PROBLEM-BASED LEARNING AND TEACHER TRAINING IN MATHEMATICS: MAKING SENSE OF WORD PROBLEMS

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Abstract

In the literature we can find many examples showing the difficulty of students in solving mathematical problems. In particular, according to many authors, the critical aspect is precisely the ability to "making sense" of the problem to be solved (see [1]).
From our point of view, it is not likely to gain an improvement of students' skills if we do not act adequately on teacher training. A first necessary condition is that teachers must be able to adequately solve the tasks that are proposed to students.
For this reason, in the "Mathematics teaching" courses for the primary school prospective teacher training program at the University of Milano-Bicocca we discuss and promote a PBL (Problem-Based learning) model. These courses combine theoretical lectures with practical activities (for example the didactic pedagogical laboratories described in [2] or [3]).
One of the main topics of such courses is precisely a discussion of the idea of "problem", as opposed to the idea of "exercise" (e.g. see [4]), and of the didactic potential of a teaching practice that makes good use of problems (e.g. see [5]).
At the end of the course, we tested prospective primary school teachers with a classic problem on proportionality combined with a typical example of pseudo-proportional problems, for which the inadequacy of the proportional model should emerge.
We proposed the task to 57 prospective teachers, asking them to give an explicit and detailed resolution of the problems and to carry out an analysis of the teaching potential of such problems for primary school pupils.
In this paper, we present a qualitative analysis of the answers collected. From such an analysis, inconsistencies between the theoretical model described at lectures and the practical application emerge for many prospective teachers. In some cases, they explicitly admit to have adapted their answers to the "didactical contract".
The observations collected allowed us to rethink some aspects of the course, and we will experiment some changes in the next academic year.
Keywords: teacher training, Problem-based learning, mathematics, proportionality

1. SENSE-MAKING AND TEACHER TRAINING

Many researchers show the difficulty of students in solving mathematical problems. Greer [6] highlights some critical issues emerging from such research. The main issue seems to be the "tendency of children to answer school mathematics word problems with apparent disregard for the reality of the situations described by the text of these problems" [6, p. 293]. Researchers impute these failures to what Schoenfeld [7] describes as a "suspension of sense-making": *school* problems are set in a somehow imaginary world and follow rules that does not necessarily have much to do with the real world in which we live. The persistence of this trend is documented by subsequent research, of which [8] provides an accurate summary.
In the literature we can find a batch of problems used to test children and analyze their arguing in problem solving. Here we list just a few (e.g. see [6]).

- There are 125 sheep and 5 dogs in a flock. How old is the shepherd?
- An army bus hold 36 soldiers. If 1128 soldiers are being bussed to their training site, how many buses are needed?
- An athlete's best time to run a mile is 4 minutes and 7 seconds. About how long would it take him to run 3 miles?

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- This year a shop sold 32 umbrellas in February. About how many do you think it will sell altogether in June, July and August this year?
- The flask is being filled from a tap at a constant rate. If the depth of the water is 2.4 cm after 10 seconds, about how deep will it be after 30 seconds? (The flask depicted is conical in shape so that the answer should be more than 7.2 cm.)
- A man wants to have a rope long enough to reach between two poles 12 meters apart, but he only has pieces of rope 1.5 metres long. How many of these pieces of rope would he need to tie together to reach between the poles?
- A cup of milk is added to a cup of popcorn. How many cups of the mixture will result?
- Carl has 5 friends and George has 6 friends. Carl and Georges decide to give a joint parti. They invite all theirs friends. All friends are present. How many friends are at the party?
- What will be the temperature of water in a container, if you pour 1 liter of water at 80° and 1 liter of water at 40° into it?
- If one orchestra can play a symphony in 1 hour, can two orchestras play the symphony in 1/2 hour?
- If 6 cats kill 6 rats in 6 minutes, how many will be needed to kill 100 rats in 50 minutes?
- Mr. Smith the butcher had 26 kg of meat in his shop, and orders 10 kg more. How much meat does he have now?
- Recently, I had a party for 20 people. I took a recipe for 4 people and multiplied each ingredient by 5. Much to my surprise, the recipe was not the same. It tasted different, and the sauce was too watery. What did I do wrong?

Pupils' suspension of sense-making materializes in different ways. Some of these problems (e.g. "the shepherd") do not admit a solution based on straight computation, but pupils provide an answer that is a result of a calculation made up applying imaginative operations. Researchers ascribe this behavior not to some cognitive deficit as such, but rather to the adherence to the *didactical contract* ("meaning the system of implicit norms, rules and expectations that reciprocally evolve between teacher and students.", e.g. see [6, p. 298]). Pupils believe they are expected to provide a numerical answer and somehow they manage to produce one. Some other of these problems imply possible constraint of the reality of the problem context (you cannot *fraction* busses), nevertheless "schoolchildren answer word problems with apparent scarce regard for whether the answers make sense when considered from the viewpoint of the real-world situation verbally described in those problems" (e.g. see [6, p. 294]). Finally, other (e.g. the "athlete") are solved without verification of which might be the most appropriate mathematical modeling. Typically, "direct proportionally is uncritically assumed by a high majority of children as an appropriate way to generate an answer" (e.g. see [6, p. 296]).
Many researchers (e.g. see [7] or [8]) remark that such suspension of sense-making indeed develops in school, as a result of schooling and of the current practice of word problems in school mathematics, and, as trainers for prospective teachers, we are particularly touched by this last remark.
There are two types of questions that we ask ourselves.
Firstly: which methodologies should we promote in our teacher training classes? We think that Problem-Based Learning (PBL) is a methodology that helps the creation of knowledge through mathematical discussion and as such is particularly suitable for the teaching/learning of mathematics. In a PBL activity teachers' key role is to promote problem solving activities by selecting suitable problems (e.g. see [2], [9] and [3]). Typically, a PBL activity is organized according to these steps (e.g. see [2], [9], [10] or [11]):

- pupils are given a problem;
- they discuss the problem and/or work on the problem in small groups, collecting information useful to solve the problem;
- all the pupils gather to compare findings and/or discuss conclusions; new problems could arise from this discussion, in this case
- pupils go back to work on the new problems, and the cycle starts again.

The validity of PBL as an instructional method has been widely studied and many researchers confirm its effectiveness in many fields (for a review e.g. see [12], [13]). In such a setup the problem given to the students to solve has a key role: the more the problem is "ill-structured", the more the activity can be effective [13, p. 15]. The validation of the given answers is a key step in the process and references to real world situation are an invaluable asset in order to promote a mathematical discussion.
Secondly: is the mathematical knowledge of the future teachers suitable for promoting a healthy attitude towards mathematics and mathematical problems in their pupils? How would prospective teachers answer to the tasks designed for school children? In the literature we can find many studies on problem-solving reasoning, mostly focusing on elementary school students. Only a few studies exist

with prospective teachers and teachers (e.g. see [14]). We have already addressed the problem in the past: in [15] we described the assignment of the “athlete” problem to a group of 173 prospective primary school teachers. Of these, a share of 113 solved the problem just applying a proportional model, 46 choose not to give any answer at all (this was an allowed option) and only 2 clearly stated that the problem has no viable solution. When questioned “did you ever try running 100 metres as fast as possible?”, many answered “we were used to reason like this in the physics assessments” or “we thought that we should refer to an ideal situation and not to a real one”.
As this final example shows, the two question we posed are strictly intertwined in our context of teacher training. On the one hand prospective teachers claim to be willing to adopt PBL and active discussion methodologies in their future work as primary school teachers, on the other hand when given “real-world” problems, they solve them without taking into account any possible constraints given by the reality of the problem context.

2. CHALLENGING PROSPECTIVE TEACHERS

The experience we describe refers to the training program for prospective primary school teacher at the University of Milano-Bicocca. Since 2011, prospective teachers have to attend a five year long combined Bachelor and Master degree. In the Milano-Bicocca implementation, this degree program includes three modules in mathematics (for a detailed description e.g. see [2], [9] or [3]). We tested prospective teachers with a pseudo-proportional problem within the *Mathematics teaching* module. In such a module lectures focus mainly on methodologies (active learning and PBL) and teaching examples (analysis of problems); moreover, students are required to attend a compulsory laboratory in which a further analysis of teaching methodologies is carried on (prospective teachers experience *teaching through a lab*, for a detailed description of one of such labs see [2]). We build the task mixing proportional and pseudo-proportional tasks using the “recipe” example, that is the last example in the sample list in section 1.. As pointed out in [6], recipe conversions are a “prototypical context” for proportional reasoning, but they are not as straightforward as they might seem. According to [6],

> “Doubling or halving the ingredients in a recipe sometimes turns out fine. However, this method doesn’t work with large increases, especially for baked goods.”

But as recipes are concerned, an extra difficulty might arise: some of the ingredient might be integer quantities, for which a proportional model does not apply. The questions we proposed were the following:

> Cinzia is helping her grandpa to cook his secret delicacies. But in order to taste them she has to make so many computations!
>
> - A good jam must contain 60% of its weight in sugar. If grandpa wants to make 160 grams of jam, how much sugar does he need?
> - In order to prepare the ricotta cheese cream you need to mix $2/3$ of ricotta cheese and $1/3$ of sugar. Grandpa uses 120 grams of ricotta cheese. How many grams of sugar does he need?
> - For the lemon cream, the recipe for 3 people requires an egg, 150 g of sugar, 80 grams of flour, half a litre of milk and the zest of a lemon. There are 5 people at home, how the ingredients should be dosed in order to make the cream for everyone?
>
> Can you help her?

The first question (we will refer to it as “Question 1” in the following) is a routine calculation with fractions, just keeping in mind that percentages are in fact fractions (a remark that sometimes puzzles primary school prospective teachers!).
The second question (“Question 2” in the following) concerns the inverse problem with respect to the first and is taken from the INVALSI (the Italian National Evaluation Service) survey database (more precisely it is among the question assigned to fifth graders in the AS 2018-19). Such question was in the lecture material but was not explicitly discussed with the prospective teachers at lectures.
In the third question (“Question 3” in the following), the fact that eggs are not easily fractionable (we actually should assume the *indivisibility* of an egg) poses the problem whether a proportional modeling is accurate. We wanted to test whether a suspension of sense-making occurred with prospective teachers, or whether they asked themselves the question “does it make sense, in a real context, to have 5/6 of an egg?”. Finally, we observe that Question 3 has been designed in order to stimulate divergent solutions.

Question 1		Question 2		Question 3				
0	1	0	1	1	2	3	4	5
2	55	8	49	0	7	36	11	3
3%	97%	14%	86 %	–	12%	63%	20%	5%

Table 1: Categorized answers to Questions 1, 2 and 3

E.g., in a real-life context, as approximation are required, one would probably just double the doses of all ingredients and make the lemon cream for 6 people ("doubling or halving the ingredients in a recipe sometimes turns out fine"). The question indeed does not ask for the cream for *exactly* 5 people but requires to have cream "for everyone".

2.1 Data collection

The text of the problem was assigned to 57 prospective primary school teachers. More precisely they were asked to

- give a detailed resolution of the problem;
- analyze the didactic potential of the problem in order to use it in a PBL laboratory session.

In order to carry out the required didactic analysis, a few hints were given to the prospective teachers:

- What mathematical knowledge should the children you suppose to assign the problem possess?
- What is the "mathematical" topic implied by the problem? How can you insert such a problem in a didactic project about such a topic? (E.g. What mathematical knowledge is required for teachers? Which kind of knowledge you can think the problem can activate in pupils? . . .)
- Describe a PBL session using such a problem, so that its role as "problem" as opposed to "exercise drill" emerges (planning of the activity, presentation of the problem text, teacher's actions, ...).

Although "resolution of the problem" and "didactical analysis of the problem" were two separate questions, prospective teachers were allowed to make connections between the two, and we expected that the explicit reference to PBL somehow influenced the problem solution they decided to include in their essay.

2.2 Analysis

All the written reply by the 57 prospective teachers were collected and analysed.
We used a mixed method approach, integrating quantitative data with a qualitative analysis.
We used Question 1 and Question 2 as indicators of the ability to perform calculation with a proportional model. Therefore, for each of such questions we assigned "0" for a wrong numerical answer and "1" for a correct numerical answer.
With respect to the Question 3, we categorized answers according to the following schema:

1. does not use a proportional modeling;
2. uses a proportional modeling, calculating the doses for 6 people;
3. uses a proportional modeling, calculating the doses for 5 people, without taking into account the nature of the ingredients (the indivisibility of an egg) (expression such as "5/3 of an egg", "slightly more than an egg and a half" are used);
4. uses a proportional modeling, calculating the doses for 5 people, with some approximations for eggs and, sometimes, for lemons;
5. uses a proportional model, calculating the doses for 5 people, with approximations for all the ingredients ("about 1 liter of milk").

The number of answers given by the prospective teachers are shown in Table 1. Computation with percentages (Question 1) is not an issue, while the reverse problem (Question 2) registers a noticeable percentage of failures.
With respect to Question 3, 63% of the prospective teachers did not take into account the real-world setup of the problem and answered on the lines of "you need 1 egg and 2/3 of an egg to prepare the cream for 5 people". We impute these kinds of answers to a suspension of sense-making, analogous to the one researchers describe for pupils. About 12% of the prospective teachers choose the "divergent solution", the one that we think shows a higher level of mathematical creativity, calculating the doses for

004	014	102	103	104	105	112	113	114	115
1	1	2	3	1	1	5	33	8	2

Table 2: Frequencies of profiles

6 people. Finally, about 20% try to deal somehow with the problem of fractioning eggs and lemons ("a big egg and a smaller egg", "the zest of a lemon does not mean exactly *one* lemon").
In order to try to show possible correlations between the ability to do the actual computation and the choice of a suitable model for Question 3, we studied the profiles given by the concatenation of the answer to the 3 questions (e.g. "102" means that the answer to Question 1 was categorized "1", the answer to Question 2 was categorized "0" and the answer to Question 2 was categorized "2"). The frequencies of the occurrences of such profiles are shown in Table 2. The number of prospective teachers involved does not allow us to carry on a proper statistical analysis. We can anyway point out that among the 9 who gave a wrong answer in Question 1 and/or Question 2 (profiles "004", "014", "102", "103", "104" and "105") the majority tend to question the proportional model and give to Question 3 an answer categorized "2" and "4".

2.2.1 Prospective teachers' arguments

As our aim was to investigate the reasoning behind the given answer, trying to understand the causes of a possible suspension of sense-making, we fully analysed the written essays handed in by the prospective teachers.
Many of the students did actually state that the indivisibility of the egg is an issue, nevertheless their arguments sound like "the problems asks for the recipe for 5 people so we need to carry out all the computation, no matter what". Some explicitly say something along such lines:

> "if the doses were asked for 6 people, then I could have just doubled everything, but ...";
>
> "in a sense-making path it would had made sense just to double the doses, but ...";
>
> "as the previous questions dealt with fractions I understood that we were not supposed to make approximations, but we were supposed to find the exact doses for 5 people"

(the latter implicitly appealing to a didactical contract). Some seems to think that reverting to the recipe for 6 people is a kind of "cheating".
There are various creative ways to deal with the indivisibility of the eggs:

> "by abstraction we need 1 whole egg and 2/3 of an other";
>
> "the teacher has to tell the pupils that the weight of one egg is between 60 and 80 grams";
>
> "a whole egg and an egg white or yolk from another egg";
>
> "a big egg and another one, about half its size".

Only a few can see the task as an opportunity to discuss and argue about possible choices they will have to make once they become teachers:

> "To overcome all these obstacles, I decide to make the cream for 6 people: by doing so, all the 5 people in the family have their share of lemon cream".

We interviewed some of the teachers that did not complete the written didactic analysis and when asked the question "as a teacher which answer would you mark as right?" they remain speechless.
What puzzled us when we analized the didactic analysis carried out by the prospective teachers is that many of the ones who gave to Question 3 an answer categorized "3" describe promoting sense-making as the aim of their proposal:

> "mathematics is not a series of abstract and meaningless formulas to be learned by hearth";
>
> "through these kinds of problems the discipline acquire a sense";
>
> "use this problem in a sense-making context so to allow for an actual building of an argument and not just the application of a rule".

Future lectures must deal with these apparent contradictions, that must be well analyzed and discussed with the prospective teachers.

3. CONCLUSIONS

Our experience leads us to conclude that a suspension of sense-making occurs in problem solving for prospective teachers as well as for primary school pupils. For prospective teachers too we attribute this suspension of sense-making to an implicit comply with the didactical contract and not to some cognitive deficit. In fact, we have been able to observe that prospective teachers with difficulties in carrying out the computations showed a greater propensity to look for divergent solutions, involving the execution of simpler calculations.
Some prospective teachers, however, show contradictions between their presumed didactic approach to teaching and the actual solution they provide to the given tasks, and this is an aspect on which a careful analysis is necessary in the future.

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