St Louis, 1995 Apr 25

Ms. chairperson, Ladies & Gentlemen

T Let me begin with thanking the Conference Organizers for accepting this contribution and also acknowledge the partial financial support of this special interest group of the Italian National Research Council

1 The subject I am going to talk about has grown out of an idea put forward by Prof T. I Seidman, which has to do with parameter identification. A few years ago he stated that "the solution of the ill-posed problems is only an intermediate construct intervening between the available data and the intended application".
Consider for instance the inverse problem of determining position dependent conductivity in the one dimensional, steady state heat equation. Conductivity a is determined from these data, temperature, source term and extra information. This is related to inverting the map F.
The conductivity, together with a new source term and appropriate boundary values for temperatures, appears in a direct (or control) problem the output of which is the result temperature, r
The final goal is to consider the composed map, relating data to results, and bring out the relevant uniqueness and stability features, if any.

2 Since the uniqueness and stability of solutions to the inverse problem play a major role, here is the abstract setting. The Data from which the unknown parameter a has to be determined, come from a map F.
This is a sufficient condition for the continuous invertibility of F:

F shall be continuous; one selects a compact parameter set $P_1$; If this property holds, then $P_1$ is transformed into a compact set $D_1$ and $F$ inverse is also continuous.

Q At this point I can state two basic questions. Namely, in view of the just stated abstract result, one is tempted to neglect state equations of any control system altogether and rely on some general, topological conditions in order to assess the properties of the composed map.
Next, by assuming that state equations carry some information, one wonders what happens to the composite map. In particular are there any circumstances, where considering the composite map is "better" than considering the identification and control maps separately?

C Answers to both questions will be sought with reference to heat equations in one and two spatial dimensions.
This is the notation: $f$ is the source term, $a$ is conductivity, which shall be strictly positive, $u$ is the known temperature (datum in the inverse problem) and $v$ is the resulting temperature from a control problem.
Many years ago Richter provided a preliminary estimate for the composite map as I understand it. He gave this stability estimate relating differences between data (i.e. second potential minus reference potential) to the difference between the second and reference results.

Indeed, this is the kind of estimate one obtains in some cases. However, there are many interesting details, which deserve attention

P1 This is an example, where estimating the stability of the composed map yields a better estimation constant than combining separate estimates.
Uniqueness of conductivity is due to a regular Cauchy problem

for the 1st order ordinary differential equation and the control problem is also supplemented by Cauchy data.
If the data of both problems comply with these constraints, then the following inequality relates differences of derivatives of results to differences of derivatives of data

If the estimates were carried out separately, then combined, one would obtain a larger constant since $k_0$ is greater than 1.

P2 Another case, where the composite map as a whole leads to a better estimate is met when the uniqueness of conductivity comes from an isolated critical point for the temperature data.
For reasons of time I skip the details and move on to the general features of this approach.
Estimates for the composed map compare to the combination of separate estimates as follows.

1- Estimates are of the same type, i.e. set in the same function spaces
2- Composed maps feature cancellation and therefore an improvement in the estimation constants only in a few special cases
3- In other cases results from the two procedures yield different constants in the estimates. The value of said constants depends on the specific data.

B A by product of this investigation is the following hyerarchy of control and identification problems, which I am not going to describe in detail.

B2 The study of composed map has produced the most interesting results in connection with two-dimensional problems.

In particular, two conditions, which yield the uniqueness of conductivity, have been studied.

These two conditions have one feature in common; they provide straight forward formulas for the spatial gradient of conductivity

The corresponding stability estimates, therefore, simplify remarkably.

One condition is uniqueness from transversal flows, which I will not discuss.

The other condition is uniqueness from collinearity

B2 This has been developed out of a paper by Ito and Kunisch, who provide the following uniqueness condition. These are what I call collinearity conditions.

From here, one easily works out this stability estimate, which also affects the gradient of $b-a$

If one seeks for the dual of the collinearity condition i.e., something which applies to the control problem one finds an overposed condition for the result potentials.

If this overposed condition is met, then we have another special case, where the estimate for the composed map simplifies

I get this inequality. In other words I can override one of the two elliptic estimates usually needed when relating the data to the results.

The elliptic estimate is needed here and applies to the gradient of $v$.

A1 So far answers to Question II have been discussed in detail, whereas Question I has been set aside. Let me return to it and conclude my argument.

Suppose that we choose to work with Lemma 1 and select relatively compact data sets to insure the continuity of the inverse map.

Now, the relative compactness of a subset of $L^p$ is equivalent to the following conditions holding simultaneously:

{the uniform $\alpha$-measurability of $g$ in any subset $B \subseteq D$ and the absolute $\alpha$-equicontinuity of the integrals of $g$ over $\mathcal{K}$}.

When one applies these conditions to e.g. the one-dimensional problem one finds that e.g. the set of second derivatives of admissible temperature data must be relatively compact.

On the other hand all estimates above have been obtained by bounding above $p$-norms of second derivatives, a constraint which DOES NOT imply relative compactness.

In other words, stability has been obtained from larger data set than required by this sufficient condition.

Also, best estimation constants have been determined. As a consequence, work at state equations has been rewarding.

Thank You for Your patience.