Szegö kernel equivariant asymptotics under Hamiltonian Lie group actions

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Abstract

Suppose that a compact and connected Lie group G acts on a complex Hodge manifold M in a holomorphic and Hamiltonian manner, and that the action linearizes to a positive holomorphic line bundle A on M. Then there is an induced unitary representation on the associated Hardy space and, if the moment map of the action is nowhere vanishing, the corresponding isotypical components are all finite dimensional. We study the asymptotic concentration behavior of the corresponding equivariant Szegö kernels near certain loci defined by the moment map.

1 Introduction

Let M be a connected complex d-dimensional projective manifold, and A an holomorphic ample line bundle on it. There exists an Hermitian metric h such that the unique covariant ∇ derivative on A that is compatible with both the complex structure and the metric has curvature $\Theta = -2\,\imath\,\omega$, where ω is a Kähler form on M. Thus the triple (M, J, ω) is a Kähler manifold, with associated Riemannian metric ρ^M and volume form $\mathrm{d}V_M := \omega^{\wedge d}/d!$.

We shall denote by $A^\vee$ the dual line bundle of A, and by $X \subset A^\vee$ the unit circle bundle; thus $X = \partial D$, where $D \subset A^\vee$ is the unit disc bundle, a strictly pseudoconvex domain. Then ∇ determines a connection 1-form α on X. If $\pi : X \to M$ is the projection, then $\mathrm{d}V_X := \frac{1}{2\,\pi}\,\alpha \wedge \pi^*(\mathrm{d}V_M)$ is a volume form. Furthermore, there is on X a natural choice of an S^1-invariant Riemannian metric ρ^X, determined by the conditions that π be a Riemannian submersion with $\ker(\alpha)$ as horizontal tangent bundle, and that the fibers of π have unit

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length. Hence $|\mathrm{d}V_X|$ is the Riemannian density of ρ^X. We shall denote by dist_X the Riemannian distance function of ρ^X.

If $H(X) \subset L^2(X)$ is the Hardy space, the orthogonal projector $\Pi : L^2(X) \to H(X)$ is known as the *Szegö projector* of X, and its distributional kernel $\Pi \in \mathcal{D}'(X \times X)$ as the *Szegö kernel* ([BS], [Z]).

Furthermore, let G be a connected compact Lie group, with Lie algebra $\mathfrak{g}$ and coalgebra $\mathfrak{g}^\vee$; we shall denote by d_G and r_G, respectively, the dimension and the rank of G. Let us assume that $\mu : G \times M \to M$ is a Hamiltonian and holomorphic action on $(M, J, 2\,\omega)$, with moment map $\Phi : M \to \mathfrak{g}^\vee$.

Then to every $\boldsymbol{\xi} \in \mathfrak{g}$ there is associated an Hamiltonian vector field $\boldsymbol{\xi}_M \in \mathfrak{X}_{ham}(M)$, which canonically lifts to a contact and CR vector field $\boldsymbol{\xi}_X \in \mathfrak{X}_{cont}(X)$, according to the law [Ko]

$$\boldsymbol{\xi}_X := \boldsymbol{\xi}_M^\sharp - \langle \Phi, \boldsymbol{\xi} \rangle\, \partial_\theta; \tag{1}$$

here notation is as follows:

1. for any vector field V on M, $V^\sharp$ denotes its its horizontal lift to X with respect to α (and similarly for tangent vectors);
2. ∂_θ is the generator of the standard circle action on X (fiber rotation).

In other words, the infinitesimal action of $\mathfrak{g}$ on M lifts to an infinitesimal contact action on X.

We shall make the stronger hypothesis that μ itself lifts to an action $\tilde{\mu} : G \times X \to X$, of which the correspondence $\boldsymbol{\xi} \mapsto \boldsymbol{\xi}_X$ is the differential. Then $\tilde{\mu}$ is a contact and CR action, and naturally determines a unitary representation of G on $H(X)$. According to the Theorem of Peter and Weyl, there is an equivariant unitary Hilbert direct sum decomposition of $H(X)$ into isotypical components corresponding to the irreducible representations (in the following, irreps) of G [St].

For a given choice of a maximal torus $T \leqslant G$ and of a set R^+ of positive roots of $\mathfrak{g}$, the irreps of G are determined by the their maximal weights, which range in the set of dominant weights. This sets up a bijective correspondence between the family $\hat{G}$ of irreps of G and a subset $\mathcal{D}^G$ of the collection $\mathcal{D}$ of all dominant weights; we have $\mathcal{D} = \mathcal{D}^G$ if G is simply connected. We shall label the irrep with maximal weight $\boldsymbol{\lambda} \in \mathcal{D}^G$ by the regular[1] half-weight $\boldsymbol{\nu} = \boldsymbol{\lambda} + \boldsymbol{\delta}$, where $\boldsymbol{\delta} := 2^{-1} \sum_{\beta \in R^+} \beta$. We shall denote by $V_{\boldsymbol{\nu}}$ the irrep corresponding to $\boldsymbol{\nu}$, and by $\chi_{\boldsymbol{\nu}} : G \to \mathbb{C}$ the corresponding character. This labeling is consistent with the philosophy of the Kirillov character formula [Ki], that we shall recall in the course of the paper. Furthermore, let us

[1] i.e., belonging to the open positive Weyl chamber

set $d_{\boldsymbol{\nu}} := \dim(V_{\boldsymbol{\nu}})$; if φ denotes any Euclidean scalar product on $\mathfrak{g}^\vee$ which is invariant uder the coadjoint action then, by the Weyl dimension formula (see, e.g., §1 of [Su], §2.5 of [V1]),

$$d_{\boldsymbol{\nu}} = \prod_{\boldsymbol{\beta}\in R^+} \frac{\varphi(\boldsymbol{\nu},\boldsymbol{\beta})}{\varphi(\boldsymbol{\delta},\boldsymbol{\beta})}. \tag{2}$$

In particular, $d_{k\,\boldsymbol{\nu}} = k^{\frac{1}{2}(d_G-r_G)}\,d_{\boldsymbol{\nu}}$. Thus, if we set $\mathcal{E}^G := \mathcal{D}^G + \boldsymbol{\delta}$, we have

$$H(X) = \bigoplus_{\boldsymbol{\nu}\in\mathcal{E}^G} H(X)^{\tilde{\mu}}_{\boldsymbol{\nu}} \tag{3}$$

where $H(X)^{\tilde{\mu}}_{\boldsymbol{\nu}}$ is the isotypical component corresponding to $V_{\boldsymbol{\nu}}$. For each $\boldsymbol{\nu}\in\mathcal{E}^G$ we have the associated equivariant Szegö projector $\Pi^{\tilde{\mu}}_{\boldsymbol{\nu}} : L^2(X) \to H(X)^{\tilde{\mu}}_{\boldsymbol{\nu}}$.

In general, $H(X)^{\tilde{\mu}}_{\boldsymbol{\nu}}$ may well be infinite-dimensional, and does not correspond to a space of holomorphic sections of any tensor power of A. Nonetheless, it follows from the theory of [GS2] that if $\mathbf{0}\not\in\Phi(M)$, then $\dim H(X)^{\tilde{\mu}}_{\boldsymbol{\nu}} < +\infty$ for every $\boldsymbol{\nu}$ (see §2 [P1]). Thus $\Pi^{\tilde{\mu}}_{\boldsymbol{\nu}}$ is a smoothing operator, so that its distributional kernel $\Pi^{\tilde{\mu}}_{\boldsymbol{\nu}} \in \mathcal{C}^\infty(X\times X)$.

We are interested in the local asymptotics of $\Pi^{\tilde{\mu}}_{k\,\boldsymbol{\nu}}$ for a fixed $\boldsymbol{\nu}\in\mathcal{E}^G$ and $k\to+\infty$ with $k\,\boldsymbol{\nu}\in\mathcal{E}^G$. The latter condition is satisfied for any k if $\boldsymbol{\delta}\in\mathcal{D}^G$, so that $\mathcal{E}^G\subset\mathcal{D}^G$, as is the case when G is simply connected.

This general theme has already been studied in specific cases ([P1], [P2], [C], [GP1], [GP2]), building on the approach developed in [Z], [BSZ] and [SZ] to the basic Fourier case where $G = S^1$, μ is trivial and $\Phi = \imath$. We refer the reader to the introductions of [P1], [P2], [GP1], [GP2] for an ampler discussion of motivation and general framing. The theme is geometrically relevant, being related to interesting geometric quotients [P3].

The results in [GP1] and [GP2] are based on the pairing of the Weyl character and integration formulae with the techniques in [Z] and [SZ]. The new ingredient here is the Kirillov character formula ([Ki], [R]) which considerably simplifies some of the arguments, and allows to deal with more general Lie groups; on the other hand, it forces restrictions on the stabilizer subgroups.

The following results are governed by the interplay between Φ and the cone over the coadjoint orbit through $\boldsymbol{\nu}$, $\mathcal{O}_{\boldsymbol{\nu}}\subset\mathfrak{g}^\vee$. As $\boldsymbol{\nu}$ is a regular element of $\mathfrak{g}^\vee$, $\mathcal{O}_{\boldsymbol{\nu}}$ is equivariantly diffeomorphic to G/T, hence it has dimension $d_G - r_G$. Let us set $\mathcal{C}(\mathcal{O}_{\boldsymbol{\nu}}) := \mathbb{R}_+\cdot\mathcal{O}_{\boldsymbol{\nu}}$.

We shall need the following hypothesis.

Assumption 1.1. We shall assume that:

1. $\mathbf{0} \notin \Phi(m)$;

2. Φ is transverse to $\mathcal{C}(\mathcal{O}_{\boldsymbol{\nu}})$ (equivalently, Φ is transverse to the ray $\mathbb{R}_+ \cdot \boldsymbol{\nu}$);

3. $M_{\mathcal{O}_{\boldsymbol{\nu}}} := \Phi^{-1}\big(\mathcal{C}(\mathcal{O}_{\boldsymbol{\nu}})\big) \neq \emptyset$.

If Assumption 1.1 holds, $M_{\mathcal{O}_{\boldsymbol{\nu}}}$ is a compact and connected G-invariant submanifold of M, of (real) codimension $r_G - 1$ (see the discussion of [GP1]). Let us set $X_{\mathcal{O}_{\boldsymbol{\nu}}} := \pi^{-1}(M_{\mathcal{O}_{\boldsymbol{\nu}}})$.

We shall also make the following assumption on the compact and connected Lie group G.

Assumption 1.2. Let $L(G) \subset \mathfrak{t}^\vee$ be the lattice of integral forms on G; then $\boldsymbol{\delta} \in L(G)$.

This condition is satisfied if G is either $U(n)$ for some $n \geq 1$, or a connected and simply connected compact semisimple Lie group. If G satisfies this assumption, then it is called *acceptable* in Harish-Chandra's terminology (§2.5 of [V1]). Under Assumption 1.2, $\mathcal{E}^G \subset \mathcal{D}^G$.

In the following we shall assume throughout that Assumptions 1.1 and 1.2 hold.

Theorem 1.1. *Suppose that* $\mathcal{O}_{\boldsymbol{\nu}} \cap \mathfrak{t}^0 = \emptyset$. *Fix* $C, \epsilon > 0$. *Then, uniformly for* $\mathrm{dist}_X\,(G \cdot x, G \cdot y) \geq C\,k^{\epsilon - \frac{1}{2}}$, *we have*

$$\Pi^{\tilde{\mu}}_{k\,\boldsymbol{\nu}}(x, y) = O\left(k^{-\infty}\right).$$

When $G = U(n)$, the previous hypothesis is satisfied by any $\boldsymbol{\nu} \in \mathcal{E}^G$ with $\sum_{j=1}^n \nu_j \neq 0$. It is never satisfied when $G = SU(2)$, but the statement of Theorem 1.1 is nonetheless true in this case, see [GP2]. More generally, let $\mu^T : T \times M \to M$ be the restriction of μ, and let $\Phi^T : M \to \mathfrak{t}^\vee$ be the moment map induced by Φ (that is, the composition of Φ with the restriction $\mathfrak{g}^\vee \to \mathfrak{t}^\vee$). If $\mathbf{0} \notin \Phi^T(M)$, then the hypothesis of Theorem 1.1 is satisfied for any $\boldsymbol{\nu}$ such that $M_{\mathcal{O}_{\boldsymbol{\nu}}} \neq \emptyset$.

Theorem 1.2. *Let us fix* $C, \epsilon > 0$, *and assume that* $\tilde{\mu}$ *is free along* $X_{\mathcal{O}}$. *Then, uniformly for*

$$\max\left\{\mathrm{dist}_X(x, X_{\mathcal{O}_{\boldsymbol{\nu}}}), \mathrm{dist}_X(y, X_{\mathcal{O}_{\boldsymbol{\nu}}})\right\} \geq C\,k^{\epsilon - \frac{1}{2}},$$

we have

$$\Pi^{\tilde{\mu}}_{k\,\boldsymbol{\nu}}(x, y) = O\left(k^{-\infty}\right).$$

If $\tilde{\mu}$ *is only generically free along* $X_{\mathcal{O}}$, *and* $X' \subset X$ *is the open subset where it is free, the same estimate holds uniformly on compact subsets of* X'.

We shall focus on the near diagonal asymptotics of $\Pi^{\tilde{\mu}}_{k\boldsymbol{\nu}}(x,x)$ for x belonging to a shrinking tubular neighborhood of $X_{\mathcal{O}_{\boldsymbol{\nu}}}$, of radius $O\left(k^{\epsilon-1/2}\right)$. Using the normal exponential map, we may parametrize such a neighborhood by a neighborhood of the zero section in the normal bundle of $X_{\mathcal{O}_{\boldsymbol{\nu}}} \subset X$, which is the pull-back of the normal bundle $N(M_{\mathcal{O}_{\boldsymbol{\nu}}}/M)$ of $M_{\mathcal{O}_{\boldsymbol{\nu}}} \subset M$. If $V_{\mathcal{O}_{\boldsymbol{\nu}}} \subset X$ is a sufficiently small neighborhood of $X_{\mathcal{O}_{\boldsymbol{\nu}}}$, we shall accordingly write the general $y \in V_{\mathcal{O}_{\boldsymbol{\nu}}}$ in additive notation as $y = x + \mathbf{v}$, for unique $x \in X_{\mathcal{O}_{\boldsymbol{\nu}}}$ and $\mathbf{v} \in N_{\pi(x)}(M_{\mathcal{O}_{\boldsymbol{\nu}}}/M)$[2].

In order to state the next Theorem, some further notation is needed.

Definition 1.1. Let φ be an Ad-invariant Euclidean product on $\mathfrak{g}$, with associated norm $\|\cdot\|_\varphi$. Let us also denote by φ the induced bi-invariant Riemannian metric on G. Clearly, φ restricts to an invariant Riemannian metric φ^T on T. We shall adopt the following notation:

1. $\mathrm{d}^\varphi V_G$: the Riemannian density on G associated to φ;
2. $\mathrm{vol}^\varphi(G) = \int_G \mathrm{d}^\varphi V_G(g)$;
3. $\mathrm{d}^\varphi V_T$: the Riemannian density on T
4. $\mathrm{vol}^\varphi(T) := \int_T \mathrm{d}^\varphi V_T(t)$.
5. For any $\boldsymbol{\gamma} \in \mathfrak{g}^\vee$,
    - $\boldsymbol{\gamma}^\varphi \in \mathfrak{g}$ is uniquely determined by the condition $\boldsymbol{\gamma} = \varphi\big(\boldsymbol{\gamma}^\varphi, \cdot\big)$;
    - $\|\boldsymbol{\gamma}\|_\varphi := \|\boldsymbol{\gamma}^\varphi\|_\varphi$;
    - $\boldsymbol{\gamma}_{\varphi,u} := \frac{1}{\|\boldsymbol{\gamma}^\varphi\|_\varphi}\,\boldsymbol{\gamma} = \frac{1}{\|\boldsymbol{\gamma}\|_\varphi}\,\boldsymbol{\gamma} \in \mathfrak{g}^\vee$;
    - $\boldsymbol{\gamma}^\varphi_u := \frac{1}{\|\boldsymbol{\gamma}^\varphi\|_\varphi}\,\boldsymbol{\gamma}^\varphi \in \mathfrak{g}$.
6. $\mathfrak{t}^{\perp_\varphi} \subseteq \mathfrak{g}$: the Euclidean orthocomplement of $\mathfrak{t}$ with respect to φ.
7. For any $\boldsymbol{\tau} \in \mathfrak{t}$, $S_{\boldsymbol{\tau}} : \mathfrak{t}^{\perp_\varphi} \to \mathfrak{t}^{\perp_\varphi}$ denotes the restriction of $\mathrm{ad}_{\boldsymbol{\tau}}$; when $\boldsymbol{\tau}$ is *regular* (as is the case when $\boldsymbol{\tau} = \boldsymbol{\nu}^\varphi$ for $\boldsymbol{\nu} \in \mathcal{E}^G$), $S_{\boldsymbol{\tau}}$ is a linear automorphism, skew-symmetric with respect to the restriction of φ.

Let us identify the coalgebra of T, $\mathfrak{t}^\vee$, with the subspace of those $\boldsymbol{\lambda} \in \mathfrak{g}^\vee$ fixed by T under the coadjoint action, and let $\mathfrak{t}^\vee_{reg} \subset \mathfrak{t}^\vee$ be the open and dense subset of those elements, called regular, that are fixed precisely by T. Hence $\mathcal{E}^G \subset \mathfrak{t}^\vee_{reg}$.

[2]We may interpret $x + \mathbf{v}$ in terms of a system of Hesenberg local coordinates on X centered at x [SZ], smoothly varying with x, see §4.

By definition, for any $m \in M_{\mathcal{O}_{\boldsymbol{\nu}}}$ there exist $h_m T \in G/T$ and $\varsigma(m) > 0$ such that

$$\Phi(m) = \varsigma(m)\,\mathrm{Coad}_{h_m}(\boldsymbol{\nu}) \in \mathrm{Coad}_{h_m}(\mathfrak{t}^\vee). \tag{4}$$

Since $\boldsymbol{\nu}$ is regular, $h_m T$ and $\varsigma(m)$ are uniquely determined, and the functions $m \in M_{\mathcal{O}_{\boldsymbol{\nu}}} \mapsto h_m T \in G/T$ and $\varsigma : M_{\mathcal{O}_{\boldsymbol{\nu}}} \to \mathbb{R}$ are smooth.

For any non-zero $\boldsymbol{\lambda} \in \mathfrak{g}^\vee$, let $\boldsymbol{\lambda}^0 = (\boldsymbol{\lambda}^\varphi)^{\perp_\varphi} \subset \mathfrak{g}$ be its annihilator hyperplane. Then

$$\Phi(m)^0 = \mathrm{Ad}_{h_m}\left(\boldsymbol{\nu}^0\right). \tag{5}$$

Let us set $\mathfrak{t}_{\boldsymbol{\nu}} := \mathfrak{t} \cap \boldsymbol{\nu}^0$, so that we have a φ-orthogonal direct sum decomposition

$$\mathfrak{t} = \mathrm{span}(\boldsymbol{\nu}^\varphi) \oplus \mathfrak{t}_{\boldsymbol{\nu}}.$$

For every $m \in M_{\mathcal{O}_{\boldsymbol{\nu}}}$, we shall set $\mathfrak{t}_m := \mathrm{Ad}_{h_m}(\mathfrak{t})$; thus $\mathfrak{t}_m$ is the unique Cartan subalgebra of $\mathfrak{g}$ containing $\Phi(m)$, or equivalently the Lie subalgebra of the (unique) maximal torus $T_m = h_m T h_m^{-1}$ stabilizing $\Phi(m)$. Thus,

$$\mathfrak{t}_m = \{\boldsymbol{\eta} \in \mathfrak{g} :\ [\boldsymbol{\eta}, \Phi(m)^\varphi] = 0\}. \tag{6}$$

Furthermore, we shall set $\mathfrak{t}'_m := \mathrm{Ad}_{h_m}(\mathfrak{t}_{\boldsymbol{\nu}}) = \mathfrak{t}_m \cap \Phi(m)^0$. More explicitly,

$$\mathfrak{t}'_m = \{\boldsymbol{\eta} \in \mathfrak{g} :\ [\boldsymbol{\eta}, \Phi(m)^\varphi] = 0,\ \langle \Phi(m), \boldsymbol{\eta}\rangle = 0\}. \tag{7}$$

Hence we have the φ-orthogonal direct sum

$$\mathfrak{t}_m = \mathrm{span}\big(\Phi(m)^\varphi\big) \oplus \mathfrak{t}'_m. \tag{8}$$

Assume $m \in M_{\mathcal{O}_{\boldsymbol{\nu}}}$. Then val_m is injective on $\Phi(m)^0$ by Remark 3.1; hence val_m is injective *a fortiori* on $\mathfrak{t}'_m$.

Definition 1.2. In the following, $m \in M_{\mathcal{O}_{\boldsymbol{\nu}}}$. We shall adopt the following notation.

1. $\mathrm{val}_m^*(\rho_m^M)$: the pull-back to $\mathfrak{g}$ of the Euclidear product $\rho_m^M = \omega_m(\cdot, J_m\cdot)$ on T_mM; ρ'_m: the restriction of $\mathrm{val}_m^*(\rho_m^M)$ to $\mathfrak{t}'_m$. Thus ρ'_m is non-degenerate (whence positive definite).

2. Given an arbitrary orthonormal basis $\mathcal{R}_m$ of $\mathfrak{t}'_m$ for the restriction of φ, let $D^\varphi(m) := M_{\mathcal{R}_m}(\rho'_m)$ be the representative matrix of ρ'_m w.r.t. $\mathcal{R}_m$, and set

   $$\mathcal{D}^\varphi(m) := \sqrt{\det D^\varphi(m)};$$

   then $\mathcal{D}^\varphi : M_{\mathcal{O}_{\boldsymbol{\nu}}} \to (0, +\infty)$ is well-defined and $\mathcal{C}^\infty$;

We can now define a $\mathcal{C}^\infty$ function $\Psi_{\boldsymbol{\nu}} : M_{\mathcal{O}_{\boldsymbol{\nu}}} \to \mathbb{R}_+$ by setting

$$\Psi_{\boldsymbol{\nu}}(m) := 2^{1+\frac{r_G-1}{2}}\,\pi \cdot \frac{1}{\|\Phi(m)\|_\varphi\,\mathcal{D}^\varphi(m)} \cdot \frac{\mathrm{vol}(\mathcal{O}_{\boldsymbol{\nu}_{\varphi,u}}, \sigma_{\boldsymbol{\nu}_{\varphi,u}})^2}{\left|\det(S_{\boldsymbol{\nu}^\varphi_u})\right|} \cdot \frac{\mathrm{vol}^\varphi(T)}{\mathrm{vol}^\varphi(G)^2}.$$

By Theorem 1.3 below, $\Psi_{\boldsymbol{\nu}}$ is actually independent of the choice of φ.

We need some further pieces of notation.

Given a real vector subspace $R \subseteq T_mM$, we shall denote by $R^{\perp_{h_m}}$ the orthocomplement of R with respect to the Hermitian structure $h_m = \rho^M_m - \imath\,\omega_m$; equivalently, $R^{\perp_{h_m}}$ is the orthocomplement of the complex subspace $R + J_m(R)$ of T_mM, and is a complex subspace of (T_mM, J_m). Clearly,

$$R^{\perp_{h_m}} = R^{\perp_{\rho^M_m}} \cap R^{\perp_{\omega_m}}. \tag{9}$$

where $R^{\perp_{\rho^M_m}}$ and $R^{\perp_{\omega_m}}$ are, respectively, the Riemannian and symplectic orthocomplements of R.

If $m \in M$, let $\mathfrak{g}_M(m) \subseteq T_mM$ be the vector subspace given by the evaluations at m of the all the vector fields on M induced by the elements of $\mathfrak{g}$. We shall see in Lemma 6.1 that for any $m \in M_{\mathcal{O}_{\boldsymbol{\nu}}}$ the normal space of $M_{\mathcal{O}_{\boldsymbol{\nu}}}$ at m satisfies

$$N_m(M_{\mathcal{O}_{\boldsymbol{\nu}}}) \subseteq J_m\big(\mathfrak{g}_M(m)\big), \quad \text{hence} \quad N_m(M_{\mathcal{O}_{\boldsymbol{\nu}}}) \cap \mathfrak{g}_M(m)^{\perp_{h_m}} = (0). \tag{10}$$

In this setting, small displacements from a fixed $x \in X$ are conveniently expressed in Heisenberg local coordinates (HLC) on X centered at x [SZ]. A choice of HLC at x gives a meaning to the expression $x+\mathbf{v}$, where $\mathbf{v} \in T_{\pi(x)}M$ has sufficiently small norm. Furthermore, the curve $\gamma_{x,\mathbf{v}} : \tau \in (-\epsilon, \epsilon) \mapsto x + \tau\,\mathbf{v}$ is horizontal at $\tau = 0$, and in fact $\gamma'_{x,\mathbf{v}}(0) = \mathbf{v}^\sharp$. More will be said in §4.

A further notational ingredient that will go into the statement of Theorem 1.3 is an invariant governing the exponential decay of various asymptotics related to Szegö kernels [SZ].

Definition 1.3. Let $\|\cdot\|$ and ω_0 be the standard norm and symplectic structures on $\mathbb{R}^{2d}$, respectively. Let us define $\psi_2 : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to \mathbb{R}$ by setting

$$\psi_2(u, v) := -\imath\,\omega_0(u, v) - \frac{1}{2}\,\|u - v\|^2.$$

A choice of Heisenberg local coordinates at x entails the choice of a unitary ismophism $T_{\pi(x)}M \cong \mathbb{C}^d$ (with the standard Hermitian structure), by means of which we shall view ψ_2 as being defined on $T_{\pi(x)}M$.

Theorem 1.3. *Assume that $x \in X_{\mathcal{O}_{\boldsymbol{\nu}}}$, and that $\tilde{\mu}$ is free at x. Set $m_x = \pi(x)$ and fix $C > 0$, $\epsilon \in (0, 1/6)$. Then, uniformly for $\mathbf{v}_j \in N_{m_x}(M_{\mathcal{O}_{\boldsymbol{\nu}}})$ and $\mathbf{w}_j \in \mathfrak{g}_M(m)^{\perp_{h_m}}$ satisfying $\|\mathbf{v}_j\|, \|\mathbf{w}_j\| \leq C\, k^\epsilon$, the following asymptotic expansion holds as $k \to +\infty$:*

$$\begin{aligned}
&\Pi^{\tilde{\mu}}_{k\,\boldsymbol{\nu}}\left(x + \frac{1}{\sqrt{k}}\,(\mathbf{v}_1 + \mathbf{w}_1), x + \frac{1}{\sqrt{k}}\,(\mathbf{v}_2 + \mathbf{w}_2)\right) \\
&\sim \quad \Psi_{\boldsymbol{\nu}}(m_x)\,\left(\frac{k}{\varsigma(m_x)\,\pi}\right)^{d+\frac{1-r_G}{2}} \cdot e^{\frac{1}{\varsigma(m_x)}\,\left[\psi_2(\mathbf{w}_1,\mathbf{w}_2) - (\|\mathbf{v}_1\|^2_{m_x} + \|\mathbf{v}_2\|^2_{m_x})\right]} \\
&\qquad \cdot \left[1 + \sum_{j\geq 1} k^{-j/2}\, P_j(m_x; \mathbf{v}_j, \mathbf{w}_j)\right],
\end{aligned}$$

where $P_j(m_x; \cdot)$ is a polynomial of degree $\leq 3\,j$ and parity j. If $X'_{\mathcal{O}_{\boldsymbol{\nu}}} \subseteq X_{\mathcal{O}_{\boldsymbol{\nu}}}$ is the open subset on which $\tilde{\mu}$ is free, the estimate holds uniformly on the compact subsets of $X'_{\mathcal{O}_{\boldsymbol{\nu}}}$.

By a Gaussian integral computation in normal Heisenberg coordinates, as in the proof of Corollary 1.3 of [P1], one can then deduce the following:

Corollary 1.1. *Assume that $\tilde{\mu}$ is free along $X_{\mathcal{O}_{\boldsymbol{\nu}}}$. Then there is an asymptotic expansion*

$$\dim H(X)^{\tilde{\mu}}_{k\boldsymbol{\nu}} \sim \left(\frac{k}{\pi}\right)^{d+1-r_G}\,\left[\delta_{\boldsymbol{\nu},0} + k^{-1}\,\delta_{\boldsymbol{\nu},1} + \ldots\right],$$

with

$$\delta_{\boldsymbol{\nu},0} := \frac{1}{2^{\frac{r_g-1}{2}}}\int_{M_{\mathcal{O}_{\boldsymbol{\nu}}}}\left[\frac{\Psi_{\boldsymbol{\nu}}(m)}{\varsigma(m)^{d+1-r_G}}\right]\,\mathrm{d}V_{M_{\mathcal{O}_{\boldsymbol{\nu}}}}(m),$$

where $\mathrm{d}V_{M_{\mathcal{O}_{\boldsymbol{\nu}}}}$ is the density on $M_{\mathcal{O}_{\boldsymbol{\nu}}}$ for the induced Riemannian metric.

In closing this introduction, we mention that there is a wider scope for the results in this paper. While our focus is on the complex projective setting, in view of the microlocal theory of almost complex Szegö kernels in [SZ] the present approach can be naturally extended to the compact symplectic category.

2 Examples

We check the statement of Theorem 4.2 against those in [P1] , [GP1], [GP2].

Example 2.1. Suppose $G = T$ is an r-dimensional torus. Let us take the standard metric φ, so that $\mathrm{vol}^{\varphi}(G) = (2\,\pi)^r$. We obtain

$$\Psi_{\boldsymbol{\nu}}(m_x) = \frac{2^{1+\frac{r-1}{2}}\,\pi}{(2\,\pi)^r} \cdot \frac{1}{\|\Phi(m)\|_{\varphi}\,\mathcal{D}^{\varphi}(m)} = \frac{1}{\left(\sqrt{2}\,\pi\right)^{r-1}}\,\frac{1}{\|\Phi(m)\|_{\varphi}\,\mathcal{D}^{\varphi}(m)}.$$

Thus Theorem 4.2 fits with Theorem 2 of [P1].

Example 2.2. Suppose $G = SU(2)$, so that $d_G = 3$, $r_G = 1$. Let $\varphi : \mathfrak{su}(2) \times \mathfrak{su}(2) \to \mathbb{R}$ be defined by

$$\varphi(A, B) := \mathrm{trace}\left(A\,\overline{B}^t\right) = \mathrm{trace}\,(A\,B)\,.$$

Let $T \leqslant G$ be the standard torus; $\mathfrak{t}$ is generated by the diagonal matrix Z with entries $\imath$, $-\imath$, which has norm $\sqrt{2}$. For $\nu \in \mathbb{Z}$, we shall denote by $\boldsymbol{\nu} \in \mathfrak{t}^{\vee}$ the weight taking value ν on Z. We have

$$\mathrm{vol}^{\varphi}(G) = 2^{3/2} \cdot (2\,\pi^2), \quad \mathrm{vol}^{\varphi}(T) = \sqrt{2} \cdot 2\,\pi.$$

For any $\nu \in \mathbb{Z}$, let $\boldsymbol{\nu}$ be the weight such that $\langle \boldsymbol{\nu}, Z\rangle = \nu$. Then

$$\boldsymbol{\nu}^{\varphi} = \frac{\nu}{2}\,Z, \quad \|\boldsymbol{\nu}\|_{\varphi} = \frac{1}{\sqrt{2}}\,\nu.$$

Furthermore,

$$\mathrm{vol}(\mathcal{O}_{\boldsymbol{\nu}}, \sigma_{\boldsymbol{\nu}}) = 2\,\pi\,\nu, \quad |\det(S_{\boldsymbol{\nu}^{\varphi}})| = \nu^2.$$

Finally, let $\lambda(m) > 0$ be defined by the condition that $\Phi(m)^{\varphi}$ be similar to $\lambda(m)\,Z$. Then $\|\Phi(m)\|_{\varphi} = \|\Phi(m)^{\varphi}\|_{\varphi} = \sqrt{2}\,\lambda(m)$ and $\varsigma(m) = \big(2\,\lambda(m)\big)/\nu$. We obtain

$$\Psi_{\boldsymbol{\nu}}(m) = 2\,\pi \cdot \frac{1}{\sqrt{2}\,\lambda(m)} \cdot 4\,\pi^2 \cdot \frac{\sqrt{2} \cdot 2\,\pi}{\left(2^{3/2} \cdot (2\,\pi^2)\right)^2} = \frac{1}{2\,\lambda(m)},$$

in agreement with [GP2].

Example 2.3. If $G = U(2)$, we have $d_G = 4$, $r_G = 2$. Let $\varphi : \mathfrak{u}(2) \times \mathfrak{u}(2) \to \mathbb{R}$ be defined as for $SU(2)$. Let $T \leqslant G$ be the standard maximal torus; then $\mathfrak{t}$ has orthonormal basis (R, S), where R and S are the diagonal matrices with diagonal entries $\begin{pmatrix}\imath & 0\end{pmatrix}$ and $\begin{pmatrix}0 & \imath\end{pmatrix}$, respectively. Let $\boldsymbol{\nu} = \nu_1\,R^* + \nu_2\,S^*$, where (R^*, S^*) is the dual basis. Then $\boldsymbol{\nu}^{\varphi} = \nu_1\,R + \nu_2\,S$.

We have in this case

$$\mathrm{vol}^{\varphi}(\mathcal{O}_{\boldsymbol{\nu}}, \sigma_{\boldsymbol{\nu}}) = 2\,\pi\,(\nu_1 - \nu_2), \quad |\det(S_{\boldsymbol{\nu}^{\varphi}})| = (\nu_1 - \nu_2)^2.$$

Furthermore,

$$\mathrm{vol}^{\varphi}(T) = (2\,\pi)^2, \quad \mathrm{vol}^{\varphi}(G) = 8\,\pi^3.$$

We obtain for $m \in M_{\mathcal{O}_{\boldsymbol{\nu}}}$

$$\Psi_{\boldsymbol{\nu}}(m) = 2^{\frac{3}{2}}\,\pi \cdot \frac{1}{\|\Phi(m)\|_{\varphi}\,\mathcal{D}^{\varphi}(m)} \cdot 4\,\pi^2 \cdot \frac{4\,\pi^2}{64\,\pi^6} = \frac{1}{\sqrt{2}\,\pi} \cdot \frac{1}{\|\Phi(m)\|_{\varphi}\,\mathcal{D}^{\varphi}(m)}, \tag{11}$$

which tallies with the front factor in the pointwise expansion in Theorem 1.4 of [GP1]. In the latter expansion the numerical factor is written in a slightly less explicit form, but replacing $V_3 = 2\,\pi^2$ it is readily seen to equal the one in (11).

3 Preliminaries

We shall adopt the following notational conventions:

1. R_j will denote a smooth real, complex or vector valued function defined in the neighborhood of the origin of some vector space, vanishing to j-th order at the origin, and allowed to vary from line to line;

2. if G acts smoothly on a manifold Z and $\boldsymbol{\xi} \in \mathfrak{g}$, $\boldsymbol{\xi}_Z$ will denote the induced vector field on Z;

3. under the same assumption, if $p \in Z$ we shall denote by $\mathrm{val}_p : \boldsymbol{\xi} \in \mathfrak{g} \to \boldsymbol{\xi}_Z(p) \in T_pZ$ the evaluation map;

4. if $m \in M$ and $\mathbf{v} \in T_mM$, we shall denote by $\|\mathbf{v}\|_m$ the norm of $\mathbf{v}$ with respect to ρ^M;

5. if $x \in X$ and $v = a\,\partial_\theta|_x + \mathbf{v}^\sharp \in T_xX$, in computations it will be convenient to set $\|v\|_x := \sqrt{a^2 + \|\mathbf{v}\|_m^2}$ (this is the norm in an obvious vertical rescaling of ρ^X).

Remark 3.1. Arguing as in §2 of [P1] (or §4.1.1 of [GP1]), one verifies that the following conditions are equivalent:

1. Assumption 1.1 holds;

2. $\tilde{\mu}$ is locally free along $X_{\mathcal{O}_{\boldsymbol{\nu}}}$;

3. for every $m \in M_{\mathcal{O}_{\boldsymbol{\nu}}}$, $\mathrm{val}_m : \mathfrak{g} \to T_mM$ is injective on the annihilator of $\Phi(m)$, that is,
$$\ker(\mathrm{val}_m) \cap \Phi(m)^0 = (0).$$

Let us define

$$\mathcal{Z}_{\boldsymbol{\nu}} := \big\{(x,y) \in X_{\mathcal{O}} \times X_{\mathcal{O}} \,:\, y \in G \cdot x\big\}. \tag{12}$$

Then $\mathcal{Z}_{\boldsymbol{\nu}}$ is a $G \times G$-invariant compact and connected submanifold of $X \times X$.

Theorem 3.1. *Uniformly on compact subsets of $(X \times X) \setminus \mathcal{Z}_{\boldsymbol{\nu}}$, we have*

$$\Pi^{\tilde{\mu}}_{k\boldsymbol{\nu}}(x,y) = O\left(k^{-\infty}\right).$$

Proof. The argument is a slight modification of the one in §3.1 and §3.2 of [GP1], based on the theory in [GS1]; hence we shall be somewhat sketchy. The ladder Szegö projector

$$\Pi_L := \bigoplus_{k=1}^{+\infty} \Pi^{\tilde{\mu}}_{k\boldsymbol{\nu}} : L^2(X) \longrightarrow \bigoplus_{k=1}^{+\infty} H(X)^{\tilde{\mu}}_{k\boldsymbol{\nu}}$$

has a distributional kernel whose wave front satisfies $\mathrm{WF}(\Pi_L) \subseteq \mathcal{Z}_{\boldsymbol{\nu}}$.

Let $K \Subset (X \times X) \setminus \mathcal{Z}_{\boldsymbol{\nu}}$. Without loss, we may assume that K is $G \times G$-invariant. There exists a $G \times G$-invariant smooth cut-off function $\varrho \geq 0$ on $X \times X$, which is identically equal to 1 on a neighborhood of K, and vanishes identically on a neighborhood of $\mathcal{Z}_{\boldsymbol{\nu}}$. Hence $\varrho \cdot \Pi_L \in \mathcal{C}^\infty(X \times X)$. Hence, we obtain a $\mathcal{C}^\infty$ function

$$F : (g,x,y) \in G \times X \times X \mapsto (\varrho \cdot \Pi_L)\left(\tilde{\mu}_{g^{-1}}(x), y\right) \in \mathbb{C}.$$

We shall set $F_{x,y} := F(\cdot, x, y) : G \to \mathbb{C}$.

Let us denote $P_{k\boldsymbol{\nu}} : L^2(X) \to L^2(X)_{k\boldsymbol{\nu}}$ the projector. Hence, $\Pi^{\tilde{\mu}}_{k\boldsymbol{\nu}} = P_{k\boldsymbol{\nu}} \circ \Pi^{\tilde{\mu}}_L$. If $\mathrm{d}^H V_G(g)$ is the Haar measure on G, this means that for $(x,y) \in X \times X$

$$\Pi^{\tilde{\mu}}_{k\boldsymbol{\nu}}(x,y) \;=\; d_{k\boldsymbol{\nu}} \int_G \overline{\chi_{k\boldsymbol{\nu}}(g)}\, \Pi_L\left(\tilde{\mu}_{g^{-1}}(x), y\right)\, \mathrm{d}^H V_G(g). \tag{13}$$

If $(x,y) \in K$, therefore,

$$\begin{aligned} \Pi^{\tilde{\mu}}_{k\boldsymbol{\nu}}(x,y) &= d_{k\boldsymbol{\nu}} \cdot \int_G \overline{\chi_{k\boldsymbol{\nu}}(g)}\, (\varrho\, \Pi_L)\left(\tilde{\mu}_{g^{-1}}(x), y\right)\, \mathrm{d}^H V_G(g) \\ &= d_{k\boldsymbol{\nu}} \cdot \mathrm{trace}\big(\mathcal{F}(F_{x,y})(k\boldsymbol{\nu} - \boldsymbol{\delta})\big), \end{aligned} \tag{14}$$

where $\mathcal{F}$ denotes the Fourier transform on G [Su], viewed as a function on $\mathcal{D}^G$. Since $d_{k\boldsymbol{\nu}} \leq C_{\boldsymbol{\nu}}\, k^{d_G - r_G}$, it suffices to show that $\mathcal{F}(F_{x,y})(k\boldsymbol{\nu} - \boldsymbol{\delta}) = O\left(k^{-\infty}\right)$ in Hilbert-Schmidt norm for $k \to +\infty$. To this end, we apply arguments in §1 of [Su].

To begin with, for any $\boldsymbol{\lambda} \in \mathcal{D}^G \setminus \{\mathbf{0}\}$ we have (see eq. (1.21) of [Su])

$$\|\mathcal{F}(F_{x,y})(\boldsymbol{\lambda})\|_{HS}^2 \leq \|\boldsymbol{\lambda}\|^{-4l} \left\|\mathcal{F}\left(\Delta_G^l F_{x,y}\right)(\boldsymbol{\lambda})\right\|_{HS}^2; \tag{15}$$

here Δ_G denotes the Laplacian (Casimir) operator on G, and $\|\cdot\|$ is the Hilbert-Schmidt norm. Hence for $k \gg 0$

$$\|\mathcal{F}(F_{x,y})(k\,\boldsymbol{\nu} - \boldsymbol{\delta})\|_{HS}^2 \leq 2\,k^{-4l}\|\boldsymbol{\nu}\|^{-4l} \left\|\mathcal{F}\left(\Delta_G^l F_{x,y}\right)(k\,\boldsymbol{\nu} - \boldsymbol{\delta})\right\|_{HS}^2; \tag{16}$$

On the other hand, by the Parceval identity (eq. (1.16) of [Su]) we also have

$$\left\|\mathcal{F}\left(\Delta_G^l F_{x,y}\right)(\boldsymbol{\lambda})\right\|_{HS}^2 \leq \frac{1}{d_{\boldsymbol{\lambda}+\boldsymbol{\delta}}} \|\Delta_G^l F_{x,y}\|_2^2 \leq \|\Delta_G^l F_{x,y}\|_2^2 \tag{17}$$

where $\|\cdot\|_2$ denotes the L^2-norm on G ($d_{\boldsymbol{\lambda}+\boldsymbol{\delta}} = d(\boldsymbol{\lambda})$ in the notation of [Su]).

Furthermore, by compactness for any $l \geq 0$ we can find $C_l > 0$ such that $\|\Delta_G^l F_{x,y}\|_2^2 \leq C_l$ for all $(x, y) \in X \times X$. Hence, by the Parceval identity, for every $\boldsymbol{\lambda} \in \mathcal{D}^G$ we have

$$\left\|\mathcal{F}\left(\Delta_G^l F_{x,y}\right)(\boldsymbol{\lambda})\right\|_{HS}^2 \leq \frac{1}{d_{\boldsymbol{\lambda}+\boldsymbol{\delta}}}\, C_l \leq C_l. \tag{18}$$

Therefore, for $k \gg 0$ we have

$$\|\mathcal{F}(F_{x,y})(k\,\boldsymbol{\nu} - \boldsymbol{\delta})\|_{HS}^2 \leq k^{-4l}\, C_l' \tag{19}$$

□

4 Proof of Theorem 1.1

In the proof of Theorem 1.1 we shall use the Weyl integration and character formulae, which we briefly recall below, referring e.g. to [V1] (§2.3-2.5) and [V2] (§4.13 and 4.14) for a detailed treatment.

Let W denote the Weyl group of $(\mathfrak{g}, \mathfrak{t})$; then W naturally acts on $\mathfrak{t}^\vee$ preserving the root lattice $L(R) \subset \mathfrak{t}^\vee$.

Let $L(G) \subset \mathfrak{t}^\vee$ be the lattice of integral forms of G. Every $\boldsymbol{\gamma} \in L(G)$ defines a character $E_{\boldsymbol{\gamma}} : T \to S^1$, and we may define

$$A_{\boldsymbol{\gamma}} := \sum_{s \in W} \epsilon(s)\, E_{s(\boldsymbol{\gamma})} : T \to \mathbb{C},$$

where $\epsilon(s) = \det(s)$ (here s is viewed as a linear map $\mathfrak{t} \to \mathfrak{t}$). In particular, since $\boldsymbol{\delta} \in L(G)$ by Assumption 1.2, we may set $\Delta := A_{\boldsymbol{\delta}}$.

Let $\mathrm{d}^H V_G$, $\mathrm{d}^H V_T$, $\mathrm{d}^H V_{G/T}$ be the Haar measures on G, T and G/T, respectively (§2.3 of [V1], §4.13 of [V2]). Then the following holds (Theorem 4.13.5 of [V2]):

Theorem 4.1. *(Weyl Integration Formula) Let us set For every L^1-function on G,*

$$\int_G f(g)\,\mathrm{d}^H V_G(g) = \frac{1}{|W|}\cdot \int_T f_T(t)\,\left|\Delta(t)\right|^2 \mathrm{d}^H V_T(t),$$

where

$$f_T(t) := \int_{G/T} f\left(g\,t\,g^{-1}\right)\,\mathrm{d}^H V_{GT}(gT).$$

Similarly, under Assumption 1.2 $A_{\boldsymbol{\nu}}$ is well-defined for any $\boldsymbol{\nu} \in \mathcal{E}^G$. Then we have (Theorem 4.14.4 of [V2]):

Theorem 4.2. *(The Weyl Character formula) On the open and dense regular locus $T' \subset T$ (defined by $\Delta \neq 0$), we have*

$$\chi_{\boldsymbol{\nu}}|_{T'} = \frac{A_{\boldsymbol{\nu}}}{\Delta}.$$

Another basic ingredient of the following arguments is the microlocal representation of Π as a Fourier integral operator introduced in [BS], and its elaboration in the projective (and sympletic) setting in [Z], [BSZ], [SZ]. We refer to the latter papers for a detailed discussion, and simply recall that the latter description has the form

$$\Pi(x,y) = \int_0^{+\infty} e^{\imath\, u\, \psi(x,y)}\, s(x,y,u)\,\mathrm{d}u, \tag{20}$$

where ψ is a complex phase function (with $\Im(\psi) \geq 0$) and s is a semiclassical symbol on $X \times X$, admitting an asymptotic expansion of the form

$$s(x,y,u) \sim \sum_{j\geq 0} s_j(x,y)\, u^{d-j}. \tag{21}$$

We shall invoke the following two properties of ψ:

1. for any $x \in X$, we have

$$\mathrm{d}_{(x,x)}\psi = (\alpha_x, -\alpha_x); \tag{22}$$

2. there exists a constant $D_X > 0$ such that ($\Im$ denoting imaginary part)

$$\Im\big(\psi(x,y)\big) \geq D_X\,\mathrm{dist}_X(x,y)^2 \qquad \forall\, x,y \in X. \tag{23}$$

A further useful tool is given by Heisenberg local coordinates on X; these are the local coordinates on X, introduced in [SZ], in which the local scaling asymptotics of Szegö kernels exhibit their universal character. Given $x \in X$, a system of local Heisenberg coordinates on X centered at x will be denoted additively, in the form $x + (\theta, \mathbf{v})$; here $\theta \in (-\pi, \pi)$ and $\mathbf{v} \in \mathbb{C}^d$ varies in an open ball centered at the origin. Fiber rotation is represented by translation in θ, and the locus $\theta = 0$ projects diffeomorphically onto its image in M, hence defines a local section of X which is horizontal at x with respect to the connection form α. Thus a system of Heisenberg local coordinates for X centered at x entails a choice of local coordinates for M centered at $m_x = \pi(x)$; actually the latter system determines a unitary isomorphism $\mathbb{C}^d \cong T_m M$ with respect to (ω_m, J_m), so when writing $x + (\theta, \mathbf{v})$ it is often assumed that $\mathbf{v} \in T_m M$ (as will be the case below). Further, when $\theta = 0$ we usually write $x + \mathbf{v}$ for $x + (0, \mathbf{v})$ (as in the statement of Theorem 1.3). Referring to (21), in Heisenberg local coordinates at x we have

$$s_0(x, x) = \frac{1}{\pi^d}. \tag{24}$$

We refer the reader to [SZ] for a detailed discussion.

Proof of Theorem 1.1. The proof is an adaptation of the argument used when $G = U(2)$ in [GP1], so we'll be somewhat sketchy.

To begin with, in view of Theorem 3.1 we may assume without loss that x and y belong to a small $S^1 \times G$-invariant neighborhood $V_{\mathcal{O}}$ of $X_{\mathcal{O}}$. Equivalently, $m_x := \pi(x)$ and $m_y := \pi(y)$ belong to a small G-invariant neighborhood of $U_{\mathcal{O}}$ of $M_{\mathcal{O}}$ in M, and . In particular, we may assume that $\tilde{\mu}$ is free on $V_{\mathcal{O}}$.

Hence $\Phi_G(m_x)$ belongs a small conic neighborhood of $\mathcal{C}(\mathcal{O}_{\boldsymbol{\nu}})$. Furthermore, replacing (x, y) by $(\tilde{\mu}_h(x), \tilde{\mu}_h(y))$ for a suitable $h \in G$, we may also assume that $\Phi_G(m_y)$ belongs to a small conic neighborhood of $\mathbb{R}_+ \cdot \boldsymbol{\nu}$. More precisely, we may assume that $\Phi_G(m_y) = \lambda_y \boldsymbol{\nu} + \beta_y$, where $\lambda_y > 0$, $\beta_y \in \boldsymbol{\nu}^0$ are smooth (here $\boldsymbol{\nu}^\perp \subset \mathfrak{g}^\vee$ is the orthcomplement of $\boldsymbol{\nu}$ with respect to, say, φ^H), and $\|\beta_y\| \ll \lambda_y$.

We have

$$\Pi_{k\boldsymbol{\nu}}(x, y) = d_{k\boldsymbol{\nu}} \int_G \overline{\chi_{k\boldsymbol{\nu}}(g)}\, \Pi\left(\tilde{\mu}_{g^{-1}}(x), y\right) \, \mathrm{d}^H V_G(g). \tag{25}$$

For a suitably small $\delta_K > 0$, let us introduce the open cover $\mathcal{V} = \{V', V''\}$ of $G \times X \times X$ given by

$$\begin{aligned} V' &:= \left\{(g, x, y) \in G \times X \times X \,:\, \mathrm{dist}_X\left(\tilde{\mu}_{g^{-1}}(x), y\right) < 2\,\delta_K\right\}, \\ V'' &:= \left\{(g, x, y) \in G \times X \times X \,:\, \mathrm{dist}_X\left(\tilde{\mu}_{g^{-1}}(x), y\right) > \delta_K\right\}. \end{aligned}$$

Let $\varrho' + \varrho'' = 1$ be a partition of unity on $G \times X \times X$ subordinate to $\mathcal{V}$. We then have

$$\Pi_{k\boldsymbol{\nu}}(x,y) = \Pi_{k\boldsymbol{\nu}}(x,y)' + \Pi_{k\boldsymbol{\nu}}(x,y)'', \tag{26}$$

where $\Pi_{k\boldsymbol{\nu}}(x,y)'$ and $\Pi_{k\boldsymbol{\nu}}(x,y)''$ are defined as in (25), except that the integrand has been multiplied by ϱ' and ϱ'', respectively. Therefore, $\mathcal{G}(g,x',y') := \varrho'' \cdot \Pi\left(\tilde{\mu}_{g^{-1}}(x'), y'\right)$ is a smooth function on $G \times X$. Hence, letting $\mathcal{G}_{x',y'} := \mathcal{G}(\cdot, x', y')$ arguing as above we have

$$\Pi_{k\boldsymbol{\nu}}(x',y')'' = d_{k\boldsymbol{\nu}}\,\mathrm{trace}\big(\mathcal{F}(\mathcal{G}_{x',y'})(k\,\boldsymbol{\nu} - \boldsymbol{\delta})\big) = O\left(k^{-\infty}\right),$$

uniformly in $(x',y') \in X \times X$. We are thus reduced to considering the asymptotics of the former summand $\Pi_{k\boldsymbol{\nu}}(x,y)'$. On the support of ϱ', y belongs to a small neighborhood of $\tilde{\mu}_{g^{-1}}(x)$; hence we may represent Π as a Fourier integral operator.

Applying Theorems 4.2 and 4.1, and recalling that $\Delta \circ \sigma = \epsilon(\sigma)\,\Delta$ for any $\sigma \in W$, we obtain

$$\begin{aligned}
\Pi_{k\boldsymbol{\nu}}(x,y) &\sim d_{k\boldsymbol{\nu}} \int_G \mathrm{d}^H V_G(g)\,\left[\overline{\chi_{k\boldsymbol{\nu}}(g)}\,\varrho'(g,x,y)\,\Pi\left(\tilde{\mu}_{g^{-1}}(x), y\right)\right] \\
&= \frac{d_{k\boldsymbol{\nu}}}{|W|} \int_T \mathrm{d}^H V_T(t) \int_{G/T} \mathrm{d}^H V_{G/T}(g\,T) \\
&\quad \left[\overline{\chi_{k\boldsymbol{\nu}}(t)}\,|\Delta(t)|^2\,\varrho'\left(g\,t\,g^{-1}, x, y\right)\,\Pi\left(\tilde{\mu}_{g\,t^{-1}g^{-1}}(x), y\right)\right] \\
&= \frac{d_{k\boldsymbol{\nu}}}{|W|} \sum_{\sigma\in W} \epsilon(\sigma) \int_T \mathrm{d}^H V_T(t) \int_{G/T} \mathrm{d}^H V_{G/T}(g\,T) \\
&\quad \left[\overline{E_{k\boldsymbol{\nu}}(t^\sigma)}\,\Delta(t)\,\varrho'\left(g\,t\,g^{-1}, x, y\right)\,\Pi\left(\tilde{\mu}_{g\,t^{-1}g^{-1}}(x), y\right)\right] \\
&= d_{k\boldsymbol{\nu}} \int_T \mathrm{d}^H V_T(t) \int_{G/T} \mathrm{d}^H V_{G/T}(g\,T) \\
&\quad \left[\overline{E_{k\boldsymbol{\nu}}(t)}\,\Delta(t)\,\varrho'\left(g\,t\,g^{-1}, x, y\right)\,\Pi\left(\tilde{\mu}_{g\,t^{-1}g^{-1}}(x), y\right)\right].
\end{aligned} \tag{27}$$

Choosing a basis of the lattice $L(G)$, we obtain an isomorphism $B : (S^1)^{r_G} \cong \mathfrak{t}/L(G) \cong T$; we shall write the general element of $(S^1)^{r_G}$ as $e^{\imath\boldsymbol{\vartheta}} = \begin{pmatrix} e^{\imath\vartheta_1} & \cdots & e^{\imath\vartheta_{r_G}} \end{pmatrix}$. Then with $\mathrm{d}\boldsymbol{\vartheta} = \mathrm{d}\vartheta_1 \cdots \mathrm{d}\vartheta_{r_G}$

$$B^*\big(\mathrm{d}V_T\big) = \frac{1}{(2\,\pi)^{r_G}}\,\mathrm{d}\boldsymbol{\vartheta}, \quad E_{k\boldsymbol{\nu}} \circ B\left(e^{\imath\boldsymbol{\vartheta}}\right) = e^{\imath\,k\,\langle\boldsymbol{\nu},\boldsymbol{\vartheta}\rangle}$$

To simplify notation, we shall simply identify T and $(S^1)^{r_G}$, and write $e^{\imath\boldsymbol{\vartheta}}$ for the corresponding element of T; with the same abuse, we shall think of

$\imath\,\boldsymbol{\vartheta}$ as an element of $\mathfrak{t} \leqslant \mathfrak{g}$. Inserting (20) in (27), we obtain

$$\begin{aligned}\Pi_{k\,\boldsymbol{\nu}}(x,y) &\sim \frac{d_{k\,\boldsymbol{\nu}}}{(2\,\pi)^{r_G}} \int_{(-\pi,\pi)^{r_G}} \mathrm{d}\boldsymbol{\vartheta} \int_{G/T} \mathrm{d}^H V_{G/T}(g\,T) \\ &\quad \left[e^{\imath\,\left[u\,\psi(\tilde{\mu}_{g\,e^{-\imath\,\boldsymbol{\vartheta}}\,g^{-1}}(x),y)-k\,\langle\boldsymbol{\nu},\boldsymbol{\vartheta}\rangle\right]}\,\mathcal{A}_k(gT,\boldsymbol{\vartheta},x,y,u)\right] \\ &= \frac{k\,d_{k\,\boldsymbol{\nu}}}{(2\,\pi)^{r_G}} \int_{(-\pi,\pi)^{r_G}} \mathrm{d}\boldsymbol{\vartheta} \int_{G/T} \mathrm{d}^H V_{G/T}(g\,T) \\ &\quad \left[e^{\imath\,k\,\Gamma(g\,T,\boldsymbol{\vartheta},u,x,y)}\,\mathcal{A}_k(gT,\boldsymbol{\vartheta},x,y,k\,u)\right]\end{aligned} \tag{28}$$

where

$$\mathcal{A}_k(gT,\boldsymbol{\vartheta},x,y,u) := \Delta\left(e^{\imath\,\boldsymbol{\vartheta}}\right)\,\varrho'\left(g\,e^{\imath\,\boldsymbol{\vartheta}}\,g^{-1},x,y\right)\,s\left(\tilde{\mu}_{g\,e^{-\imath\,\boldsymbol{\vartheta}}\,g^{-1}}(x),y,u\right),$$

$$\Gamma(g\,T,\boldsymbol{\vartheta},u,x,y) := u\,\psi(\tilde{\mu}_{g\,e^{-\imath\,\boldsymbol{\vartheta}}\,g^{-1}}(x),y) - \langle\boldsymbol{\nu},\boldsymbol{\vartheta}\rangle.$$

On the support of ϱ' we have $\tilde{\mu}_{g\,e^{-\imath\,\boldsymbol{\vartheta}}\,g^{-1}}(x) \sim y$. In view of (22), in any given coordinate system therefore

$$\mathrm{d}_{\left(\tilde{\mu}_{g\,e^{-\imath\,\boldsymbol{\vartheta}}\,g^{-1}}(x),y\right)}\psi \sim (\alpha_y,-\alpha_y). \tag{29}$$

On the other hand, by Lemma 2.10 of [P1] in Heisenberg local coordinates centered at x we have

$$\begin{aligned}&\tilde{\mu}_{g\,e^{-\imath\,\boldsymbol{\vartheta}}\,g^{-1}}(x) = \tilde{\mu}_{e^{-\mathrm{Ad}_g(\imath\,\boldsymbol{\vartheta})}}(x) \\ &= x + \Big(\big\langle\Phi(m_x),\mathrm{Ad}_g(\imath\,\boldsymbol{\vartheta})\big\rangle + R_3(\boldsymbol{\vartheta}), -\mathrm{Ad}_g(\imath\,\boldsymbol{\vartheta})_M(m_x) + R_2(\boldsymbol{\vartheta})\Big) \\ &= x + \Big(\big\langle\Phi\big(\mu_{g^{-1}}(m_x)\big),\imath\,\boldsymbol{\vartheta}\big\rangle + R_3(\boldsymbol{\vartheta}), -\mathrm{Ad}_g(\imath\,\boldsymbol{\vartheta})_M(m_x) + R_2(\boldsymbol{\vartheta})\Big).\end{aligned} \tag{30}$$

It follows that on the support of ϱ'

$$\partial_{\boldsymbol{\vartheta}}\Gamma(g\,T,\boldsymbol{\vartheta},u,x,y) \sim u\;\mathrm{Ad}_{g^{-1}}\Phi(m_y)\big|_{\mathfrak{t}} - \boldsymbol{\nu}. \tag{31}$$

Since $\mathcal{O}_{\boldsymbol{\nu}} \cap \mathfrak{t}^0 = \emptyset$, there exists $r_0 > 0$ such that, with $\Phi^T(m) = \Phi(m)|_{\mathfrak{t}}$,

$$\|\,\Phi^T(m)\big|_{\mathfrak{t}}\,\| \geq r_0, \quad \forall m \in M_{\mathcal{O}}.$$

Hence, if $U_{\mathcal{O}}$ is a sufficiently small open neighborhood of $M_{\mathcal{O}}$, then

$$\big\|\Phi^T(m)\big\| \geq \frac{1}{2}\,r_0, \quad \forall m \in U_{\mathcal{O}}.$$

This applies to $\mathrm{Ad}_{g^{-1}}\Phi(m_y) = \Phi \circ \mu_{g^{-1}}(m_y)$. It then follows from (31) (arguing as in the proof of Lemma 5.3 of [GP1]) that the following holds:

Lemma 4.1. *Suppose $D \gg 0$, and let $\rho \in \mathcal{C}_c^\infty\big((1/(2\,D), 2\,D)\big)$ be such a bump function that $\rho \equiv 1$ on $(1/D, D)$. Then only a rapidly decreasing contribution to the asymptotics of (28) is lost, if the integrand is multiplied by $\rho(t)$.*

Hence we may assume that integration in u is compactly supported. The proof is the completed by iteratively integrating by parts in $\mathrm{d}u$, as in the proof of Proposition 5.2 of [GP1].

□

5 Proof of Theorem 1.2

In the proof of Theorem 1.2, we shall rely on the Kirillov character formula ([Ki], [R]), which we briefly recall.

Let $\exp_G : \boldsymbol{\xi} \in \mathfrak{g} \mapsto e^{\boldsymbol{\xi}} \in G$ be the exponential map of G, and let $\mathfrak{g}' \subseteq \mathfrak{g}$ and $G' \subseteq G$ be open neighborhoods of the origin $\mathbf{0} \in \mathfrak{g}$ and of the unit $1_G \in G$, respectively, such that $\exp_G$ restricts to a diffeomorphism $\mathfrak{g}' \to G'$. We may find an Ad-invariant Euclidean product φ^H on $\mathfrak{g}$, such that $\mathrm{d}^H V_G(g)$ is the Riemannian density associated to the induced bi-invariant Riemannian metric on G, which with abuse of notation we shall also denote by φ^H. Let $\mathrm{d}^H \boldsymbol{\xi}$ be the Lebesgue measure on $\mathfrak{g}$ induced by φ^H. Let the $\mathcal{C}^\infty$ function $\mathcal{P} : \mathfrak{g}' \to (0, +\infty)$ be defined by the equality

$$\exp_G^*(\mathrm{d}^H V_G) = \mathcal{P}^2\, \mathrm{d}^H \boldsymbol{\xi}.$$

Clearly $\mathcal{P}(0) = 1$.

Let us set $n_G := (d_G - r_G)/2$. Then for every $\boldsymbol{\nu} \in \mathcal{E}^G$ and $\boldsymbol{\xi} \in \mathfrak{g}'$ we have

$$\dim(\mathcal{O}_{\boldsymbol{\nu}}) = \dim(G/T) = d_G - r_G = 2\, n_G.$$

Furthermore, let us denote by $\sigma_{\boldsymbol{\nu}}$ the Kostant-Kirillov symplectic structure on $\mathcal{O}_{\boldsymbol{\nu}}$, so that $\sigma_{\boldsymbol{\nu}}^{n_G}/n_G!$ is the symplectic volume form on $\mathcal{O}_{\boldsymbol{\nu}}$. In the following we shall set

$$\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}}} := \frac{\sigma_{\boldsymbol{\nu}}^{n_G}}{n_G!}, \quad \mathrm{vol}\big(\mathcal{O}_{\boldsymbol{\nu}}\big) := \int_{\mathcal{O}_{\boldsymbol{\nu}}} \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}}}.$$

The Kirillov character formula then says that

$$\chi_{\boldsymbol{\nu}}\left(e^{\boldsymbol{\xi}}\right) = \frac{1}{(2\,\pi)^{n_{\boldsymbol{\nu}}}}\, \frac{1}{\mathcal{P}(\boldsymbol{\xi})} \int_{\mathcal{O}_{\boldsymbol{\nu}}} e^{\imath\, \langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle}\, \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}}}(\boldsymbol{\lambda}) \quad (\boldsymbol{\xi} \in \mathfrak{g}'). \tag{32}$$

Given (32) and (2), setting $\boldsymbol{\xi} = \mathbf{0}$ we get

$$d_{\boldsymbol{\nu}} = \frac{\mathrm{vol}\big(\mathcal{O}_{\boldsymbol{\nu}}\big)}{(2\,\pi)^{n_G}} \quad \Rightarrow \quad \mathrm{vol}\big(\mathcal{O}_{\boldsymbol{\nu}}\big) = (2\,\pi)^{n_G} \prod_{\boldsymbol{\beta} \in R^+} \frac{\varphi(\boldsymbol{\nu}, \boldsymbol{\beta})}{\varphi(\boldsymbol{\delta}, \boldsymbol{\beta})}. \tag{33}$$

For every $k \geq 1$, by a rescaling we obtain

$$\chi_{k\boldsymbol{\nu}}\left(e^{\boldsymbol{\xi}}\right) = \left(\frac{k}{2\,\pi}\right)^{n_G} \frac{1}{\mathcal{P}(\boldsymbol{\xi})} \int_{\mathcal{O}_{\boldsymbol{\nu}}} e^{\imath\, k\,\langle\boldsymbol{\lambda},\boldsymbol{\xi}\rangle} \,\frac{\sigma_{\boldsymbol{\nu}}^{n_G}(\boldsymbol{\lambda})}{n_G!} \quad (\boldsymbol{\xi} \in \mathfrak{g}'). \tag{34}$$

In particular,

$$d_{k\boldsymbol{\nu}} = \left(\frac{k}{2\,\pi}\right)^{n_G} \mathrm{vol}\big(\mathcal{O}_{\boldsymbol{\nu}}\big). \tag{35}$$

Proof of Theorem 1.2. We may assume without loss that $\epsilon \in (0, 1/6)$. Furthermore, it suffices to prove the Theorem when $x = y$, since by the Cauchy-Schwartz inequality

$$\left|\Pi^{\tilde{\mu}}_{k\boldsymbol{\nu}}(x, y)\right| \leq \Pi^{\tilde{\mu}}_{k\boldsymbol{\nu}}(x, x)^{\frac{1}{2}}\, \Pi^{\tilde{\mu}}_{k\boldsymbol{\nu}}(y, y)^{\frac{1}{2}},$$

and on the other hand $\Pi^{\tilde{\mu}}_{k\boldsymbol{\nu}}(x, x)^{\frac{1}{2}}$ may be seen to satisfy an *a priori* polynomial bound in k, by adapting the arguments in §5.1.2 of [GP1]. By Theorem 3.1, we need only consider the case where x belongs to a small $S^1\times G$-invariant neighborhood $V_{\mathcal{O}_{\boldsymbol{\nu}}} \subseteq X$ of $X_{\mathcal{O}_{\boldsymbol{\nu}}}$; in particular, we may assume without loss that $\tilde{\mu}$ is free on $V_{\mathcal{O}_{\boldsymbol{\nu}}}$. Furthermore, we may replace x by $\tilde{\mu}_g(x)$ for any given $g \in G$, and assume that $\Phi(m_x) = \lambda_x\,\boldsymbol{\nu} + \beta_x$, where $\lambda_x > 0$, $\beta_x \in \boldsymbol{\nu}^\perp$, and $\|\beta_x\| \ll \lambda_x$.

Let us start from (25) with $x = y$:

$$\Pi_{k\boldsymbol{\nu}}(x, x) = d_{k\boldsymbol{\nu}} \int_G \overline{\chi_{k\boldsymbol{\nu}}(g)}\, \Pi\left(\tilde{\mu}_{g^{-1}}(x), x\right)\, \mathrm{d}^H V_G(g). \tag{36}$$

Let us set, for some small $\varepsilon_1 > 0$,

$$\begin{aligned} W' &:= \left\{(g, x) \in G \times X \,:\, \mathrm{dist}_X\left(\tilde{\mu}_g(x), x\right) < 2\,\varepsilon_1\right\} \\ W'' &:= \left\{(g, x) \in G \times X \,:\, \mathrm{dist}_X\left(\tilde{\mu}_g(x), x\right) > \varepsilon_2\right\}. \end{aligned}$$

Let $\rho' + \rho'' = 1$ be a partition of unity on $G \times X$ subordinate to the open cover $\{W', W''\}$. Then an argument as in the proof of Theorem 3.1 shows that only a rapidly decreasing contribution to the asymptotics of (36) is lost, if the integrand is multiplied by ρ''. Hence we are reduced to considering the asymptotics of

$$\Pi_{k\boldsymbol{\nu}}(x, x)' := d_{k\boldsymbol{\nu}} \int_G \rho'(g, x)\, \overline{\chi_{k\boldsymbol{\nu}}(g)}\, \Pi\left(\tilde{\mu}_{g^{-1}}(x), x\right)\, \mathrm{d}^H V_G(g). \tag{37}$$

Since $\tilde{\mu}$ is free on $V_{\mathcal{O}_{\boldsymbol{\nu}}}$, the partial function $\rho'_x := \rho'(\cdot, x)$ is supported on a small open neighborhood of the unit $1_G \in G$, which we may assume to be diffeomorphic to an open neighborhood of $\mathbf{0} \in \mathfrak{g}$ by the exponential map. Hence on the same neighborhood we may set $g = e^{\boldsymbol{\xi}}$ and express $\chi_{k\boldsymbol{\nu}}(g)$ by the Kirillov character formula (34). Furthermore, since $\tilde{\mu}_{g^{-1}}(x) \sim x$ for $\rho'_x(g) \neq 0$, we may also replace Π by its description as an FIO (20). After the rescaling $u \mapsto k\,u$, we obtain

$$\begin{aligned}
&\Pi_{k\boldsymbol{\nu}}(x,x) \sim \Pi_{k\boldsymbol{\nu}}(x,x)' \qquad (38)\\
&\sim \;\; k\,d_{k\boldsymbol{\nu}} \cdot \left(\frac{k}{2\,\pi}\right)^{n_G} \int_0^{+\infty} \mathrm{d}u \int_{\mathfrak{g}'} \mathrm{d}^H\boldsymbol{\xi} \int_{\mathcal{O}_{\boldsymbol{\nu}}} \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}}}(\boldsymbol{\lambda}) \left[e^{\imath\,k\,\Gamma_x(u,\boldsymbol{\xi},\boldsymbol{\lambda})}\,\mathcal{A}_{x,k}(u,\boldsymbol{\xi},\boldsymbol{\lambda})\right]
\end{aligned}$$

where

$$\Gamma_x(u,\boldsymbol{\xi},\boldsymbol{\lambda}) := u\,\psi\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}}}(x),x\right) - \langle\boldsymbol{\lambda},\boldsymbol{\xi}\rangle$$
$$\mathcal{A}_{x,k}(u,\boldsymbol{\xi},\boldsymbol{\lambda}) := \rho'_x\left(e^{\boldsymbol{\xi}}\right)\,\mathcal{P}(\boldsymbol{\xi})\,s\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}}}(x),x,k\,u\right).$$

Since $\tilde{\mu}_{e^{-\boldsymbol{\xi}}}(x) \sim x$ on the support of $\mathcal{A}_{x,k}$, we have in local coordinates $\mathrm{d}_{(\tilde{\mu}_{e^{-\boldsymbol{\xi}}}(x),x)}\psi \sim (\alpha_x,-\alpha_x)$. Hence $\partial_{\boldsymbol{\xi}}\Gamma_x(u,\boldsymbol{\xi},\boldsymbol{\lambda}) \sim u\,\Phi(m_x) - \boldsymbol{\lambda}$. We then have an analogue of Lemma 4.1, so that integration in $\mathrm{d}u$ may be assumed to be compactly supported. We express this by multiplying the amplitude in (38) by a bump function $\rho = \rho(u)$ compactly supported in $(1/D, D)$ for some $D \gg 0$.

Let $\gamma \in \mathcal{C}^\infty(\mathfrak{g})$ be ≥ 0, supported on a ball of radius 2 centered at the origin (say with respect to φ^H) and $\equiv 1$ on a ball of radius 1 centered at the origin. Let us define $\gamma_k \in \mathcal{C}^\infty_c(\mathfrak{g})$ for $k = 1, 2, \ldots$ by setting

$$\gamma_k(\boldsymbol{\xi}) := \gamma\left(k^{1/2-\epsilon}\,\boldsymbol{\xi}\right).$$

Let $\Pi_{k\boldsymbol{\nu}}(x,x)_1$ and $\Pi_{k\boldsymbol{\nu}}(x,x)_2$ be given by the second line of (38) multiplied by, respectively, γ_k and $1-\gamma_k$.

Lemma 5.1. $\Pi_{k\boldsymbol{\nu}}(x,x)_2 = O\left(k^{-\infty}\right)$ *as* $k \to +\infty$.

Proof of Lemma 5.1. On the support of $1-\gamma_k$, we have $\|\boldsymbol{\xi}\|^H \geq k^{\epsilon-\frac{1}{2}}$ in φ^H-norm. Hence for a certain constant $r_0 > 0$ depending only on the choice of an invariant open neighborhood $V_{\mathcal{O}} \subseteq X$ of $X_{\mathcal{O}}$ we have

$$\mathrm{dist}_X\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}}}(x),x\right) \geq r_0\,k^{\epsilon-\frac{1}{2}} \qquad \left(x \in V_{\mathcal{O}},\,\boldsymbol{\xi} \in \mathrm{supp}(1-\gamma_{D_1,k})\right). \qquad (39)$$

Hence by (23)

$$\left|\partial_u\Gamma_x(u,\boldsymbol{\xi},\boldsymbol{\lambda})\right| = \left|\psi\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}}}(x),x\right)\right| \geq \left|\Im\psi\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}}}(x),x\right)\right| \geq D_X\,r_0^2\,k^{2\epsilon-1}. \qquad (40)$$

Iteratively integrating by parts in $\mathrm{d}u$ then implies the statement, since at each step we introduce a factor $O\left(k^{-2\,\epsilon}\right)$ (see Proposition 5.2 of [GP1]). □

Thus we are reduced to considering the asymptotics of $\Pi_{k\,\boldsymbol{\nu}}(x,x)_1$. On the support of γ_k we have $\|\boldsymbol{\xi}\|^H \leq 2\,k^{\epsilon-\frac{1}{2}}$. Let us operate the rescaling $\boldsymbol{\xi} \mapsto \boldsymbol{\xi}/\sqrt{k}$, and rewrite $\Pi_{k\,\boldsymbol{\nu}}(x,x)_1$ as

$$\begin{aligned}\Pi_{k\,\boldsymbol{\nu}}(x,x)_1 &= k^{1-d_G/2}\,d_{k\,\boldsymbol{\nu}}\cdot\left(\frac{k}{2\,\pi}\right)^{n_G}\int_{1/D}^{D}\mathrm{d}u\int_{\mathfrak{g}'}\mathrm{d}^H\boldsymbol{\xi}\int_{\mathcal{O}_{\boldsymbol{\nu}}}\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}}}(\boldsymbol{\lambda})\\ &\left[e^{\imath\,k\,\Gamma_x(u,\boldsymbol{\xi}/\sqrt{k},\boldsymbol{\lambda})}\,\mathcal{A}_{x,k}(u,\boldsymbol{\xi}/\sqrt{k},\boldsymbol{\lambda})\,\rho(u)\,\gamma\left(k^{-\epsilon}\,\boldsymbol{\xi}\right)\right]. \end{aligned} \tag{41}$$

Now integration in $\boldsymbol{\xi}$ is on an expanding ball of radius $O\left(k^{\epsilon}\right)$ centered at the origin. In Heisenberg local coordinates at x, by Lemma 2.10 of [P1] we have

$$\begin{aligned}&\tilde{\mu}_{e^{-\boldsymbol{\xi}/\sqrt{k}}}(x) \\ &= x+\left(\frac{1}{\sqrt{k}}\big\langle\Phi(m_x),\boldsymbol{\xi}\big\rangle\big)+R_3\left(\frac{1}{\sqrt{k}}\boldsymbol{\xi}\right),-\frac{1}{\sqrt{k}}\boldsymbol{\xi}_M(m_x)+R_2\left(\frac{1}{\sqrt{k}}\boldsymbol{\xi}\right)\right).\end{aligned} \tag{42}$$

As in the proof of Theorem 1 of [P1], using the expansions in §3 of [SZ] one gets

$$\begin{aligned}&\Gamma_x\left(u,\boldsymbol{\xi}/\sqrt{k},\boldsymbol{\lambda}\right) \\ &= \frac{1}{\sqrt{k}}\left\langle u\,\Phi(m_x)-\boldsymbol{\lambda},\boldsymbol{\xi}\right\rangle+\frac{\imath\,u}{2\,k}\,\|\boldsymbol{\xi}_X(x)\|^2+u\,R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}\right)\,e^{\imath\,\varsigma_{x,k}(\boldsymbol{\xi}/\sqrt{k})},\end{aligned} \tag{43}$$

where $\varsigma_{x,k}(\boldsymbol{\xi})=\big\langle\Phi(m_x),\boldsymbol{\xi}\big\rangle\big)+R_3(\boldsymbol{\xi})$. Hence we rewrite (41) as follows

$$\begin{aligned}\Pi_{k\,\boldsymbol{\nu}}(x,x)_1 &= k^{1-d_G/2}\,d_{k\,\boldsymbol{\nu}}\cdot\left(\frac{k}{2\,\pi}\right)^{n_G}\int_{1/D}^{D}\mathrm{d}u\int_{\mathfrak{g}'}\mathrm{d}^H\boldsymbol{\xi}\int_{\mathcal{O}_{\boldsymbol{\nu}}}\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}}}(\boldsymbol{\lambda})\\ &\left[e^{\imath\,\sqrt{k}\,\Upsilon_x(u,\boldsymbol{\xi},\boldsymbol{\lambda})}\,\mathcal{B}_{x,k}(u,\boldsymbol{\xi}/\sqrt{k},\boldsymbol{\lambda})\,\gamma\left(D_1\,k^{-\epsilon}\,\boldsymbol{\xi}\right)\right], \end{aligned} \tag{44}$$

where now

$$\begin{aligned}\Upsilon_x(u,\boldsymbol{\xi},\boldsymbol{\lambda}) &:= \left\langle u\,\Phi(m_x)-\boldsymbol{\lambda},\boldsymbol{\xi}\right\rangle\\ \mathcal{B}_{x,k}(u,\boldsymbol{\xi}/\sqrt{k},\boldsymbol{\lambda}) &:= e^{-\frac{u}{2}\,\|\boldsymbol{\xi}_X(x)\|^2}\cdot\rho(u)\,\mathcal{A}_{x,k}(u,k^{-1/2}\,\boldsymbol{\xi},\boldsymbol{\lambda})\,e^{u\,k\,R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}\right)}\,e^{\imath\,\varsigma_{x,k}(\boldsymbol{\xi}/\sqrt{k})}\end{aligned}$$

We have $\boldsymbol{\xi}_X(x)\neq 0$ if $x\in V_{\mathcal{O}_{\boldsymbol{\nu}}}$, $\boldsymbol{\xi}\neq 0$.

Under the present transversality assumption (Assumption 1.1), there exists $s_0>0$ such that

$$\big\|\Phi(m)-\boldsymbol{\lambda}\big\|\geq s_0\cdot\mathrm{dist}_M(m,M_{\mathcal{O}}),\qquad \forall\,m\in M,\ \forall\boldsymbol{\lambda}\in\mathcal{C}(\mathcal{O}_{\boldsymbol{\nu}}).$$

Therefore, in the situation of the Theorem,

$$\big\|\partial_{\boldsymbol{\xi}}\Upsilon_x(u,\boldsymbol{\xi},\boldsymbol{\lambda})\big\|=\big\|u\,\Phi(m_x)-\boldsymbol{\lambda}\big\|\geq\frac{s_0}{D}\cdot C\,k^{\epsilon-\frac{1}{2}},\quad\forall\,\boldsymbol{\lambda}\in\mathcal{O}_{\boldsymbol{\nu}},\quad\forall\,u\in\left(\frac{1}{D},D\right).$$

The statement of the Theorem then follows by iteratively integrating by parts in $\mathrm{d}\boldsymbol{\xi}$, since each step introduces a factor $O\left(k^{-\epsilon}\right)$. □

6 Proof of Theorem 1.3

Before delving into the proof of Theorem 1.3, let us make the following remarks.

Suppose $x \in X$, and let $\varpi_x : (\theta, \mathbf{v}) \mapsto x + (\theta, \mathbf{v})$ be a system of Heisenberg local coordinates at x. Then ϖ_x induces an isomorphism $T_x X \cong \mathbb{R} \times \mathbb{R}^{2n}$, in terms of which we can give a meaning to the expression $x + \upsilon$, when $\upsilon \in T_x X$ is small. For some $c_1 > c_2 > 0$ we have

$$c_2 \|\upsilon_1 - \upsilon_2\| \leq \mathrm{dist}_X(x + \upsilon_1, x + \upsilon_2) \leq c_1 \|\upsilon_1 - \upsilon_2\| \tag{45}$$

if $\upsilon_j \sim \mathbf{0}$.

Let $\mathfrak{t}'_m$ be as in (7); we have the following characterization of the normal bundle $N(M_{\mathcal{O}_{\boldsymbol{\nu}}})$ to $M_{\mathcal{O}_{\boldsymbol{\nu}}}$ in M, which can be proved by minor adaptations of the arguments used in Lemma 4.2 and Step 4.3 of [GP1].

Lemma 6.1. *For any $m \in M_{\mathcal{O}_{\boldsymbol{\nu}}}$, $N_m(M_{\mathcal{O}_{\boldsymbol{\nu}}}) = J_m(\mathfrak{t}'_m)$.*

Furthermore, we may identify the normal bundle of $X_{\mathcal{O}_{\boldsymbol{\nu}}} \subseteq X$, $N(X_{\mathcal{O}_{\boldsymbol{\nu}}})$, with the pull-back of $N(M_{\mathcal{O}_{\boldsymbol{\nu}}})$; even more explicitly, for every $x \in X_{\mathcal{O}_{\boldsymbol{\nu}}}$ we have with $m_x = \pi(x)$

$$N_x(X_{\mathcal{O}_{\boldsymbol{\nu}}}) = N_{m_x}(X_{\mathcal{O}_{\boldsymbol{\nu}}})^\sharp.$$

Hence there is an orthogonal direct sum

$$N_{m_x}(X_{\mathcal{O}_{\boldsymbol{\nu}}})^\sharp \oplus \mathfrak{g}_X(x) \oplus \mathfrak{g}_M(m_x)^{\perp_h \sharp} \subseteq T_x X. \tag{46}$$

We then have the following consequence, whose proof is omitted.

Lemma 6.2. *Suppose $x \in X_{\mathcal{O}_{\boldsymbol{\nu}}}$, and choose a system of Heisenberg local coordinates at x. Then there exists $\delta > 0$ such that for any choice of $\boldsymbol{\xi} \in \mathfrak{g}$, $\mathbf{v}_j \in N_{m_x}(X_{\mathcal{O}_{\boldsymbol{\nu}}})$, $\mathbf{w}_j \in \mathfrak{g}_M(m_x)^{\perp_h}$ of sufficiently small norm we have*

$$\mathrm{dist}_X\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}}}\big(x + (\mathbf{v}_1 + \mathbf{w}_1)\big), x + (\mathbf{v}_2 + \mathbf{w}_2)\right) \geq \delta \|\boldsymbol{\xi}\|^{\varphi}.$$

Furthermore, δ may be chosen uniformly on $X_{\mathcal{O}_{\boldsymbol{\nu}}}$.

Proof of Theorem 1.3. We may replace x by $\tilde{\mu}_h(x)$ for a suitable $h \in G$, and assume without loss that $\Phi(m_x) = \varsigma(m_x)\,\boldsymbol{\nu}$. Hence, we may assume that

$$\mathfrak{t}_m = \mathfrak{t}, \quad \mathfrak{t}'_m = \mathfrak{t}_{\boldsymbol{\nu}} \tag{47}$$

(see (6), (7)). Let us set

$$x_{j,k} := \frac{1}{\sqrt{k}}\,(\mathbf{v}_j + \mathbf{w}_j) \qquad (j = 1, 2;\ k > 0).$$

and replace (36) by

$$\Pi_{k\,\boldsymbol{\nu}}(x_{1,k},x_{2,k}) = d_{k\,\boldsymbol{\nu}}\int_G \overline{\chi_{k\,\boldsymbol{\nu}}(g)}\,\Pi\left(\tilde{\mu}_{g^{-1}}(x_{1,k}),x_{2,k}\right)\,\mathrm{d}^H V_G(g). \tag{48}$$

Given that $x_{j,k}\to x$, the argument leading to (37) now implies $\Pi_{k\,\boldsymbol{\nu}}(x_{1,k},x_{2,k})\sim \Pi_{k\,\boldsymbol{\nu}}(x_{1,k},x_{2,k})'$, where

$$\Pi_{k\,\boldsymbol{\nu}}(x_{1,k},x_{2,k})' := d_{k\,\boldsymbol{\nu}}\int_G \rho'(g,x)\,\overline{\chi_{k\,\boldsymbol{\nu}}(g)}\,\Pi\left(\tilde{\mu}_{g^{-1}}(x_{1,k}),x_{2,k}\right)\,\mathrm{d}^H V_G(g). \tag{49}$$

We then obtain in place of (38)

$$\begin{aligned}\Pi_{k\,\boldsymbol{\nu}}(x_{1,k},x_{2,k}) &\sim \Pi_{k\,\boldsymbol{\nu}}(x_{1,k},x_{2,k})' \\ &\sim k\,d_{k\,\boldsymbol{\nu}}\cdot\left(\frac{k}{2\,\pi}\right)^{n_G}\int_0^{+\infty}\mathrm{d}u\int_{\mathfrak{g}'}\mathrm{d}^H\boldsymbol{\xi}\int_{\mathcal{O}_{\boldsymbol{\nu}}}\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}}}(\boldsymbol{\lambda})\left[e^{\imath\,k\,\Gamma_{x,k}(u,\boldsymbol{\xi},\boldsymbol{\lambda})}\,\mathcal{B}_{x,k}(u,\boldsymbol{\xi},\boldsymbol{\lambda})\right]\end{aligned} \tag{50}$$

where

$$\Gamma_{x,k}(u,\boldsymbol{\xi},\boldsymbol{\lambda}) := u\,\psi\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}}}(x_{1,k}),x_{2,k}\right) - \langle\boldsymbol{\lambda},\boldsymbol{\xi}\rangle \tag{51}$$

$$\mathcal{B}_{x,k}(u,\boldsymbol{\xi},\boldsymbol{\lambda}) := \rho'_x\left(e^{\boldsymbol{\xi}}\right)\,\mathcal{P}(\boldsymbol{\xi})\cdot s\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}}}(x_{1,k}),x_{2,k},k\,u\right). \tag{52}$$

Since $\tilde{\mu}_{e^{-\boldsymbol{\xi}}}(x_{1,k})\sim x_{2,k}$ on the support of $\mathcal{B}_{x,k}$, by the same argument used in the proof of Theorem 1.2 we may multiply the integrand in (50) by the same cut-off function $\rho=\rho(u)$ without affecting the asymptotics, so as to assume that integration in $\mathrm{d}u$ is supported in $(1/D,D)$ for some $D\gg 0$.

In view of Lemma 6.2, we have for $k\gg 0$

$$\mathrm{dist}_X\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}}}\big(x_{1,k}\big)\big),x_{2,k})\right)\geq\delta\,\|\boldsymbol{\xi}\|^{\varphi}.$$

Using this, we obtain an obvious analogue of (39), so that we can reprove Lemma 5.1 in the present setting. Rescaling in $\boldsymbol{\xi}$, we obtain in place of (41):

$$\begin{aligned}&\Pi_{k\,\boldsymbol{\nu}}(x_{1,k},x_{2,k}) \\ &\sim k^{1-d_G/2}\,d_{k\,\boldsymbol{\nu}}\cdot\left(\frac{k}{2\,\pi}\right)^{n_G}\int_{1/D}^{D}\mathrm{d}u\int_{\mathfrak{g}}\mathrm{d}^H\boldsymbol{\xi}\int_{\mathcal{O}_{\boldsymbol{\nu}}}\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}}}(\boldsymbol{\lambda}) \\ &\qquad\left[e^{\imath\,k\,\Gamma_{x,k}(u,\boldsymbol{\xi}/\sqrt{k},\boldsymbol{\lambda})}\,\mathcal{B}_{x,k}(u,\boldsymbol{\xi}/\sqrt{k},\boldsymbol{\lambda})\,\rho(u)\,\gamma\left(k^{-\epsilon}\,\boldsymbol{\xi}\right)\right].\end{aligned} \tag{53}$$

In view of Corollary 2.2 of [P1], and using that $\omega_{m_x}\big(\boldsymbol{\xi}_M(m),\mathbf{w}_1\big)=0$, in place of (42) we have

$$\tilde{\mu}_{e^{-\boldsymbol{\xi}/\sqrt{k}}}(x_{1,k}) = x + \left(\Theta_{k,1},V_{k,1}\right). \tag{54}$$

where

$$\begin{aligned}\Theta_{k,1} &= \Theta_k(x, \boldsymbol{\xi}, \mathbf{v}_1, \mathbf{w}_1) \qquad (55)\\ &= \frac{1}{\sqrt{k}} \big\langle \Phi(m_x), \boldsymbol{\xi}) \big\rangle + \frac{1}{k}\, \omega_{m_x}\big(\boldsymbol{\xi}_M(m_x), \mathbf{v}_1\big) + R_3\left(\frac{1}{\sqrt{k}}\,\boldsymbol{\xi}, \frac{1}{\sqrt{k}}\,\mathbf{v}_1, \frac{1}{\sqrt{k}}\,\mathbf{w}_1\right),\end{aligned}$$

$$\begin{aligned}V_{k,1} &= V_k(x, \boldsymbol{\xi}, \mathbf{v}_1, \mathbf{w}_1) \qquad (56)\\ &= \frac{1}{\sqrt{k}}\big(\mathbf{v}_1 + \mathbf{w}_1 - \boldsymbol{\xi}_M(m_x)\big) + R_2\left(\frac{1}{\sqrt{k}}\,\boldsymbol{\xi}, \frac{1}{\sqrt{k}}\,\mathbf{v}_1, \frac{1}{\sqrt{k}}\,\mathbf{w}_1\right).\end{aligned}$$

We then have (see §3 of [SZ])

$$\begin{aligned}& u\,\psi\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}/\sqrt{k}}}(x_{1,k}), x_{2,k}\right) \qquad (57)\\ &= \imath\, u\,\left[1 - e^{\imath\,\Theta_k}\right] - \imath\, u\,\psi_2\left(V_{k,1}, \frac{1}{\sqrt{k}}\,(\mathbf{v}_2 + \mathbf{w}_2)\right) + u\,R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}, \frac{\mathbf{v}_j}{\sqrt{k}}, \frac{\mathbf{w}_j}{\sqrt{k}}\right).\end{aligned}$$

We have

$$\begin{aligned}\imath\, u\,\left[1 - e^{\imath\,\Theta_{k,1}}\right] &= u\,\Theta_{k,1} + \frac{\imath\, u}{2}\,\Theta_{k,1}^2 + u\,R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}, \frac{\mathbf{v}_1}{\sqrt{k}}, \frac{\mathbf{w}_1}{\sqrt{k}}\right) \qquad (58)\\ &= \frac{u}{\sqrt{k}}\big\langle \Phi(m_x), \boldsymbol{\xi}) \big\rangle + \frac{u}{k}\,\omega_{m_x}\big(\boldsymbol{\xi}_M(m_x), \mathbf{v}_1\big) + \frac{\imath\, u}{2\,k}\big\langle \Phi(m_x), \boldsymbol{\xi}\big\rangle^2\\ &\quad + u\,R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}, \frac{\mathbf{v}_1}{\sqrt{k}}, \frac{\mathbf{w}_1}{\sqrt{k}}\right),\end{aligned}$$

and

$$\begin{aligned}&\psi_2\left(V_{k,1}, \frac{1}{\sqrt{k}}\,(\mathbf{v}_2 + \mathbf{w}_2)\right) \qquad (59)\\ &= \frac{1}{k}\left[-\imath\,\omega_{m_x}\big(\mathbf{v}_1 + \mathbf{w}_1 - \boldsymbol{\xi}_M(m_x), \mathbf{v}_2 + \mathbf{w}_2\big) - \frac{1}{2}\,\big\|\big(\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{w}_1 - \mathbf{w}_2) - \boldsymbol{\xi}_M(m_x)\big\|_{m_x}^2\right]\\ &\quad + R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}, \frac{\mathbf{v}_j}{\sqrt{k}}, \frac{\mathbf{w}_j}{\sqrt{k}}\right).\end{aligned}$$

Since $\mathbf{w}_j \in \mathfrak{g}_M(m)^{\perp_{h_m}}$, we have $\omega_m\big(\boldsymbol{\xi}_M(m_x), \mathbf{w}_2\big) = 0$ for any $\boldsymbol{\xi} \in \mathfrak{g}$.

Lemma 6.3. *If $m \in M_{\mathcal{O}_\nu}$ and $\mathbf{v}_j \in N_m(M_{\mathcal{O}_\nu})$, then $\omega_m(\mathbf{v}_1, \mathbf{v}_2) = 0$.*

Proof of Lemma 6.3. By Lemma 6.1, there are $\boldsymbol{\eta}_j \in \mathfrak{t}'_m \subseteq \mathfrak{t}_m$ such that $\mathbf{v}_j = J_m\big(\boldsymbol{\eta}_{j\,M}(m)\big)$ (given our previous reduction we may assume $\mathfrak{t}_m = \mathfrak{t}$). Hence $\omega_m(\mathbf{v}_1, \mathbf{v}_2) = \omega_m\big(\boldsymbol{\eta}_{1\,M}(m), \boldsymbol{\eta}_{2\,M}(m)\big)$. On the other hand, μ restricts to a Hamiltonian action of the maximal torus T_m, and therefore the vector fields $\boldsymbol{\eta}_M$, with $\boldsymbol{\eta} \in \mathfrak{t}$, are all in symplectic involution. Hence $\omega\big(\boldsymbol{\eta}_{1M}, \boldsymbol{\eta}_{2M}\big) \equiv 0$. Hence $\omega_m(\mathbf{v}_1, \mathbf{v}_2) = 0$. □

Given (46) and Lemma 6.3, in view of Definition 1.3 we may rewrite (59) as

$$\begin{aligned}
&\psi_2\left(V_{k,1}, \frac{1}{\sqrt{k}}\,(\mathbf{v}_2+\mathbf{w}_2)\right) \qquad (60)\\
&= \frac{1}{k}\left[\psi_2(\mathbf{w}_1,\mathbf{w}_2) - \frac{1}{2}\left\|\mathbf{v}_1-\mathbf{v}_2\right\|^2_{m_x} + \imath\,\omega_{m_x}\big(\boldsymbol{\xi}_M(m_x),\mathbf{v}_2\big) - \frac{1}{2}\left\|\boldsymbol{\xi}_M(m_x)\right\|^2_m\right]\\
&\quad + R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}, \frac{\mathbf{v}_j}{\sqrt{k}}, \frac{\mathbf{w}_j}{\sqrt{k}}\right).
\end{aligned}$$

Hence, (57) may be rewritten

$$\begin{aligned}
&u\,\psi\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}/\sqrt{k}}}(x_{1,k}), x_{2,k}\right) \qquad (61)\\
&= \frac{u}{\sqrt{k}}\left\langle\Phi(m_x),\boldsymbol{\xi})\right\rangle + \frac{u}{k}\,\omega_{m_x}\big(\boldsymbol{\xi}_M(m_x),\mathbf{v}_1\big) + \frac{\imath\,u}{2\,k}\left\langle\Phi(m_x),\boldsymbol{\xi}\right\rangle^2\\
&\quad + u\,R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}, \frac{\mathbf{v}_1}{\sqrt{k}}, \frac{\mathbf{w}_1}{\sqrt{k}}\right)\\
&\quad - \frac{\imath\,u}{k}\left[\psi_2(\mathbf{w}_1,\mathbf{w}_2) - \frac{1}{2}\left\|\mathbf{v}_1-\mathbf{v}_2\right\|^2_{m_x} + \imath\,\omega_{m_x}\big(\boldsymbol{\xi}_M(m_x),\mathbf{v}_2\big) - \frac{1}{2}\left\|\boldsymbol{\xi}_M(m_x)\right\|^2_{m_x}\right]\\
&\quad + u\,R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}, \frac{\mathbf{v}_j}{\sqrt{k}}, \frac{\mathbf{w}_j}{\sqrt{k}}\right).
\end{aligned}$$

Whence

$$\begin{aligned}
&\imath\,k\,u\,\psi\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}/\sqrt{k}}}(x_{1,k}), x_{2,k}\right) \qquad (62)\\
&= \imath\sqrt{k}\,u\left\langle\Phi(m_x),\boldsymbol{\xi}\right\rangle + \imath\,u\,\omega_{m_x}\big(\boldsymbol{\xi}_M(m_x),\mathbf{v}_1+\mathbf{v}_2\big) - \frac{u}{2}\left\langle\Phi(m_x),\boldsymbol{\xi}\right\rangle^2\\
&\quad + u\left[\psi_2(\mathbf{w}_1,\mathbf{w}_2) - \frac{1}{2}\left\|\mathbf{v}_1-\mathbf{v}_2\right\|^2_{m_x} - \frac{1}{2}\left\|\boldsymbol{\xi}_M(m_x)\right\|^2_{m_x}\right]\\
&\quad + \imath\,u\,k\,R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}, \frac{\mathbf{v}_j}{\sqrt{k}}, \frac{\mathbf{w}_j}{\sqrt{k}}\right)\\
&= \imath\sqrt{k}\,u\left\langle\Phi(m_x),\boldsymbol{\xi}\right\rangle + \imath\,u\,\omega_{m_x}\big(\boldsymbol{\xi}_M(m_x),\mathbf{v}_1+\mathbf{v}_2\big) - \frac{u}{2}\left\|\boldsymbol{\xi}_X(x)\right\|^2_x\\
&\quad + u\left[\psi_2(\mathbf{w}_1,\mathbf{w}_2) - \frac{1}{2}\left\|\mathbf{v}_1-\mathbf{v}_2\right\|^2_{m_x}\right]\\
&\quad + \imath\,u\,k\,R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}, \frac{\mathbf{v}_j}{\sqrt{k}}, \frac{\mathbf{w}_j}{\sqrt{k}}\right).
\end{aligned}$$

In view of (51)

$$\begin{aligned}
&\imath\, k\, \Gamma_{x,k}(u, \boldsymbol{\xi}/\sqrt{k}, \boldsymbol{\lambda}) \qquad (63)\\
&= \imath\, k \left[u\, \psi\left(\tilde{\mu}_{e^{-\boldsymbol{\xi}/\sqrt{k}}}(x_{1,k}), x_{2,k} \right) - \left\langle \boldsymbol{\lambda}, \frac{1}{\sqrt{k}}\, \boldsymbol{\xi} \right\rangle \right]\\
&= \imath\, \sqrt{k}\, \big\langle u\, \Phi(m_x) - \boldsymbol{\lambda}, \boldsymbol{\xi} \big\rangle + \imath\, u\, \omega_m\big(\boldsymbol{\xi}_M(m), \mathbf{v}_1 + \mathbf{v}_2\big) - \frac{u}{2}\, \|\boldsymbol{\xi}_X(x)\|_x^2\\
&\quad + u\, \left[\psi_2(\mathbf{w}_1, \mathbf{w}_2) - \frac{1}{2}\, \|\mathbf{v}_1 - \mathbf{v}_2\|_m^2 \right]\\
&\quad + \imath\, u\, k\, R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}, \frac{\mathbf{v}_j}{\sqrt{k}}, \frac{\mathbf{w}_j}{\sqrt{k}} \right)
\end{aligned}$$

Let $\mathrm{d}^{\varphi}\boldsymbol{\xi}$ be the Lebesgue measure on $\mathfrak{g}$ associated to φ. Then

$$\mathrm{d}^H\boldsymbol{\xi} = \mathrm{vol}^{\varphi}(G)^{-1}\, \mathrm{d}^{\varphi}\boldsymbol{\xi}. \qquad (64)$$

Using this and (35), (53) may be rewritten as an oscillatory integral in $\sqrt{k}$ with a real phase, in the form

$$\begin{aligned}
&\Pi_{k\,\boldsymbol{\nu}}(x_{1,k}, x_{2,k}) \qquad (65)\\
&\sim \frac{\mathrm{vol}\big(\mathcal{O}_{\boldsymbol{\nu}}\big)}{\mathrm{vol}^{\varphi}(G)} \cdot k^{1-d_G/2} \left(\frac{k}{2\,\pi} \right)^{2\, n_G}\\
&\quad \cdot \int_{1/D}^{D} \mathrm{d}u \int_{\mathfrak{g}} \mathrm{d}^{\varphi}\boldsymbol{\xi} \int_{\mathcal{O}_{\boldsymbol{\nu}}} \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}}}(\boldsymbol{\lambda}) \left[e^{\imath\, \sqrt{k}\, \Upsilon_x(u, \boldsymbol{\xi}, \boldsymbol{\lambda})}\, \mathcal{C}_{x,k}(u, \boldsymbol{\xi}, \boldsymbol{\lambda}; \mathbf{v}_j, \mathbf{w}_j) \right],
\end{aligned}$$

with phase Υ_x and amplitude $\mathcal{C}_{x,k}$ given by, respectively,

$$\Upsilon_x(u, \boldsymbol{\xi}, \boldsymbol{\lambda}) := \big\langle u\, \Phi(m_x) - \boldsymbol{\lambda}, \boldsymbol{\xi} \big\rangle, \qquad (66)$$

$$\begin{aligned}
\mathcal{C}_{x,k}(u, \boldsymbol{\xi}, \boldsymbol{\lambda}; \mathbf{v}_j, \mathbf{w}_j) \;:=\; & e^{\imath\, u\, \omega_m\left(\boldsymbol{\xi}_M(m), \mathbf{v}_1 + \mathbf{v}_2\right) - \frac{u}{2}\, \|\boldsymbol{\xi}_X(x)\|_x^2 + u\left[\psi_2(\mathbf{w}_1, \mathbf{w}_2) - \frac{1}{2}\, \|\mathbf{v}_1 - \mathbf{v}_2\|_{m_x}^2\right]}\\
& \cdot \mathcal{B}'_{x,k}(u, \boldsymbol{\xi}/\sqrt{k}, \boldsymbol{\lambda}) \qquad (67)
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{B}'_{x,k}(u, \boldsymbol{\xi}/\sqrt{k}, \boldsymbol{\lambda}) \qquad (68)\\
&= e^{\imath\, u\, k\, R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}, \frac{\mathbf{v}_j}{\sqrt{k}}, \frac{\mathbf{w}_j}{\sqrt{k}} \right)} \cdot \mathcal{B}_{x,k}(u, \boldsymbol{\xi}/\sqrt{k}, \boldsymbol{\lambda})\, \rho(u)\, \gamma\left(k^{-\epsilon}\, \boldsymbol{\xi}\right).
\end{aligned}$$

There exists $r_{\boldsymbol{\nu}} > 0$, depending only on $\boldsymbol{\nu}$, such that $\|\boldsymbol{\xi}_X(x)\|_x \geq r_{\boldsymbol{\nu}}\, \|\boldsymbol{\xi}\|^{\varphi}$, $\forall\, x \in X_{\mathcal{O}_{\boldsymbol{\nu}}}$. Furthermore, Taylor expansion at the origin yields an asymptotic expansion of the form

$$\mathcal{B}'_{x,k}(u, \boldsymbol{\xi}/\sqrt{k}, \boldsymbol{\lambda}) \sim \gamma\left(k^{-\epsilon}\, \boldsymbol{\xi}\right) \cdot \sum_{j \geq 0} k^{d-j/2}\, P_j(m_x, u; \boldsymbol{\xi}, \mathbf{v}_j, \mathbf{w}_j), \qquad (69)$$

where $P_j(m_x, u; \cdot, \cdot, \cdot)$ is a polynomial of degree $\leq 3\,j$, and parity j (recall that $\|\boldsymbol{\xi}\|^{\varphi}$, $\|\mathbf{v}_j\|$, $\|\mathbf{w}_j\| \leq C'\,k^{\epsilon}$ for some fixed $C' > 0$, and that $\epsilon \in (0, 1/6)$). In view of (21) and (24),

$$P_j(m_x, u; \boldsymbol{\xi}, \mathbf{v}_j, \mathbf{w}_j) = \left(\frac{u}{\pi}\right)^d. \tag{70}$$

The expansion may be integrated term by term.

Recall that we have reduced to the case where $\Phi(m_x) \in \mathbb{R}_+\,\boldsymbol{\nu}$, hence $\Phi(m_x) = \varsigma(m_x)\,\boldsymbol{\nu}$ (see (4)).

Lemma 6.4. *Let $\mathcal{O}' \Subset \mathcal{O}'' \subset \mathcal{O}_{\boldsymbol{\nu}}$ be suitably small neighborhoods of $\boldsymbol{\nu}$. Let $\varrho_{\boldsymbol{\nu}} \in \mathcal{C}^{\infty}_c(\mathcal{O}'')$ be $\mathcal{C}^{\infty}$, ≥ 0, and $\equiv 1$ on $\mathcal{O}'$. Then the asymptotics of (65) are unchanged, it the integrand is multiplied by $\varrho_{\boldsymbol{\nu}}(\boldsymbol{\lambda})$.*

Proof of Lemma 6.4. Since the adjoint action is unitary, $\mathcal{O}_{\boldsymbol{\nu}} \cap \mathbb{R}_+\,\boldsymbol{\nu} = \{\boldsymbol{\nu}\}$. Hence, by (66) there exists $a_0 > 0$ such that

$$\|\partial_{\boldsymbol{\xi}} \Upsilon_x\| = \|u\,\Phi(m_x) - \boldsymbol{\lambda}\|^{\varphi} \geq a_0,$$

for all $u > 0$ and $\boldsymbol{\lambda} \in \mathrm{supp}(1 - \varrho_{\boldsymbol{\nu}})$. The claim follows integrating by parts in $\boldsymbol{\xi}$, which is legitimate in view of the cut-off and the exponential factor. □

In the following, we shall redefine $\mathcal{C}_{x,k}$ implicitly incorporating the factor $\varrho_{\boldsymbol{\nu}}(\boldsymbol{\lambda})$, so that integration in $\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}}}(\boldsymbol{\lambda})$ is over $\mathcal{O}'' \subset \mathcal{O}_{\boldsymbol{\nu}}$.

We have an equivariant diffeomorphism

$$\beta : g\,T \in G/T \mapsto g \cdot \boldsymbol{\nu} := \mathrm{Coad}_g(\boldsymbol{\nu}) \in \mathcal{O}_{\boldsymbol{\nu}}. \tag{71}$$

Let $\mathrm{d}^H V_{G/T}$ be the Haar measure on G/T; then

$$\beta^*\big(\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}}}\big) = \mathrm{vol}(\mathcal{O}_{\boldsymbol{\nu}})\,\mathrm{d}^H V_{G/T}. \tag{72}$$

Furthermore, in view of the factor $\beta^*(\varrho_{\boldsymbol{\nu}})$ which is left implicit, integration over $\mathcal{O}'' \subset \mathcal{O}_{\boldsymbol{\nu}}$ in (65) gets replaced by integration over a small neighborhood of $e_G\,T \in G/T$, according to (72). Let us introduce local coordinates on G/T near $e_G\,T$ by composing the projection $\pi_{G/T} : G \to G/T$ with the restriction of the exponential map of G to the Euclidean orthocomplement $\mathfrak{t}^{\perp_{\varphi}} \subset \mathfrak{g}$ of $\mathfrak{t}$ w.r.t. φ:

$$E : \boldsymbol{\gamma} \in \mathfrak{t}^{\perp_{\varphi}} \mapsto e^{\boldsymbol{\gamma}}\,T \in G/T.$$

When restricted to a small open neighborhood of the origin, E is a diffeomorphism onto its image, hence a local chart for G/T centered at $e_G\,T$; we have an isomorphism $\mathfrak{t}^{\perp_\varphi} \cong T_{e_G T}(G/T)$. The Lebesgue measure on $\mathfrak{t}^{\perp_\varphi}$ associated to the restriction φ' of φ will be denoted $\mathrm{d}^\varphi \boldsymbol{\gamma}$. Being T-invariant, φ' determines an equivariant Riemannian metric on G/T, whose associated Riemannian density and volume will be denoted $\mathrm{d}^\varphi V_{G/T}$ and $\mathrm{vol}^\varphi(G/T)$, respectively. Then

$$E^*(\mathrm{d}^\varphi V_{G/T}) = \mathcal{R}(\boldsymbol{\gamma})\,\mathrm{d}^\varphi \boldsymbol{\gamma}, \quad \mathcal{R}(\boldsymbol{\gamma}) = 1 + R_1(\boldsymbol{\gamma}). \tag{73}$$

Viewing G as a principal T-bundle over G/T, by fiber integration one obtains $\mathrm{vol}^\varphi(G/T) = \mathrm{vol}^\varphi(G)/\mathrm{vol}^\varphi(T)$. Clearly,

$$\mathrm{d}^\varphi V_{G/T} = \mathrm{vol}^\varphi(G/T)\,\mathrm{d}^H V_{G/T}. \tag{74}$$

Hence, if we view $\beta \circ E$, restricted to a small neighborhood of the origin in $\mathfrak{t}^{\perp_\varphi}$ as a local coordinate chart on $\mathcal{O}_{\boldsymbol{\nu}}$, we obtain by (73) and (74):

$$(\beta \circ E)^*(\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}}}) = \mathrm{vol}(\mathcal{O}_{\boldsymbol{\nu}})\,E^*(\mathrm{d}^H V_{G/H}) = \mathrm{vol}(\mathcal{O}_{\boldsymbol{\nu}})\,\frac{\mathrm{vol}^\varphi(T)}{\mathrm{vol}^\varphi(G)}\,\mathcal{R}(\boldsymbol{\gamma})\,\mathrm{d}^\varphi\boldsymbol{\gamma}. \tag{75}$$

With these substitutions, recalling (71) we may rewrite (65) in the following manner:

$$\begin{aligned}\Pi_{k\,\boldsymbol{\nu}}(x_{1,k}, x_{2,k}) \quad \sim \quad & \mathrm{vol}\big(\mathcal{O}_{\boldsymbol{\nu}}\big)^2 \cdot \frac{\mathrm{vol}^\varphi(T)}{\mathrm{vol}^\varphi(G)^2} \cdot k^{1-d_G/2}\left(\frac{k}{2\,\pi}\right)^{2\,n_G} \\ & \int_{1/D}^{D} \mathrm{d}u \int_{\mathfrak{g}} \mathrm{d}^\varphi \boldsymbol{\xi} \int_{\mathfrak{t}^{\perp_\varphi}} \mathrm{d}^\varphi\,\boldsymbol{\gamma}\,\left[e^{\imath\,\sqrt{k}\,\Upsilon_x(u,\boldsymbol{\xi},e^{\boldsymbol{\lambda}}\cdot\boldsymbol{\nu})}\,\mathcal{C}_{x,k}\left(u, \boldsymbol{\xi}, e^{\boldsymbol{\gamma}}\cdot\boldsymbol{\nu}; \mathbf{v}_j, \mathbf{w}_j\right)\right].\end{aligned} \tag{76}$$

Let us set $g \cdot \boldsymbol{\nu} := \mathrm{Ad}_g(\boldsymbol{\xi})$, for $g \in G$ and $\boldsymbol{\xi} \in \mathfrak{g}$. We have:

$$e^{\boldsymbol{\gamma}} \cdot \boldsymbol{\nu}^\varphi = \boldsymbol{\nu}^\varphi + [\boldsymbol{\gamma}, \boldsymbol{\nu}^\varphi] + R_2(\boldsymbol{\gamma}) = \boldsymbol{\nu}^\varphi - [\boldsymbol{\nu}^\varphi, \boldsymbol{\gamma}] + R_2(\boldsymbol{\gamma}).$$

Since $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda}^\varphi$ intertwines the coadjoint and adjoint actions, (66) may be rewritten as follows:

$$\begin{aligned}\Upsilon_x(u, \boldsymbol{\xi}, e^{\boldsymbol{\gamma}} \cdot \boldsymbol{\nu}) \quad &= \quad \big\langle u\,\Phi(m_x) - e^{\boldsymbol{\gamma}} \cdot \boldsymbol{\nu}, \boldsymbol{\xi}\big\rangle \\ &= \quad \varphi\left(u\,\Phi(m_x)^\varphi - e^{\boldsymbol{\gamma}} \cdot \boldsymbol{\nu}^\varphi, \boldsymbol{\xi}\right) \\ &= \quad \varphi\left(u\,\Phi(m_x)^\varphi - \boldsymbol{\nu}^\varphi + [\boldsymbol{\nu}^\varphi, \boldsymbol{\gamma}] + R_2(\boldsymbol{\gamma}), \boldsymbol{\xi}\right) \\ &= \quad \varphi\left(\big(u\,\varsigma(m_x) - 1\big)\,\boldsymbol{\nu}^\varphi + [\boldsymbol{\nu}^\varphi, \boldsymbol{\gamma}] + R_2(\boldsymbol{\gamma}), \boldsymbol{\xi}\right);\end{aligned} \tag{77}$$

here $R_2(\boldsymbol{\gamma})$ is real-valued. In terms of the φ-orthogonal direct sum decompositions

$$\mathfrak{t} := \mathrm{span}\,(\boldsymbol{\nu}^\varphi) \cap \mathfrak{t}_{\boldsymbol{\nu}}, \quad \mathfrak{g} = \mathrm{span}\,(\boldsymbol{\nu}^\varphi) \oplus \mathfrak{t}_{\boldsymbol{\nu}} \oplus \mathfrak{t}^{\perp_\varphi},$$

we shall write the general element of $\mathfrak{g}$ as

$$\boldsymbol{\xi} = s\,\boldsymbol{\nu}_u^{\varphi} + \boldsymbol{\xi}' + \boldsymbol{\xi}'', \quad \text{where} \quad s \in \mathbb{R},\, \boldsymbol{\xi}' \in \mathfrak{t}_{\boldsymbol{\nu}},\, \boldsymbol{\xi}'' \in \mathfrak{t}^{\perp_{\varphi}}.$$

Furthermore, we may introduce orthnormal basis of $\mathfrak{t}_{\boldsymbol{\nu}}$ and $\mathfrak{t}^{\perp_{\varphi}}$ w.r.t. φ, so as to unitarily identify $\mathfrak{t}_{\boldsymbol{\nu}} \cong \mathbb{R}^{r_G-1}$ and $\mathfrak{t}^{\perp_{\varphi}} \cong \mathbb{R}^{2\,n_{\boldsymbol{\nu}}}$. Let $Z_{\boldsymbol{\nu}^{\varphi}}$ the skew-symmetric and non-degenerate matrix representing $S_{\boldsymbol{\nu}^{\varphi}}$ w.r.t. the given orthonormal basis of $\mathfrak{t}^{\perp_{\varphi}}$. Then (77) may be rewritten:

$$\begin{aligned}
\Upsilon_{x,\boldsymbol{\xi}'}(u,s,\boldsymbol{\xi}'',\boldsymbol{\gamma}) &:= \Upsilon_x(u,\boldsymbol{\xi},e^{\boldsymbol{\gamma}}\cdot\boldsymbol{\nu}) \qquad (78)\\
&= \varphi\left(\big(u\,\varsigma(m_x)-1\big)\,\boldsymbol{\nu}^{\varphi} + [\boldsymbol{\nu}^{\varphi},\boldsymbol{\gamma}] + R_2(\boldsymbol{\gamma}),\, s\,\boldsymbol{\nu}_u^{\varphi} + \boldsymbol{\xi}' + \boldsymbol{\xi}''\right)\\
&= s\,\big(u\,\varsigma(m_x)-1\big)\,\|\boldsymbol{\nu}^{\varphi}\|_{\varphi} - \varphi\left(\boldsymbol{\gamma},[\boldsymbol{\nu}^{\varphi},\boldsymbol{\xi}'']\right) + \varphi\left(R_2(\boldsymbol{\gamma}),\, s\,\boldsymbol{\nu}_u^{\varphi} + \boldsymbol{\xi}' + \boldsymbol{\xi}''\right)\\
&= s\,\big(u\,\varsigma(m_x)-1\big)\,\|\boldsymbol{\nu}^{\varphi}\|_{\varphi} - \boldsymbol{\gamma}^t\, Z_{\boldsymbol{\nu}^{\varphi}}\,\boldsymbol{\xi}'' + R_2(\boldsymbol{\gamma})^t\,\left(s\,\boldsymbol{\nu}_u^{\varphi} + \boldsymbol{\xi}' + \boldsymbol{\xi}''\right).
\end{aligned}$$

We write (76) in the form

$$\begin{aligned}
\Pi_{k\,\boldsymbol{\nu}}(x_{1,k},x_{2,k}) \;\sim\; & \mathrm{vol}\big(\mathcal{O}_{\boldsymbol{\nu}}\big)^2 \cdot \frac{\mathrm{vol}^{\varphi}(T)}{\mathrm{vol}^{\varphi}(G)^2} \cdot k^{1-d_G/2}\left(\frac{k}{2\,\pi}\right)^{2\,n_G} \qquad (79)\\
& \cdot \int_{\mathfrak{t}_{\boldsymbol{\nu}}} \mathrm{d}^{\varphi}\boldsymbol{\xi}'\,\left[\mathcal{I}_{x,k}\left(\boldsymbol{\xi}';\mathbf{v}_j,\mathbf{w}_j\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}_{x,k}\left(\boldsymbol{\xi}';\mathbf{v}_j,\mathbf{w}_j\right) \;:=\; & \int_{1/D}^{D}\mathrm{d}u \int_{-\infty}^{+\infty}\mathrm{d}s \int_{\mathfrak{t}^{\perp_{\varphi}}}\mathrm{d}^{\varphi}\boldsymbol{\xi}'' \int_{\mathfrak{t}^{\perp_{\varphi}}}\mathrm{d}^{\varphi}\boldsymbol{\gamma}\\
& \left[e^{\imath\sqrt{k}\,\Upsilon_{x,\boldsymbol{\xi}'}(u,s,\boldsymbol{\xi}'',\boldsymbol{\gamma})}\,\mathcal{C}_{x,k}\left(u,\boldsymbol{\xi},e^{\boldsymbol{\gamma}}\cdot\boldsymbol{\nu};\mathbf{v}_j,\mathbf{w}_j\right)\right]. \qquad (80)
\end{aligned}$$

We view $\mathcal{I}_{x,k}\left(\boldsymbol{\xi}'\right)$ as an oscillatory integral depending on the parameter $\boldsymbol{\xi}'$, with real phase $\Upsilon_{x,\boldsymbol{\xi}'}$. Using that $Z_{\boldsymbol{\nu}^{\varphi}}$ is non-degenerate, and that $\boldsymbol{\gamma}$ is small in norm, one obtains the following.

Lemma 6.5. *For any $\boldsymbol{\xi}' \in \mathfrak{t}_{\boldsymbol{\nu}}$, $\Upsilon_{x,\boldsymbol{\xi}'}$ has a unique critical point, given by*

$$P_0 = (u_0,s_0,\boldsymbol{\xi}_0'',\boldsymbol{\gamma}_0) = \left(\frac{1}{\varsigma(m_x)},0,\mathbf{0},\mathbf{0}\right).$$

Hence $\Upsilon_{x,\boldsymbol{\xi}'}(P_0)=0$. The Hessian matrix at the critical point is

$$H_{P_0}(\Upsilon_{x,\boldsymbol{\xi}'}) = \begin{pmatrix} 0 & \varsigma(m_x)\,\|\boldsymbol{\nu}^{\varphi}\|_{\varphi} & \mathbf{0}^t & \mathbf{0}^t\\ \varsigma(m_x)\,\|\boldsymbol{\nu}^{\varphi}\|_{\varphi} & 0 & \mathbf{0}^t & \mathbf{0}^t\\ \mathbf{0} & \mathbf{0} & [0] & Z_{\boldsymbol{\nu}^{\varphi}}\\ \mathbf{0} & \mathbf{0} & -Z_{\boldsymbol{\nu}^{\varphi}} & \partial^2_{\boldsymbol{\gamma},\boldsymbol{\gamma}}\Upsilon_{x,\boldsymbol{\xi}'}\big|_{P_0}\end{pmatrix},$$

where $[0]$ *denotes the zero matrix of order* $(2\,n_G) \times (2\,n_G)$*. Hence, its determinant and signature are*

$$\det\big(H_{P_0}(\Upsilon_{x,\boldsymbol{\xi}'})\big) = -\varsigma(m_x)^2\ \|\boldsymbol{\nu}^{\varphi}\|_{\varphi}^2\ \det(Z_{\boldsymbol{\nu}^{\varphi}})^2, \quad \mathrm{sign}\big(H_{P_0}(\Upsilon_{x,\boldsymbol{\xi}'})\big) = 0.$$

In particular, P_0 *is a non-degenerate critical point.*

Integrating by parts in $\mathrm{d}\boldsymbol{\gamma}$ shows that the asymptotics of (80) are unchanged, if the integrand is multiplied by a cut-off function in $\boldsymbol{\xi}''$, compactly supported and identically equal to 1 near the origin.

We can apply the stationary phase Lemma. Recalling (67)-(70), we obtain for (80) an asymptotic expansion of the form

$$\begin{aligned}\mathcal{I}_{x,k}\left(\boldsymbol{\xi}'; \mathbf{v}_j, \mathbf{w}_j\right) \ \sim\ & \gamma\left(k^{-\epsilon}\,\boldsymbol{\xi}'\right)\cdot\left(\frac{2\,\pi}{k^{1/2}}\right)^{1+d_G-r_G}\cdot\frac{e^{\frac{1}{\varsigma(m_x)}\left[\psi_2(\mathbf{w}_1,\mathbf{w}_2)-\frac{1}{2}\|\mathbf{v}_1-\mathbf{v}_2\|^2_{m_x}\right]}}{\varsigma(m_x)\ \|\boldsymbol{\nu}^{\varphi}\|_{\varphi}\ \det(Z_{\boldsymbol{\nu}^{\varphi}})}\\ & \cdot\frac{k^d}{\varsigma(m_x)^d\,\pi^d}\,e^{\frac{1}{\varsigma(m_x)}\left[\imath\,\omega_{m_x}\left(\boldsymbol{\xi}'_M(m),\mathbf{v}_1+\mathbf{v}_2\right)-\frac{1}{2}\|\boldsymbol{\xi}'_M(m_x)\|^2_{m_x}\right]}\\ & \cdot\left[1+\sum_{j\geq 1}k^{-j/2}\,P'_j(m_x;\boldsymbol{\xi}',\mathbf{v}_j,\mathbf{w}_j)\right],\end{aligned} \tag{81}$$

where again $P_j(m_x;\cdot,\cdot,\cdot)$ is a polynomial of degree $\leq 3\,j$ and parity j. We have replaced $\|\boldsymbol{\xi}'_X(x)\|_x$ by $\|\boldsymbol{\xi}'_M(m_x)\|_{m_x}$ in view of the fact that $\langle\Phi_M(m_x),\boldsymbol{\xi}'\rangle = 0$ since $\boldsymbol{\xi}' \in \mathfrak{t}^{\perp_{\varphi}}$, so that $\boldsymbol{\xi}'_X(x) = \boldsymbol{\xi}'_M(m_x)^{\sharp}$.

The final expansion is obtained by inserting (81) in (79) and integrating term by term. The front cut-off, in view of the Gaussian type exponential, may be omitted without affecting the asymptotics. The j-th summand in (81), $j \geq 0$, contributes by a factor given by the Gaussian type integral

$$k^{-j/2}\int_{\mathfrak{t}_{\boldsymbol{\nu}}}\mathrm{d}^{\varphi}\boldsymbol{\xi}'\ \left[e^{\frac{1}{\varsigma(m_x)}\left[\imath\,\omega_{m_x}\left(\boldsymbol{\xi}'_M(m_x),\mathbf{v}_1+\mathbf{v}_2\right)-\frac{1}{2}\left\|\boldsymbol{\xi}'_M(m_x)\right\|^2_{m_x}\right]}\ P'_j(m_x;\boldsymbol{\xi}',\mathbf{v}_j,\mathbf{w}_j)\right], \tag{82}$$

where we set $P_0 = 1$.

We compute the leading order term. Recall that we have fixed an orthonormal basis of $\mathfrak{t}'_m = \mathfrak{t}_{\boldsymbol{\nu}} \cong \mathbb{R}^{r_G-1}$ (47). Let $D^{\varphi}(m_x)$ be as in Definition 1.2, and let $P^{\varphi}(m_x)$ denote its positive definite square root. Furthermore, by Lemma 6.1 there exist unique $\boldsymbol{\upsilon}_j \in \mathfrak{t}_{\boldsymbol{\nu}}$ such that $\mathbf{v}_j = J_{m_x}\big(\boldsymbol{\upsilon_j}_M(m_x)\big)$. Let

$\langle\cdot,\cdot\rangle_{st}$ denote the standard scalar product on $\mathbb{R}^{r_G-1}$, then

$$\begin{aligned}
&\varsigma(m_x)^{-1}\,\omega_{m_x}\big(\boldsymbol{\xi}'_M(m),\mathbf{v}_1+\mathbf{v}_2\big) \qquad (83)\\
&= \varsigma(m_x)^{-1}\,\omega_{m_x}\Big(\boldsymbol{\xi}'_M(m_x),J_{m_x}\big(\boldsymbol{\upsilon_1}_M(m_x)\big)+J_{m_x}\big(\boldsymbol{\upsilon_2}_M(m_x)\big)\Big)\\
&= \varsigma(m_x)^{-1}\,\rho^M_{m_x}\Big(\boldsymbol{\xi}'_M(m_x),\boldsymbol{\upsilon}_{1M}(m_x)+\boldsymbol{\upsilon}_{2M}(m_x)\Big)\\
&= \varsigma(m_x)^{-1}\,\boldsymbol{\xi}'^T\,D^\varphi(m_x)\,(\boldsymbol{\upsilon}_1+\boldsymbol{\upsilon}_2)\\
&= \Big\langle\varsigma(m_x)^{-1/2}\,P^\varphi(m_x)\,\boldsymbol{\xi}',\varsigma(m_x)^{-1/2}\,P^\varphi(m_x)\,(\boldsymbol{\upsilon}_1+\boldsymbol{\upsilon}_2)\Big\rangle_{st}.
\end{aligned}$$

Similarly, if $\|\cdot\|$ is the standard Euclidean norm then

$$\varsigma(m_x)^{-1}\,\|\boldsymbol{\xi}'_M(m_x)\|^2_{m_x}=\left\|\varsigma(m_x)^{-1/2}\,P^\varphi(m_x)\,\boldsymbol{\xi}'\right\|^2. \qquad (84)$$

Hence, setting $\boldsymbol{\eta}=\varsigma(m_x)^{-1/2}\,P^\varphi(m_x)\,\boldsymbol{\xi}'$, we obtain

$$\begin{aligned}
&\int_{\mathfrak{t}_\nu} \mathrm{d}^\varphi\boldsymbol{\xi}'\left[e^{\frac{1}{\varsigma(m_x)}\left[\imath\,\omega_{m_x}\left(\boldsymbol{\xi}'_M(m_x),\mathbf{v}_1+\mathbf{v}_2\right)-\frac{1}{2}\left\|\boldsymbol{\xi}'_M(m_x)\right\|^2_{m_x}\right]}\right]\\
&= \frac{\varsigma(m_x)^{\frac{r_G-1}{2}}}{\det\left(P^\varphi(m_x)\right)}\int_{\mathbb{R}^{r_G-1}}\mathrm{d}\boldsymbol{\eta}\left[e^{\imath\left\langle\boldsymbol{\eta},\varsigma(m_x)^{-1/2}\,P^\varphi(m_x)\,(\boldsymbol{\upsilon}_1+\boldsymbol{\upsilon}_2)\right\rangle_{st}-\frac{1}{2}\|\boldsymbol{\eta}\|^2}\right]\\
&= \frac{(2\,\pi)^{\frac{r_G-1}{2}}\,\varsigma(m_x)^{\frac{r_G-1}{2}}}{\mathcal{D}^\varphi(m)}\,e^{-\frac{1}{2\,\varsigma(m_x)}\,\|\mathbf{v}_1+\mathbf{v}_2\|^2_m}. \qquad (85)
\end{aligned}$$

Plugging (85) into (81) and then in (79) we obtain the leading order term in the statement of the Theorem. The other terms can be handled similarly.

□

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