Some properties of a class of Network Games with strategic complements or substitutes*

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Abstract. We investigate a class of parametric network games which encompasses both the cases of strategic complements and strategic substitutes. In the case of a bounded strategy space, we derive a representation formula for the unique Nash equilibrium. We also prove a comparison result between the Nash equilibrium and the social optimum, and then compute the price of anarchy for some simple test problems.

1 Introduction

This chapter investigates some aspects of a class of Network Games, within the framework developed in the seminal paper [3], in a socio-economic context. For an excellent review on this topic the interested reader can refer to [10]. Here, we recall that the peculiarity of this approach is that each player is identified with the node of a graph and players that can interact directly are connected through links of the graph. The so called *peers* of a given player can influence her action, according to their proximity in the network of relationships. The influence of peers on a given neighbour player can be of two different types. Roughly speaking, for a given player, if an increase of the action of her peers causes an increase of the player's action we say that the peers act as strategic complements, if an increase of the action of her peers causes a decrease of her action we say that the peers act as strategic substitutes. In order to keep the analysis at a reasonable level of complexity, authors have mainly focused on games with strategic complements or with strategic substitutes, where the type of interaction is the same for all players. The graph structure has thus a prominent role in modelling the interactions among the various players who can represent different kinds of socio-economic agents, depending on the specific application.

As is common in social and economic game-theoretical models, two important concepts are the Nash equilibrium and the social optimum (or welfare) of the game which, in the above mentioned papers, were connected to graph-algebraic quantities. In particular, in the case of interior solution a very interesting representation formula has been derived in the seminal paper by Ballester et al. [3], which involves the so called Katz-Bonacich centrality measure [6]. As a matter of fact, a large number of papers devoted to this topic have focused on the case of interior solution and unbounded strategy space, utilizing classical game-theoretical

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methods, i.e., the best response approach. Only very recently some authors have framed the topic of Network Games in the theory of variational inequalities, although the variational inequality approach to Nash equilibrium problems was initiated by Gabay and Moulin [9] more than forty years ago. In this respect, we refer the reader to the interesting paper by Parise and Ozdaglar [15], which although comprehensive in many respects, such as uniqueness and sensitivity of equilibrium, does not focus on the Katz-Bonacich representation of the solution or on the comparison with the social optimum. On the other hand, in [16] the authors started to generalize some classical results to the case where some components of the solution lay on the boundary, while in [17] the case of a generalized Nash equilibrium has been treated for the first time, within the Network Games framework. The variational inequality approach has also been applied to a game with global complementarities and global congestion in [19].

In this work, we extend the results in [16] where we considered the standard quadratic reference model with strategic complements. Specifically, the paper is structured as follows. In the subsequent Section 2 we provide some basic material on graph theory and define the class of network games with strategic complements and substitutes. Moreover, we recall the definition of Nash equilibrium of a game and its relationship with variational inequalities. Section 3 is devoted to the investigation of a class of parametric quadratic utility functions considered in [1] which encompasses both the classes of strategic complements and substitutes. For both classes we derive a Katz-Bonacich-like representation formula in the case where the solution has some boundary components. Moreover, in the case of strategic complements, by exploiting the sequential best-response dynamics, we compare the components of the unique Nash equilibrium of the game and the unique social optimum, proving that the Nash equilibrium is component-wise less than or equal to the social optimal solution. Section 4 is devoted to illustrate our findings by means of some numerical experiments, and we also analyse the so called *price of anarchy*. We touch upon possible future developments in the small concluding section.

2 Basics on Network Games and variational inequalities

In Network Games players are represented by the nodes of a graph (V, E), where V is the sets of nodes and E is the set of arcs formed by ordered pairs of nodes (v, w). In the case where, for all arcs in the network, (v, w) and (w, v) are the same, and there are neither multiple arcs connecting the same pair of nodes, nor loops, the graph is called undirected and simple. In our model we allow for asymmetric relationships between pairs of players, hence we will consider directed graphs.

Two nodes v and w are said to be adjacent if they are connected by an arc, i.e., if (v, w) or (w, v) is an arc. The information about the adjacency of nodes can be stored in the adjacency matrix G whose elements g_{ij} are equal to 1 if (v_i, v_j) is an arc, 0 otherwise. We will also consider the more general case where each arc is given a non-negative weight w_{ij}. In this case, G is called the weighted adjacency matrix of the graph. G is thus an asymmetric and zero-diagonal matrix. Given a node v, the nodes connected to v with an arc are called the *neighbours* of v. A *walk* in the graph g is a finite sequence of the form $v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \ldots, e_{j_k}, v_{j_k}$, which consists of alternating nodes and arc of the graph, such that $v_{i_{t-1}}$ and v_{i_t} are end nodes of e_{j_t}. In the case of an unweighted graph, the *length* of a walk is simply the number of its arcs. Let us remark that, in a walk, it is allowed to visit a node or go through an arc more than once. The indirect connections between any two nodes in the graph are described by means of the powers of the adjacency matrix G. Indeed, for an unweighted graph, without loops and multiple arcs, it can be proved that the element $g_{ij}^{[k]}$ of G^k gives the number of walks of length k between v_i and v_j.

In the sequel, the set of players will be denoted by $\{1, 2, \ldots, n\}$ instead of $\{v_1, v_2, \ldots, v_n\}$. We denote with $A_i \subset \mathbb{R}$ the action space of player i, while $A = A_1 \times \cdots \times A_n$. A vector $x = (x_1, \ldots, x_n) \in A$ is called a *profile*. We also use the common notations $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $x = (x_i, x_{-i})$ when we wish to distinguish the action of player i from the action of all the other players. Each player i is endowed with a payoff function $u_i : A \to \mathbb{R}$ that she wishes to maximize. The notation $u_i(x, G)$ is often utilized when one wants to emphasize that the utility of player i also depends on the actions taken by her neighbours in the graph.

We now recall the definition of a Nash equilibrium, which is one of the most common solution concept in Game Theory.

Definition 1. *An action profile $x^* \in A$ is a Nash equilibrium iff for each $i \in \{1, \ldots, n\}$:*

$$u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*), \qquad \forall\ x_i \in A_i. \tag{1}$$

Another quantity of interest, in particular in socio-economic application, is the *Welfare* associated to each action profile:

$$W(x) := \sum_{i=1}^{n} u_i(x). \tag{2}$$

In the case where the function W has a unique maximizer x^{so} over A (called social optimum), and the game has a unique Nash equilibrium x^*, it is interesting to compute the ratio:

$$\gamma = \frac{W(x^*)}{W(x^{so})}, \tag{3}$$

which, in similar models, is known as the *price of anarchy* (see, e.g., [20]).

As mentioned in the introduction, it is convenient, for tractability reasons, to consider games where the neighbours of a player influence the player's behaviour in the same direction for all players. We make this concept precise with the help of the marginal utility function.

Definition 2. *The network game has the property of strategic complements if:*

$$\frac{\partial^2 u_i}{\partial x_j \partial x_i}(x) > 0, \qquad \forall\ (i, j) :\ g_{ij} \neq 0,\ \forall\ x \in A.$$

Definition 3. *The network game has the property of strategic substitutes if:*

$$\frac{\partial^2 u_i}{\partial x_j \partial x_i}(x) < 0, \qquad \forall\ (i, j) :\ g_{ij} \neq 0,\ \forall\ x \in A.$$

The variational inequality approach to Nash equilibrium problems is recalled in the following theorem. For an account of variational inequalities the interested reader can refer to [12, 14].

Theorem 1. *For each $i \in \{1, \dots, n\}$, let u_i be a continuously differentiable function on A and $u_i(\cdot, x_{-i})$ be concave with respect to its own action x_i, for each $x_{-i} \in A_{-i}$. Moreover, let A be closed and convex. Then, x^* is a Nash equilibrium if and only if it solves the variational inequality $VI(T, A)$: find $x^* \in A$ such that*

$$T(x^*)^\top (x - x^*) \geq 0, \qquad \forall\, x \in A, \tag{4}$$

where the operator

$$[T(x)]^\top := -\left(\frac{\partial u_1}{\partial x_1}(x), \dots, \frac{\partial u_n}{\partial x_n}(x)\right) \tag{5}$$

is also called the pseudo-gradient of the game.

We recall here some useful monotonicity properties.

Definition 4. *An operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone on A iff:*

$$[T(x) - T(y)]^\top (x - y) \geq 0, \qquad \forall\, x, y \in A.$$

If the equality holds only when $x = y$, T is said to be strictly monotone on A.
T is said to be τ-strongly monotone on A iff there exists $\tau > 0$ such that

$$[T(x) - T(y)]^\top (x - y) \geq \tau \|x - y\|^2, \qquad \forall\, x, y \in A.$$

Remark 1. *For linear operators on $\mathbb{R}^n$ the two concepts of strict and strong monotonicity coincide and are equivalent to the positive definiteness of the corresponding matrix.*

Conditions that ensure the unique solvability of a variational inequality problem are given by the following theorem (see, e.g. [14]).

Theorem 2. *If $K \subset \mathbb{R}^n$ is a compact convex set and $T : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on K, then the variational inequality problem $VI(T, K)$ admits at least one solution. In the case that K is unbounded, existence of a solution may be established under the following coercivity condition:*

$$\lim_{\|x\| \to +\infty} \frac{[T(x) - T(x_0)]^\top (x - x_0)}{\|x - x_0\|} = +\infty,$$

for $x \in K$ and some $x_0 \in K$.
Furthermore, the solution is unique if T is strictly monotone on K.

3 The parametric quadratic model

Let $A_i = [0, L_i]$ for any $i \in \{1, \dots, n\}$, hence $A = [0, L_1] \times \ldots \times [0, L_n]$. The payoff of player i is given by:

$$u_i(x) = -\frac{\beta}{2} x_i^2 + \alpha_i x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} f_{ij}(\alpha) x_i x_j, \qquad \alpha, \beta > 0. \tag{6}$$

The last term describes the interaction between player i and her neighbours. The coefficient α_i describes the type of agent, and in some economic applications (see e.g. [7]) can be interpreted as the household's parental capital. If, for all $j \neq i$ and a fixed value of $\alpha = (\alpha_1, \dots, \alpha_n)$,

$f_{ij}(\alpha) \geq 0$ holds, then the associated game falls in the class of games with strategic complements; if, for all $j \neq i$ and a fixed value of α, $f_{ij}(\alpha) \leq 0$ holds, it falls in the class of games with strategic substitutes. The pseudo-gradient's components of this game are easily computed as:

$$T_i(x) = \beta x_i - \alpha_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} f_{ij}(\alpha) x_j, \qquad i \in \{1, \ldots, n\},$$

which can be written in compact form as

$$T(x) = [\beta I - \mathcal{F}(\alpha)]x - \alpha, \tag{7}$$

where $\mathcal{F}(\alpha)$ is a zero-diagonal matrix whose off-diagonal entries are equal to $f_{ij}(\alpha)$, and is called the *interaction matrix*.

Throughout the paper we posit the symmetry assumption on the interaction matrix:

$$f_{ij}(\alpha) = f_{ji}(\alpha), \qquad \forall\ \alpha,\ \forall\ i, j \in \{1, \ldots, n\},\ i \neq j. \tag{S}$$

Remark 2. *Under the symmetry assumption (S), the game under consideration also falls in the class of potential games according to the definition introduced by Monderer and Shapley [13]. Indeed, a potential function is given by:*

$$P(x) = \sum_{i=1}^{n} u_i(x) - \frac{1}{2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} f_{ij}(\alpha) x_i x_j.$$

Applying a result of Monderer and Shapley to our case, we obtain in general, that the solutions of the problem $\max_{x \in A}\ P(x)$ *form a subset of the solution set of the Nash game. Therefore, if both problems have a unique solution it follows that they are equivalent.*

We will seek Nash equilibrium points by solving the variational inequality:

$$T(x^*)^\top (x - x^*) \geq 0, \qquad \forall\ x \in A. \tag{8}$$

The following lemma characterizes the monotonicity of F given in (7).

Lemma 1. *(a) Fix $\alpha > 0$ and let $f_{ij}(\alpha) \geq 0$ for any i, j. The matrix $\beta I - \mathcal{F}(\alpha)$ is positive definite iff*

$$\beta > \lambda_{max}(\mathcal{F}(\alpha)) = \rho(\mathcal{F}(\alpha)) \tag{9}$$

where $\lambda_{max}(\mathcal{F}(\alpha))$is the maximum eigenvalue of $\mathcal{F}(\alpha)$, and $\rho(\mathcal{F}(\alpha))$ its spectral radius.

(b) Let $f_{ij}(\alpha) \leq 0$ for any i, j. The matrix $\beta I - \mathcal{F}(\alpha)$ is positive definite iff

$$\beta > \lambda_{max}(\mathcal{F}(\alpha)) \tag{10}$$

or, equivalently, $\lambda_{min}(-\mathcal{F}(\alpha)) > -\beta$.

Moreover, the condition $\beta > \rho(\mathcal{F}(\alpha))$ is, in general, stronger than the two equivalent conditions above.

Proof.

(a) Let$f_{ij}(\alpha) \geq 0$ and recall that if M is a non-negative symmetric matrix, a consequence of the Perron-Frobenius Theorem is that $\rho(M) = \lambda_{max}(M)$. Furthermore,

$$\lambda_{max}(M) = \max_{x \neq 0} \frac{x^\top M x}{x^\top x}.$$

The matrix $\beta I - \mathcal{F}(\alpha)$ is positive definite iff $x^\top [\beta I - \mathcal{F}(\alpha)] x > 0$ for any $x \neq 0$, that is

$$\beta > \frac{x^\top \mathcal{F}(\alpha) x}{x^\top x}, \qquad \forall\, x \neq 0,$$

which is equivalent to $\beta > \lambda_{max}(\mathcal{F}(\alpha))$, and, as a consequence of the Perron-Frobenius theorem, we finally get that $\beta I - \mathcal{F}(\alpha)$ is positive definite iff $\beta > \lambda_{max}(\mathcal{F}(\alpha)) = \rho(\mathcal{F}(\alpha))$. We notice that this condition also ensures that the matrix $I - \frac{1}{\beta}\mathcal{F}(\alpha)$ is non singular and its inverse matrix can be expanded in a power series according to Lemma 2 below.

(b) Following the same reasoning as in the non-negative case, we get that the matrix $\beta I - \mathcal{F}(\alpha)$ is positive definite if and only if $\beta > \max_{x \neq \mathbf{0}} \frac{x^\top \mathcal{F}(\alpha) x}{x^\top x} = \lambda_{max}(\mathcal{F}(\alpha)) = -\lambda_{min}(-\mathcal{F}(\alpha))$. However, the condition $\beta > \rho(\mathcal{F}(\alpha))$ is stronger because

$$\rho(\mathcal{F}(\alpha)) \geq |\lambda_{max}(\mathcal{F}(\alpha))| \geq \lambda_{max}(\mathcal{F}(\alpha)).$$

□

In the next lemma we recall a well known result about series of matrices.

Lemma 2 (see, e.g., [2]). *Let M be a square matrix and consider the series $\sum_{p=0}^{\infty} M^p$. The series converges provided that $\lim_{p\to\infty} M^p = 0$, which is equivalent to $\rho(M) < 1$. In such case the matrix $I - M$ is non singular and we have the power series expansion $(I - M)^{-1} = \sum_{p=0}^{\infty} M^p$.*

We now introduce a centrality measure of networks, known as the Katz-Bonacich vector, (see, e.g., [6]), which allows for an interesting interpretation of the Nash equilibrium of network games. Although we confine our analysis to the symmetric case, we give here the definition for the general case of a general matrix G, with entries g_{ij}. Such a matrix can be thought of as the adjacency matrix of a weighted directed graph. The case of an undirected network without self loops is characterized by $g_{ij} = g_{ji}, j \neq i, g_{ii} = 0$, and if G is a $0-1$ matrix, the graph in unweighted.

Definition 5. *Let w be a non negative vector. The weighted vector of Katz-Bonacich, of parameter ϕ, in the graph is given by:*

$$b_w(G, \phi) = [I - \phi G]^{-1} w = \sum_{p=0}^{\infty} \phi^p G^p w. \tag{11}$$

The inverse exists and can be expressed by the series above if the condition $\phi\rho(G) < 1$ is satisfied. We also recall that, if $G \geq 0$, a theorem on non-negative matrices ensures that $[I - \phi G]^{-1}$ is non-negative too. In the simplest case of a $0-1$ adjacency matrix, Indeed, the (i, j) entry, $g_{ij}^{[p]}$, of the matrix G^p gives the number of walks of length p between nodes i and

j, and if $w = (1, \ldots, 1)$, $b_{w,i}(G, \phi)$ counts the total number of walks in the graph, which start at node i, exponentially damped by ϕ. In the general case, the weight of the links are taken into account, and paths reaching an arbitrary node j are pondered by w_j.

The importance of the Katz-Bonacich vector stems from the fact that, when the strategy space is $\mathbb{R}^n_+$, it is related in a simple manner to the unique Nash equilibrium of the game. Indeed, the relation given in [3] can be extended in a straightforward fashion to the case of the utility functions (6) as follows.

Theorem 3. *Let $A = \mathbb{R}^n_+$ and consider the utility functions defined in* (6)*, with $f_{ij} \geq 0$ for any i, j. Moreover, let $\beta > \rho(\mathcal{F}(\alpha))$. Then, the unique Nash equilibrium x^* is interior and given by:*

$$x^* = \frac{1}{\beta}\left[I - \frac{1}{\beta}\mathcal{F}(\alpha)\right]^{-1} \alpha = \sum_{p=0}^{\infty} \frac{1}{\beta^{p+1}} [\mathcal{F}(\alpha)]^p \alpha = \frac{1}{\beta} b_w \left(\mathcal{F}(\alpha), \frac{1}{\beta}\right). \tag{12}$$

We now recall a Proposition due to [1] which, under some additional assumptions, provides a sufficient condition for $\beta > \rho(\mathcal{F}(\alpha))$ to be true. This condition involves a smallness condition on the variance of α, and roughly speaking, means that a low variability of the types of players, given by α, entails a unique Nash equilibrium.

Proposition 1. *For a given type profile α, consider the game defined in* (6) *with interaction terms given by $f_{ij}(\alpha) = h_i(\alpha) f(\alpha_i - \alpha_j)$ or $f_{ij}(\alpha) = h_i(\alpha) f(|\alpha_i - \alpha_j|)$, where $f : \mathbb{R} \to \mathbb{R}$ is non-expansive and there is $\delta_0 \in \mathbb{R}$ such that $f(\delta_0) = 0$. Then, this is a game with complementarities that admits a unique equilibrium if the standard deviation of types σ_α satisfies the following inequality:*

$$\beta > n\, h(k)\, (\sqrt{2}\, \sigma_\alpha + |\delta_0|),$$

where $h(k) = \max_{i=1,\ldots,n} |h_i(\alpha)|$.

Proof. The proof can be found in [1] but we warn the reader that the formula therein differs from ours for the missing coefficient n multiplying δ_0 (probably due to a misprint). $\square$

We now assume that the strategies of each player have an upper bound and derive a Katz-Bonacich type representation of the solution, in the case where exactly k components take on their maximum value.

Theorem 4. *Let u_i be defined as in* (6)*, $\beta > \rho(\mathcal{F}(\alpha))$, $x_i \in [0, L_i]$ for any $i \in \{1, \ldots, n\}$ and x^* be the unique Nash equilibrium of the game.*

(a) Assume that $f_{ij}(\alpha) \geq 0$ for any $i, j \in \{1, \ldots, n\}$. We then have that $x_i^* > 0$ for any $i \in \{1, \ldots, n\}$. Moreover, assume that exactly k components of x^* take on their maximum value: $x^*_{i_1} = L_{i_1}, \ldots, x^*_{i_k} = L_{i_k}$, and denote with $\tilde{x}^* = (\tilde{x}^*_{i_{k+1}}, \ldots, \tilde{x}^*_{i_n})$ the subvector of the non-boundary components of x^*. We then get:

$$\tilde{x}^* = [\beta I_{n-k} - \mathcal{F}_1(\alpha)]^{-1} w = b_w \left(\mathcal{F}_1(\alpha)), \frac{1}{\beta}\right), \tag{13}$$

where $\mathcal{F}_1(\alpha)$ is the submatrix obtained from $\mathcal{F}(\alpha)$ choosing the rows $i_{k+1}, \ldots, i_n$ and the columns $i_{k+1}, \ldots, i_n$; $\mathcal{F}_2(\alpha)$ is the submatrix obtained from $\mathcal{F}(\alpha)$ choosing the rows $i_{k+1}, \ldots, i_n$ and the columns $i_1, \ldots, i_k$; $w = \alpha_{n-k} + \mathcal{F}_2(\alpha)\, L$ and $L = (L_{i_1}, \ldots, L_{i_k})$, $\alpha_{n-k} = (\alpha_{i_{k+1}}, \ldots, \alpha_{i_n})$.

(b) Assume now that $f_{ij}(\alpha) \le 0$ for any $i, j \in \{1, \dots, n\}$, and there are no zero components of the solution x^*, while exactly k components of x^* take on their maximum value. Then, formula (13) *also applies to this case. Moreover, $\tilde{x}^*$ can alternatively be expressed as:*

$$\tilde{x}^* = \frac{1}{\beta}\left[I_{n-k} + \frac{1}{\beta}\mathcal{F}_1(\alpha)\right] b_w\left(\mathcal{F}_1^2(\alpha), \frac{1}{\beta^2}\right). \tag{14}$$

Proof. (a) The Nash equilibrium $x^*$of the game solves the variational inequality

$$\sum_{i=1}^{n} T_i(x^*)^\top (x_i - x_i^*) \ge 0, \qquad \forall\, x \in A, \tag{15}$$

where $A = [0, L_1] \times \ldots \times [0, L_n]$. Let us assume that there exists l such that $x_l^* = 0$, and choose in (15) $x = (x_1^*, \dots, x_{l-1}^*, L_l, x_{l+1}^*, \dots, x_n^*) \in A$. With this choice, (15) reads:

$$0 \le T_l(x^*)x_l = \left(-\sum_{j\neq l}^{n} f_{lj}(\alpha)x_j^* - \alpha_i\right) L_l < 0$$

which yields a contradiction. Thus, $x_i^* > 0$ for any $i = 1, \dots, n$.

Let $\tilde{A}$ denote the face of A obtained intersecting A with the hyperplanes: $x_{i_1} = L_{i_1}, \dots, x_{i_k} = L_{i_k}$. Moreover, let $\tilde{x} = (x_{i_{k+1}}, \dots, x_{i_n})$, $\tilde{x}^* = (\tilde{x}_{i_{k+1}}^*, \dots, \tilde{x}_{i_n}^*)$ and $\tilde{T} : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ such that $\tilde{T}_{i_l}(\tilde{x})$ is obtained by fixing $x_{i_1} = L_{i_1}, \dots, x_{i_k} = L_{i_k}$ in $T_{i_l}(a)$. We consider now the restriction of (15) to $\tilde{A}$, which reads:

$$\sum_{l=k+1}^{n} \tilde{T}_{i_l}(\tilde{x}^*)(\tilde{x}_{i_l} - \tilde{x}_{i_l}^*) \ge 0, \qquad \forall\, \tilde{x} \in \tilde{A}.$$

Since we are assuming that exactly k components of the solution x^* reach their upper bounds, it follows that $\tilde{x}^*$ lies in the interior of $\tilde{A}$, hence

$$\tilde{T}(\tilde{x}^*) = 0,$$

which can be written explicitly as:

$$\beta x_{i_l}^* - \sum_{m=k+1}^{n} f_{i_l i_m}(\alpha)x_{i_m}^* = \alpha_{i_l} + \sum_{m=1}^{k} f_{i_l i_m}(\alpha)L_{i_m}, \;\; l = k+1, \dots, n,$$

which yields:

$$[\beta I_{n-k} - \mathcal{F}_1(\alpha)]\tilde{x}^* = \alpha_{n-k} + \mathcal{F}_2(\alpha)L. \tag{16}$$

Because the matrix $[\beta I_{n-k} - \mathcal{F}_1(\alpha)]$ is non singular, the thesis is proved.

(b) To prove (14), divide both sides of (16) by β and multiply by the matrix $[I_{n-k} + \frac{1}{\beta}\mathcal{F}_1(\alpha)]$ to get:

$$\left[I_{n-k} - \frac{1}{\beta^2}\mathcal{F}_1^2(\alpha)\right] = \frac{1}{\beta}\left[I_{n-k} + \frac{1}{\beta}\mathcal{F}_1(\alpha)\right](\alpha_{n-k} + \mathcal{F}_2(\alpha)L),$$

whence

$$\begin{aligned}
\tilde{x}^* &= \frac{1}{\beta}\left[I_{n-k} - \frac{1}{\beta^2}\mathcal{F}_1^2(\alpha)\right]^{-1}\left[I_{n-k} + \frac{1}{\beta}\mathcal{F}_1(\alpha)\right](\alpha_{n-k} + \mathcal{F}_2(\alpha)L) \\
&= \left[I_{n-k} + \frac{1}{\beta}\mathcal{F}_1(\alpha)\right]\left[I_{n-k} - \frac{1}{\beta^2}\mathcal{F}_1^2(\alpha)\right]^{-1}(\alpha_{n-k} + \mathcal{F}_2(\alpha)L) \\
&= \frac{1}{\beta}\left[I_{n-k} + \frac{1}{\beta}\mathcal{F}_1(\alpha)\right] b_w\left(\mathcal{F}_1^2(\alpha), \frac{1}{\beta^2}\right).
\end{aligned}$$

Formula (14) admits the following interpretation, which is better illustrated in case of interior solution, where it reads:

$$x^* = \frac{1}{\beta}\left[I + \frac{1}{\beta}\mathcal{F}(\alpha)\right] b_w\left(\mathcal{F}^2(\alpha), \frac{1}{\beta^2}\right).$$

Indeed, it is evident in this case that our solution is obtained by transforming, through the matrix $[I + \frac{1}{\beta}\mathcal{F}(\alpha)]$, the solution of an auxiliary game with strategic complements associated to the interaction matrix $\mathcal{F}^2(\alpha)$.

□

The following result shows a relationship between the social optimum and the Nash equilibrium of the game, in the case of strategic complements.

Theorem 5. *Assume that u_i are defined as in (6), $f_{ij}(\alpha) \geq 0$ for any $i, j \in \{1, \dots, n\}$, $\beta > 2\rho(\mathcal{F}(\alpha))$, and $x_i \in [0, L_i]$ for any $i \in \{1, \dots, n\}$. Then,*

$$x_i^* \leq x_i^{so} \qquad \forall\, i \in \{1, \dots, n\}, \tag{17}$$

where x^* is the Nash equilibrium and x^{so} is the social optimum of the game.

Proof. Since $\beta > 2\rho(\mathcal{F}(\alpha))$ and the welfare function reads

$$W(x) = -\frac{1}{2}x^\top\left[\beta I - 2\mathcal{F}(\alpha)\right]x + \alpha^\top x,$$

Lemma 1 guarantees that there exists a unique Nash equilibrium x^* and a unique social optimum x^{so}. Moreover, x^{so} satisfies the following Karush-Kuhn-Tucker system for some multiplier vectors $\lambda, \mu \in \mathbb{R}_+^n$:

$$\begin{aligned}
&\beta x_i^{so} - 2\sum_{\substack{j=1 \\ j\neq i}}^{n} f_{ij}(\alpha)x_j^{so} - \alpha_i - \lambda_i + \mu_i = 0 && i = 1, \dots, n, \\
&x_i^{so} \geq 0, \qquad \lambda_i \geq 0, \qquad \lambda_i x_i^{so} = 0 && i = 1, \dots, n, \\
&x_i^{so} \leq L_i, \qquad \mu_i \geq 0, \qquad \mu_i(x_i^{so} - L_i) = 0 && i = 1, \dots, n.
\end{aligned}$$

It is easy to check that the above system is equivalent to the following system:

$$
\begin{aligned}
x_i^{so} &= \max\left\{0,\ \min\left\{L_i,\ \frac{1}{\beta}\left[\alpha_i + 2\sum_{\substack{j=1\\ j\neq i}}^{n} f_{ij}(\alpha)x_j^{so}\right]\right\}\right\}\\
&= \min\left\{L_i,\ \frac{1}{\beta}\left[\alpha_i + 2\sum_{\substack{j=1\\ j\neq i}}^{n} f_{ij}(\alpha)x_j^{so}\right]\right\} \qquad i = 1, \ldots, n,
\end{aligned}
$$

where the last equality holds since $\alpha, \beta > 0$ and $f_{ij}(\alpha), x^{so} \geq 0$.

Given any strategy profile $x = (x_i, x_{-i})$, the best response of player i to rivals' strategies x_{-i} is given by

$$
B_i(x_{-i}) = \arg\max_{x_i \in [0, L_i]} u_i(\cdot, x_{-i}) = \min\left\{L_i,\ \frac{1}{\beta}\left[\alpha_i + \sum_{\substack{j=1\\ j\neq i}}^{n} f_{ij}(\alpha)x_j\right]\right\}.
$$

We now consider the sequential best response dynamics starting from the social optimum x^{so}, that is the sequence $\{x^k\}$ defined as follows:
$x^0 = x^{so}$,
$x^1 = \big(B_1(x^0_{-1}),\ x^0_2,\ x^0_3,\ \ldots\ ,\ x^0_n\big)$,
$x^2 = \big(B_1(x^0_{-1}),\ B_2(x^1_{-2}),\ x^0_3,\ \ldots\ ,\ x^0_n\big)$,
$\ldots$
$x^n = \big(B_1(x^0_{-1}),\ B_2(x^1_{-2}),\ B_3(x^2_{-3}),\ \ldots\ ,\ B_n(x^{n-1}_n)\big)$,
$x^{n+1} = \big(B_1(x^n_{-1}),\ B_2(x^1_{-2}),\ \ldots\ ,\ B_n(x^{n-1}_n)\big)$,
$x^{n+2} = \big(B_1(x^n_{-1}),\ B_2(x^{n+1}_{-2}),\ B_3(x^2_{-3}),\ \ldots\ ,\ B_n(x^{n-1}_n)\big), \ldots.$

We note that

$$
\begin{aligned}
x_1^1 = B_1\left(x^0_{-1}\right) &= \min\left\{L_1,\ \frac{1}{\beta}\left[\alpha_1 + \sum_{\substack{j=1\\ j\neq 1}}^{n} f_{1j}(\alpha)x_j^0\right]\right\}\\
&\leq \min\left\{L_1,\ \frac{1}{\beta}\left[\alpha_1 + 2\sum_{\substack{j=1\\ j\neq 1}}^{n} f_{1j}(\alpha)x_j^0\right]\right\} = x_1^0,
\end{aligned}
$$

hence $x^1 \leq x^0$. Moreover, we have

$$\begin{aligned} x_2^2 = B_2\left(x_{-2}^1\right) &= \min\left\{L_2,\ \frac{1}{\beta}\left[\alpha_2 + \sum_{\substack{j=1\\j\neq 2}}^n f_{2j}(\alpha)x_j^1\right]\right\} \\ &\leq \min\left\{L_2,\ \frac{1}{\beta}\left[\alpha_2 + \sum_{\substack{j=1\\j\neq 2}}^n f_{2j}(\alpha)x_j^0\right]\right\} \\ &\leq \min\left\{L_2,\ \frac{1}{\beta}\left[\alpha_2 + 2\sum_{\substack{j=1\\j\neq 2}}^n f_{2j}(\alpha)x_j^0\right]\right\} = x_2^0 = x_2^1, \end{aligned}$$

hence $x^2 \leq x^1$. Similarly, we can prove that $x^n \leq x^{n-1} \leq \cdots \leq x^1 \leq x^0$. Furthermore, we get

$$\begin{aligned} x_1^{n+1} = B_1(x_{-1}^n) &= \min\left\{L_1,\ \frac{1}{\beta}\left[\alpha_1 + \sum_{\substack{j=1\\j\neq 1}}^n f_{1j}(\alpha)x_j^n\right]\right\} \\ &\leq \min\left\{L_1,\ \frac{1}{\beta}\left[\alpha_1 + \sum_{\substack{j=1\\j\neq 1}}^n f_{1j}(\alpha)x_j^0\right]\right\} \\ &= B_1\left(x_{-1}^0\right) = x_1^n, \end{aligned}$$

hence $x^{n+1} \leq x^n$, and

$$\begin{aligned} x_2^{n+2} = B_2(x_{-2}^{n+1}) &= \min\left\{L_2,\ \frac{1}{\beta}\left[\alpha_2 + \sum_{\substack{j=1\\j\neq 2}}^n f_{2j}(\alpha)x_j^{n+1}\right]\right\} \\ &\leq \min\left\{L_2,\ \frac{1}{\beta}\left[\alpha_2 + \sum_{\substack{j=1\\j\neq 2}}^n f_{2j}(\alpha)x_j^1\right]\right\} \\ &= B_2\left(x_{-2}^1\right) = x_2^{n+1}, \end{aligned}$$

thus $x^{n+2} \leq x^{n+1}$. Following the same argument as before, we can prove that $x^{k+1} \leq x^k$ for any $k \in \mathbb{N}$ and hence, in particular, $x^k \leq x^{so}$ holds for any k. Since the potential function P is strongly concave, the sequence $\{x^k\}$ converges to the unique Nash equilibrium x^* (see, e.g., [5, Proposition 3.9]), and hence $x^* \leq x^{so}$. □

We remark that inequality (17) does not hold in general in the case of strategic substitutes, as the example in the next section shows.

4 Numerical experiments

In this section, we show a numerical example for the parametric quadratic game described in Section 3.

Example 1. We consider a game with $n = 5$ players, where $L_i = L = 1$ for any $i \in \{1, \dots, n\}$, $\alpha = (1, 2, 1, 2, 1)$, $\beta = 2.5$ and the interaction matrix is given by

$$f_{ij}(\alpha) = B|\alpha_i - \alpha_j| \qquad \forall\ i, j = 1, \dots, n.$$

We consider two cases: $B = 0.5$ (strategic complements) and $B = -0.5$ (strategic substitutes). In both cases the spectral radius of the matrix $\mathcal{F}(\alpha)$ results to be $\rho(\mathcal{F}(\alpha)) \simeq 1.2247$. Since $\beta > 2\rho(\mathcal{F}(\alpha))$, there exists a unique Nash equilibrium and a unique social optimum. Table 1 shows the unconstrained Nash equilibrium (assuming $L = +\infty$, given by formula (12)), the constrained Nash equilibrium (assuming $L = 1$) and the social optimum in the case $B = 0.5$.

Table 1: Case $B = 0.5$: unconstrained Nash equilibrium, constrained Nash equilibrium (assuming $L = 1$) and social optimum for Example 1.

Player	Unconstrained NE	Constrained NE	Social Optimum
1	0.9474	0.8000	1.0000
2	1.3684	1.0000	1.0000
3	0.9474	0.8000	1.0000
4	1.3684	1.0000	1.0000
5	0.9474	0.8000	1.0000

Figure 1 shows the price of anarchy of the Nash equilibrium for different values of L and β, in the case $B = 0.5$. The results suggest that the price of anarchy is a non-increasing function

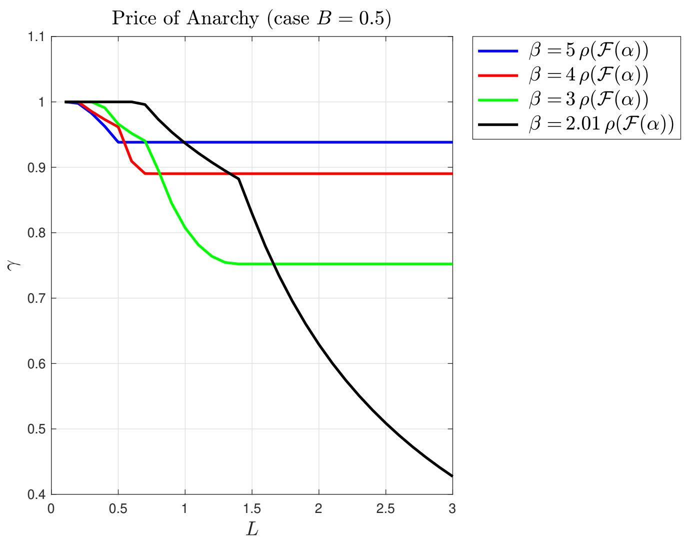

Figure 1: Case $B = 0.5$: Price of Anarchy for different values of L and β.

of L; it is constant when either L is small enough (i.e., the Nash equilibrium coincides with the social optimum) or greater than some threshold (i.e., the Nash equilibrium and the social optimum are both interior to the feasible region); the larger the value of β, the larger the asymptotic value of the price of anarchy is.

The case $B = -0.5$ with strategic substitutes is analysed in Table 2 and Figure 2. In particular, Table 2 shows that in the case of strategic substitutes, neither the inequality (17) between the Nash equilibrium and the social optimum nor the opposite inequality applies.

Table 2: Case $B = -0.5$: unconstrained Nash equilibrium, constrained Nash equilibrium (assuming $L = 1$) and social optimum for Example 1.

Player	Unconstrained NE	Constrained NE	Social Optimum
1	0.1053	0.1053	0.0000
2	0.7368	0.7368	0.8000
3	0.1053	0.1053	0.0000
4	0.7368	0.7368	0.8000
5	0.1053	0.1053	0.0000

On the other hand, Figure 2 suggests that in the case of strategic substitutes, the price of anarchy is not in general a non-increasing function of L (contrary to the case of strategic complements), while, as in the case of strategic complements, the price of anarchy is constant when either L is small enough or greater than some threshold, and the larger the value of β, the larger its asymptotic value is.

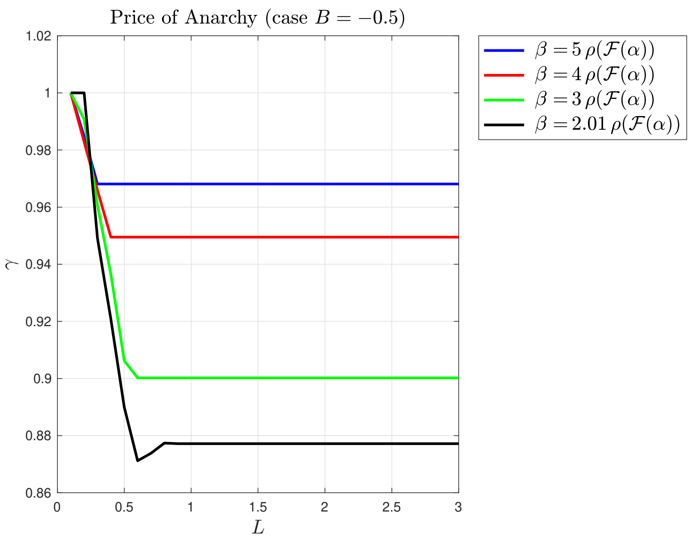

Figure 2: Case $B = -0.5$: Price of Anarchy for different values of L and β.

5 Conclusions and future research perspectives

In this work we carried on our research program of applying the variational inequality approach to network game problems. We investigated a class of parametric quadratic utility functions for which we obtained a Katz-Bonacich-like representation formula for the unique solution, and studied both theoretically and numerically the price of anarchy. Future research will concern the investigation of games with nonlinear utility function, and the inclusion of random data in the model (see, e.g., [11, 18]). Moreover, the topic of generalized Nash equilibrium problems on networks will be investigated using the theoretical results developed in [8].

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