Research Article

Csaba Farkas, Alessio Fiscella and Patrick Winkert*

Singular Finsler Double Phase Problems with Nonlinear Boundary Condition

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Abstract: In this paper, we study a singular Finsler double phase problem with a nonlinear boundary condition and perturbations that have a type of critical growth, even on the boundary. Based on variational methods in combination with truncation techniques, we prove the existence of at least one weak solution for this problem under very general assumptions. Even in the case when the Finsler manifold reduces to the Euclidean norm, our work is the first one dealing with a singular double phase problem and nonlinear boundary condition.

Keywords: Anisotropic Double Phase Operator, Critical Type Exponent, Existence Results, Minkowski Space, Nonlinear Boundary Condition, Singular Problems

MSC 2010: 35J15, 35J62, 58B20, 58J60

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1 Introduction

In this paper, we consider singular Finsler double phase problems with nonlinear boundary condition. The Finsler double phase operator is defined by

$$\operatorname{div}(A(u)) := \operatorname{div}(F^{p-1}(\nabla u)\nabla F(\nabla u) + \mu(x)F^{q-1}(\nabla u)\nabla F(\nabla u)) \tag{1.1}$$

for $u \in W^{1,\mathcal{H}}(\Omega)$ with $W^{1,\mathcal{H}}(\Omega)$ being the Musielak–Orlicz–Sobolev space and F being a positive homogeneous function such that $F \in C^\infty(\mathbb{R}^N \setminus \{0\})$ and the Hessian matrix $\nabla^2(F^2/2)(x)$ is positive definite for all $x \neq 0$. Furthermore, μ is a nonnegative bounded function and $1 < p < q < N$. If F coincides with the Euclidean norm, that is, $F(\xi) = (\sum_{i=1}^N |\xi_i|^2)^{1/2}$ for $\xi \in \mathbb{R}^N$, then (1.1) reduces to the usual double phase operator given by

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u). \tag{1.2}$$

Also, if $\mu \equiv 0$ or $\inf_{\overline{\Omega}} \mu > 0$, then (1.2) (similarly (1.1)) reduces to the (Finsler) p-Laplacian or the (Finsler) (q,p)-Laplacian, respectively. The study of such operators and corresponding energy functionals are motivated by physical phenomena; see, for example, the work of Zhikov [59] (see also the monograph of Zhikov, Kozlov and Oleĭnik [37]) in order to describe models for strongly anisotropic materials. Related functionals to (1.2) have been studied intensively with respect to regularity properties of local minimizers; see the works

***Corresponding author: Patrick Winkert,** Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany, e-mail: winkert@math.tu-berlin.de. https://orcid.org/0000-0003-0320-7026
Csaba Farkas, Department of Mathematics and Computer Science, Sapientia Hungarian University of Transylvania, Târgu Mureş, Romania, e-mail: farkascs@ms.sapientia.ro
Alessio Fiscella, Departamento de Matemática, Universidade Estadual de Campinas, IMECC, Rua Sérgio Buarque de Holanda, 651, Campinas, SP CEP 13083-859, Brazil, e-mail: fiscella@ime.unicamp.br. https://orcid.org/0000-0001-6281-4040

of Baroni, Colombo and Mingione [4–6], Baroni, Kuusi and Mingione [7], Byun and Oh [10], Colombo and Mingione [14, 15], De Filippis and Palatucci [18], Marcellini [41, 42], Ok [43, 44], Ragusa and Tachikawa [52] and the references therein.

On the other hand, the minimization of the functional $E_F\colon H_0^1(\Omega) \to \mathbb{R}$ defined by

$$E_F(u) = \int_\Omega F^2(\nabla u)\,\mathrm{d}x \quad \text{for } u \in H_0^1(\Omega),$$

under certain constraints on perimeter or volume occurs in many subjects of mathematical physics. Here the minimizer corresponds to an optimal shape (or configuration) of anisotropic tension-surface. The minimization of the functional E_F describes, for example, the specific polyhedral shape of crystal structures in solid crystals with sufficiently small grains, as shown by Dinghas [22] and Taylor [54]. It is clear that E_F is the energy functional to the Finsler Laplacian given by

$$\Delta_F u = \operatorname{div}(F(\nabla u)\nabla F(\nabla u)). \tag{1.3}$$

The Finsler Laplacian, given in (1.3), has been studied by several authors in the last decade. We refer, for example, to the papers of Cianchi and Salani [12] and Wang and Xia [55], both dealing with the Serrin-type overdetermined anisotropic problem, or to Farkas, Fodor and Kristály [27] who studied a sublinear Dirichlet problem of this type. Related works concerning anisotropic phenomena can be found in the works of Bellettini and Paolini [8], Belloni, Ferone and Kawohl [9], Della Pietra and Gavitone [20], Della Pietra, di Blasio and Gavitone [19], Della Pietra, Gavitone and Piscitelli [21], Farkas [26], Farkas, Kristály and Varga [28], Ferone and Kawohl [30] and the references therein.

In this paper, we combine the effect of a Finsler manifold and a double phase operator along with a singular term and a nonlinear boundary condition. More precisely, we study the following problem:

$$\begin{cases} -\operatorname{div}(\mathcal{A}(u)) + u^{p-1} + \mu(x)u^{q-1} = u^{p^*-1} + \lambda(u^{\gamma-1} + g_1(x,u)) & \text{in } \Omega, \\ \mathcal{A}(u)\cdot\nu = u^{p_*-1} + g_2(x,u) & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$, $N \ge 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$, $\nu(x)$ is the outer unit normal of Ω at the point $x \in \partial\Omega$, λ is a positive parameter and the following assumptions hold true:

(H1) $0 < \gamma < 1$ and

$$1 < p < q < N, \quad q < p^*, \quad 0 \le \mu(\cdot) \in L^\infty(\Omega). \tag{1.5}$$

(H2) $g_1\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $g_2\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and there exist $1 < \theta_1 < p \le \nu_1 < p^*$, $p < \nu_2 < p_*$ as well as nonnegative constants a_1, a_2, b_1 such that

$$\begin{aligned} g_1(x,s) &\le a_1 s^{\nu_1-1} + b_1 s^{\theta_1-1} && \text{for a.a. } x \in \Omega \text{ and for all } s \ge 0, \\ g_2(x,s) &\le a_2 s^{\nu_2-1} && \text{for a.a. } x \in \partial\Omega \text{ and for all } s \ge 0, \end{aligned}$$

where p^* and p_* are the critical exponents to p given by

$$p^* := \frac{Np}{N-p} \quad \text{and} \quad p_* := \frac{(N-1)p}{N-p}. \tag{1.6}$$

(H3) The function $F\colon \mathbb{R}^N \to [0,\infty)$ is a positively homogeneous Minkowski norm with finite reversibility

$$r_F = \max_{w \ne 0} \frac{F(-w)}{F(w)}.$$

Because we are looking for positive solutions and hypothesis (H2) concerns the positive semiaxis $\mathbb{R}_+ = [0,\infty)$, without any loss of generality, we may assume that $g_1(x,s) = g_2(x,s) = 0$ for all $s \le 0$ and for a.a. $x \in \Omega$ or $x \in \partial\Omega$, respectively. Moreover, note that we always have $r_F \ge 1$; see for example Farkas, Kristály and Varga [28]. It is clear that the Euclidean norm has finite reversibility. Finally, we observe that (1.5) implies that $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^q(\Omega)$ compactly, as shown in Section 2.

Definition 1.1. A function $u \in W^{1,\mathcal{H}}(\Omega)$ is called a weak solution of problem (1.4) if $u^{\gamma-1}\varphi \in L^1(\Omega)$, $u > 0$ for a.a. $x \in \Omega$ and if

$$\int_\Omega (F^{p-1}(\nabla u)\nabla F(\nabla u) + \mu(x)F^{q-1}(\nabla u)\nabla F(\nabla u)) \cdot \nabla\varphi \,\mathrm{d}x + \int_\Omega u^{p-1}\varphi \,\mathrm{d}x + \int_\Omega \mu(x)u^{q-1}\varphi \,\mathrm{d}x$$
$$= \int_\Omega u^{p^*-1}\varphi \,\mathrm{d}x + \lambda \int_\Omega (u^{\gamma-1} + g_1(x,u))\varphi \,\mathrm{d}x + \int_{\partial\Omega} (u^{p_*-1} + g_2(x,u))\varphi \,\mathrm{d}\sigma$$

is satisfied for all $\varphi \in W^{1,\mathcal{H}}(\Omega)$.

From hypotheses (H1)–(H3), we know that the definition of a weak solution is well defined.

The main result in this paper is the following theorem.

Theorem 1.2. *Let hypotheses* (H1)–(H3) *be satisfied. Then there exists $\lambda_* > 0$ such that for every $\lambda \in (0, \lambda_*)$ problem* (1.4) *has a nontrivial weak solution.*

To the best of our knowledge, this is the first work on a singular double phase problem with nonlinear boundary condition even in the Euclidean case, that is, when $F(\xi) = (\sum_{i=1}^N |\xi_i|^2)^{1/2}$ for $\xi \in \mathbb{R}^N$. The novelty of our paper is not only due to the combination of the Finsler double phase operator with a singular term and nonlinear boundary condition. Indeed, in (1.4) we also deal with a type of critical Sobolev nonlinearities, even on the boundary, related to the lower exponent p, as explained in (1.6). Such critical terms make the study of compactness of the energy functional related to (1.4) more intriguing, since the embeddings $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$ are not compact. We overcome these difficulties with a local analysis on a suitable closed convex subset of $W^{1,\mathcal{H}}(\Omega)$ combined with a truncation argument.

We point out that p^* and p_* are not the critical exponents to the space $W^{1,\mathcal{H}}(\Omega)$. Indeed, from Fan [24] we know that $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{\mathcal{H}_*}(\Omega)$ is continuous while $\mathcal{H}_*$ is the Sobolev conjugate function of $\mathcal{H}$; see also Crespo-Blanco, Gasiński, Harjulehto and Winkert [16, Definition 2.18 and Proposition 2.18]. So far it is not known how $\mathcal{H}_*$ explicitly looks like in the double phase setting. For the moment, p^* and p_* seem to be the best exponents (probably not optimal) and only continuous (in general noncompact) embeddings from $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$ are available. So we call it "types of critical growth".

For singular double phase problems with Dirichlet boundary condition there exists only a few works. Recently, Liu, Dai, Papageorgiou and Winkert [40] studied the singular problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) = a(x)u^{-\gamma} + \lambda u^{r-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Based on the fibering method along with the Nehari manifold, the existence of at least two weak solutions with different energy sign is shown; see also [17] for the corresponding Neumann problem. Furthermore, under a different treatment, Chen, Ge, Wen and Cao [11] considered problems of type (1.7) and proved the existence of a weak solution having negative energy. Finally, the existence of at least one weak solution to the singular problem

$$\begin{aligned} -\operatorname{div}(A(u)) &= u^{p^*-1} + \lambda(u^{\gamma-1} + g(u)) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

has been shown by the first and the third author in [29]. The current paper can be seen as a nontrivial extension of the one in [29] to the case of a nonlinear boundary condition including type of critical growth. In particular, we are able to cover the situation when $1 < p < 2$ and/or $1 < q < 2$, which has not been considered in [29] where $2 \le p < q$.

Also, for the p-Laplacian or the (q,p)-Laplacian only a few works exist involving singular terms and Neumann/Robin boundary conditions. We refer to Papageorgiou, Rădulescu and Repovš [47, 48] for singular homogeneous Neumann p-Laplace problems and for singular Robin (q,p)-Laplacian problems, respectively. Existence results for singular Neumann–Laplace problems have been obtained by Lei [38] based on variational and perturbation methods.

Finally, the reader can find existence results for double phase problems without singular term in the papers of Colasuonno and Squassina [13], El Manouni, Marino and Winkert [23], Fiscella [31], Fiscella and Pinamonti [32], Gasiński and Papageorgiou [33], Gasiński and Winkert [34–36], Liu and Dai [39], Papageorgiou, Rădulescu and Repovš [46], Perera and Squassina [51], Zeng, Bai, Gasiński and Winkert [56, 58] and the references therein. For related works dealing with certain types of double phase problems, we refer to the works of Bahrouni, Rădulescu and Winkert [1], Barletta and Tornatore [3], Faraci and Farkas [25], Papageorgiou, Rădulescu and Repovš [45], Papageorgiou and Winkert [50] and Zeng, Bai, Gasiński and Winkert [57].

2 Preliminaries

In this section, we are going to mention the main facts about the Minkowski space $(\mathbb{R}^N, F)$ and the properties about Musielak–Orlicz–Sobolev spaces.

To this end, let $F\colon \mathbb{R}^N \to [0,\infty)$ be a positively homogeneous Minkowski norm, that is, F is a positive homogeneous function such that $F \in C^\infty(\mathbb{R}^N \setminus \{0\})$ and the Hessian matrix $\nabla^2(F^2/2)(x)$ is positive definite for all $x \neq 0$. We point out that the pair $(\mathbb{R}^N, F)$ is the simplest not necessarily reversible Finsler manifold whose flag curvature is identically zero, the geodesics are straight lines and the intrinsic distance between two points $x, y \in \mathbb{R}^N$ is given by

$$d_F(x,y) = F(y-x).$$

The pair $(\mathbb{R}^N, d_F)$ is a quasi-metric space and in general it holds $d_F(x,y) \neq d_F(y,x)$.

The so-called Randers metric is a typical example for a Minkowski norm with finite reversibility, which is given by

$$F(x) = \sqrt{\langle Ax, x\rangle} + \langle b, x\rangle,$$

where A is a positive definite and symmetric $(N \times N)$-type matrix and $b = (b_i) \in \mathbb{R}^N$ is a fixed vector such that $\sqrt{\langle A^{-1}b, b\rangle} < 1$. Note that

$$r_F = \frac{1 + \sqrt{\langle A^{-1}b, b\rangle}}{1 - \sqrt{\langle A^{-1}b, b\rangle}}.$$

The pair $(\mathbb{R}^N, F)$ is often called Randers space which describes the electromagnetic field of the physical space-time in general relativity; see Randers [53]. They are deduced as the solution of the Zermelo navigation problem.

In the next proposition we recall some basic properties of F; see Bao, Chern and Shen [2, Section 1.2].

Proposition 2.1. *Let $F\colon \mathbb{R}^N \to [0,\infty)$ be a positively homogeneous Minkowski norm. Then the following assertions hold true:*
(i) *Positivity: $F(x) > 0$ for all $x \neq 0$.*
(ii) *Convexity: F and F^2 are strictly convex.*
(iii) *Euler's theorem: $x \cdot \nabla F(x) = F(x)$ and*

$$\nabla^2(F^2/2)(x)x \cdot x = F^2(x) \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}.$$

(iv) *Homogeneity: $\nabla F(tx) = \nabla F(x)$ and*

$$\nabla^2 F^2(tx) = \nabla^2 F^2(x) \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\} \text{ and for all } t > 0.$$

Furthermore, $L^r(\Omega)$ and $L^r(\Omega;\mathbb{R}^N)$ stand for the usual Lebesgue spaces endowed with the norm $\|\cdot\|_r$ for $1 \leq r < \infty$. The corresponding Sobolev spaces are denoted by $W^{1,r}(\Omega)$ and $W^{1,r}_0(\Omega)$ equipped with the norms

$$\|u\|_{1,r,F} = \|F(\nabla u)\|_r + \|u\|_r \quad \text{and} \quad \|u\|_{1,r,0,F} = \|F(\nabla u)\|_r,$$

respectively.

On the boundary $\partial\Omega$ of Ω, we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ and denote by $L^r(\partial\Omega)$ the boundary Lebesgue space with norm $\|\cdot\|_{r,\partial\Omega}$. We know that the trace mapping

$$W^{1,r}(\Omega) \to L^{\tilde{r}}(\partial\Omega)$$

is compact for $\tilde{r} < r_*$ and continuous for $\tilde{r} = r_*$, where r_* is the critical exponent of r on the boundary given by

$$r_* = \begin{cases} \dfrac{(N-1)r}{N-r} & \text{if } r < N, \\ \text{any } \ell \in (r, \infty) & \text{if } r \geq N. \end{cases}$$

For simplification, we will avoid the notation of the trace operator throughout the paper.

Let us now introduce the Musielak–Orlicz–Sobolev spaces. For this purpose, let $\mathcal{H}\colon \Omega \times [0,\infty) \to [0,\infty)$ be the function defined by

$$(x,t) \mapsto t^p + \mu(x)t^q,$$

where (1.5) is satisfied. Then the Musielak–Orlicz space $L^{\mathcal{H}}(\Omega)$ is defined by

$$L^{\mathcal{H}}(\Omega) = \{u \mid u\colon \Omega \to \mathbb{R} \text{ is measurable and } \rho_{\mathcal{H}}(u) < \infty\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\Big\{\tau > 0 : \rho_{\mathcal{H}}\Big(\frac{u}{\tau}\Big) \leq 1\Big\},$$

where the modular function $\rho_{\mathcal{H}}\colon L^{\mathcal{H}}(\Omega) \to \mathbb{R}$ is given by

$$\rho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega (|u|^p + \mu(x)|u|^q)\,\mathrm{d}x.$$

From Colasuonno and Squassina [13, Proposition 2.14], we know that the space $L^{\mathcal{H}}(\Omega)$ is a reflexive Banach space.

Furthermore, we define the seminormed space

$$L^q_\mu(\Omega) = \Big\{u \mid u\colon \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega \mu(x)|u|^q\,\mathrm{d}x < \infty\Big\},$$

which is endowed with the seminorm

$$\|u\|_{q,\mu} = \Big(\int_\Omega \mu(x)|u|^q\,\mathrm{d}x\Big)^{\frac{1}{q}}.$$

Similarly, we define $L^q_\mu(\Omega;\mathbb{R}^N)$ with the seminorm $\|F(\,\cdot\,)\|_{q,\mu}$.

The Musielak–Orlicz–Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) = \{u \in L^{\mathcal{H}}(\Omega) : F(\nabla u) \in L^{\mathcal{H}}(\Omega)\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H},F} = \|F(\nabla u)\|_{\mathcal{H}} + \|u\|_{\mathcal{H}}.$$

Finally, we mention the main embedding results between Musielak–Orlicz–Sobolev spaces and usual Lebesgue and Sobolev spaces. We refer to Gasiński and Winkert [36, Proposition 2.2] or Crespo-Blanco, Gasiński, Harjulehto and Winkert [16, Proposition 2.17].

Proposition 2.2. *Let* (1.5) *be satisfied and let p^* and p_* be the critical exponents to p; see* (1.6). *Then the following embeddings hold:*

(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *and* $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ *are continuous for all* $r \in [1,p]$.
(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *is continuous for all* $r \in [1,p^*]$ *and compact for all* $r \in [1,p^*)$.
(iii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\partial\Omega)$ *is continuous for all* $r \in [1,p_*]$ *and compact for all* $r \in [1,p_*)$.
(iv) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^q_\mu(\Omega)$ *is continuous.*
(v) $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is continuous.*

Let $B\colon W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$ be the nonlinear operator defined by

$$\langle B(u), \varphi\rangle_{\mathcal{H},F} := \int_\Omega (F^{p-1}(\nabla u)\nabla F(\nabla u) + \mu(x)F^{q-1}(\nabla u)\nabla F(\nabla u)) \cdot \nabla\varphi \,\mathrm{d}x, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{H},F}$ is the duality pairing between $W^{1,\mathcal{H}}(\Omega)$ and its dual space $W^{1,\mathcal{H}}(\Omega)^*$. The operator

$$B\colon W^{1,\mathcal{H}}(\Omega) \to W^{1,\mathcal{H}}(\Omega)^*$$

has the following properties (see Crespo-Blanco, Gasiński, Harjulehto and Winkert [16, Proposition 3.4 (ii)]) by taking the properties of F into account.

Proposition 2.3. *The operator B defined by* (2.1) *is bounded, continuous and monotone (hence maximal monotone).*

3 Proof of the Main Result

Let $J_\lambda\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ be the functional given by

$$\begin{aligned} J_\lambda(u) = {} & \frac{1}{p}\|F(\nabla u)\|_p^p + \frac{1}{q}\|F(\nabla u)\|_{q,\mu}^q + \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|u\|_{q,\mu}^q - \frac{1}{p^*}\|u_+\|_{p^*}^{p^*} \\ & - \frac{\lambda}{\gamma}\int_\Omega (u_+)^\gamma \,\mathrm{d}x - \lambda\int_\Omega G_1(x,u_+)\,\mathrm{d}x - \frac{1}{p_*}\|u_+\|_{p_*,\partial\Omega}^{p_*} - \int_{\partial\Omega} G_2(x,u_+)\,\mathrm{d}\sigma, \end{aligned}$$

where $u_\pm = \max(\pm u, 0)$ and

$$G_1(x,s) = \int_0^s g_1(x,t)\,\mathrm{d}t \quad \text{as well as} \quad G_2(x,s) = \int_0^s g_2(x,t)\,\mathrm{d}t.$$

Due to the presence of the singular term, it is easy to see that J_λ is not C^1.

Throughout the paper, we denote by c_{p^*} and c_{p_*} the inverses of the Sobolev embedding constants of $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$, respectively. This means, in particular,

$$(c_{p^*})^{-1} = \inf_{\substack{u\in W^{1,p}(\Omega),\\ u\neq 0}} \frac{\|u\|_{1,p,F}}{\|u\|_{p^*}} \quad \text{and} \quad (c_{p_*})^{-1} = \inf_{\substack{u\in W^{1,p}(\Omega),\\ u\neq 0}} \frac{\|u\|_{1,p,F}}{\|u\|_{p_*,\partial\Omega}}. \tag{3.1}$$

Moreover, we define the function $\Psi\colon (0,\infty) \to \mathbb{R}$ given by

$$\Psi(s) := \frac{1}{p2^{p-1}r_F^p} - \frac{2^{p^*-1}c_{p^*}^{p^*}}{p^*}s^{p^*-p} - \frac{2^{p_*-1}c_{p_*}^{p_*}}{p_*}s^{p_*-p}, \tag{3.2}$$

where $r_F = \max_{w\neq 0}\frac{F(-w)}{F(w)}$ is finite by (H3). Since Ψ is strictly decreasing, we know there exists a unique $\varrho^* > 0$ such that $\Psi(\varrho^*) = 0$. In addition, $\Psi(s) \geq 0$ for all $s \in (0,\varrho^*)$.

We start with the study of the functional $I\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{p}\|F(\nabla u)\|_p^p + \frac{1}{q}\|F(\nabla u)\|_{q,\mu}^q + \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|u\|_{q,\mu}^q - \frac{1}{p^*}\|u\|_{p^*}^{p^*} - \frac{1}{p_*}\|u\|_{p_*,\partial\Omega}^{p_*}.$$

The next proposition shows the sequentially weakly lower semicontinuity of the functional

$$I\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$$

on closed convex subsets of $W^{1,\mathcal{H}}(\Omega)$.

Proposition 3.1. *Let hypotheses* (H1)–(H3) *be satisfied. For every $\varrho \in (0,\varrho^*)$ the restriction of I to the closed convex set B_ϱ, which is given by*

$$B_\varrho := \{u \in W^{1,\mathcal{H}}(\Omega) : \|u\|_{1,p,F} \leq \varrho\},$$

is sequentially weakly lower semicontinuous.

Proof. Let $\varrho \in (0,\varrho^*)$ and let $\{u_n\}_{n\in\mathbb{N}} \subseteq B_\varrho$ be such that $u_n \rightharpoonup u$ in $W^{1,\mathcal{H}}(\Omega)$. We are going to prove that

$$\liminf_{n\to\infty}(I(u_n)-I(u)) \geq 0.$$

For $\kappa \geq 1$ we consider the truncation functions $T_\kappa, R_\kappa \colon \mathbb{R}\to\mathbb{R}$ given by

$$T_\kappa(s)=\begin{cases}-\kappa & \text{if } s<-\kappa,\\ s & \text{if } -\kappa\leq s\leq\kappa,\\ \kappa & \text{if } s>\kappa,\end{cases}\qquad R_\kappa(s)=\begin{cases}s+\kappa & \text{if } s<-\kappa,\\ 0 & \text{if } -\kappa\leq s\leq\kappa,\\ s-\kappa & \text{if } s>\kappa.\end{cases}$$

Note that $T_\kappa(s)+R_\kappa(s)=s$ for all $s\in\mathbb{R}$.

First, we observe that

$$\begin{aligned}\|F(\nabla u)\|_p^p &= \int_{\{|u|\leq k\}} F^p(\nabla u)\,\mathrm{d}x + \int_{\{|u|>k\}} F^p(\nabla u)\,\mathrm{d}x\\ &= \int_{\{|u|\leq k\}} F^p(\nabla(T_\kappa(u)))\,\mathrm{d}x + \int_{\{|u|>k\}} F^p(\nabla(R_\kappa(u)))\,\mathrm{d}x\\ &= \|F(\nabla(T_\kappa(u)))\|_p^p + \|F(\nabla(R_\kappa(u)))\|_p^p.\end{aligned} \tag{3.3}$$

The same argument leads to

$$\|F(\nabla u)\|_{q,\mu}^q = \|F(\nabla(T_\kappa(u)))\|_{q,\mu}^q + \|F(\nabla(R_\kappa(u)))\|_{q,\mu}^q. \tag{3.4}$$

Since $\|\cdot\|_p$ is sequentially weakly lower semicontinuous and considering that

$$F(\nabla(T_\kappa(u_n))) \rightharpoonup F(\nabla(T_\kappa(u))) \quad \text{in } L^q_\mu(\Omega),$$

due to the weak convergence of $u_n \rightharpoonup u$ in $W^{1,\mathcal{H}}(\Omega)$, for every $\kappa\geq 1$ we have

$$\begin{cases}\liminf\limits_{n\to\infty}\Big(\frac{1}{p}\|F(\nabla(T_\kappa(u_n)))\|_p^p - \frac{1}{p}\|F(\nabla(T_\kappa(u)))\|_p^p\Big)\geq 0,\\ \lim\limits_{n\to\infty}\Big(\frac{1}{q}\|F(\nabla(T_\kappa(u_n)))\|_{q,\mu}^q - \frac{1}{q}\|F(\nabla(T_\kappa(u)))\|_{q,\mu}^q\Big)=0.\end{cases} \tag{3.5}$$

Applying the triangle inequality for the Minkowski norm F (see Bao, Chern and Shen [2, Theorem 1.2.2]) along with the convexity of the function $s\mapsto s^r$, $r>1$, we get the following inequality:

$$\frac{1}{2^{r-1}r_F^r}F^r(w_1-w_2)-2F^r(w_2)\leq F^r(w_1)-F^r(w_2) \quad \text{for all } w_1,w_2\in\mathbb{R}^N. \tag{3.6}$$

From (3.6), by taking $w_1=\nabla(R_\kappa(u_n))$ and $w_2=\nabla(R_\kappa(u))$, respectively, we get

$$\begin{cases}\|F(\nabla(R_\kappa(u_n)))\|_p^p-\|F(\nabla(R_\kappa(u)))\|_p^p \geq \frac{1}{2^{p-1}r_F^p}\|F(\nabla(R_\kappa(u_n))-\nabla(R_\kappa(u)))\|_p^p - 2\|F(\nabla(R_\kappa(u)))\|_p^p,\\ \|F(\nabla(R_\kappa(u_n)))\|_{q,\mu}^q-\|F(\nabla(R_\kappa(u)))\|_{q,\mu}^q \geq \frac{1}{2^{q-1}r_F^q}\|F(\nabla(R_\kappa(u_n))-\nabla(R_\kappa(u)))\|_{q,\mu}^q - 2\|F(\nabla(R_\kappa(u)))\|_{q,\mu}^q.\end{cases} \tag{3.7}$$

On the other hand, by the Brezis–Lieb lemma (see, e.g., Papageorgiou and Winkert [49, Lemma 4.1.22], we have

$$\begin{cases}\liminf\limits_{n\to\infty}(\|u_n\|_{p^*}^{p^*}-\|u\|_{p^*}^{p^*}) = \liminf\limits_{n\to\infty}\|u_n-u\|_{p^*}^{p^*},\\ \liminf\limits_{n\to\infty}(\|u_n\|_{p_*,\partial\Omega}^{p_*}-\|u\|_{p_*,\partial\Omega}^{p_*}) = \liminf\limits_{n\to\infty}\|u_n-u\|_{p_*,\partial\Omega}^{p_*}.\end{cases} \tag{3.8}$$

Claim. $\|h\|_p^p \geq \|R_\kappa(h)\|_p^p$ for all $h\in W^{1,\mathcal{H}}(\Omega)$ and for all $\kappa\geq 1$.

First, we have

$$\begin{aligned}\|h\|_p^p &= \|T_\kappa(h)+R_\kappa(h)\|_p^p\\ &= \int_{\{h<-\kappa\}}|-\kappa+R_\kappa(h)|^p\,\mathrm{d}x + \int_{\{|h|\leq\kappa\}}|u+R_\kappa(h)|^p\,\mathrm{d}x + \int_{\{h>\kappa\}}|\kappa+R_\kappa(h)|^p\,\mathrm{d}x\\ &\geq \int_{\{h<-\kappa\}}|-\kappa+R_\kappa(h)|^p\,\mathrm{d}x + \int_{\{h>\kappa\}}|R_\kappa(h)|^p\,\mathrm{d}x.\end{aligned} \tag{3.9}$$

Applying the inequality

$$|w_2|^p > |w_1|^p + p|w_1|^{p-2}w_1(w_2 - w_1) \quad \text{for all } w_1, w_2 \in \mathbb{R}^N,$$

with $w_2 = R_\kappa(h) - \kappa$ and $w_1 = R_\kappa(h)$, we get

$$\begin{aligned} \int\limits_{\{h<-\kappa\}} |-\kappa + R_\kappa(h)|^p \,dx &\geq \int\limits_{\{h<-\kappa\}} [|R_\kappa(h)|^p + p|R_\kappa(h)|^{p-2}R_\kappa(h)\cdot(-\kappa)]\,dx \\ &\geq \int\limits_{\{h<-\kappa\}} |R_\kappa(h)|^p \,dx \end{aligned} \tag{3.10}$$

since $R_\kappa(h) < 0$ if $h < -\kappa$. Combining (3.9) and (3.10) leads to

$$\|u\|_p^p \geq \int\limits_{\{h<-\kappa\}} |R_\kappa(h)|^p \,dx + \int\limits_{\{h>\kappa\}} |R_\kappa(h)|^p \,dx = \|R_\kappa(h)\|_p^p$$

because $R_\kappa(h) = 0$ if $|h| \leq \kappa$. This proves the claim.

Thus, we may apply the Brezis–Lieb lemma along with the claim in order to obtain

$$\begin{aligned} \liminf_{n\to\infty}(\|u_n\|_p^p - \|u\|_p^p) &= \liminf_{n\to\infty}\|u_n - u\|_p^p \\ &\geq \liminf_{n\to\infty}\|R_\kappa(u_n) - R_\kappa(u)\|_p^p \\ &\geq \frac{1}{2^{p-1}r_F^p}\liminf_{n\to\infty}\|R_\kappa(u_n) - R_\kappa(u)\|_p^p \end{aligned} \tag{3.11}$$

since $r_F \geq 1$, and so $2^{p-1}r_F^p \geq 1$.

Note that

$$\begin{cases} \|F(\nabla(R_\kappa(u)))\|_p^p \to 0 & \text{as } \kappa \to \infty, \\ \|F(\nabla(R_\kappa(u)))\|_{q,\mu}^q \to 0 & \text{as } \kappa \to \infty, \\ \|u_n\|_{q,\mu}^q \to \|u\|_{q,\mu}^q & \text{as } n \to \infty. \end{cases} \tag{3.12}$$

The last convergence in (3.12) follows from Proposition 2.2 (ii) since $q < p^*$ and due to the boundedness of $\mu(\cdot)$, as given in (1.5).

Hence, for κ large enough, taking (3.3)–(3.5), (3.7), (3.8), (3.11) and (3.12) into account, we have that

$$\begin{aligned} \liminf_{n\to\infty}(I(u_n) - I(u)) \geq \liminf_{n\to\infty}\Big(&\frac{1}{p2^{p-1}r_F^p}\|R_\kappa(u_n) - R_\kappa(u)\|_{1,p,F}^p \\ &- \frac{1}{p^*}\|u_n - u\|_{p^*}^{p^*} - \frac{1}{p_*}\|u_n - u\|_{p_*,\partial\Omega}^{p_*}\Big). \end{aligned} \tag{3.13}$$

We observe that

$$\begin{cases} \|u_n - u\|_{p^*}^{p^*} \leq 2^{p^*-1}\|T_\kappa(u_n) - T_\kappa(u)\|_{p^*}^{p^*} + 2^{p^*-1}\|R_\kappa(u_n) - R_\kappa(u)\|_{p^*}^{p^*}, \\ \|u_n - u\|_{p_*,\partial\Omega}^{p_*} \leq 2^{p_*-1}\|T_\kappa(u_n) - T_\kappa(u)\|_{p_*,\partial\Omega}^{p_*} + 2^{p_*-1}\|R_\kappa(u_n) - R_\kappa(u)\|_{p_*,\partial\Omega}^{p_*}. \end{cases} \tag{3.14}$$

By Lebesgue's dominated convergence theorem, we get that

$$\lim_{n\to\infty}\|T_\kappa(u_n) - T_\kappa(u)\|_{p^*}^{p^*} = 0 \quad \text{and} \quad \lim_{n\to\infty}\|T_\kappa(u_n) - T_\kappa(u)\|_{p_*,\partial\Omega}^{p_*} = 0. \tag{3.15}$$

Finally, combining (3.13)–(3.15), we arrive at

$$\begin{aligned} \liminf_{n\to\infty}(I(u_n) - I(u)) \geq \liminf_{n\to\infty}\Big(&\frac{1}{p2^{p-1}r_F^p}\|R_\kappa(u_n) - R_\kappa(u)\|_{1,p,F}^p \\ &- \frac{2^{p^*-1}}{p^*}\|R_\kappa(u_n) - R_\kappa(u)\|_{p^*}^{p^*} - \frac{2^{p_*-1}}{p_*}\|R_\kappa(u_n) - R_\kappa(u)\|_{p_*,\partial\Omega}^{p_*}\Big). \end{aligned}$$

By using this along with (3.1) and the fact that $\psi(s) \geq 0$ for all $s \in (0, \varrho^*)$ (see (3.2)), it follows that

$$\begin{aligned}\liminf_{n\to\infty}(I(u_n) - I(u)) &\geq \liminf_{n\to\infty}\Big(\frac{1}{p2^{p-1}r_F^p}\|R_\kappa(u_n) - R_\kappa(u)\|_{1,p,F}^p - \frac{2^{p^*-1}c_{p^*}^{p^*}}{p^*}\|R_\kappa(u_n) - R_\kappa(u)\|_{1,p,F}^{p^*}\\ &\quad - \frac{2^{p_*-1}c_{p_*}^{p_*}}{p_*}\|R_\kappa(u_n) - R_\kappa(u)\|_{1,p,F}^{p_*}\Big)\\ &\geq \liminf_{n\to\infty}(\|R_\kappa(u_n) - R_\kappa(u)\|_{1,p,F}^p \Psi(\varrho)) \geq 0,\end{aligned}$$

which proves the assertion of the proposition. □

Taking into account assumption (H2) together with the compact embeddings $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r_1}(\Omega)$ for $r_1 < p^*$ and $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r_2}(\partial\Omega)$ for $r_2 < p_*$ (see Proposition 2.2 (ii) and (iii)), it is quite standard to prove that the functional

$$u \mapsto \frac{\lambda}{\gamma}\int_\Omega (u_+)^\gamma\,\mathrm{d}x + \lambda\int_\Omega G(x, u_+)\,\mathrm{d}x + \int_{\partial\Omega} G_2(x, u_+)\,\mathrm{d}\sigma$$

is sequentially weakly lower semicontinuous on $W^{1,\mathcal{H}}(\Omega)$ for every $\lambda > 0$. This fact along with Proposition 3.1 leads to the following corollary.

Corollary 3.2. *Let hypotheses* (H1)–(H3) *be satisfied. For every $\lambda > 0$ and for every $\varrho \in (0, \varrho^*)$, the restriction of J_λ to the closed convex set B_ϱ is sequentially weakly lower semicontinuous.*

Now we are going to prove Theorem 1.2. For this purpose, we introduce the functionals $I_1\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ and $I_2\colon L^{\mathcal{H}}(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned}I_1(u) = -\frac{1}{q}\|F(\nabla u)\|_{q,\mu}^q - \frac{1}{q}\|u\|_{q,\mu}^q + \frac{1}{p^*}\|u_+\|_{p^*}^{p^*} + \frac{\lambda}{\gamma}\int_\Omega (u_+)^\gamma\,\mathrm{d}x\\ + \lambda\int_\Omega G_1(x, u_+)\,\mathrm{d}x + \frac{1}{p_*}\|u_+\|_{p_*,\partial\Omega}^{p_*} + \int_{\partial\Omega} G_2(x, u_+)\,\mathrm{d}\sigma\end{aligned}$$

and

$$I_2(u) = \frac{1}{p^*}\|u_+\|_{p^*}^{p^*} + \frac{\lambda}{\gamma}\int_\Omega (u_+)^\gamma\,\mathrm{d}x + \lambda\int_\Omega G(x, u_+)\,\mathrm{d}x + \frac{1}{p_*}\|u_+\|_{p_*,\partial\Omega}^{p_*} + \int_{\partial\Omega} G_2(x, u_+)\,\mathrm{d}\sigma.$$

Proof of Theorem 1.2. Let $\lambda > 0$ and let $\varrho \in (0, \varrho^*)$ be as in Corollary 3.2. First, we define

$$\varphi_\lambda(\varrho) := \inf_{\|u\|_{1,p,F}<\varrho} \frac{\sup_{B_\varrho} I_1 - I_1(u)}{\varrho^p - \|u\|_{1,p,F}^p} \quad \text{and} \quad \psi_\lambda(\varrho) := \sup_{B_\varrho} I_1.$$

Claim. There exist $\lambda, \varrho > 0$ small enough such that

$$\varphi_\lambda(\varrho) < \frac{1}{p}. \tag{3.16}$$

In order to prove (3.16), it is enough to find $\lambda, \varrho > 0$ such that

$$\inf_{\xi<\varrho} \frac{\psi_\lambda(\varrho) - \psi_\lambda(\xi)}{\varrho^p - \xi^p} < \frac{1}{p}. \tag{3.17}$$

Taking $\xi = \varrho - \varepsilon$ for some $\varepsilon \in (0, \varrho)$, we easily see that

$$\begin{aligned}\frac{\psi_\lambda(\varrho) - \psi_\lambda(\xi)}{\varrho^p - \xi^p} &= \frac{\psi_\lambda(\varrho) - \psi_\lambda(\varrho - \varepsilon)}{\varrho^p - (\varrho - \varepsilon)^p}\\ &= \frac{\psi_\lambda(\varrho) - \psi_\lambda(\varrho - \varepsilon)}{\varepsilon} \cdot \frac{-\frac{\varepsilon}{\varrho}}{\varrho^{p-1}[(1 - \frac{\varepsilon}{\varrho})^p - 1]}.\end{aligned}$$

Therefore, if we pass to the limit as $\varepsilon \to 0$, then (3.17) holds if

$$\limsup_{\varepsilon\to 0^+} \frac{\psi_\lambda(\varrho)-\psi_\lambda(\varrho-\varepsilon)}{\varepsilon} < \varrho^{p-1} \tag{3.18}$$

is satisfied.

Thus we have to verify (3.18) to get our claim. First, note that

$$\begin{aligned}\frac{1}{\varepsilon}|\psi_\lambda(\varrho)-\psi_\lambda(\varrho-\varepsilon)| &= \frac{1}{\varepsilon}\Big|\sup_{v\in B_1} I_1(\varrho v) - \sup_{v\in B_1} I_1((\varrho-\varepsilon)v)\Big|\\ &\le \frac{1}{\varepsilon}\sup_{v\in B_1}|I_1(\varrho v)-I_1((\varrho-\varepsilon)v)|\\ &\le \frac{1}{\varepsilon}\sup_{v\in B_1}\Big|\frac{(\varrho-\varepsilon)^q-\varrho^q}{q}[\|F(\nabla v)\|_{q,\mu}^q+\|v\|_{q,\mu}^q]+I_2(\varrho v)-I_2((\varrho-\varepsilon)v)\Big|.\end{aligned}$$

The growth conditions in (H2), along with the continuous embeddings $W^{1,p}(\Omega)\hookrightarrow L^{p^*}(\Omega)$ as well as $W^{1,p}(\Omega)\hookrightarrow L^{p_*}(\partial\Omega)$, yield

$$\begin{aligned}\frac{\psi_\lambda(\varrho)-\psi_\lambda(\varrho-\varepsilon)}{\varepsilon} &\le \frac{1}{\varepsilon}\sup_{\|v\|_{1,p,F}\le 1}\int_\Omega\Big|\int_{(\varrho-\varepsilon)v_+(x)}^{\varrho v_+(x)}[t^{p^*-1}+\lambda t^{\gamma-1}+\lambda g_1(x,t)]\,dt\Big|\,dx\\ &\quad+\frac{1}{\varepsilon}\sup_{\|v\|_{1,p,F}\le 1}\int_{\partial\Omega}\Big|\int_{(\varrho-\varepsilon)v_+(x)}^{\varrho v_+(x)}[t^{p_*-1}+g_2(x,t)]\,dt\Big|\,d\sigma\\ &\le \frac{c_{p^*}^{p^*}}{p^*}\Big|\frac{\varrho^{p^*}-(\varrho-\varepsilon)^{p^*}}{\varepsilon}\Big|+\lambda\frac{c_{p^*}^{\gamma}|\Omega|^{\frac{p^*-\gamma}{p^*}}}{\gamma}\Big|\frac{\varrho^{\gamma}-(\varrho-\varepsilon)^{\gamma}}{\varepsilon}\Big|\\ &\quad+\lambda a_1\frac{c_{p^*}^{\nu_1}|\Omega|^{\frac{p^*-\nu_1}{p^*}}}{\nu_1}\Big|\frac{\varrho^{\nu_1}-(\varrho-\varepsilon)^{\nu_1}}{\varepsilon}\Big|+\lambda b_1\frac{c_{p^*}^{\theta_1}|\Omega|^{\frac{p^*-\theta_1}{p^*}}}{\theta_1}\Big|\frac{\varrho^{\theta_1}-(\varrho-\varepsilon)^{\theta_1}}{\varepsilon}\Big|\\ &\quad+\frac{c_{p_*}^{p_*}}{p_*}\Big|\frac{\varrho^{p_*}-(\varrho-\varepsilon)^{p_*}}{\varepsilon}\Big|+a_2\frac{c_{p_*}^{\nu_2}|\partial\Omega|^{\frac{p_*-\nu_2}{p_*}}}{\nu_2}\Big|\frac{\varrho^{\nu_2}-(\varrho-\varepsilon)^{\nu_2}}{\varepsilon}\Big|.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0^+}\frac{\psi_\lambda(\varrho)-\psi_\lambda(\varrho-\varepsilon)}{\varepsilon} &\le c_{p^*}^{p^*}\varrho^{p^*-1}+\lambda c_{p^*}^{\gamma}|\Omega|^{\frac{p^*-\gamma}{p^*}}\varrho^{\gamma-1}+\lambda a_1 c_{p^*}^{\nu_1}|\Omega|^{\frac{p^*-\nu_1}{p^*}}\varrho^{\nu_1-1}\\ &\quad+\lambda b_1 c_{p^*}^{\theta_1}|\Omega|^{\frac{p^*-\theta_1}{p^*}}\varrho^{\theta_1-1}+c_{p_*}^{p_*}\varrho^{p_*-1}+a_2 c_{p_*}^{\nu_2}|\partial\Omega|^{\frac{p_*-\nu_2}{p_*}}\varrho^{\nu_2-1}.\end{aligned}$$

Now, we consider the function $\Lambda\colon(0,\infty)\to\mathbb{R}$ given by

$$\Lambda(s)=\frac{s^{p-\gamma}-c_{p^*}^{p^*}s^{p^*-\gamma}-c_{p_*}^{p_*}s^{p_*-\gamma}-a_2c_{p_*}^{\nu_2}|\partial\Omega|^{\frac{p_*-\nu_2}{p_*}}s^{\nu_2-\gamma}}{c_{p^*}^{\gamma}|\Omega|^{\frac{p^*-\gamma}{p^*}}+a_1c_{p^*}^{\nu_1}|\Omega|^{\frac{p^*-\nu_1}{p^*}}s^{\nu_1-\gamma}+b_1c_{p^*}^{\theta_1}|\Omega|^{\frac{p^*-\theta_1}{p^*}}s^{\theta_1-\gamma}}.$$

We easily see that $\lim_{s\to 0}\Lambda(s)=0$, and from L'Hospital's rule we verify that $\lim_{s\to\infty}\Lambda(s)=-\infty$. Moreover, since $\nu_2>p$ (see (H2)) and due to the continuity of Λ, we know that there exists $s_0>0$ small enough such that $\Lambda(s)>0$ for all $s\in(0,s_0)$. Hence, we find $s_{\max}>0$ such that

$$\Lambda(s_{\max})=\max_{s>0}\Lambda(s).$$

Let us set

$$\lambda_* := \Lambda(\min\{s_{\max},\varrho^*\}).$$

If we now take $\lambda<\lambda_*$ and $\varrho<\min\{s_{\max},\varrho^*\}$, then (3.18) is satisfied, and so (3.16). This proves the claim.

From the claim we know that there exists an element $\hat{u} \in W^{1,\mathcal{H}}(\Omega)$ with $\|\hat{u}\|_{1,p,F} \le \varrho$ such that

$$J_\lambda(\hat{u}) < \frac{1}{p}\varrho^p - I_1(u_1) \quad \text{for all } u_1 \in B_\varrho. \tag{3.19}$$

From Corollary 3.2 we know that $J_\lambda|_{B_\varrho}$ is sequentially weakly lower semicontinuous. Therefore,

$$J_\lambda\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$$

restricted to B_ϱ has a global minimizer $u \in W^{1,\mathcal{H}}(\Omega)$ with $\|u\|_{1,p,F} \le \varrho$. Suppose that $\|u\|_{1,p,F} = \varrho$. Then we have from (3.19) that

$$J_\lambda(u) = \frac{1}{p}\varrho^p - I_1(u) > J_\lambda(\hat{u}),$$

which is a contradiction. We conclude that $u \in B_\varrho$ is a local minimizer for J_λ with $\|u\|_{1,p,F} < \varrho$ for $\lambda < \lambda_*$.

We claim that $u \neq 0$. Let $v \in W^{1,\mathcal{H}}(\Omega)$ be such that $v > 0$ and let $t > 0$. Then we have

$$\begin{aligned} J_\lambda(tv) = {} & \frac{t^p}{p}\|F(\nabla v)\|_p^p + \frac{t^q}{q}\|F(\nabla v)\|_{q,\mu}^q + \frac{t^p}{p}\|v\|_p^p + \frac{t^q}{q}\|v\|_{q,\mu}^q - \frac{t^{p^*}}{p^*}\|v\|_{p^*}^{p^*} \\ & - \lambda\frac{t^\gamma}{\gamma}\int_\Omega v^\gamma \,\mathrm{d}x - \lambda\frac{a_1 t^{\nu_1}}{\nu_1}\|v\|_{\nu_1}^{\nu_1} - \lambda\frac{b_1 t^{\theta_1}}{\theta_1}\|v\|_{\theta_1}^{\theta_1} - \frac{t^{p_*}}{p_*}\|v\|_{p_*,\partial\Omega}^{p_*} - \frac{a_2 t^{\nu_2}}{\nu_2}\|v\|_{\nu_2,\partial\Omega}^{\nu_2}, \end{aligned}$$

which implies $J_\lambda(tv) < 0$ for $t > 0$ sufficiently small. Thus, $u \neq 0$.

Let us now prove that $u \in W^{1,\mathcal{H}}(\Omega)$ is nonnegative a.e. in Ω. First, we observe that $u + tu_- \in B_\varrho$ and $(u + tu_-)_+ = u_+$ for $t > 0$ sufficiently small. Using this fact, we have

$$\begin{aligned} 0 & \le \frac{J_\lambda(u + tu_-) - J_\lambda(u)}{t} \\ & = \frac{1}{p}\int_\Omega \frac{F^p(\nabla(u + tu_-)) - F^p(\nabla u)}{t}\,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)\frac{F^q(\nabla(u + tu_-)) - F^q(\nabla u)}{t}\,\mathrm{d}x \\ & \quad + \frac{1}{p}\int_\Omega \frac{|u + tu_-|^p - |u|^p}{t}\,\mathrm{d}x + \frac{1}{q}\int_\Omega \mu(x)\frac{|u + tu_-|^q - |u|^q}{t}\,\mathrm{d}x. \end{aligned}$$

From this, we conclude

$$\begin{aligned} 0 & \le \lim_{t\to 0^+} \frac{J_\lambda(u + tu_-) - J_\lambda(u)}{t} \\ & = \int_\Omega F^{p-1}(\nabla u)\nabla F(\nabla u)\cdot \nabla u_- \,\mathrm{d}x + \int_\Omega \mu(x)F^{q-1}(\nabla u)\nabla F(\nabla u)\cdot\nabla u_-\,\mathrm{d}x \\ & \quad + \int_\Omega |u|^{p-2}uu_-\,\mathrm{d}x + \int_\Omega \mu(x)|u|^{q-2}uu_-\,\mathrm{d}x. \end{aligned}$$

However, from Proposition 2.1 (iii) we know that

$$\begin{aligned} \int_\Omega F(\nabla u)^{p-1}\nabla F(\nabla u)\cdot\nabla u_-\,\mathrm{d}x & = -\int_\Omega F^{p-1}(\nabla u_-)\nabla F(\nabla u_-)\cdot\nabla u_-\,\mathrm{d}x \\ & = -\|F(\nabla u_-)\|_p^p \end{aligned}$$

and

$$\int_\Omega \mu(x)F^{q-1}(\nabla u)\nabla F(\nabla u)\cdot\nabla u_-\,\mathrm{d}x = -\|F(\nabla u_-)\|_{q,\mu}^q.$$

This leads to

$$\begin{aligned} 0 & \le \lim_{t\to 0}\frac{J_\lambda(u + tu_-) - J_\lambda(u)}{t} \\ & = -\|F(\nabla u_-)\|_p^p - \|F(\nabla u_-)\|_{q,\mu}^q - \|u_-\|_p^p - \|u_-\|_{q,\mu}^q \le 0. \end{aligned}$$

Therefore, $u_- = 0$, and so $u \ge 0$ a.e. in Ω.

Let us now show that u is positive in Ω. We argue indirectly and suppose there is a set C with positive measure such that $u = 0$ in C. Let $\varphi \in W^{1,\mathcal{H}}(\Omega)$ with $\varphi > 0$ and let $t > 0$ small enough such that $u + t\varphi \in B_\sigma$ and $(u + t\varphi)^\gamma > u^\gamma$ a.e. in Ω. We obtain

$$
\begin{aligned}
0 &\leq \frac{J_\lambda(u+t\varphi) - J_\lambda(u)}{t} \\
&= \frac{1}{p}\frac{\|F(\nabla(u+t\varphi))\|_p^p - \|F(\nabla u)\|_p^p}{t} + \frac{1}{q}\frac{\|F(\nabla(u+t\varphi))\|_{q,\mu}^q - \|F(\nabla u)\|_{q,\mu}^q}{t} \\
&\quad + \frac{1}{p}\frac{\|u+t\varphi\|_p^p - \|u\|_p^p}{t} + \frac{1}{q}\frac{\|u+t\varphi\|_{q,\mu}^q - \|u\|_{q,\mu}^q}{t} - \frac{1}{p^*}\frac{\|u+t\varphi\|_{p^*}^{p^*} - \|u\|_{p^*}^{p^*}}{t} \\
&\quad - \frac{\lambda}{\gamma t^{1-\gamma}}\int_C \varphi^\gamma \,\mathrm{d}x - \frac{\lambda}{\gamma}\int_{\Omega\setminus C}\frac{(u+t\varphi)^\gamma - u^\gamma}{t}\,\mathrm{d}x - \lambda\int_\Omega \frac{G_1(x,u+t\varphi) - G_1(x,u)}{t}\,\mathrm{d}x \\
&\quad - \frac{1}{p_*}\frac{\|u+t\varphi\|_{p_*,\partial\Omega}^{p_*} - \|u\|_{p_*,\partial\Omega}^{p_*}}{t} - \int_{\partial\Omega}\frac{G_2(x,u+t\varphi) - G_2(x,u)}{t}\,\mathrm{d}\sigma \\
&< \frac{1}{p}\frac{\|F(\nabla(u+t\varphi))\|_p^p - \|F(\nabla u)\|_p^p}{t} + \frac{1}{q}\frac{\|F(\nabla(u+t\varphi))\|_{q,\mu}^q - \|F(\nabla u)\|_{q,\mu}^q}{t} \\
&\quad + \frac{1}{p}\frac{\|u+t\varphi\|_p^p - \|u\|_p^p}{t} + \frac{1}{q}\frac{\|u+t\varphi\|_{q,\mu}^q - \|u\|_{q,\mu}^q}{t} - \frac{1}{p^*}\frac{\|u+t\varphi\|_{p^*}^{p^*} - \|u\|_{p^*}^{p^*}}{t} \\
&\quad - \frac{\lambda}{\gamma t^{1-\gamma}}\int_C \varphi^\gamma \,\mathrm{d}x - \lambda\int_\Omega \frac{G_1(x,u+t\varphi) - G_1(x,u)}{t}\,\mathrm{d}x \\
&\quad - \frac{1}{p_*}\frac{\|u+t\varphi\|_{p_*,\partial\Omega}^{p_*} - \|u\|_{p_*,\partial\Omega}^{p_*}}{t} - \int_{\partial\Omega}\frac{G_2(x,u+t\varphi) - G_2(x,u)}{t}\,\mathrm{d}\sigma.
\end{aligned}
$$

This yields

$$0 \leq \frac{J_\lambda(u+t\varphi) - J_\lambda(u)}{t} \to -\infty \quad \text{as } t \to 0^+,$$

a contradiction. Hence, $u > 0$ a.e. in Ω.

Next we want to show that

$$u^{\gamma-1}\varphi \in L^1(\Omega) \quad \text{for all } \varphi \in W^{1,\mathcal{H}}(\Omega) \tag{3.20}$$

and

$$
\begin{aligned}
&\int_\Omega (F(\nabla u)^{p-1} + \mu(x)F(\nabla u)^{q-1})\nabla F(\nabla u)\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega u^{p-1}\varphi\,\mathrm{d}x + \int_\Omega \mu(x)u^{q-1}\varphi\,\mathrm{d}x - \int_\Omega u^{p^*-1}\varphi\,\mathrm{d}x \\
&\quad - \lambda\int_\Omega u^{\gamma-1}\varphi\,\mathrm{d}x - \lambda\int_\Omega g_1(x,u)\varphi\,\mathrm{d}x - \int_{\partial\Omega} u^{p_*-1}\varphi\,\mathrm{d}\sigma - \int_{\partial\Omega} g_2(x,u)\varphi\,\mathrm{d}\sigma \geq 0
\end{aligned} \tag{3.21}
$$

for all $\varphi \in W^{1,\mathcal{H}}(\Omega)$ with $\varphi \geq 0$.

Now, we choose $\varphi \in W^{1,\mathcal{H}}(\Omega)$ with $\varphi \geq 0$ and fix a decreasing sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq (0,1]$ such that $\lim_{n\to\infty} t_n = 0$. It is clear that the functions

$$h_n(x) = \frac{(u(x) + t_n\varphi(x))^\gamma - u(x)^\gamma}{t_n}, \quad n \in \mathbb{N},$$

are measurable and nonnegative. Moreover, we have

$$\lim_{n\to\infty} h_n(x) = \gamma u(x)^{\gamma-1}\varphi(x) \quad \text{for a.a. } x \in \Omega.$$

Applying Fatou's lemma gives

$$\int_\Omega u^{\gamma-1}\varphi\,\mathrm{d}x \leq \frac{1}{\gamma}\liminf_{n\to\infty}\int_\Omega h_n\,\mathrm{d}x. \tag{3.22}$$

Then, for $n \in \mathbb{N}$ large enough, we obtain

$$\begin{aligned}
0 &\leq \frac{J_\lambda(u+t\varphi)-J_\lambda(u)}{t}\\
&= \frac{1}{p}\frac{\|F(\nabla(u+t_n\varphi))\|_p^p-\|F(\nabla u)\|_p^p}{t_n}+\frac{1}{q}\frac{\|F(\nabla(u+t_n\varphi))\|_{q,\mu}^q-\|F(\nabla u)\|_{q,\mu}^q}{t_n}\\
&\quad+\frac{1}{p}\frac{\|u+t_n\varphi\|_p^p-\|u\|_p^p}{t_n}+\frac{1}{q}\frac{\|u+t_n\varphi\|_{q,\mu}^q-\|u\|_{q,\mu}^q}{t_n}-\frac{1}{p^*}\frac{\|u+t_n\varphi\|_{p^*}^{p^*}-\|u\|_{p^*}^{p^*}}{t_n}\\
&\quad-\frac{\lambda}{\gamma}\int_\Omega h_n\,\mathrm{d}x-\lambda\int_\Omega\frac{G_1(x,u+t_n\varphi)-G_1(x,u)}{t_n}\,\mathrm{d}x\\
&\quad-\frac{1}{p_*}\frac{\|u+t_n\varphi\|_{p_*,\partial\Omega}^{p_*}-\|u\|_{p_*,\partial\Omega}^{p_*}}{t_n}-\int_{\partial\Omega}\frac{G_2(x,u+t_n\varphi)-G_2(x,u)}{t_n}\,\mathrm{d}\sigma.
\end{aligned}$$

Passing to the limit as $n\to\infty$ in the inequality above and using (3.22), we derive (3.20) and have

$$\begin{aligned}
\lambda\int_\Omega u^{\gamma-1}\varphi\,\mathrm{d}x &\leq \int_\Omega(F(\nabla u)^{p-1}+\mu(x)F(\nabla u)^{q-1})\nabla F(\nabla u)\cdot\nabla\varphi\,\mathrm{d}x\\
&\quad+\int_\Omega u^{p-1}\varphi\,\mathrm{d}x+\int_\Omega\mu(x)u^{q-1}\varphi\,\mathrm{d}x-\int_\Omega u^{p^*-1}\varphi\,\mathrm{d}x\\
&\quad-\lambda\int_\Omega u^{\gamma-1}\varphi\,\mathrm{d}x-\lambda\int_\Omega g_1(x,u)\varphi\,\mathrm{d}x-\int_{\partial\Omega}u^{p_*-1}\varphi\,\mathrm{d}\sigma-\int_{\partial\Omega}g_2(x,u)\varphi\,\mathrm{d}\sigma,
\end{aligned}$$

which shows (3.21). Note that it is sufficient to prove the integrability in (3.20) for nonnegative test functions $\varphi\in W^{1,\mathcal{H}}(\Omega)$.

Now, let $\varepsilon\in(0,1)$ be such that $(1+t)u\in B_\sigma$ for all $t\in[-\varepsilon,\varepsilon]$. Note that the function $\beta(t):=J_\lambda((1+t)u)$ has a local minimum in zero. We apply again Proposition 2.1 (iii) in order to get

$$\begin{aligned}
0=\beta'(0)&=\lim_{t\to0}\frac{J_\lambda((1+t)u)-J_\lambda(u)}{t}\\
&=\|F(\nabla u)\|_p^p+\|F(\nabla u)\|_{q,\mu}^q+\|u\|_p^p+\|u\|_{q,\mu}^q-\|u\|_{p^*}^{p^*}\\
&\quad-\lambda\int_\Omega u^\gamma\,\mathrm{d}x-\lambda\int_\Omega g_1(x,u)u\,\mathrm{d}x-\|u\|_{p_*,\partial\Omega}^{p_*}-\int_{\partial\Omega}g_2(x,u)u\,\mathrm{d}\sigma.
\end{aligned}\tag{3.23}$$

Finally, we need to show that u is a positive weak solution of (1.4). To this end, let $v\in W^{1,\mathcal{H}}(\Omega)$ and take the test function $\varphi=(u+\varepsilon v)_+\in W^{1,\mathcal{H}}(\Omega)$ in (3.21). Taking (3.23) into account, we have

$$\begin{aligned}
0&\leq\int_{\{u+\varepsilon v\geq0\}}(F^{p-1}(\nabla u)+\mu(x)F^{q-1}(\nabla u))\nabla F(\nabla u)\cdot\nabla(u+\varepsilon v)\,\mathrm{d}x\\
&\quad+\int_{\{u+\varepsilon v\geq0\}}(u^{p-1}+\mu(x)u^{q-1})(u+\varepsilon v)\,\mathrm{d}x-\int_\Omega u^{p^*-1}(u+\varepsilon v)\,\mathrm{d}x\\
&\quad-\lambda\int_\Omega u^{\gamma-1}(u+\varepsilon v)\,\mathrm{d}x-\lambda\int_\Omega g_1(x,u)(u+\varepsilon v)\,\mathrm{d}x-\int_{\partial\Omega}(u^{p_*-1}+g_2(x,u))(u+\varepsilon v)\,\mathrm{d}\sigma\\
&=\|F(\nabla u)\|_p^p+\|F(\nabla u)\|_{q,\mu}^q+\|u\|_p^p+\|u\|_{q,\mu}^q-\|u\|_{p^*}^{p^*}-\lambda\int_\Omega u^\gamma\,\mathrm{d}x\\
&\quad-\lambda\int_\Omega g_1(x,u)u\,\mathrm{d}x-\|u\|_{p_*,\partial\Omega}^{p_*}-\int_{\partial\Omega}g_2(x,u)u\,\mathrm{d}\sigma\\
&\quad+\varepsilon\int_\Omega F^{p-1}(\nabla u)\nabla F(\nabla u)\cdot\nabla v\,\mathrm{d}x+\varepsilon\int_\Omega\mu(x)F^{q-1}(\nabla u)\nabla F(\nabla u)\cdot\nabla v\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\int_\Omega u^{p-1}v\,dx+\varepsilon\int_\Omega \mu(x)u^{q-1}v\,dx-\varepsilon\int_\Omega u^{p^*-1}v\,dx-\varepsilon\lambda\int_\Omega u^{\gamma-1}v\,dx\\
&-\varepsilon\lambda\int_\Omega g_1(x,u)v\,dx-\varepsilon\int_{\partial\Omega} u^{p_*-1}v\,d\sigma-\varepsilon\int_{\partial\Omega} g_2(x,u)v\,d\sigma\\
&-\int_{\{u+\varepsilon v<0\}} F^p(\nabla u)\,dx-\varepsilon\int_{\{u+\varepsilon v<0\}} F^{p-1}(\nabla u)\nabla F(\nabla u)\cdot\nabla v\,dx\\
&-\int_{\{u+\varepsilon v<0\}} \mu(x)F^q(\nabla u)\,dx-\varepsilon\int_{\{u+\varepsilon v<0\}} \mu(x)F^{q-1}(\nabla u)\nabla F(\nabla u)\cdot\nabla v\,dx\\
&-\int_{\{u+\varepsilon v<0\}} u^{p-1}(u+\varepsilon v)\,dx-\int_{\{u+\varepsilon v<0\}} \mu(x)u^{q-1}(u+\varepsilon v)\,dx\\
&+\int_{\{u+\varepsilon v<0\}} u^{p^*-1}(u+\varepsilon v)\,dx+\lambda\int_{\{u+\varepsilon v<0\}} u^{\gamma-1}(u+\varepsilon v)\,dx\\
&+\lambda\int_{\{u+\varepsilon v<0\}} g_1(x,u)(u+\varepsilon v)\,dx+\int_{\partial\Omega} u^{p_*-1}(u+\varepsilon v)\,d\sigma+\int_{\partial\Omega} g_2(x,u)(u+\varepsilon v)\,d\sigma\\
\le{}&\varepsilon\Big[\int_\Omega (F^{p-1}(\nabla u)+\mu(x)F^{q-1}(\nabla u))\nabla F(\nabla u)\cdot\nabla v\,dx+\int_\Omega u^{p-1}v\,dx\\
&+\int_\Omega \mu(x)u^{q-1}v\,dx-\int_\Omega u^{p^*-1}v\,dx-\lambda\int_\Omega u^{\gamma-1}v\,dx-\lambda\int_\Omega g_1(x,u)v\,dx-\int_{\partial\Omega} u^{p_*-1}v\,d\sigma-\int_{\partial\Omega} g_2(x,u)v\,d\sigma\Big]\\
&-\varepsilon\int_{\{u+\varepsilon v<0\}} F^{p-1}(\nabla u)\nabla F(\nabla u)\cdot\nabla v\,dx-\varepsilon\int_{\{u+\varepsilon v<0\}} \mu(x)F^{q-1}(\nabla u)\nabla F(\nabla u)\cdot\nabla v\,dx\\
&-\varepsilon\int_{\{u+\varepsilon v<0\}} u^{p-1}v\,dx-\varepsilon\int_{\{u+\varepsilon v<0\}} \mu(x)u^{q-1}v\,dx. \qquad (3.24)
\end{aligned}
$$

Note that the measure of the set $\{u+\varepsilon v<0\}$ goes to 0 as $\varepsilon\to 0$. Hence,

$$
\begin{aligned}
\int_{\{u+\varepsilon v<0\}} F^{p-1}(\nabla u)\nabla F(\nabla u)\nabla v\,dx \to 0 \quad &\text{as } \varepsilon\to 0,\\
\int_{\{u+\varepsilon v<0\}} \mu(x)F^{q-1}(\nabla u)\nabla F(\nabla u)\nabla v\,dx \to 0 \quad &\text{as } \varepsilon\to 0,\\
\int_{\{u+\varepsilon v<0\}} u^{p-1}v\,dx \to 0 \quad &\text{as } \varepsilon\to 0,\\
\int_{\{u+\varepsilon v<0\}} \mu(x)u^{q-1}v\,dx \to 0 \quad &\text{as } \varepsilon\to 0.
\end{aligned}
$$

Therefore, dividing the inequality (3.24) by ε and passing to the limit as $\varepsilon\to 0$, we conclude that

$$
\begin{aligned}
&\int_\Omega (F(\nabla u)^{p-1}+\mu(x)F(\nabla u)^{q-1})\nabla F(\nabla u)\cdot\nabla v\,dx+\int_\Omega u^{p-1}v\,dx+\int_\Omega \mu(x)u^{q-1}v\,dx\\
&\qquad-\int_\Omega u^{p^*-1}v\,dx-\lambda\int_\Omega u^{\gamma-1}v\,dx-\lambda\int_\Omega g_1(x,u)v\,dx-\int_{\partial\Omega} u^{p_*-1}v\,d\sigma-\int_{\partial\Omega} g_2(x,u)v\,d\sigma\ge 0.
\end{aligned}
$$

Since $v\in W^{1,\mathcal{H}}(\Omega)$ was arbitrary chosen, we see from the last inequality that equality must hold. Therefore, $u\in W^{1,\mathcal{H}}(\Omega)$ is a weak solution of problem (1.4) in the sense of Definition 1.1. □

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