Braess' paradox: A cooperative game-theoretic point of view*

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Abstract

Braess' paradox is a classical result in the theory of congestion games. It motivates theoretically why adding a resource (e.g., an arc) to a network may sometimes worsen, rather than improve, the overall network performance. Differently from previous literature, which studies Braess' paradox in a non-cooperative game-theoretic setting, in this work a framework is proposed to investigate its occurrence by exploiting cooperative games with transferable utility (TU games) on networks. In this way, instead of focusing on the marginal contribution to the network utility provided by the insertion of an arc when a single initial scenario is considered, the arc average marginal utility with respect to various initial scenarios, i.e., its Shapley value in a suitably-defined TU game, is evaluated. It is shown that, for choices of the utility function of the TU game modeling congestion, there are cases for which the Shapley value associated with an arc is negative, meaning that its average marginal contribution to the network utility is negative.

Keywords: transportation networks, TU games, Braess' paradox, traffic assignment, user equilibrium, system optimum.

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1 Introduction

Braess' paradox [7, 8][1] highlights why adding one resource to a network may in some cases worsen, rather than improve, the overall network performance (similarly, its removal may improve such performance). Among others, mathematical studies about Braess' paradox, which is a classical result in the theory of congestion games [25], regarded issues such as establishing necessary and sufficient conditions for its occurrence [43], investigating its probability of occurrence in random networks [47], determining its dependence on demand [28], and establishing upper bounds on the loss in performance due to the insertion of a link [39]. For any positive integer n, a special graph with n vertices (known as n-th Braess' graph [38]) exists, for which one can prove that the phenomenon (stated in the form of a performance increase after link removal) occurs.

Although originally (and still typically) investigated in the context of transportation networks, Braess' paradox is relevant also in several other contexts (see the recent review [30]), such as in telecommunications networks and Internet [31, 39], mechanical and electrical networks [12], metabolic networks [27], and sports analytics (in the latter, the paradox corresponds to an increase in team performance after removing one of its players [41]). The occurrence of the phenomenon in real transportation networks was observed several times (especially after link removal) and announced in the press. Well-known cases were reported, e.g., in New York and Seoul [30]. A heuristic methodology to detect, in real road networks, connections that are associated with the occurrence of Braess' paradox was presented in [3].

The paradox is typically studied through non-cooperative game theory, via the two so-called *Wardrop's principles* [48], later coined in [15], respectively, as *user optimality* and *system optimality*. An even earlier qualitative formulation of such principles was given by Pigou in [37, p. 194]. According to the historical account on Braess' paradox reported in [29], the first rigorous mathematical formulation of the conditions described by Wardrop's principles was provided in [6], more than one decade before the formulation of Braess' paradox. The first algorithms for the computation of the traffic pattern in a general network, according to each of the two principles proposed by Wardrop, were provided in [15], quite simultaneously with the formulation of Braess' paradox. The paradox is related to the concept of *price of anarchy* [18]: the players (e.g., the users of a road network), being driven by the pursuit of maximizing their own individual interests, tend to reduce social welfare. Sometimes, as an undesired consequence, they even fail to maximize such interests, when compared to the case in which they behave in a more collaborative way. When several resources are added at the same time to a network, however, such a non-cooperative approach, which does not take into account every potential interaction among the resources in all the possible contingent situations, has a limitation. It does not allow one to quantify the average marginal contribution (be it positive or negative) of each resource to the overall network performance. This suggests to investigate the phenomenon in a cooperative setting.

This work aims to study the occurrence of Braess' paradox exploiting an approach based on *cooperative games with transferable utility* (*TU games*) on graphs. For transportation networks such games were defined in [22]. The players can be either network nodes or arcs. In the approach proposed in this paper, the characteristic function of the game is defined in terms of an appropriate congestion measure over subgraphs associated with subsets of arcs (the case of nodes instead of arcs can be considered, too, with no substantial change). The measure is computed by solving an instance of the classical user equilibrium problem. Then, the *Shapley value* (a well-known cooperative game-theoretic power index [1, 2]) of an arc (or node) is used as a measure of its importance. It was shown in [21, 22, 24] that a Shapley value-based measure of importance has several advantages with respect to other classical such measures (e.g., degree, closeness, betweenness, and eigenvalue centrality, as measures of node importance), since the latter are not typically able to take into account simultaneously several features of the network under study (instead, for the Shapley value-based measure, such features are embedded in the characteristic function of the TU game). Moreover, instead of focusing only on a specific coalitional scenario (e.g., by evaluating the importance of a resource with regards to the whole network), the Shapley value-based approach determines an average with respect to several coalitional scenarios (e.g., by considering also different cases for which distinct subsets of resources are not functioning, hence they are removed from the network).

[1] The work [7] was originally published in German. Its English translation is available in [8].

The contribution of this work is threefold:

1. Differently from previous literature on Braess' paradox, instead of focusing on the marginal contribution to the network utility provided by the insertion of an arc (or node) when a single initial scenario is considered, its average marginal utility with respect to various initial scenarios (i.e., its Shapley value in a suitable TU game) is evaluated;

2. In contrast to [22, 24], the goal consists in identifying situations in which the Shapley value of an arc/node is negative, hence its insertion into the network has a negative average marginal value. This indicates a corresponding degradation of the average network performance, therefore the unsuitability of such an inclusion;

3. Using as a test bed Braess' original example [7], the one in which the paradox was first discovered, the appearance of a negative Shapley value (as a cooperative version of Braess' paradox) is shown for the arc of the network associated with Braess' paradox (i.e., the one whose inclusion in the network causes a degradation of network performance).

The paper is organized as follows. Section 2 provides a background on cooperative games with transferable utility (TU games) and transportation network cooperative games. Section 3 defines the two utility functions that are adopted in the work, based, respectively, on the user equilibrium and the system optimum. Section 4 reports extensive numerical results showing the occurrence, in transportation network TU games, of a negative Shapley value as a cooperative version of Braess' paradox. Section 5 contains some conclusions and pointers to extensions.

2 Preliminaries

2.1 Cooperative games with transferable utility

A *cooperative game with transferable utility*, *TU game* for short, is a pair (N, v) defined as follows: N is the set of players, which is called the *grand coalition*; each subset $S \subseteq N$ is a *(sub)coalition*. The real-valued mapping $v : 2^N \to \mathbb{R}$ (where 2^N is the power set of N, i.e., the set of all its subsets), such that $v(\emptyset) = 0$, is called *utility function* (also called *characteristic function*). It assigns to each coalition S the value $v(S)$, which represents the utility that can be achieved jointly by the players in S, without any contribution from the players in $N \setminus S$. The quantity $v(N)$ is the utility of the grand coalition and, for each player i, $v(\{i\})$ is the utility of the player i without entering any coalition with more than one player.

In TU games, the utilities can be transferred from one player to another one without any loss (e.g., by means of a common "currency", which is valued equally by all the players). TU games can be studied by using suitable *solution concepts*. Each of them is a criterion for dividing the total utility $v(N)$ of the game among individual players. Examples of solution concepts are the core, the nucleolus, and the power indices, such as the Banzhaf power index and the Shapley value (see, e.g., [36]). Each power index corresponds to a way of allocating the total utility in a "fair way" among the players. The larger the amount of utility allocated to a player by a power index, the greater the importance of the player in the grand coalition. In this work, we use the *Shapley value*, defined as

$$Sh(i) = \sum_{S \subseteq N} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} [v(S) - v(S \setminus \{i\})], \qquad \forall\, i \in N, \tag{1}$$

where $v(S) - v(S \setminus \{i\})$ is the marginal contribution[2] of player i when one moves from the coalition $S \setminus \{i\}$ to S. Hence, the Shapley value $Sh(i)$ represents the average marginal contribution of each player across

[2] In this work, the marginal contribution of each player i is defined as $v(S) - v(S \setminus \{i\})$, i.e., with respect to a coalition S from which that player is removed (if initially present, otherwise the two coalitions S and $S \setminus \{i\}$ are identical); equivalently, one could have defined it as $v(S \cup \{i\}) - v(S)$, i.e., with respect to a coalition S to which it is added (if initially absent, otherwise the two coalitions $S \cup \{i\}$ and S are identical).

all possible coalitions, when players, starting from the empty coalition (i.e., the one with no players), enter the grand coalition randomly (in such a way that all orders are equally likely). Such an interpretation of the Shapley value allows to apply this concept as a measure of player's importance even in contexts where the players of the game cannot be modeled as rational decision makers. This is the case, e.g., of features in supervised machine learning problems [13] and genes in microarray games [26].

Moreover, we recall here the definitions of some properties of TU games that will be investigated in the rest of the paper, for two specific TU games proposed in the work (see, e.g., [10, 36] for additional discussions about such properties). The game (N, v) is

- subadditive if $v(S \cup T) \leq v(S) + v(T)$ holds for any pair of disjoint coalitions $S, T \subseteq N$;
- superadditive if $v(S \cup T) \geq v(S) + v(T)$ holds for any pair of disjoint coalitions $S, T \subseteq N$;
- monotonic if $v(S) \leq v(T)$ holds for any $S, T \subseteq N$ such that $S \subset T$;
- convex if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ holds for any $S, T \subseteq N$;
- cohesive if $v(N) \geq \sum_{i=1}^{k} v(S_i)$ holds for any partition $\{S_1, \ldots, S_k\}$ of N.

Notice that any convex game is superadditive and any superadditive game is cohesive.

2.2 Transportation network cooperative games

Consider a directed graph $G = (V, A)$, where V is the set of nodes, $A \subseteq V \times V$ is the set of arcs and W is the set of Origin-Destination pairs. In our game theoretic approach, we consider the set N of players to be a subset of (critical) arcs (i.e., arcs that are the focus of the analysis). The graph associated to a generic coalition $S \subseteq N$ is denoted by

$$G(S) := (V,\ S \cup (A \setminus N)), \tag{2}$$

that is, it contains the arcs of coalition S and the arcs that are not players of the game (N, v). Nodes can represent intersections, points of interest, transportation centroids, stops, stations, critical transportation components, etc. Arcs can represent physical connections between pairs of nodes, i.e., roads, rails, parts of routes or entire routes, etc. The utility function can be exploited to represent a wide range of transportation attributes, such as connectivity, distance, travel time saving, assigned demand, etc. The *transportation network* (*TN*) can also be redefined in such a way that its nodes represent the arcs of the original network, i.e., the physical transportation segments (road, rail, etc.).

In [22], a *transportation network cooperative game* (*TNc game*) was defined by choosing the utility function of a TU game on the basis of the TN topology and some network attributes. Then, the vector of the Shapley values of the players, represented therein by the network nodes, was used to evaluate the node centrality. Theoretical properties of the resulting centrality measure were investigated in [20, 22]. Unfortunately, the utility functions defined in [22] are not suitable to model congestion, as they refer to situations in which the cost associated with each arc is given exogenously. Here, instead, we are interested in taking congestion into account and considering the arcs as players. At the cost of some computational burden, the utility of every coalition can be defined in terms of the equilibrium flows of an associated traffic assignment problem [35].

Since in the next sections we will define TNc games where some players may have negative marginal contributions, we will distinguish between the positive part of the Shapley value of player i, defined as

$$Sh^{+}(i) := \sum_{S \subseteq N} \frac{(|S|-1)!\ (|N|-|S|)!}{|N|!} \max\{0, v(S) - v(S \setminus \{i\})\}, \tag{3}$$

and the negative part of the Shapley value of i, defined as

$$Sh^{-}(i) := \sum_{S \subseteq N} \frac{(|S|-1)!\ (|N|-|S|)!}{|N|!} \min\{0, v(S) - v(S \setminus \{i\})\}. \tag{4}$$

Hence, the Shapley value of each player i is equal to the sum of its positive and negative parts, i.e.,

$$Sh(i) = Sh^{+}(i) + Sh^{-}(i). \tag{5}$$

3 Two TNc games based on equilibrium flows

We introduce the following notations and definitions for equilibrium flows. We denote by d_w the *demand* on an *O/D pair* w and by d the *vector of the demands*. A *path* is represented by p, whereas x_p is the nonnegative *flow on the path* p in G, and x is the *vector of path flows*. Similarly, $x_p(S)$ is the flow on the path p in $G(S)$, and $x(S)$ is the vector of the path flows in $G(S)$. We represent by f_a the nonnegative *flow on the arc* a, and by f the *vector of arc flows*. P_w denotes the set of *acyclic paths* joining the O/D pair w, whereas $P_w(S)$ is the set of paths joining the O/D pair w in $G(S)$, and P (respectively, $P(S)$) is the set of all paths joining the O/D pairs in G (respectively, $G(S)$). We denote by $c_a(f)$ the *cost* on the arc a associated with the flow vector f. This cost can be represented, e.g., by travel time or "pollution" (see Sections 3.1 and 3.2 for more details on this issue). Δ is the *incidence arc-path matrix*, whose elements are defined as $\Delta_{a,p} := 1$ if arc a belongs to path p and 0 otherwise. Finally, $C_p(x)$ (respectively, $C_p(x(S))$ represents the *user cost* on path p in G (respectively, $G(S)$), which is the sum of the costs on the arcs of such path, whereas $C(x)$ (respectively, $C(x(S))$ is the vector of the costs on the paths in G (respectively, $G(S)$). The vector of arc flows f and the vector of path flows x are linked by the following relationship: $f_a = \sum_{p \in P} \Delta_{a,p} x_p$. A path flow x is *feasible* if it satisfies the demands, i.e., $\sum_{p \in P_w} x_p = d_w$ for any O/D pair w.

3.1 TNc game based on the user equilibrium

The work hypothesis that we make to define a utility function based on the user equilibrium is the following: the drivers have perfect knowledge of the travel costs on the network and choose the best route according to the so-called *Wardrop's first principle* [48], i.e., "no driver can unilaterally reduce his/her travel cost by shifting to another route" (see also Nash equilibrium [33]). This provides a deterministic user equilibrium.

We define $\mathcal{F}$ as the set of coalitions $S \subseteq N$ such that for each O/D pair $w \in W$ there exists at least one path in $G(S)$ connecting w. For any coalition $S \in \mathcal{F}$, a feasible path flow vector $\bar{x}(S)$ is called a *Wardrop equilibrium* in the network $G(S)$ if for every O/D pair $w \in W$ and every $p \in P_w(S)$ one has

$$C_p(\bar{x}(S)) \begin{cases} = \lambda_w(S) & \text{if } \bar{x}_p(S) > 0, \\ \geq \lambda_w(S) & \text{if } \bar{x}_p(S) = 0, \end{cases} \tag{6}$$

where $\lambda_w(S)$ is the "equilibrium disutility" for the O/D pair w. Finding a Wardrop equilibrium $\bar{x}(S)$ and the corresponding disutility λ amounts to solving the following *variational inequality* [14, 42]:

$$\langle C(\bar{x}(S)), x(S) - \bar{x}(S) \rangle \geq 0, \qquad \text{for any feasible path flow } x(S). \tag{7}$$

The total cost (or total travel time) associated to the Wardrop equilibrium $\bar{x}(S)$ is defined as

$$TC^{ue}(S) = \sum_{w \in W} \sum_{p \in P_w(S)} \bar{x}_p(S) C_p(\bar{x}(S)). \tag{8}$$

Definition 3.1. *We define the utility function $v^{ue} : 2^N \to \mathbb{R}$ by considering two cases:*

(a) If the empty coalition $\emptyset \in \mathcal{F}$ (i.e., all coalitions belong to $\mathcal{F}$), then

$$v^{ue}(S) := TC^{ue}(\emptyset) - TC^{ue}(S), \qquad \forall\, S \subseteq N. \tag{9}$$

(b) If the empty coalition $\emptyset \notin \mathcal{F}$ (i.e., some coalitions belong to $\mathcal{F}$ and others do not), then

$$v^{ue}(S) := \begin{cases} \left[\max\limits_{S' \in \mathcal{F}_m} TC^{ue}(S') \right] - TC^{ue}(S) & \text{if } S \in \mathcal{F}, \\ 0 & \text{if } S \notin \mathcal{F}, \end{cases} \qquad \forall\, S \subseteq N, \tag{10}$$

where $\mathcal{F}_m \subseteq \mathcal{F}$ is the subset of minimally connected *coalitions, i.e., the ones for which the removal of any player makes the resulting coalition not belong to $\mathcal{F}$.*

The interpretation of the utility function v^{ue} is the following. In case (a), the utility of S is the cost saving with respect to the empty coalition. In case (b), when the whole demand is served, the utility of S is the cost saving with respect to the maximum cost of a minimally connected coalition; otherwise (i.e., when not the whole demand is served), it equals 0. Subtracting from the maximum cost the costs associated with the coalitions is required in order to transform disutilities (costs) into utilities (savings).

We now analyze some basic properties of the TNc game (N, v^{ue}):

1. In general, the game (N, v^{ue}) is not monotonic as the following counterexample shows.

   Example 3.2. *Consider the TNc game (N, v^{ue}) on the network introduced by Braess in [7], in which the paradox was first discovered (see Figure 1). Let $V = \{1,2,3,4\}$, $N = A = \{q,r,s,t,u\}$ and $W = \{(1,4)\}$ with traffic demand $d = 6$. The arc cost functions (as functions of their respective arc*

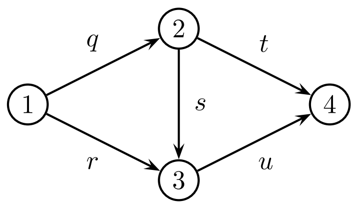


Figure 1: Braess network.

   flows) are defined as follows:

$$\begin{aligned} c_q(f) &= 10f_q, \\ c_r(f) &= f_r + 50, \\ c_s(f) &= f_s + 10, \\ c_t(f) &= f_t + 50, \\ c_u(f) &= 10f_u. \end{aligned} \tag{11}$$

   There are three paths connecting the O/D pair: $p_1 = (q,t)$, $p_2 = (r,u)$, $p_3 = (q,s,u)$. Hence, the path cost functions are as follows:

$$\begin{aligned} C_1(x) &= 11x_1 + 10x_3 + 50, \\ C_2(x) &= 11x_2 + 10x_3 + 50, \\ C_3(x) &= 10x_1 + 10x_2 + 21x_3 + 10, \end{aligned} \tag{12}$$

   where $x = (x_1, x_2, x_3)$ denotes the path flow vector.

   Since the empty coalition $\emptyset \notin \mathcal{F}$ (see Definition 3.1b), we need to analyze the set $\mathcal{F}_m$ of minimally connected *coalitions, i.e., $\mathcal{F}_m = \{\{q,t\},\{r,u\},\{q,s,u\}\}$, to compute the utility of any coalition. The total costs of minimally connected coalitions are as follows:*

$$\begin{array}{lll} S_1 = \{q,t\} & \bar{x}(S_1) = (6,0,0) & TC^{ue}(S_1) = 6 \cdot 116 = 696, \\ S_2 = \{r,u\} & \bar{x}(S_2) = (0,6,0) & TC^{ue}(S_2) = 6 \cdot 116 = 696, \\ S_3 = \{q,s,u\} & \bar{x}(S_3) = (0,0,6) & TC^{ue}(S_3) = 6 \cdot 136 = 816, \end{array}$$

   hence $\max_{S' \in \mathcal{F}_m} TC^{ue}(S') = 816$. Let now consider the coalitions $S = \{q,r,t,u\}$, $T = \{q,r,s,t,u\}$ and compute their utilities (see (10)):

$$\begin{array}{llll} S = \{q,r,t,u\} & \bar{x}(S) = (3,3,0) & TC^{ue}(S) = 498 & v^{ue}(S) = 816 - 498 = 318, \\ T = \{q,r,s,t,u\} & \bar{x}(T) = (2,2,2) & TC^{ue}(T) = 552 & v^{ue}(T) = 816 - 552 = 264. \end{array}$$

Since $S \subset T$ and $v^{ue}(S) > v^{ue}(T)$, the game (N, v^{ue}) is not monotonic.

2. In general, the game (N, v^{ue}) is not cohesive. In fact, if we consider Example 3.2 with the partition $\{S_1, S_2\}$ of N, where $S_1 = \{q, r, t, u\}$ and $S_2 = \{s\}$, then

$$v^{ue}(N) = 264 < 318 = v^{ue}(S_1) + v^{ue}(S_2).$$

3. In general, the game (N, v^{ue}) is not subadditive. In fact, if we consider Example 3.2 with the disjoint coalitions $S = \{q\}$ and $T = \{t\}$, then $v^{ue}(S) = v^{ue}(T) = 0$ and

$$v^{ue}(S \cup T) = 120 > 0 = v^{ue}(S) + v^{ue}(T).$$

4. In general, the game (N, v^{ue}) is not superadditive, because it is not cohesive.

5. In general, the game (N, v^{ue}) is not convex, because it is not superadditive.

3.2 TNc game based on the system optimum

To introduce a utility function based on the system optimum, we make the following hypothesis: the drivers behave according to the so-called *Wardrop's second principle* [48], i.e., "drivers cooperate with one another in order to minimize the total system travel time". We can think of this as a situation in which congestion is minimized by instructing the drivers about which routes to use, and supposing that they follow the suggestion. Although this is not realistic from the point of view of the behavior of drivers, it may be useful in transportation network planning to design networks that try to achieve a "social optimum" (e.g., minimum pollution, or cost).

For any coalition $S \in \mathcal{F}$, the optimum flow in the above sense amounts to solving the following nonlinear programming problem [35, Section 2.4]:

$$\begin{cases} \min \displaystyle\sum_{a \in S \cup (A \setminus N)} f_a \, c_a(f) \\ \displaystyle\sum_{p \in P_w(S)} x_p(S) = d_w & \forall\, w \in W, \\ f_a = \displaystyle\sum_{p \in P(S)} \Delta_{a,p} x_p(S) & \forall\, a \in S \cup (A \setminus N), \\ x(S) \geq 0. \end{cases} \tag{13}$$

Let us denote by $\hat{x}(S)$ the *system optimum* path flow, i.e., the optimal solution of the above nonlinear program, and by $\hat{f}(S)$ the corresponding system optimum arc flow, which can be expressed in terms of the matrix Δ as $\hat{f}(S) = \Delta \hat{x}(S)$.

The total cost (or total travel time) associated to the system optimum $\hat{x}(S)$ is defined as

$$TC^{so}(S) = \sum_{w \in W} \sum_{p \in P_w(S)} \hat{x}_p(S) C_p(\hat{x}(S)). \tag{14}$$

Definition 3.3. *We define the utility function $v^{so} : 2^N \to \mathbb{R}$ by considering two cases:*

(a) If the empty coalition $\emptyset \in \mathcal{F}$ (i.e., all coalitions belong to $\mathcal{F}$), then

$$v^{so}(S) := TC^{so}(\emptyset) - TC^{so}(S), \qquad \forall\, S \subseteq N. \tag{15}$$

(b) If the empty coalition $\emptyset \notin \mathcal{F}$ (i.e., some coalitions belong to $\mathcal{F}$ and others do not), then

$$v^{so}(S) := \begin{cases} \left[\displaystyle\max_{S' \in \mathcal{F}_m} TC^{so}(S') \right] - TC^{so}(S) & \text{if } S \in \mathcal{F}, \\ 0 & \text{if } S \notin \mathcal{F}, \end{cases} \qquad \forall\, S \subseteq N. \tag{16}$$

We now analyze some basic properties of the TNc game (N, v^{so}):

1. **The game** (N, v^{so}) **is monotonic**, as the following proof shows.

   Proof. Consider any two coalitions $S, T \subseteq N$, such that $S \subseteq T$, and distinguish among the following three cases:

   a) If $S, T \notin \mathcal{F}$, then $v^{so}(S) = v^{so}(T) = 0$;

   b) If $S, T \in \mathcal{F}$, then $TC^{so}(S) \geq TC^{so}(T)$, hence $v^{so}(S) \leq v^{so}(T)$;

   c) If $S \notin \mathcal{F}$ and $T \in \mathcal{F}$, then there exists a minimally connected coalition $U \subseteq T$. Hence, we have

   $$v^{so}(T) = \left[\max_{S' \in \mathcal{F}_m} TC^{so}(S')\right] - TC^{so}(T) \geq TC^{so}(U) - TC^{so}(T) \geq 0 = v^{so}(S), \tag{17}$$

   where the first equality follows from (16) *and the last inequality follows from the inclusion $U \subseteq T$.* ☐

   Since the game (N, v^{so}) is monotonic, Braess' paradox and negative Shapley values of arcs cannot occur.

2. In general, the game (N, v^{so}) is not cohesive, as the following counterexample shows.

   Example 3.4. *Consider the TNc game (N, v^{so}) on the network described in Example 3.2 with the same arc and path cost functions, while the traffic demand $d = 10$.*

   The total costs of minimally connected coalitions are as follows:

   $$\begin{array}{lll} S_1 = \{q, t\} & \hat{x}(S_1) = (10, 0, 0) & TC^{so}(S_1) = 1600, \\ S_2 = \{r, u\} & \hat{x}(S_2) = (0, 10, 0) & TC^{so}(S_2) = 1600, \\ S_3 = \{q, s, u\} & \hat{x}(S_3) = (0, 0, 10) & TC^{so}(S_3) = 2200, \end{array}$$

   hence $\max_{S' \in \mathcal{F}_m} TC^{so}(S') = 2200$. If we consider the partition $\{S_1, S_2\}$ of N, where $S_1 = \{q, s, t\}$ and $S_2 = \{r, u\}$, then we have:

   $$\begin{array}{lll} \hat{x}(S_1) = (10, 0, 0) & TC^{so}(S_1) = 1600 & v^{so}(S_1) = 600, \\ \hat{x}(S_2) = (0, 10, 0) & TC^{so}(S_2) = 1600 & v^{so}(S_2) = 600; \\ \hat{x}(N) = (5, 5, 0) & TC^{so}(N) = 1050 & v^{so}(N) = 1150. \end{array}$$

   Since

   $$v^{so}(N) = 1150 < 1200 = v^{so}(S_1) + v^{so}(S_2),$$

   the game (N, v^{so}) is not cohesive.

3. In general, the game (N, v^{so}) is not subadditive. In fact, if we consider Example 3.4 with the disjoint coalitions $S = \{q\}$ and $T = \{t\}$, then $v^{so}(S) = v^{so}(T) = 0$ and

   $$v^{so}(S \cup T) = 600 > 0 = v^{so}(S) + v^{so}(T).$$

4. In general, the game (N, v^{so}) is not superadditive, because it is not cohesive.

5. In general, the game (N, v^{so}) is not convex, because it is not superadditive.

4 Numerical results

In this section, we report some numerical results about the two proposed TNc games on the Braess' original network shown in Figure 1, where $V = \{1, 2, 3, 4\}$, $A = \{q, r, s, t, u\}$, $W = \{(1, 4)\}$, and the arc and path cost functions are defined as in (11)–(12). We analyze the two TNc games based on the user equilibrium and the system optimum, where the set N of players coincides with the set A of arcs. For any value of the traffic demand $d \geq 0$, we compute in closed form the user equilibrium and the system optimum, the utility of each coalition, the marginal contribution of each arc to each coalition, and the Shapley value of each arc. In particular, we show that in the TNc game (N, v^{ue}) the marginal contribution of arc s to the grand coalition is negative when $d \in (80/31, 80/9)$ (i.e., Braess' paradox occurs in that interval of demand), while the Shapley value of arc s (i.e., its average marginal contribution to all coalitions) is negative in a subinterval of $(80/31, 80/9)$, specifically for $d \in (400/133, 1624/197)$ (see Section 4.1 and in particular Figure 3 for a comparison of the two cases).

4.1 TNc game based on the user equilibrium

In order to find the utility of each coalition S, we need to distinguish among the four following cases for the traffic demand: $d \in [0, 40/11]$, $d \in [40/11, 4]$, $d \in [4, 80/9]$ and $d \geq 80/9$.

Tables 1–4 in the Appendix report, for each coalition $S \in \mathcal{F}$, the corresponding user equilibrium $\bar{x}(S)$, the total cost $TC^{ue}(S)$, the utility $v^{ue}(S)$, and the marginal contribution of each arc belonging to S. Minimally connected coalitions in $\mathcal{F}_m$, their maximum total cost and the negative marginal contributions of arc s are reported in frames. Tables 1–4 do not show coalitions $S \notin \mathcal{F}$, because their utility and the marginal contribution of their arcs are zero. Table 1 shows that, when $d \in (0, 40/11]$, arc s gives a positive marginal contribution to coalitions $\{q, r, s, u\}$, $\{q, s, t, u\}$ and $\{q, s, u\}$, since it allows satisfying the demand through path (q, s, u) having a positive utility (of course, for $d = 0$ one obtains always a zero utility). On the other hand, its marginal contribution to the grand coalition N is reported in a frame because it is positive if $d \in (0, 80/31)$, while it is negative if $d \in (80/31, 40/11]$, since the total cost corresponding to the user equilibrium $(0, 0, d)$ of the grand coalition N is larger than the total cost corresponding to the user equilibrium $(d/2, d/2, 0)$ of the coalition $N \setminus \{s\}$. Table 2 shows that, when $d \in [40/11, 4]$, arc s gives a negative marginal contribution to the grand coalition, while on the same interval it gives a positive marginal contribution to coalitions $\{q, r, s, u\}$, $\{q, s, t, u\}$, and on the interval $[40/11, 4)$ it provides a positive marginal contribution to the coalition $\{q, s, u\}$. Table 3 shows that, when $d \in [4, 80/9)$, arc s gives a negative marginal contribution to the grand coalition, while it gives a positive marginal contribution to coalitions $\{q, r, s, u\}$ and $\{q, s, t, u\}$, whereas the utility of the coalition $\{q, s, u\}$ is zero, as well as the marginal contribution of arc s to that coalition. Table 4 shows that, when $d \geq 80/9$, arc s gives a zero marginal contribution to the grand coalition and to the coalition $\{q, s, u\}$, while it gives a positive marginal contribution to coalitions $\{q, r, s, u\}$ and $\{q, s, t, u\}$.

It follows from Tables 1–4 that the Shapley values of the arcs are given by the following explicit formulas:

$$Sh(q) = Sh(u) = \begin{cases} d(1600 - 367d)/120 & \text{if } d \in [0, 40/11], \\ d(261d - 604)/156 & \text{if } d \in [40/11, 4], \\ d(703d - 2372)/156 & \text{if } d \in [4, 80/9], \\ d(233d - 580)/60 & \text{if } d \geq 80/9, \end{cases} \tag{18}$$

$$Sh(r) = Sh(t) = \begin{cases} 11d^2/40 & \text{if } d \in [0, 40/11], \\ 11d(47d - 152)/208 & \text{if } d \in [40/11, 4], \\ 3d(311d - 1112)/208 & \text{if } d \in [4, 80/9], \\ 3d(103d - 280)/80 & \text{if } d \geq 80/9, \end{cases} \tag{19}$$

$$Sh(s) = \begin{cases} d(400 - 133d)/30 & \text{if } d \in [0, 40/11], \\ d(93d - 1208)/312 & \text{if } d \in [40/11, 4], \\ d(197d - 1624)/312 & \text{if } d \in [4, 80/9], \\ d(d + 40)/120 & \text{if } d \geq 80/9. \end{cases} \tag{20}$$

We note that, due to the symmetry of the arc cost functions, arcs q and u have the same Shapley value for any demand. The same fact holds for arcs r and t.

Figure 2 shows the Shapley value of each arc as a function of the traffic demand. Specifically, the Shapley value of arc s is positive for $d \in (0, 400/133)$, negative for $d \in (400/133, 1624/197)$, with a minimum equal to $-82418/7683 \simeq -10.73$ achieved at $d = 812/197 \simeq 4.12$, and positive for $d > 1624/197$. The Shapley values of the other arcs are positive for any $d > 0$.

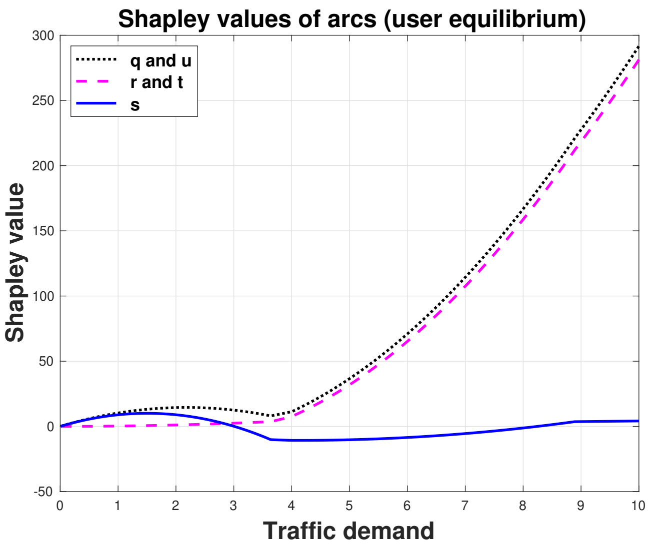


Figure 2: TNc game based on the user equilibrium: Shapley values of the arcs as functions of the traffic demand.

Figure 3 compares the marginal contribution of arc s to the grand coalition and the Shapley value (i.e., the average marginal contribution) of arc s. Notice that, as anticipated at the beginning of Section 4, Braess' paradox occurs (i.e., the marginal contribution of s to the grand coalition is negative) when $d \in (80/31, 80/9)$, while the Shapley value of s is negative in the subinterval $d \in (400/133, 1624/197) \subset (80/31, 80/9)$. This difference is due to the fact that arc s gives a positive marginal contribution to coalitions $\{q, r, s, u\}$, $\{q, s, t, u\}$ and a positive (or, in some cases, zero) marginal contribution to the coalition $\{q, s, u\}$.

Figures 4 and 5 give a graphical representation on the network of the absolute and normalized Shapley values of arcs, respectively. Six different values of the demand ($d = 1$, 3, 4, 6, 8 and 10) have been chosen. For

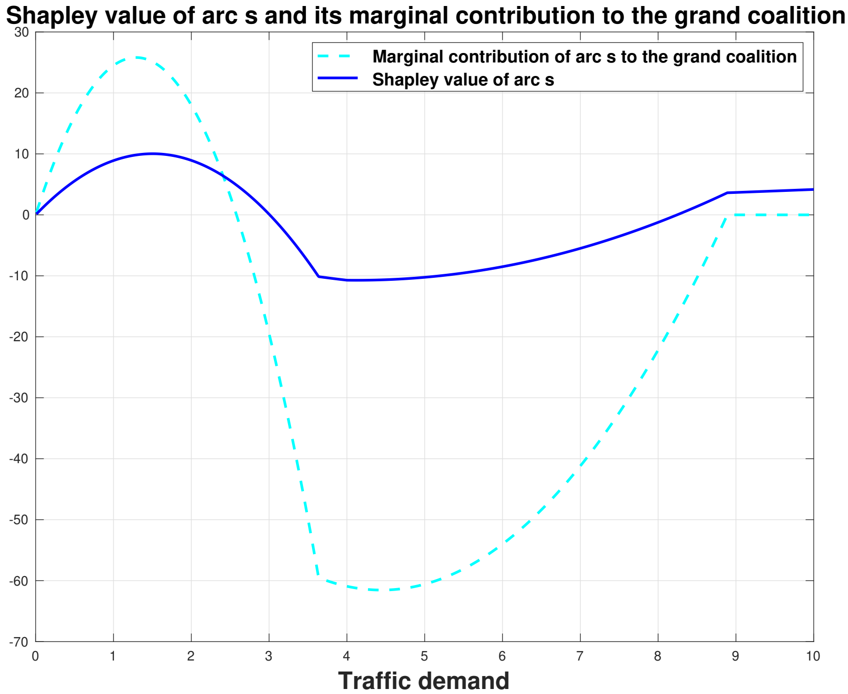


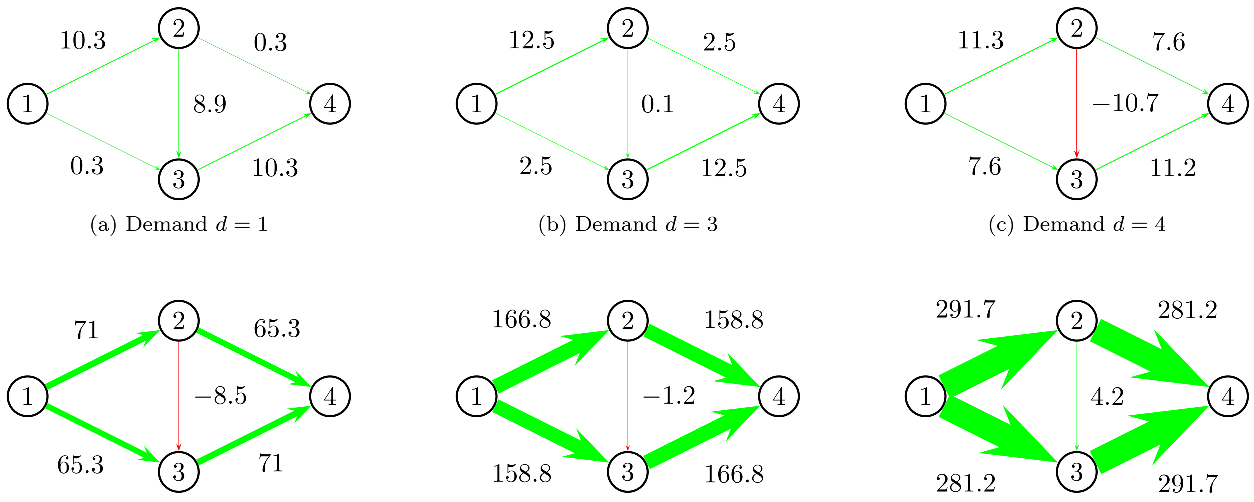
Figure 3: TNc game based on the user equilibrium: Shapley value of arc s and its marginal contribution to the grand coalition.

each of such values, each arc in the network is colored in green if its Shapley value is positive and in red if its Shapley value is negative. In Figure 4, the thickness of each arc i is proportional to its absolute Shapley value $|Sh(i)|$, while in Figure 5 it is proportional to its normalized Shapley value $|Sh(i)|/\sum_{j\in N}|Sh(j)|$. Figures 2 and 4 clearly show that, when $d \in [4, 10]$, the importance of arcs different from s grows significantly from about 10 to about 300, while the importance of arc s varies in the smaller range $[-10.7,\ 4.2]$. On the other hand, Figures 2, 4 and 5 highlight that, for any $d > 0$, arcs q and u are more important than arcs r and t, and in Figures 4 and 5 this gap is more evident for $d \in \{1, 3\}$. Moreover, in Figures 4 and 5, the importance of arc s is comparable to that of the other arcs for $d \in \{1, 3\}$, while for $d \in \{4, 6, 8, 10\}$ arc s is much less important than the other arcs. Notice that Braess' paradox occurs for $d \in (80/31, 80/9) \simeq (2.58, 8.89)$, hence for $d = 3$ the marginal contribution of s to the grand coalition is negative, but its Shapley value is positive (about 0.1).

As for the positive and negative parts of Shapley values (see (3)-(4)), we remark that, for any $d > 0$, $Sh(i) = Sh^+(i) > 0$ and $Sh^-(i) = 0$ hold for any arc $i \neq s$, while for arc s we have

$$Sh^+(s) = \begin{cases} \dfrac{d(400-133d)}{30} & \text{if } d \in \left[0, \dfrac{80}{31}\right], \\ \dfrac{4d(4-d)}{3} & \text{if } d \in \left[\dfrac{80}{31}, \dfrac{40}{11}\right], \\ \dfrac{d(200-39d)}{120} & \text{if } d \in \left[\dfrac{40}{11}, 4\right], \\ \dfrac{d(d+40)}{120} & \text{if } d \in \left[4, \dfrac{80}{9}\right], \\ \dfrac{d(d+40)}{120} & \text{if } d \geq 80/9, \end{cases} \qquad Sh^-(s) = \begin{cases} 0 & \text{if } d \in \left[0, \dfrac{80}{31}\right], \\ \dfrac{d(80-31d)}{10} & \text{if } d \in \left[\dfrac{80}{31}, \dfrac{40}{11}\right], \\ \dfrac{9d(9d-80)}{130} & \text{if } d \in \left[\dfrac{40}{11}, 4\right], \\ \dfrac{9d(9d-80)}{130} & \text{if } d \in \left[4, \dfrac{80}{9}\right], \\ 0 & \text{if } d \geq 80/9. \end{cases} \tag{21}$$

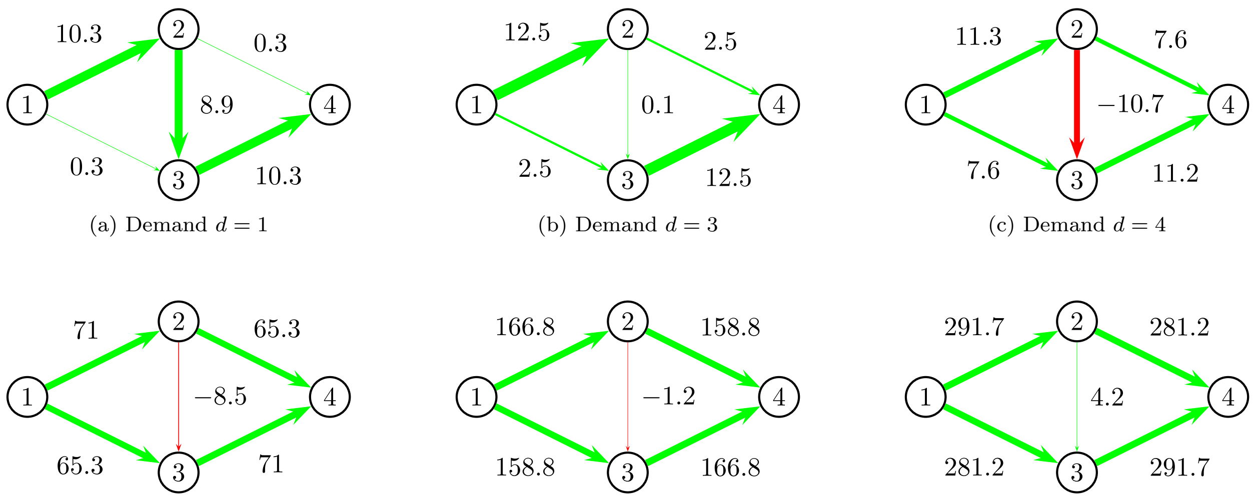
Figure 6 shows the positive and negative parts of the Shapley value of arc s as functions of the traffic demand.



(a) Demand $d = 1$ (b) Demand $d = 3$ (c) Demand $d = 4$

(d) Demand $d = 6$ (e) Demand $d = 8$ (f) Demand $d = 10$

Figure 4: TNc game based on the user equilibrium: graphical representation of the Shapley values of the arcs for six values of the traffic demand. Each arc i is colored in green if $Sh(i) > 0$ and in red if $Sh(i) < 0$. The thickness of arc i is proportional to $|Sh(i)|$. The Shapley value $Sh(i)$ is indicated on each arc i. Braess' paradox occurs for $d \in \{3, 4, 6, 8\}$. Notice that for $d = 3$ the marginal contribution of s to the grand coalition is negative, but its Shapley value is positive.



(a) Demand $d = 1$ (b) Demand $d = 3$ (c) Demand $d = 4$

(d) Demand $d = 6$ (e) Demand $d = 8$ (f) Demand $d = 10$

Figure 5: TNc game based on the user equilibrium: graphical representation of the Shapley values of the arcs for six values of the traffic demand. Each arc i is colored in green if $Sh(i) > 0$ and in red if $Sh(i) < 0$. For each network, the thickness of arc i is proportional to the ratio $|Sh(i)|/\sum_{j \in N} |Sh(j)|$. The Shapley value $Sh(i)$ is indicated on each arc i. Braess' paradox occurs for $d \in \{3, 4, 6, 8\}$. Notice that for $d = 3$ the marginal contribution of s to the grand coalition is negative, but its Shapley value is positive.

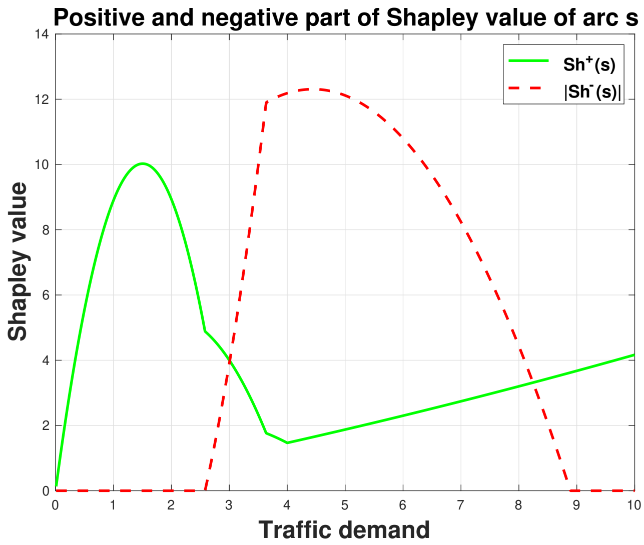


Figure 6: TNc game based on the user equilibrium: positive and negative parts of the Shapley value of arc s as functions of the traffic demand.

Notice that arc s does not give any negative marginal contribution if $d \in [0, 80/31]$; it gives both positive and negative marginal contributions if $d \in (80/31, 80/9)$, while it gives only positive marginal contributions if $d \geq 80/9$. In particular, there exist demand values such that the following three cases are verified:

1. $Sh^+(s) > 0 = Sh^-(s)$, e.g., for $d = 1$ we have $Sh^+(s) \simeq 8.9$ and $Sh^-(s) = 0$;
2. $Sh^+(s) > |Sh^-(s)| > 0$, e.g., for $d = 2.6$ we have $Sh^+(s) \simeq 4.9$ and $Sh^-(s) \simeq -0.2$;
3. $|Sh^-(s)| > Sh^+(s) > 0$, e.g., for $d = 4$ we have $Sh^+(s) \simeq 1.5$ and $Sh^-(s) \simeq -12.2$.

4.2 TNc game based on the system optimum

In order to analyze the system optimum, we need to distinguish among the four following cases: $d \in [0, 20/11]$, $d \in [20/11, 4]$, $d \in [4, 40/9]$ and $d \geq 40/9$. Tables 5–8 in the Appendix report, for each coalition $S \in \mathcal{F}$, the corresponding system optimum $\hat{x}(S)$, the total cost $TC^{so}(S)$, the utility $v^{so}(S)$, and the marginal contribution of each arc belonging to S. Tables 5–8 do not show coalitions $S \notin \mathcal{F}$ because their utility and the marginal contribution of their arcs are zero. Since the game (N, v^{so}) is monotonic[3] (see Section 3.2), arc s does not give any negative marginal contribution. Moreover, for $d \in (0, 4)$, it gives positive marginal contributions to the grand coalition and to coalitions $\{q, r, s, u\}$, $\{q, s, t, u\}$ and $\{q, s, u\}$, for $d \in [4, 40/9)$, it gives positive marginal contribution to the grand coalition and to $\{q, r, s, u\}$, $\{q, s, t, u\}$, while for $d \geq 40/9$, it gives positive marginal contribution only to $\{q, r, s, u\}$ and $\{q, s, t, u\}$.

[3]The investigation of the monotonicity property made in Sections 3.1 and 3.2, and the numerical results reported in Section 4.1 for the TNc game based on the user equilibrium, show clearly that the absence of such a property is a necessary (but not sufficient) condition for the occurrence of a negative Shapley value.

It follows from Tables 5–8 that the Shapley values of the arcs are given by the following explicit formulas:

$$Sh(q) = Sh(u) = \begin{cases} \dfrac{d(1600-367d)}{120} & \text{if } d \in \left[0, \dfrac{20}{11}\right], \\ \dfrac{87}{52}d^2 - \dfrac{151}{39}d + \dfrac{610}{39} & \text{if } d \in \left[\dfrac{20}{11}, 4\right], \\ \dfrac{703}{156}d^2 - \dfrac{593}{39}d + \dfrac{610}{39} & \text{if } d \in \left[4, \dfrac{40}{9}\right], \\ \dfrac{233}{60}d^2 - \dfrac{29}{3}d + \dfrac{10}{3} & \text{if } d \geq \dfrac{40}{9}, \end{cases} \tag{22}$$

$$Sh(r) = Sh(t) = \begin{cases} \dfrac{11}{40}d^2 & \text{if } d \in \left[0, \dfrac{20}{11}\right], \\ \dfrac{19(11d-20)^2}{1040} + \dfrac{11}{40}d^2 & \text{if } d \in \left[\dfrac{20}{11}, 4\right], \\ \dfrac{933}{208}d^2 - \dfrac{417}{26}d + \dfrac{95}{13} & \text{if } d \in \left[4, \dfrac{40}{9}\right], \\ \dfrac{309}{80}d^2 - \dfrac{21}{2}d - 5 & \text{if } d \geq \dfrac{40}{9}, \end{cases} \tag{23}$$

$$Sh(s) = \begin{cases} \dfrac{d(400-133d)}{30} & \text{if } d \in \left[0, \dfrac{20}{11}\right], \\ \dfrac{31}{104}d^2 - \dfrac{151}{39}d + \dfrac{610}{39} & \text{if } d \in \left[\dfrac{20}{11}, 4\right], \\ \dfrac{(d+20)^2}{120} + \dfrac{(9d-40)^2}{130} & \text{if } d \in \left[4, \dfrac{40}{9}\right], \\ \dfrac{(d+20)^2}{120} & \text{if } d \geq \dfrac{40}{9}. \end{cases} \tag{24}$$

Figure 7 shows the Shapley value of each arc as a function of the traffic demand. Similarly to the TNc game based on the user equilibrium, for any $d > 0$, the Shapley value of arcs q and u is greater than that of arcs r and t, while, the TNc game based on the user equilibrium, for any $d > 0$, the Shapley value of arc s is always positive.

Figures 8 and 9 give a graphical representation on the network of the absolute and normalized Shapley values of the arcs, respectively for six different values of the demand. Since all the Shapley values are positive, arcs are colored in green. In Figure 8, the thickness of each arc i is proportional to its Shapley value $Sh(i)$, while in Figure 9 it is proportional to its normalized Shapley value $Sh(i)/\sum_{j\in N} Sh(j)$. Figures 7 and 8 clearly show that the importance of arcs different from s is increasing with respect to the demand. Figures 7, 8 and 9 highlight that, for any $d > 0$, arcs q and u are more important than arcs r and t, and in Figures 8 and 9 this fact is more evident for small values of d. Moreover, it easily follows from Figure 7 that the relative importance of arc s is decreasing with respect to the demand.

4.3 Summing up: The role of arc s in the two TNc games

Figure 10 compares the Shapley value of arc s in the two TNc games with utility functions v^{ue} and v^{so}, respectively. For any $d > 0$, the Shapley value of s in (N, v^{so}) is positive (i.e., the cooperative version of Braess' paradox does not occur), while the one in (N, v^{ue}) is positive for $d \in (0, 400/133) \cup (1624/197, +\infty)$ and negative (i.e., the cooperative version of Braess' paradox occurs) for $d \in (400/133, 1624/197)$. In particular, the Shapley value of arc s is the same in both games when $d \in [0, 20/11]$, since in this case the

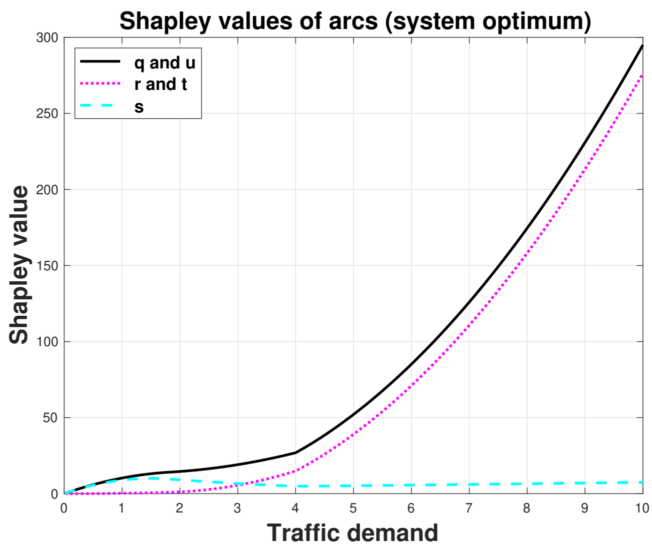
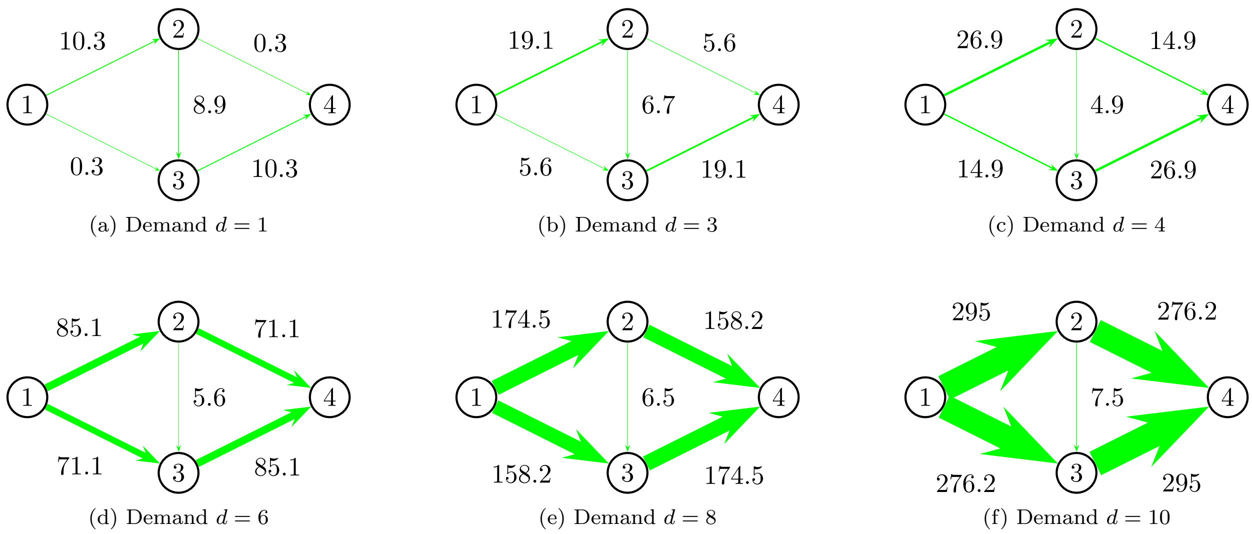


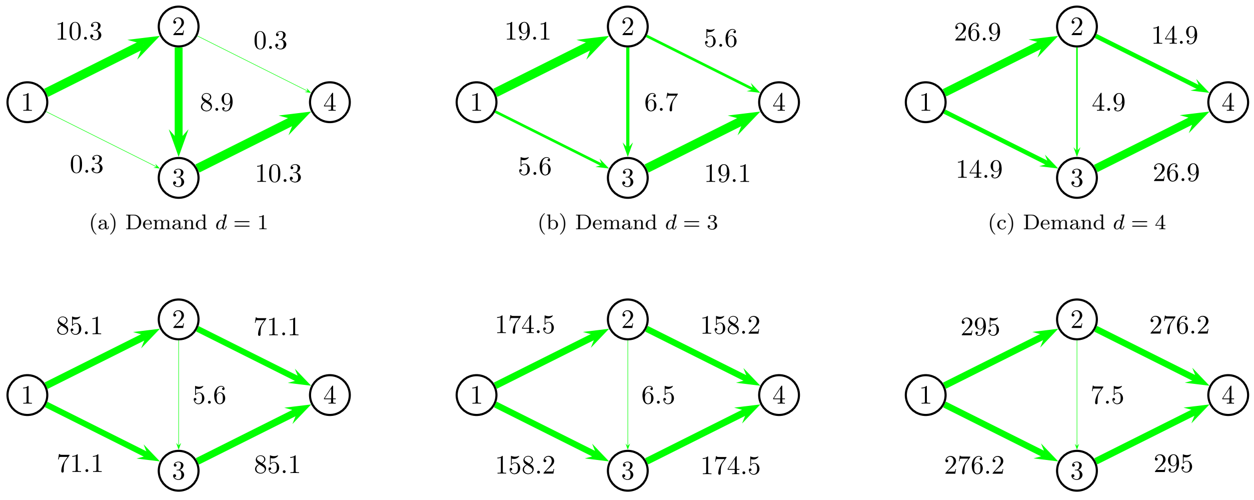
Figure 7: TNc game based on the system optimum: Shapley values of the arcs as functions of the traffic demand.



(a) Demand $d = 1$ (b) Demand $d = 3$ (c) Demand $d = 4$

(d) Demand $d = 6$ (e) Demand $d = 8$ (f) Demand $d = 10$

Figure 8: TNc game based on the system optimum: graphical representation of the Shapley values of the arcs for six values of the traffic demand. Each arc is colored in green since its Shapley value (indicated on the arc) is positive. The thickness of each arc i is proportional to $Sh(i)$.



(a) Demand $d = 1$

(b) Demand $d = 3$

(c) Demand $d = 4$

(d) Demand $d = 6$

(e) Demand $d = 8$

(f) Demand $d = 10$

Figure 9: TNc game based on the system optimum: graphical representation of the Shapley values of the arcs for six values of the traffic demand. Each arc is colored green since its Shapley value (indicated on the arc) is positive. For each network, the thickness of arc i is proportional to the ratio $Sh(i)/\sum_{j \in N} Sh(j)$.

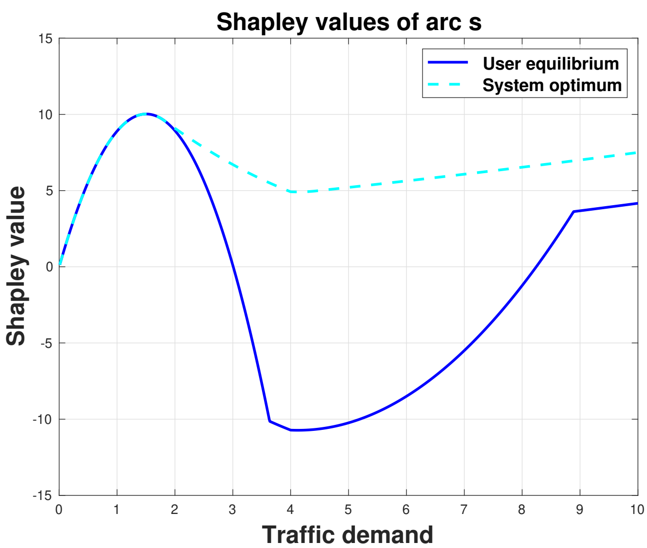


Figure 10: Shapley value of arc s in the TNc games based on the user equilibrium and the system optimum.

user equilibrium coincides with the system optimum for each coalition, while for any $d > 20/11$ the Shapley value of arc s in (N, v^{so}) is larger than the one obtained in (N, v^{ue}).

The reason for this gap is shown in Table 9 in the Appendix, where the marginal contributions of arc s to coalitions $\{q, r, s, t, u\}$, $\{q, r, s, t\}$, $\{q, s, t, u\}$ and $\{q, s, u\}$ are reported for the two utility functions v^{ue} and v^{so}. The other coalitions are omitted because, for them, the marginal contributions of s are equal to zero. When $d \in [0, 20/11]$, the marginal contributions of s to any coalition are equal for v^{ue} and v^{so}, while for $d > 20/11$ there exists at least a coalition such that its marginal contributions in the two games are different. In particular, the marginal contributions of s to the grand coalition are different for $d \in (20/11, 80/9)$; for the coalitions $\{q, r, s, t\}$ and $\{q, s, t, u\}$ the marginal contributions of s are different for $d > 20/11$, while the marginal contributions of s to coalition $\{q, s, u\}$ are always equal. Moreover, Table 9 clearly shows that the classical version of Braess' paradox (i.e., the marginal contribution of s to the grand coalition is negative) occurs for $d \in (80/31, 80/9)$.

Finally, we recall that relationships between the user equilibrium and the system optimum, together with situations in which one can prove that they coincide, are reported, e.g., in [40, Section 3.5]. Typically, the two solutions become increasingly dissimilar as the demand (then, the congestion level) increases.

5 Conclusions and extensions

We proposed a framework to investigate Braess' paradox from a cooperative game-theoretical point of view. This is done by considering the arc average marginal utility with respect to various initial scenarios, i.e., its Shapley value in a transferable-utility game defined on networks. The utility function is defined in such a way as to model congestion. More specifically, it is defined in terms of either user equilibrium or system optimum.

The results of the numerical analysis performed on the Braess' original example show the appearance of a cooperative version of Braess' paradox for a certain range of values of the demand, when the utility function based on the user equilibrium is adopted to model the TNc game. Instead, no such possibility arises when the utility function based on the system optimum is used.

Among possible extensions of the analysis, we mention the following. The first is of computational nature. Although in some particular cases one can compute the Shapley values in polynomial time (see, e.g., [1, 44, 45]), in general its calculation involves an effort that grows exponentially with the number of players [19, 32]. Hence, to investigate the possibility of having negative Shapley values for large networks, methodologies for approximate computation need to be adopted or developed (see, e.g., [2]), such as those based on Monte-Carlo methods [9]. In [21], Monte-Carlo approximations of the Shapley values were obtained for a particular class of transportation network cooperative games, which was developed to evaluate the relative importance of public transport transfers. In the present context, the issue of approximate computation is even more important since the evaluation of the utility of each coalition can be computationally demanding by itself, requiring a variational inequality/optimization problem to be solved for each single evaluation. When closed form solutions cannot be found, one has to resort to suitable algorithms; e.g., for both cases, the ones reported in [4, 5, 17, 23, 34].

A second direction of improvement regards the analysis of the stochastic case, taking into account, e.g., the stochastic models of traffic assignment considered in [16].

A third direction consists in using the Shapley value-based approach considered in this work to provide a game-theoretical measure of both *positive* and *negative* importance of a resource in a network. This would be useful for network design purposes, and could be achieved by adopting a non-monotonic game formulation whose characteristic function models congestion (e.g., the utility function based on the user equilibrium).

Finally, an envisaged development is the exploitation of the proposed approach for the study of network resilience [11, 46], in particular for the assessment of the effects of failures of the links, taking into account their failure probabilities.

Appendix

In this appendix, several tables are included. They are related to the numerical results reported in Section 4. Tables 1–4 report closed-form expressions of various quantities of interest for the TNc game based on the user equilibrium, distinguishing, respectively, among the cases $d \in [0, 40/11]$, $d \in [40/11, 4]$, $d \in [4, 80/9]$, and $d \geq 80/9$. Tables 5–8 provide analogous expressions for the TNc game based on the system optimum, distinguishing, respectively, among the cases $d \in [0, 20/11]$, $d \in [20/11, 4]$, $d \in [4, 40/9]$, and $d \geq 40/9$. Table 9 reports a comparison of the two games, focusing on arc s. See the captions of the tables and Section 4 for further details and comments about them.

Table 1: TNc game based on the user equilibrium, for $d \in [0, 40/11]$. User equilibrium, total cost, utility, and marginal contribution of arcs are reported, for each coalition $S \in \mathcal{F}$. The minimally connected coalitions in $\mathcal{F}_m$, their maximum total cost and the negative marginal contributions of arc s are reported in frames.

Coalition	User equilibrium	Total cost	Utility	Marginal contribution of arc				
S	$\bar{x}(S)$	$TC^{ue}(S)$	$v^{ue}(S)$	q	r	s	t	u
$\{q,r,s,t,u\}$	$(0,0,d)$	$d(21d+10)$	$10d(4-d)$	$10d(4-d)$	0	$d(80-31d)/2$	0	$10d(4-d)$
$\{q,r,s,t\}$	$(d,0,0)$	$d(11d+50)$	0	0	0	0	0	–
$\{q,r,s,u\}$	$(0,0,d)$	$d(21d+10)$	$10d(4-d)$	$10d(4-d)$	0	$10d(4-d)$	–	$10d(4-d)$
$\{q,r,t,u\}$	$(d/2,d/2,0)$	$d(11d/2+50)$	$11d^2/2$	$11d^2/2$	$11d^2/2$	–	$11d^2/2$	$11d^2/2$
$\{q,r,t\}$	$(d,0,0)$	$d(11d+50)$	0	0	0	–	0	–
$\{q,r,u\}$	$(0,d,0)$	$d(11d+50)$	0	0	0	–	–	0
$\{q,s,t,u\}$	$(0,0,d)$	$d(21d+10)$	$10d(4-d)$	$10d(4-d)$	–	$10d(4-d)$	0	$10d(4-d)$
$\{q,s,t\}$	$(d,0,0)$	$d(11d+50)$	0	0	–	0	0	–
$\{q,s,u\}$	$(0,0,d)$	$d(21d+10)$	$10d(4-d)$	$10d(4-d)$	–	$10d(4-d)$	–	$10d(4-d)$
$\{q,t,u\}$	$(d,0,0)$	$d(11d+50)$	0	0	–	–	0	0
$\{q,t\}$	$(d,0,0)$	$d(11d+50)$	0	0	–	–	0	–
$\{r,s,t,u\}$	$(0,d,0)$	$d(11d+50)$	0	–	0	0	0	0
$\{r,s,u\}$	$(0,d,0)$	$d(11d+50)$	0	–	0	0	–	0
$\{r,t,u\}$	$(0,d,0)$	$d(11d+50)$	0	–	0	–	0	0
$\{r,u\}$	$(0,d,0)$	$d(11d+50)$	0	–	0	–	–	0

Table 2: TNc game based on the user equilibrium, for $d \in [40/11, 4]$. User equilibrium, total cost, utility, and marginal contribution of arcs are reported, for each coalition $S \in \mathcal{F}$. The minimally connected coalitions in $\mathcal{F}_m$, their maximum total cost and the negative marginal contributions of arc s are reported in frames.

Coalition	User equilibrium	Total cost	Utility	Marginal contribution of arc				
S	$\bar{x}(S)$	$TC^{ue}(S)$	$v^{ue}(S)$	q	r	s	t	u
$\{q,r,s,t,u\}$	$\left(\frac{11d-40}{13}, \frac{11d-40}{13}, \frac{80-9d}{13}\right)$	$\frac{d(31d+1010)}{13}$	$\frac{8d(14d-45)}{13}$	$\frac{8d(14d-45)}{13}$	$\frac{121d(11d-40)}{156}$	$\boxed{\frac{9d(9d-80)}{26}}$	$\frac{121d(11d-40)}{156}$	$\frac{8d(14d-45)}{13}$
$\{q,r,s,t\}$	$(d,0,0)$	$d(11d+50)$	0	0	0	0	0	–
$\{q,r,s,u\}$	$\left(0, \frac{11d-40}{12}, \frac{d+40}{12}\right)$	$\frac{d(131d+560)}{12}$	$\frac{d(d+40)}{12}$	$\frac{d(d+40)}{12}$	$\frac{11d(11d-40)}{12}$	$\frac{d(d+40)}{12}$	–	$\frac{d(d+40)}{12}$
$\{q,r,t,u\}$	$(d/2,d/2,0)$	$d(11d/2+50)$	$11d^2/2$	$11d^2/2$	$11d^2/2$	–	$11d^2/2$	$11d^2/2$
$\{q,r,t\}$	$(d,0,0)$	$d(11d+50)$	0	0	0	–	0	–
$\{q,r,u\}$	$(0,d,0)$	$d(11d+50)$	0	0	0	–	–	0
$\{q,s,t,u\}$	$\left(\frac{11d-40}{12}, 0, \frac{d+40}{12}\right)$	$\frac{d(131d+560)}{12}$	$\frac{d(d+40)}{12}$	$\frac{d(d+40)}{12}$	–	$\frac{d(d+40)}{12}$	$\frac{11d(11d-40)}{12}$	$\frac{d(d+40)}{12}$
$\{q,s,t\}$	$(d,0,0)$	$d(11d+50)$	0	0	–	0	0	–
$\boxed{\{q,s,u\}}$	$(0,0,d)$	$d(21d+10)$	$10d(4-d)$	$10d(4-d)$	–	$10d(4-d)$	–	$10d(4-d)$
$\{q,t,u\}$	$(d,0,0)$	$d(11d+50)$	0	0	–	–	0	0
$\boxed{\{q,t\}}$	$(d,0,0)$	$\boxed{d(11d+50)}$	0	0	–	–	0	–
$\{r,s,t,u\}$	$(0,d,0)$	$d(11d+50)$	0	–	0	0	0	0
$\{r,s,u\}$	$(0,d,0)$	$d(11d+50)$	0	–	0	0	–	0
$\{r,t,u\}$	$(0,d,0)$	$d(11d+50)$	0	–	0	–	0	0
$\boxed{\{r,u\}}$	$(0,d,0)$	$\boxed{d(11d+50)}$	0	–	0	–	–	0

Table 3: TNc game based on the user equilibrium, for $d \in [4, 80/9]$. User equilibrium, total cost, utility, and marginal contribution of arcs are reported, for each coalition $S \in \mathcal{F}$. The minimally connected coalitions in $\mathcal{F}_m$, their maximum total cost and the negative marginal contributions of arc s are reported in frames.

Coalition	User equilibrium	Total cost	Utility	Marginal contribution of arc				
S	$\bar{x}(S)$	$TC^{ue}(S)$	$v^{ue}(S)$	q	r	s	t	u
$\{q,r,s,t,u\}$	$\left(\frac{11d-40}{13}, \frac{11d-40}{13}, \frac{80-9d}{13}\right)$	$\frac{d(31d+1010)}{13}$	$\frac{22d(11d-40)}{13}$	$\frac{8d(14d-45)}{13}$	$\frac{121d(11d-40)}{156}$	$\boxed{\frac{9d(9d-80)}{26}}$	$\frac{121d(11d-40)}{156}$	$\frac{8d(14d-45)}{13}$
$\{q,r,s,t\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\{q,r,s,u\}$	$\left(0, \frac{11d-40}{12}, \frac{d+40}{12}\right)$	$\frac{d(131d+560)}{12}$	$\frac{11d(11d-40)}{12}$	$\frac{d(d+40)}{12}$	$\frac{11d(11d-40)}{12}$	$\frac{d(d+40)}{12}$	0	$\frac{11d(11d-40)}{12}$
$\{q,r,t,u\}$	$(d/2,d/2,0)$	$d((11d)/2+50)$	$(d(31d-80))/2$	$11d^2/2$	$11d^2/2$	0	$11d^2/2$	$11d^2/2$
$\{q,r,t\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\{q,r,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\{q,s,t,u\}$	$\left(\frac{11d-40}{12}, 0, \frac{d+40}{12}\right)$	$\frac{d(131d+560)}{12}$	$\frac{11d(11d-40)}{12}$	$\frac{11d(11d-40)}{12}$	0	$\frac{d(d+40)}{12}$	$\frac{11d(11d-40)}{12}$	$\frac{d(d+40)}{12}$
$\{q,s,t\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\boxed{\{q,s,u\}}$	$(0,0,d)$	$\boxed{d(21d+10)}$	0	0	0	0	0	0
$\{q,t,u\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\boxed{\{q,t\}}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\{r,s,t,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\{r,s,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\{r,t,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\boxed{\{r,u\}}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$

Table 4: TNc game based on the user equilibrium, for $d \geq 80/9$. User equilibrium, total cost, utility, and marginal contribution of arcs are reported, for each coalition $S \in \mathcal{F}$. The minimally connected coalitions in $\mathcal{F}_m$ and their maximum total cost are reported in frames.

Coalition	User equilibrium	Total cost	Utility	Marginal contribution of arc				
S	$\bar{x}(S)$	$TC^{ue}(S)$	$v^{ue}(S)$	q	r	s	t	u
$\{q,r,s,t,u\}$	$(d/2, d/2, 0)$	$d(11d/2+50)$	$d(31d-80)/2$	$11d^2/2$	$5d(13d-8)/12$	0	$5d(13d-8)/12$	$11d^2/2$
$\{q,r,s,t\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\{q,r,s,u\}$	$\left(0, \frac{11d-40}{12}, \frac{d+40}{12}\right)$	$\frac{d(131d+560)}{12}$	$\frac{11d(11d-40)}{12}$	$\frac{d(d+40)}{12}$	$\frac{11d(11d-40)}{12}$	$\frac{d(d+40)}{12}$	0	$\frac{11d(11d-40)}{12}$
$\{q,r,t,u\}$	$(d/2, d/2, 0)$	$d((11d)/2+50)$	$(d(31d-80))/2$	$(11d^2)/2$	$(11d^2)/2$	0	$(11d^2)/2$	$(11d^2)/2$
$\{q,r,t\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\{q,r,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\{q,s,t,u\}$	$\left(\frac{11d-40}{12}, 0, \frac{d+40}{12}\right)$	$\frac{d(131d+560)}{12}$	$\frac{11d(11d-40)}{12}$	$\frac{11d(11d-40)}{12}$	0	$\frac{d(d+40)}{12}$	$\frac{11d(11d-40)}{12}$	$\frac{d(d+40)}{12}$
$\{q,s,t\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\boxed{\{q,s,u\}}$	$(0,0,d)$	$\boxed{d(21d+10)}$	0	0	0	0	0	0
$\{q,t,u\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\boxed{\{q,t\}}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\{r,s,t,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\{r,s,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\{r,t,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\boxed{\{r,u\}}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$

Table 5: TNc game based on the system optimum, for $d \in [0, 20/11]$. System optimum, total cost, utility and marginal contribution of arcs are reported, for each coalition $S \in \mathcal{F}$. The minimally connected coalitions in $\mathcal{F}_m$ and their maximum total cost are reported in frames.

Coalition	System Optimum	Total cost	Utility	Marginal contribution of arc				
S	$\hat{x}(S)$	$TC^{so}(S)$	$v^{so}(S)$	q	r	s	t	u
$\{q,r,s,t,u\}$	$(0,0,d)$	$d(21d+10)$	$10d(4-d)$	$10d(4-d)$	0	$d(80-31d)/2$	0	$10d(4-d)$
$\{q,r,s,t\}$	$(d,0,0)$	$d(11d+50)$	0	0	0	0	0	0
$\{q,r,s,u\}$	$(0,0,d)$	$d(21d+10)$	$10d(4-d)$	$10d(4-d)$	0	$10d(4-d)$	0	$10d(4-d)$
$\{q,r,t,u\}$	$(d/2,d/2,0)$	$d(11d/2+50)$	$11d^2/2$	$11d^2/2$	$11d^2/2$	0	$11d^2/2$	$11d^2/2$
$\{q,r,t\}$	$(d,0,0)$	$d(11d+50)$	0	0	0	0	0	0
$\{q,r,u\}$	$(0,d,0)$	$d(11d+50)$	0	0	0	0	0	0
$\{q,s,t,u\}$	$(0,0,d)$	$d(21d+10)$	$10d(4-d)$	$10d(4-d)$	0	$10d(4-d)$	0	$10d(4-d)$
$\{q,s,t\}$	$(d,0,0)$	$d(11d+50)$	0	0	0	0	0	0
$\boxed{\{q,s,u\}}$	$(0,0,d)$	$d(21d+10)$	$10d(4-d)$	$10d(4-d)$	0	$10d(4-d)$	0	$10d(4-d)$
$\{q,t,u\}$	$(d,0,0)$	$d(11d+50)$	0	0	0	0	0	0
$\boxed{\{q,t\}}$	$(d,0,0)$	$\boxed{d(11d+50)}$	0	0	0	0	0	0
$\{r,s,t,u\}$	$(0,d,0)$	$d(11d+50)$	0	0	0	0	0	0
$\{r,s,u\}$	$(0,d,0)$	$d(11d+50)$	0	0	0	0	0	0
$\{r,t,u\}$	$(0,d,0)$	$d(11d+50)$	0	0	0	0	0	0
$\boxed{\{r,u\}}$	$(0,d,0)$	$\boxed{d(11d+50)}$	0	0	0	0	0	0

Table 6: TNc game based on the system optimum, for $d \in [20/11, 4]$. System optimum, total cost, utility and marginal contribution of arcs are reported, for each coalition $S \in \mathcal{F}$. The minimally connected coalitions in $\mathcal{F}_m$ and their maximum total cost are reported in frames.

Coalition	System Optimum	Total cost	Utility	Marginal contribution of arc				
S	$\hat{x}(S)$	$TC^{so}(S)$	$v^{so}(S)$	q	r	s	t	u
$\{q,r,s,t,u\}$	$\left(\frac{11d-20}{13}, \frac{11d-20}{13}, \frac{40-9d}{13}\right)$	$\frac{31d^2+1010d-800}{13}$	$\frac{112d^2-360d+800}{13}$	$\frac{112d^2-360d+800}{13}$	$\frac{11(11d-20)^2}{156}$	$\frac{(9d-40)^2}{26}$	$\frac{11(11d-20)^2}{156}$	$\frac{112d^2-360d+800}{13}$
$\{q,r,s,t,\}$	$(d,0,0)$	$d(11d+50)$	0	0	0	0	0	0
$\{q,r,s,u\}$	$\left(0, \frac{11d-20}{12}, \frac{d+20}{12}\right)$	$\frac{131d^2+560d-400}{12}$	$\frac{(d+20)^2}{12}$	$\frac{(d+20)^2}{12}$	$\frac{(11d-20)^2}{12}$	$\frac{(d+20)^2}{12}$	0	$\frac{(d+20)^2}{12}$
$\{q,r,t,u\}$	$(d/2,d/2,0)$	$d((11d)/2+50)$	$(11d^2)/2$	$(11d^2)/2$	$(11d^2)/2$	0	$(11d^2)/2$	$(11d^2)/2$
$\{q,r,t\}$	$(d,0,0)$	$d(11d+50)$	0	0	0	0	0	0
$\{q,r,u\}$	$(0,d,0)$	$d(11d+50)$	0	0	0	0	0	0
$\{q,s,t,u\}$	$\left(\frac{11d-20}{12}, 0, \frac{d+20}{12}\right)$	$\frac{131d^2+560d-400}{12}$	$\frac{(d+20)^2}{12}$	$\frac{(d+20)^2}{12}$	0	$\frac{(d+20)^2}{12}$	$\frac{(11d-20)^2}{12}$	$\frac{(d+20)^2}{12}$
$\{q,s,t\}$	$(d,0,0)$	$d(11d+50)$	0	0	0	0	0	0
$\boxed{\{q,s,u\}}$	$(0,0,d)$	$d(21d+10)$	$10d(4-d)$	$10d(4-d)$	0	$10d(4-d)$	0	$10d(4-d)$
$\{q,t,u\}$	$(d,0,0)$	$d(11d+50)$	0	0	0	0	0	0
$\boxed{\{q,t\}}$	$(d,0,0)$	$\boxed{d(11d+50)}$	0	0	0	0	0	0
$\{r,s,t,u\}$	$(0,d,0)$	$d(11d+50)$	0	0	0	0	0	0
$\{r,s,u\}$	$(0,d,0)$	$d(11d+50)$	0	0	0	0	0	0
$\{r,t,u\}$	$(0,d,0)$	$d(11d+50)$	0	0	0	0	0	0
$\boxed{\{r,u\}}$	$(0,d,0)$	$\boxed{d(11d+50)}$	0	0	0	0	0	0

Table 7: TNc game based on the system optimum, for $d \in [4, 40/9]$. System optimum, total cost, utility and marginal contribution of arcs are reported, for each coalition $S \in \mathcal{F}$. The minimally connected coalitions in $\mathcal{F}_m$ and their maximum total cost are reported in frames.

Coalition	System Optimum	Total cost	Utility	Marginal contribution of arc				
S	$\hat{x}(S)$	$TC^{so}(S)$	$v^{so}(S)$	q	r	s	t	u
$\{q,r,s,t,u\}$	$\left(\frac{11d-20}{13}, \frac{11d-20}{13}, \frac{40-9d}{13}\right)$	$\frac{31d^2+1010d-800}{13}$	$\frac{2(11d-20)^2}{13}$	$\frac{112d^2-360d+800}{13}$	$\frac{11(11d-20)^2}{156}$	$\frac{(9d-40)^2}{26}$	$\frac{11(11d-20)^2}{156}$	$\frac{112d^2-360d+800}{13}$
$\{q,r,s,t\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\{q,r,s,u\}$	$\left(0, \frac{11d-20}{12}, \frac{d+20}{12}\right)$	$\frac{131d^2+560d-400}{12}$	$\frac{(11d-20)^2}{12}$	$\frac{(d+20)^2}{12}$	$\frac{(11d-20)^2}{12}$	$\frac{(d+20)^2}{12}$	0	$\frac{(11d-20)^2}{12}$
$\{q,r,t,u\}$	$(d/2,d/2,0)$	$d((11d)/2+50)$	$(d(31d-80))/2$	$(11d^2)/2$	$(11d^2)/2$	0	$(11d^2)/2$	$(11d^2)/2$
$\{q,r,t\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\{q,r,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\{q,s,t,u\}$	$\left(\frac{11d-20}{12}, 0, \frac{d+20}{12}\right)$	$\frac{131d^2+560d-400}{12}$	$\frac{(11d-20)^2}{12}$	$\frac{(11d-20)^2}{12}$	0	$\frac{(d+20)^2}{12}$	$\frac{(11d-20)^2}{12}$	$\frac{(d+20)^2}{12}$
$\{q,s,t\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\boxed{\{q,s,u\}}$	$(0,0,d)$	$\boxed{d(21d+10)}$	0	0	0	0	0	0
$\{q,t,u\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\boxed{\{q,t\}}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\{r,s,t,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\{r,s,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\{r,t,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\boxed{\{r,u\}}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$

Table 8: TNc game based on the system optimum, for $d \geq 40/9$. System optimum, total cost, utility and marginal contribution of arcs are reported, for each coalition $S \in \mathcal{F}$. The minimally connected coalitions in $\mathcal{F}_m$ and their maximum total cost are reported in frames.

Coalition	System Optimum	Total cost	Utility	Marginal contribution of arc				
S	$\hat{x}(S)$	$TC^{so}(S)$	$v^{so}(S)$	q	r	s	t	u
$\{q,r,s,t,u\}$	$\left(\frac{d}{2},\frac{d}{2},0\right)$	$d(11d/2+50)$	$d(31d-80)/2$	$11d^2/2$	$\frac{65d^2-40d-400}{12}$	0	$\frac{65d^2-40d-400}{12}$	$11d^2/2$
$\{q,r,s,t\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\{q,r,s,u\}$	$\left(0,\frac{11d-20}{12},\frac{d+20}{12}\right)$	$\frac{131d^2+560d-400}{12}$	$\frac{(11d-20)^2}{12}$	$\frac{(d+20)^2}{12}$	$\frac{(11d-20)^2}{12}$	$\frac{(d+20)^2}{12}$	0	$\frac{(11d-20)^2}{12}$
$\{q,r,t,u\}$	$(d/2,d/2,0)$	$d(11d/2+50)$	$d(31d-80)/2$	$11d^2/2$	$11d^2/2$	0	$11d^2/2$	$11d^2/2$
$\{q,r,t\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\{q,r,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\{q,s,t,u\}$	$\left(\frac{11d-20}{12},0,\frac{d+20}{12}\right)$	$\frac{131d^2+560d-400}{12}$	$\frac{(11d-20)^2}{12}$	$\frac{(11d-20)^2}{12}$	0	$\frac{(d+20)^2}{12}$	$\frac{(11d-20)^2}{12}$	$\frac{(d+20)^2}{12}$
$\{q,s,t\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\boxed{\{q,s,u\}}$	$(0,0,d)$	$\boxed{d(21d+10)}$	0	0	0	0	0	0
$\{q,t,u\}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\boxed{\{q,t\}}$	$(d,0,0)$	$d(11d+50)$	$10d(d-4)$	$10d(d-4)$	0	0	$10d(d-4)$	0
$\{r,s,t,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\{r,s,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\{r,t,u\}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$
$\boxed{\{r,u\}}$	$(0,d,0)$	$d(11d+50)$	$10d(d-4)$	0	$10d(d-4)$	0	0	$10d(d-4)$

Table 9: Marginal contributions of arc s in the TNc games (N, v^{ue}) and (N, v^{so}). The negative marginal contributions are reported in frames.

Demand	$\{q,r,s,t,u\}$		$\{q,r,s,t\}$		$\{q,s,t,u\}$		$\{q,s,u\}$	
	v^{ue}	v^{so}	v^{ue}	v^{so}	v^{ue}	v^{so}	v^{ue}	v^{so}
$\left[0, \frac{20}{11}\right]$	$\frac{d(80-31d)}{2}$	$\frac{d(80-31d)}{2}$	$10d(4-d)$	$10d(4-d)$	$10d(4-d)$	$10d(4-d)$	$10d(4-d)$	$10d(4-d)$
$\left[\frac{20}{11}, \frac{80}{31}\right]$	$\frac{d(80-31d)}{2}$	$\frac{(9d-40)^2}{26}$	$10d(4-d)$	$\frac{(d+20)^2}{12}$	$10d(4-d)$	$\frac{(d+20)^2}{12}$	$10d(4-d)$	$10d(4-d)$
$\left[\frac{80}{31}, \frac{40}{11}\right]$	$\boxed{\frac{d(80-31d)}{2}}$	$\frac{(9d-40)^2}{26}$	$10d(4-d)$	$\frac{(d+20)^2}{12}$	$10d(4-d)$	$\frac{(d+20)^2}{12}$	$10d(4-d)$	$10d(4-d)$
$\left[\frac{40}{11}, 4\right]$	$\boxed{\frac{9d(9d-80)}{26}}$	$\frac{(9d-40)^2}{26}$	$\frac{d(d+40)}{12}$	$\frac{(d+20)^2}{12}$	$\frac{d(d+40)}{12}$	$\frac{(d+20)^2}{12}$	$10d(4-d)$	$10d(4-d)$
$\left[4, \frac{40}{9}\right]$	$\boxed{\frac{9d(9d-80)}{26}}$	$\frac{(9d-40)^2}{26}$	$\frac{d(d+40)}{12}$	$\frac{(d+20)^2}{12}$	$\frac{d(d+40)}{12}$	$\frac{(d+20)^2}{12}$	0	0
$\left[\frac{40}{9}, \frac{80}{9}\right]$	$\boxed{\frac{9d(9d-80)}{26}}$	0	$\frac{d(d+40)}{12}$	$\frac{(d+20)^2}{12}$	$\frac{d(d+40)}{12}$	$\frac{(d+20)^2}{12}$	0	0
$\left[\frac{80}{9}, +\infty\right)$	0	0	$\frac{d(d+40)}{12}$	$\frac{(d+20)^2}{12}$	$\frac{d(d+40)}{12}$	$\frac{(d+20)^2}{12}$	0	0

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