A separation approach to vector quasi-equilibrium problems: Saddle point and gap function*

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Abstract. The image space approach is applied to the study of vector quasi-equilibrium problems. Exploiting separation arguments in the image space, Lagrangian-type optimality conditions and gap functions for vector quasi-equilibrium problems are derived.

Keywords: Vector quasi-equilibrium problems, image space, separation.

1 Introduction

The theory of equilibrium problems provides a general framework for the analysis of several topics in optimization: from the classical optimality conditions for constrained extremum problems to the equilibrium conditions for network flow, economic and mechanical engineering problems [3, 7, 14]. Recently, equilibrium problems, that were first introduced in a scalar form, have been generalized to the vector case, following similar developments in the field of variational inequalities [12, 6].

In this paper, by making use of the image space analysis, we consider a separation approach to a vector quasi-equilibrium problem (for short, $VQEP$) and deepen the theory of Lagrangian-type optimality conditions and gap functions [4, 15] associated with a $VQEP$ which consists in finding $x^* \in K(x^*)$ such that:

$$f(x^*, y) \not\leq_{C\setminus\{0\}} 0, \quad \forall y \in K(x^*),$$

where $f : X \times X \longrightarrow \mathbb{R}^p$, $K : X \to 2^X$, X is a Banach space, and C is a convex cone in $\mathbb{R}^p$ such that clC is a pointed cone; in the definition of $VQEP$ we have used the notation: $x \not\geq_C y$ iff $x - y \notin C$. When $K(x) \equiv K$ is a constant multifunction, then $VQEP$ is called vector equilibrium problem and denoted by VEP; when $p = 1$, $C = \mathbb{R}_+$ and $K(x) \equiv K$, then $VQEP$ collapses to the classic equilibrium problem (EP):

$$\text{find} \quad x^* \in K \quad s.t. \quad f(x^*, y) \geq 0, \quad \forall y \in K.$$

It is easy to see that vector optimization problems and vector variational inequalities can be formulated as a $VQEP$ choosing a suitable function $f(x, y)$.
Recall that, given the vector optimization problem:

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$$\min_{C\setminus\{0\}} h(x) \quad s.t. \quad x \in K, \qquad (P)$$

where $h : X \longrightarrow \mathbb{R}^p$, $x^* \in K$ is said a vector minimum point (for short v.m.p.) for (P) iff the following system is impossible:

$$h(x^*) - h(y) \in C \setminus \{0\}, \quad y \in K.$$

The quasi (Stampacchia) vector variational inequality ($QVVI$) consists in finding

$$x^* \in K(x^*) \quad s.t. \quad F(x^*)(y - x^*) \not\leq_{C\setminus\{0\}} 0, \quad \forall y \in K(x^*),$$

where $F : X \longrightarrow \mathbb{R}^{p\times n}$.

$QVVI$ collapses to the vector variational inequality when $K(x) \equiv K$.

The quasi Minty vector variational inequality ($QMVVI$) consists in finding $x^* \in K(x^*)$ such that

$$F(y)(x^* - y) \not\geq_{C\setminus\{0\}} 0, \quad \forall y \in K(x^*).$$

The following result is an immediate consequence of the definition of an optimal solution of a vector optimization problem and the definitions of $QVVI$ and $QMVVI$.

Proposition 1.1. *1. Let $f(x, y) := h(y) - h(x)$, then x^* is a v.m.p. for (P) iff x^* is a solution to VEP.*

2. Let $f(x, y) := F(x)(y - x)$; then x^* is a solution of QVVI iff x^* is a solution to $VQEP$.

3. Let $f(x, y) := F(y)(y - x)$; then x^* is a solution of MQVVI iff x^* is a solution to $VQEP$.

The image space analysis has shown to be a unifying scheme for studying constrained extremum problems, variational inequalities, and, more generally, can be applied to any kind of problem, say it P, that can be expressed under the form of the impossibility of a parametric system [8].

In this approach, the impossibility of such a system is reduced to the disjunction of two suitable subsets $\mathcal{K}$ and $\mathcal{H}$ of the image space associated with P. $\mathcal{K}$ is defined by the image of the functions involved in P, while $\mathcal{H}$ is a convex cone that depends only on the type of conditions (equalities, inequalities, etc) on the class of problems to which P belongs. The disjunction between $\mathcal{K}$ and $\mathcal{H}$ can be proved by showing that they lie in two disjoint level sets of a suitable separating functional, which leads one to obtain Lagrangian-type optimality conditions for P.

In this paper, we aim at applying the image space approach to the analysis of $VQEP$, and, in particular, of saddle point conditions and gap functions associated with $VQEP$. The gap function approach has been widely studied in the field of variational inequalities (see e.g. [4, 3, 23]). One of the main feature of such kind of approach is that variational inequalities can be equivalently formulated in terms of an optimization problem. Recently, gap function theory has been extended to VVI and $VQEP$ [3, 15]. A gap function for $VQEP$ $\phi : K^0 \longrightarrow \mathbb{R}$ is a function that is non negative, for every $x \in K^0 := \{x \in X : x \in K(x)\}$, and that fulfils the condition $\phi(x) = 0$ if and only if x is a solution of $VQEP$. It is immediate to see that solving $VQEP$ is equivalent to find a minimum point of ϕ on the set K^0, provided that the optimal value of ϕ is zero. An analogous definition has been given in [15, 11] for a generalized $VQEP$, the only difference being in the fact that ϕ is non positive for every $x \in K^0$. The analysis developped in [15, 11] is concerned with particular classes of generalized $VQEP$ that, anyway, do not collapse, as particular cases, to the class considered in the present paper.

The paper is organized as follows. In Section 2, we will analyse the general features of the image space approach for $VQEP$ giving particular attention to linear separation arguments. In Section 3, we will characterize the linear separation, in the image space, in terms of saddle point and Kuhn-Tucker type conditions for a suitable Lagrangian associated with $VQEP$. In Section 4, following the approach introduced in [7], we will show how the separations tecniques in the image space, allow us to define a gap function for a $VQEP$.

We remark that an important peculiarity of our analysis is that we explicitely define the constraint mapping $K : X \longrightarrow 2^X$ by

$$K(x) := \{y \in X : g(x, y) \in D\}, \tag{1}$$

where $g : X \times X \longrightarrow \mathbb{R}^m$, D is a closed convex cone in $\mathbb{R}^m$. Such a definition allows us to have a more handly expression of the gap function, from the computational point of view.

We recall the main notations and definitions that will be used in the sequel. The closure, the interior,the relative interior, the boundary, the relative boundary and the convex hull of a set $M \subseteq \mathbb{R}^n$ are denoted by clM, $int\ M$, $ri\ M$, $bd\ M$, $rbd\ M$, and $conv\ M$, respectively. $affM$ and $linM$ will denote the smallest affine variety and the smallest subspace in $\mathbb{R}^n$ that contains M, respectively.

Let $y \in \mathbb{R}^p$, $y := (y_1, \dots, y_p)$; $y_{(1^-)} := (y_2, \dots, y_p)$,
$y_{(i^-)} := (y_1, \dots, y_{i-1},\ y_{i+1}, \dots, y_p)$, $i = 2, \dots, p-1$, $y_{(p^-)} := (y_1, \dots, y_{p-1})$.
$\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^p$, $y \geq 0$ iff $y_i \geq 0$, $i = 1, \dots, p$. $\mathbb{R}^p_+ := \{x \in \mathbb{R}^p : x \geq 0\}$.

Let $D \subseteq \mathbb{R}^m$ be a convex cone, the positive polar of D is the set $D^* := \{x^* \in \mathbb{R}^m\ :\ \langle x^*, x\rangle \geq 0,\ \forall x \in D\}$. A closed convex cone D is said pointed if $D \cap (-D) = \{0\}$.

Let $g : X \longrightarrow \mathbb{R}^m$. g is said D–function on the convex set $K \subseteq X$ iff:

$$g(\lambda x_1 + (1-\lambda)x_2) - \lambda g(x_1) - (1-\lambda)g(x_2) \ \in D, \quad \forall x_1, x_2 \in K,\ \forall \lambda \in (0, 1).$$

We observe that if $D = \mathbb{R}^m_+$, then a D–function is a componentwise concave function. It is easy to show that if g is a D–function on K, then the set $g(K) - D$ is convex.

2 A separation approach to vector quasi-equilibrium problems

The image space analysis can be applied everytime the problem, we want to deal with, is expressed under the form of the impossibility of a suitable generalized system. In particular, if we let $K(x)$ be defined by (1), then $x^* \in X$ is a solution to $VQEP$ iff $x^* \in K(x^*)$ and the following system is impossible:

$$-f(x^*, y) \in C \setminus \{0\}, \quad g(x^*, y) \in D, \quad y \in X, \tag{$S(x^*)$}$$

The space $\mathbb{R}^{p+m}$ in which the function $(f(x, \cdot), g(x, \cdot))$ runs, is called the image space (for short, IS) associated with $VQEP$. By Proposition 1.1, it is immediate to see that vector optimization problems and quasi vector variational inequalities can be formulated as the impossibility of the system $S(x^*)$ choosing a suitable function $f(x, y)$.

The impossibility of $S(x^*)$ is stated by means of separation arguments in the IS, proving that two suitable subsets of the IS lie in disjoint level sets of a separating functional.

Let us consider the following subsets of the IS:

$$\mathcal{K}(x^*) := \{(u, v) \in \mathbb{R}^{p+m} :\ u = -f(x^*, y), \quad v = g(x^*, y), \quad y \in X\},$$

$$\mathcal{H} := \{(u, v) \in \mathbb{R}^{p+m} :\ u \in C \setminus \{0\}, \quad v \in D\}.$$

$\mathcal{K}(x^*)$ is called the *image* associated with $VQEP$ (or, equivalently, to $S(x^*)$). The impossibility of $S(x^*)$ can be formulated in terms of the disjunction of the sets $\mathcal{K}(x^*)$ and $\mathcal{H}$.

Proposition 2.1. *$x^* \in X$ is a solution to $QVEP$ iff $x^* \in K(x^*)$ and*

$$\mathcal{K}(x^*) \cap \mathcal{H} = \emptyset. \tag{2}$$

Let us introduce the set $\mathcal{E}(x^*) := \mathcal{K}(x^*) - cl\mathcal{H}$, which is called the *extended image* associated with $VQEP$. The extended image plays a key role in the image space analysis: first of all, it provides an equivalent formulation of the optimality condition (2).

Proposition 2.2. *If the cone $\mathcal{H}$ fulfils the condition $\mathcal{H} = \mathcal{H} + cl\mathcal{H}$, then (2) is equivalent to the condition*

$$\mathcal{E}(x^*) \cap \mathcal{H} = \emptyset. \tag{3}$$

Proof. It is a consequence of the following relations:

$$\mathcal{E}(x^*) - \mathcal{H} = \mathcal{K}(x^*) - cl\mathcal{H} - \mathcal{H} = \mathcal{K}(x^*) - (cl\mathcal{H} + \mathcal{H}) = \mathcal{K}(x^*) - \mathcal{H}.$$

□

Remark 2.1. In [2] it has been proved that if C is an open or closed convex cone, then $\mathcal{H} = \mathcal{H} + cl\mathcal{H}$, provided that D is a closed convex cone.
Moreover, it is known ([8], Lemma 3.1) that $\mathcal{E}(x^*)$ is a convex set when $g(x^*, \cdot)$ is a D–function and $-f(x^*, \cdot)$ is a (clC)–function, for a fixed $x^* \in X$.

Condition (3) can be proved showing that $\mathcal{E}(x^*)$ and $\mathcal{H}$ lie in two disjoint level sets of a suitable functional; when such a functional can be chosen linear we say that $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation.

Definition 2.1. *The sets $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation iff $\exists(\mu^*, \lambda^*) \in C^* \times D^*$, $(\mu^*, \lambda^*) \neq 0$, such that*

$$\langle \mu^*, u \rangle + \langle \lambda^*, v \rangle \leq 0, \qquad \forall (u, v) \in \mathcal{E}(x^*), \tag{4}$$

or,equivalently,

$$\langle \mu^*, -f(x^*, y) \rangle + \langle \lambda^*, g(x^*, y) \rangle \leq 0, \quad \forall y \in X. \tag{5}$$

The equivalence between (4) and (5) is proved by next result that shows that a linear functional separates $\mathcal{K}(x^*)$ and $\mathcal{H}$ iff it separates $\mathcal{E}(x^*)$ and $\mathcal{H}$.

Proposition 2.3. *Let $(\mu^*, \lambda^*) \in C^* \times D^*$, $(\mu^*, \lambda^*) \neq 0$, Then (4) is equivalent to*

$$\langle \mu^*, u \rangle + \langle \lambda^*, v \rangle \leq 0, \qquad \forall (u, v) \in \mathcal{K}(x^*). \tag{6}$$

Proof. Suppose that (6) holds. Let $(h_1, h_2) \in \mathcal{H}$. Since

$$\langle \mu^*, -h_1 \rangle + \langle \lambda^*, -h_2 \rangle \leq 0,$$

then

$$\langle \mu^*, u - h_1 \rangle + \langle \lambda^*, v - h_2 \rangle \leq 0, \qquad \forall (u, v) \in \mathcal{K}(x^*),$$

and (4) holds.
It is obvious that (4) implies (6), since $\mathcal{K}(x^*) \subseteq \mathcal{E}(x^*)$. □

The following result has been proved in [10]. We recall that $\mathcal{E}(x^*)$ and $\mathcal{H}$ are said to be *properly linearly separated* if they admit a linear separation, and, moreover, they are not both contained in the separating hyperplane.

Theorem 2.1. *$\mathcal{E}(x^*)$ and $\mathcal{H}$ are properly linearly separable iff*

$$0 \notin ri\ conv(\mathcal{E}(x^*)) \tag{7}$$

The existence of a separating hyperplane doesn't guarantee that $\mathcal{E}(x^*) \cap \mathcal{H} = \emptyset$. In order to ensure the disjunction of the two sets, some restrictions on the choice of the multipliers (μ^*, λ^*) must be imposed.

Proposition 2.4. *Let clC be a pointed cone and assume that the sets $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation.*
i) If $\mu^* \in int\ C^*$ then $\mathcal{E}(x^*) \cap \mathcal{H} = \emptyset$.
ii) Suppose that C is an open cone. If $\mu^* \neq 0$ then $\mathcal{E}(x^*) \cap \mathcal{H} = \emptyset$.

Proof. We recall (see e.g. [1]) that $cl\ C$ is pointed iff $int\ C^* \neq \emptyset$ and that

$$int\ C^* = \{x^* \in C^* : \langle x, x^* \rangle > 0, \quad \forall x \in cl\ C, x \neq 0\}.$$

i) Ab absurdo, suppose that $\mathcal{E}(x^*) \cap \mathcal{H} \neq \emptyset$. This implies that $\mathcal{K}(x^*) \cap \mathcal{H} \neq \emptyset$ and, therefore, $\exists z \in K(x^*)$ such that $-f(x^*, z) \in C \setminus \{0\}$. Then, taking into account that $\mu^* \in int\ C^*$, we have

$$0 < \langle \mu^*, -f(x^*, z) \rangle \ \leq \langle \mu^*, -f(x^*, z) \rangle + \langle \lambda^*, g(x^*, z) \rangle \leq 0, \tag{8}$$

which is impossible.
ii) Ab absurdo, suppose that $\mathcal{E}(x^*) \cap \mathcal{H} \neq \emptyset$. Following the proof of part *i)*, $\exists z \in K(x^*)$ such that $-f(x^*, z) \in C \ = int\ C$. Then, taking into account that $\mu^* \neq 0$, we have (8) which is impossible. □

Remark 2.2. In particular, if we define $f(x, y) := h(y) - h(x), K(x) \equiv K, C$ convex cone (resp. C open convex cone), and $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation with $\mu > 0$ (resp. $\mu \neq 0$), then x^* is a v.m.p. for (P).

We aim at establishing sufficient conditions that guarantee that the hypotheses of the Proposition 2.4 are fulfilled.
Let us recall the following preliminary result due to Hiriart-Urruty and Lemarechal [13].

Proposition 2.5. *Let B be a convex set in $\mathbb{R}^m$ and $x \in rbd(B)$. Then B admits a supporting hyperplane at x and its normal vector belongs to aff$(B - x)$.*

Theorem 2.2.

1. *Let $C := \mathbb{R}^p_+$, $int\ D \neq \emptyset$ and suppose that the sets $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation.*

   Assume that, for every $i = 1, \dots, p$, the following system is possible:

   $$-f_{i^-}(x^*, y) > 0, \quad g(x^*, y) \in int\ D, \quad y \in X. \qquad S_i(x^*)$$

   then in (5) we can suppose that $\mu^* > 0$.

2. *Suppose that the sets $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a proper linear separation. If*

   $$0 \in ri\ conv(g(x^*, X) - D), \tag{9}$$

   then we can suppose that $\mu^* \neq 0$ in (5).

Proof. *1.* Ab absurdo, suppose that, $\exists i \in \{1, \ldots, p\}$ such that $\mu_i^* = 0$; then $(\mu_{i^-}^*, \lambda^*) \neq 0$ and, since $S_i(x^*)$ is possible, $\exists \bar{y} \in X$ such that

$$0 < \langle \mu_{i^-}^*, -f_{i^-}(x^*, \bar{y}) \rangle + \langle \lambda^*, g(x^*, \bar{y}) \rangle \leq -\langle \mu_i^*, -f_i(x^*, \bar{y}) \rangle = 0,$$

which is absurd.
2. By Theorem 2.1 , proper linear separation is equivalent to the condition

$$0 \notin ri\ conv(\mathcal{E}(x^*)).$$

Since $0 \in \mathcal{E}(x^*)$ then $0 \in rbd[conv(\mathcal{E}(x^*))]$. Applying Proposition 2.5, we obtain that there exist $(\mu^*, \lambda^*) \in aff[conv(\mathcal{E}(x^*))]$, $(\mu^*, \lambda^*) \neq 0$, such that

$$\langle \mu^*, u \rangle + \langle \lambda^*, v \rangle \leq 0, \qquad \forall (u, v) \in \mathcal{E}(x^*). \tag{10}$$

Since $0 \in \mathcal{E}(x^*)$ then $(\mu^*, \lambda^*) \in lin[conv(\mathcal{E}(x^*))]$.

Ab absurdo, suppose that $\mu^* = 0$. Then

$$\lambda^* \in lin[(conv(g(x^*, X) - D)], \tag{11}$$

and (10) implies

$$\langle \lambda^*, v \rangle \leq 0, \quad \forall v \in conv(g(x^*, X) - D). \tag{12}$$

By (9), there exists a neighbourhood U of $0 \in \mathbb{R}^m$ such that

$$V := U \cap lin[conv(g(x^*, X) - D)] \subseteq conv[g(x^*, X) - D].$$

Taking into account (11), we obtain that $\gamma\lambda^* \in V$ for $|\gamma| < \epsilon$, sufficiently small. Since $V \subseteq conv[g(x^*, X) - D]$, by (12),

$$\gamma \langle \lambda^*, \lambda^* \rangle \leq 0, \quad \forall \gamma : \ |\gamma| < \epsilon,$$

which is impossible, for $\lambda^* \neq 0$. □

Remark 2.3. The condition given in statement *1* of Theorem 2.2 has been also considered in [16] in a slightly different form. The assumption (9) in statement *2* is a generalization of the Slater condition for scalar optimization problems [17, 18]. If $int\ D \neq \emptyset$ and $g(x^*, \cdot)$ is a $D-$function, then (9) is equivalent to assume that there exists $\bar{y} \in X$ such that

$$g(x^*, \bar{y}) \in int\ D.$$

3 Saddle point conditions

In this section, following the line considered in [2] we will characterize the linear separation, in the image space, in terms of a saddle point condition of the Lagrangian function associated with $VQEP$ (or, equivalently, to the system $S(x^*)$), defined by $L : C^* \times D^* \times X \longrightarrow \mathbb{R}$,

$$L(x^*; \mu, \lambda, y) := \langle \mu, f(x^*, y) \rangle - \langle \lambda, g(x^*, y) \rangle.$$

Proposition 3.1. *Suppose that $f(x^*, x^*) = 0$. Then $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation and $g(x^*, x^*) \in D$, iff $\exists (\mu^*, \lambda^*) \in C^* \times D^*$, $(\mu^*, \lambda^*) \neq 0$, such that (μ^*, λ^*, x^*) is a saddle point for $L(x^*; \mu, \lambda, y)$ on $(C^* \times D^*) \times X$.*

Proof. Suppose that $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation. ¿From (5) we obtain that $\langle \lambda^*, g(x^*, x^*)\rangle \le 0$, which implies that $\langle \lambda^*, g(x^*, x^*)\rangle = 0$, since $g(x^*, x^*) \in D$ and $\lambda^* \in D^*$. Therefore

$$0 = L(x^*; \mu^*, \lambda^*, x^*) \ \le L(x^*; \mu^*, \lambda^*, y), \quad \forall y \in X.$$

It remains to show that $L(x^*; \mu, \lambda, x^*) \le 0, \quad \forall(\mu, \lambda) \in (C^* \times D^*)$. We observe that

$$L(x^*; \mu, \lambda, x^*) = -\langle \lambda, g(x^*, x^*)\rangle$$

which is non positive, $\forall \lambda \in D^*$, and the necessity part of the statement is proved.

Sufficiency. Suppose that (μ^*, λ^*, x^*) is a saddle point for $L(x^*; \mu, \lambda, y)$ on $(C^* \times D^*) \times X$, that is

$$-\langle \lambda, g(x^*, x^*)\rangle \le \ -\langle \lambda^*, g(x^*, x^*)\rangle \le \langle \mu^*, f(x^*, y)\rangle - \langle \lambda^*, g(x^*, y)\rangle, \quad \forall(\mu, \lambda, y) \in (C^* \times D^*) \times X.$$

First of all we prove that $g(x^*, x^*) \in D$. Ab absurdo suppose that $g(x^*, x^*) \notin D = (D^*)^*$; then $\exists \bar{\lambda} \in D^*$ such that $\langle \bar{\lambda}, g(x^*, x^*)\rangle < 0$. Since D^* is a cone, then

$$\alpha\bar{\lambda} \in D^*, \ \forall \alpha \ge 0 \quad \text{and} \quad -\alpha\langle \bar{\lambda}, g(x^*, x^*)\rangle \longrightarrow +\infty, \quad \alpha \longrightarrow +\infty;$$

this contradicts the first inequality in the saddle point condition.
Computing the first inequality for $\lambda = 0$, we obtain $\langle \lambda^*, g(x^*, x^*)\rangle \le 0$ and, therefore, $\langle \lambda^*, g(x^*, x^*)\rangle = 0$. The second inequality coincides with (5) and the proposition is proved. □

Remark 3.1. We observe that the saddle value, $L(x^*; \mu^*, \lambda^*, x^*)$, is equal to zero. This property will be useful in Section 4, for the analysis of a gap function associated with $VQEP$.

Proposition 3.2. *Assume that $f(x^*, \cdot)$ and $g(x^*, \cdot)$ are differentiable at x^* and $f(x^*, x^*) = 0$; If (μ^*, λ^*, x^*) is a saddle point for $L(x^*; \mu, \lambda, y)$ on $(C^* \times D^*) \times X$ then it is a solution of the following system (S)*

$$\begin{cases} \nabla_y L(x^*; \mu, \lambda, y) = 0 \\ \langle \lambda, g(x^*, y)\rangle = 0 \\ g(x^*, y) \in D, \mu \in C^*, \ \lambda \in D^*, \ y \in X. \end{cases} \tag{13}$$

Proof. Suppose that (μ^*, λ^*, x^*) is a saddle point for $L(x^*; \mu, \lambda, y)$ on $(C^* \times D^*) \times X$, that is

$$-\langle \lambda, g(x^*, x^*)\rangle \le -\langle \lambda^*, g(x^*, x^*)\rangle \le \langle \mu^*, f(x^*, y)\rangle - \langle \lambda^*, g(x^*, y)\rangle], \quad \forall(\mu, \lambda, y) \in (C^* \times D^*) \times X.$$

As in the proof of the sufficiency in Proposition 3.1, we can show that $g(x^*, x^*) \in D$.

Computing the first inequality for $\lambda = 0$, we obtain $\langle \lambda^*, g(x^*, x^*)\rangle \le 0$ and, therefore,

$$\langle \lambda^*, g(x^*, x^*)\rangle = 0. \tag{14}$$

The second inequality implies that x^* is a global minimum point of $L(x^*; \mu^*, \lambda^*, y)$, since $f(x^*, x^*) = 0$. Then

$$\nabla_y L(x^*; \mu^*, \lambda^*, x^*) = 0. \tag{15}$$

(15), (14) and the relation $(\mu^*, \lambda^*) \in (C^* \times D^*)$, allow us to complete the proof. □

Proposition 3.3. *Assume that*

1. *$-f(x^*, \cdot)$ is a (cl C)–function, differentiable at x^*, and such that $f(x^*, x^*) = 0$;*
2. *$g(x^*, \cdot)$ is a D–function, differentiable at x^*;*

If (μ^*, λ^*, x^*) is a solution of system (S) then it is a saddle point for $L(x^*; \mu, \lambda, y)$ on $(C^* \times D^*) \times X$.

Proof. Suppose that (μ^*, λ^*, x^*) is a solution of (S). By the assumptions *1* and *2*, it is easy to prove that $L(x^*; \mu^*, \lambda^*, y)$ is a convex function in the variable y, so that $\nabla_y L(x^*; \mu^*, \lambda^*, x^*) = 0$ implies that

$$L(x^*; \mu^*, \lambda^*, x^*) \leq L(x^*; \mu^*, \lambda^*, y), \quad \forall y \in X.$$

Taking into account the complementarity relation $\langle \lambda^*, g(x^*, x^*) \rangle = 0$ and the condition $\lambda \in D^*$, we obtain

$$-\langle \lambda, g(x^*, x^*) \rangle \leq \; -\langle \lambda^*, g(x^*, x^*) \rangle, \quad \forall (\mu, \lambda) \in (C^* \times D^*),$$

and the statement is proved. □

Corollary 3.1. *Suppose that the hypotheses 1 and 2 of Proposition 3.3 hold.*
Then (μ^*, λ^*, x^*) is a saddle point for $L(x^*; \mu, \lambda, y)$ on $(C^* \times D^*) \times X$ iff it is a solution of (S).

Coupling Theorem 2.2 with the results obtained in the present section, it is possible to obtain necessary and sufficient Lagrangian–type optimality conditions for $VQEP$.

Theorem 3.1. *Let $f(x^*, x^*) = 0$, $-f(x^*, \cdot)$ be a (clC)-function, and $g(x^*, \cdot)$ be a D–function.*

1. *Assume that $C := \mathbb{R}^p_+$, $int\ D \neq \emptyset$, and that, for every $i = 1, \ldots, p$, the system $S_i(x^*)$, defined in 1 of Theorem 2.2, is possible; then $x^* \in X$ is a solution of $VQEP$ iff $\exists (\mu^*, \lambda^*) \in (C^* \times D^*)$, $(\mu^*, \lambda^*) \neq 0$, such that (μ^*, λ^*, x^*) is a saddle point for $L(x^*; \mu, \lambda, y)$ on $(C^* \times D^*) \times X$.*

2. *Assume that C is an open convex cone and that the following conditions hold:*

$$int\ \mathcal{E}(x^*) \neq \emptyset, \tag{16}$$

$$0 \in ri\ (g(x^*, X) - D); \tag{17}$$

   then $x^* \in X$ is a solution of $VQEP$ iff $\exists (\mu^*, \lambda^*) \in (C^* \times D^*)$, $(\mu^*, \lambda^*) \neq 0$, such that (μ^*, λ^*, x^*) is a saddle point for $L(x^*; \mu, \lambda, y)$ on $(C^* \times D^*) \times X$.

Proof. *1.* Suppose that x^* is a solution of $VQEP$. Then (3) holds. Since $-f(x^*, \cdot)$ is a (clC)-function and $g(x^*, \cdot)$ is a D–function, then the set $\mathcal{E}(x^*)$ is convex (see Remark 2.1). Therefore $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation. By Proposition 3.1, we have that $\exists (\mu^*, \lambda^*) \in C^* \times D^*$ such that (μ^*, λ^*, x^*) is a saddle point for the Lagrangian function $L(x^*; \mu, \lambda, y)$ on $(C^* \times D^*) \times X$.
Vice versa, let (μ^*, λ^*, x^*) be a saddle point for $L(x^*; \mu, \lambda, y)$. By Proposition 3.1, $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation. Taking into account part *1* of Theorem 2.2, we have that $\mu^* > 0$. Part *i)* of Proposition 2.4 ensures that (3) holds and, therefore, x^* is a solution of $VQEP$.

2. As in the proof of *1*, it can be shown that if x^* is a solution to $VQEP$, then there exists $(\mu^*, \lambda^*) \in C^* \times D^*$ such that (μ^*, λ^*, x^*) is a saddle point for the Lagrangian function $L(x^*; \mu, \lambda, y)$ on $(C^* \times D^*) \times X$. Vice versa, let (μ^*, λ^*, x^*) be a saddle point for $L(x^*; \mu, \lambda, y)$. By Proposition 3.1, $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation, i.e. (5) holds. Condition (16) guarantees that the linear separation is proper.

Since $g(x^*, \cdot)$ is a $D-$function, then the set $g(x^*, X) - D$ is convex, and, by *2* of Theorem 2.2, we obtain that (5) is fulfilled with $\mu^* \neq 0$. By *ii)* of Proposition 2.4 , we have that x^* is a solution to $VQEP$. □

Remark 3.2. Note that, if C is an open convex cone and $int\ D \neq \emptyset$, then (16) is fulfilled. Actually, $int\ \mathcal{H} = C \times int\ D \neq \emptyset$, which implies that $int\ \mathcal{E}(x^*) \neq \emptyset$.

If we further assume that the functions $f(x^*, \cdot)$ and $g(x^*, \cdot)$ are differentiable at x^*, then, by Theorem 3.1 and Corollary 3.1, we obtain a Kuhn-Tucker-type condition for $VQEP$.

Theorem 3.2. *Suppose that the hypotheses 1 and 2 of Proposition 3.3 hold.*

i) Assume that $C := \mathbb{R}^p_+$, int $D \neq \emptyset$, and that, for every $i = 1, \dots, p$, the system $S_i(x^*)$, defined in 1 of Theorem 2.2, is possible; then $x^* \in X$ is a solution of $VQEP$ iff $\exists (\mu^*, \lambda^*) \in (C^* \times D^*)$, $(\mu^*, \lambda^*) \neq 0$, such that (μ^*, λ^*, x^*) is a solution of (S).

ii) Assume that C is an open convex cone and that the conditions (16) and (17) hold;

then $x^* \in X$ is a solution of $VQEP$ iff $\exists (\mu^*, \lambda^*) \in (C^* \times D^*)$, $(\mu^*, \lambda^*) \neq 0$, such that (μ^*, λ^*, x^*) is a solution of (S).

4 A gap function for a vector quasi-equilibrium problem

Let us introduce the definition of gap function associated with $VQEP$, which generalizes those existing in the literature for EP and VEP (see e.g. [19, 3]). Let $K^0 := \{x \in X : x \in K(x)\}$.

Definition 4.1. *A function $\phi : K^0 \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a gap function for $VQEP$ iff*
i) $\phi(x) \geq 0, \quad \forall x \in K^0$;
ii) $\phi(x) = 0$ if and only if x is a solution of $VQEP$.

We remark that a similar definition, where ϕ is required to be non positive, has been introduced in [15] for a generalized $VQEP$.

Consider the following function $\psi : X \longrightarrow \mathbb{R}$:

$$\psi(x) := \min_{(\mu,\lambda) \in S} \sup_{y \in X} [-\langle \mu, f(x,y) \rangle \ + \ \langle \lambda, g(x,y) \rangle],$$

where $S := \{(\mu, \lambda) \in (C^* \times D^*) : \|(\mu, \lambda)\|_s = 1\}$ and $\| \cdot \|_s$ is the s-norm in $\mathbb{R}^{p+m}$.
Let $\Omega := \{x \in K^0 : \psi(x) = 0\}$. The saddle point condition, that characterizes the separation in the image space (see the Proposition 3.1), allows us to prove that $\psi(x)$ is a gap function for $VQEP$.

Theorem 4.1. *Let $f(x, x) = 0, \forall x \in K^0$ and $-f(x^*, \cdot)$ be a (clC)–function, $g(x^*, \cdot)$ be a D–function, for every $x^* \in \Omega$.*

1. *Assume that $C := \mathbb{R}^p_+$, int $D \neq \emptyset$, and that, for every $i = 1, \dots, p$ and $\forall x^* \in \Omega$, the following system is possible*
$$-f_{i^-}(x^*, y) > 0, \quad g(x^*, y) \in int\ D, \quad y \in X; \qquad S_i(x^*)$$
then $\psi(x)$ is a gap function for $VQEP$.

2. *Assume that C is an open convex cone and that, $\forall x^* \in \Omega$, (16) and (17) hold; then $\psi(x)$ is a gap function for $VQEP$.*

Proof. It is easy to prove that $\psi(x) \geq 0, \ \ \forall x \in K^0$; in fact, if $(\mu, \lambda) \in (C^* \times D^*)$, then

$$\langle \mu, f(x, x) \rangle + \langle \lambda, g(x, x) \rangle = \langle \lambda, g(x, x) \rangle \geq 0.$$

1. Suppose that x^* is a solution to $VQEP$. Since $-f(x^*, \cdot)$ is a (clC)–function and $g(x^*, \cdot)$ is a D–function, then the set $\mathcal{E}(x^*)$ is convex (see Remark 2.1). Therefore $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation. Without loss of generality we can suppose that the coefficients of the separating hyperplane $(\mu^*,\lambda^*) \in S$. From Proposition 3.1, we have that (μ^*,λ^*,x^*) is a saddle point for $L(x^*;\mu,\lambda,y) := \langle \mu, f(x^*,y)\rangle - \langle \lambda, g(x^*,y)\rangle$ on $(C^* \times D^*) \times X$ and the saddle value $L(x^*;\mu^*,\lambda^*,x^*) = 0$ (see Remark 3.1). Recalling that the saddle point condition can be characterized by suitable minimax problems [22], we have

$$\min_{(\mu,\lambda)\in C^*\times D^*} \sup_{y\in X}[-\langle \mu, f(x^*,y)\rangle + \langle \lambda, g(x^*,y)\rangle] = L(x^*;\mu^*,\lambda^*,x^*) = 0. \tag{18}$$

Since $(\mu^*,\lambda^*) \in S$, taking into account (18), we obtain that $\psi(x^*) = 0$.

Vice versa, suppose that $\psi(x^*) = 0$. Then $\exists(\mu^*,\lambda^*) \in S$, such that

$$-\langle \mu^*, f(x^*,y)\rangle + \langle \lambda^*, g(x^*,y)\rangle \le 0, \quad \forall y \in X. \tag{19}$$

Hence, $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation. ¿From Theorem 2.2, the possibility of the system $S_i(x^*)$ for $i = 1,\dots,p$, implies that $\mu^* > 0$. By *i)* of Proposition 2.4 , we obtain that x^* is a solution to $VQEP$.

2. As in the proof of *1*, it can be shown that if x^* is a solution to $VQEP$, then $\psi(x^*) = 0$.

Vice versa, suppose that $\psi(x^*) = 0$, so that (19) holds and $\mathcal{E}(x^*)$ and $\mathcal{H}$ admit a linear separation. Condition (16) guarantees that the linear separation is proper.

Since $g(x^*,\cdot)$ is a $D-$function, then the set $g(x^*,X) - D$ is convex, and, by *2* of Theorem 2.2, we obtain that (19) is fulfilled with $\mu^* \neq 0$. By *ii)* of Proposition 2.4 , we have that x^* is a solution to $VQEP$. □

Remark 4.1. We observe that $h_x(\mu,\lambda) := \sup_{y\in X}[\langle \mu, f(x,y)\rangle + \langle \lambda, g(x,y)\rangle]$, being the supremum of a collection of linear functions, is a convex function, so that $\psi(x) = \min_{(\mu,\lambda)\in S} h_x(\mu,\lambda)$ is the optimal value of a parametric problem on a compact set, with a convex objective function.

Theorem 4.1 generalizes Theorem 5.1 of [20] and Theorem 2 of [21]. Next corollary extends the above mentioned results to $QVVI$ and $QMVVI$.

Corollary 4.1. *In the hypotheses of Theorem 4.1, with*

$$f(x,y) := F(x)(y-x) \quad \text{or} \quad f(x,y) := F(y)(y-x),$$

$\psi(x)$ is a gap function for $QVVI$ and $QMVVI$, respectively.

Example 4.1. Let $X := \mathbb{R}$, $C := \mathbb{R}^2_+, D := \mathbb{R}_+$, $f(x,y) := (f_1(x,y), f_2(x,y))$, where

$$f_1(x,y) := xy - x^2, \quad f_2(x,y) := y^2 - x^2,$$

and $g(x,y) := xy - 2x$. In the present case, $K^0 = \{x \in \mathbb{R} : x \le 0 \,\cup\, x \ge 2\}$ and

$$\psi(x) := \min_{(\mu_1,\mu_2,\lambda)\in S} \sup_{y\in\mathbb{R}}[\mu_1(x^2 - xy) + \mu_2(x^2 - y^2) + \lambda(xy - 2x)], \tag{20}$$

where we have set $S := \{(\mu_1,\mu_2,\lambda) \in \mathbb{R}^3_+ : \mu_1 + \mu_2 + \lambda = 1\}$.

Let us compute, in (20), the supremum with respect to $y \in \mathbb{R}$. We obtain

$$\sup_{y\in\mathbb{R}}[\mu_1(x^2 - xy) + \mu_2(x^2 - y^2) + \lambda(xy - 2x)] =$$

$$\begin{cases} \dfrac{x^2(\lambda-\mu_1)^2}{4\mu_2} + (\mu_1+\mu_2)x^2 - 2\lambda x, & \text{if } \mu_2 \neq 0, \\ (\mu_1+\mu_2)x^2 - 2\lambda x, & \text{if } \mu_2 = 0, \lambda = \mu_1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us consider the case where $\mu_2 \neq 0$. We immediately observe that $\psi(x) = 0$ iff $x = 0$ or

$$x = \frac{8\lambda\mu_2}{(\lambda-\mu_1)^2 + 4\mu_1\mu_2 + 4\mu_2^2}. \tag{21}$$

Note that $x^* = 0 \in K^0$, so that it is a solution of $VQEP$ and any $(\mu_1, \mu_2, \lambda) \in S$, with $\mu_2 \neq 0$, is a vector of multipliers associated with x^*.

Let us consider the solutions given by (21). Dividing (21) by μ_2^2 and setting

$$\frac{\lambda}{\mu_2} = k_1, \quad \frac{\mu_1}{\mu_2} = k_2,$$

we obtain

$$x = \frac{8k_1}{(k_1-k_2)^2 + 4k_2 + 4}. \tag{22}$$

By imposing the condition $x \geq 2$, we have

$$4k_1 \geq (k_1-k_2)^2 + 4k_2 + 4$$

which is equivalent to

$$(k_1 - k_2 - 2)^2 \leq 0,$$

that implies $k_1 - k_2 = 2$ or, equivalently,

$$\lambda - \mu_1 - 2\mu_2 = 0.$$

Therefore, the set of multipliers associated with the solution (21) fulfils the system

$$\begin{cases} \lambda + \mu_1 + \mu_2 = 1 \\ \lambda - \mu_1 - 2\mu_2 = 0 \\ \mu_1 \geq 0, \mu_2 > 0, \lambda \geq 0 \end{cases},$$

that is,

$$\lambda = \frac{\mu_2+1}{2}, \quad \mu_1 = \frac{1-3\mu_2}{2}, \quad 0 < \mu_2 \leq \frac{1}{3}.$$

Substituting the previous relations in (21), we obtain $x = 2$, which belongs to the set K^0 and therefore it is a solution of $VQEP$.

Let us consider the case where $\mu_2 = 0, \lambda = \mu_1$. In such a case

$$\sup_{y\in\mathbb{R}} [\mu_1(x^2 - xy) + \mu_2(x^2 - y^2) + \lambda(xy - 2x)] = \mu_1 x(x-2).$$

Therefore, we obtain once more the solutions $x = 0$, $x = 2$ with related multipliers given by $\mu_1 = \lambda = \frac{1}{2}, \quad \mu_2 = 0$.

The gap function ψ that we have analysed in the present section, in general, is not differentiable. Following the line adopted in [4, 24], adding a suitable regularizing term $H(x, y) : X \times X \longrightarrow \mathbb{R}$ to the function $\langle \mu, f(x, y)\rangle + \langle \lambda, g(x, y)\rangle$, it is possible to obtain a directionally differentiable gap function for $VQEP$.

Further extensions can be obtained by means of nonlinear scalarization methods for VEP (see e.g. [3]).

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