Journal of Optimization Theory and Applications (2026) 208:119
https://doi.org/10.1007/s10957-026-02951-1

On a Robust Approach to "split" Feasibility Problems: Solvability and Global Error Bound Conditions

Amos Uderzo[1]

Received: 5 May 2025 / Accepted: 6 February 2026

Abstract

In the present paper, a robust approach to a special class of convex feasibility problems is considered. By techniques of convex and variational analysis, conditions for the existence of robust feasible solutions and related error bounds are investigated. This is done by reformulating the robust counterpart of a split feasibility problem as a set-valued inclusion, a problem format for which one can take profit from the solvability and stability theory that has been recently developed. As a result, a sufficient condition for solution existence and error bounds is established in terms of problem data and discussed through several examples. A specific focus is devoted to error bound conditions in the case in which the robust counterpart of split feasibility problems is polyhedral. Connections with some of the existent error bound conditions for the original split feasibility problem are also discussed.

Keywords Feasibility problem · Robust optimization · Solution existence · Error bounds · Subdifferential calculus · Variational principles.

Mathematics Subject Classification: 49J53 · 65K10 · 90C25

1 Introduction

In the present paper, after [11] by "split" feasibility problem the following mathematical question is meant: given a matrix $A \in \mathbf{M}_{m\times n}(\mathbb{R})$ and two closed convex sets $C \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$,

$$\text{find } x \in C \quad \text{such that} \quad Ax \in Q. \tag{SFP}$$

Communicated by Tuyen Van Nguyen.

✉ Amos Uderzo
amos.uderzo@unimib.it

[1] Department of Mathematics and Applications, Università degli studi di Milano-Bicocca, Milan, Italy

This kind of problems was firstly considered in [11, Section 6.1] in the context of iterative projection methods for solving more general convex feasibility problems, the latter meaning to find a point in the nonempty intersection of finitely many closed convex sets (see, for instance, [4, 5]). As it was recognized soon, the format (SFP) is relevant from the viewpoint of both theory and applications. It allows to formalize classic constraint systems in mathematical programming and in affine complementarity problems. Besides, (SFP) emerges as a model of interest in applications to engineering, in particular to the area of signal processing, such as phase retrieval, image reconstruction and computed tomography issues (see, for instance, [20]) and to inverse problems (see [12]). As a consequence, a good deal of investigations has been carried out on various topics related to (SFP) (see, among the others, [10, 21, 22, 32, 42]). In the present paper, by elaborating on the format (SFP), the case of problems affected by data uncertainty is addressed. In handling real-world models leading to (SFP), there are several reasons that motivate such an analysis perspective. To summarize, they include the impossibility to neglect effects of errors, which unavoidably enter any process of measuring/estimating parameters that describe a real-world system. Another type of error also occurring stems from the impossibility to implement a solution with the same precision degree to which it is computed, making nominal solution inadequate as a matter of fact to any practical purpose. In the need to address the phenomenon of data uncertainty, several methodologies have been developed in optimization. The investigations exposed in this paper follow the approach proposed in [7, 8, 26, 36], which since the late 90s is attracting the interest of many researchers (see, among the others, [6, 9, 14, 25] and references therein). Roughly speaking, the philosophy behind this approach consists in regarding constraint systems as “hard" (violations of constraints can not be admitted at all by the decision maker), while uncertainty is handled without making any reference to its stochastic nature (the probability distribution of random data could be unknown or only partially known, the constraint fulfilment in probability could not suit the aforementioned “hardness"). According to such an attitude, it must be considered as a feasible solution only those solutions, which are able to satisfy the constraint system, whatever the realizations of uncertain parameters are. This leads to a sort of “immunization against uncertainty", which appears to be suitable in decision-making environments, where a conservative attitude towards the constraint fulfilment is required (interested readers can find some of them discussed in detail in [7, 8]). In consideration of the relevance of convex feasibility problems in both theory and application contexts, it becomes important to consider the effect of data uncertainty on various topics of their analysis, such as solution existence and error bounds. In recent times, engineering and systems biology models are involving bigger and bigger amounts of data. A proportional amount of uncertainty is then expected to affect the related convex feasibility problems. Thus, considering a robust approach to convex feasibility problems affected by uncertainty becomes a way to cope with a realistic feature of the problem, while proposing a solution concept, which is able to edging the impact of such a feature. Further theoretical reasons motivating the interest in a robust approach to feasibility problems deal with connections with possible generalizations of (SFP) (see Remark 2).

In what follows, a set Ω is assumed to collect all possible different scenarios (or states of the world), whose multiplicity causes the uncertainty phenomenon affecting the problem data. With reference to split feasibility problems, the present analysis focuses on the uncertainty of the entries of A, disregarding at present the possible uncertainty related to C and Q. The realization of the scenario ω in terms of entries of A is denoted by A_ω. The uncertainty set collecting all possible values of the matrix data consequently becomes $\mathscr{U} = \{A_\omega \ : \ \omega \in \Omega\}$. Thus, the resulting robust counterpart to (SFP) amounts to

$$\text{find } x \in C \quad \text{such that} \quad A_\omega x \in Q, \quad \forall \omega \in \Omega. \tag{RSFP}$$

The aim of the present paper is to conduct a study of solvability and error bounds for (RSFP) relying on basic tools from convex and variational analysis. In order to perform a related technique of analysis, let us introduce the set-valued map $\mathscr{A}_{\mathscr{U}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, defined by

$$\mathscr{A}_{\mathscr{U}}(x) = \{y \in \mathbb{R}^m \ : \ y = A_\omega x, \ A_\omega \in \mathscr{U}\}. \tag{1}$$

Through the above map, (RSFP) can be reformulated as the following set-valued inclusion problem

$$\text{find } x \in C \quad \text{such that} \quad \mathscr{A}_{\mathscr{U}}(x) \subseteq Q. \tag{2}$$

This kind of generalized (set-inclusive) equations was addressed, independently of the robust optimization approach, already in [37], but its appearance in optimization contexts can be traced back at least to [36]. In the above setting, the set of all solutions of (RSFP) (henceforth *robust feasible solutions*) is given by

$$\mathrm{S}_{C,\mathscr{U},Q} = \{x \in C \ : \ \mathscr{A}_{\mathscr{U}}(x) \subseteq Q\} = C \cap \mathscr{A}_{\mathscr{U}}^{+1}(Q) = C \cap \bigcap_{\omega \in \Omega} A_\omega^{-1}(Q), \tag{3}$$

where the notation $\mathscr{F}^{+1}(S)$ indicates the core of S through $\mathscr{F}$, i.e. one the two possible ways, in which the inverse image through a set-valued map $\mathscr{F}$ of a given subset S of the range space is meant.

By global error bound for (RSFP) or (2), any situation is meant in which there exists $\sigma > 0$ such that

$$\operatorname{dist}\left(x; \mathrm{S}_{C,\mathscr{U},Q}\right) \le \sigma \rho(x), \quad \forall x \in \mathbb{R}^n,$$

where ρ denotes a residual function, i.e. any function $\rho : \mathbb{R}^n \longrightarrow [0, +\infty]$ working in such a way that

$$x \in \mathrm{S}_{C,\mathscr{U},Q} \qquad \text{iff} \qquad \rho(x) = 0,$$

whose values can be computed by problem data. The study of error bounds for various kinds of problems (e.g. scalar inequality systems and cone constraints, variational inequalities and complementarity problems, equilibrium problems) is a topic widely explored in the course of the last decades, because of connections with profound themes in variational analysis (regularity phenomena, implicit functions, and Lipschitz behaviours of maps) and the broad range of applications in different areas, such as penalty function methods in mathematical programming, implementation and convergence analysis of numerical methods for solving optimization problems. As a result, the existing literature on the subject is vast (in the impossibility to provide an exhaustive account, [15, 16, 31, 41] may be indicated as a source for a bibliographical search). In spite of this, to the best of his knowledge, the author has no information about any attempt to study error bounds specific for (RSFP) in the recorded literature. Conditions for error bounds for split feasibility problems in general normed spaces in have been considered in [40]. Local error bounds for nonlinear split feasibility problems have been investigated in [17]. A stability/sensitivity analysis of split feasibility problems has been recently conducted by variational analysis techniques, in various settings, in [21, 22]. Nevertheless, as pointed out in [7], the methodology for handling data perturbation proposed by stability/sensitivity analysis "is aimed at completely different questions". Indeed, instead of being concerned with the effects of the uncertainty set as a whole on the very feasible solution concept (robust feasibility), the analysis in that framework deals with reactions to (often small) perturbations near a reference value of the classic solution set. The reader should notice that this essential difference reflects on the different use of set-valued maps arising in the two methodologies.

The contents of the paper are arranged according to the following scheme. In Section 2 various preliminary elements (discussions about the standing assumptions, needed tools from convex and variational analysis, the main technique of investigation) are presented. Section 3 exposes the main result of the paper. In the same section, the reader will find several examples and comments aimed at illustrating some features of such a result. In Section 4 a specific focus on error bound conditions for the robust counterparts of split feasibility problems in the polyhedral case is discussed. A section for conclusions to be drawn completes the contents.

The basic notations used throughout the paper are standard. $\mathbb{R}$ denotes the field of real numbers, while $\mathbb{N}$ denotes the subset of natural numbers. $\mathbb{R}^n$ stands for the space of vectors with n real components, $\mathbb{R}^n_+$ the nonnegative orthant in $\mathbb{R}^n$, with $\mathbf{0}$ denoting the null vector. In any Euclidean space, $\|\cdot\|$ stands for the Euclidean norm. Given a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\pm\infty\}$, $\mathrm{dom}\,\varphi = \varphi^{-1}(\mathbb{R})$ denotes its domain and, if $\ell \in \mathbb{R}$, $[\varphi \le \ell] = \{x \in \mathbb{R}^n : \varphi(x) \le \ell\}$ denotes its sublevel set, whereas $[\varphi > \ell] = \{x \in \mathbb{R}^n : \varphi(x) > \ell\}$ denotes its strict superlevel set. If S is a subset of $\mathbb{R}^n$, $\mathrm{dist}\,(x; S) = \inf_{z \in S} \|z - x\|$ denotes that distance of a point $x \in \mathbb{R}^n$ from S, with the convention that $\mathrm{dist}\,(x; \varnothing) = +\infty$. Consistently, if $r \ge 0$, $\mathrm{B}\,[S, r] = [\mathrm{dist}\,(\cdot; S) \le r]$ indicates the r-enlargement of the set S, with radius r. In particular, if $S = \{x\}$, $\mathrm{B}\,[x, r]$ denotes the closed ball with center x and radius r. In this context, $\mathrm{B}\,[\mathbf{0}, 1]$ and $\mathrm{bd}\,\mathrm{B}\,[\mathbf{0}, 1]$ will be simply indicated by $\mathbb{B}$ and $\mathbb{S}$, respectively. The symbol $\mathrm{bd}\,S$, $\mathrm{int}\,S$ and $\overline{\mathrm{co}}\,S$ indicate the boundary, the topological interior and the convex closure of S, respectively. Whenever S is a vector subspace of an Euclidean space, by $S^\perp$ its orthogonal complement is denoted. Given a pair of subsets, say A and S, the excess of A over S is indicated by $\mathrm{exc}(A; S) = \sup_{a \in A} \mathrm{dist}\,(a; S)$, whereas their Hausdorff-Pompeiu distance by $\mathrm{haus}(A; S) = \max\{\mathrm{exc}(A; S), \mathrm{exc}(S; A)\}$. Given a set-valued map $\mathscr{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $\mathrm{dom}\,\mathscr{F} = \{x \in \mathbb{R}^n : \mathscr{F}(x) \ne \varnothing\}$ denotes the effective domain of $\mathscr{F}$. The vector space $\mathbf{M}_{m\times n}(\mathbb{R})$, consisting of all $m \times n$ matrices with real entries, will be equipped with the operator norm, indicated by $\|\cdot\|_{\mathbf{M}}$. Given $A \in \mathbf{M}_{m\times n}(\mathbb{R})$, $A^\top$ stands for the transposed matrix of A and $\ker A$ stands for its kernel.

The acronyms l.s.c. and p.h., standing for lower semicontinuous and positively homogeneous, respectively, will be also used. The meaning of further symbols employed in subsequent sections will be explained contextually to their introduction.

2 Preliminaries

2.1 Standing Assumptions on the Problem Data

In this section, the assumptions standing throughout the paper are formulated and discussed.

• Assumptions on the (certain) data C and Q:

(a_0) $\varnothing \neq C \subseteq \mathbb{R}^n$ and $\varnothing \neq Q \subseteq \mathbb{R}^m$ are both closed and convex sets.

• Assumptions on uncertain data: In the main result of the paper, for enabling to make use of certain convex analysis constructions, Ω will be supposed to be a separated compact topological space. Given an uncertainty set $\mathscr{U} = \{A_\omega \colon \omega \in \Omega\}$, it is convenient to introduce the map $\mathbf{A} : \Omega \longrightarrow (\mathbb{R}^m)^{\mathbb{R}^n}$ by setting

$$\mathbf{A}(\omega) = A_\omega.$$

The following assumptions are made on $\mathscr{U}$ via the map $\mathbf{A}$:

(a_1) $\mathbf{A}(\omega) \in \mathbf{M}_{m\times n}(\mathbb{R})$, for every $\omega \in \Omega$;
(a_2) the set $\mathbf{A}(\Omega) = \mathscr{U}$ is a compact and convex subset of $\mathbf{M}_{m\times n}(\mathbb{R})$.

Assumption (a_1) means that, even in the presence of uncertainty, the robust counterpart of the split feasibility problem maintains a linear structure with respect to the decision variable x.

Assumption (a_2) indicates that no specific structure is imposed on the dependence of A_ω on ω (the decision maker knows only that the scenario $\omega \in \Omega$ yields the "revealed matrix" A_ω, not how it does), apart from the aforementioned properties of $\mathscr{U}$. In other terms, unlike $\mathscr{U}$, the map $\mathbf{A}$ may be not among the problem data. Nonetheless, in dealing with concrete instances of (RSFP) and examples to be worked out, the explicit form of $\mathbf{A}$ will be given.

Remark 1 Notice that assumptions (a_1) and (a_2) are satisfied whenever Ω is a compact convex subset of a topological vector space $\mathbb{V}$ over the field $\mathbb{R}$ and the map $\mathbf{A} : \mathbb{V} \longrightarrow \mathbf{M}_{m\times n}(\mathbb{R})$ is linear (or affine) and continuous. A specific example of such a circumstance is discussed below in details.

Example 1 (*Birkhoff polytope*) Let $n = m = 3$ and let $\mathbb{V} = \mathbb{R}^4$ be equipped with its usual Euclidean space structure. Consider the map $\mathbf{A} : \mathbb{R}^4 \longrightarrow \mathbf{M}_{3\times 3}(\mathbb{R})$ defined by

$$\mathbf{A}(\omega) = \begin{pmatrix} \omega_1 & \omega_2 & 1-\omega_1-\omega_2 \\ \omega_3 & \omega_4 & 1-\omega_3-\omega_4 \\ 1-\omega_1-\omega_3 & 1-\omega_2-\omega_4 & \omega_1+\omega_2+\omega_3+\omega_4-1 \end{pmatrix},$$

where $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in \mathbb{R}^4$. In this setting, let us define Ω to be the following compact polyhedron (polytope)

$$\Omega = \left\{ \omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in \mathbb{R}^4_+ \ : \ \begin{array}{l} \omega_1 + \omega_2 \leq 1 \\ \omega_1 + \omega_3 \leq 1 \\ \omega_2 + \omega_4 \leq 1 \\ \omega_3 + \omega_4 \leq 1 \\ \omega_1 + \omega_2 + \omega_3 + \omega_4 \geq 1 \end{array} \right\} \subseteq [0, 1]^4.$$

The map $\mathbf{A}$ is clearly affine and therefore $\mathbf{A}(\Omega)$ is a compact convex subset (actually a polytope) of $\mathbf{M}_{3\times 3}(\mathbb{R})$. It is possible to see, indeed, that by construction of $\mathbf{A}$ and Ω, the set $\mathbf{A}(\Omega)$ coincides with Birkhoff polytope

$$\begin{aligned} \mathbf{B}_3 = \{A = (a_{i,j})_{i,j=1,2,3} \in \mathbf{M}_{3\times 3}(\mathbb{R}) : \ & \sum_{i=1}^{3} a_{i,j} = 1, \quad \forall j = 1, 2, 3, \\ & \sum_{j=1}^{3} a_{i,j} = 1, \quad \forall i = 1, 2, 3, \\ & a_{i,j} \geq 0, \quad \forall i, j = 1, 2, 3\} \end{aligned}$$

consisting of all the 3-dimensional doubly stochastic matrices (see, for instance, [3, Chapter II.5]). Clearly, the above example can be generalized to an arbitrary dimension $n \in \mathbb{N} \setminus \{0, 1\}$, by a proper choice of the polytope Ω in the space $\mathbb{R}^{(n-1)^2}$, in such a way that $\mathbf{A}(\Omega) = \mathbf{B}_n$.

Of course, assumptions (a_1) and (a_2) may happen to be satisfied even though Ω is neither convex nor compact, with $\mathbf{A} : \Omega \longrightarrow \mathbf{M}_{m\times n}(\mathbb{R})$ being not affine, as illustrated in the next example.

Example 2 Let $n = m = 1$, let $\Omega = (-\infty, -1] \cup \{0\} \cup [1, +\infty) \subseteq \mathbb{R}$ and let $\mathbf{A} : \mathbb{R} \longrightarrow \mathbf{M}_{1\times 1}(\mathbb{R}) \cong \mathbb{R}$ be defined by

$$\mathbf{A}(\omega) = \frac{\omega}{\omega^2 + 1}.$$

In such an event, one has $\mathbf{A}(\Omega) = [-1/2, 1/2]$, which is compact and convex.

Remark 2 It is worth mentioning that, in the particular case in which $\mathscr{U}$ is a polyhedral subset of $\mathbf{M}_{m\times n}(\mathbb{R})$, problem (RSFP) can be cast in a generalized class of split feasibility problems known as split feasibility problems with multiple output sets (see, for instance, [33]). Given a natural number $k > 1$, k nonempty, closed convex sets $Q_i \subseteq \mathbb{R}^{m_i}$ and matrices $A_i \in \mathbf{M}_{m_i \times n}(\mathbb{R})$, with $i \in \{1, \ldots, k\}$, the latter consists in finding $x \in C$ such that

$$A_i x \in Q_i, \quad \forall i \in \{1, \ldots, k\}.$$

To see how, it suffices to observe that, as $\mathscr{U}$ is polyhedral, the set of its extreme points is finite, so it is possible to define k to be its cardinality and then employ the Krein-Milman theorem (see the proof of Theorem 3). In doing so, one has to set $Q_i = Q$

for every $i \in \{1, \ldots, k\}$. To the best of the author's knowledge, the analysis of split feasibility problems with multiple output sets has focused so far on the convergence of iterative solution methods, whereas error bound conditions have not been considered yet.

2.2 Convex Analysis Tools

In this section, some basic concepts from convex analysis are recalled in view of their subsequent employment. These mainly deal with the well-known calculus for convex sets and functions. Given a nonempty convex set $C \subseteq \mathbb{R}^n$ and $\bar{x} \in C$, denote by

$$\mathrm{N}(\bar{x}; C) = \{v \in \mathbb{R}^n \ : \ \langle v, x - \bar{x} \rangle \leq 0, \quad \forall x \in C\}$$

the normal cone to C at $\bar{x}$. By the very definition, it is clear that if C is in particular a closed cone, then one has $\mathrm{N}(\bar{x}; C) \subseteq \mathrm{N}(\mathbf{0}; C) = C^{\ominus}$, where $C^{\ominus}$ stands for the negative dual cone to C.

Given a convex function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\bar{x} \in \operatorname{dom} \varphi$, the symbol

$$\partial\varphi(\bar{x}) = \{v \in \mathbb{R}^n \ : \ \langle v, x - \bar{x} \rangle \leq \varphi(x) - \varphi(\bar{x}), \quad \forall x \in \mathbb{R}^n\}$$

denotes the Moreau-Rockafellar subdifferential of φ at $\bar{x}$. In particular, for a convex function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ it is $\operatorname{dom} \varphi = \mathbb{R}^n$, so, since a convex function defined on a finite-dimensional Euclidean space is known to be continuous on the interior of its domain, then $\partial\varphi(x) \neq \varnothing$ for every $x \in \mathbb{R}^n$ (see [30, Theorem 3.39(b)]), namely $\operatorname{dom} \partial\varphi = \mathbb{R}^n$. The following calculus rules will be employed in the sequel:

- sum rule: if $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\vartheta : \mathbb{R}^n \longrightarrow \mathbb{R}$ are convex functions and $\bar{x} \in \mathbb{R}^n$, then the following equality holds

$$\partial\left(\varphi + \vartheta\right)(\bar{x}) = \partial\varphi(\bar{x}) + \partial\vartheta(\bar{x}) \tag{4}$$

  (see [30, Theorem 3.48]).
- chain rule: if $\Lambda \in \mathbf{M}_{m \times n}(\mathbb{R})$, $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a convex function, and $\bar{x} \in \mathbb{R}^n$, then it holds

$$\partial(\varphi \circ \Lambda)(\bar{x}) = \Lambda^\top \left(\partial\varphi(\Lambda\bar{x})\right) \tag{5}$$

  (see [30, Theorem 3.55]).
- Ioffe-Tikhomirov rule: let $(\varXi, \tau)$ be a separated compact topological space and let $\varphi_\xi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function for every $\xi \in \varXi$. Define $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ as being

$$\varphi(x) = \sup_{\xi \in \varXi} \varphi_\xi(x).$$

If the function $\varphi_\xi(x) : \Xi \longrightarrow \mathbb{R} \cup \{+\infty\}$, given by $\xi \mapsto \varphi_\xi(x)$, is u.s.c. on Ξ, $\bar{x} \in \mathrm{dom}\, f$ and φ_ξ is continuous at $\bar{x}$ for every $\xi \in \Xi$, then it holds

$$\partial\varphi(\bar{x}) = \overline{\mathrm{co}} \left(\bigcup_{\xi \in \Xi_{\bar{x}}} \partial\varphi_\xi(\bar{x}) \right), \tag{6}$$

where $\Xi_{\bar{x}} = \{\xi \in \Xi : \varphi_\xi(\bar{x}) = \varphi(\bar{x})\}$ indicates the clean-up index set (see [43, Theorem 2.4.18]).

Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed, convex set and $\bar{x} \in \mathbb{R}^n$. In this case, the function $x \mapsto \mathrm{dist}\,(x; C)$ turns out to be convex, with $\mathrm{dom}\,\mathrm{dist}\,(\cdot; C) = \mathbb{R}^n$. Since this function is going to play a certain role in the analysis of the topic under study, it is convenient to recall the form taken by its subdifferential, namely

$$\partial\mathrm{dist}\,(\cdot; C)\,(\bar{x}) = \begin{cases} \mathrm{N}(\bar{x}; C) \cap \mathbb{B} & \text{if } \bar{x} \in C, \\ \left\{ \dfrac{\bar{x} - \Pi_C(\bar{x})}{\mathrm{dist}\,(\bar{x}; C)} \right\} & \text{if } \bar{x} \notin C, \end{cases} \tag{7}$$

where $\Pi_C : \mathbb{R}^n \longrightarrow C$ denotes the standard Euclidean projector onto C, i.e. $\Pi_C(x)$ stands for the unique element of C such that $\|x - \Pi_C(x)\| = \mathrm{dist}\,(x; C)$ (see [30, Corollary 3.79]). In both the cases it is $\partial\mathrm{dist}\,(\cdot; C)\,(\bar{x}) \subseteq \mathrm{N}(\bar{x}; C) \cap \mathbb{B}$.

2.3 A Subdifferential Condition for Solvability and Error Bounds for Convex Inequalities

The basic technique employed to ensure solution existence for (RSFP) and, at the same time, to establish error bounds relies on the following proposition. Details of its proof are provided for the sake of completeness.

Proposition 1 *Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex function. Assume that $[\varphi > 0] \neq \varnothing$ and that*

$$\tau = \inf_{x \in [\varphi > 0]} \mathrm{dist}\,(\mathbf{0}; \partial\varphi(x)) = \inf\{\|v\| \ : \ v \in \partial\varphi(x), \quad x \in [\varphi > 0]\} > 0. \tag{8}$$

Then, $[\varphi \leq 0] \neq \varnothing$ and

$$\mathrm{dist}\,(x; [\varphi \leq 0]) \leq \frac{\varphi(x)}{\tau}, \quad \forall x \in [\varphi > 0].$$

Proof Notice that, since $\mathrm{dom}\,\varphi = \mathbb{R}^n$, then φ is continuous on $\mathbb{R}^n$, which is clearly a complete Banach space. Thus, both the assertions in the thesis can be established by applying [2, Theorem 2.8] and by expressing the strong slope of φ in term of its subdifferential, according to the dual representation established, for instance, in [15, Theorem 5(i)]. □

The above and similar, even more general, results are standard and widely employed tools in variational analysis, which ultimately rest upon the Ekeland/Bishop-Phelps variational principle or some of its equivalent reformulations.

2.4 Functional Characterization of $\mathrm{S}_{C,\mathscr{U},Q}$

Throughout this paper, following [37], in order to express error bounds for (RSFP) the residual function $\nu_{\mathscr{A}_\mathscr{U}} : \mathbb{R}^n \longrightarrow [0, +\infty]$, defined by

$$\nu_{\mathscr{A}_\mathscr{U}}(x) = \mathrm{exc}(\mathscr{A}_\mathscr{U}(x); Q) + \mathrm{dist}\,(x; C) \tag{9}$$

will be employed.

The next proposition shows that $\nu_{\mathscr{A}_\mathscr{U}}$ can actually play the role of a residual function for (RSFP).

Proposition 2 *Given a* (RSFP)*, under the standing assumptions* $(a_0) - (a_2)$*, the following equalities hold:*

$$\mathrm{S}_{C,\mathscr{U},Q} = \nu^{-1}_{\mathscr{A}_\mathscr{U}}(0) = [\nu_{\mathscr{A}_\mathscr{U}} \leq 0]. \tag{10}$$

Proof As for (10), if $x \in \mathrm{S}_{C,\mathscr{U},Q}$, then one has $\mathrm{exc}(\mathscr{A}_\mathscr{U}(x); Q) = \mathrm{dist}\,(x; C) = 0$, so it is true that $x \in [\nu_{\mathscr{A}_\mathscr{U}} \leq 0]$. Vice versa, if $x \in [\nu_{\mathscr{A}_\mathscr{U}} \leq 0]$, as $\mathrm{exc}(\mathscr{A}_\mathscr{U}(x); Q)$ and $\mathrm{dist}\,(x; C)$ are both nonnegative, then it must be $\mathrm{exc}(\mathscr{A}_\mathscr{U}(x); Q) = \mathrm{dist}\,(x; C) = 0$. Since C and Q are closed sets, these equalities imply $\mathscr{A}_\mathscr{U}(x) \subseteq Q$ and $x \in C$, which means $x \in \mathrm{S}_{C,\mathscr{U},Q}$. □

In order to enhance the result presented in Section 3, it is convenient to provide an equivalent reformulation of the above residual function, which is based on the support function associated to a given convex set, i.e. given $D \subseteq \mathbb{R}^m$, the function $\varsigma\,(\cdot; D) : \mathbb{R}^m \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\varsigma\,(v; D) = \sup_{y \in D} \langle y, v \rangle.$$

Proposition 3 *Let a problem* (RSFP) *satisfy the assumptions* $(a_0) - (a_2)$*. Then, it holds*

$$\nu_{\mathscr{A}_\mathscr{U}}(x) = \sup_{u \in \mathbb{B}} [\varsigma\,(u; \mathscr{A}_\mathscr{U}(x)) - \varsigma\,(u; Q)] + \mathrm{dist}\,(x; C)\,, \quad \forall x \in \mathbb{R}^n.$$

Proof Fix an arbitrary $x \in \mathbb{R}^n$. Clearly, on account of definition (9), it suffices to show that

$$\mathrm{exc}(\mathscr{A}_\mathscr{U}(x); Q) = \sup_{u \in \mathbb{B}} [\varsigma\,(u; \mathscr{A}_\mathscr{U}(x)) - \varsigma\,(u; Q)].$$

According to the definition of excess, one has

$$\mathrm{exc}(\mathscr{A}_\mathscr{U}(x); Q) = \sup_{y \in \mathscr{A}_\mathscr{U}(x)} \min_{q \in Q} \|y - q\| = \sup_{y \in \mathscr{A}_\mathscr{U}(x)} \min_{q \in Q} \max_{u \in \mathbb{B}} \langle u, y - q \rangle.$$

Since $\mathbb{B}$ is compact and the inner product is bilinear and continuous, it is possible to apply the Sion's minimax theorem (see [35]), in such a way to obtain

$$\begin{aligned}\mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x); Q) &= \sup_{y\in\mathscr{A}_{\mathscr{U}}(x)} \max_{u\in\mathbb{B}} \min_{q\in Q}\langle u, y-q\rangle = \sup_{y\in\mathscr{A}_{\mathscr{U}}(x)} \max_{u\in\mathbb{B}} [\langle u, y\rangle - \varsigma(u; Q)] \\ &= \sup_{u\in\mathbb{B}}[\varsigma(u; \mathscr{A}_{\mathscr{U}}(x)) - \varsigma(u; Q)],\end{aligned}$$

thereby completing the proof. □

The present analysis is devised in such a way to take advantage from specific properties of $v_{\mathscr{A}_{\mathscr{U}}}$, which are inherited by the particular form taken by the set-valued map $\mathscr{A}_{\mathscr{U}}$. To this purpose, recall that a set-valued map $\mathscr{F}:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be

- p.h. on a cone $D \subseteq \mathbb{R}^n$ if

$$\mathscr{F}(\lambda x) = \lambda\mathscr{F}(x), \quad \forall\lambda > 0,\ \forall x \in D;$$

- concave on the convex set $D \subseteq \mathbb{R}^n$ if

$$\mathscr{F}(tx_1 + (1-t)x_2) \subseteq t\mathscr{F}(x_1) + (1-t)\mathscr{F}(x_2), \quad \forall x_1,\ x_2 \in D,\ \forall t \in [0, 1];$$

- Lipschitz continuous on a set $D \subseteq \mathbb{R}^n$ if there exists $l \geq 0$ such that

$$\mathrm{haus}(\mathscr{F}(x_1); \mathscr{F}(x_2)) \leq l\|x_1 - x_2\|, \quad \forall x_1,\ x_2 \in D.$$

Lemma 1 (Properties of $\mathscr{A}_{\mathscr{U}}$) *Let a set-valued map $\mathscr{A}_{\mathscr{U}}:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be defined as in (1). Under the standing assumptions $(a_0)-(a_2)$ the following assertions hold:*

(i) $\mathrm{dom}\,\mathscr{A}_{\mathscr{U}} = \mathbb{R}^n$*;*
(ii) $\mathscr{A}_{\mathscr{U}}$ *takes convex and compact values;*
(iii) $\mathscr{A}_{\mathscr{U}}(\mathbf{0}) = \{\mathbf{0}\}$*;*
(iv) $\mathscr{A}_{\mathscr{U}}$ *is p.h. and concave on* $\mathbb{R}^n$*;*
(v) $\mathscr{A}_{\mathscr{U}}$ *is Lipschitz continuous on* $\mathbb{R}^n$*.*

Proof (i) The equality trivially follows from formula (1) and the nonemptiness of Ω.

(ii) Fix an arbitrary $x \in \mathbb{R}^n$ and consider the evaluation map $\mathrm{v}_x : \mathbf{M}_{m\times n}(\mathbb{R}) \longrightarrow \mathbb{R}^m$, which is defined as

$$\mathrm{v}_x(A_\omega) = A_\omega x.$$

Observe that, owing to (a_1), v_x is a linear map acting between finite-dimensional Euclidean spaces and, as such, it is continuous. Since it is $\mathscr{A}_{\mathscr{U}}(x) = \mathrm{v}_x(\mathbf{A}(\Omega))$ and, according to assumption (a_2), $\mathbf{A}(\Omega)$ is compact, then also its image through v_x is compact. Since, according to assumption (a_2), $\mathbf{A}(\Omega)$ is convex, also $\mathrm{v}_x(\mathbf{A}(\Omega))$ is convex.

(iii) By assumption (a_1) one has $A_\omega\mathbf{0} = \mathbf{0}$ for every $\omega \in \Omega$, whence the equality in the assertion.

(iv) Both the mentioned properties are straightforward consequences of the linearity of each map A_ω.

(v) As $\mathbf{A}_\omega$ is a compact subset of $\mathbf{M}_{m\times n}(\mathbb{R})$, it must be bounded, so there exists $\beta > 0$ such that

$$\|A_\omega\|_{\mathbf{M}} \le \beta, \quad \forall \omega \in \Omega. \tag{11}$$

Take arbitrary $x_1,\ x_2 \in \mathbb{R}^n$ and $y \in \mathscr{A}_{\mathscr{U}}(x_1)$. Then, it must be $y = A_{\omega_1}x_1$ for some $\omega_1 \in \Omega$. Then, by linearity of A_{ω_1} from inequality (11) one obtains

$$\begin{aligned}\operatorname{dist}(y; \mathscr{A}_{\mathscr{U}}(x_2)) = \operatorname{dist}\big(A_{\omega_1}x_1; \mathscr{A}_{\mathscr{U}}(x_2)\big) &= \inf_{\omega\in\Omega} \|A_{\omega_1}x_1 - A_\omega x_2\| \\ &\le \|A_{\omega_1}x_1 - A_{\omega_1}x_2\| \le \|A_{\omega_1}\|_{\mathbf{M}}\|x_1 - x_2\| \\ &\le \beta\|x_1 - x_2\|.\end{aligned}$$

Thus, by arbitrariness of $y \in \mathscr{A}_{\mathscr{U}}(x_1)$ it is possible to deduce

$$\begin{aligned}\operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x_1); \mathscr{A}_{\mathscr{U}}(x_2)) &= \sup_{y\in\mathscr{A}_{\mathscr{U}}(x_1)} \operatorname{dist}(y; \mathscr{A}_{\mathscr{U}}(x_2)) \\ &\le \beta\|x_1 - x_2\|, \quad \forall x_1,\ x_2 \in \mathbb{R}^n.\end{aligned}$$

Interchanging the role of $x_1,\ x_2 \in \mathbb{R}^n$ leads to

$$\operatorname{haus}(\mathscr{A}_{\mathscr{U}}(x_1); \mathscr{A}_{\mathscr{U}}(x_2)) \le \beta\|x_1 - x_2\|, \quad \forall x_1,\ x_2 \in \mathbb{R}^n.$$

This completes the proof. □

Remark 3 Lemma 1 implies that $\mathscr{A}_{\mathscr{U}}$ belongs to the class of set-valued maps called fans after [23], in particular to the subclass of the compactly generated ones. Like convex processes in the sense of Rockafellar (see [34, Section 39]), they can be regarded as the simplest (set-valued) generalization of linear operators. Unlike convex processes, they typically fail to have convex graphs.

The properties of $\nu_{\mathscr{A}_{\mathscr{U}}}$ (inherited from those of $\mathscr{A}_{\mathscr{U}}$) needed for the subsequent analysis are gathered in the next lemma.

Lemma 2 (Properties of $\nu_{\mathscr{A}_{\mathscr{U}}}$) *Let the function $\nu_{\mathscr{A}_{\mathscr{U}}} : \mathbb{R}^n \longrightarrow [0, +\infty]$ be defined as in (9). Under the standing assumptions $(a_0) - (a_2)$ the following assertions hold:*

(i) $\operatorname{dom} \nu_{\mathscr{A}_{\mathscr{U}}} = \mathbb{R}^n$;
(ii) $\nu_{\mathscr{A}_{\mathscr{U}}}$ *is convex on* $\mathbb{R}^n$;
(iii) $\nu_{\mathscr{A}_{\mathscr{U}}}$ *is Lipschitz continuous on* $\mathbb{R}^n$.

If, in addition, C and Q are cones, then

(iv) $\nu_{\mathscr{A}_{\mathscr{U}}}$ *is p.h..*

Proof Observe first that function $x \mapsto \operatorname{dist}(x; C)$ is Lipschitz continuous on $\mathbb{R}^n$ and, as C is convex, convex on $\mathbb{R}^n$. Thus, it suffices to prove that all the above assertions hold true for the function $x \mapsto \operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x); Q)$.

(i) According to Lemma 1(ii), $\mathscr{A}_{\mathscr{U}}(x)$ is compact and, as such, bounded, for every $x \in \mathbb{R}^n$. Thus, by the continuity of function $y \mapsto \operatorname{dist}(y; Q)$ one has $\operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x); Q) < +\infty$, for every $x \in \mathbb{R}^n$.

(ii) In the light of Lemma 1(iv), it is true that

$$\mathscr{A}_{\mathscr{U}}(tx_1 + (1-t)x_2) \subseteq t\mathscr{A}_{\mathscr{U}}(x_1) + (1-t)\mathscr{A}_{\mathscr{U}}(x_2), \quad \forall x_1,\ x_2 \in \mathbb{R}^n, \forall t \in [0,1].$$

Then, by virtue of the convexity of function $y \mapsto \mathrm{dist}\,(y; Q)$, it follows

$$\begin{aligned}
&\mathrm{exc}(\mathscr{A}_{\mathscr{U}}(tx_1 + (1-t)x_2); Q) \\
&= \sup_{y \in \mathscr{A}_{\mathscr{U}}(tx_1+(1-t)x_2)} \mathrm{dist}\,(y; Q) \\
&\leq \sup_{y \in t\mathscr{A}_{\mathscr{U}}(x_1)+(1-t)\mathscr{A}_{\mathscr{U}}(x_2)} \mathrm{dist}\,(y; Q) \\
&= \sup_{\substack{y_1 \in \mathscr{A}_{\mathscr{U}}(x_1) \\ y_2 \in \mathscr{A}_{\mathscr{U}}(x_2)}} \mathrm{dist}\,(ty_1 + (1-t)y_2; Q) \\
&\leq \sup_{\substack{y_1 \in \mathscr{A}_{\mathscr{U}}(x_1) \\ y_2 \in \mathscr{A}_{\mathscr{U}}(x_2)}} [t\mathrm{dist}\,(y_1; Q) + (1-t)\mathrm{dist}\,(y_2; Q)] \\
&= t \sup_{y_1 \in \mathscr{A}_{\mathscr{U}}(x_1)} \mathrm{dist}\,(y_1; Q) + (1-t) \sup_{y_2 \in \mathscr{A}_{\mathscr{U}}(x_2)} \mathrm{dist}\,(y_2; Q) \\
&= t\mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x_1); Q) + (1-t)\mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x_2); Q), \\
&\qquad \forall x_1,\ x_2 \in \mathbb{R}^n, \forall t \in [0,1].
\end{aligned}$$

(iii) By recalling that, if A and B are arbitrary subsets of $\mathbb{R}^m$, then $\mathrm{exc}(A; Q) \leq \mathrm{exc}(A; B) + \mathrm{exc}(B; Q)$, in the light of Lemma 1(v) one obtains

$$\begin{aligned}
\mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x_1); Q) &\leq \mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x_1); \mathscr{A}_{\mathscr{U}}(x_2)) + \mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x_2); Q) \\
&\leq \beta\|x_1 - x_2\| + \mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x_2); Q), \quad \forall x_1,\ x_2 \in \mathbb{R}^n,
\end{aligned}$$

whence

$$\mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x_1); Q) - \mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x_2); Q) \leq \beta\|x_1 - x_2\|, \quad \forall x_1,\ x_2 \in \mathbb{R}^n.$$

By interchanging the role of x_1 and x_2, one can deduce the Lipschitz continuity of $x \mapsto \mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x); Q)$.

(iv) In the light of Lemma 1(iv), it is true that

$$\mathscr{A}_{\mathscr{U}}(tx) = t\mathscr{A}_{\mathscr{U}}(x), \quad \forall x \in \mathbb{R}^n,\ \forall t > 0.$$

So it suffices to recall that, under the additional hypotheses, functions $x \mapsto \mathrm{dist}\,(x; C)$ and $y \mapsto \mathrm{dist}\,(y; Q)$ are p.h. and hence

$$\begin{aligned}
\mathrm{exc}(\mathscr{A}_{\mathscr{U}}(tx); Q) &= \sup_{y \in \mathscr{A}_{\mathscr{U}}(tx)} \mathrm{dist}\,(y; Q) = \sup_{y \in t\mathscr{A}_{\mathscr{U}}(x)} \mathrm{dist}\,(y; Q) \\
&= \sup_{y \in \mathscr{A}_{\mathscr{U}}(x)} \mathrm{dist}\,(ty; Q) = t \sup_{y \in \mathscr{A}_{\mathscr{U}}(x)} \mathrm{dist}\,(y; Q) \\
&= t\mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x); Q), \quad \forall x \in \mathbb{R}^n,\ \forall t > 0.
\end{aligned}$$

This completes the proof. □

Remark 4 (i) It is worth pointing out that assertion (iii) in Lemma 2 is not a mere consequence of assertion (ii). Indeed, the convexity of a function whose domain is $\mathbb{R}^n$ can only entail a local Lipschitz behaviour for such a function. Assertion (iii) says that $\nu_{\mathscr{A}_{\mathscr{U}}}$ is globally Lipschitz continuous, with a constant β (the same Lipschitz constant as $\mathscr{A}_{\mathscr{U}}$) valid all over $\mathbb{R}^n$.

(ii) All the properties of $\nu_{\mathscr{A}_{\mathscr{U}}}$ mentioned in Lemma 2 can be established in an alternative way, by exploiting well-known properties of the support function and the representation of $\nu_{\mathscr{A}_{\mathscr{U}}}$ provided in Proposition 3.

2.5 Subtransversality of Sets

The set C plays the role of a geometric constraint for a problem (RSFP). A geometric qualification for collections of sets, which enables one to treat separately the constraint C and the set-valued inclusion problem $\mathscr{A}_{\mathscr{U}}(x) \subseteq Q$ relies on the concept of subtransversality. This concept, which can be regarded as a metric generalization of the notion of transversality, as it is classically understood in differential topology (see, for instance, [19, Chapter 3]), formalizes a certain "good" mutual arrangement of several sets in space. It emerged as a regularity property under different names in different topics within variational analysis, such as subdifferential and normal calculus, Lipschitzian behaviour of set-valued maps (calmness and metric subregularity), weak sharp minima (see [5, 24, 27]). Its development was definitely stimulated by investigations in the convergence theory of projection methods for feasibility problems (see [28]). Recall that a pair of subsets $S_1,\ S_2$ of an Euclidean space is said to be *subtransversal* at $\bar{s} \in S_1 \cap S_2$ if there exist positive constants η and δ such that

$$[S_1 + (\eta r)\mathbb{B}] \cap [S_2 + (\eta r)\mathbb{B}] \cap \mathrm{B}[\bar{s}, \delta] \subseteq (S_1 \cap S_2) + r\mathbb{B}, \quad \forall r \in [0, \delta).$$

This property admits the following equivalent metric reformulation, which will be exploited in the sequel: the pair $S_1,\ S_2$ is subtransversal at $\bar{s}$ iff there exist $\kappa,\ \delta > 0$ such that

$$\mathrm{dist}\,(x; S_1 \cap S_2) \le \kappa\,[\mathrm{dist}\,(x; S_1) + \mathrm{dist}\,(x; S_2)]\,, \quad \forall x \in \mathrm{B}[\bar{s}, \delta] \tag{12}$$

(see [28, Theorem 1(ii)]). The geometric idea behind the above metric inequality is transparent: "if you are close to both the sets of the pair, then the intersection cannot be too far away" to quote [5]. Referring the reader to [5, 24, 27, 28] for major details, for the purposes of the present analysis it is useful to recall some situations, in which this property takes place. If $\bar{s} \in \mathrm{int}\,(S_1 \cap S_2)$, then the pair $S_1,\ S_2$ is subtransversal at $\bar{s}$. If S_1 and S_2 are closed and convex and $\mathbf{0} \in \mathrm{int}\,(S_1 - S_2)$, the pair $S_1,\ S_2$ is subtransversal at each point $\bar{s} \in S_1 \cap S_2$. Whenever S_1 and S_2 are both polyhedral sets (namely, they can be expressed as the intersection of finitely many closed halfspaces) and $\bar{s}$ is any element in their intersection, the inequality in (12) becomes globally true (δ can be taken arbitrarily positive) (see [24, Theorem 8.35]).

3 Conditions for Solvability and Error Bounds

Before exposing the main result of the paper, it is worthwhile to summarize general properties enjoyed by the solution set to (RSFP) (robust feasible solutions) in the adopted setting.

Proposition 4 *Under the assumptions* $(a_0) - (a_2)$*,*

(i) $\mathrm{S}_{C,\mathscr{U},Q}$ *is a closed convex (possibly empty) set;*
(ii) if, in addition, C *and* Q *are cones,* $\mathrm{S}_{C,\mathscr{U},Q}$ *is also a cone (thereby, nonempty).*

Proof (i) This assertion is a straightforward consequence of Proposition 2 and Lemma 2(ii) and (iii).

(ii) In this case, as Q is a closed cone, $A_\omega^{-1}(Q)$ is a closed cone for every $\omega \in \Omega$. Therefore, $\mathscr{A}_{\mathscr{U}}^{+}(Q) = \bigcap_{\omega\in\Omega} A_\omega^{-1}(Q)$ is a closed cone. Since so is C, the claim follows from the characterization of $\mathrm{S}_{C,\mathscr{U},Q}$ in (3). □

Let us consider the partition $\{R_i \subseteq \mathbb{R}^n \;\; i = 1, 2, 3\}$ of the set $[\nu_{\mathscr{A}_{\mathscr{U}}} > 0]$, which is defined as follows

$$R_1 = \{x \in \mathbb{R}^n \;:\; \mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x); Q) > 0, \quad \mathrm{dist}\,(x; C) > 0\},$$

$$R_2 = \{x \in \mathbb{R}^n \;:\; \mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x); Q) = 0, \quad \mathrm{dist}\,(x; C) > 0\},$$

and

$$R_3 = \{x \in \mathbb{R}^n \;:\; \mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x); Q) > 0, \quad \mathrm{dist}\,(x; C) = 0\}.$$

In what follows, given $x \in \mathbb{R}^n$ it is convenient to collect in the clean-up set, henceforth denoted by

$$\Omega_x = \{\omega \in \Omega \;:\; \mathrm{dist}\,(A_\omega x; Q) = \mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x); Q)\},$$

all those values of ω for which $A_\omega x$ is a farthest element in $\mathscr{A}_{\mathscr{U}}(x)$ from Q. Notice that, as $\mathscr{A}_{\mathscr{U}}(x)$ is nonempty and compact, it follows that $\Omega_x \neq \varnothing$, for every $x \in \mathbb{R}^n$. From the robustness point of view Ω_x singles out all those uncertain scenarios (and corresponding matrices), which are accountable for the worst violation of the constraint system.

In view of the statement of the next result, let us introduce the following constants associated with the data of a given (RSFP):

$$\tau_1 = \inf\left\{\|v\| \;:\; v \in \left[\overline{\mathrm{co}} \bigcup_{\omega\in\Omega_x} A_\omega^\top \frac{A_\omega x - \Pi_Q(A_\omega x)}{\mathrm{dist}\,(A_\omega x; Q)}\right] + \frac{x - \Pi_C(x)}{\mathrm{dist}\,(x; C)}, \quad x \in R_1\right\}$$

$$\tau_2 = \inf\left\{\|v\| \;:\; v \in \left[\overline{\mathrm{co}} \bigcup_{\omega\in\Omega} A_\omega^\top(\mathrm{N}(A_\omega x; Q) \cap \mathbb{B})\right] + \frac{x - \Pi_C(x)}{\mathrm{dist}\,(x; C)}, \quad x \in R_2\right\}$$

and

$$\tau_3 = \inf\left\{ \|v\| : v \in \left[\overline{\text{co}} \bigcup_{\omega\in\Omega_x} A_\omega^\top \frac{A_\omega x - \Pi_Q(A_\omega x)}{\text{dist}\,(A_\omega x; Q)} \right] + (\mathrm{N}(x; C) \cap \mathbb{B}), \quad x \in R_3 \right\}.$$

To understand the meaning of the above constants, it is helpful to remark that

$$\frac{A_\omega x - \Pi_Q(A_\omega x)}{\text{dist}\,(A_\omega x; Q)} \in \mathrm{N}(\Pi_Q(A_\omega x); Q) \cap \mathbb{S}, \quad \forall \omega \in \Omega_x, \; \forall x \in R_1 \cup R_3,$$

and

$$\frac{x - \Pi_C(x)}{\text{dist}\,(x; C)} \in \mathrm{N}(\Pi_C(x); C) \cap \mathbb{S}, \quad \forall x \in R_1 \cup R_2.$$

On the basis of the above constructions, define

$$\tau_{\mathscr{U}} = \min\{\tau_1, \tau_2, \tau_3\}. \tag{13}$$

One is now in a position to establish a condition for the existence of robust feasible solutions and global error bounds for (RSFP), which are expressed via the residual function $\nu_{\mathscr{A}_{\mathscr{U}}}$.

Theorem 1 *With reference to* (RSFP)*, let assumptions* $(a_0) - (a_2)$ *be satisfied. If* Ω *is a separated compact topological space and* $\tau_{\mathscr{U}} > 0$*, then* $\mathrm{S}_{C,\mathscr{U},Q} \neq \varnothing$ *and the following estimate holds*

$$\text{dist}\left(x; \mathrm{S}_{C,\mathscr{U},Q}\right) \le \tau_{\mathscr{U}}^{-1} \nu_{\mathscr{A}_{\mathscr{U}}}(x), \quad \forall x \in \mathbb{R}^n. \tag{14}$$

Proof In the light of Proposition 2 the characterization $\mathrm{S}_{C,\mathscr{U},Q} = [\nu_{\mathscr{A}_{\mathscr{U}}} \le 0]$ holds. Consequently, observe that in the case $[\nu_{\mathscr{A}_{\mathscr{U}}} > 0] = \varnothing$ it happens that $\mathrm{S}_{C,\mathscr{U},Q} = \mathbb{R}^n$, so the estimate in (14) becomes trivially true. To the contrary, suppose now that $[\nu_{\mathscr{A}_{\mathscr{U}}} > 0] \neq \varnothing$. By Lemma 2(ii) $\nu_{\mathscr{A}_{\mathscr{U}}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex. Thus, all the assertions in the thesis can be achieved by applying Proposition 1, upon showing that $\tau_{\mathscr{U}} > 0$ implies the condition in (8) to be actually satisfied, with $\tau = \tau_{\mathscr{U}}$. So, take an arbitrary $x \in [\nu_{\mathscr{A}_{\mathscr{U}}} > 0]$. If this is true, one (and only one) of the following cases must take place:

• Case $x \in R_1$: according to the sum rule for subdifferential (4), one has

$$\partial \nu_{\mathscr{A}_{\mathscr{U}}}(x) = \partial \text{exc}(\mathscr{A}_{\mathscr{U}}(\cdot); Q)(x) + \partial \text{dist}\,(\cdot; C)\,(x). \tag{15}$$

By the Ioffe-Tikhomirov rule (6), which can be invoked under the assumption made on Ω, one finds

$$\begin{aligned}\partial \mathrm{exc}(\mathscr{A}_{\mathscr{U}}(\cdot); Q)(x) &= \partial \left(\sup_{\omega\in\Omega} \mathrm{dist}\,(A_\omega(\cdot)); Q) \right)(x) \\ &= \partial \left(\sup_{\omega\in\Omega} \mathrm{dist}\,(\cdot; Q) \circ A_\omega \right)(x) \\ &= \overline{\mathrm{co}} \left(\bigcup_{\omega\in\Omega_x} \partial \mathrm{dist}\,(\cdot; Q)\,(A_\omega x) \right).\end{aligned}$$

Then, the chain rule for the subdifferential calculus (5) enables one to write

$$\partial \mathrm{dist}\,(\cdot; Q)\,(A_\omega x) = A_\omega^\top \left(\partial \mathrm{dist}\,(\cdot; Q)\,(A_\omega x)\right).$$

Since $A_\omega x \notin Q$, because $\omega \in \Omega_x$ and $\mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x); Q) > 0$, then according to formula (7) it results in

$$\partial \mathrm{dist}\,(\cdot; Q)\,(A_\omega x) = \left\{ \frac{A_\omega x - \Pi_Q(A_\omega x)}{\mathrm{dist}\,(A_\omega x; Q)} \right\}.$$

Again, as $x \notin C$, the same subdifferential rule gives

$$\partial \mathrm{dist}\,(\cdot; C)\,(x) = \left\{ \frac{x - \Pi_C(x)}{\mathrm{dist}\,(x; C)} \right\}. \tag{16}$$

Thus, if $x \in R_1$ one obtains

$$\partial \nu_{\mathscr{A}_{\mathscr{U}}}(x) = \left[\overline{\mathrm{co}} \bigcup_{\omega\in\Omega_x} A_\omega^\top \frac{A_\omega x - \Pi_Q(A_\omega x)}{\mathrm{dist}\,(A_\omega x; Q)} \right] + \frac{x - \Pi_C(x)}{\mathrm{dist}\,(x; C)}.$$

By taking into account the definition of τ_1, the last equality allows one to say that if $v \in \partial \nu_{\mathscr{A}_{\mathscr{U}}}(x)$ it must be $\|v\| \geq \tau_1 \geq \tau_{\mathscr{U}} > 0$, which shows that the condition in (8) turns out to be satisfied.

• Case $x \in R_2$: notice that formula (15) still holds, whereas, since $\mathrm{exc}(\mathscr{A}_{\mathscr{U}}(\cdot); Q)(x) = 0$, now it happens that $A_\omega x \in Q$, for every $\omega \in \Omega$. In other words, $\Omega_x = \Omega$. In such an event, by formula (7) one obtains

$$\partial \mathrm{dist}\,(\cdot; Q)\,(A_\omega x) = \mathrm{N}(A_\omega x; Q) \cap \mathbb{B},$$

while equality (16) is still valid. As a consequence, if $x \in R_2$ one obtains

$$\partial \nu_{\mathscr{A}_{\mathscr{U}}}(x) = \left[\overline{\mathrm{co}} \bigcup_{\omega\in\Omega} A_\omega^\top \left(\mathrm{N}(A_\omega x; Q) \cap \mathbb{B}\right) \right] + \frac{x - \Pi_C(x)}{\mathrm{dist}\,(x; C)}.$$

Thus, by recalling the definition of τ_2, one deduces that if $v \in \partial \nu_{\mathscr{A}_{\mathscr{U}}}(x)$ it must be $\|v\| \geq \tau_2 \geq \tau_{\mathscr{U}} > 0$, saying that the condition in (8) is again satisfied.
• Case $x \in R_3$: by proceeding in the same manner as in the case $x \in R_1$, one obtains

$$\partial \mathrm{exc}(\mathscr{A}_{\mathscr{U}}(\cdot); Q)(x) = \overline{\mathrm{co}} \bigcup_{\omega \in \Omega_x} A_\omega^\top \frac{A_\omega x - \Pi_Q(A_\omega x)}{\mathrm{dist}\,(A_\omega x; Q)}.$$

Since in the present case it is $x \in C$, by taking into account the formula in (7) (second case), one finds

$$\partial \nu_{\mathscr{A}_{\mathscr{U}}}(x) = \left[\overline{\mathrm{co}} \bigcup_{\omega \in \Omega_x} A_\omega^\top \frac{A_\omega x - \Pi_Q(A_\omega x)}{\mathrm{dist}\,(A_\omega x; Q)} \right] + (\mathrm{N}(x; C) \cap \mathbb{B}).$$

Therefore, according to the definition of τ_3, one can state that if $v \in \partial \nu_{\mathscr{A}_{\mathscr{U}}}(x)$ it must be $\|v\| \geq \tau_3 \geq \tau_{\mathscr{U}} > 0$. Once again the condition in (8) turns out to be satisfied.

This completes the proof. □

The next example shows that the positivity condition on $\tau_{\mathscr{U}}$ in Theorem 1 is essential already for getting the existence of robust feasible solutions.

Example 3 Let $n = 1$, $m = 2$, and (RSFP) be defined by the data $\Omega = [-1, 1] \subseteq \mathbb{R}$, $C = [1, +\infty)$, $Q = [0, +\infty) \times \{0\}$, with $\mathbf{A} : [-1, 1] \longrightarrow \mathbf{M}_{2\times 1}(\mathbb{R})$ given by

$$\mathbf{A}(\omega) = \begin{pmatrix} 1 \\ \omega \end{pmatrix}, \qquad \omega \in [-1, 1].$$

Assumptions $(a_0) - (a_2)$ are evidently fulfilled. The reformulation as a set-valued inclusion problem amounts to

$$\text{find } x \in [1, +\infty) \text{ such that } \quad \mathscr{A}_{\mathscr{U}}(x) \subseteq [0, +\infty) \times \{0\},$$

where

$$\mathscr{A}_{\mathscr{U}}(x) = \mathrm{co} \left\{ \begin{pmatrix} x \\ x \end{pmatrix}, \begin{pmatrix} x \\ -x \end{pmatrix} \right\}.$$

It is readily seen that

$$\mathrm{S}_{C,\mathscr{U},Q} = \mathscr{A}_{\mathscr{U}}^{+1}([0, +\infty) \times \{0\}) \cap [1, +\infty) = \{0\} \cap [1, +\infty) = \varnothing.$$

With reference to the partition $\{R_i \subseteq \mathbb{R}\ i = 1, 2, 3\}$, whenever taking $x \in (0, 1) \subset R_1$, one finds $\Omega_x = \{-1,\ 1\}$. In the case $\omega = 1$, one has

$$\frac{A_1 x - \Pi_{[0,+\infty)\times\{0\}}(A_1 x)}{\mathrm{dist}\,(A_1 x; [0, +\infty) \times \{0\})} = \frac{\begin{pmatrix} x \\ x \end{pmatrix} - \begin{pmatrix} x \\ 0 \end{pmatrix}}{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

whereas in the case $\omega = -1$, one has

$$\frac{A_{-1}x - \Pi_{[0,+\infty)\times\{0\}}(A_{-1}x)}{\operatorname{dist}(A_{-1}x; [0, +\infty) \times \{0\})} = \frac{\begin{pmatrix} x \\ -x \end{pmatrix} - \begin{pmatrix} x \\ 0 \end{pmatrix}}{x} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

Consequently, one obtains

$$A_1^\top \frac{A_1 x - \Pi_{[0,+\infty)\times\{0\}}(A_1 x)}{\operatorname{dist}(A_1 x; [0, +\infty) \times \{0\})} = (1 \;\; 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

and similarly

$$A_{-1}^\top \frac{A_{-1}x - \Pi_{[0,+\infty)\times\{0\}}(A_{-1}x)}{\operatorname{dist}(A_{-1}x; [0, +\infty) \times \{0\})} = (1 \;\; -1) \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 1.$$

Since for $x \in R_1$ it is

$$\frac{x - \Pi_{[1,+\infty)}(x)}{\operatorname{dist}(x; [1, +\infty))} = \frac{x-1}{1-x} = -1,$$

it results in

$$\begin{aligned} \overline{\mathrm{co}} \bigcup_{\omega \in \{-1, 1\}} A_\omega^\top \frac{A_\omega x - \Pi_{[0,+\infty)\times\{0\}}(A_\omega x)}{\operatorname{dist}(A_\omega x; [0, +\infty) \times \{0\})} &+ \frac{x - \Pi_{[1,+\infty)}(x)}{\operatorname{dist}(x; [1, +\infty))} \\ &= \{1\} + \{-1\} = \{0\}. \end{aligned}$$

Thus one sees that $\tau_1 = \tau_{\mathscr{U}} = 0$, that is the condition $\tau_{\mathscr{U}} > 0$ fails to be satisfied.

As a further comment on Theorem 1, it must be remarked that it provides a condition for solution existence and error bounds, which is in general only sufficient, as illustrated by the next example.

Example 4 Let $n = 1$, $m = 2$, and (RSFP) be defined by the data $\Omega = [0, 1] \subseteq \mathbb{R}$, $C = [0, +\infty) = \mathbb{R}_+$, $Q = \mathbb{R} \times (-\infty, 0]$, with $\mathbf{A} : [0, 1] \longrightarrow \mathbf{M}_{2\times 1}(\mathbb{R})$ given by

$$\mathbf{A}(\omega) = \begin{pmatrix} 1 \\ \omega \end{pmatrix}, \qquad \omega \in [0, 1].$$

The set-valued inclusion problem reformulation amounts to

$$\text{find } x \in [0, +\infty) \text{ such that } \quad \mathscr{A}_{\mathscr{U}}(x) \subseteq \mathbb{R} \times (-\infty, 0],$$

where

$$\mathscr{A}_{\mathscr{U}}(x) = \mathrm{co} \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ x \end{pmatrix} \right\}.$$

For such an instance of (RSFP), as it is $\mathscr{A}^{+1}_{[0,1]}(\mathbb{R}\times(-\infty,0])=(-\infty,0]$, one finds

$$\mathrm{S}_{C,\mathscr{U},Q}=(-\infty,0]\cap[0,+\infty)=\{0\}.$$

In the present case, $[\nu_{\mathscr{A}_{\mathscr{U}}}>0]=R_2\cup R_3$, being $R_1=\varnothing$, with $R_2=(-\infty,0)$ and $R_3=(0,+\infty)$. If $x\in R_2$, one has

$$\frac{x-\Pi_{\mathbb{R}_+}(x)}{\mathrm{dist}\,(x;\mathbb{R}_+)}=-1,$$

and

$$\mathrm{N}(A_\omega x;\mathbb{R}\times(-\infty,0])=\begin{cases}\{\mathbf{0}\} & \text{if } \omega\in(0,1],\\ \mathbb{R}\times[0,+\infty) & \text{if } \omega=0.\end{cases}$$

Consequently, as it is

$$A_\omega^\top\left(\mathrm{N}(A_\omega x;\mathbb{R}\times(-\infty,0])\cap\mathbb{B}\right)=\begin{cases}\{\mathbf{0}\} & \text{if } \omega\in(0,1],\\ [-1,1] & \text{if } \omega=0,\end{cases}$$

one obtains

$$\begin{aligned}\left[\overline{\mathrm{co}}\bigcup_{\omega\in[0,1]}A_\omega^\top(\mathrm{N}(A_\omega x;\mathbb{R}\times(-\infty,0])\cap\mathbb{B})\right]+\frac{x-\Pi_{\mathbb{R}_+}(x)}{\mathrm{dist}\,(x;\mathbb{R}_+)}&=\overline{\mathrm{co}}\,\{0,\ [-1,1]\}+\{-1\}\\ &=[-1,1]+\{-1\}\\ &=[-2,0].\end{aligned}$$

It follows that $\tau_{\mathscr{U}}=\tau_2=0$, the positivity condition on $\tau_{\mathscr{U}}$ thereby being violated. On the other hand, since it is

$$\mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x);\mathbb{R}\times(-\infty,0])=\left\{\begin{matrix}0 & \text{if } x\le 0,\\ x & \text{if } x>0\end{matrix}\right\}=\max\{x,\ 0\}$$

and $\mathrm{dist}\,(x;\mathbb{R}_+)=-\min\{x,\ 0\}$, it is true that

$$\mathrm{dist}\left(x;\mathrm{S}_{C,\mathscr{U},Q}\right)=|x|\le\max\{x,\ 0\}-\min\{x,\ 0\},\quad\forall x\in\mathbb{R}.$$

So a global error bound for the current instance of (RSFP) takes place, with $\tau=1$, even in the violation of the condition on $\tau_{\mathscr{U}}$.

Example 5 Let $n=m=2$ and consider the instance of (RSFP) defined by the data $\Omega=[0,1]\subseteq\mathbb{R}$, $C=\mathbb{R}^2$, $Q=-\mathbb{R}^2_+=\mathbb{R}^2_-$, with $\mathbf{A}:[0,1]\longrightarrow\mathbf{M}_{2\times2}(\mathbb{R})$ being given by

$$\mathbf{A}(\omega)=\begin{pmatrix}\omega & 1-\omega\\ 1-\omega & \omega\end{pmatrix},\qquad\omega\in[0,1].$$

A robust solution to the corresponding (RSFP) is thereby any $x \in \mathbb{R}^2$ satisfying the inequality system

$$\begin{cases} \omega x_1 + (1-\omega)x_2 \le 0 \\ (1-\omega)x_1 + \omega x_2 \le 0 \end{cases} \qquad \forall \omega \in [0, 1]. \tag{17}$$

In the present case $\mathscr{U} = \mathbf{A}([0, 1])$ turns out to be the Birkhoff polytope $\mathbf{B}_2$ of all the doubly stochastic matrices in $\mathbf{M}_{2\times 2}(\mathbb{R})$, whose extreme elements are the two permutation matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(see, for instance, [3, Birkhoff-von Neumann Theorem]). According to the Krein-Milman representation of $\mathbf{B}_2$, one can write

$$\mathbf{A}(\omega) = \omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1-\omega) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \omega \in [0, 1]$$

and hence it results in

$$A_\omega x = \left[\omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1-\omega) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \omega \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (1-\omega) \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}. \tag{18}$$

Notice that $\binom{x_2}{x_1}$ is the vector symmetric to $x = \binom{x_1}{x_2}$ with respect to the symmetry ax $x_2 = x_1$. Thus, it is plain to see that

$$\mathscr{A}_{\mathscr{U}}(x) = \left\{ \omega \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (1-\omega) \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \ : \ \omega \in [0, 1] \right\} = \operatorname{co} \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \right\}. \tag{19}$$

Clearly, $\mathscr{A}_{\mathscr{U}}$ is single-valued iff its argument is $x = t(1, 1)$, with $t \in \mathbb{R}$. The solution set $\mathrm{S}_{C,\mathscr{U},Q}$ of the problem in consideration coincides with the solution set of the set-valued inclusion problem

$$\text{find } x \in \mathbb{R}^2 \text{ such that } \quad \mathscr{A}_{\mathscr{U}}(x) \subseteq \mathbb{R}^2_-.$$

By recalling the representations in (18) and (19) as well as their plain geometrical meaning (otherwise, by elementary reasoning on the system (17)), it is readily seen that

$$\mathrm{S}_{C,\mathscr{U},Q} = \mathbb{R}^2_-.$$

Assumption $(a_0) - (a_2)$ are clearly satisfied by the problem data (in particular, as for (a_2) remember Example 1), while Ω, equipped with the topological structure induced by $\mathbb{R}$, is separated and compact. So, in order to apply Theorem 1, it remains to check that condition $\tau_{\mathscr{U}} > 0$ is valid. To do so, observe first that, since for this problem it is $R_1 = R_2 = \varnothing$, then one has $\tau_1 = \tau_2 = +\infty$ and therefore $\tau_{\mathscr{U}} = \tau_3$. To estimate this constant, it is useful to observe that, since it is

$$A_\omega x \in \operatorname{co} \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \right\}, \quad \forall \omega \in [0, 1], \ \forall x \in \mathbb{R}^2,$$

then for every $x \in \mathbb{R}^2 \backslash \mathbb{R}^2_-$ one obtains

$$a_\omega x = \frac{A_\omega x - \Pi_{\mathbb{R}^2_-}(A_\omega x)}{\operatorname{dist}\left(A_\omega x; \mathbb{R}^2_-\right)} = \begin{cases} \binom{1}{0} & \text{if } (A_\omega x)_1 > 0,\ (A_\omega x)_2 \le 0 \\ \frac{A_\omega x}{\|A_\omega x\|} & \text{if } (A_\omega x)_1 > 0,\ (A_\omega x)_2 > 0 \\ \binom{0}{1} & \text{if } (A_\omega x)_1 \le 0,\ (A_\omega x)_2 > 0, \end{cases}$$

where it is $A_\omega x = ((A_\omega x)_1,\ (A_\omega x)_2)$. It follows

$$\{a_\omega x \in \mathbb{R}^2 \ :\ \omega \in [0, 1],\ x \in \mathbb{R}^2 \backslash \mathbb{R}^2_-\} = \mathbb{S} \cap \mathbb{R}^2_+. \tag{20}$$

On account of the representation in (19), one can establish that for every $x \in \mathbb{R}^2 \backslash \mathbb{R}^2_-$ it holds

$$\operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x); \mathbb{R}^2_-) = \begin{cases} \|x\|, & \text{if } x_1 > 0,\ x_2 > 0 \\ \max\{x_1,\ x_2\} & \text{if } x_1 > 0,\ x_2 \le 0 \ \text{ or } \ x_1 \le 0,\ x_2 > 0, \end{cases}$$

while it is

$$\begin{aligned} \Omega_x &= \left\{\tilde{\omega} \in [0, 1] :\ \operatorname{dist}\left(A(\tilde{\omega}, x); \mathbb{R}^2_-\right) = \max_{\omega \in [0,1]} \operatorname{dist}\left(A_\omega x; \mathbb{R}^2_-\right)\right\} \\ &= \begin{cases} \{0,\ 1\} & \text{if } x \in \mathbb{R}^2 \backslash \mathbb{R}^2_-,\ x_1 \ne x_2 \\ [0,1] & \text{if } x \in \mathbb{R}^2 \backslash \mathbb{R}^2_-,\ x_1 = x_2. \end{cases} \end{aligned}$$

The reader should notice that, since it is $\operatorname{dist}\left(\cdot\,; \mathbb{R}^2\right) \equiv 0$, then

$$\nu_{\mathscr{A}_{\mathscr{U}}}(x) = \operatorname{exc}(\mathscr{A}_{\mathscr{U}}(\cdot); \mathbb{R}^2_-), \quad \forall x \in \mathbb{R}^2 \backslash \mathbb{R}^2_-, \tag{21}$$

so $\nu_{\mathscr{A}_{\mathscr{U}}}$ is actually p.h. and convex, consistently with what established in Lemma 2 (iii) and (iv).

What has been observed about Ω_x implies that, in computing τ_3, one needs to consider only the extreme elements of $\mathbf{A}([0, 1]) = \mathbf{B}_2$, namely

$$A_0^\top = A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_1^\top = A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By symmetry of the set $\mathbb{S} \cap \mathbb{R}^2_+$ with respect to the ax $x_1 = x_2$, one sees

$$A_0^\top \left(\mathbb{S} \cap \mathbb{R}^2_+\right) = \mathbb{S} \cap \mathbb{R}^2_+,$$

whereas it is evident that

$$A_1^\top \left(\mathbb{S} \cap \mathbb{R}^2_+\right) = \mathbb{S} \cap \mathbb{R}^2_+.$$

From (20) it follows that for every $x \in \mathbb{R}^2 \backslash \mathbb{R}^2_-$ it holds

$$\bigcup_{\omega \in \Omega_x} A_\omega^\top a_\omega x = \left\{ \begin{pmatrix} a_1(\omega, x) \\ a_2(\omega, x) \end{pmatrix}, \begin{pmatrix} a_2(\omega, x) \\ a_1(\omega, x) \end{pmatrix} \right\},$$

where $a_\omega x = (a_1(\omega, x),\ a_2(\omega, x))$, and hence

$$\bigcup_{x \in \mathbb{R}^2 \backslash \mathbb{R}^2_-} \left[\overline{\mathrm{co}} \bigcup_{\omega \in \Omega_x} A_\omega^\top a_\omega x \right] = \overline{\mathrm{co}} \left(\mathbb{S} \cap \mathbb{R}^2_+ \right) = \{ v = (v_1, v_2) \in \mathbb{B} \ : \ v_1 + v_2 \geq 1 \}.$$

Since it is

$$\mathrm{N}(x; \mathbb{R}^2) \cap \mathbb{B} = \{\mathbf{0}\},$$

from the above considerations it is possible to achieve the estimate

$$\tau_3 = \inf\{\|v\| \ : \ v = (v_1, v_2) \in \mathbb{B} \ : \ v_1 + v_2 \geq 1\} = \left\| \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right\| = \frac{1}{\sqrt{2}} > 0.$$

This shows that hypothesis $\tau_{\mathscr{U}} > 0$ is valid for the problem under study, thereby enabling to apply Theorem 1. It has been already seen that $\mathrm{S}_{C,\mathscr{U},Q} \neq \varnothing$. Let us check the validity of the estimate in (14). It is readily seen that

$$\begin{aligned} \mathrm{dist}\left(x; \mathrm{S}_{C,\mathscr{U},Q}\right) &= \mathrm{dist}\left(x; \mathbb{R}^2_-\right) \\ &= \begin{cases} \|x\|, & \text{if } x_1 > 0,\ x_2 > 0, \\ \max\{x_1,\ x_2\} & \text{if } x_1 > 0,\ x_2 \leq 0 \ \text{ or } \ x_1 \leq 0,\ x_2 > 0, \end{cases} \\ &= \mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x); \mathbb{R}^2_-), \quad \forall x \in \mathbb{R}^2 \backslash \mathbb{R}^2_-. \end{aligned}$$

By remembering the equality in (21), it is possible to conclude that for the problem under study the estimate

$$\mathrm{dist}\left(x; \mathrm{S}_{C,\mathscr{U},Q}\right) \leq \sqrt{2} \nu_{\mathscr{A}_{\mathscr{U}}}(x), \quad \forall x \in \mathbb{R}^2$$

is actually valid.

Remark 5 The reader should notice that the key condition $\tau_{\mathscr{U}} > 0$ in Theorem 1 is fully expressed in terms of problem data. From the computational viewpoint, it involves metric projections (onto C and Q), which can be easily calculated in many concrete cases (such as closed balls, half-spaces, orthants), matrix transposition (not inversion) and normal cone constructions. The computational complexity of the latter operation depends on the geometrical properties of the underlying sets C and Q. Again, in many concrete cases (e.g. for sublevel sets of given convex functionals and their intersection, balls, orthants) formulae for the explicit representation of the corresponding normal cones are available (see, for instance, [30]).

In order to deepen the understanding of the condition provided by Theorem 1 it may be helpful to consider which error bound condition can be derived from it in the special case card $\Omega = 1$, as to say in the absence of uncertainty, with (RSFP) thereby collapsing to (SFP). In such a setting, the solution set to (SFP) will be denoted by $\mathrm{S}_{C,A,Q}$. In the present event, one has $\mathscr{A}_{\mathscr{U}}(x) = \{Ax\}$ and hence $\mathrm{exc}(\mathscr{A}_{\mathscr{U}}(x); Q) = \mathrm{dist}\,(Ax; Q)$. Consequently, the partition $\{R_i \subseteq \mathbb{R}^n \;\; i = 1, 2, 3\}$ of $[\nu_{\mathscr{A}_{\mathscr{U}}} > 0]$, takes the form

$$R_1 = \{x \in \mathbb{R}^n \;:\; \mathrm{dist}\,(Ax; Q) > 0, \quad \mathrm{dist}\,(x; C) > 0\},$$

$$R_2 = \{x \in \mathbb{R}^n \;:\; \mathrm{dist}\,(Ax; Q) = 0, \quad \mathrm{dist}\,(x; C) > 0\},$$

and

$$R_3 = \{x \in \mathbb{R}^n \;:\; \mathrm{dist}\,(Ax; Q) > 0, \quad \mathrm{dist}\,(x; C) = 0\}.$$

In the proof of the next proposition, the constants $\tau_1,\ \tau_2,\ \tau_3$, related to the partition $\{R_i \subseteq \mathbb{R}^n \;\; i = 1, 2, 3\}$ play the same role as in the robust case.

Proposition 5 *With reference to* (SFP)*, suppose that the following set of conditions holds:*

(i) $A^\top \left(\mathrm{N}(\Pi_Q(Ax); Q) \cap \mathbb{S}\right) \cap [-\mathrm{N}(\Pi_C(x); C) \cap \mathbb{S}] = \varnothing, \quad \forall x \in R_1$*;*
(ii) $A^\top (\mathrm{N}(Ax; Q) \cap \mathbb{B}) \cap [-\mathrm{N}(\Pi_C(x); C) \cap \mathbb{S}] = \varnothing, \quad \forall x \in R_2$*;*
(iii) $A^\top \left(\mathrm{N}(\Pi_Q(Ax); Q) \cap \mathbb{S}\right) \cap [-\mathrm{N}(x; C) \cap \mathbb{B}] = \varnothing, \quad \forall x \in R_3$*.*

Then, $\mathrm{S}_{C,A,Q} \neq \varnothing$ *and there exists* $\tau > 0$ *such that*

$$\mathrm{dist}\left(x; \mathrm{S}_{C,A,Q}\right) \leq \tau^{-1} \left[\mathrm{dist}\,(Ax; Q) + \mathrm{dist}\,(x; C)\right], \quad \forall x \in \mathbb{R}^n. \tag{22}$$

Proof Take an arbitrary $x \in \mathbb{R}_1$. Then condition (i) is equivalent to the fact that

$$\mathbf{0} \notin A^\top \left(\mathrm{N}(\Pi_Q(Ax); Q) \cap \mathbb{S}\right) + [\mathrm{N}(\Pi_C(x); C) \cap \mathbb{S}]\,. \tag{23}$$

Since $\mathrm{N}(\Pi_Q(Ax); Q) \cap \mathbb{S}$ is a compact set, so is its image $A^\top \left(\mathrm{N}(\Pi_Q(Ax); Q) \cap \mathbb{S}\right)$ through the continuous map $A^\top$. Thus, as a Minkowski sum of two compact sets, $A^\top \left(\mathrm{N}(\Pi_Q(Ax); Q) \cap \mathbb{S}\right) + [\mathrm{N}(\Pi_C(x); C) \cap \mathbb{S}]$ is, in particular, closed. As a consequence, the exclusion in (23) entails

$$\inf\{\|v\| \,:\; v \in A^\top \left(\mathrm{N}(\Pi_Q(Ax); Q) \cap \mathbb{S}\right) + [\mathrm{N}(\Pi_C(x); C) \cap \mathbb{S}]\} > 0.$$

Then, by recalling that

$$\frac{Ax - \Pi_Q(Ax)}{\mathrm{dist}\,(Ax; Q)} \in \mathrm{N}(\Pi_Q(Ax); Q) \cap \mathbb{S},$$

the last inequality implies that $\tau_1 > 0$.

By a similar reasoning, it is possible to show that, in the case $x \in R_2$, since condition (ii) holds, it must be $\tau_2 > 0$ and, in the case $x \in R_3$, since condition (iii) holds, it must be $\tau_3 > 0$. Thus, by setting $\tau = \min\{\tau_1,\ \tau_2,\ \tau_3\}$, one finds that the subdifferential condition (8) in Proposition 1 is satisfied. It remains to observe that, in the current setting, one has

$$\nu_{\mathscr{A}_{\mathscr{U}}}(x) = \mathrm{dist}\,(Ax; Q) + \mathrm{dist}\,(x; C)\,, \quad \forall x \in \mathbb{R}^n,$$

in order to achieve the thesis. □

Remark 6 Proposition 5 provides a sufficient condition for solvability of (SFP) and global error bounds, which is based on projections onto normal cones. Notice that the estimate in (22) can be written, in particular, as

$$\mathrm{dist}\left(x; \mathrm{S}_{C,A,Q}\right) \leq \tau^{-1}\mathrm{dist}\,(Ax; Q)\,, \quad \forall x \in C. \tag{24}$$

Conditions ensuring error bounds for (SFP) have been established also in [39, 40] in a Hilbert space setting, by means of the notion of bounded linear regularity, which was introduced in [39, Definition 2.2]. In the current notations, that property postulates for any $r > 0$, such that $\mathrm{S}_{C,A,Q} \cap (r\mathbb{B}) \neq \varnothing$, the existence of $\gamma_r > 0$ such that

$$\mathrm{dist}\left(x; \mathrm{S}_{C,A,Q}\right) \leq \gamma_r^{-1}\mathrm{dist}\,(Ax; Q)\,, \quad \forall x \in C \cap (r\mathbb{B}). \tag{25}$$

More precisely, in [39, Proposition 2.5] several (qualification) conditions are established in terms of C, A, and Q, which guarantee the aforementioned property for a problem (SFP) to hold. While (22) and (25) refer exactly to the same problem, some features make them apparently different. Whereas inequality (22) provides a global error bound (τ is the same all over $\mathbb{R}^n$), in the very definition of bounded linear regularity the constant γ_r may happen to depend on the radius $r > 0$. Besides, the set of conditions (i)–(iii) in Proposition 5 (which can guarantee also solution existence) is formulated in the domain space $\mathbb{R}^n$, whereas conditions (i)–(v) in [39, Proposition 2.5] are all formulated in the range space ($\mathbb{R}^m$, according to the current finite-dimensional setting). These distinguishing features reflect the different approach, by which the error bounds in (24) and in (25) can be obtained. Indeed, if putting problem (SFP) in the more general framework of classical generalized equations, one can glimpse behind an error bound a manifestation of a certain regularity property for set-valued maps. Following [13, Chapter 3.8], a set-valued map $\mathscr{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be metrically subregular at $\bar{x} \in \mathbb{R}^n$ for $\bar{y} \in \mathbb{R}^m$ if there exist positive constants δ and κ such that

$$\mathrm{dist}\left(x; \mathscr{F}^{-1}(\bar{y})\right) \leq \kappa\,\mathrm{dist}\,(\bar{y}; \mathscr{F}(x))\,, \quad \forall x \in \mathrm{B}\,[\bar{x}, \delta]\,.$$

Now, given a problem (SFP), by setting $\mathscr{F}_{C,A,Q} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ as being

$$\mathscr{F}_{C,A,Q}(x) = (x, Ax) - (C \times Q),$$

one has that $\bar{x} \in \mathrm{S}_{C,A,Q}$ iff $\mathbf{0} \in \mathscr{F}_{C,A,Q}(\bar{x})$. On the other hand, by equipping the cartesian product $\mathbb{R}^n \times \mathbb{R}^m$ with the sum distance, the metric subregularity property of $\mathscr{F}_{C,A,Q}$ at $\bar{x} \in \mathrm{S}_{C,A,Q}$ for $\mathbf{0}$ yields

$$\begin{aligned}\operatorname{dist}\left(x; \mathrm{S}_{C,A,Q}\right) &= \operatorname{dist}\left(x; \mathscr{F}^{-1}_{C,A,Q}(\mathbf{0})\right) \leq \kappa \operatorname{dist}\left(\mathbf{0}; \mathscr{F}_{C,A,Q}(x)\right) \\ &= \kappa \inf_{(c,v)\in C\times Q} \|(x, Ax) - (c, v)\| \\ &= \kappa \left[\operatorname{dist}(x; C) + \operatorname{dist}(Ax; Q)\right], \quad \forall x \in \mathrm{B}\left[\bar{x}, \delta\right].\end{aligned}$$

The metric subregularity property was deeply investigated all during the last decades, so various conditions ensuring this property to hold can be found in the variational analysis literature (see, for instance, [13, 24]). A particular case of interest, both for theoretical reasons and broad range of applications, takes place when C and Q are polyhedrons. In such an event, $\mathscr{F}_{C,A,Q}$ becomes a convex polyhedral map, i.e. its graph is a polyhedral convex subset of $\mathbb{R}^n \times \mathbb{R}^m$. Notice that if $\mathscr{F}_{C,A,Q}$ is convex polyhedral, so is also $\mathscr{F}^{-1}_{C,A,Q}$, sharing the same graph. For this class of maps a well-known result in variational analysis ensures the metric subregularity property at each point of their graph (see [24, Theorem 8.34]). This explains why condition (i) in [39, Proposition 2.5] works. More in general, under the standing assumptions on the data of (SFP), $\mathscr{F}_{C,A,Q}$ has still convex graph. The condition (ii) in [39, Proposition 2.5] reads: exist $u_0 \in C$ and $v_0 \in Q$ such that $Au_0 = v_0 \in \operatorname{int} Q$, that is

$$\mathbf{0} = Au_0 - v_0 \in Au_0 - \operatorname{int} Q = \operatorname{int}(Au_0 - Q) \subseteq \operatorname{int}(A(C) - Q),$$

which implies $\mathbf{0} \in \operatorname{int}(A(C) - Q)$. By the celebrated Robinson-Ursescu theorem for maps with convex graph (see [13, Theorem 5B.4]), such a condition (known as Robinson regularity condition) is equivalent to the metric regularity property of the following set-valued map

$$x \rightrightarrows \begin{cases} Ax - Q & \text{if } x \in C, \\ \varnothing & \text{if } x \notin C \end{cases}$$

at $\bar{x} \in \mathrm{S}_{C,A,Q}$ for $\mathbf{0}$. Such a property, in turn, implies the metric subregularity of the same map, leading to (25).

In contrast, the error bounds in (22) and (24) are obtained by specializing an approach involving set-valued inclusion problems (2), which can not be cast in the standard generalized equation format (see discussions in [37, 38]). Their behaviour is substantially different from that of classical generalized equations, as considered, for instance, in [13, 29]. As a matter of fact, since $\mathscr{A}_{\mathscr{U}}$ fails to be a convex set-valued map (remember Remark 3), the aforementioned connections with regularity properties of set-valued maps are dramatically broken. Thus, the robust approach as here proposed can only rely on the convexity property of the residual function $\nu_{\mathscr{A}_{\mathscr{U}}}$ and subdifferential calculus. Consistently, conditions (i)–(iii) in Proposition 5 are formulated in the dual space of $\mathbb{R}^n$.

In the special case in which C and Q are both closed convex cones, clearly one has $\mathrm{S}_{C,A,Q} \neq \varnothing$. For this case, it is possible to provide an error bound condition, which is formulated independently of the partition of $\mathbb{R}^n \setminus \mathrm{S}_{C,A,Q}$. A fact which is employed in the proof of the next result is pointed out in the remark below.

Remark 7 Let K_1 and K_2 be two cones such that $K_1 \cap (-K_2) = \{\mathbf{0}\}$. Then, for every $r > 0$ it must be

$$\mathbf{0} \notin K_1 + (K_2 \setminus \mathrm{int}\, r\mathbb{B}) .$$

Indeed, if it were $\mathbf{0} \in K_1 + (K_2 \setminus \mathrm{int}\, r_0\mathbb{B})$ for some $r_0 > 0$, then there would exist $k_1 \in K_1$ and $k_2 \in K_2 \setminus \{\mathbf{0}\}$ such that $\mathbf{0} = k_1 + k_2$. This would imply $k_1 = -k_2 \neq \mathbf{0}$ and hence $K_1 \cap (-K_2) \neq \{\mathbf{0}\}$.

Theorem 2 *With reference to* (SFP)*, suppose that:*

(i) $C \subseteq \mathbb{R}^n$ *and* $Q \subseteq \mathbb{R}^m$ *are (nonempty) closed, convex cones;*
(ii) $\ker A^\top \cap Q^\ominus = \{\mathbf{0}\}$*;*
(iii) $A^\top \left(Q^\ominus\right) \cap \left(-C^\ominus\right) = \{\mathbf{0}\}$.

Then there exists $\tau > 0$ *such that the following estimate holds*

$$\mathrm{dist}\left(x; \mathrm{S}_{C,A,Q}\right) \leq \tau^{-1}[\mathrm{dist}\,(Ax; Q) + \mathrm{dist}\,(x; C)], \quad \forall x \in \mathbb{R}^n. \tag{26}$$

Proof Observe that, on account of Remark 7, if taking the two cones $K_1 = A^\top \left(Q^\ominus\right)$ and $K_2 = C^\ominus$, hypothesis (iii) implies

$$\mathbf{0} \notin A^\top \left(Q^\ominus\right) + \left[C^\ominus \setminus \mathrm{int}\, \mathbb{B}\right],$$

which, in turn, passing to subsets, entails

$$\mathbf{0} \notin A^\top \left(Q^\ominus \cap \mathbb{B}\right) + \left[C^\ominus \cap \mathbb{S}\right]. \tag{27}$$

Hypothesis (iii) is also equivalent to

$$C^\ominus \cap \left[-A^\top \left(Q^\ominus\right)\right] = \{\mathbf{0}\}.$$

Thus, if taking now $K_1 = C^\ominus$ and $K_2 = A^\top \left(Q^\ominus\right)$, it implies

$$\mathbf{0} \notin \left[A^\top \left(Q^\ominus\right) \setminus \mathrm{int}\, r\mathbb{B}\right] + C^\ominus, \quad \forall r > 0,$$

which, again passing to subsets, entails

$$\mathbf{0} \notin \left[A^\top \left(Q^\ominus\right) \setminus \mathrm{int}\, r\mathbb{B}\right] + \left(C^\ominus \cap \mathbb{B}\right), \quad \forall r > 0. \tag{28}$$

Notice that owing to hypothesis (ii) it is $\mathbf{0} \notin A^\top \left(Q^\ominus \cap \mathbb{S}\right)$, otherwise it would exist $q \in Q^\ominus$, with $q \neq \mathbf{0}$, such that $A^\top q = \mathbf{0}$, so that $q \in \ker A^\top \cap Q^\ominus \neq \{\mathbf{0}\}$. Therefore, since $A^\top\left(Q^\ominus \cap \mathbb{S}\right)$ is compact, then for some $r_0 > 0$ it must be

$$A^\top\left(Q^\ominus \cap \mathbb{S}\right) \subseteq A^\top\left(Q^\ominus\right) \setminus \operatorname{int} r_0\mathbb{B}.$$

By combining the last inclusion with the exclusion in (28), one obtains

$$\mathbf{0} \notin A^\top\left(Q^\ominus \cap \mathbb{S}\right) + \left[C^\ominus \cap \mathbb{B}\right]. \tag{29}$$

Notice that both the sets

$$A^\top\left(Q^\ominus \cap \mathbb{B}\right) + \left[C^\ominus \cap \mathbb{S}\right] \quad \text{and} \quad A^\top\left(Q^\ominus \cap \mathbb{S}\right) + \left[C^\ominus \cap \mathbb{B}\right]$$

are compact, as a sum of two compact sets. As a result, their union

$$D = \left\{A^\top\left(Q^\ominus \cap \mathbb{B}\right) + \left[C^\ominus \cap \mathbb{S}\right]\right\} \bigcup \left\{A^\top\left(Q^\ominus \cap \mathbb{S}\right) + \left[C^\ominus \cap \mathbb{B}\right]\right\}$$

is still compact and, by virtue of the exclusion in (27) and (29), one finds that $\mathbf{0} \notin D$. Consequently, it must be

$$\tau = \inf\{\|v\| \ : \ v \in D\} > 0. \tag{30}$$

Now, on the basis of the above considerations, let us show that, in the particular case in which card $\Omega = 1$, the condition $\tau_{\mathscr{U}} > 0$ is actually satisfied. Observe that if it is $x \in R_1$, since it is $\mathrm{N}(\Pi_Q(Ax); Q) \subseteq Q^\ominus$, then

$$A^\top \frac{Ax - \Pi_Q(Ax)}{\operatorname{dist}(Ax; Q)} + \frac{x - \Pi_C(x)}{\operatorname{dist}(x; C)} \in A^\top\left(Q^\ominus \cap \mathbb{B}\right) + \left[C^\ominus \cap \mathbb{S}\right];$$

if it is $x \in R_2$, then

$$A^\top\left(\mathrm{N}(Ax; Q) \cap \mathbb{B}\right) + \frac{x - \Pi_C(x)}{\operatorname{dist}(x; C)} \subseteq A^\top\left(Q^\ominus \cap \mathbb{B}\right) + \left[C^\ominus \cap \mathbb{S}\right];$$

if it is $x \in R_3$, then

$$A^\top \frac{Ax - \Pi_Q(Ax)}{\operatorname{dist}(Ax; Q)} + \left(\mathrm{N}(x; C) \cap \mathbb{B}\right) \subseteq A^\top\left(Q^\ominus \cap \mathbb{S}\right) + \left[C^\ominus \cap \mathbb{B}\right].$$

By remembering the definition of τ_1, τ_2, τ_3, and $\tau_{\mathscr{U}}$, from the above inclusions it is possible to deduce

$$\tau_{\mathscr{U}} \geq \tau > 0.$$

Notice furthermore that, by hypothesis (i) and card $\Omega = 1$, assumptions $(a_0) - (a_2)$ are trivially fulfilled. Thus the thesis follows from Theorem 1. $\square$

Remark 8 The reader should notice that hypothesis (ii) of Theorem 2 is fulfilled, in particular, whenever A is onto ($n \geq m$ and A with full rank), inasmuch as in this case one has $\ker(A^\top) = (A(\mathbb{R}^n))^\perp = \{\mathbf{0}\}$.

Hypothesis (iii) of Theorem 2 is fulfilled, in particular, whenever (SFP) is unconstrained, i.e. $C = \mathbb{R}^n$.

4 Error Bounds in the Polyhedral Case

In order for complementing the result exposed in Section 3, the present section focuses on global error bound conditions in the special case in which the data defining a (RSFP) happen to be polyhedral. To do so, throughout the current section, assumption (a_0) is enhanced as follows:

(a_0^+) $\varnothing \neq C \subseteq \mathbb{R}^n$ is closed and convex, and $\{\mathbf{0}\} \neq Q \subsetneqq \mathbb{R}^m$ is a closed, convex, pointed cone.

The following concept, introduced in [37], turned out to be effective in addressing solvability and stability issues within the context of the set-valued inclusion problems.

Definition 1 (*Q-increase property*) Let $\mathscr{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map, let $\{\mathbf{0}\} \neq Q \subsetneqq \mathbb{R}^m$ be a closed, convex cone, and let $x_0 \in \operatorname{dom} \mathscr{F}$. The map $\mathscr{F}$ is said to be *(metrically) Q-increasing* at x_0 if there exist $\alpha > 1$ and $\delta > 0$ such that

$$\forall r \in (0, \delta] \quad \exists z \in \mathrm{B}[x_0, r] \ : \ \mathrm{B}[\mathscr{F}(z), \alpha r] \subseteq \mathrm{B}[\mathscr{F}(x_0) + Q, r]. \tag{31}$$

The value

$$\operatorname{inc}(\mathscr{F}; Q; x_0) = \sup\{\alpha > 1 : \exists \delta > 0 \text{ for which (31) holds}\} \tag{32}$$

is called *exact bound of Q-increase* of $\mathscr{F}$ at x_0. If $\mathscr{F}$ is Q-increasing at each $x_0 \in \mathbb{R}^n$, then it is said to be Q-increasing on $\mathbb{R}^n$.

Examples of entire classes of set-valued maps enjoying the above property can be found in [37, 38], along with further comments about connections with the decrease principle in variational analysis.

Remark 9 In the sequel, the fact will be exploited that, whenever a set-valued map $\mathscr{F}$ is Q-increasing at x_0 while it is $\mathscr{F}(x_0) \not\subseteq Q$, then, if α and δ are as in Definition 1, for every $r \in (0, \delta]$ the inclusion in (31) must be necessarily satisfied by $z \in \mathrm{B}[x_0, r] \backslash \{x_0\}$ (see [38, Remark 4]).

Let us introduce below a qualification condition on which the current approach to the study of error bounds relies: with reference to the set-valued inclusion problem (1), the map $\mathscr{A}_{\mathscr{U}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to satisfy a *Slater-like qualification condition* if

$$\operatorname{int} Q \neq \varnothing \quad \text{and} \quad \exists u \in \mathbb{B} \ : \ \mathscr{A}_{\mathscr{U}}(u) \subseteq \operatorname{int} Q. \tag{SCQ}$$

The next lemma states that the qualification condition (SCQ) provides a sufficient condition for the map $\mathscr{A}_{\mathscr{U}}$ to be Q-increasing on $\mathbb{R}^n$ and, at the same time, an uniform estimate from below of its Q-increase exact bound.

Lemma 3 *Let $\mathscr{A}_{\mathscr{U}}$ be defined as in (1). Under the assumptions $(a_0^+) - (a_2)$, if (SCQ) holds then $\mathscr{A}_{\mathscr{U}}$ is Q-increasing on $\mathbb{R}^n$ and there exists $\alpha_{\mathscr{A}_{\mathscr{U}}} > 1$ such that*

$$\mathrm{inc}(\mathscr{A}_{\mathscr{U}}; Q; x) \geq \alpha_{\mathscr{A}_{\mathscr{U}}}, \quad \forall x \in \mathbb{R}^n.$$

Proof Let $u \in \mathbb{R}^n$ be as in (SCQ). According to Lemma 1(ii) the set $\mathscr{A}_{\mathscr{U}}(u)$ is compact. As a consequence, there exists $\eta > 0$ such that

$$\mathscr{A}_{\mathscr{U}}(u) + \eta\mathbb{B} \subseteq Q. \tag{33}$$

Take an arbitrary $x \in \mathbb{R}^n$ and $r > 0$ and set $z = x + ru$. Then, it is $z \in \mathrm{B}[x, r]$ and, by the concavity and the positive homogeneity property of $\mathscr{A}_{\mathscr{U}}$ ensured by Lemma 1(iv), on the account of the inclusion in (33) one finds

$$\begin{aligned}
&\mathrm{B}\left[\mathscr{A}_{\mathscr{U}}(z), (\eta+1)r\right] \\
&= \mathscr{A}_{\mathscr{U}}(x + ru) + (\eta+1)r\mathbb{B} \subseteq \mathscr{A}_{\mathscr{U}}(x) + r\mathscr{A}_{\mathscr{U}}(u) + \eta r\mathbb{B} + r\mathbb{B} \\
&= \mathscr{A}_{\mathscr{U}}(x) + r\left(\mathscr{A}_{\mathscr{U}}(u) + \eta\mathbb{B}\right) + r\mathbb{B} \subseteq \mathscr{A}_{\mathscr{U}}(x) + rQ + r\mathbb{B} \\
&= \mathrm{B}\left[\mathscr{A}_{\mathscr{U}}(x) + Q, r\right].
\end{aligned}$$

Thus, by setting $\alpha_{\mathscr{A}_{\mathscr{U}}} = \eta + 1 > 1$, the above inclusion shows that $\mathscr{A}_{\mathscr{U}}$ fulfils the condition in (31), with $\alpha = \alpha_{\mathscr{A}_{\mathscr{U}}}$ and any $\delta > 0$. By recalling the definition in (32), one can conclude that $\mathrm{inc}(\mathscr{A}_{\mathscr{U}}; Q; x) \geq \alpha_{\mathscr{A}_{\mathscr{U}}}$, thereby completing the proof. $\square$

The role of the qualification condition (SCQ) is explained by the next proposition. For its proof, it is convenient to recall the following basic properties of the excess operator: let $Q \subseteq \mathbb{R}^m$ be a closed, convex cone; then

(p_1) $\forall S \subseteq \mathbb{R}^m$, $\mathrm{exc}(S + Q; Q) = \mathrm{exc}(S; Q)$;

(p_2) $\forall r > 0$ and $\forall S \subseteq \mathbb{R}^m$, such that it is $\mathrm{exc}(S; Q) > 0$, it holds $\mathrm{exc}(\mathrm{B}[S, r]; Q) = \mathrm{exc}(S; Q) + r$

(for their proofs, the reader is referred to [37, Remark 2.1(iv)] and [37, Lemma 2.2], respectively).

Proposition 6 *With reference to (RSFP), under the assumptions $(a_0^+) - (a_2)$ suppose that:*

(i) C and $\mathscr{A}_{\mathscr{U}}^{+1}(Q)$ are polyhedral;
(ii) the qualification condition (SCQ) is satisfied.

Then, if $\mathrm{S}_{C,\mathscr{U},Q} \neq \varnothing$, there exists $\tau > 0$ such that the following estimate holds

$$\mathrm{dist}\left(x; \mathrm{S}_{C,\mathscr{U},Q}\right) \leq \tau^{-1}\nu_{\mathscr{A}_{\mathscr{U}}}(x), \quad \forall x \in \mathbb{R}^n.$$

Proof By virtue of Lemma 3, hypothesis (ii) guarantees that $\mathscr{A}_{\mathscr{U}}$ is Q-increasing on $\mathbb{R}^n$. Let $\alpha \in (1, \alpha_{\mathscr{A}_{\mathscr{U}}})$, where $\alpha_{\mathscr{A}_{\mathscr{U}}}$ as in Lemma 3. Let us show that this property yields the following implication, which is a sort of Caristi-like property, as is meant in the context of variational analysis (see, for instance, [1] and [18, Chapter I.2]):

$$\begin{gathered}\forall x_0 \notin \mathscr{A}_{\mathscr{U}}^{+1}(Q) \quad \exists \hat{x} \in \mathbb{R}^n \backslash \{x_0\} \\ \operatorname{exc}(\mathscr{A}_{\mathscr{U}}(\hat{x}); Q) + (\alpha - 1)\|\hat{x} - x_0\| \leq \operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x_0); Q).\end{gathered} \tag{34}$$

Indeed, notice that by continuity of function $x \mapsto \operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x); Q)$ (remember the argument in the proof of Lemma 2(iii)) and the fact that $\operatorname{exc}(\mathscr{A}_{\mathscr{U}}; x_0) > 0$, there exists $\delta > 0$ such that

$$\operatorname{exc}(\mathscr{A}_{\mathscr{U}}; x) > 0, \quad \forall x \in \mathrm{B}\,[x_0, \delta]\,.$$

By the Q-increase property of $\mathscr{A}_{\mathscr{U}}$ at x_0, if taking $r \in (0, \delta)$, there exists $z \in \mathrm{B}\,[x_0, r]$, such that

$$\mathrm{B}\,[\mathscr{A}_{\mathscr{U}}(z), \alpha r] \subseteq \mathrm{B}\,[\mathscr{A}_{\mathscr{U}}(x_0) + Q, r]\,. \tag{35}$$

It is to be noticed that, since $\mathscr{A}_{\mathscr{U}}(x_0) \not\subseteq Q$, then, according to what was observed in Remark 9, it must be $z \in \mathbb{R}^n \backslash \{x_0\}$. By exploiting the properties (p_1) and (p_2) of the excess operator and the inclusion in (35), which is valid because $z \in \mathrm{B}\,[x_0, \delta]$ and hence $\operatorname{exc}(\mathscr{A}_{\mathscr{U}}(z); Q) > 0$, one obtains

$$\begin{aligned}\operatorname{exc}(\mathscr{A}_{\mathscr{U}}(z); Q) &= \operatorname{exc}(\mathrm{B}\,[\mathscr{A}_{\mathscr{U}}(z), \alpha r]\,; Q) - \alpha r \\ &\leq \operatorname{exc}(\mathrm{B}\,[\mathscr{A}_{\mathscr{U}}(x_0) + Q, r]\,; Q) - \alpha r \\ &= \operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x_0) + Q; Q) + r - \alpha r \\ &= \operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x_0); Q) - (\alpha - 1) r.\end{aligned}$$

By taking into account that $\|z - x_0\| \leq r$, from the above inequality it follows

$$\operatorname{exc}(\mathscr{A}_{\mathscr{U}}(z); Q) + (\alpha - 1)\|z - x_0\| \leq \operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x_0); Q),$$

which shows the validity of (34). Such a Caristi-like condition, by virtue of the Bishop-Phelps/Ekeland variational principle (see, for instance, [18, Chapter I.2]), implies the existence of $x_\alpha \in \mathbb{R}^n$ such that

$$\operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x_\alpha); Q) = 0 \tag{36}$$

and

$$\|x_\alpha - x_0\| \leq \frac{\operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x_0); Q)}{\alpha - 1}. \tag{37}$$

By combining (36) and (37), one obtains

$$\operatorname{dist}\left(x_0; \mathscr{A}_{\mathscr{U}}^{+1}(Q)\right) \leq \|x_\alpha - x_0\| \leq \frac{\operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x_0); Q)}{\alpha - 1}.$$

By the arbitrariness of x_0, the above argument enables to achieve the following estimate

$$\operatorname{dist}\left(x; \mathscr{A}_{\mathscr{U}}^{+1}(Q)\right) \leq \frac{\operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x); Q)}{\alpha - 1}, \quad \forall x \in \mathbb{R}^n, \tag{38}$$

in the case $x \in \mathscr{A}_{\mathscr{U}}^{+1}(Q)$ the above inequality being trivially true. Now, by virtue of hypothesis (i), the metric reformulation of the global subtransversality of pairs of polyhedral sets given in (12) allows one to write for a proper $\kappa > 0$

$$\begin{aligned} \operatorname{dist}\left(x; \mathrm{S}_{C,\mathscr{U},Q}\right) &= \operatorname{dist}\left(x; C \cap \mathscr{A}_{\mathscr{U}}^{+1}(Q)\right) \\ &\leq \kappa \left[\operatorname{dist}(x; C) + \operatorname{dist}\left(x; \mathscr{A}_{\mathscr{U}}^{+1}(Q)\right)\right], \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

Thus, by taking into account the estimate in (38), it results in

$$\operatorname{dist}\left(x; \mathrm{S}_{C,\mathscr{U},Q}\right) \leq \kappa \left[\operatorname{dist}(x; C) + \frac{\operatorname{exc}(\mathscr{A}_{\mathscr{U}}(x); Q)}{\alpha - 1}\right] \leq \tau^{-1} \nu_{\mathscr{A}_{\mathscr{U}}}(x), \quad \forall x \in \mathbb{R}^n,$$

provided that one takes $\tau \leq \left(\kappa \max\{1, (\alpha - 1)^{-1}\}\right)^{-1}$. This completes the proof. □

The hypotheses of Proposition 6 are formulated only partially in terms of problem data. In order to derive from it a more satisfactory condition for error bounds, one can refer to a quantitative form of regularity for linear maps. To this aim, recall that a linear map, represented by $A \in \mathbf{M}_{m\times n}(\mathbb{R})$, is said to be covering (at a linear rate) if there exists $\sigma > 0$ such that $A\mathbb{B} \supseteq \sigma\mathbb{B}$, in which case the quantity

$$\operatorname{cov}(A) = \sup\{\sigma > 0 \ : \ A\mathbb{B} \supseteq \sigma\mathbb{B}\}$$

is called exact covering bound of A. Roughly speaking, it provides a measure of how much A is surjective, with $\operatorname{cov}(A) = 0$ signaling the failure of such a property. Historically, this particular manifestation of the regularity of a map played a crucial role in understanding the deep connections between different topics of variational analysis (see [24, 29]). For the purposes of the present analysis, it suffices to recall that the following characterization of the exact covering bound holds:

$$\operatorname{cov}(A) = \operatorname{dist}\left(\mathbf{0}; A^\top \mathbb{S}\right) = \min_{u \in \mathbb{S}} \|A^\top u\| \tag{39}$$

(see [29, Corollary 1.58]).

Theorem 3 (Error bound under polyhedral assumptions) *With reference to* (RSFP), *under the assumptions* $(a_0^+) - (a_2)$*, suppose that:*

(i) C*,* Q *and* $\mathbf{A}(\Omega) = \mathscr{U}$ *are polyhedral;*
(ii) $\operatorname{cov}(\mathscr{A}_{\mathscr{U}}) = \inf_{\omega\in\Omega} \operatorname{cov}(A_\omega) > 0$*;*
(iii) $\operatorname{int}\left(\bigcap_{\omega\in\Omega} A_\omega^{-1}(Q)\right) \neq \varnothing$.

Then, if $\mathrm{S}_{C,\mathscr{U},Q} \neq \varnothing$*, there exists* $\tau > 0$ *such that the following estimate holds*

$$\operatorname{dist}\left(x; \mathrm{S}_{C,\mathscr{U},Q}\right) \leq \tau^{-1} \nu_{\mathscr{A}_{\mathscr{U}}}(x), \quad \forall x \in \mathbb{R}^n. \tag{40}$$

Proof According to hypothesis (i), as a polytope, by the Krein-Milman theorem $\mathscr{U}$ can be expressed as

$$\mathscr{U} = \overline{\mathrm{co}}\, \{A_1, \ldots, A_k\},$$

where $k \in \mathbb{N}\backslash\{0\}$, by a proper choice of $A_i \in \mathscr{U}$. Therefore, one readily sees that

$$\mathscr{A}_{\mathscr{U}}^{+1}(Q) = \bigcap_{i=1}^{k} A_i^{-1}(Q). \tag{41}$$

Notice that, since Q is polyhedral and hence each set $A_i^{-1}(Q)$ is polyhedral as well, for $i = 1, \ldots, k$, so is the intersection in (41). This fact makes the hypothesis (i) of Proposition 6 satisfied.

By hypothesis (iii), there exist $u \in \mathbb{R}^n$ and $\epsilon > 0$ such that

$$u + \epsilon\mathbb{B} \subseteq \bigcap_{\omega \in \Omega} A_\omega^{-1}(Q).$$

Since Q is pointed, it is possible to assume that $u \neq \mathbf{0}$. Indeed, otherwise, if it is $\epsilon\mathbb{B} \subseteq \bigcap_{\omega\in\Omega} A_\omega^{-1}(Q)$, then for some $v \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ it must be

$$A_\omega v \in Q \qquad \text{and} \qquad A_\omega v \in -Q, \qquad \forall \omega \in \Omega,$$

whence $A_\omega v = \mathbf{0} \in Q$, for every $\omega \in \Omega$. It follows that $\mathbf{0} \neq v \in \mathscr{A}_{\mathscr{U}}^{+1}(Q)$ and

$$v + \epsilon\mathbb{B} \subseteq \mathscr{A}_{\mathscr{U}}^{+1}(Q) + \mathscr{A}_{\mathscr{U}}^{+1}(Q) = \mathscr{A}_{\mathscr{U}}^{+1}(Q).$$

Furthermore, since $\mathscr{A}_{\mathscr{U}}^{+1}(Q)$ is a cone, it is possible to assume that $u \in \mathbb{B}$. According to hypothesis (ii), if $\sigma \in (0, \operatorname{cov}(\mathscr{A}_{\mathscr{U}}))$, by linearity of any A_ω one has

$$A_\omega(\epsilon\mathbb{B}) \supseteq \sigma\epsilon\mathbb{B}, \quad \forall \omega \in \Omega,$$

which implies

$$A_\omega u + \sigma\epsilon\mathbb{B} \subseteq A_\omega(u + \epsilon\mathbb{B}) \subseteq Q, \quad \forall \omega \in \Omega.$$

The last inclusion says that

$$\mathscr{A}_{\mathscr{U}}(u) + \sigma\epsilon\mathbb{B} \subseteq Q,$$

meaning that the qualification condition (SCQ) is satisfied. All its hypotheses being satisfied, Proposition 6 can be invoked to achieve the assertion in the thesis. □

Remark 10 When problem (RSFP) collapses to (SFP), assumption (ii) in Theorem 3 would read as A being onto, while assumption (iii) would read as $\operatorname{int} A^{-1}(Q) \neq \varnothing$, so they appear to be redundant in comparison with condition (i) in [39, Proposition 2.5].

Example 6 If considering the problem in Example 5, which falls in the polyhedral case, it happens that, while as seen Theorem 1 does apply, Theorem 3 can not be applied, because hypothesis (ii) fails to be satisfied. Indeed, it is readily seen that $\mathrm{cov}(\mathbf{A}(1/2)) = 0$ and, consequently, $\mathrm{cov}(\mathscr{A}_{\mathscr{U}}) = 0$. This reveals that the conditions proposed in Theorem 3 are only sufficient for the validity of error bounds. It is also worth observing that, for the problem at the issue, it holds

$$\mathrm{int}\left(\bigcap_{\omega\in[0,1]} A_\omega^{-1}(\mathbb{R}^2_-)\right) = \mathrm{int}\left(A_0^{-1}(\mathbb{R}^2_-) \cap A_1^{-1}(\mathbb{R}^2_-)\right) = \mathrm{int}\,\mathbb{R}^2_- \neq \varnothing,$$

so hypothesis (iii) of Theorem 3 turns out to be satisfied.

Example 7 Let $n = m = 2$ and let a (RSFP) be defined by the data $\Omega = [0, 1] \subseteq \mathbb{R}$, $C = \mathbb{R}^2_+$, $Q = \mathbb{R}^2_-$, with $\mathbf{A} : [0, 1] \longrightarrow \mathbf{M}_{2\times 2}(\mathbb{R})$ being given by

$$\mathbf{A}(\omega) = \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix} = \omega A_0 + (1-\omega)A_1, \quad \omega \in [0, 1].$$

where

$$A_0 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In other words, the Krein-Milman representation of $\mathscr{U}$ is given by $\mathscr{U} = \overline{\mathrm{co}}\,\{A_0,\ A_1\}$. Thus, the introduced (RSFP) falls in the polyhedral case, thereby fulfilling hypothesis (i) of Theorem 3. Since it is

$$A_0^{-1}(\mathbb{R}^2_-) = \{x \in \mathbb{R}^2\ :\ x_1 \le x_2 \le 0\} \qquad \text{and} \qquad A_1^{-1}(\mathbb{R}^2_-) = \mathbb{R}^2_-,$$

it results in

$$\mathscr{A}_{\mathscr{U}}^{+1}(\mathbb{R}^2_-) = \{x \in \mathbb{R}^2\ :\ x_1 \le x_2 \le 0\}.$$

Consequently, one finds

$$\mathrm{S}_{C,\mathscr{U},Q} = \mathbb{R}^2_+ \cap \mathscr{A}_{\mathscr{U}}^{+1}(\mathbb{R}^2_-) = \{\mathbf{0}\}.$$

One readily sees that assumptions $(a_0^+) - (a_2)$ are satisfied by the problem at hand. Since it is

$$\mathrm{int}\left(\bigcap_{\omega\in[0,1]} A_\omega^{-1}(\mathbb{R}^2_-)\right) = \mathrm{int}\left(A_0^{-1}(\mathbb{R}^2_-) \cap A_1^{-1}(\mathbb{R}^2_-)\right) = \{x \in \mathbb{R}^2 : \ x_1 < x_2 < 0\}$$
$$\neq \varnothing,$$

hypothesis (iii) of Theorem is 3 fulfilled. As for hypothesis (ii), observe that, according to (39), it is

$$\begin{aligned}
\mathrm{cov}(A_\omega) &= \min_{u\in\mathbb{S}} \|A_\omega^\top u\| = \min_{u=(u_1,u_2)\in\mathbb{S}} \left\| \begin{pmatrix} u_1 \\ -\omega u_1 + u_2 \end{pmatrix} \right\| \\
&= \min_{\theta\in[0,2\pi]} \sqrt{\cos^2\theta + (-\omega\cos\theta + \sin\theta)^2} \\
&= \min_{\theta\in[0,2\pi]} \sqrt{1+\omega^2\cos^2\theta - 2\omega\sin\theta\cos\theta}.
\end{aligned}$$

By minimizing the continuous function $(\omega, \theta) \mapsto \psi(\omega, \theta)$, defined by

$$\psi(\omega,\theta) = \omega^2\cos^2\theta - 2\omega\sin\theta\cos\theta,$$

over the compact box $[0, 1] \times [0, 2\pi]$, one finds

$$\min_{(\omega,\theta)\in[0,1]\times[0,2\pi]} \psi(\omega,\theta) = \psi(1,\theta_*) \geq -\frac{3}{4},$$

where $\theta_* \in [\pi/4, \pi/3]$. It follows

$$\mathrm{cov}(\mathscr{A}_{\mathscr{U}}) = \inf_{\omega\in[0,1]} \mathrm{cov}(A_\omega) \geq \sqrt{1+\psi(1,\theta_*)} \geq \frac{1}{2} > 0,$$

thereby showing that hypothesis (ii) is satisfied. Thus, Theorem 3 can be applied to the present (RSFP). Let us check the validity of the global error bounds prescribed by this theorem. It is clear that

$$\mathrm{dist}\left(x; \mathrm{S}_{C,\mathscr{U},Q}\right) = \mathrm{dist}\,(x; \mathbf{0}) = \|x\|, \quad \forall x \in \mathbb{R}^2.$$

On the other hand, one has

$$\mathrm{dist}\left(x; \mathbb{R}^2_+\right) = \begin{cases} \|x\|, & \text{if } x \in \mathbb{R}^2_-, \\ 0, & \text{if } x \in \mathbb{R}^2_+, \\ -\min\{x_1, x_2\} & \text{if } x_1 < 0,\ x_2 > 0 \text{ or } x_1 > 0,\ x_2 < 0, \end{cases}$$

and

$$\mathrm{dist}\left(x; \mathscr{A}_{\mathscr{U}}^{+1}(\mathbb{R}^2_-)\right) = \begin{cases} \|x\|, & \text{if } x \in \mathbb{R}^2_+, \text{ or } x_1 > 0,\ x_2 \geq -x_1 \\ x_2, & \text{if } x_1 < 0,\ x_2 > 0, \\ \dfrac{|x_1 - x_2|}{\sqrt{2}}, & \text{if } x_2 < |x_1|, \\ 0, & \text{if } x_1 \leq x_2 \leq 0. \end{cases}$$

Consequently, by summing up the above functions one obtains

$$\mathrm{dist}\left(x;\mathbb{R}^2_+\right)+\mathrm{dist}\left(x;\mathscr{A}^{+1}_{\mathscr{U}}(\mathbb{R}^2_-)\right)=\begin{cases}\|x\|, & \text{if } x\in\mathbb{R}^2_+,\\ -\min\{x_1,\,x_2\}+\|x\| & \text{if } x_2<0,\\ & \quad x_2\ge -x_1,\\ -\min\{x_1,\,x_2\}+\dfrac{|x_1-x_2|}{\sqrt{2}}, & \text{if } x_1\ge 0,\\ & \quad x_2\le -x_1,\\ \|x\|+\dfrac{|x_1-x_2|}{\sqrt{2}}, & \text{if } x_1\le 0,\\ & \quad x_2\le x_1,\\ \|x\|, & \text{if } x_1\le x_2\le 0,\\ -\min\{x_1,\,x_2\}+x_2, & \text{if } x_1<0,\\ & \quad x_2>0.\end{cases}$$

Since it holds

$$\|x\|\le\sqrt{2}\max\{|x_1|,\,|x_2|\},\quad \forall x\in\mathbb{R}^2,$$

it is possible to conclude that, for the problem under discussion, the global error bound estimate in (40) holds true with $\tau^{-1}\ge\sqrt{2}$.

5 Conclusions

The contents of this paper describe a methodology for dealing with split feasibility problems, when some of their data happen to be affected by uncertainty. This methodology follows a robust approach to convex optimization problems under uncertainty. Evidences are brought of the fact that the recently developed set-valued inclusion theory offers a comfortable framework, where to analyse solution existence and error bound issues by techniques of convex and variational analysis. The condition established for solvability of the robust counterpart of a split feasibility problem and for global error bounds for the associated solution set are expressed in terms of metric projections and classic constructions from convex analysis, on the basis of the problem data. The fact that the main result is only a sufficient condition for error bounds leaves space for further investigations on the issue. Another direction for future research work deals with the case of (RSFP), for which also the data C and Q are affected by uncertainty. A different development of the proposed analysis should address a more general class of robust split feasibility problems, where the assumption of boundedness on $\mathscr{U}$ in (a_2) is removed. Instead, a complete rethinking of the approach should be required if assumption (a_1) is dropped out, leading to nonlinear split feasibility problems. Nonetheless, even in such a more general setting, the author is confident that an approach via set-valued inclusions could afford useful insights into the question.

Acknowledgements The author thanks Professor Y. Censor from the University of Haifa for useful discussions on the contents of the paper. The author wishes to thank as well both the anonymous referees for carefully reading the manuscript, and for their relevant remarks and comments, which contributed to improve effectively the quality of the paper.

Funding Open access funding provided by Università degli Studi di Milano - Bicocca within the CRUI-CARE Agreement.

References

1. Arutyunov, A.V., Zhukovskiy, E.S., Zhukovskiy, S.E.: Caristi-like condition and the existence of minima of mappings in partially ordered spaces. J. Optim. Theory Appl. **180**(1), 48–61 (2019)
2. Azé, D., Corvellec, J.-N.: Variational Methods in Classical Open Mapping Theorems. J. Convex Analysis **13**, 477–488 (2006)
3. Barvinok, A.: A Course in Convexity. American Mathematical Society, Providence, RI (2002)
4. Bauschke, H.H., Borwein, J.M.: On the convergence of von Neumann's alternating projection algorithm for two sets. Set-Valued Anal. **1**(2), 185–212 (1993)
5. Bauschke, H.H., Borwein, J.M.: On projection algorithms for solving convex feasibility problems. SIAM Rev. **38**(3), 367–426 (1996)
6. Ben-Tal, A., Boyd, S., Nemirovski, A.: Extending scope of robust optimization: Comprehensive robust counterparts of uncertain problems. Math. Progr. Series B **107**(1–2), 63–89 (2006)
7. Ben-Tal, A., Ghaoui, L.E., Nemirovski, A.: Robust optimization. Princeton University Press, Princeton (2009)
8. Ben-Tal, A., Nemirovski, A.: Robust convex optimization. Math. Oper. Res. **23**, 769–805 (1998)
9. Ben-Tal, A., Nemirovski, A.: Robust Optimization - methodology and applications. Math. Progr. Series B **92**, 453–480 (2002)
10. Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Prob. **18**(2), 441–453 (2002)
11. Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in a product space. Numer. Algorithms **8**, 221–239 (1994)
12. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Prob. **21**(6), 2071–2084 (2005)
13. Dontchev, A.L., Rockafellar, R.T.: Implicit functions and solution mappings. A view from variational analysis, Second edition, Springer Series in Operations Research and Financial Engineering, Springer, New York (2014)
14. El Ghaoui, L., Oustry, F., Lebret, H.: Robust solutions to uncertain semidefinite programs. SIAM J. on Optimization **9**, 33–52 (1998)
15. Fabian, M.J., Henrion, R., Kruger, A.Y., Outrata, J.V.: Error bounds: Necessary and sufficient conditions. Set-Valued Var. Anal. **18**(2), 121–149 (2010)
16. Facchinei, F., Pang, J.-S.: Finite-dimensional variational inequalities and complementarity problems. Vol. I, Springer Series in Operations Research, Springer-Verlag, New York (2003)

17. Gao, T., Shen, W., Wang, J.: Local error bound property for nonlinear split feasibility problems in Banach spaces. J. Nonlinear Convex Anal. **24**(5), 1129–1139 (2023)
18. Granas, A., Dugundji, J.: Fixed Point Theory. Springer, New-York (2003)
19. Hirsch, M.W.: Differential topology, Graduate Texts in Mathematics, No. 33. Springer-Verlag, New York-Heidelberg (1976)
20. Hurt, N.E.: Phase retrieval and zero crossings. Mathematical methods in image reconstruction. Mathematics and its Applications, 52, Kluwer Academic Publishers Group, Dordrecht (1989)
21. Huong, V.T., Xu, H.-K., Yen, N.D.: Stability analysis of split equality and feasibility problems. J. Glob. Optim. **92**(2), 411–429 (2025)
22. Huong, V.T., Xu, H.-K., Yen, N.D.: Stability of nonhomogeneous split equality and split feasibility problems with possibly nonconvex constraint sets, to appear in Optimization (2025)
23. Ioffe, A.D.: Nonsmooth analysis: Differential calculus of nondifferentiable mappings. Trans. Amer. Math. Soc. **266**(1), 1–56 (1981)
24. Ioffe, A.D.: Variational analysis of regular mappings. Theory and applications, Springer Monographs in Mathematics, Springer, Cham (2017)
25. Khan, A.A., Tammer, C., Zălinescu, C.: Set-valued optimization. An introduction with applications, Vector Optimization, Springer, Heidelberg (2015)
26. Kouvelis, P., Yu, G.: Robust discrete optimization and its applications. Nonconvex Optimization and its Applications, 14, Kluwer Academic Publishers, Dordrecht (1997)
27. Kruger, A.Y., Luke, D.R., Thao, N.H.: About subtransversality of collections of sets. Set-Valued Var. Anal. **25**(4), 701–729 (2017)
28. Kruger, A.Y., Luke, D.R., Thao, N.H.: Set regularities and feasibility problems. Math. Program. Ser. B **168**(1–2), 279–311 (2018)
29. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation. I. Basic theory. Springer-Verlag, Berlin (2006)
30. Mordukhovich, B.S., Nam, N.M.: Convex Analysis and Beyond, vol. 1. Basic theory. Springer, Cham (2022)
31. Pang, J.S.: Error bounds in mathematical programming. Math. Program. Ser. B **79**, 299–332 (1997)
32. Qu, B., Liu, B.: The split feasibility problem and its solution algorithm, Math. Probl. Eng. art. ID 4123168 (2018)
33. Reich, S., Truong, M.T., Mai, T.N.H.: The split feasibility problem with multiple output sets in Hilbert spaces. Optim. Lett. **14**(8), 2335–2353 (2020)
34. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (1970)
35. Sion, M.: On general minimax theorems. Pacific J. Math. **8**, 171–176 (1958)
36. Soyster, A.L.: Convex programming with set-inclusive constraints and applications to inexact linear programming. Operation Res. **21**, 1154–1157 (1973)
37. Uderzo, A.: On some generalized equations with metrically C-increasing mappings: solvability and error bounds with applications to optimization. Optimization **68**(1), 227–253 (2019)
38. Uderzo, A.: On some global implicit function theorems for set-valued inclusions with applications to parametric vector optimization. Optimization **74**(17), 4669–4700 (2025)
39. Wang, J.H., Hu, Y., Li, C., Yao, J.-C.: Linear convergence of CQ algorithms and applications in gene regulatory network inference. Inverse Prob. **33**(5), 055017 (2017)
40. Wang, J.H., Gao, T.M., Li, C., Yang, X.Q.: Relative regularity conditions and linear regularity properties for split feasibility problems in normed linear spaces. J. Math. Anal. Appl. **532**, 127982 (2024)
41. Wu, Z., Ye, J.J.: On error bounds for lower semicontinuous functions. Math. Program. Ser. A **92**, 301–314 (2002)
42. Xu, H.K.: Iterative methods for the split feasibility problem in infite-dimensional Hilbert spaces. Inverse Probl. **26**, 105018 (2010)
43. Zălinescu, C.: Convex Analysis in General Vector Spaces. World Scientific, Singapore (2002)