} \mathscr{L} = 2$.*

Proof. One has the Lyndon-Hochschild-Serre Spectral Sequence

$$E_2^{pq} = H^p(\mathscr{L}, H^q(R)) \Rightarrow H^{p+q}(\mathscr{G}).$$

Since R is a proper subalgebra of the surface Lie algebra $\mathscr{G}$, it is free, and hence $H^q(R) = 0$ for $q \geq 2$, proving that E_2 collapses at E_3. This means that the second page E_2 is

$$\begin{array}{ccccc}
\cdots & \cdots & \cdots & \cdots & \iddots \\
0 & 0 & 0 & 0 & \cdots \\
H^1(R)^{\mathscr{L}} & H^1(\mathscr{L}, H^1(R)) & H^2(\mathscr{L}, H^1(R)) & H^3(\mathscr{L}, H^1(R)) & \cdots \\
k & H^1(\mathscr{L}) & H^2(\mathscr{L}) & H^3(\mathscr{L}) & \cdots
\end{array}$$

Since $\operatorname{cd} \mathscr{G} = 2$, the differential $H^1(\mathscr{L}, H^1(R)) \to H^3(\mathscr{L})$ must be surjective. □

Lemma 9.17. *Let A be a graded k-algebra and let M be a graded A-module. Then, M is free if, and only if, $\operatorname{Tor}_1^A(M, k) = 0$.*

Proof. Consider a minimal free A-resolution

$$\cdots \to F_1 \to F_0 \to M \to 0$$

of M, i.e., the induced maps $F_i \otimes_A k \to F_{i-1} \otimes_A k$ are zero $\forall i \geq 1$. In particular, $\mathrm{Tor}_1^A(M,k) \simeq F_1 \otimes_A k$. It follows that $\mathrm{Tor}_1^A(M,k) = 0$ if, and only if, $F_1 \otimes_A k = 0$, which happens precisely when $F_1 = 0$, by Nakayama Lemma. Plus, $F_1 = 0$ iff $M \simeq F_0$ is free. □

Lemma 9.18. *Let $\mathscr{L}$ be a graded Lie algebra and M a graded $\mathscr{U}(\mathscr{L})$-module. Then, M is free if, and only if, M^* is injective.*

Proof. One has the isomorphism $\mathrm{Tor}_1^{\mathscr{U}(\mathscr{L})}(M,k)^* \simeq \mathrm{Ext}^1_{\mathscr{U}(\mathscr{L})}(k, M^*)$. Now, M is free if, and only if, $\mathrm{Tor}_1^{\mathscr{U}(\mathscr{L})}(M,k) = 0$, and M^* is free, i.e., $M^* \simeq \mathrm{Hom}_k(\mathscr{U}(\mathscr{L}), X)$ for some vector space X, if, and only if, $\mathrm{Ext}^1_{\mathscr{U}(\mathscr{L})}(k, M^*) = 0$. □

Lemma 9.19. *Let $0 \to R \to \mathscr{F} \to \mathscr{L} \to 0$ be a minimal extension (i.e., $R \leq \mathscr{F}'$) of a Lie algebra $\mathscr{L}$ with a free Lie algebra R. Then, the following are equivalent:*

(1) *$R/[R,R]$ is a free $\mathscr{U}(\mathscr{L})$-module.*
(2) $\mathrm{cd}\,\mathscr{L} \leq 2$.

Proof. From the proof Lemma 9.16, it follows that $H^3(\mathscr{L}) \simeq H^1(\mathscr{L}, H^1(R))$. Since $H^1(R) = (R/[R,R])^*$, one has that $R/[R,R]$ is free iff $(R/[R,R])^*$ is injective, which happens iff $H^1(\mathscr{L}, H^1(R)) = \mathrm{Ext}^1_{\mathscr{U}(\mathscr{L})}(k, (R/[R,R])^*) = 0$. □

10. Higher dimensional Poincaré duality

In the previous section we have seen many properties of surface Lie algebras which can be applied to PD^n Lie algebras for any n. In this section, we will recall such results and see what being BK implies for the class of PD^n Lie algebras.

Proposition 10.1. *Let A be a duality k-algebra of dimension n and of type FP.*

Let D be the right A-module $\mathrm{Ext}^n_A(k,A) = H^n(A,A)$. Then for every left A-module M the following homological duality holds

$$H^i(A,M) \simeq \mathrm{Tor}^A_{n-i}(D,M).$$

The argument to see that proper subalgebras of surface Lie algebras are free can be generalised to higher dimensional Poincaré duality.

Lemma 10.2. *Let $\mathscr{L}$ be any k-Lie algebra and let $\mathscr{M}$ be a proper subalgebra. Then, $\mathrm{Hom}_{\mathscr{U}(\mathscr{L})}(k, \mathrm{ind}_{\mathscr{M}}^{\mathscr{L}} k) = 0$, i.e., the $\mathscr{L}$-module $\mathrm{ind}_{\mathscr{M}}^{\mathscr{L}} k = \mathscr{U}(\mathscr{L}) \otimes_{\mathscr{U}(\mathscr{M})} k$ has no $\mathscr{L}$-fixed points.*

Proof. Put $A = \mathscr{U}(\mathscr{L})$ and $B = \mathscr{U}(\mathscr{M})$. Let $\alpha : k \to A \otimes_B k$ be an A-linear map. As k is a simple A-module, the map α is determined by $\alpha(1)$. Complete a k-basis $\{y_i,\ i \in I\}$ of $\mathscr{M}$ to a k-basis $\{y_i, z_j,\ i \in I,\ j \in J\}$ for $\mathscr{L}$. By PBW Theorem 2.1, B has k-basis $\{y_{i_1} \cdots y_{i_n} |\ i_1, \ldots, i_n \in I,\ n \in \mathbb{N}_0\}$ and it is infinite-codimensional in A, whose k-basis can be suitably chosen to be $\{z_{j_1}\cdots z_{j_m}y_{i_1}\cdots y_{i_n} \mid i_1,\dots,i_n\in I,\ j_1\dots,j_m\in J,\ m,n\in\mathbb{N}_0, m\neq 0\}$. Note now that the induced module can be written as the following quotient of A:

$$A\otimes_B k\simeq A/AB_+ =: A_B,$$

via the map $a\otimes x\mapsto xa+AB_+$. With respect to the above isomorphism, if $\alpha(1)=u+AB_+$, one can choose $u\in A_+$ to be of the form

$$u=\sum r_{j_1,\dots,j_r}z_{j_1}\cdots z_{j_r}.$$

Finally,

$$0=\alpha(z_j\cdot 1)=z_j\alpha(1),\ j\in J$$

implies $u=\sum r_{j_1,\dots,j_r}z_jz_{j_1}\cdots z_{j_r}\in AB_+$, i.e., $\alpha(1)=0$. □

Proposition 10.3. *Let $\mathscr{L}$ be a PD^n Lie algebra. If $\mathscr{M}$ is a proper subalgebra of $\mathscr{L}$, then $\operatorname{cd}\mathscr{M}<n=\operatorname{cd}\mathscr{L}$.*

Proof. Again we argue as for 2-dimensional Poincaré duality Lie algebras. Recall that $\mathscr{M}$ has cohomological dimension $<n$ iff $H_n(\mathscr{M},k)=0$.

By the homological version of Eckmann-Shapiro Lemma (see [56]),

$$\operatorname{Tor}_n^{\mathscr{U}(\mathscr{M})}(k,k)\simeq\operatorname{Tor}_n^{\mathscr{U}(\mathscr{L})}(k,\operatorname{ind}_{\mathscr{M}}^{\mathscr{L}}k).$$

By Proposition 9.1, since $D=H^n(\mathscr{L},\mathscr{U}(\mathscr{L}))$ is the trivial module, one has an isomorphism $\operatorname{Tor}_n^{\mathscr{U}(\mathscr{L})}(k,\operatorname{ind}_{\mathscr{M}}^{\mathscr{L}}k)\simeq H^0(\mathscr{L},\operatorname{ind}_{\mathscr{M}}^{\mathscr{L}}k)=\operatorname{Ext}^0_{\mathscr{U}(\mathscr{L})}(k,\operatorname{ind}_{\mathscr{M}}^{\mathscr{L}}k)$, which yields, by applying Lemma 10.2, $H_n(\mathscr{M},k)=0$. □

Theorem 10.4 (Proposition 5.10 of [47])**.** *Let $\mathscr{L}$ be a graded Lie algebra. Then, $\mathscr{L}$ is PD^n if, and only if, its cohomology ring $H^\bullet(\mathscr{L},k)$ satisfies PD^n.*

10.1. **Properties of PD^n Lie algebras which are Bloch-Kato.** In this subsection, we consider the Koszul and the BK property for Poincaré-duality Lie algebras.

We start by showing that for standard Lie algebras, being PD^n almost implies the Koszul property.

Proposition 10.5. *Let A be a standard PD^n algebra. If $H^{n,n}(A,k)\neq 0$, then A is Koszul.*

Proof. Since $H^n(A)\simeq k$ and $H^{n,n}(A)\neq 0$, one has $H^n(A)=H^{n,n}(A)$.

The multiplication mapping $H^{i,\bullet}(A)\otimes H^{n-i,\bullet}(A)\to H^n(A)\simeq k$ is non-degenerate. Since $H^{n-i,\bullet}$ is concentrated in degree $\geq n-i$, if $x\in H^{i,j}(\mathscr{L})$ with $j>i$, then $x\cdot H^{n-i,\bullet}\subseteq H^{n,j+\bullet}(\mathscr{L})=0$, proving that $x=0$. □

Proposition 10.6. *Let $\mathscr{L}$ be an N-Koszul Lie algebra of PD^n. If $n\leq 2N$, then $\mathscr{L}$ is Koszul.*

Proof. Consider the non-degenerate pairing

$$(\_) \smile (\_) : H^i(\mathscr{L}) \otimes H^{n-i}(\mathscr{L}) \to H^n(\mathscr{L})$$

induced by the cup product. Since $n \leq 2N$, we have $n - N \leq N$, and hence $H^{n-N}(\mathscr{L})$ is concentrated in degree $N - n$. It follows that $H^n(\mathscr{L}) = H^N(\mathscr{L}) \smile H^{n-N}(\mathscr{L}) = H^{N,N}(\mathscr{L}) \smile H^{n-N,n-N}(\mathscr{L}) \subseteq H^{n,n}(\mathscr{L})$, i.e., $H^n(\mathscr{L})$ is concentrated in degree n. By Lemma 10.5, since $\mathscr{L}$ is PD^n and $H^{n,n}(\mathscr{L}) \neq 0$, we conclude that $\mathscr{L}$ is Koszul. □

In particular, quadratic PD^n Lie algebras are Koszul for $n \leq 4$.

Lemma 10.7. *Let A be a graded-commutative (or commutative) k-algebra that* satisfies *PD^{n+1}. Then, for each non-zero degree-*1 *element $x \in A_1$, the quotient $A/(0 : x)$ satisfies Poincaré duality in dimension n.*

Proof. The pairing $A_1 \otimes A_n \to k$ is non-degenerate, and hence $a \in A_n \mapsto xa$ is surjective, from which follows that its kernel, that is the degree n component of $(0 : x)$, has codimension 1 in A_n. By setting $\bar{A} = A/(0 : x)$, one thus sees that $\bar{A}_n$ is 1-dimensional.

Let $a \in A_i$ be such that $\bar{a}\bar{b} = 0$ for all $b \in A_{n-i}$. This means that $\forall b \in A_{n-i}$, $ab \in I = (0 : x)$, that is $abx = 0$. Since $A_{i+1} \otimes A_{n-i} \to A_{n+1}$ is non degenerate, if $axb = 0$ for all $b \in A_{n-i}$, then $ax = 0$, i.e., $a \in (0 : x) = I$, and hence $\bar{a} = 0$. It follows that $\bar{A}_i \otimes \bar{A}_{n-i} \to \bar{A}_n$ is a non-degenerate pairing. □

Recall that if A is a graded-commutative universally Koszul algebra, then, the annihilator $(0 : x)$ of a degree-1 element x is a 1-generated ideal of A, i.e., $(0 : x) \in \mathcal{L}(A)$ (see Proposition 4.12). This can be used to prove the following:

Corollary 10.8. *Let $\mathscr{L}$ be a BK Lie algebra. Assume $\mathscr{L}$ is PD^{n+1}. Then, for every $0 \neq x \in \mathscr{L}_1$, there is a* 1*-generated Lie subalgebra $\mathscr{M}$ of $\mathscr{L}$ that is a PD^n Lie algebra and $x \notin \mathscr{M}$.*

Proof. Put $A = H^\bullet(\mathscr{L})$ and choose any element $\xi \in A_1 \setminus x^\perp$, where

$$x^\perp = \left\{\eta \in \mathscr{L}_1^* = A_1 \;\middle|\; \eta(x) = 0\right\}.$$

Then, $\mathscr{M} = \langle (0 : \xi)_1^\perp \rangle$ is a standard (thus Koszul) subalgebra of $\mathscr{L}$ that does not contain x, as $\xi(x) \neq 0$. Eventually, notice that, by the proof of Theorem 4.15, $\mathscr{M}$ has cohomology $A/(0 : \xi)$, which satisfies PD^n. □

A partial converse to this result holds

Proposition 10.9. *Let A be a universally Koszul graded-commutative algebra satisfying Poincaré duality in dimension $n + 1$. If I is a* 1*-generated ideal of A such that A/I satisfies Poincaré duality in dimension n and $(0 : I)_1 \neq 0$, then there exists $x \in A_1$ such that $I = (0 : x)$.*

Proof. Take $0 \neq x \in (0:I)_1$ and consider the ideal $(0:x)$. Since $xI = 0$, one has $I \subseteq (0:x)$.

We claim that $I = (0:x)$. If not, the two ideals differ already in degree 1, so let $y \in (0:x)_1 \setminus I$. Since A/I satisfies Poincaré duality in dimension n, $(A/I)_2 = (A/I)_1 \cdot (y + I)$, and hence $(A/(0:x))_2 = 0$ which contradicts Lemma 10.7. □

Corollary 10.10. *Let $\mathscr{L}$ be a Bloch-Kato Lie algebra of PD^{n+1} and let $\mathscr{M}$ be a proper standard subalgebra of $\mathscr{L}$. Let I be the ideal of $H^\bullet(\mathscr{L})$ such that $H^\bullet(\mathscr{M}) = H^\bullet(\mathscr{L})/I$ and assume that $(0:I)_1 \neq 0$. Then $\mathscr{M}$ has cohomological dimension n and it contains a PD^n Lie subalgebra.*

Proof. Since I is a 1-generated ideal of $A = H^\bullet(\mathscr{L})$, if $0 \neq x \in (0:I)_1$, then $xI = 0$, and hence $I \subseteq (0:x)$. Let $\mathscr{N}$ be the Lie subalgebra of $\mathscr{L}$ with cohomology algebra $A/(0:x)$; then $\mathscr{N}$ is PD^n. Moreover, A/I is universally Koszul and $A/(0:x)$ is one of its quotients by 1-generated ideals, proving that $A/I \to A/(0:x)$ is a Koszul homomorphism. It follows that $\mathscr{N}$ is a subalgebra of $\mathscr{M}$ and hence $n = \operatorname{cd}\mathscr{N} \leq \operatorname{cd}\mathscr{M} < n+1$, proving that $\mathscr{M}$ has cohomological dimension n. □

Lemma 10.11. *Let A be a graded-commutative algebra and let ξ and η be two degree-1 elements of A. Assume that A satisfies Poincaré duality in dimension $n+1$. If $(0:\xi) = (0:\eta)$ then ξ and η are linearly dependent.*

Proof. Consider the two linear mappings $\phi_\xi, \phi_\eta : A_n \to A_{n+1} \simeq k$ given by $\phi_\xi(\theta) = \theta\xi$ and $\phi_\eta(\theta) = \theta\eta$. Hence, we can see those linear maps as linear 1-forms of A_n, i.e., $\phi_\xi, \phi_\eta \in A_n^*$.

By hypothesis, $(0:\xi) = (0:\eta)$ and hence $\ker\phi_\xi = \ker\phi_\eta$.

We can assume $\xi \neq 0 \neq \eta$ and hence $\ker\phi_\xi \neq A_n \neq \ker\phi_\eta$, since $A_1 \otimes A_n \to A_{n+1} = k$ is non-degenerate. Let $\theta \in A_n$ be such that $\phi_\xi(\theta) = r \neq 0$ and $\phi_\eta(\theta) = 1$. Hence, one can decompose A_n into the direct sum $\mathrm{Span}(\theta) \oplus \ker\phi_\xi$, so that, every element of A_n can be written as $\rho = s\theta + \kappa$, where $s \in k$ and $\kappa \in \ker\phi_\xi$. It follows

$$\phi_\xi(\rho) = s\phi_\xi(\theta) = sr = sr\phi_\eta(\theta) = r\phi_\eta(\rho).$$

Since $r\phi_\eta = \phi_{r\eta}$, from $\phi_\xi - \phi_{r\eta} = \phi_{\xi - r\eta} = 0$, there follows $\xi = r\eta$. □

From this follows

Corollary 10.12. *Let $\mathscr{L}$ be a PD^{n+1} Lie algebra that is BK. For all $\xi \in \mathscr{L}_1^* = H^1(\mathscr{L})$, denote by $\mathscr{M}_\xi$ the 1-generated subalgebra of $\mathscr{L}$ generated by $(0:\xi)_1^\perp \leq \mathscr{L}_1$. Then $\mathscr{M}_\xi$ is a PD^n Lie algebra.*

Moreover, for all $\xi, \eta \in H^1(\mathscr{L})$, if $\mathscr{M}_\xi = \mathscr{M}_\eta$ then ξ and η are linearly independent. In other words, the assignment

$$[\xi] \in \mathbb{P}(H^1(\mathscr{L})) \mapsto \mathscr{M}_\xi = \langle (0:\xi)_1^\perp \rangle \leq \mathscr{L}$$

with values into the set of PD^n *subalgebras of* $\mathscr{L}$ *is injective, where* $\mathbb{P}(V)$ *denotes the projective space of the vector space* V*.*

Proof. If $\mathscr{M}_\xi = \mathscr{M}_\eta$, then $(0 : \xi)_1 = (0 : \eta)_1$ and, since the annihilators of degree 1 elements of $H^\bullet(\mathscr{L})$ are generated in degree 1 as ideals, we conclude that $(0 : \xi) = (0 : \eta)$, proving that ξ and η need to be linearly dependent by Lemma 10.11. □

This results suggests

Question 10.13. *Let* $\mathscr{L}$ *be a BK &* PD^{n+1} *Lie algebra and let* $\mathscr{M}$ *be a standard* PD^n *subalgebra of* $\mathscr{L}$*. Does there exist a cohomology class* $\xi \in H^1(\mathscr{L})$ *such that the kernel of the restriction* $H^\bullet(\mathscr{L}) \to H^\bullet(\mathscr{M})$ *is the annihilator* $(0 : \xi)$*? Equivalently, let* I *be a* 1*-generated ideal of a graded-commutative, universally Koszul algebra* A *satisfying* PD^{n+1}*. If* A/I *satisfies* PD^n*, is it true that* $(0 : I)_1 \neq 0$*?*

10.2. **$\mathbf{PD}^n$ Lie algebras of Elementary Type.** Let $\mathcal{E}$ be the class consisting of all quadratic Lie algebras with at most 2 defining relations. Hence, $\mathcal{E}$ is a closed BK-class.

Proposition 10.14. *Let* $\mathscr{L}$ *be a non-abelian* $\mathcal{E}$*-ET Lie algebra and let* $n \geq 2$*. Then,* $\mathscr{L}$ *is* PD^n *if, and only if, there are some integers* d *and* $\ell = n - 2$*, such that* $\mathscr{L} \simeq \mathscr{G}_{2d} \sqcap k^\ell$*.*

Proof. Notice that the only quadratic Poincaré duality Lie algebras of dimension 2 are the surface Lie algebras. Hence, we can argue by induction on $n \geq 2$.

If $\mathscr{L}$ is a free product of standard Lie algebras, $\mathscr{L} = \mathscr{A} \amalg \mathscr{B}$, then $H^n(\mathscr{L}, k) = H^n(\mathscr{A}) \oplus H^n(\mathscr{B})$. Since $H^n(\mathscr{L})$ is 1-dimensional, one of its two factors, say $H^n(\mathscr{A})$, must vanish. If $i \leq n - 1$ is the cohomological dimension of $\mathscr{A}$, then $H^i(\mathscr{A}) \cdot H^{n-i}(\mathscr{L}) \subseteq H^n(\mathscr{A}) = 0$, proving that $\mathscr{L}$ is not PD^n.

Suppose $\mathscr{L} = \mathscr{A} \sqcap k$. Then, $H^\bullet(\mathscr{L}) \simeq H^\bullet(\mathscr{A}) \wedge \Lambda(x)$. It follows that $(x) = (0 : x)$, so that $H^\bullet(\mathscr{A}) \simeq H^\bullet(\mathscr{L})/(0 : x)$, and hence, by Lemma 10.7, $\mathscr{A}$ is PD^{n-1}. By induction, $\mathscr{A}$ is the direct sum of a surface Lie algebra and an abelian one, and so is $\mathscr{L}$. □

This suggests the following.

Question 10.15. *Let* $\mathscr{L}$ *be a* PD^n *Lie algebra,* $n \geq 3$*. If* $\mathscr{L}$ *is BK, does* $\mathscr{L}$ *contain a non-trivial degree* 1 *central element?*

If the question has positive answer, then we deduce a complete classification of BK Lie algebras of Poincaré duality: they are all of the shape $\mathscr{L} \simeq \mathscr{G}_{2d} \sqcap k^\ell$. In particular, $\mathscr{L}$ would belong to the class of $\mathcal{E}$-ET Lie algebras.

10.3. **Cohomologically homogeneous algebras.** We have seen in Corollary 10.8 that all BK & PD^n Lie algebras have (standard) subalgebras of cohomological dimension i for every $i \leq n$. This occurrence suggests an interesting (and perhaps uninvestigated) phenomenon.

Definition 10.16. *A Lie algebra of finite cohomological dimension n is **cohomologically homogeneous** if, for all $1 \leq i \leq n$, there exists a subalgebra of cohomological dimension i.*

An analogous definition may be introduced for (pro-p) groups, yet no example of a non-cohomologically homogeneous group is known to the author.

If a Lie algebra $\mathscr{L}$ has $\operatorname{cd}\mathscr{L} = n \leq 2$, then clearly $\mathscr{L}$ is cohomologically homogeneous, for any cyclic Lie algebra is free, and hence of cohomological dimension 1.

The cohomological dimension of a RAAG Lie algebra on a graph Γ (see Example 11.2) is easily seen to be equal to the maximal number of pairwise adjacent vertices in Γ, i.e., the clique-number of Γ. Hence, RAAG Lie algebras are cohomologically homogeneous. As noticed above, the same is true for BK & PD^n Lie algebras. Though, the hypothesis of being of Poincaré-duality is not necessary, as the following result shows.

Proposition 10.17. *Let $\mathscr{L}$ be a BK Lie algebra. Then, $\mathscr{L}$ is cohomologically homogeneous.*

In order to prove this, we use the following

Lemma 10.18. *If $\mathscr{L}$ is a quadratic Lie algebra of finite cohomological dimension, then maximal standard subalgebras of $\mathscr{L}$ have cohomological dimension $\geq \operatorname{cd}\mathscr{L} - 1$.*

Proof. The proof is similar to that of Corollary 3.16. By Lemma 3.11, we can decompose $\mathscr{L}$ into the HNN-extension $\mathrm{HNN}_\phi(\mathscr{M}, t)$, for some maximal standard subalgebra $\mathscr{M}$ of $\mathscr{L}$.

By Bass-Serre Theory for Lie algebras, one has the exact sequence of $\mathscr{U}(\mathscr{L})$-modules

$$0 \to \operatorname{ind}_{\mathscr{A}}^{\mathscr{L}}(k) \to \operatorname{ind}_{\mathscr{M}}^{\mathscr{L}}(k) \to k \to 0$$

which yields the cohomological exact sequence

$$\operatorname{Ext}^i_{\mathscr{U}(\mathscr{L})}(\operatorname{ind}_{\mathscr{A}}^{\mathscr{L}}(k), k) \to \operatorname{Ext}^{i+1}_{\mathscr{U}(\mathscr{L})}(k, k) \to \operatorname{Ext}^{i+1}_{\mathscr{U}(\mathscr{L})}(\operatorname{ind}_{\mathscr{M}}^{\mathscr{L}}(k), k)$$

By Eckmann-Shapiro, the latter reads

$$H^i(\mathscr{A}) \to H^{i+1}(\mathscr{L}) \to H^{i+1}(\mathscr{M}).$$

In particular, since $\operatorname{cd}\mathscr{A} \leq \operatorname{cd}\mathscr{M}$, one has $\operatorname{cd}\mathscr{L} \leq \operatorname{cd}\mathscr{M} + 1$. □

Proof of Prop. 10.17. By the Lemma, any maximal standard subalgebra $\mathscr{M}$ of $\mathscr{L}$ has $\mathrm{cd} \geq \mathrm{cd}\,\mathscr{L} - 1$. Moreover, since $\mathscr{M}$ is quadratic as well, one can construct a sequence $(\mathscr{M}^{(i)})$ of quadratic subalgebras of $\mathscr{L}$ such that $\mathscr{M}^{(0)} = \mathscr{L}$, $\mathscr{M}^{(i+1)}$ is a maximal standard subalgebra of $\mathscr{M}^{(i)}$ and $\mathrm{cd}\,\mathscr{M}^{(i+1)} \geq \mathrm{cd}\,\mathscr{M}^{(i)} - 1$. Since $\mathscr{L}$ is finitely generated in degree 1, one has $\mathscr{M}^{(j)} = 0$ for some $j \gg 0$. It follows that for some i, one has $\mathrm{cd}\,\mathscr{M}^{(i)} = \mathrm{cd}\,\mathscr{L} - 1$.

Now, we argue by induction on $\mathrm{cd}\,\mathscr{L}$. If $\mathrm{cd}\,\mathscr{L} \leq 2$ we have seen above that $\mathscr{L}$ is cohomologically homogeneous. By induction, $\mathscr{M}^{(i)}$ is cohomologically homogeneous and $\mathrm{cd}\,\mathscr{M}^{(i)} = \mathrm{cd}\,\mathscr{L} - 1$, and so is $\mathscr{L}$. □

Notice that the proof shows that, for a quadratic Lie algebra $\mathscr{L}$ to be cohomologically homogeneous, it is enough that it contains a sequence $(\mathscr{M}^{(i)})$ of quadratic subalgebras such that $\mathscr{M}^{(0)} = \mathscr{L}$ and $\mathscr{M}^{(i+1)}$ is a maximal standard subalgebra of $\mathscr{M}^{(i)}$, i.e., that $\mathscr{L}$ has a *quadratic filtration*.

Nevertheless, having a quadratic (or even Koszul) filtration is not equivalent to being BK, as the following example shows.

Example 10.19. *Let $\mathscr{L}$ be the Lie algebra as of Example 4.3-(4), i.e.,*

$$\mathscr{L} = \left\langle x_1, x_2, x_3, x_4 \,\middle|\, [x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_1] \right\rangle .$$

Put $\mathscr{M}^{(i)} = \langle x_1, \ldots, x_{4-i} \rangle$. It is easy to see that $\mathscr{M}^{(i)}$ is a retract of $\mathscr{L}$, and hence is quadratic. However, $\mathscr{L}$ is not BK.

One can also see that such a filtration consists of Koszul algebras.

Although BK Lie algebras are far from being finite-dimensional (except when they are abelian), they share a feature with them.

Proposition 10.20. *Let $\mathfrak{g}$ be a finite dimensional $\mathbb{N}$-graded Lie algebra. Then, $\mathrm{cd}\,\mathfrak{g} < \infty$ and $\mathfrak{g}$ is cohomologically homogeneous.*

Proof. The fact that finite-dimensional Lie algebras have finite cohomological dimension is classical and relies on the realizability of the cohomology ring $H^\bullet(\mathfrak{g}, k)$ as that of the finite dimensional Chevalley-Eilenberg differential algebra.

Let $\mathfrak{g}$ be minimally generated by elements v_i, t and let I be the ideal of $\mathfrak{g}$ generated by v_i. The sequence $0 \to I \to \mathscr{L} \to \langle t \rangle \to 0$ is split exact, so that $\mathfrak{g} = I \rtimes t$. By using the long-exact sequence induced by Theorem 3.12, $\mathrm{cd}\,\mathfrak{g} \leq \mathrm{cd}\,I + 1$. If $\mathrm{cd}\,I = \mathrm{cd}\,\mathfrak{g}$, one can repeat the argument, which stops by finite dimensionality of $\mathfrak{g}$. By induction on the dimension of $\mathfrak{g}$, one can easily conclude. □

One can also introduce a slightly different notion of cohomologically homogeneous Lie algebras only involving the diagonal part of the cohomology ring, which has the advantage of allowing one to treat all graded Lie algebras just as if they were quadratic .

Definition 10.21. *Let $\mathscr{L}$ be a Lie algebra. We say that $\mathscr{L}$ has diagonal-cohomology dimension* $\operatorname{dcd}\mathscr{L} \leq n$ *if* $dH^{n+1}(\mathscr{L}, k) = H^{n+1,n+1}(\mathscr{L}, k) = 0$. *We say that $\mathscr{L}$ is* ***diagonal-cohomologically homogeneous*** *if, for all* $i \leq \operatorname{dcd}\mathscr{L}$, *there exists a subalgebra $\mathscr{M}$ of $\mathscr{L}$ with* $\operatorname{dcd}\mathscr{M} = i$.

Notice that there is no obvious group-theoretic counterpart to such a phenomenon, as there is no natural way to define the “diagonal part” of the group cohomology. However, this can be achieved by considering the maximal 1-generated subalgebra of the cohomology ring $H^\bullet(G, R)$ for some ring R, which yet may not be quadratic.

Recall that the diagonal part of the cohomology equals the Koszul dual of the quadratic cover $\mathrm{q}\mathscr{L}$ of $\mathscr{L}$. In particular, the diagonal-cohomology dimension of $\mathscr{L}$ and $\mathrm{q}\mathscr{L}$ coincide. Moreover, $\operatorname{dcd}\mathscr{L} = \max\left\{n \geq 0 \mid \mathrm{q}\mathscr{L}^!_n \neq 0\right\}$.

Proposition 10.22. *All finitely generated graded Lie algebras are diagonal-cohomologically homogeneous.*

Proof. First of all, notice that the diagonal-cohomological dimension is finite, as $(\mathrm{q}\mathscr{L})^!$ is a finite dimensional algebra. Put $n = \operatorname{dcd}\mathscr{L} < \infty$ and argue by induction on n. If $n = 1$, there is nothing to prove.

So let $n > 1$. By induction, it is enough to prove that $\mathscr{L}$ contains a subalgebra of diagonal-cohomology dimension $n - 1$.

Consider the quadratic cover $\mathrm{q}\mathscr{L}$ and let $\mathscr{M}$ be a maximal standard subalgebra of $\mathrm{q}\mathscr{L}$. Then, as in Lemma 10.18, there is a decomposition $\mathrm{q}\mathscr{L} = \mathrm{HNN}_\phi(\mathscr{M}, t)$, with $\phi : \mathscr{A} \to \mathscr{M}$, and a long exact sequence

$$\cdots \to H^{i,j+1}(\mathscr{M}) \to H^{ij}(\mathscr{A}) \to H^{i+1,j+1}(\mathrm{q}\mathscr{L}) \to H^{i+1,j+1}(\mathscr{M}) \to \dots$$

If $H^{n,n}(\mathscr{M}) = 0$, then $H^{n-1,n-1}(\mathscr{A})$ surjects onto $H^{n,n}(\mathscr{L}) \neq 0$, and hence $\operatorname{dcd}\mathscr{A} \geq n - 1$. Eventually, since the composite functor $(\mathrm{q}\_)^!$ sends monomorphims to epimorphisms, we deduce that $\operatorname{dcd}\mathscr{A} \leq \operatorname{dcd}\mathscr{M} \leq n - 1$.

Denote by $\pi : \mathrm{q}\mathscr{L} \to \mathscr{L}$ the quadratic cover of $\mathscr{L}$. Since π is an isomorphism in degree 1 and 2, we recover a natural isomorphism $\mathrm{q}\pi(\mathscr{M}) \simeq \mathrm{q}\mathscr{M}$, and hence we can identify such Lie algebras. In turn, this shows that $n - 1 = \operatorname{dcd}\mathscr{M} = \operatorname{dcd}\mathrm{q}\pi(\mathscr{M}) = \operatorname{dcd}\pi(\mathscr{M})$, and hence $\pi(\mathscr{M})$ is the subalgebra of $\mathscr{L}$ we were looking for. ∎

§ III. Combinatorial structures

In this last part of the work we will give many examples of Lie algebras defined by some combinatorial structure and, for each class, we will provide a characterisation of BK Lie algebras in terms of the defining combinatorial structure.

11. Graph product of Lie algebras

In this section we will consider a classical construction which allows one to encode in a combinatorial object the ingredients for defining some Lie algebras. Using graphs, one can define large classes of Lie algebras for which one can tell properties of the Lie algebra just by looking at the structure of the underlying graph.

By a **graph**, we will always mean a finite simplicial graph, i.e., a pair $\Gamma = (V, E)$ where V is a set, the vertex-set, and E is a set of 2-element subsets of V, the edge-set. An induced subgraph of Γ is a pair (V', E') where $V' \subseteq V$ and E' is the set of pairs $\{v, w\} \in E$ where $v, w \in V'$.

Definition 11.1. *Let $\mathcal{C}$ be a class of Lie algebras and let $\Gamma = (V, E)$ be a graph. A $\mathcal{C}$-labelling of Γ is a function $v \mapsto \mathscr{L}_v$ that associates an element of $\mathcal{C}$ to every vertex of Γ. To a $\mathcal{C}$-labelled graph Γ one can associate a Lie algebra, the **graph product**,*

$$\mathscr{P}_\Gamma = \left\langle \mathscr{L}_v : v \in \mathscr{V}(\Gamma) \,\middle|\, R \right\rangle$$

where R is the set consisting of the relations of each Lie algebra $\mathscr{L}_v$ ($v \in V$) and the elements $[x, y]$, for every $x \in \mathscr{L}_v$ and $y \in \mathscr{L}_{v'}$, whenever $\{v, v'\}$ is an edge in Γ.

Example 11.2. ***Right-angled Artin (graded) Lie algebras*** *(or RAAG) are the graph products of arbitrary $\{k\}$-labelled graphs, i.e., graphs labelled with the class consisting only of the standard abelian Lie algebra k of dimension 1. For instance, the RAAG Lie algebra of the square graph is that of Example 4.3-(4). Explicitly, if $\Gamma = (V, E)$ is a graph, we define*

$$\mathscr{L}_\Gamma = \left\langle x_v : \ v \in V \,\middle|\, [x_v, x_w] : \ \{v, w\} \in E \right\rangle .$$

Henceforth, we will identify the vertex set V of Γ with the (canonical) generating system of $\mathscr{L}_\Gamma$.

*By Theorem 4.15 and [7, Thm. 1.2], it follows that a RAAG Lie algebra is BK if, and only if, its underlying graph is of **elementary type** (or trivially perfect), i.e., it does not contain neither the square graph nor the line of length 3 as induced subgraphs. This amounts to saying that for the class of RAAG Lie algebras, being BK is equivalent to belonging to ET($\mathcal{T}$), where $\mathcal{T} = \{0\}$ is the trivial class.*

In section 13 we will realise such a class of Lie algebras by using a different kind of combinatorial structure. RAAG Lie algebras have also been treated in [58].

Let $\mathscr{M}$ and $\mathscr{L}$ be two Lie algebras. Recall that $\mathscr{M}$ is said to be a **retract** of $\mathscr{L}$ if there is a pair of maps $\iota : \mathscr{M} \to \mathscr{L}$ and $\sigma : \mathscr{L} \to \mathscr{M}$ such that the composition $\sigma \circ \iota$ is the identity on $\mathscr{M}$. In that case, ι is injective and σ is surjective. The map σ is called retraction and we also say it is a split surjection. This amounts to saying that the identity of $\mathscr{M}$ splits as $\mathscr{M} \to \mathscr{L} \to \mathscr{M}$.

Now, consider the induced morphism $\iota^* : H^\bullet(\mathscr{L}) \to H^\bullet(\mathscr{M})$; it is the restriction homomorphism corresponding to the embedding $\mathscr{M} \overset{\iota}{\hookrightarrow} \mathscr{L}$. Since $(\sigma \circ \iota)^* = \iota^* \circ \sigma^*$ is the identity on $H^\bullet(\mathscr{M})$, the restriction ι^* is surjective.

Lemma 11.3. *Let Γ be a C-labelled graph and let Δ be an induced subgraph. Then, $\mathscr{P}_\Delta$ is naturally a retract of $\mathscr{P}_\Gamma$.*

Proof. It is easy to see that the identity maps $\mathscr{L}_v \to \mathscr{L}_v$ ($v \in \Delta$) induce a Lie algebra homomorphism $\iota : \mathscr{P}_\Delta \to \mathscr{P}_\Gamma$. On the other hand, the identity maps $\mathscr{L}_v \to \mathscr{L}_v$ ($v \in \Delta$) and the zero maps $\mathscr{L}_w \to 0$ ($w \notin \Delta$) induce a morphism $\sigma : \mathscr{P}_\Gamma \to \mathscr{P}_\Delta$ such that $\sigma \circ \iota$ is the identity on $\mathscr{P}_\Delta$. □

As an application, we prove that there is a "Tits' alternative" on the standard 2-generator subalgebras of the RAAG Lie algebra $\mathscr{L}_\Gamma$.

Proposition 11.4. *Let Γ be a finite graph and let x and y be elements of degree 1 in the RAAG Lie algebra $\mathscr{L}_\Gamma$. Then, $\langle x, y\rangle$ is either free or abelian.*

Proof. If Γ is a single vertex, there is nothing to prove.

Assume Γ has at least 2 vertices. Denote by Γ_x the induced subgraph of Γ spanned by the vertices which appear with non-trivial coefficient in the expression of x with respect to the canonical basis. Similarly define Γ_y.

If all vertices v in Γ_x and w in $\Gamma_y \setminus \{x\}$ are adjacent in Γ, then $[x, y] = 0$.

Assume that $v \in \Gamma_x$ and $w \in \Gamma_y \setminus \{x\}$ are not adjacent and let Δ be the induced subgraph of Γ on the vertices v, w. Consider the natural epimorphism $\pi : \mathscr{L}_\Gamma \to \mathscr{L}_\Delta$. Since $\pi|_{\langle x,y\rangle}$ is surjective and $\mathscr{L}_\Delta$ is free, it follows that $\langle x, y\rangle \simeq \mathscr{L}_\Delta$ is also free. □

Recall that, given a simplicial graph Γ and a vertex v, one can define the following subgraphs:

(1) The **star** of v in Γ, denoted $\mathrm{St}_\Gamma(v)$, is the induced subgraph of Γ spanned by all the vertices at distance at most 1 from v;
(2) The **link** of v in Γ, denoted $\mathrm{Lk}_\Gamma(v)$, is the induced subgraph of Γ spanned by all the vertices adjacent to v.

Let Γ be a $\mathcal{C}$-labelled graph. If $\mathcal{C}$ is the class of all Koszul Lie algebras, we say that Γ is a Koszul-labelled graph (or a **K-graph**).

Lemma 11.5. *Let Γ be a K-graph. Then, $\mathscr{P}_\Gamma$ is Koszul.*

Proof. We argue by induction on the number of vertices n. If $n = 1$, there is nothing to prove. Let $n \geq 2$. If Γ is complete, then $\mathscr{P}_\Gamma$ is the direct sum of all $\mathscr{L}_v$ and hence it is Koszul.

Let $n > 2$ and suppose that Γ is not complete. Fix two non adjacent vertices $v, v' \in V$. Denote by Λ and Λ' the induced $\mathcal{C}$-labelled graphs obtained by removing respectively v and v' from Γ. Denote by Δ the intersection of Λ and Λ'. Then, $\mathscr{P}_\Gamma$ is clearly isomorphic to the free product of $\mathscr{P}_\Lambda$ and $\mathscr{P}_{\Lambda'}$ with amalgamated subalgebra $\mathscr{P}_\Delta$. Since by induction all these algebras are Koszul, the same is true for their amalgamated free product, by Lemma 3.13 □

The latter proves — without using Fröberg theorem [13] — that all right-angled Artin Lie algebras are Koszul.

For a K-graph Λ define P_Λ to be the graded-commutative algebra generated by $(H^\bullet(\mathscr{L}_v))_{v\in\mathscr{V}(\Lambda)}$ subjected to the relations

$$xy = 0$$

whenever $x \in A_v$ and $y \in A_w$ where $\{v, w\}$ is an edge in Λ.

Proposition 11.6. *Let Γ be K-graph. Then*

$$H^\bullet(\mathscr{P}_\Gamma, k) \simeq P_{\Gamma^{op}},$$

where Γ^{op} is the opposite graph of Γ with set of edges complementary with that of E.

Proof. Since $\mathscr{P}_\Gamma$ is Koszul, $H^\bullet(\mathscr{P}_\Gamma) \simeq \mathscr{P}_\Gamma^!$ and the result follows just by linear algebra. □

Theorem 11.7. *Let Γ be a K-graph. Then, $\mathscr{P}_\Gamma$ is BK if, and only if, all the followings hold:*

(0) *$\mathscr{L}_v$ is BK for all $v \in \mathscr{V}(\Gamma)$.*
(1) *Γ is a graph of elementary type (i.e., it does not contain neither a square graph nor a line of length 3 as induced subgraphs),*
(2) *Let Δ_+ be the subgraph spanned by those vertices v such that $\mathscr{L}_v$ is not abelian. Then Δ_+ is totally disconnected.*
(3) *If $\mathscr{L}_v$ is not abelian, then the link of v is complete.*

Proof. Condition (0) clearly needs to be satisfied.

Notice that $\mathscr{P}_\Gamma$ always contains the right-angled Artin Lie algebra $\mathscr{L}_\Gamma$ as a standard subalgebra. It follows that if Γ is not of elementary type, then $\mathscr{P}_\Gamma$ is not BK, by Example 11.2.

If $\mathscr{L}_v$ and $\mathscr{L}_{v'}$ are not abelian for two adjacent vertices v and v', then $\mathscr{L}_v \sqcap \mathscr{L}_{v'}$ is a standard subalgebra of $\mathscr{P}_\Gamma$ and hence $\mathscr{P}_\Gamma$ is not Bloch-Kato.

Assume that $\mathscr{L}_v$ is not abelian and $\mathrm{Lk}_\Gamma(v)$ is not complete. If $\mathscr{L}$ is the graph product on $\mathrm{Lk}_\Gamma(v)$, then $\mathscr{P}_{\mathrm{St}_\Gamma(v)} = \mathscr{L}_v \sqcap \mathscr{L}$ is a standard subalgebra of $\mathscr{P}_\Gamma$. Since $\mathscr{L}$ is not abelian, $\mathscr{P}_\Gamma$ is not BK, by Proposition 4.4.

Conversely, assume (1), (2) and (3). We argue by induction on the number of vertices. The base case is trivial. We may now assume that Γ is connected, and that Γ' is a C-labelled graph such that $\mathscr{P}_{\Gamma'}$ is BK and $\Gamma = \nabla(\Gamma')$, the *cone* on Γ' (for the definition of the cone, see Section 13.4).

Notice that conditions (1),(2) and (3) are inherited by induced subgraphs. Hence, by induction, $\mathscr{P}_{\Gamma'}$ is BK.

If $\mathscr{L}_v$ is abelian, then $\mathscr{P}_\Gamma = \mathscr{P}_{\Gamma'} \sqcap \mathscr{L}_v$ is BK.

If $\mathscr{L}_v$ is not abelian, then, by (2), $\mathscr{L}_w$ is abelian for all vertices w of Γ', and by (3), the subgraph $\Gamma' = \mathrm{Lk}_\Gamma(v)$ is complete. It follows that $\mathscr{P}_\Gamma$ is the direct sum of $\mathscr{L}_v$ with an abelian Lie algebra.

This construction proves that $\mathscr{P}_\Gamma$ belongs to $\mathrm{ET}(\mathcal{C})$. □

12. Holonomy Lie algebras

In this section we introduce holonomy Lie algebras of hyperplane arrangements. The study of the BK property for such Lie algebras is the aim of the next section, where we introduce a new type of construction which both generalises the above-mentioned holonomies, as well as RAAG Lie algebras.

12.1. Hyperplane arrangements.

Definition 12.1. *Let $\mathbb{F}$ be a fixed field and ℓ a natural number. By a* ***hyperplane arrangement*** *in $V = \mathbb{F}^\ell$ we mean a finite set $\mathcal{A}$ of linear subspaces $H \leq V$ of dimension $\ell - 1$.*

In literature such objects are often called *central* arrangements, in order to distinguish them from *affine* arrangements, whose hyperplanes need not to be vector subspaces of V. With $\mathcal{A}$, one can associate its **intersection lattice** $\mathbb{L}(\mathcal{A})$ whose underlying set is given by all possible intersections of (possibly empty) subsets of $\mathcal{A}$. The partial order in $\mathbb{L}(\mathcal{A})$ is opposite to the natural inclusion order, so that $\hat{0} := \mathbb{F}^\ell$ (corresponding to the null intersection) and $\hat{1} := \bigcap \{H \mid H \in \mathcal{A}\}$ are the least and greatest elements in $\mathbb{L}(\mathcal{A})$, respectively. The lattice $\mathbb{L}(\mathcal{A})$ is ranked, that is all chains $(V < L_1 < \cdots < L_m = L)$ from V to any element $L \in \mathbb{L}(\mathcal{A})$ have the same length. The length m of such a chain is the **rank** of L. For a ranked lattice $\mathbb{L}$, and integers $i < j$, one may consider the ranked poset $\mathbb{L}_{ij}$ consisting of all elements of rank $r \in \{i, i+1, \ldots, j\}$.

In order to study hyperplane arrangements, one may define some algebraic structures. For instance, if $\mathbb{F} = \mathbb{C}$ then the complement of $\mathcal{A}$ is a manifold (being an open subset of $\mathbb{C}^\ell$) and thus one may study its fundamental group or its cohomology algebra.

By works of Brieskorn's [5], it turns out that the cohomology with complex coefficients of the complement X of a complex hyperplane arrangement only depends on its combinatorial structure, i.e., on the lattice $\mathbb{L}(\mathcal{A})$. In fact, such a cohomology ring is the Orlik-Solomon algebra of the hyperplane arrangement.

Another important algebraic structure associated with a hyperplane arrangement is the *holonomy Lie algebra.* For an arbitrary manifold M and a field k, the holonomy k-Lie algebra can be defined as the quotient of the tensor algebra on the first homology group $H_1(M,k)$ by the image of the dual of the cup-product $H_2(M,k) \to H_1(M,k) \wedge H_1(M,k)$.

Kohno [20] also proved that, for a complex arrangement, the graded $\mathbb{C}$-Lie algebra associated to the lower central series of the fundamental group of X is precisely the holonomy Lie algebra of X. He also gave a presentation of such a Lie algebra in terms of the intersection lattice.

In a similar fashion, one can define such objects for hyperplane arrangement defined over arbitrary fields.

The holonomy Lie algebra is a quadratic Lie algebra, and its quadratic dual is the quadratic cover of the Orlik-Solomon algebra [43]. However, in general the Orlik-Solomon algebra needs not to be quadratic, but it is known that it is when the arrangement is supersolvable. In that case, the Orlik-Solomon algebra is Koszul. In fact, there is no known non-supersolvable arrangement with Koszul Orlik-Solomon algebra, yet quadratic Orlik-Solomon algebras are not necessarily Koszul (see [43] and Example 14.9).

12.2. **The holonomy Lie algebra of a hyperplane arrangement.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement in $V = \mathbb{F}^\ell$.

For every i, let $\alpha_i \in V^*$ be a linear form such that $\ker \alpha_i = H_i$. Thus, α_i is uniquely defined up to the multiplication by a non-zero scalar. In particular, giving a hyperplane arrangement in V is equivalent to giving a finite set of points in the projective space $\mathbb{P}(V^*)$ of the dual V^*.

Define the **holonomy algebra** $\mathscr{U}(\mathcal{A})$ of $\mathcal{A}$ over k as the standard k-algebra given by the following presentation

(12.1)

$$\left\langle x_1, \dots, x_n \,\middle|\, \left[x_i, \sum_{j \in X} x_j\right] : \; X \subseteq \mathcal{A} \text{ maximal s.t. } i \in X, \; \dim \bigcap_{j \in X} H_j = \ell - 2 \right\rangle$$

Note that $\mathscr{U}(\mathcal{A})$ is a quadratic k-algebra. If $\operatorname{char} k = 0$, we denote by $\mathscr{L}(\mathcal{A})$ the Lie algebra of primitive elements of $\mathscr{U}(\mathcal{A})$ (see the beginning of Section 5.1), that is the Lie algebra given by the presentation as in 12.1, but in the category of k-Lie algebras. If $\operatorname{char} k > 0$, the primitive elements of $\mathscr{U}(\mathcal{A})$ naturally form a restricted p-Lie algebra and we denote by $\mathscr{L}(\mathcal{A})$ the ordinary Lie subalgebra generated by elements of degree 1. As we have seen in Section 5, if p is odd, one can treat the restricted p-Lie algebras in the same way.

Let $(\mathbb{L}(\mathcal{A}), \preceq)$ be the intersection lattice of $\mathcal{A}$. For an integer r, set

$$\mathbb{L}_r(\mathcal{A}) = \left\{ L \in \mathbb{L}(\mathcal{A}) \;\middle|\; \dim L = \ell - r \right\}.$$

The elements of $\mathbb{L}_r(\mathcal{A})$ are called **flats of rank** r. For a flat $L \in \mathbb{L}(\mathcal{A})$, consider the open interval $\mathcal{A}_L =]0, L[= [0, L] \setminus \{0, L\}$ in $\mathbb{L}(\mathcal{A})$, e.g., $\mathcal{A}_L = \left\{ H \in \mathcal{A} \;\middle|\; H \preceq L \right\}$ if L has rank 2. The holonomy algebra can thus be presented as the algebra generated by a set $\{x_H\}$ in bijection with the set $\mathcal{A}$ and subject to the relations

$$\left[x_H, \sum_{K \in \mathcal{A}_L} x_K \right] : \; L \in \mathbb{L}_2(\mathcal{A}), \; H \in \mathcal{A}_L.$$

It was shown by Kohno [20] that, for a complex arrangement $\mathcal{A}$, there is a natural isomorphism of the complex holonomy Lie algebra $\mathscr{L}(\mathcal{A})$ with the graded Lie algebra $\mathbb{C} \otimes_{\mathbb{Z}} \operatorname{gr} \pi_1(X, x)$, where X is the complement of the arrangement in $\mathbb{C}^\ell$.

12.3. **Orlik-Solomon algebra.** In complete analogy with the presentation discovered by Brieskorn for the Orlik-Solomon algebra of a complex hyperplane arrangement, one can define the so-called Orlik-Solomon algebra $A(\mathcal{A})$ associated with $\mathcal{A}$.

Fix a hyperplane arrangement $\mathcal{A}$ defined by linear forms $\alpha_1, \ldots, \alpha_n \in V^*$ and denote by $H_i = \ker \alpha_i$ the hyperplane defined by α_i. The **Orlik-Solomon algebra** over k of $\mathcal{A}$ is the quotient of the exterior algebra on $x_1, \ldots, x_n$ by the ideal generated by

$$\sum_{j=1}^{p} (-1)^j x_{i_1} \ldots \hat{x}_{i_j} \ldots x_{i_p}$$

where $\{\alpha_{i_1}, \ldots, \alpha_{i_p}\}$ is linearly dependent in V^*, i.e., $\dim(H_{i_1} \cap \cdots \cap H_{i_p}) \neq \ell - p$. Notice that $A(\mathcal{A})$ is a connected graded-commutative k-algebra.

The Orlik-Solomon algebra is a useful object that allows one to treat by algebraic means the combinatorics of a hyperplane arrangement.

Theorem 12.2 (Brieskorn [5]). *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$. Then the complex Orlik-Solomon algebra $A(\mathcal{A})$ is isomorphic with the cohomology ring $H^\bullet(X, \mathbb{C})$ of the space*

$$X = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$$

However, in the present article, we focus only on the quadratic part of the Orlik-Solomon algebra for its connection with the holonomy algebra, which was first noticed by Shelton and Yuzvinsky in [43].

Theorem 12.3. *Let $\mathcal{A}$ be a hyperplane arrangement over any field $\mathbb{F}$ and let $\mathrm{q}A$ be the quadratic cover of $A(\mathcal{A})$. Then*

$$\mathscr{U}(\mathcal{A})^! \simeq \mathrm{q}A$$

As an example, we now show that every abelian Lie algebra is the holonomy Lie algebra of some hyperplane arrangement.

Let $\mathcal{A}$ and $\mathcal{B}$ be hyperplane arrangements in $\mathbb{F}^{\ell}$ and $\mathbb{F}^{\ell'}$, respectively. Define the **product** $\mathcal{A} \times \mathcal{B}$ as the set of hyperplanes $\hat{H} = H \times \mathbb{F}^{\ell'}$ and $\hat{H}' = \mathbb{F}^{\ell} \times H'$, where $H \in \mathcal{A}$ and $H' \in \mathcal{B}$.

Lemma 12.4. *If $\mathcal{A}$ and $\mathcal{B}$ are hyperplane arrangements, then $\mathscr{L}(\mathcal{A} \times \mathcal{B}) = \mathscr{L}(\mathcal{A}) \sqcap \mathscr{L}(\mathcal{B})$. In particular, abelian Lie algebras are holonomy Lie algebras of some arrangement.*

Proof. Let α_H be the defining polynomial of $H \in \mathcal{A} \cup \mathcal{B}$. For $H \in \mathcal{A}$, let $\hat{\alpha}_H = \alpha_H \times \mathbb{F}^{\ell'} : \mathbb{F}^{\ell+\ell'} \to \mathbb{F}$ be the extension of α_H to the space $\mathbb{F}^{\ell} \times \mathbb{F}^{\ell'}$. For $H' \in \mathcal{B}$, define $\hat{\alpha}_{H'}$ similarly.

The sets $\{\hat{\alpha}_H \mid H \in \mathcal{A}\}$ and $\{\hat{\alpha}_{H'} \mid H' \in \mathcal{B}\}$ are mutually independent. In particular, for $H_1, H_2 \in \mathcal{A}$ and $H' \in \mathcal{B}$, the codimension of $\hat{H}_1 \cap \hat{H}_2 \cap \hat{H}'$ is 3. Thus, $\forall H \in \mathcal{A}$, $H' \in \mathcal{B}$, the set $(\mathcal{A} \times \mathcal{B})_L$, where $L = \hat{H} \cap \hat{H}'$, consists of just the 2 hyperplanes $\hat{H}$ and $\hat{H}'$.

This shows that each generator corresponding to some hyperplane $\hat{H}$, $H \in \mathcal{A}$, commutes with every generator corresponding to elements of $\mathcal{B}$.

The second part of the statement thus follows by induction from the fact that the abelian Lie algebra of dimension 1 is the holonomy of any hyperplane arrangement consisting of a single hyperplane. □

12.4. **Supersolvable arrangements.** Supersolvable arrangements have been deeply studied in recent years (e.g., [43], [12], [17]) because they provide well-behaved algebraic and combinatorial structures.

Given a hyperplane arrangement $\mathcal{A}$, one defines a **circuit** as a sequence of hyperplanes $(H_{i_1}, \dots, H_{i_p})$ such that their functionals $\{\alpha_{i_1}, \dots, \alpha_{i_p}\}$ form a minimal dependent set, i.e., $\{\alpha_{i_1}, \dots, \hat{\alpha}_{i_j}, \dots, \alpha_{i_p}\}$ is linearly independent $\forall j = 1, \dots, p$. A **broken circuit** is a sequence $(H_{i_1}, \dots, H_{i_p})$ such that $i_1 < \cdots < i_p$ and $(H_{i_1}, \dots, H_{i_p}, H_j)$ is a circuit for some $j > i_p$. Finally the arrangement $\mathcal{A}$ is called **supersolvable** if every minimal broken circuit consists of two hyperplanes. As it was proved by Terao [52], this is equivalent to requiring that the intersection lattice has a maximal flag of modular elements and also to being **fiber-type**, i.e., there is a filtration $\emptyset \subsetneq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_\ell = \mathcal{A}$ satisfying the following statements:

(1) The rank of $\mathcal{A}_i$ is i, and
(2) For all hyperplanes $H, H' \in \mathcal{A}_i$, there is some $j < i$ and a hyperplane $V \in \mathcal{A}_j$ such that $H \cap H' \cap V$ has codimension 2.

Moreover, the numbers $d_i = |\mathcal{A}_i \setminus \mathcal{A}_{i-1}|$ are called the **exponents** of the arrangements.

Theorem 12.5 (Shelton-Yuzvinsky [43])**.** *If $\mathcal{A}$ is supersolvable, then $\mathscr{U}(\mathcal{A})$ and $A(\mathcal{A})$ are Koszul algebras and Koszul dual to each other. Moreover, the Hilbert* *series of $A(\mathcal{A})$ can be written in terms of the exponents d_i of $\mathcal{A}$ as*

$$H_{A(\mathcal{A})}(-t) = \prod_{i=1}^{\ell}(1 - d_i t).$$

In particular, we see that the eigenvalues of the holonomy Lie algebra of a supersolvable hyperplane arrangement are all real numbers, and hence we get the following constraints on its first Betti numbers by Proposition 6.9.

Corollary 12.6. *If $\mathcal{A}$ is a supersolvable hyperplane arrangement of rank ℓ, then*

$$b_2(\mathscr{L}(\mathcal{A})) \leq \frac{\ell - 1}{2\ell} b_1(\mathscr{L}(\mathcal{A}))^2.$$

13. Generalised holonomies

Recall that an abstract simplicial complex is a pair $C = (V, \Phi)$ where V is the set of **vertices** and Φ is a set of subsets of V, the **simplices** of C, containing all the singletons of elements of V and that is closed under taking subsets. By abuse of notation, we often denote by C the set of vertices.

For a hyperplane arrangement $\mathcal{A}$ one can define the following simplicial complex $C(\mathcal{A}) = (C, \Phi)$ with vertex set $C = \mathcal{A}$. The simplices of $C(\mathcal{A})$ are the subsets of $\mathcal{A}_L$ for $L \in \mathbb{L}_2(\mathcal{A})$, i.e., the set of simplices is

$$\Phi = \bigcup_{L \in \mathbb{L}_2(\mathcal{A})} \mathscr{P}(\mathcal{A}_L),$$

where $\mathscr{P}(\_)$ denotes the power-set functor. In particular, the set of **maximal simplices** is $\max \Phi = \{\mathcal{A}_L \mid L \in \mathbb{L}_2(\mathcal{A})\}$. Notice that there exist arrangements with different intersection lattices but with the same associated complexes. Indeed, $C(\mathcal{A})$ is the simplicial atomic complex on the poset $\mathbb{L}_{1,2}(\mathcal{A}) = \mathbb{L}_1(\mathcal{A}) \cup \mathbb{L}_2(\mathcal{A})$ (see [61, §3, sec. 3.1]) and it only depends on the low-rank flats of $\mathcal{A}$.

If C is a simplicial complex, we introduce a Lie algebra in terms of generators and relations depending on C:

$$\mathscr{L}_C := \left\langle C \,\middle|\, [x, x_K] : \ K \in \max \Phi, \ x \in K \right\rangle,$$

where $x_K = \sum_{y \in K} y$. Notice that if $C = C(\mathcal{A})$, then the maximal simplices correspond to the $\mathcal{A}_L$'s, $L \in \mathbb{L}_2(\mathcal{A})$, and hence $\mathscr{L}_C = \mathscr{L}(\mathcal{A})$. We are thus led to call $\mathscr{L}_C$ the **generalised holonomy** Lie algebra associated with C.

For instance, if C is 1-dimensional, $\mathscr{L}_C$ is the RAAG Lie algebra (Example 11.2) associated with the 1-skeleton of C.

13.1. **Admissible complexes.** Let C be a simplicial complex. If maximal simplices intersect on edges or greater simplices, the associated Lie algebra may be abelian even if the complex has no maximal simplex of dimension 1. For instance, the Lie algebra associated with the complex on $\{1, 2, 3, 4\}$ with maximal simplices $\{1, 2, 4\}, \{2, 3, 4\}, \{1, 3, 4\}$ is abelian, and hence it is isomorphic with that associated with the complete graph on 4 vertices. Indeed, the three simplices introduce 6 independent (quadratic) relations and the free Lie algebra on 4 generators has dimension 6 in degree 2, so that no Lie bracket survives in the quotient.

We would like to avoid such a situation. Call a complex **admissible** if the intersection of any two maximal simplices consists of at most one vertex. Such complexes generalise those appearing in hyperplane arrangements. In fact, if $\mathcal{A}$ is a hyperplane arrangement, $C(\mathcal{A})$ is an admissible complex with complete 1-skeleton. However, different complexes still may yield to isomorphic associated Lie algebras, e.g., the 2-simplex and the line of length 2 have the same associated Lie algebra, namely, the direct product of the free Lie algebra on two elements and the 1-dimensional Lie algebra. Though, abelian Lie algebras correspond to unique complexes (see Corollary 13.13).

The just defined class of generalised holonomy Lie algebras on admissible complexes extends both right-angled Artin Lie algebras (those associated to 1-dimensional complexes, or simplicial graphs) and holonomy Lie algebras of hyperplane arrangements. Indeed, 1-dimensional complexes are trivially admissible and if $\mathcal{A}$ is a hyperplane arrangement, for L and L' distinct flats of rank 2, the intersection $\mathcal{A}_L \cap \mathcal{A}_{L'}$ – which corresponds to the intersection of two maximal simplices – has at most one element.

In Section 14.3, we will prove that this is an actual generalisation of holonomy Lie algebras of hyperplane arrangements.

Example 13.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two hyperplane arrangements. Then the complex associated to the product arrangement $\mathcal{A} \times \mathcal{B}$ is the 1-dimensional join of those associated to $\mathcal{A}$ and $\mathcal{B}$, namely, it is the complex on vertex set $C(\mathcal{A}) \sqcup C(\mathcal{B})$ with additional edges between each pair a and b, where a and b are any vertex of $C(\mathcal{A})$ and $C(\mathcal{B})$ respectively. It follows that $\mathscr{L}(\mathcal{A} \times \mathcal{B}) = \mathscr{L}_{C(\mathcal{A}\times\mathcal{B})}$ is the direct sum of $\mathscr{L}(\mathcal{A})$ and $\mathscr{L}(\mathcal{B})$.*

Given an arbitrary hyperplane arrangement, we have concocted a simplicial complex whose associated Lie algebra is isomorphic with the holonomy Lie algebra of the arrangement; moreover, such a complex is admissible and has a complete 1-skeleton. One may thus wonder whether one can make the converse association.

For this, we fix an arbitrary *infinite* field $\mathbb{F}$, and let C be an arbitrary finite admissible simplicial complex with complete 1-skeleton. Let $M = \max C$ be the set of maximal simplices of C and consider a family of injective maps $\phi_K : K \to \mathbb{F}$ from the vertices of the simplex $K \in M$ into the field $\mathbb{F}$. For each simplex $K \in M$, one can consider the line arrangement

$$\mathcal{B}_K = \left\{ h_K(v) \;\middle|\; v \in K \right\}$$

in $\mathbb{F}^2$ with coordinates x^K and y^K, where $h_K(v) = \{(x^K, y^K) \mid x^K = \phi_K(v)y^K\}$. Let $\mathcal{B}$ be the cartesian product of all $\mathcal{B}_K$'s. By intersecting $\mathcal{B}$ with all hyperplanes $x^K - \phi_K(v)y^K = x^{K'} - \phi_{K'}(v)y^{K'}$, for any vertex v belonging to distinct simplices $K, K' \in M$, one obtains and arrangement $\mathcal{A}_C$ consisting of the same number of hyperplanes as the number of vertices of C. Plus, it is easy to see that the complex associated with $\mathcal{A}_C$ is isomorphic with C.

Notice that we have assumed the field $\mathbb{F}$ to be infinite just to ensure that there exist injective maps from simplices of any dimension into $\mathbb{F}$. Otherwise, it is enough to assume that $\mathbb{F}$ has at least n elements, where n is the number of vertices of the largest simplex in C.

We have thus proved that every rank 2 matroid is $\mathbb{F}$-representable for any field $\mathbb{F}$ with enough elements.

Lemma 13.2. *Let $\mathcal{A}$ be a hyperplane arrangement over an arbitrary field k. Then, the center of $\mathscr{L}(\mathcal{A})$ is nontrivial.*

Proof. We identify the canonical generators of $\mathscr{L}(\mathcal{A})$ with the hyperplanes of $\mathcal{A}$ and with the vertices of the complex $C(\mathcal{A})$. Consider the element

$$z = \sum_{x \in \mathcal{A}} x \in \mathscr{L}(\mathcal{A}).$$

Fix a hyperplane y; if $x \in \mathcal{A}$, then the edge $\{x, y\}$ is contained in a single maximal simplex K of $C(\mathcal{A})$. Moreover, any two maximal simplices containing y only intersect in y itself, as $C(\mathcal{A})$ is an admissible complex. Therefore,

$$z = y + \sum_K (x_K - y),$$

where $x_K = \sum_{x \in K} x$, and the sum is taken over all maximal simplices K containing y. Hence

$$[y, z] = \sum_K [y, x_K - y] = 0.$$

□

13.2. **Strongly induced subcomplexes and retractions.** A subset $\mathcal{B}$ of a hyperplane arrangement $\mathcal{A}$ is called a **subarragement** of $\mathcal{A}$. In general, the holonomy of a subarrangement does not embed into that of $\mathcal{A}$. For instance, consider the arrangement $\mathcal{A}$ consisting of the three hyperplanes H_{ij} defined by the linear forms $h_{ij} = z_i - z_j$, $1 \le i < j \le 3$, i.e., $H_{ij} = \{(z_1, z_2, z_3) \mid z_i - z_j = 0\}$ (it is called the braid arrangement of rank 2). Then

$$\mathscr{L}(\mathcal{A}) = \langle x_1, x_2, x_3 \,|\, [x_i, x_1 + x_2 + x_3] : \ i = 1, 2 \rangle .$$

If $\mathcal{B} = \{H_{12}, H_{13}\}$, then $\mathscr{L}(\mathcal{B}) = \langle x_1, x_2 \,|\, [x_1, x_2] \rangle$ does not embed into $\mathscr{L}(\mathcal{A})$ in a natural way. However, notice that $\mathscr{L}(\mathcal{B})$ is a quotient of $\mathscr{L}(\mathcal{A})$.

In fact we can easily derive a general phenomenon: If $\mathcal{B}$ is a subarrangement of $\mathcal{A}$, then there is a natural surjective Lie algebra homomorphism $\mathscr{L}(\mathcal{A}) \to$

$\mathscr{L}(\mathcal{B})$ whose kernel is generated (as an ideal) in degree 1 by those generators corresponding to hyperplanes that do not belong to $\mathcal{B}$.

From the point of view of simplicial complexes, this fact can be translated by saying that the simplicial complex of $\mathcal{B}$ is the maximal subcomplex of $C(\mathcal{A})$ whose vertex-set is $\mathcal{B}$.

Lemma 13.3. *Let C be a simplicial complex and let X be a set of vertices of C. Let D be the maximal subcomplex of C with vertex-set X. Then the identity map on X induces a well defined surjective morphism $\sigma : \mathscr{L}_C \to \mathscr{L}_D$, by setting $\sigma(y) = 0$, for any vertex $y \notin X$ of C.*

Proof. For a generator x corresponding to a vertex of C, set $\sigma(x)$ to be x if x is a vertex of D, and zero otherwise. We prove now that it extends to a Lie algebra homomorphism $\mathscr{L}_C \to \mathscr{L}_D$, i.e., for every maximal simplex K of C and every element x in K, one has

$$[\sigma(x), \sigma(x_K)] = 0. \tag{13.1}$$

Let K be a maximal simplex of C and let x be one of its vertices. Then, $K \cap D$ either is empty or it is a maximal simplex of D.

In the first case, $\sigma(x) = 0$ and hence 13.1 trivially holds. If $K \cap D$ is a maximal simplex of D, one has $\sigma(x_K) = x_{K\cap D}$, which commutes with $\sigma(x)$. □

Nevertheless, in this work we are interested in subalgebras and not in quotients, as the definition of Bloch-Kato Lie algebras suggests. The most naïve attempt to get subalgebras from subcomplexes is that of inverting the map σ. This leads to the definition of strongly induced subcomplexes.

Definition 13.4. *Let C be a simplicial complex and let X be a set of vertices. The maximal subcomplex D having X as the set of vertices is said to be **strongly induced** in C if, for all pairs of distinct vertices $x, y \in X$, every maximal simplex of C containing x and y is a simplex of D.*

We can characterise strongly induced subcomplexes as follows.

Lemma 13.5. *Let C be an admissible complex. Let D be a subcomplex of C whose 1-skeleton is induced in that of C. Then the following are equivalent:*

(1) *D is strongly induced in C;*
(2) *$\forall x, y \in D$, $x \neq y$*

$$\{x, y\} \subseteq K \in \max C \implies K \in \max D$$

(3) *$\max D \subseteq \max C$.*

Proof. (2) is precisely the definition of strongly induced subcomplex, so we shall only prove the equivalence of the latter two conditions.

(2)⇒(3): Let K be a maximal simplex of D. We may assume K contains at least two different vertices x and y. Now, let H be the maximal simplex of C

containing K. Then $x, y \in H$ and hence $H \in \max D$. By maximality of K, we get $K = H \in \max C$.

(3)$\Rightarrow$(2): Let x and y be different vertices in D and assume that there they are contained in some maximal simplex K of C. For the 1-skeleton of D is induced in the 1-skeleton of C, the two vertices x and y are adjacent in D. So, let K' be a maximal simplex of D containing x and y. By (2), one has $K' \in \max C$.

Eventually, notice that K and K' are maximal simplices of C having two different vertices in common, and hence they need to be equal, proving that $K = K' \in \max D$. □

As said above, strongly induced subcomplexes are quite interesting from the algebraic point of view as they provide an embedding of their associated Lie algebra into that associated to the whole complex. In fact, such an embedding admits a left inverse, that is a retraction.

We get the following:

Lemma 13.6. *Let C be a simplicial complex and let D be a strongly induced subcomplex. Then $\mathscr{L}_D$ is naturally a retract subalgebra of $\mathscr{L}_C$ and the restriction $H^\bullet(\mathscr{L}_C) \to H^\bullet(\mathscr{L}_D)$ is surjective. If moreover $\mathscr{L}_C$ is Koszul, then $\mathscr{L}_D$ is Koszul.*

Proof. Analogously to Lemma 13.3, we have a surjection $\sigma : \mathscr{L}_C \to \mathscr{L}_D$, with $\sigma(x) = \chi_D(x)x$, for every vertex x, where χ_D is the indicator function on the vertices of D.

Denote by ι the inclusion $D \hookrightarrow C$. If K is a maximal simplex of D, then it is a maximal simplex of C and hence, for every $x \in K$, one has $[\iota(x), \iota(x_K)] = [x, x_K] = 0$, proving that ι extends to a Lie algebra morphism $\iota : \mathscr{L}_D \to \mathscr{L}_C$.

Since in degree 1 the composite $\sigma\iota$ is the identity on D, it is the identity in each degree and this makes $\mathscr{L}_D$ a retract of $\mathscr{L}_C$.

Assume now $\mathscr{L}_C$ to be Koszul. Then, the bigraded cohomology ring $H^{\bullet,\bullet}(\mathscr{L}_C)$ is concentrated on the diagonal. Since the restriction ι^* preserves both the degrees, $\mathscr{L}_D$ has diagonal cohomology. □

In the next paragraph we will study the embeddability problem for a different kind of subcomplexes.

13.3. **Induced subcomplexes.** As the name suggests, strongly induced subcomplexes are defined by a rather strong property and each simplicial complex contains very few of such subcomplexes. In this section, we will define a weaker notion of subcomplex which allows one to study the BK property for the associated holonomy Lie algebra.

Let X be a set of vertices of a simplicial complex C. The **induced subcomplex spanned by** X in C is the subcomplex D of C with X as set of vertices and such that a set $S \subseteq X$ containing at least two vertices is a maximal simplex in D iff S is a maximal simplex in C. We will denote by $C(X)$ the induced subcomplex of C spanned by X, and a subcomplex of C is called **induced** if it is equal to some $C(X)$.

Remark 13.7. *(1) All strongly induced subcomplexes are induced. In order to see that the converse does not hold, consider a 2-simplex C, with vertices x, y, z. The induced subcomplex D spanned by x and y is disconnected and it is not strongly induced because the maximal simplex C contains x and y but it is not contained in D.*

(2) Though, notice that if D is induced in C, then there is a natural morphism $\mathscr{L}_D \to \mathscr{L}_C$, as the maximal simplices of D are maximal simplices of C. However, a priori, $\mathscr{L}_D$ is no longer a retract of $\mathscr{L}_C$.

Proposition 13.8. *There exists an admissible complex C and a set of vertices X such that $\mathscr{L}_C$ is Koszul and $\mathscr{L}_{C(X)}$ does not embed into $\mathscr{L}_C$.*

Proof. Let C be the complex with set of vertices $\{1,2,3,4\}$ and maximal simplices $\{1,2,3\},\{1,4\},\{3,4\}$. Then

$$\mathscr{L}_C = \left\langle x_1, x_2, x_3, x_4 \,\middle|\, [x_1, x_2 + x_3], [x_3, x_1 + x_2], [x_1, x_4], [x_3, x_4] \right\rangle.$$

Let $X = \{2,3,4\}$. In the Lie subalgebra $\langle x_2, x_3, x_4 \rangle$ spanned by the generators corresponding to vertices X, the relation $[x_4, [x_3, x_2]] = 0$ holds. In particular, we see that such subalgebra is not quadratic, and its quadratic cover is $\mathscr{L}_{C(X)}$.

Notice that $\mathscr{L}_C$ is isomorphic with the RAAG Lie algebra associated with the square graph Γ, i.e., the direct sum of two abelian Lie algebras of rank 2, and hence it is Koszul. Indeed, the relations of $\mathscr{L}_C$ may be rewritten as

$$[x_1, y], [x_3, y], [x_1, x_4], [x_3, x_4]$$

where $y = x_1 + x_2 + x_3$. □

An admissible complex C is said to have the **embedding property** if $\mathscr{L}_{C(X)}$ embeds into $\mathscr{L}_C$ whenever X is a set of vertices of C.

We prove the following

Proposition 13.9. *Let C be an admissible complex with the embedding property, let z be a vertex of C and consider the complementary set X of vertices of z in C.*

(1) *If z is adjacent to every vertex of C, then*

$$\mathscr{L}_C = \mathscr{L}_{C(X)} \rtimes \langle z \rangle = \mathrm{HNN}(\mathscr{L}_{C(X)}, z)$$

(2) *Otherwise, let C_z be the induced subcomplex of C spanned by z and the vertices that are adjacent to z, i.e., the star of z, and let X_z be the set of vertices of C_z different from z. Then,*

$$\mathscr{L}_C \simeq \mathscr{L}_{C_z} \coprod_{\mathscr{L}_{C(X_z)}} \mathscr{L}_{C(X)}$$

Proof. 1) The Lie subalgebra $\langle X\rangle$ is an ideal of $\mathscr{L}_C$, as, $\forall x \in X$,

$$[z,x] = \sum_{y\in K\setminus\{z\}} [x,y] \in \langle X\rangle,$$

where K is the unique maximal simplex containing x and z.

We recover the extension $0 \to \langle X\rangle \to \mathscr{L}_C \to \langle z\rangle \to 0$. In fact, such an extension is trivial, as the quotient $\mathscr{L}/\langle X\rangle$ is 1-dimensional, and hence there is a Lie algebra section $\langle z\rangle \to \mathscr{L}_C$. This proves that $\mathscr{L}_C = \langle X\rangle \rtimes \langle z\rangle = \mathrm{HNN}_\phi(\langle X\rangle, z)$, where $\phi(x) = \sum_{y\in K\setminus\{z\}}[x,y]$, for $x \in X$, if K is the maximal simplex containing both x and z. Also, $\langle X\rangle \simeq \mathscr{L}_{C(X)}$, since C has the embedding property.

2) Since C has the embedding property, $\mathscr{L}_{C(X_z)}$ is naturally a subalgebra of both $\mathscr{L}_{C(X)}$ and $\mathscr{L}_{C_z}$. Also, the diagram

$$\begin{array}{ccc}
\mathscr{L}_{C(X_z)} & \longrightarrow & \mathscr{L}_{C(X)} \\
\downarrow & & \downarrow \\
\mathscr{L}_{C_z} & \longrightarrow & \mathscr{L}_C
\end{array}$$

commutes. We claim that the diagram is a pushout diagram, proving that

$$\mathscr{L}_C \simeq \mathscr{L}_{C(X)} \coprod_{\mathscr{L}_{C(X_z)}} \mathscr{L}_{C_z}$$

Indeed, let $\mathscr{A} = \mathscr{L}_{C(X)}$, $\mathscr{B} = \mathscr{L}_{C_z}$ and $\mathscr{C} = \mathscr{L}_{C(X_z)}$. Let $\mathscr{L} = \mathscr{L}_C$.

Then the above diagram reads

$$\begin{array}{ccc}
\mathscr{C} & \xrightarrow{\gamma_\alpha} & \mathscr{A} \\
{\scriptstyle\gamma_\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\
\mathscr{B} & \xrightarrow[\beta]{} & \mathscr{L}
\end{array}$$

Let $\alpha' : \mathscr{A} \to \mathscr{M}$ and $\beta' : \mathscr{B} \to \mathscr{M}$ be morphisms such that $\alpha'\gamma_\alpha = \beta'\gamma_\beta$.

For an element $x \in C$ set $\phi(x)$ to be $\beta'(z)$ if $x = z$ and $\alpha'(x)$, for $x \neq z$ otherwise.

Then, in degree 1, one has $\phi\alpha = \alpha'$ and $\phi\beta = \beta'$.

We want to prove that ϕ yields a well defined Lie algebra homomorphism $\mathscr{L} \to \mathscr{M}$. So, let $K \in \max C$. If $z \notin K$, then

$$[\phi(x), \phi(x_K)] = [\alpha'(x), \alpha'(x_K)] = \alpha'[x, x_K] = 0,$$

for $x \in K$.

Suppose that $z \in K$. Then, if $x \in K$, one has $\phi(x) = \alpha'(x) = \alpha'\gamma_\alpha(x) = \beta'\gamma_\beta(x) = \beta'(x)$, if $x \neq z$, and $\phi(z) = \beta'(z)$ otherwise, so that, for every $x \in K$, one has

$$[\phi(x), \phi(x_K)] = [\beta'(x), \beta'(x_K)] = \beta'[x, x_K] = 0.$$

Thus ϕ defines a homomorphism $\mathscr{L} \to \mathscr{M}$. In fact, ϕ is the unique homomorphism satisfying $\phi'\alpha = \alpha'$ and $\phi'\beta = \beta'$: If $\phi' : \mathscr{L} \to \mathscr{M}$ is another such homomorphism, as $\mathscr{L}$ is generated in degree 1, one has $\phi'_1 = \phi_1$, and hence $\phi \equiv \phi'$. □

Without the assumption that C has the embedding property, one can only decompose $\mathscr{L}_C$ into HNN-extension and amalgamated free product on the subalgebras generated by the corresponding vertices.

By using Corollary 13.9 in combination with Lemmata 3.15 and 3.13, one can easily prove by induction on the number of vertices:

Proposition 13.10. *If C is a complex with the embedding property, then $\mathscr{L}_C$ is Koszul.*

Example 13.11. *Take the non-admissible complex $\{1,2,3,4\}$ with maximal simplices*

$$\{1,2,4\},\{1,3,4\}.$$

Then

$$[x_1, x_2 + x_4] - [x_1, x_3 + x_4] = [x_1, x_2 - x_3]$$

namely, the sum of relations corresponding to simplices containing x_4 yields an element of $\langle x_1, x_2, x_3\rangle$. In an admissible complex, this situation cannot occur, as in the present example 1 and 4 belong to different maximal simplices, so that their corresponding generators do not appear in a common relation.

Lemma 13.12. *Let C be a finite admissible complex and let X be a set of vertices of C. Then $\mathscr{L}_{C(X)}$ is the quadratic cover of $\langle X\rangle$. In other words, the natural map $\mathscr{L}_{C(X)} \to \mathscr{L}_C$ is injective up to degree 2.*

Proof. The quadratic cover of $\langle X\rangle$ is the quadratic Lie algebra generated by X and with relations $R \cap \mathscr{F}(X)_2$, where $\mathscr{F}(X)$ is the free Lie algebra on X and R is the degree-2 relation space of $\mathscr{L}_C$. □

We are now able to prove that the restriction to admissible complexes actually tells apart abelian Lie algebras.

Corollary 13.13. *Let C be an admissible complex. Then, $\mathscr{L}_C$ is abelian if, and only if, C is a complete graph.*

Proof. First, notice that the generalised holonomy Lie algebra of a simplex K is abelian if, and only if, K is a graph. In fact, the only graph that is also a simplex is a single edge.

Assume that $\mathscr{L}_C$ is abelian and let K be a simplex of maximal dimension in C. The Lie subalgebra generated by K is abelian by hypothesis, and hence quadratic. Since $\mathscr{L}_K$ is the quadratic cover of the Lie subalgebra generated by the vertices of K they must be isomorphic. In particular, $\mathscr{L}_K$ is abelian and K has dimension ≤ 1. Thus, C is a graph. □

Recall that, by Proposition 13.10, if C is a complex with the embedding property, then $\mathscr{L}_C$ is Koszul.

Remark 13.14. *Notice that in Proposition 13.8 we constructed a complex whose associated Lie algebra is Koszul but it does not have the embedding property. In Example 14.9, we will show that there exist holonomy Lie algebras which are not Koszul, and we will explicitly show that the associated complex does not have the embedding property.*

We can also state a partial converse to Theorem 13.10.

Proposition 13.15. *If $\mathscr{L}_C$ is BK, then C has the embedding property.*

Proof. Since C is admissible, the Lie subalgebra $\langle X\rangle$ generated by a subset $X \subset C$ has quadratic cover $\mathscr{L}_{C(X)}$, by Lemma 13.12. Since $\mathscr{L}_C$ is Bloch-Kato, the subalgebra $\langle X\rangle$ is quadratic, and hence isomorphic with $\mathscr{L}_{C(X)}$, by Fact 4.6. □

However, every 1-dimensional complex has the embedding property. Since there exist RAAG Lie algebras which are not BK, this proves that it is not necessary for the Lie algebra to be Bloch-Kato in order that the complex has the embedding property.

This argument shows that for an admissible complex C having the embedding property is strictly weaker than $\mathscr{L}_C$ being Bloch-Kato and strictly stronger than being Koszul.

In the next subsections we will deal with the problem of detecting BK Lie algebras among the class of generalised holonomies. The strategy is similar to that of [48] and [2]: one first finds a small class of simplicial complexes with non-BK holonomy and then prove that those complexes that do not contain such bad induced subcomplexes can be obtained by performing distinguished constructions that reflect into elementary constructions at the Lie algebra level.

13.4. **Simplicial complexes of elementary type.** We start by giving some non-standard terminology for simplicial complexes. An edge in an admissible complex C is said to be **proper** if it is a maximal simplex. Two simplices are properly connected in C if there is some proper edge between two vertices of the simplices. A path of edges in C is called proper if its edges are proper.

Let C be a simplicial complex and let X be a set of vertices of C. Define the **partial cone** $\nabla_X(C)$ as the complex with set of vertices $C \amalg \{*\}$ and such that its maximal simplices are those of C and the proper edges $\{*, v\}$, for $v \in X$. Notice that this definition is not standard, as the dimension of the cone is meant not to increase: one only adds simplices of dimension 1 to the complex, in contrast with the classical definition of cone.

Henceforth, we will call $\nabla_X(C)$ a partial cone only if it is not a cone nor a disjoint union, i.e., $\emptyset \neq X \subsetneq C$. On the other hand, we will call $\nabla_C(C)$ the cone on C, and we will simply denote it by $\nabla(C)$.

We now introduce three classes of admissible complexes that give rise to bad-behaved (in a "Bloch-Kato sense") Lie algebras.

Definition 13.16. *The following will be called* ***forbidden complexes****:*

(1) *Partial cones on simplices of dimension* ≥ 2*, i.e.,* $\nabla_X(\Delta^n)$*, where* $n \geq 2$ *and* $\emptyset \neq X \subsetneq \Delta^n$*;*
(2) *Two simplices of dimension* ≥ 2 *intersecting in just one vertex, i.e.,* $\Delta^n \vee_v \Delta^m$*, where* $n, m \geq 2$*;*
(3) *The square graph and the line of length* 3*.*

Lemma 13.17. *If* C *is a forbidden complex, then* $\mathscr{L}_C$ *is not BK.*

Proof. For any C in the list of forbidden complexes, we construct a non-quadratic standard subalgebra $\mathscr{M}$ of $\mathscr{L}_C$.

(1) Let $C = \nabla_X(\Delta^n)$, for $n \geq 2$ and $\emptyset \neq X \subsetneq \Delta$, and let v_0 be the tip of the cone. Let x_K be the sum of the elements corresponding to the vertices of Δ^n. Choose $x \in X$ and $z \in C \setminus X$. We claim that $\mathscr{M} = \langle x_K, x, z + v_0 \rangle$ is not quadratic.

Indeed, if $a = x_K$, $b = x$ and $c = z + v_0$, then $[a, [b, c]] = 0$. By contradiction, we assume $\mathscr{M}$ to be quadratic. Since, $\mathscr{M}_2 = \mathrm{Span}\{[b, c], [a, c]\}$ has dimension 2, and $[a, b] = 0$, one recovers a presentation $\mathscr{M} \simeq \left\langle a, b, c \,\middle|\, [a, b] \right\rangle$. But then the relation $[a, [b, c]]$ does not hold in $\mathscr{M}$.

(2) Let z be the common vertex of the two simplices and let x and y be sum of the vertices $\neq z$ of Δ^n and Δ^m respectively. Let $x_1 \neq z$ be a vertex of Δ^n. Then the Lie subalgebra generated by $z, x, x_1 + y$ is not quadratic, as

$$\begin{aligned} &[z, [x_1 + y, z + x]] = \\ =&[z, [x_1, z + x]] + [z, [y, z + x]] = \\ =&[z, [y, z + x]] = [z, [y, x]] = \\ =&[[z, y], x] + [y, [z, x]] = 0 \end{aligned}$$

Indeed, $[z, z + x] = 0$ but $[z, x_1 + y] = [z, x_1] \neq 0$.

(3) For the square graph, see Example 4.3-(4). For the line of length 3, see [7]. □

We now introduce a notion of complex of elementary type that mimics and generalises the one that was defined for simplicial graphs in [48].

Definition 13.18. *An admissible complex* C *is said to be of* ***elementary type*** *(or ET) if it can be obtained from simplices of any dimension by applying (total) cones and disjoint unions.*

Notice that $\mathscr{L}_{\nabla(C)} \simeq \mathscr{L}_C \sqcap k$ and $\mathscr{L}_{C \amalg C'} \simeq \mathscr{L}_C \amalg \mathscr{L}_{C'}$. Therefore, the Lie algebras associated with complexes of elementary type are of $\mathcal{T}$-elementary type, where $\mathcal{T}$ is the trivial class, and hence they are BK. Moreover, by Example 11.2, $\mathscr{L}_C$ is a RAAG Lie algebra.

Theorem 13.19. *Let C be an admissible complex. Then C is of ET iff it does not contain any forbidden complex as an induced subcomplex.*

Proof. 1) Let C be admissible without induced subcomplexes isomorphic to any of the forbidden complexes.

We argue by induction on the number of vertices of C. If C consists of just one simplex, then it is of ET.

So, suppose, by induction, that for any $X \subsetneq C$, the induced hull $C(X)$ of X in C is of ET. It is then enough to assume C to be connected.

If C is 1-dimensional, then, by [48], C is a graph of elementary type.

So suppose C has dimension ≥ 2 and has more than one maximal simplex.

Let K be a maximal simplex of dimension ≥ 2.

Since C is connected and properly contains K, there is an element x at distance 1 from K. Let $v \in K$ such that $e = \{x, v\}$ is an edge. Consider $X = C \backslash \{v\}$ and set $C_v = C(X)$.

It is clear that e is proper, for otherwise there would be two different intersecting high dimensional maximal simplices in C.

Since C does not contain partial cones, $C(K \cup \{x\})$ is a total cone on K, i.e., every vertex of K is properly adjacent to x. We have thus proved that any vertex x' adjacent to K is the tip of a cone on K. In particular, C_v is connected and, by induction, it is of ET. Since C_v is not a simplex, it must be a cone, namely, there exists a vertex $v_0 \neq v$ such that $\{v_0, x\}$ is an edge for all $x \neq v, v_0$. Now, since induced subcomplexes of type (1) are forbidden in C, $\{v_0, v\}$ must be a proper edge, and hence $C = \nabla(C(C \setminus v_0))$ is of ET, by induction.

2) For the converse we may use the fact that, as we noticed above, the holonomy of complexes of ET are BK. So, let C be an admissible complex containing an induced subcomplex $C(X)$ isomorphic with one of the three forbidden complexes. Then $\mathscr{M} = \mathscr{L}_{C(X)}$ either is a 1-generated subalgebra of $\mathscr{L}_C$ or it is not quadratic. By Lemma 13.17, $\mathscr{M}$ is not Bloch-Kato. □

13.5. **The BK property for generalised holonomies.** We can now prove without effort the main result of the section.

Theorem 13.20. *Let C be an admissible complex. Then $\mathscr{L}_C$ is BK if, and only if, C is of ET.*

Proof. If C is of ET, then $\mathscr{L}_C$ is Bloch-Kato.

If C is not ET, then there is some X such that $C(X)$ is a forbidden subcomplex. Also, $C(X)$ is an induced subcomplex in C, proving that $\langle X \rangle$ is either not quadratic, or it is naturally embedded into $\mathscr{L}_C$ and isomorphic with $\mathscr{L}_{C(X)}$. Since $\mathscr{L}_{C(X)}$ is not Bloch-Kato, the same is true for $\mathscr{L}_C$. □

Lemma 13.21. *Let Γ be a graph of elementary type. Then all standard Lie subalgebras of $\mathscr{L}_\Gamma$ are right-angled Artin Lie algebras.*

Proof. We translate the pro-p group theoretic proof of [48] to our scope.

We prove it by induction on the number of vertices of Γ. If Γ consists of a single vertex, the result is clear. Suppose that Γ has more than one vertex. Decompose Γ into its connected components Γ_i, $1 \le i \le k$.

If $k \ge 2$, then every component Γ_i is of ET, and hence, by induction, every subalgebra of $\mathscr{L}_{\Gamma_i}$ is a right-angled Artin Lie algebra. Let $\mathscr{M}$ be a 1-generated subalgebra of $\mathscr{L}_\Gamma$; by the Kurosh subalgebra theorem, $\mathscr{M}$ is a free product of a free Lie algebra and subalgebras of the $\mathscr{L}_{\Gamma_i}$'s, and thus it is a right-angled Artin Lie algebra. Explicitly, it is the algebra on the graph obtained from the disjoint union of the graphs corresponding to the intersections $\mathscr{M} \cap \mathscr{L}_{\Gamma_i}$, and a finite number of isolated vertices, equal to the rank of the free part of $\mathscr{M}$.

Now suppose that Γ is connected, and thus it is a cone $\nabla(\Gamma') = \Gamma$, with tip v. Then $\mathscr{L}_\Gamma = \langle v\rangle \times \mathscr{L}_{\Gamma'}$. Let $\phi : \mathscr{L}_\Gamma \to \mathscr{L}_{\Gamma'}$ be the natural projection. Let $\mathscr{M}$ be a 1-generated subalgebra of $\mathscr{L}_\Gamma$. Then we have the following central extension

$$0 \to \mathscr{M} \cap \langle v\rangle \to \mathscr{M} \to \phi(\mathscr{M}) \to 0.$$

We claim that this sequence splits. Since Γ' is ET, by induction $\phi(\mathscr{M})$ is a RAAG Lie algebra, say $\phi(\mathscr{M}) = \mathscr{L}_\Delta$. Let u be a vertex of Δ, and choose $m \in \mathscr{M}$ such that $\phi(m) = u$. As $u \in \mathscr{L}_{\Gamma'}$, one has $\phi(u) = u$, and hence, $u - m \in \ker\phi \le \langle v\rangle$. This means that there is a scalar $\alpha_u \in k$ such that $u + \alpha_u v = m \in \mathscr{M}$. Define $\rho_1 : \phi(\mathscr{M})_1 \to \mathscr{M}_1$ by linearly extending $\rho_1(u) = u + \alpha_u v$, $\forall u \in V(\Delta)$. For $x = \sum_{u\in V(\Delta)} r_u u \in \phi(\mathscr{M})_1 = \mathrm{Span}_k V(\Delta)$, put $\alpha_x = \sum_{u\in V(\Delta)} r_u\alpha_u$, so that

$$\rho_1(x) = x + \alpha_x v,\ \forall x \in \phi(\mathscr{M}).$$

Since v is in the center of $\mathscr{L}_\Gamma$, for $\{x, x'\} \in E(\Gamma')$, we have

$$[\rho_1(x), \rho_1(x')] = [x + \alpha_x v, x' + \alpha_{x'} v] = [x, x'] = 0,$$

whence ρ_1 extends to a well-defined Lie algebra homomorphism $\rho : \phi(\mathscr{M}) \to \mathscr{M}$ Moreover, $\phi\rho(u) = \phi(u + \alpha_u v) = \phi(u) = u$, for every $u \in V(\Delta)$, and thus $\phi\rho = \mathrm{Id}_{\phi(\mathscr{M})}$, i.e., ρ is a section. Now, since $\mathscr{M} \cap \langle v\rangle$ is contained in the center of $\mathscr{M}$, we have $\mathscr{M} = (\mathscr{M} \cap \langle v\rangle) \times \phi(\mathscr{M})$. If $\mathscr{M} \cap \langle v\rangle = 0$, then $\mathscr{M} \le \mathscr{L}_\Delta$ is a RAAG Lie algebra by induction; otherwise, $\mathscr{M}$ contains v, and hence we have $\mathscr{M} \le \mathscr{L}_{\nabla(\Delta)}$. □

Corollary 13.22. *Let C be an admissible complex. Then the following are equivalent:*

(1) *$\mathscr{L}_C$ is BK;*

(2) *If $\mathscr{M}$ is a standard subalgebra of $\mathscr{L}_C$, then there is a graph Γ such that $\mathscr{M} \simeq \mathscr{L}_\Gamma$.*

Proof. Let $\mathscr{L}_C$ be Bloch-Kato. Then, $\mathscr{L}_C$ is a RAAG Lie algebra and C is of ET. We may thus assume that C has dimension 1 and that it does not contain as induced subgraphs neither the square graph nor the line of length 3. By Lemma 13.21, it follows that all standard subalgebras of $\mathscr{L}_C$ are RAAG Lie algebras.

Suppose $\mathscr{L}_C$ is not Bloch-Kato. Then, by Theorem 4.5, it contains a non-quadratic standard subalgebra, which cannot be a generalised holonomy. □

We conclude the present section by stating the graded-commutative analogue of the above characterisation of BK holonomies in terms of universal Koszulity.

Theorem 13.23. *If C is admissible, then the quadratic dual of $\mathscr{L}_C$ is the graded-commutative algebra given by the following presentation in the category of graded-commutative algebras:*

$$A_C := \mathscr{L}_C^! = \left\langle C \,\middle|\, ab = \partial(xyz) = 0 : \ \{a,b\} \notin \mathscr{E},\ \{x,y,z\} \text{ is a simplex of } C \right\rangle,$$

where $\partial(xyz) = xy - xz + yz$ and $\mathscr{E}$ is the set of proper edges of C.

Notice that, in case C is the associated complex to a hyperplane arrangement $\mathcal{A}$, the algebra A_C is the quadratic cover of the Orlik-Solomon algebra of $\mathcal{A}$. In case the Orlik-Solomon algebra is quadratic we say that $\mathcal{A}$ is a quadratic arrangement.

It then follows by Theorem 4.15 and Theorem 12.3:

Theorem 13.24. *If C is an admissible complex, then the following are equivalent:*

(1) *C is a complex of elementary type;*
(2) *$\mathscr{L}_C$ is BK;*
(3) *A_C is universally-Koszul.*

In particular, if $\mathcal{A}$ is a quadratic arrangement, then the Orlik-Solomon algebra of $\mathcal{A}$ is universally Koszul if, and only if, $C(\mathcal{A})$ is of elementary type.

14. Applications

14.1. **Arrangements with BK holonomies.** Without effort, Theorem 13.20 may be specialised to holonomy Lie algebras of hyperplane arrangements. It turns out that holonomy Lie algebras of hyperplane arrangements are rarely BK.

Theorem 14.1. *Let $\mathcal{A}$ be a hyperplane arrangement over a field $\mathbb{F}$. Then, the following are equivalent:*

(1) *The holonomy k-Lie algebra $\mathscr{L}(\mathcal{A})$ is Bloch-Kato;*

(2) *There is at most one rank-2 flat $L \in \mathbb{L}_2(\mathcal{A})$ such that $|\mathcal{A}_L| \geq 3$.*

Proof. Identify the elements of $\mathcal{A}$ with the standard generators of $\mathscr{L}(\mathcal{A})$ and put $C = C(\mathcal{A})$.

Let L and L' be distinct flats of rank 2 in $\mathbb{L}(\mathcal{A})$. Assume that the corresponding sets of hyperplanes $\mathcal{A}_L$ and $\mathcal{A}_{L'}$ have more than 2 elements.

Assume first that there exists a (unique) hyperplane Z in the intersection $\mathcal{A}_L \cap \mathcal{A}_{L'}$. If for all hyperplanes $H \supset L$ and $H' \supset L'$ different from Z, one has $|\mathcal{A}_{H \cap H'}| \geq 3$, then the induced subcomplex spanned by $\mathcal{A}_L \cup \mathcal{A}_{L'}$ is the connected sum of the corresponding simplices, and hence C is not of elementary type. Otherwise, there is a hyperplane $H \supset L$ such that the set $X = \left\{ H' \in \mathcal{A}_{L'} \;\middle|\; |\mathcal{A}_{H,H'}| \leq 2 \right\}$ is not empty. Moreover, $Z \notin X$, so that $\emptyset \neq X \subsetneq \mathcal{A}_{L'}$ and the induced subcomplex spanned by $\{H\} \cup \mathcal{A}_{L'}$ is the partial cone $\nabla_X(C(\mathcal{A}_{L'}))$.

Now assume that $\mathcal{A}_M \cap \mathcal{A}_{M'} = \emptyset$ for all rank-2 flats M and M' with $|\mathcal{A}_M|, |\mathcal{A}_{M'}| \geq 3$. Put $\mathcal{A}_L = \{H_1, \dots, H_n\}$ and $\mathcal{A}_{L'} = \{H'_1, \dots, H'_m\}$, $m, n \geq 3$. By assumption, $\mathcal{A}_{H_i \cap H'_j} = \{H_i, H'_j\}$, for all i, j, so that the induced subcomplex of C spanned by the vertices H_i $(1 \leq i \leq n)$ and H_j $(1 \leq j \leq m)$ is the join of two simplices of dimension at least 2. In particular, the induced subcomplex $C(\{H_1, H_2, H'_1, H'_2\})$ is the line of length 3.

Conversely, suppose that $\mathcal{A}$ has at most one rank-2 flat L with $|\mathcal{A}_L| \geq 3$. The associated complex is thus an iterated total cone over a simplex, so that C is of elementary type. □

14.2. **Graphic arrangements.** Let $\Gamma = (V, E)$ be a graph with vertex set $V = \{1, \dots, \ell\}$, and let $\mathbb{F}$ be a field. For $1 \leq i < j \leq \ell$, let $\alpha_{ij} = x_i - x_j$ be a linear form in $\mathbb{F}^\ell$. These data give rise to a hyperplane arrangement

$$\mathcal{A}_\Gamma = \left\{ \ker \alpha_{ij} \;\middle|\; \{i, j\} \in E,\ i < j \right\},$$

the **graphic arrangement** associated with the graph Γ.

For instance, the complete graph on ℓ vertices yields the **braid arrangement** $\mathcal{A}_{\ell-1}$ of rank $\ell - 1$. Notice that (non-necessarily induced) subgraphs give rise to subarrangements. Since all graphs are subgraphs of the complete ones, any graphic arrangement is a subarrangement of some $\mathcal{A}_{\ell-1}$. The simplicial complex associated with $\mathcal{A}_3$ is depicted in Figure 1.

If Γ_1, Γ_2 are graphs, fix vertices v_1, v_2 of Γ_1 and Γ_2, respectively. Consider the following two graphs obtained as unions of the Γ_i's

$$\Gamma = \Gamma_1 \sqcup \Gamma_2, \text{ the disjoint union, and}$$
$$\Gamma' = \Gamma_1 \vee_{v_1 = v_2} \Gamma_2, \text{ the connected sum.}$$

It is easy to see that the holonomy Lie algebras of $\mathcal{A}_\Gamma$ and $\mathcal{A}_{\Gamma'}$ are isomorphic. In particular, since we are only interested in Lie algebras, and not in the underlying arrangement, one can always assume Γ to be connected.

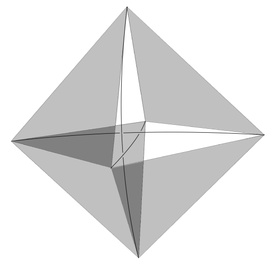


FIGURE 1. The simplicial complex $C(\mathcal{A}_3)$.

For graphic arrangements, one can tell whether the Orlik-Solomon algebra is Koszul just by looking at the graph ([44, Thm. 6.4], [50]).

Theorem 14.2. *Let Γ be a graph and let $A(\Gamma)$ be the Orlik-Solomon algebra of the graphic arrangement $\mathcal{A}_\Gamma$. Then, the following are equivalent:*

(1) *Γ is chordal;*
(2) *$\mathcal{A}_\Gamma$ is supersolvable;*
(3) *$A(\Gamma)$ is Koszul;*
(4) *$A(\Gamma)$ is a quadratic algebra.*

Remark 14.3. *The implication (2)$\Rightarrow$(3) is true for any arrangement by [43].*

We can now apply the procedure for constructing the associated complex to graphic arrangements. If Γ is a graph, then $C_\Gamma := C(\mathcal{A}_\Gamma)$ is a complex with the edges of Γ as a set of vertices. Moreover, the 1-skeleton of C_Γ is complete and a set of edges $\{e_1, e_2, e_3\}$ is a simplex of C_Γ if, and only if, $\bigcup e_i$ span a clique of Γ.

Indeed, with the same notations as above, the rank of the intersection of three hyperplanes $H_{i_s j_s} := \ker(x_{i_s} - x_{j_s})$ $(s = 1, 2, 3)$ is 2 iff the set $\{i_1, j_1, i_2, j_2, i_3, j_3\}$ only contains three elements. In particular, the dimension of the associated complex $C(\mathcal{A}_\Gamma)$ is at most 2.

Corollary 14.4. *Let Γ be a graph. Then $\mathscr{L}(\mathcal{A}_\Gamma)$ is abelian if, and only if, Γ is* triangle-free*, i.e., it has no 3-cliques.*

Proof. Recall that the associated complex to a hyperplane arrangement has a complete 1-skeleton. Since the associated complex C_Γ is admissible, $\mathscr{L} = \mathscr{L}(\mathcal{A}_\Gamma)$ is abelian iff C_Γ is a (complete) graph, by Corollary 13.13. By definition, C_Γ is a graph iff Γ has no 3-clique. □

Remark 14.5. *From this result, it is easy to construct hyperplane arrangements such that their intersection lattices only agree up to rank 2. For this, consider the square graph Σ and the disjoint union Γ of four segments. Then, $A(\Gamma)$ is quadratic and $A(\Sigma)$ is not, so that the intersection lattices are different. However, the corresponding holonomy Lie algebras of the hyperplane arrangements are isomorphic, in fact abelian, proving that the intersection lattices are equal up to rank 2.*

In [43], the authors say that no example of a non-supersolvable arrangement with a Koszul holonomy algebra is known. Since, for a graph Γ, the graphic arrangement $\mathcal{A}_\Gamma$ is supersolvable precisely when Γ is chordal, we get a plethora of examples of non-supersolvable arrangements with Koszul holonomy. Indeed, if Γ is triangle-free and not chordal, then $\mathscr{L}(\mathcal{A}_\Gamma)$ is abelian, thus Koszul, yet $\mathcal{A}_\Gamma$ is not supersolvable.

From Corollary 14.4 it also follows that Koszulity of $\mathscr{L}(\mathcal{A}_\Gamma)$ occurs in two opposite cases, namely, when Γ is either chordal or triangle-free. The only graphs which are both triangle-free and chordal are forests (i.e., disjoint unions of trees). We suspect that all graphic arrangements have Koszul holonomy.

Lemma 14.6. *Let C be a 2-dimensional complex with complete 1-skeleton and let $\mathscr{L}$ be its generalised holonomy Lie algebra. If $x_1, \dots, x_n$ are the canonical generators of $\mathscr{L}$ corresponding to the vertices of C, let $\mathscr{M}$ be the subalgebra of $\mathscr{L}$ generated by the first $n-1$ elements $x_1, \dots, x_{n-1}$. Then, $\mathscr{M}$ is an ideal of $\mathscr{L}$.*

Proof. By abuse of notation, we will identify the canonical generators of $\mathscr{L}$ with the vertices of C. For all $i = 1, \dots, n-1$, there exists a vertex $x_j \neq x_n$ (possibly equal to x_i) such that $\{x_i, x_j, x_n\}$ is a maximal simplex. Hence, $[x_n, x_i] = [x_i, x_j] \in \mathscr{M}$. □

From Theorem 14.1, one can easily deduce the following.

Corollary 14.7. *Let Γ be a graph and let $\mathscr{L}$ be the holonomy Lie algebra of the graphic arrangement $\mathcal{A}_\Gamma$. Then, $\mathscr{L}$ is BK if, and only if, Γ contains at most a single 3-clique.*

Also, Theorem 14.2 implies the following characterisation of universally Koszul Orlik-Solomon algebras.

Corollary 14.8. *Let Γ be a graph and let $A(\Gamma)$ be the Orlik-Solomon algebra of the hyperplane arrangement $\mathcal{A}_\Gamma$. Then, $A(\Gamma)$ is universally Koszul if, and only if, Γ is chordal with at most one triangle.*

Proof. Let B be the quadratic cover of $A(\Gamma)$. Then, $A(\Gamma)$ is universally Koszul if, and only if, $B = A(\Gamma)$ and B is universally Koszul.

By Theorem 4.15 and Corollary 14.7, since $B = \mathscr{L}^!_{\mathcal{A}_\Gamma}$, B is universally Koszul if, and only if, Γ contains at most one triangle.

In turn, by Theorem 14.2, $B = A(\Gamma)$ if, and only if, Γ is chordal. □

14.3. **An example.** The goal is to prove that the class of generalised holonomy Lie algebras associated with simplicial complexes is not just the union of those of RAAG Lie algebras and holonomies of hyperplane arrangements.

Moreover, neither the class of holonomy Lie algebras of hyperplane arrangements nor the class of RAAG Lie algebras contains the other one. In fact, not all RAAG Lie algebras are holonomy Lie algebras of central hyperplane arrangements, as the disconnected 2-vertex complex gives rise to a non-abelian free Lie algebra and no holonomy Lie algebra can have that shape. For the converse, it is enough to notice that there exist non-Koszul holonomy Lie algebras, as the following example shows.

Example 14.9 (Example 5.1 of [43]). *Consider the arrangement $\mathcal{A}$ given by the following 1-forms*

$$x,\ y,\ z,\ x+y,\ x+z \text{ and } y+z$$

in $\mathbb{F}^3$. We claim that $\mathscr{L}(\mathcal{A})$ is not a right-angled Artin Lie algebra.

The simplicial complex associated to $\mathcal{A}$ has complete 1-skeleton on the vertex set $\{1,2,3,4,5,6\}$. The only maximal simplices of dimension ≥ 2 are $\{1,2,4\}$, $\{1,3,5\}$ and $\{2,3,6\}$. This allows one to compute the quadratic dual of $\mathscr{L}(\mathcal{A})$: it is the graded-commutative algebra given by the presentation

$$A = \mathscr{L}(\mathcal{A})^! = \left\langle x_1,\ldots,x_6 \,\middle|\, \partial(x_1x_2x_4), \partial(x_1x_3x_5), \partial(x_2x_3x_6)\right\rangle.$$

One can thus compute the Hilbert series of A:

$$A(t) = \sum_{i\geq 0} (\dim A_i)t^i = 1 + 6t + 12t^2 + 8t^3 + t^4$$

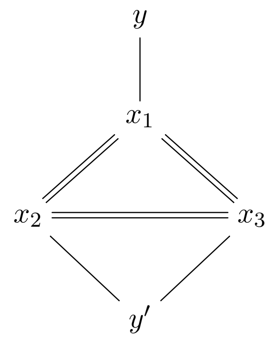
Now, if $\mathscr{L}(\mathcal{A})$ was Koszul, then, by Fröberg's formula, the Hilbert series of $\mathscr{U}(\mathscr{L}(\mathcal{A}))$ should be equal to $1/A(-t)$. By computing the series expansion of $1/A(-t)$ at $t=0$, one recovers that the coefficient of t^{13} is negative, proving that it cannot be the Hilbert series of any graded space. It follows that $\mathscr{L}(\mathcal{A})$ is not Koszul, and hence it is not a right-angled Artin Lie algebra.

Notice that the associated complex does not have the embedding property: The induced subcomplex D spanned by the vertices 1, 2 and 3 is totally disconnected, so that $\mathscr{L}_D$ is free. However, the subalgebra of $\mathscr{L}(\mathcal{A})$ generated by the elements a_i corresponding to the vertices $i = 1,2,3$ has at least one minimal defining relation. Indeed,

$$[a_1,[a_2,a_3]] = [a_1,[a_6,a_2]] = [a_6,[a_1,a_2]] = [a_6,[a_4,a_1]] = 0.$$

Also notice that there is no graphic arrangement with the same holonomy as the arrangement $\mathcal{A}$.

Consider the 2-dimensional complex $C = \{x_1, x_2, x_3, y, y'\}$ where the maximal simplices are $\{y, x_1\}$, $\{x_1, x_2, x_3\}$, $\{x_2, y'\}$ and $\{x_3, y'\}$, i.e.,



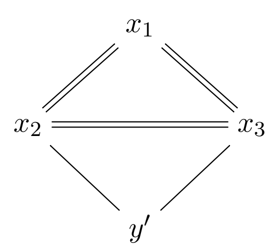
One can thus minimally present $\mathscr{L}_C$ by 5 generators and 5 relations. Suppose $\mathscr{L}_C$ is isomorphic with the RAAG Lie algebra defined by a graph Γ.

Thus Γ has 5 vertices and 5 edges. If Γ is disconnected, then it is the disjoint union of a diamond graph and an isolated vertex and hence it is of ET. Since $\mathscr{L}_C$ is not Bloch-Kato, the graph Γ must be connected.

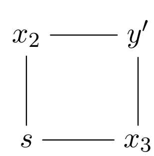
One can compute the low degrees dimensions of $\mathscr{L}_C$ and of $\mathscr{L}_\Gamma$, where Γ ranges through the 5 non-isomorphic graphs with 5 vertices and 5 edges. It turns out that the only graphs Γ for which these dimensions agree are the pentagon and the square with a tail of length 1.

(1) Suppose Γ is the pentagon and $\mathscr{L}_\Gamma \simeq \mathscr{L}_C$.

Consider the induced hull of $\{y', x_1, x_2, x_3\}$ in C:



The Lie algebra of such a complex is isomorphic with the RAAG Lie algebra associated with the square:



where $s = x_1 + x_2 + x_3$.

In particular, $\mathscr{L}_{C_4}$ is a subalgebra of $\mathscr{L}_{C_5}$, which is a contradiction, as the following Lemma shows.

Lemma 14.10. *If $n \geq 3$, then $\mathscr{L}_{C_n}$ does not embed into $\mathscr{L}_{C_{n+1}}$.*

Proof. Present the two Lie algebras as follows

$$\mathscr{M} = \langle v_1, \dots, v_n \,|\, [v_i, v_{i+1}] \rangle$$
$$\mathscr{L} = \langle x_1, \dots, x_{n+1} \,|\, [x_i, x_{i+1}] \rangle$$

where the indices are taken mod n and mod $n+1$, respectively, and let $\phi : \mathscr{M} \to \mathscr{L}$ be an injective homomorphism.

Set $y_i = \phi(v_i) = \sum_j \lambda_{ij} x_j$, for $i = 1, \dots, n$. Since $\dim \mathscr{L}_1 = \dim \mathscr{M}_1 + 1$, there is a vertex x_i such that $\{y_1, \dots, y_n, x_i\}$ is a basis for $\mathscr{L}_1$. We may suppose $i = 1$.

Then, $\mathscr{L}_1 = \operatorname{Span} x_1 \oplus \phi(\mathscr{M}_1)$ and hence

$$\mathscr{L}_2 = [\phi(\mathscr{M}_1), \phi(\mathscr{M}_1)] + [\phi(\mathscr{M}_1), x_1].$$

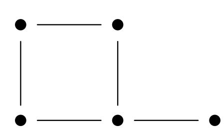
Now, $\mathscr{M}_2 \simeq [\phi(\mathscr{M}_1), \phi(\mathscr{M}_1)]$ has dimension $\binom{n}{2} - n$. Since $[x_j, x_1] = 0$ for $j \in \{1, 2, n+1\}$, there holds

$$[y_i, x_1] = \lambda_{i3}[x_3, x_1] + \cdots + \lambda_{i,n}[x_n, x_1] = \sum_{j=3}^{n} [\lambda_{ij} x_j, x_1],$$

and one sees that $[\phi(\mathscr{M}_1), x_1]$ has dimension $\leq n-2$. It follows that $\dim \mathscr{L}_2 \leq \binom{n}{2} - n + n - 2$ (notice that $\binom{n}{2} - 2$ is non-negative for $n \geq 3$).

On the other hand, $\dim \mathscr{L}_2 = \binom{n+1}{2} - n - 1$, and since $\binom{n+1}{2} - n - 1$ is always greater than $\binom{n}{2} - 2$, we get a contradiction.[2] □

(2) Suppose now that $\mathscr{L}_C$ is the RAAG Lie algebra associated with the tailed square Γ:



Notice that the induced hull Λ of x_1, x_2, y, y' in C is the disjoint union of two segments

We claim that $\mathscr{L}_\Lambda = k^2 \amalg k^2$ does not embed into $\mathscr{L}_\Gamma$.

Indeed, let $\phi : \mathscr{L}_\Lambda \to \mathscr{L}_\Gamma$ be a monomorphism. Let $x \in \Gamma$ be the vertex of valency 3 and let y be the opposite vertex in the square. Then $\mathscr{L} = \mathscr{L}_\Gamma/([x, y])$ is the RAAG associated with a graph of ET. Let $\pi : \mathscr{L}_\Gamma \to \mathscr{L}$ be the natural projection and let $\mathscr{M} = \mathscr{L}_\Lambda / \ker(\pi\phi)$. Notice that $\pi\phi$ induces an injective morphism of $\mathscr{M}$ into $\mathscr{L}$ and hence $\mathscr{M}$ is of ET.

Case 1. If $\ker \pi\phi = 0$, then $\mathscr{M} \simeq \mathscr{L}_\Lambda$ is a subalgebra of $\mathscr{L}$. Since $\pi(x)$ is central in $\mathscr{L}$ and $\mathscr{M}$ has trivial center, one deduces that $\pi(x) \notin \mathscr{M}$. Moreover, one may suppose that $\pi(x)$ does not appear with non-zero coefficient in any generator of $\mathscr{M}$, as $\mathscr{M}/(\pi(x))$ is isomorphic with $\mathscr{M}$.

[2]One has $A = \binom{n+1}{2} - n - 1 = \frac{1}{2}(n+1)(n-2)$ and $B = \binom{n}{2} - 2 = \frac{1}{2}(n^2 - n - 4)$:

$$2(A - B) = n^2 - n - 2 - (n^2 - n - 4) = 2.$$

This amounts to saying that $\mathscr{M}$ is a subalgebra of $\mathscr{L}/(\pi(x))$ and, by dimension arguments, it turns out that they are isomorphic. Nevertheless, $\mathscr{L}/(\pi(x))$ is the RAAG Lie algebra on the graph having an isolated vertex and hence different from Λ, on which $\mathscr{M}$ is defined.

Case 2. If $\ker \pi\phi \neq 0$, then $\mathscr{M}$ admits a minimal presentation consisting of 4 generators and 3 degree-2 relations, as $\ker \pi\phi = \phi^{-1} \ker \pi$. Indeed, $\mathscr{M} = \mathscr{L}_\Lambda / I$, where I is the intersection of the ideal generated by $[x, y]$ and $\mathscr{L}_\Lambda$, and hence, as $\mathscr{M}$ is quadratic, I is generated by the preimage of $[x, y]$ in $\mathscr{L}_\Lambda$.

In particular, $\mathscr{M}$ does not contain any surface Lie algebra as a 1-generated subalgebra, and hence it is isomorphic with a RAAG Lie algebra defined on a graph Δ with 4 vertices and 3 edges.

Moreover, Δ must be isomorphic with the claw graph $K_{1,3}$ or with the opposite graph of the latter.

We may present $\mathscr{M}$ in the following way:

$$\mathscr{M} = \langle v_1, v_2, w_1, w_2 \mid [v_1, v_2], [w_1, w_2], r \rangle$$

where $r = \sum_{ij} \alpha_{ij} [v_i, w_j]$.

Now, if we set $w_i' = \alpha_{i1} w_1 + \alpha_{i2} w_2$, the relation r becomes

$$r = [v_1, w_1'] + [v_2, w_2'].$$

If w_1' and w_2' were linearly dependent, then $\mathscr{M}$ would be the right-angled Artin Lie algebra on the line of length 3, that is not of elementary type.

Thus, w_1' and w_2' are linearly independent, and the presentation of $\mathscr{M}$ can be read as

$$\langle v_1, v_2, w_1, w_2 \mid [v_1, v_2], [w_1, w_2], [v_1, w_1] + [v_2, w_2] \rangle$$

Since $\mathscr{M}$ has trivial center, it is not isomorphic with the RAA Lie algebra of the claw graph. (If $z = \sum_i (\alpha_i v_i + \beta_i w_i)$ is central, then $[z, w_1] = 0$ and $[z, w_1] = 0$ imply $\alpha_i = 0$ and β_i, respectively.)

It remains to prove that $\mathscr{M}$ is not isomorphic with the RAAG Lie algebra $\mathscr{A} = \mathscr{L}_{K_3} \amalg k$ associated with the disjoint union of a triangle and an isolated vertex. Let $\mathscr{H}$ be an abelian 1-generated subalgebra of $\mathscr{A}$ of dimension ≥ 2. By the Kurosh subalgebra theorem, $\mathscr{H}$ must be contained in $\mathscr{L}_{K_3}$.

Suppose $\mathscr{M}$ coincides with $\mathscr{A}$. Since $[v_1, v_2] = [w_1, w_2] = 0$, one has $\langle v_1, v_2 \rangle, \langle w_1, w_2 \rangle \leq \mathscr{L}_{K_3}$, which is a contradiction.

14.4. **Bestvina-Brady Lie algebras.** In the realm of group theory, numerous constructions draw inspiration from graphs. One of these is the Bestvina-Brady group associated with a graph Γ, which is defined as a particular normal subgroup of the right-angled Artin group G_Γ. Such groups provided the first examples of non-finitely presentable groups of type FP_2. Here, we present the Lie-theoretic analogue of such structures, which has already been studied in [19]. Nevertheless, for graded Lie algebras, there is no distinction between finite presentability and type FP_2 (see [58]).

Let Γ be a graph. Define the morphism $f : \mathscr{L}_\Gamma \to k$ by putting $f(v) = 1$, for every vertex $v \in V(\Gamma)$. Denote by $\mathscr{B}_\Gamma$ the kernel of the morphism f. In particular, $\mathscr{L}_{\geq 2} = [\mathscr{L}, \mathscr{L}] \subseteq \mathscr{B}_\Gamma$. We call $\mathscr{B}_\Gamma$ the **Bestvina-Brady Lie algebra** associated with the graph Γ.

In [19], the authors prove that the geometry of the flag complex of Γ determines cohomological finiteness properties for $\mathscr{B}_\Gamma$. In particular, they prove the following.

Theorem 14.11. *Let Γ be a finite graph. Then, $\mathscr{B}_\Gamma$ is of type FP_n if, and only if, the flag complex Δ_Γ of Γ is $(n-1)$-acyclic over k, i.e., $\tilde{H}_i(\Delta_\Gamma, k) = 0$ for $0 \leq i \leq n-1$. In particular, $\mathscr{B}_\Gamma$ is finitely generated precisely when Γ is connected. In that case, $\mathscr{B}_\Gamma$ is a standard subalgebra of $\mathscr{L}_\Gamma$.*

Proposition 14.12. *Let Γ be a finite connected graph. Then, $\mathscr{B}_\Gamma$ is a free Lie algebra of rank $|V(\Gamma)| - 1$ if, and only if, Γ is a tree.*

Proof. First assume that $\Gamma = T$ is a tree. We argue by induction on the number of vertices of T. The result clearly holds when T consists of a single vertex, for $\mathscr{B}_{\{v\}} = 0$. Suppose that T has at least 2 vertices. Since T is a finite tree, it contains a vertex v of valency 1 and we set $e = \{v, w\} \in E(T)$ for the unique edge starting from v. If T_0 is the induced subtree of T spanned by the vertices $\neq v$ of T, then $\mathscr{B}_{T_0}$ is free and $\mathscr{L}_T = \mathrm{HNN}_\phi(\mathscr{L}_{T_0}, v)$, where $\phi : \langle w \rangle \to 0 \subset \mathscr{L}_{T_0}$ is the zero derivation. Since $\mathscr{B}_{T_0} = \mathscr{B}_T \cap \mathscr{L}_{T_0}$ and $w \notin \mathscr{B}_{T_0}$, it follows that $\mathscr{B}_T$ is free by [22, Thm. 4].

Now, assume that Γ is not a tree. Since Γ is connected, it contains an induced n-cycle for some $n \geq 3$.

If $n = 3$, then $\operatorname{cd} \mathscr{L}_\Gamma \geq 3$. Now, notice that $\mathscr{L}_\Gamma = \mathscr{B}_\Gamma \rtimes k$, and hence, by Corollary 3.16, $\operatorname{cd} \mathscr{B}_\Gamma \geq \operatorname{cd} \mathscr{L}_\Gamma - 1 \geq 2$, proving that $\mathscr{B}_\Gamma$ is not free.

Assume now $n \geq 4$ and let Δ be an induced n-cycle in Γ. By Theorem 14.11, $\mathscr{B}_\Delta$ is a standard subalgebra of $\mathscr{L}_\Delta$ but it is not of type FP_2, hence it is not a free Lie algebra. Since $\mathscr{B}_\Delta = \mathscr{B}_\Gamma \cap \mathscr{L}_\Delta$, we deduce that $\mathscr{B}_\Gamma$ contains a non-free subalgebra, proving that $\mathscr{B}_\Gamma$ is not free, too. □

Alternative proof. Let Γ be a finite connected graph with n vertices and m edges. It is a tree precisely when $n = m + 1$.

If Γ is a tree, then $\mathscr{B}_\Gamma$ is finitely presented by Theorem 14.11. Since $\mathscr{L}_\Gamma = \mathscr{B}_\Gamma \rtimes k$, it follows that $b_2(\mathscr{B}_\Gamma) = b_2(\mathscr{L}_\Gamma) - b_1(\mathscr{B}_\Gamma) = m - (n-1) = 0$, and hence $\mathscr{B}_\Gamma$ is free by Proposition 2.10.

Conversely, if $\mathscr{B}_\Gamma$ is free of rank $n-1$, then it is of type FP and $m = b_2(\mathscr{L}_\Gamma) = b_2(\mathscr{B}_\Gamma) + b_1(\mathscr{B}_\Gamma) = 0 + n - 1$, i.e., Γ is a tree. □

In the same fashion, one can define an analogue of Bestvina-Brady Lie algebras for generalised holonomies. So, let C be any (finite) simplicial complex and consider the morphism $\chi = \chi_C : \mathscr{L}_C \to k$ sending any vertex v of C to $1 \in k$. The **Bestvina-Brady Lie algebra** of the complex C is thus $\mathscr{B}_C := \ker \chi$. If

$\mathcal{A}$ is a hyperplane arrangement, then the Bestvina-Brady Lie algebra of $\mathcal{A}$ is $\mathscr{B}_{\mathcal{A}} := \mathscr{B}_{C(\mathcal{A})}$.

Corollary 14.13. *Let C be a simplicial complex and assume $\mathscr{L}_C$ to be Koszul. Then, the Bestvina-Brady Lie algebra $\mathscr{B}_C$ is of type FP if, and only if, it is Koszul.*

In particular, when $C = \Gamma$ is a graph, then the Bestvina-Brady Lie algebra $\mathscr{B}_\Gamma$ is Koszul if, and only if, the flag complex of Γ is acyclic over k.

Proof. Koszul Lie algebras are clearly of type FP.

Assume $\mathscr{B}_C$ to be of type FP. Since $\mathscr{L}_C$ is Koszul and it splits as $\mathscr{B}_C \rtimes \langle t \rangle$, if $\mathscr{B}_C$ is of type FP, then it is Koszul by Corollary 3.20.

When C is a graph Γ, the RAAG Lie algebra $\mathscr{L}_\Gamma$ is Koszul, and hence $\mathscr{B}_\Gamma$ is Koszul if, and only if, it is of type FP, which happens precisely when the flag complex of Γ is acyclic over k by Theorem 14.11. □

In turn, the above arguments show that, for a graph Γ, the Bestvina-Brady Lie algebra $\mathscr{B}_\Gamma$ is n-Koszul if, and only if, the flag complex of Γ is $(n-1)$-acyclic.

Proposition 14.14. *Let k be a field of characteristic zero, and let C be a (finite, admissible) simplicial complex. Then, the Bestvina-Brady k-Lie algebra $\mathscr{B}_C$ is finitely generated precisely when C is connected.*

Proof. First assume C not to be connected. Let Γ be the 1-skeleton of C and consider the Bestvina-Brady Lie algebra of Γ, which is not finitely generated by Theorem 14.11. Since $\mathscr{L}_C$ naturally surjects on $\mathscr{L}_\Gamma$, we recover a surjection $\mathscr{B}_C \to \mathscr{B}_\Gamma$, whence $\mathscr{B}_C$ is not finitely generated.

Assume now C to be connected. We will construct a connected graph $\Lambda = \Lambda_C$ and a surjective homomorphism $\mathscr{L}_\Lambda \to \mathscr{L}_C$ inducing a surjection $\mathscr{B}_\Lambda \to \mathscr{B}_C$.

Denote by M the set of maximal simplices of C of dimension at least 2. The vertex-set of the graph Λ is the union of M and the vertex-set of C. Now, in case x is a vertex of C, $\{x, y\}$ is an edge of Λ iff either y is a vertex of C and $\{x, y\}$ is a proper edge of C, or y is a simplex of dimension ≥ 2 and x is one of its vertices. It follows that Λ is connected and hence $\mathscr{B}_\Lambda$ is finitely generated.

We now define a map $\phi : \mathscr{L}_\Lambda \to \mathscr{L}_C$ by prescribing the image of degree 1 elements. If x is a vertex of C, then we declare $\phi(x) = x$; if $K \in M$, then we put

$$\phi(K) = \sum_{v \in K} \frac{v}{|K|}.$$

Notice that the assumption on the characteristic of the field k only plays a role in defining ϕ.

In order that ϕ be a well defined homomorphism, it suffices to notice that if $K \in M$ and $v \in K$, then $[\phi(K), \phi(v)] = \frac{1}{|K|}[x_K, v] = 0$, where $x_K = \sum_{v' \in K} v'$.

Eventually, notice that $\chi_C = \chi_\Lambda \circ \phi$, so that we recover a surjective homomorphism $\mathscr{B}_\Lambda \to \mathscr{B}_C$, which proves that $\mathscr{B}_C$ is finitely generated. □

Since the associated complexes of hyperplane arrangements have complete 1-skeleton, it follows that the complex Bestvina-Brady Lie algebra of any arrangement is finitely generated.

Remark 14.15. *The so-defined graph Λ_C, where C is the associated complex to $\mathcal{A}$, is isomorphic to the Hasse diagram of the poset $\mathbb{L}_{1,2}(\mathcal{A}) = \mathbb{L}_1(\mathcal{A}) \cup \mathbb{L}_2(\mathcal{A})$.*

In fact, for hyperplane arrangements, one can say more.

Corollary 14.16. *Let $\mathcal{A}$ be a hyperplane arrangement, and let k be a field such that its characteristic does not divide the number of hyperplanes in $\mathcal{A}$. Then, the Bestvina-Brady Lie algebra $\mathscr{B}_{\mathcal{A}}$ over k is quadratic, and it is Koszul (resp. BK) if, and only if, $\mathscr{L}(\mathcal{A})$ is.*

Proof. By Lemma 13.2, the element of $\mathscr{L}(\mathcal{A})$ given by the sum z of all hyperplanes is a central element of degree 1 and hence, for any vector space decomposition $(\mathscr{L}(\mathcal{A}))_1 = V \oplus kz$, the holonomy $\mathscr{L}(\mathcal{A})$ splits into the direct sum $\mathscr{L}(\mathcal{A}) = \langle V \rangle \sqcap \langle z \rangle$. Since $z \notin \mathscr{B}_{\mathcal{A}}$, we can take $V = (\mathscr{B}_{\mathcal{A}})_1$, and hence $\mathscr{L}(\mathcal{A}) = \mathscr{B}_{\mathcal{A}} \sqcap \langle z \rangle$. In particular, $\mathscr{B}_{\mathcal{A}} \simeq \mathscr{L}(\mathcal{A})/(z)$ is a (finitely-generated) quadratic Lie algebra. Moreover, since $H^{\bullet,\bullet}(\mathscr{L}(\mathcal{A})) = H^{\bullet,\bullet}(\mathscr{B}_{\mathcal{A}}) \wedge \Lambda_\bullet(z^*)$, the Lie algebra $\mathscr{B}_{\mathcal{A}}$ is Koszul (resp. BK) iff $\mathscr{L}(\mathcal{A})$ is. □

Proposition 14.17. *Let Γ be a graph and let G_Γ and $\mathscr{L}_\Gamma$ be respectively the right-angled Artin (pro-p) group and the RAAG Lie algebra associated with Γ. Then, the following are equivalent:*

(1) *G_Γ is coherent;*
(2) *$\mathscr{L}_\Gamma$ is Bogomolov;*
(3) *Γ is chordal.*

For a proof, see [18] for the Lie algebra case, [48] for the pro-p case and [9] for the abstract case. In particular, it follows from Corollary 13.22 that if C is a complex of ET, then $\mathscr{L}_C$ is coherent.

By using a combination of Proposition 14.17 and Corollary 14.13, one easily deduces the following.

Corollary 14.18. *If Γ is a connected chordal graph, then $\mathscr{B}_\Gamma$ is Koszul. In particular, flag complexes of connected chordal graphs are acyclic over any field.*

Now, since the flag complex Δ_Γ of Γ is finite, its integral homology groups are finitely generated abelian groups. The Universal Coefficient Theorem implies that $H_i(\Delta_\Gamma, \mathbb{Z}) \otimes k$ embeds into $H_i(\Delta_\Gamma, k) = 0$, for any field k, and hence Δ_Γ is homologically trivial over $\mathbb{Z}$. Therefore, we deduce the following graph-theoretic result.

Theorem 14.19. *The flag complex of a finite connected chordal graph is acyclic.*

The latter result had already been proved in [30], albeit through a completely different, and rather combinatorial proof.

Corollary 14.20. *Let Γ be a finite graph and suppose that the RAAG Lie algebra $\mathscr{B}_\Gamma$ is BK. Then, Γ is chordal and $\mathscr{B}_\Gamma$ is Bogomolov.*

Proof. First notice that, if Γ is chordal, then $\mathscr{B}'_\Gamma = \mathscr{L}'_\Gamma$ is a free Lie algebra, by Proposition 14.17, i.e., $\mathscr{B}_\Gamma$ is Bogomolov.

Suppose now that Γ is not chordal. Then, there is an induced cycle Θ of length at least 4 in Γ. As $\mathscr{B}_\Theta$ is not of type FP_2 by Theorem 14.11, and it is a standard subalgebra of $\mathscr{B}_\Gamma$, it follows that $\mathscr{B}_\Gamma$ is not BK. □

In particular, Conjecture 4.22 holds true within the class of Bestvina-Brady Lie algebras, too. In the following example we see that the opposite implication in the latter result does not hold.

Example 14.21. *Consider the graph Γ obtained by taking the cone on an arbitrary graph Γ' that is chordal but not of elementary type. Then, Γ is chordal and one has $\mathscr{B}_\Gamma \simeq \mathscr{L}_{\Gamma'}$, which is Bogomolov but not BK as Γ' is not a graph of elementary type.*

From this example it follows that, if $\mathscr{B}_\Gamma$ is BK, then Γ cannot contain an induced copy of $\nabla(L_3)$, the cone on a line of length 3. Chordal graphs without such an induced subgraph are called **Ptolemaic graphs** and they are distance-hereditary graphs, meaning that induced subgraphs are isometrically embedded into the graph. The name comes from the fact that for every four vertices x, u, v, w in the graph, one has

$$d(u,v)d(w,x) \leq d(u,w)d(v,x) + d(u,x)d(v,w),$$

that is Ptolemy's inequality in Euclidean spaces [36].

Conjecture 14.22. *Let Γ be a finite graph. Then, $\mathscr{B}_\Gamma$ is BK if, and only if, Γ is Ptolemaic.*

Conclusion

In this work, we have deeply studied quadratic Lie algebras from the cohomological point of view. While doing so, we discovered that HNN-extensions are a powerful tool to deduce information about subalgebras of such Lie algebras. We examined their usage in the study of several test benches, like RAAG Lie algebras and holonomy Lie algebras of hyperplane arrangements, as well as their unified version, the generalised holonomies.

We have also shown that the classes of Bloch-Kato Lie algebras and of Bloch-Kato pro-p groups share several common features. Though, we found a Bloch-Kato Lie algebra that is not of elementary type in the sense of I. Efrat's elementary type conjecture, and this suggests the potential existence of pro-p groups that, while satisfying the Bloch-Kato property, do not fall under the class of groups of elementary type.

Additionally, a number of open questions stemmed from this work, most of which we are collecting here.

Questions.

(1) *If $\mathscr{L}$ is a BK Lie algebra, is the derived subalgebra $\mathscr{L}'$ a free Lie algebra? (Conj. 4.22)*
(2) *Is the center of a Koszul Lie algebra always concentrated in degree* 1*? (Conj. 6.2)*
(3) *Is any quadratic Lie algebra embeddable into a Koszul one? (Question 7.3)*
(4) *Are the standard PD^n subalgebras of a PD^{n+1} Bloch-Kato Lie algebra $\mathscr{L}$ classified by the points of the projective space of $\mathscr{L}_1$? (Question 10.13)*
(5) *If $\mathscr{L}$ is a BK & PD^n Lie algebra with $n \geq 3$, do there exist non-trivial central elements of $\mathscr{L}$ in degree* 1*? (Quest. 10.15)*
(6) *If Γ is a Ptolemaic graph, then is the Bestvina-Brady Lie algebra $\mathscr{B}_\Gamma$ a BK Lie algebra? (Conj. 14.22)*

Conclusione

In questo lavoro, abbiamo approfondito lo studio delle algebre di Lie quadratiche dal punto di vista coomologico. Nel frattempo, abbiamo scoperto che le estensioni HNN sono un potente strumento per dedurre informazioni sulle sottoalgebre di tali algebre di Lie. Abbiamo esaminato il loro utilizzo nello studio di diversi casi, come le algebre di Lie RAAG e le algebre di Lie di olonomia degli arrangiamenti di iperpiani, così come la loro versione unificata, le olonomie generalizzate.

Abbiamo anche dimostrato che le classi delle algebre di Lie di Bloch-Kato e dei gruppi pro-p di Bloch-Kato condividono diverse caratteristiche comuni. Tuttavia, abbiamo trovato un'algebra di Lie di Bloch-Kato che non è di tipo elementare nel senso della congettura di tipo elementare di I. Efrat, suggerendo potenzialmente l'esistenza di gruppi pro-p che, pur soddisfacendo la proprietà di Bloch-Kato, non rientrano nella classe dei gruppi di tipo elementare.

Inoltre, da questo lavoro sono emerse numerose domande, la maggior parte delle quali sono raccolte in seguito.

Domande.

(1) *Se $\mathscr{L}$ è un'algebra di Lie di Bloch-Kato, la sua sottoalgebra derivata $\mathscr{L}'$ è un'algebra di Lie libera? (Cong. 4.22)*
(2) *Il centro di un'algebra di Lie di Koszul è sempre concentrato in grado 1? (Cong. 6.2)*
(3) *È possibile immergere qualsiasi algebra di Lie quadratica in una di tipo Koszul? (Question 7.3)*
(4) *Le sottoalgebre standard PD^n di un'algebra di Lie $\mathscr{L}$ BK & PD^{n+1} sono classificate dai punti dello spazio proiettivo di $\mathscr{L}_1$? (Quest. 10.13)*
(5) *Se $\mathscr{L}$ è un'algebra di Lie di BK & PD^n con $n \geq 3$, esistono elementi centrali non banali di $\mathscr{L}$ di grado 1? (Quest. 10.15)*
(6) *Se Γ è un grafo di Tolemaico, allora l'algebra di Lie di Bestvina-Brady $\mathscr{B}_\Gamma$ è un'algebra di Lie di BK? (Conj. 14.22)*

Conclusión

En este trabajo, nos centramos en estudiar a fondo las álgebras de Lie cuadráticas desde el punto de vista cohomológico. Al hacerlo, descubrimos que las extensiones HNN son una herramienta poderosa para obtener información sobre las subálgebras de dichas álgebras de Lie. Examinamos su uso en el estudio de varios casos, como las álgebras de Lie RAAG y las álgebras de Lie de holonomía de arreglos de hiperplanos, así como su versión unificada, las holonomías generalizadas.

También demostramos que las clases de álgebras de Lie de Bloch-Kato y de grupos pro-p de Bloch-Kato comparten varias características comunes. Sin embargo, encontramos una álgebra de Lie de Bloch-Kato que no es de tipo elemental en el sentido de la conjetura de tipo elemental de I. Efrat, lo que sugiere la posible existencia de grupos pro-p que, aunque cumplen con la propiedad de Bloch-Kato, no pertenecen a la clase de los grupos de tipo elemental.

Además, este trabajo ha dado lugar a varias preguntas abiertas, la mayoría de las cuales recompilamos aquí.

Preguntas.

(1) *Si $\mathscr{L}$ es un álgebra de Lie de Bloch-Kato, ¿su subálgebra derivada $\mathscr{L}'$ es un álgebra de Lie libre? (Conj. 4.22)*
(2) *¿El centro de un álgebra de Lie de Koszul siempre está concentrado en grado 1? (Conj. 6.2)*
(3) *¿Es posible encajar cualquier álgebra de Lie cuadrática en una de tipo Koszul? (Quest. 7.3)*
(4) *¿Las subálgebras PD^n de una álgebra de Lie $\mathscr{L}$ BK & PD^{n+1} están clasificadas por los puntos del espacio proyectivo de $\mathscr{L}_1$? (Quest. 10.13)*
(5) *Si $\mathscr{L}$ es un álgebra de Lie BK & PD^n con $n \geq 3$, ¿existen elementos centrales no triviales de $\mathscr{L}$ de grado 1? (Quest. 10.15)*
(6) *Si Γ es un grafo de Ptolomeo, ¿el álgebra de Lie de Bestvina-Brady $\mathscr{B}_\Gamma$ es un álgebra de Lie de Bloch-Kato? (Conj. 14.22)*

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