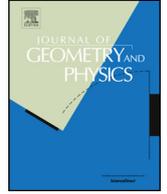




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## On the singularities of the Szegő kernels on CR orbifolds

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## ABSTRACT

In this paper we study the microlocal properties of the Szegő kernel of a given compact connected orientable CR orbifold, whose Kohn Laplacian has closed range. This last assumption is satisfied if certain geometric conditions hold true, as in the smooth case. Furthermore, we explain how to generalize a CR version of quantization commutes with reduction to orbifolds. As an application, we give a pure analytic proof of Kodaira-Bailey theorem.

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### 1. Introduction

The study of Szegő kernels on CR manifolds plays an important role in CR/Complex analysis and geometry. Boutet de Monvel and Sjöstrand [3] showed that the Szegő kernel on a strongly pseudo-convex CR manifold is a complex Fourier integral operator. Boutet de Monvel and Sjöstrand’s result has a profound impact in CR/Complex analysis and geometry. In [16], the authors calculate the second coefficient of the asymptotic expansion of Boutet de Monvel and Sjöstrand. When  $X$  is a CR orbifold, the study of the associated Szegő kernel on  $X$  is a very natural question and could have some application in CR/Complex geometry and geometric quantization theory. For example, in [17] and [13], the authors studied quantization commutes with reduction for strongly pseudo-convex CR manifolds under the assumption that the group action  $G$  is free near  $\mu^{-1}(0)$ ,  $\mu$  is the associated momentum map, see also [9]. When the group action  $G$  is locally free near  $\mu^{-1}(0)$ , we need to understand the behavior of the Szegő kernels on CR orbifolds. Furthermore, there have been quite some interesting works recently on the Hardy spaces and the Szegő projection on non-smooth domains, for example for quotient domains; we refer to [5] and references therein. The main goal of this paper is to study the associated Szegő kernel on a CR orbifold  $X$  with non-degenerate Levi form. As applications, we give a pure analytic proof of Kodaira-Bailey theorem and explain how to generalize a CR version of quantization commutes with reduction to orbifolds.

We now formulate our main results. We refer the reader to Section 2 for some notations and terminology used here. Let  $(X, T^{1,0}X)$  be a compact orientable connected CR orbifold of dimension  $2n + 1$ ,  $n \geq 1$ . In Section 2 we recall the basic material concerning orbifolds, we give definitions for the spaces of  $(0, q)$ -forms with  $L^2(X)$  coefficients and for the Kohn Laplacian  $\square_b^{(q)}$  on the orbifold  $X$ . We can then define abstractly the Szegő projector for  $X$  to be the projector

$$\Pi^{(q)} : L^2_{(0,q)}(X) \rightarrow \text{Ker} \square_b^{(q)}.$$

The main aim of this paper is to prove the following result, which is a generalization of Theorem 1.2 in [11].

**Theorem 1.1.** *Assume that the Levi form is non-degenerate of constant signature  $(n_-, n_+)$  on  $X$ . Suppose  $\square_b^{(q)}$  has closed range. Then*

$$\Pi^{(q)} \equiv \Pi_- + \Pi_+$$

where  $\Pi_{\pm} \equiv 0$  if  $q \neq n_{\pm}$ . Thus we fix  $q = n_-$  and we have  $\text{WF}'(\Pi_-) = \text{diag}(\Sigma^- \times \Sigma^-)$  and furthermore  $\text{WF}'(\Pi_+) = \text{diag}(\Sigma^+ \times \Sigma^+)$  if  $q = n_- = n_+$ , where

$$\Sigma^{\pm} = \{(x, \lambda \omega_0(x)) \in T^*X : \lambda \gtrless 0\},$$

and

$$\text{WF}'(\Pi_{\pm}) = \{(x, \xi, y, -\eta) \in T^*X \times T^*X; (x, \xi, y, \eta) \in \text{WF}(\Pi_{\pm})\};$$

$\text{WF}(\Pi_{\pm})$  denotes the wave front set of  $\Pi_{\pm}$  in the sense of Hörmander. Moreover,  $\Pi_{\pm}$  has the following micro-local expression: Let

$$\Pi_{\pm}(x, y) \in \mathcal{D}'(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

be the distribution kernel of  $\Pi_{\pm}$ . Consider an open set  $U \subset X$  and an orbifold chart  $(\tilde{U}, G_U) \rightarrow U$ . We have

$$\Pi_{\pm}(x, y) = \sum_{g \in G_U} \left( \int_0^{+\infty} e^{it \varphi_{\pm}(\tilde{x}, g \cdot \tilde{y})} s_{\pm}(\tilde{x}, g \cdot \tilde{y}, t) dt + F_{\pm}(\tilde{x}, g \cdot \tilde{y}) \right) \text{ on } U \times U, \tag{1}$$

where  $\pi(\tilde{x}) = x$ ,  $\pi(\tilde{y}) = y$ ,  $\pi : \tilde{U} \rightarrow U$  is the natural projection,  $F_{\pm}$  are  $G_U$ -invariant smoothing operators on  $\tilde{U}$  and

$$s_{\pm}(\tilde{x}, \tilde{y}, t) \sim \sum_{j=0}^{\infty} s_{\pm}^j(\tilde{x}, \tilde{y}) t^{n-j}$$

in  $S^n_{1,0}(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ ,

$$s_+(\tilde{x}, \tilde{y}, t) = 0 \text{ if } q \neq n_+$$

and

$$s_-^0(\tilde{x}_0, \tilde{x}_0) = \frac{1}{2} \pi^{-n-1} |\det \mathcal{L}_{\tilde{x}_0}| \tau_{\tilde{x}_0, n_-}, \tilde{x}_0 \in \tilde{U},$$

if  $q = n_- = n_+$ ,

$$s_+^0(\tilde{x}_0, \tilde{x}_0) = \frac{1}{2} \pi^{-n-1} |\det \mathcal{L}_{\tilde{x}_0}| \tau_{\tilde{x}_0, n_+}, \tilde{x}_0 \in \tilde{U},$$

the phase functions satisfy

$$\begin{aligned} \varphi_+, \varphi_- &\in C^\infty(\tilde{U} \times \tilde{U}), \quad \text{Im } \varphi_\pm(\tilde{x}, \tilde{y}) \geq 0, \\ \varphi_-(\tilde{x}, \tilde{x}) &= 0, \quad \varphi_-(\tilde{x}, \tilde{y}) \neq 0 \text{ if } \tilde{x} \neq \tilde{y}, \\ d_x \varphi_-(\tilde{x}, \tilde{y})|_{\tilde{x}=\tilde{y}} &= -\omega_0(\tilde{x}), \quad d_y \varphi_-(\tilde{x}, \tilde{y})|_{\tilde{x}=\tilde{y}} = \omega_0(\tilde{x}), \\ -\overline{\varphi_+}(\tilde{x}, \tilde{y}) &= \varphi_-(\tilde{x}, \tilde{y}), \end{aligned}$$

$s_\pm$  and the phase functions  $\varphi_\pm$  can be taken to be  $G_U$ -invariant which means  $s_\pm(g \cdot \tilde{x}, g \cdot \tilde{y}) = s_\pm(\tilde{x}, \tilde{y})$ ,  $\varphi_\pm(g \cdot \tilde{x}, g \cdot \tilde{y}) = \varphi_\pm(\tilde{x}, \tilde{y})$ , for every  $g \in G_U$ .

Here,  $\mathcal{L}_{\tilde{x}_0}$  denotes the Levi form of  $X$  at  $\tilde{x}_0$ ,  $\det \mathcal{L}_{\tilde{x}_0} = \lambda_1(\tilde{x}_0) \cdots \lambda_n(\tilde{x}_0)$ ,  $\lambda_j(\tilde{x}_0)$ ,  $j = 1, \dots, n$ , are the eigenvalues of  $\mathcal{L}_{\tilde{x}_0}$  with respect to the given Hermitian metric on  $\mathbb{C}TX$  and  $\tau_{\tilde{x}_0, n_\pm}$  are given by (16).

It should be mentioned that the phase  $\varphi_\pm$  is the same as the description of Szegő kernel for lower energy forms on  $\tilde{U}$  in [15]. Furthermore, recall that if  $n_- = n_+$  or  $|n_- - n_+| > 2$ , then  $\square_b^{(q)}$  has closed range, where  $q = n_-$ .

**Remark 1.1.** The phase  $\varphi_\pm$  is not unique. For a better understanding, we explain the reason why the distributional kernel is well-defined even if the phase function is not  $G_U$ -invariant. Suppose we define a distributional kernel  $A$  on  $\tilde{U}$  via an oscillatory integral in the following way. For each  $G_U$ -invariant  $f \in \Omega_c^{0,q}(\tilde{U})$  we write

$$(A f)(\tilde{x}) := \frac{1}{|G_U|} \sum_{g \in G_U} \int_{\tilde{U}} \int_0^{+\infty} e^{it \varphi(\tilde{x}, g \cdot \tilde{y})} a(\tilde{x}, g \cdot \tilde{y}, t) f(\tilde{y}) dt dV_{\tilde{U}}(\tilde{y})$$

where  $\varphi$  is a  $G_U$ -invariant phase function and

$$a \in S_{\text{cl}}^n(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

is a  $G_U$ -invariant symbol. Now, we can always change variables  $t \mapsto h(x, y)t$  in  $t$ , where  $h \in C^\infty(\tilde{U} \times \tilde{U}, \mathbb{R}_+)$  is a not necessarily  $G_U$ -invariant function on  $\tilde{U} \times \tilde{U}$ . Thus, we obtain the same distributional kernel  $A$ , which is a well-defined in the sense of [19, (6.3)], but now the phase function  $h\varphi$  is not  $G_U$ -invariant in general. Actually it is a non-trivial fact that  $s_\pm$  and  $\varphi_\pm$  can be taken to be  $G_U$ -invariant (see Theorem 2.5).

**Remark 1.2.** In general, we can take any equivalent phase  $\varphi_\pm$  in (1). If the phase  $\varphi_\pm$  and  $s_\pm$  are not  $G_U$ -invariant, we can prove that there are

$$F_\pm(\tilde{x}, \tilde{y}), \hat{F}_\pm(\tilde{x}, \tilde{y}) \in C^\infty(\tilde{U} \times \tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

such that

$$\begin{aligned} &\sum_{g \in G_U} \left( \int_0^{+\infty} e^{it \varphi_\pm(\tilde{x}, g \cdot \tilde{y})} s_\pm(\tilde{x}, g \cdot \tilde{y}, t) dt + F_\pm(\tilde{x}, g \cdot \tilde{y}) \right) \\ &= \sum_{h, g \in G_U} \frac{1}{|G_U|} \left( \int_0^{+\infty} e^{it \varphi_\pm(h \cdot \tilde{x}, g \cdot \tilde{y})} s_\pm(h \cdot \tilde{x}, g \cdot \tilde{y}, t) dt + \hat{F}_\pm(h \circ \tilde{x}, g \cdot \tilde{y}) \right), \end{aligned} \tag{2}$$

where  $\hat{F}_\pm$  are smoothing operators on  $\tilde{U}$  ( $\hat{F}_\pm$  can be different from  $F_\pm$ ). From (2), the oscillatory integral

$$\sum_{g \in G_U} \left( \int_0^{+\infty} e^{it \varphi_\pm(\tilde{x}, g \cdot \tilde{y})} s_\pm(\tilde{x}, g \cdot \tilde{y}, t) dt + F_\pm(\tilde{x}, g \cdot \tilde{y}) \right)$$

defines a well-defined operator in the sense of [19, (6.3)] even  $\varphi_\pm, s_\pm$  are not  $G_U$ -invariant.

The main idea of the proof is the following: on  $\tilde{U}$ , by using the approximate Szegő kernel  $S(\tilde{x}, \tilde{y})$  constructed in [11], we define a distributional kernel

$$\hat{S}(\tilde{x}, \tilde{y}) := \sum_{g \in G_U} S(\tilde{x}, g \cdot \tilde{y}).$$

We then show that  $\hat{S}$  is microlocally self-adjoint and it behaves as the identity when restricted to  $\text{Ker} \square_b^{(q)}$ , up to smoothing contributions. Moreover, we can prove that  $\hat{S}$  is equal to the projector  $\Pi^{(q)}$  up to some smoothing operator, if  $\square_b^{(q)}$  has closed range. This is a consequence of the properties of  $\hat{S}$  just mentioned and a theorem of functional analysis.

We assume that  $X$  admits a CR compact Lie group action  $G$ ,  $\dim G = d$ . We do not assume that  $X$  admits a transversal and  $S^1$  action. We will work with Assumption 2.1 below and suppose that  $X$  is strongly pseudoconvex. We refer the reader to Section 2.3 and Section 3.1 for the notations and terminology used here. We will assume that 0 is a regular value of the moment map  $\mu$  and thus  $\mu^{-1}(0)$  is a suborbifold of  $X$ . Put  $X_G := \mu^{-1}(0)/G$ . Note that both  $X$  and  $X_G$  are orbifolds. Let  $\square_{b, X_G}^{(0)}$  be the Kohn Laplacian on the orbifold  $X_G$ . Let

$$(\text{Ker} \square_b^{(0)})^G := \left\{ u \in \text{Ker} \square_b^{(0)}; g^*u = u, \forall g \in G \right\}.$$

As an application of Theorem 1.1, we generalize a CR version of quantization commutes with reduction to orbifolds. Note that the following theorem is the orbifold version of [17, Theorem 1.2] (see also [13]).

**Theorem 1.2.** *There is a map*

$$\sigma : (\text{Ker} \square_b^{(0)})^G \subset L^2(X) \rightarrow \text{Ker} \square_{b, X_G}^{(0)} \subset L^2(X_G)$$

such that  $\sigma$  is Fredholm, that is,

- (i)  $\text{Ker} \sigma \cap (\text{Ker} \square_b^{(0)})^G$  is a finite dimensional subspace of  $C^\infty(X)$ ,
- (ii)  $\text{Coker} \sigma \cap \text{Ker} \square_{b, X_G}^{(0)}$  is a finite dimensional subspace of  $C^\infty(X_G)$ .

We refer the reader to (34) for the explicit formula for  $\sigma$ .

Let us recall prior literature. First there is the paper [24], which deals with the Szegő kernel on an orbifold circle bundle. The semi-classical behavior of Bergman kernels on complex orbifolds was studied by Dai, Liu and Ma [6]. Our approach is closer in spirit to [6] where an explicit expression for an operator on a local chart  $(\tilde{U}, G_U) \rightarrow U$  is given. This is also motivated by previous results concerning  $G$ -invariant Szegő kernel for locally free  $G$ -action on CR manifolds we obtained in [8], in this case the CR reduction is an orbifold.

It is known that a compact connected strongly pseudoconvex CR orbifold  $X$  with a transversal CR  $S^1$  action can be identified with the circle orbibundle of a positive holomorphic orbifold line bundle  $(L, h) \rightarrow M$  over a compact complex Hodge orbifold  $M$ , see [4]. By this identification the  $k$ -th Fourier component of the Szegő kernel function and the Bergman kernel function for the  $k$ -th tensor power of the line bundle  $L$  are equal up to a constant factor of  $2\pi$ . Hence in Section 4 we obtained a phase function version of the Bergman kernels expansions on orbifolds (see [18, Theorems 5.4.10, 5.4.11] or [6, Theorem 1.4] for another version).

## 2. Preliminaries

### 2.1. Orbifolds

We recall basic definitions we need about orbifolds, for a more precise discussion see [1], [18] and references therein. Define  $\mathcal{M}_s$  to be the category whose objects are pairs  $(M, G)$ , where  $M$  is a connected smooth manifold and  $G$  is a finite group acting smoothly and effectively on  $M$ . A morphism  $\Phi : (M, G) \rightarrow (M', G')$  is family of open embeddings  $\varphi : M \rightarrow M'$  such that:

- i) For each  $\varphi \in \Phi$  there exists an injective group homomorphism  $\lambda_\varphi : G \rightarrow G'$  satisfying

$$\varphi(g \cdot x) = \lambda_\varphi(g) \cdot \varphi(x), \quad x \in M, g \in G.$$

- ii) For  $g \in G'$ ,  $\varphi \in \Phi$ , if  $(g\varphi)(M) \cap \varphi(M) \neq \emptyset$ , then  $g \in \lambda_\varphi(G)$ .
- iii) For  $\varphi \in \Phi$ , we have  $\Phi = \{g\varphi, g \in G'\}$ .

Let  $X$  be a paracompact Hausdorff space and let  $\mathcal{U}$  be a covering of  $X$  consisting of connected open subsets. We assume  $\mathcal{U}$  satisfies the condition: for any  $x \in U \cap V$ ,  $U, V \in \mathcal{U}$ , there is  $W \in \mathcal{U}$  such that  $x \in W \subset U \cap V$ . Then we say that the topological space  $X$  has an orbifold structure if

- i) for each  $U \in \mathcal{U}$  there exists  $(\tilde{U}, G_U) \in \mathcal{M}_s$  and a ramified covering  $(\tilde{U}, G_U) \rightarrow U$  such that  $U$  is homeomorphic to  $\tilde{U}/G_U$ , where  $\tilde{U}/G_U$  is endowed with the quotient topology;
- ii) for any  $U, V \in \mathcal{U}$ ,  $U \subset V$ , there exists a morphism  $\varphi_{V,U} : (\tilde{U}, G_U) \rightarrow (\tilde{V}, G_V)$  which covers the inclusion  $U \subset V$  and satisfies  $\varphi_{WU} = \varphi_{WV} \circ \varphi_{VU}$  for any  $U, V, W \in \mathcal{U}$  such that  $U \subset V \subset W$ .

Let us denote by  $\pi_U : (\tilde{U}, G_U) \rightarrow U$  the ramified covering. We recall that for each  $p \in X$  one can always find local coordinates  $x = (x_1, \dots, x_n)$  on  $(\tilde{U}, G_U)$ ,  $\tilde{U} \subseteq \mathbb{R}^n$ , and  $\tilde{p}$  with  $\pi_U(\tilde{p}) = p$  such that  $x(\tilde{p}) \equiv 0 \in \mathbb{R}^n$  is a fixed point of  $G_U$  and  $G_U$  acts linearly on  $\mathbb{R}^n$ . We call these coordinates *standard coordinates* at  $p$ . If  $|G_U| > 1$  then we say that  $p$  is a singular point, otherwise we say that  $p$  is regular.

A *vector orbibundle*  $(E, p)$  on an orbifold  $X$  is an orbifold  $E$  together with a continuous projection  $p : E \rightarrow X$  such that for  $U \in \mathcal{U}$  there exists a  $G_U$ -equivariant lift  $\tilde{p} : \tilde{E}_U \rightarrow \tilde{U}$  defining a trivial vector bundle such that the diagram

$$\begin{array}{ccc} (\tilde{E}_U, G_U^E) & \xrightarrow{\tilde{p}} & (\tilde{U}, G_U) \\ \downarrow & & \downarrow \\ E_U & \xrightarrow{p} & U \end{array}$$

commutes. As before, when we say that  $\tilde{p} : \tilde{E}_U \rightarrow \tilde{U}$  is  $G_U$ -equivariant we mean that there exists an injective group homomorphism  $\lambda_p : G_U^E \rightarrow G_U$  such that

$$\tilde{p}(g \cdot e) = \lambda_p(g) \cdot \tilde{p}(e) \quad e \in \tilde{E}_U, g \in G_U^E;$$

if  $\lambda_p$  is a group isomorphism, we say that the vector orbibundle is *proper*.

A smooth section of a vector orbibundle  $(E, p)$  over  $X$  is a smooth map  $s : X \rightarrow E$  such that  $p \circ s = 1_X$ , we denote the space of smooth sections as  $C^\infty(X, E)$ . We recall that a *smooth map*  $f : X \rightarrow Y$  between orbifolds is a continuous map between the underlying topological spaces such that for each  $U \in \mathcal{U}$  there exists a local lift  $(\tilde{f}_U, \bar{f}_U)$  such that  $\tilde{f}_U : \tilde{U} \rightarrow \tilde{V}$  is smooth,  $\bar{f}_U : G_U \rightarrow G_V$  is a group homomorphism, the diagram

$$\begin{array}{ccc} (\tilde{U}, G_U) & \xrightarrow{\tilde{f}_U} & (\tilde{V}, G_V) \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & V \end{array}$$

commutes and  $\tilde{f}_U$  is  $\bar{f}_U$ -equivariant:

$$\tilde{f}_U(g \cdot x) = \bar{f}_U(g) \tilde{f}_U(x), \quad g \in G_U, x \in \tilde{U}.$$

The tangent orbibundle  $TX$  is defined by gluing together the bundles defined over the charts

$$(T\tilde{U}, G_U) \rightarrow (\tilde{U}, G_U),$$

where the action of  $G_U$  on  $T_{\tilde{x}}\tilde{U}$  is induced by differentiating the action on  $\tilde{U}$ , dg. It has a natural structure of proper vector orbibundle. The smooth sections of it are called vector fields.

We can then define the *differential* of a smooth map  $f : X \rightarrow Y$  to be the smooth map  $df : TX \rightarrow TY$  such that locally for a given  $x \in U$ ,  $f(x) \in V$ , and *standard coordinates* at  $x$  we have a commutative diagram

$$\begin{array}{ccc} (T_{\tilde{x}}\tilde{U}, G_U) & \xrightarrow{d_{\tilde{x}}\tilde{f}} & (T_{\tilde{f}(\tilde{x})}\tilde{V}, G_V) \\ \downarrow & & \downarrow \\ T_x U & \xrightarrow{d_x f} & T_{f(x)} V \end{array}$$

where  $d_{\tilde{x}}\tilde{f}$  is  $\bar{f}_U$ -equivariant. We say that  $f$  is an *immersion* at  $x \in X$  if  $df$  is injective, so this means that there is a local lift  $\tilde{f}_U : \tilde{U} \rightarrow \tilde{V}$  such that both  $d_{\tilde{x}}\tilde{f}$  and  $\bar{f}_U$  are injective.

There are different ways to construct new vector bundles out of old, see [21] p. 29 and also [1] p. 14. Given any continuous functor  $\mathcal{F}$  from vector spaces to vector spaces we obtain a new orbibundle  $\mathcal{F}(TX) \rightarrow X$  with fibers  $\mathcal{F}(T_{\tilde{x}}\tilde{U})/G_U$ . In particular, this allows us to construct the cotangent orbibundle  $T^*X$  and exterior powers  $\wedge^q T^*X$ . We recall the definition of sub-orbibundle that we will require later to define  $T^{1,0}X$ . A *sub-orbibundle*  $(F, p|_F)$  of a vector orbibundle  $(E, p)$  is a suborbifold of  $E$  being a vector bundle on  $X$ . We recall that an orbifold  $Y$  is a *suborbifold* of an orbifold  $X$  if there exists an immersion  $\iota : Y \rightarrow X$ , we identify  $Y$  with its image in  $X$ .

**Example 2.1.** Consider an action of a finite group  $G$  on  $\mathbb{R}^{2n+1}$ . A form on the orbifold  $\mathbb{R}^{2n+1}/G$  is a  $G$ -invariant form on  $\mathbb{R}^{2n+1}$ .

At this point we can give the definition of exterior derivative. Consider a chart  $(\tilde{U}, G_U) \rightarrow U$  of an orbifold  $X$ , a  $k$ -form  $\omega$  defines a  $G_U$ -invariant  $k$ -form  $\omega_U$  on  $\tilde{U}$ , locally this is given by

$$\omega_U := f_U dx^{j_1} \wedge dx^{j_2} \wedge \dots \wedge dx^{j_k} \quad \text{on } \tilde{U} \subseteq \mathbb{R}^n,$$

where  $f_U$  is a  $G_U$ -invariant function on  $\tilde{U}$ . The exterior derivative of  $\omega$  is given locally by

$$d\omega_U := \sum_{i=1}^n \frac{\partial f_U}{\partial x^i} dx^i \wedge dx^J \quad \text{on } \tilde{U} \subseteq \mathbb{R}^n,$$

where  $J$  is the multi-index  $J = (j_1, \dots, j_n)$ . It is easy to check that it defines a  $(k + 1)$ -form which is invariant under the action of  $G_U$ .

If  $X$  is an oriented  $n$  dimensional orbifold, we define the integral of a smooth section  $\omega$  of the orbibundle  $\Lambda^n T^*X$  as follows. Suppose that  $\text{supp}(\omega)$  is compactly supported on  $U \in \mathcal{U}$ , we define

$$\int_U \omega := \frac{1}{|G_U|} \int_{\tilde{U}} \tilde{\omega}_U.$$

In the general case, for every open cover of  $X$  by positively oriented charts,  $(U_i, G_{U_i})$ , we have a partition of unity,  $(\rho_i)_{i \in I}$  subordinate to this cover. We define,

$$\int_X \omega := \sum_{i \in I} \frac{1}{|G_{U_i}|} \int_{\tilde{U}_i} \tilde{\rho}_i \tilde{\omega}_{U_i}. \tag{3}$$

It remains to show that it does not depend on the choice of open cover and on the choice of partition of unity. Now, given another open cover  $((V_j, G_{V_j}))_{j \in J}$  and an associated partition of unity  $(\sigma_j)_{j \in J}$ , since  $\text{supp}(\rho_i \sigma_j \omega) \subseteq U_i \cap V_j$  we have

$$\frac{1}{|G_{U_i}|} \int_{\tilde{U}_i} \tilde{\rho}_i \tilde{\sigma}_{j,i} \tilde{\omega}_{U_i} = \frac{1}{|G_{V_j}|} \int_{\tilde{V}_j} \tilde{\rho}_{i,j} \tilde{\sigma}_j \omega_{V_j}$$

where  $\tilde{\sigma}_{j,i}$  is defined on  $\tilde{U}_i$  as the composition between  $\sigma_j : U_i \rightarrow \mathbb{R}$  and the covering  $\tilde{U}_i \rightarrow U_i$ , in a similar way  $\tilde{\rho}_{i,j}$  is defined. Now,

$$\begin{aligned} \sum_{i \in I} \frac{1}{|G_{U_i}|} \int_{\tilde{U}_i} \tilde{\rho}_i \tilde{\omega}_{U_i} &= \sum_{i \in I} \sum_{j \in J} \frac{1}{|G_{U_i}|} \int_{\tilde{U}_i} \tilde{\rho}_i \tilde{\sigma}_{j,i} \tilde{\omega}_{U_i} \\ &= \sum_{i \in I} \sum_{j \in J} \frac{1}{|G_{V_j}|} \int_{\tilde{V}_j} \tilde{\rho}_{i,j} \tilde{\sigma}_j \omega_{V_j} \\ &= \sum_{j \in J} \frac{1}{|G_{V_j}|} \int_{\tilde{V}_j} \tilde{\sigma}_j \tilde{\omega}_{V_j}, \end{aligned}$$

where we use the fact that

$$\sum_{j \in J} \tilde{\sigma}_{j,i} = 1, \quad \sum_{i \in I} \tilde{\rho}_{i,j} = 1.$$

### 2.2. CR orbifolds

In this section, we recall the definition of CR orbifolds, we also refer the reader to [7, Section 5]. For background concerning CR manifolds we refer to [11], and [17] and references therein. Moreover, the presentation here about CR geometry is taken from [17, Section 2]. Let  $X$  be a compact, connected and orientable orbifold of dimension  $2n + 1$ ,  $n \geq 1$ , a CR structure on  $X$  is a sub-orbibundle  $T^{1,0}X$  of rank  $n$  of the complexified orbifold tangent bundle  $TX \otimes \mathbb{C}$  (which is defined locally for each  $x \in X$  by taking  $T_x \tilde{U} \otimes \mathbb{C}$ ), satisfying  $T^{1,0}X \cap T^{0,1}X = \{0\}$ , where  $T^{0,1}X = \overline{T^{1,0}X}$ , and  $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ , where  $\mathcal{V}$  is the space of smooth sections of the orbibundle  $T^{0,1}X$ .

There is a unique sub-orbibundle  $HX$  of  $TX$  such that  $HX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$ , i.e.  $HX$  is the real part of  $T^{1,0}X \oplus T^{0,1}X$ . Let  $J : HX \rightarrow HX$  be the complex structure map given by  $J(u + \bar{u}) = \iota u - \iota \bar{u}$ , for every  $u \in T^{1,0}X$ . By complex linear extension of  $J$  to  $TX \otimes \mathbb{C}$ , the  $\iota$ -eigenspace of  $J$  is  $T^{1,0}X = \{V \in HX \otimes \mathbb{C}; JV = \iota V\}$ . We shall also write  $(X, HX, J)$  to denote a CR orbifold.

We fix a real non-vanishing 1-form  $\omega_0 \in C^\infty(X, T^*X)$  so that  $\langle \omega_0(x), u \rangle = 0$ , for every  $u \in H_x X$ , for every  $x \in X$ . For each  $x \in X$ , we define a quadratic form on  $HX$  by

$$\mathcal{L}_x(U, V) = \frac{1}{2} d\omega_0(JU, V), \quad \forall U, V \in H_x X.$$

We extend  $\mathcal{L}$  to  $HX \otimes \mathbb{C}$  by complex linear extension. Then, for  $U, V \in T_x^{1,0}X$ ,

$$\mathcal{L}_x(U, \bar{V}) = \frac{1}{2}d\omega_0(JU, \bar{V}) = -\frac{1}{2i}d\omega_0(U, \bar{V}). \tag{4}$$

The Hermitian quadratic form  $\mathcal{L}_x$  on  $T_x^{1,0}X$  is called Levi form at  $x$ . This means that for each local coordinate patch  $(\tilde{U}, G_U) \rightarrow U$  there exists a  $G_U$ -invariant Hermitian quadratic form on  $(T_{\tilde{x}}\tilde{U}, G_U)$ . In this paper, we always assume that the Levi form  $\mathcal{L}$  on  $T^{1,0}X$  is non-degenerate of constant signature  $(n_-, n_+)$  on  $X$ , where  $n_-$  denotes the number of negative eigenvalues of the Levi form and  $n_+$  denotes the number of positive eigenvalues of the Levi form. Let  $R \in \mathcal{C}^\infty(X, TX)$  be the non-vanishing vector field determined by

$$\omega_0(R) = -1, \quad d\omega_0(R, \cdot) \equiv 0 \text{ on } TX. \tag{5}$$

Note that  $X$  is a contact orbifold with contact form  $\omega_0$ , contact plane  $HX$  and  $R$  is the Reeb vector field.

Here, a Hermitian metric  $\langle \cdot | \cdot \rangle$  on  $TX \otimes \mathbb{C}$  so that  $T^{1,0}X \perp T^{0,1}X$  means that  $\langle \cdot | \cdot \rangle$  is defined locally on each local chart  $U$  as  $G_U$ -invariant Hermitian metric on  $T_{\tilde{x}}\tilde{U} \otimes \mathbb{C}$ ; from now on when it is clear how to define a geometric structure on an orbundle we will not always recall the definition. Furthermore, notice that every Hermitian metric  $h'$  on  $\tilde{U}$  gives rise to a  $G_U$ -invariant metric  $h_U$  on  $\tilde{U}$  by averaging over the group:

$$h_U(v, w) := \sum_{g \in G_U} h'(d_{\tilde{x}}g(v), d_{\tilde{x}}g(w)), \quad v, w \in T_{\tilde{x}}\tilde{U}.$$

For  $u \in TX \otimes \mathbb{C}$ , we write  $|u|^2 := \langle u | u \rangle$ . We always consider a Hermitian metric  $\langle \cdot | \cdot \rangle$  on  $\mathbb{C}TX$  so that  $T^{1,0}X$  is orthogonal to  $T^{0,1}X$ ,  $\langle u | v \rangle$  is real if  $u, v$  are real tangent vectors,  $\langle R | R \rangle = 1$  and  $R$  is orthogonal to  $T^{1,0}X \oplus T^{0,1}X$ .

Denote by  $T^{*1,0}X$  and  $T^{*0,1}X$  the dual bundles of  $T^{1,0}X$  and  $T^{0,1}X$ , respectively. They can be identified with sub-orbundles of the complexified cotangent orbundle  $T^*X \otimes \mathbb{C}$ . Define the vector orbundle of  $(0, q)$ -forms by  $T^{*0,q}X := \wedge^q T^{*0,1}X$ . The Hermitian metric  $\langle \cdot | \cdot \rangle$  on  $TX \otimes \mathbb{C}$  induces, by duality, a Hermitian metric on  $T^*X \otimes \mathbb{C}$  and also on the orbundles of  $(0, q)$  forms  $T^{*0,q}X, q = 0, 1, \dots, n$ . We shall also denote all these induced metrics by  $\langle \cdot | \cdot \rangle$ . For  $u \in T^{*0,q}X$ , we write  $|u|^2 := \langle u | u \rangle$ . Note that we have the point-wise orthogonal decompositions:

$$\begin{aligned} T^*X \otimes \mathbb{C} &= T^{*1,0}X \oplus T^{*0,1}X \oplus \{\lambda\omega_0; \lambda \in \mathbb{C}\}, \\ TX \otimes \mathbb{C} &= T^{1,0}X \oplus T^{0,1}X \oplus \{\lambda R; \lambda \in \mathbb{C}\}. \end{aligned}$$

Let  $U$  be an open set of  $X$ . Let  $\Omega^{0,q}(U)$  denote the space of smooth sections of  $T^{*0,q}X$  over  $U$  and let  $\Omega_c^{0,q}(U)$  be the subspace of  $\Omega^{0,q}(U)$  whose elements have compact support in  $U$ ; more precisely an element  $u$  in  $\Omega_c^{0,q}(U)$  has a lift  $\tilde{u} \in \Omega_c^{0,q}(\tilde{U})$  which is  $G_U$ -invariant. The metric on  $TX \otimes \mathbb{C}$  induces a metric on  $\Lambda^q(T^*X \otimes \mathbb{C})$ , so we have a well-defined orthogonal projector

$$\pi^{0,q} : \Lambda^q(T^*X \otimes \mathbb{C}) \rightarrow T^{*0,q}X.$$

We can define the tangential Cauchy-Riemann operator  $\bar{\partial}_b$  by composing with the exterior differential

$$\bar{\partial}_b = \pi^{0,q+1} \circ d : \Omega^{0,q}(X) \rightarrow \Omega^{0,q+1}(X),$$

thus, on a local chart  $(\tilde{U}, G_U) \rightarrow U$ ,  $\bar{\partial}_b$  acts on  $G_U$ -invariant functions on  $\tilde{U}$  and commutes with the action of  $G_U$  on  $\tilde{U}$ . Let  $dV_X(x)$  be the volume form on  $X$  induced by the Hermitian metric  $\langle \cdot | \cdot \rangle$ . For a local chart  $(\tilde{U}, G_U) \rightarrow U$ , let  $dV_{\tilde{U}}(x)$  be the volume form on  $\tilde{U}$  induced by the Hermitian metric  $\langle \cdot | \cdot \rangle$ . The natural global  $L^2$  inner product  $(\cdot | \cdot)$  on  $\Omega^{0,q}(X)$  induced by  $dV_X(x)$  and  $\langle \cdot | \cdot \rangle$  is given by

$$(u | v) := \int_X \langle u(x) | v(x) \rangle dV_X(x), \quad u, v \in \Omega^{0,q}(X).$$

We denote by  $L^2_{(0,q)}(X)$  the completion of  $\Omega^{0,q}(X)$  with respect to  $(\cdot | \cdot)$ . It should be clear that the elements in  $L^2_{(0,q)}(X)$  are  $L^2$  sections of the orbundle  $\Lambda^q(T^*X \otimes \mathbb{C})$ , the integral appearing in the definition of the  $L^2$  inner product  $(\cdot | \cdot)$  is meant to be interpreted as in the previous section, see (3). We extend  $(\cdot | \cdot)$  to  $L^2_{(0,q)}(X)$  in the standard way. For  $f \in L^2_{(0,q)}(X)$ , we denote  $\|f\|^2 := (f | f)$ . We extend  $\bar{\partial}_b$  to  $L^2_{(0,r)}(X), r = 0, 1, \dots, n$ , by

$$\bar{\partial}_b : \text{Dom } \bar{\partial}_b \subset L^2_{(0,r)}(X) \rightarrow L^2_{(0,r+1)}(X),$$

where  $\text{Dom } \bar{\partial}_b := \{u \in L^2_{(0,r)}(X); \bar{\partial}_b u \in L^2_{(0,r+1)}(X)\}$  and, for any  $u \in L^2_{(0,r)}(X)$ ,  $\bar{\partial}_b u$  is defined in the sense of distributions. We also write

$$\bar{\partial}_b^* : \text{Dom } \bar{\partial}_b^* \subset L^2_{(0,r+1)}(X) \rightarrow L^2_{(0,r)}(X)$$

to denote the Hilbert space adjoint of  $\bar{\partial}_b$  in the  $L^2$  space with respect to  $(\cdot | \cdot)$ . Let  $\square_b^{(q)}$  denote the Gaffney extension of the Kohn Laplacian given by

$$\text{Dom } \square_b^{(q)} = \left\{ s \in L^2_{(0,q)}(X); s \in \text{Dom } \bar{\partial}_b \cap \text{Dom } \bar{\partial}_b^*, \bar{\partial}_b s \in \text{Dom } \bar{\partial}_b^*, \bar{\partial}_b^* s \in \text{Dom } \bar{\partial}_b \right\}, \tag{6}$$

$$\square_b^{(q)} s = \bar{\partial}_b \bar{\partial}_b^* s + \bar{\partial}_b^* \bar{\partial}_b s \text{ for } s \in \text{Dom } \square_b^{(q)}.$$

### 2.3. Group actions on orbifolds

Consider a compact connected Lie group action  $G$ . A smooth action of  $G$  on an orbifold  $X$  is a continuous action  $\cdot$  between the underlying topological spaces such that for each  $g \in G$  the map  $g \cdot : X \rightarrow X$  is smooth by mean of definition in Section 2.1: this means that locally  $(G_U, \tilde{U}) \rightarrow U$ , for each  $g \in G$ , there exists a local lift  $(\tilde{g}_{\cdot U}, \bar{g}_{\cdot U})$  such that  $\tilde{g}_{\cdot U} : \tilde{U} \rightarrow \tilde{U}$  is smooth,  $\bar{g}_{\cdot U} : G_U \rightarrow G_U$  is a group homomorphism, the diagram

$$\begin{array}{ccc} (\tilde{U}, G_U) & \xrightarrow{\tilde{g}_{\cdot U}} & (\tilde{V}, G_V) \\ \downarrow & & \downarrow \\ U & \xrightarrow{g_{\cdot}} & V \end{array}$$

commutes and  $\tilde{g}_{\cdot U}$  is  $\bar{g}_{\cdot U}$ -equivariant:

$$\tilde{g}_{\cdot U}(g \cdot x) = \bar{g}_{\cdot U}(g) \tilde{g}_{\cdot U}(x), \quad g \in G_U, x \in \hat{U}.$$

In this work, when  $X$  admits a compact Lie group action  $G$ , we suppose that

**Assumption 2.1.** We assume  $g^* \omega_0 = \omega_0$  on  $X$  and  $g_* J = J g_*$  on  $HX$ , for every  $g \in G$ , where  $g^*$  and  $g_*$  denote the pull-back map and push-forward map of  $G$ , respectively.

Let  $\mathfrak{g}$  denote the Lie algebra of  $G$  and  $e$  denotes the identity element in  $G$ . For a moment denote by  $f : G \times X \rightarrow X$  the action, so that  $f(g, x) = g \cdot x$ . Fix  $x \in X$ , define

$$\xi_X(x) = d_e f(\cdot, x)(\xi).$$

For any  $\xi \in \mathfrak{g}$ ,  $\xi_X$  is the so called vector field on  $X$  induced by  $\xi$ .

By the motivation explained in the introduction, in Theorem 4.1, we assume that  $X$  admits a CR and transversal  $S^1$ -action which is locally free,  $T \in C^\infty(X, TX)$  denotes the global real vector field given by the infinitesimal circle action. In this case, we will take  $T$  to be our Reeb vector field  $R$ . Note that in Theorem 1.1, we do not assume that  $X$  admits a group action.

### 2.4. Operators on orbifolds

Let us recall the definitions of kernels and operators on orbifolds given in [19, Section 6]. By the formulas [19, (6.3) and (6.6)] we have

$$\mathcal{K}(x, x') = \sum_{g \in G_U} (1, g^{-1}) \cdot \tilde{\mathcal{K}}(\tilde{x}, g \cdot \tilde{x}')$$

where  $\mathcal{K} : C^\infty_0(\tilde{U}, \tilde{E}) \rightarrow C^\infty(\tilde{U}, \tilde{E})$  is an operator acting on sections of a proper orbifold vector bundle  $E$ . Here,  $(1, g^{-1}) \cdot$  denotes the action on  $\tilde{E}_{\tilde{x}} \times \tilde{E}_{\tilde{x}'}$ . Now, in this paper we shall consider operators acting on smooth functions on the orbifold  $X$ , thus the bundle  $E$  and the action  $(1, g^{-1}) \cdot$  are trivial. Now, we shall recall more precisely the definitions that we need.

Let  $U \subset X$  be an open set and let  $E$  be a vector orbibundle over  $U$ . As smooth case, let  $\mathcal{D}'(\tilde{U}, \tilde{E})^{G_U}$  denote the space of  $G_U$ -invariant distribution sections over  $\tilde{U}$  and let  $\mathcal{E}'(\tilde{U}, \tilde{E})^{G_U}$  denote the space of  $G_U$ -invariant distribution sections over  $\tilde{U}$  whose elements have compact support in  $\tilde{U}$ . Let  $C^\infty(\tilde{U}, \tilde{E})^{G_U}$  denote the space of  $G_U$ -invariant smooth sections over  $\tilde{U}$  and let  $C^\infty_c(\tilde{U}, \tilde{E})^{G_U}$  denote the space of  $G_U$ -invariant smooth sections over  $\tilde{U}$  whose elements have compact support in  $\tilde{U}$ . For  $s \in \mathbb{R}$ , let  $H^s_{\text{comp}}(\tilde{U}, \tilde{E})$  denote the Sobolev space of  $G_U$ -invariant sections over  $\tilde{U}$  of order  $s$  whose elements have compact support in  $\tilde{U}$ . Put

$$H^s_{\text{loc}}(\tilde{U}, \tilde{E})^{G_U} := \left\{ u \in \mathcal{D}'(\tilde{U}, \tilde{E})^{G_U}; \chi u \in H^s_{\text{comp}}(\tilde{U}, \tilde{E})^{G_U}, \forall \chi \in C^\infty_c(\tilde{U}) \right\}.$$

Let  $H^s(X, E)$  denote the Sobolev space of sections over  $X$  of order  $s$ . Let  $F$  be another vector orbibundle over  $U$ . We write  $E \boxtimes F^*$  to denote the vector orbibundle over  $U \times U$  with fiber over  $\tilde{U} \times \tilde{U}$  consisting of the linear maps from  $\tilde{F}_{\tilde{y}}$  to  $\tilde{E}_{\tilde{y}}$ .

Let  $H : \mathcal{C}_c^\infty(U, E) \rightarrow \mathcal{D}'(U, E)$  be a continuous operator and let

$$H(x, y) \in \mathcal{D}'(U \times U, E \boxtimes E^*)$$

be the distribution kernel of  $H$ . We say that  $H$  is smoothing on  $U$  if

$$H(x, y) \in \mathcal{C}^\infty(U \times U, E \boxtimes E^*).$$

When  $H$  is smoothing on  $U$ , we write  $H \equiv 0$  (on  $U$ ) or  $H(x, y) \equiv 0$  (on  $U$ ). More precisely, let  $(\tilde{U}, G_U) \rightarrow U$  be an orbifold chart. The notation  $H \equiv 0$  means that we can find a

$$\tilde{H}(x, y) \in \mathcal{C}^\infty(\tilde{U} \times \tilde{U}, \tilde{E} \boxtimes \tilde{E}^*)$$

such that

$$H(x, y) = \sum_{g \in G_U} \tilde{H}(\tilde{x}, g \cdot \tilde{y}) \equiv 0,$$

for all  $(x, y) \in U \times U$ , where  $\pi(\tilde{x}) = x$ ,  $\pi(\tilde{y}) = y$ ,  $\pi : \tilde{U} \rightarrow U$  is the natural projection and  $\tilde{E}$  is the lifting of  $E$ . Let

$$G(x, y) \in \mathcal{D}'(U \times U, E \boxtimes E^*).$$

We will write  $G$  to denote the continuous operator  $G : \mathcal{C}_c^\infty(U, E) \rightarrow \mathcal{D}'(U, E)$  with distribution kernel  $G(x, y)$ .

Let

$$H : \mathcal{C}^\infty(X, T^{*0,q}X) \rightarrow \mathcal{D}'(X, T^{*0,q}X)$$

be a continuous operator. We say that  $H$  is a complex Fourier integral operator if, let  $(\tilde{U}, G_U) \rightarrow U$  be any orbifold chart of  $X$ , there is a complex Fourier integral operator  $\tilde{H}_U : \Omega^{0,q}(\tilde{U}) \rightarrow \Omega^{0,q}(\tilde{U})$  has a distribution kernel

$$\tilde{H}_U(\tilde{x}, \tilde{y}) \in \mathcal{D}'(\tilde{U} \times \tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

such that

$$H(x, y) \equiv \sum_{g \in G_U} \tilde{H}_U(\tilde{x}, g \cdot \tilde{y})$$

for  $(x, y) \in U \times U$ , where  $\pi(\tilde{x}) = x$ ,  $\pi(\tilde{y}) = y$ ,  $\pi : \tilde{U} \rightarrow U$  is the natural projection. Furthermore, we shall often write the distribution  $\tilde{H}$  as an oscillatory integral

$$H(x, y) := \sum_{g \in G_U} \left( \int_0^{+\infty} e^{it\varphi(\tilde{x}, g \cdot \tilde{y})} a(\tilde{x}, g \cdot \tilde{y}, t) dt \right) \text{ on } U \times U$$

where  $a$  is a symbol in the sense of Hörmander, to be described in the next paragraph, and  $\varphi$  is a  $G_U$ -invariant complex phase function as in [20].

Let us recall the definition of Hörmander symbol space. Let  $\tilde{U} \subset \mathbb{R}^{2n+1}$  be a local coordinate patch with local coordinates  $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{2n+1})$ . We recall the following definition.

**Definition 2.1.** For  $m \in \mathbb{R}$ ,

$$S_{1,0}^m(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

is the space of all smooth functions

$$a \in \mathcal{C}^\infty(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

such that, for all compact  $K \Subset \tilde{U} \times \tilde{U}$  and all  $\alpha, \beta \in \mathbb{N}_0^{2n+1}$ ,  $\gamma \in \mathbb{N}_0$ , there is a constant  $C_{\alpha,\beta,\gamma} > 0$  such that

$$\left| \partial_x^\alpha \partial_y^\beta \partial_t^\gamma a(\tilde{x}, \tilde{y}, t) \right| \leq C_{\alpha,\beta,\gamma} (1 + |t|)^{m-\gamma}, \text{ for every } (\tilde{x}, \tilde{y}, t) \in K \times \mathbb{R}_+, t \geq 1.$$

Put

$$S^{-\infty}(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*) := \bigcap_{m \in \mathbb{R}} S_{1,0}^m(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*).$$

Let

$$a_j \in S_{1,0}^{m_j}(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),$$

$j = 0, 1, 2, \dots$  with  $m_j \rightarrow -\infty$ , as  $j \rightarrow \infty$ . Then there exists

$$a \in S_{1,0}^{m_0}(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

unique modulo  $S^{-\infty}$ , such that

$$a - \sum_{j=0}^{k-1} a_j \in S_{1,0}^{m_k}(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

for  $k = 0, 1, 2, \dots$

If  $a$  and  $a_j$  have the properties listed above, then we write

$$a \sim \sum_{j=0}^{\infty} a_j \quad (\text{in the space } S_{1,0}^{m_0}(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)).$$

Furthermore, we write

$$s(\tilde{x}, \tilde{y}, t) \in S_{cl}^m(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

if

$$s(\tilde{x}, \tilde{y}, t) \in S_{1,0}^m(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

and

$$s(\tilde{x}, \tilde{y}, t) \sim \sum_{j=0}^{\infty} s^j(x, y) t^{m-j} \text{ in } S_{1,0}^m(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),$$

$$s^j(\tilde{x}, \tilde{y}) \in C^\infty(\tilde{U} \times \tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j \in \mathbb{N}_0.$$

We simply write  $S_{1,0}^m$  to denote

$$S_{1,0}^m(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),$$

$m \in \mathbb{R} \cup \{-\infty\}$ .

The previous definition can be adapted in the setting of orbi-vector bundles as follows. Let  $E$  be a vector orbibundle over  $X$  and let  $H : C^\infty(X, E) \rightarrow \mathcal{D}'(X, E)$  be a continuous operator with distribution kernel  $H(x, y) \in \mathcal{D}'(X \times X, E \boxtimes E^*)$ . Locally, let  $\tilde{E}$  be the vector bundle over  $\tilde{U}$ . Let  $m \in \mathbb{R}$ ,  $0 \leq \rho, \delta \leq 1$ , then  $S_{\rho,\delta}^m(T^*\tilde{U}, \tilde{E})$  denotes the Hörmander symbol space on  $T^*\tilde{U}$  with values in  $\tilde{E}$  of order  $m$  type  $(\rho, \delta)$  and  $S_{cl}^m(T^*\tilde{U}, \tilde{E})$  denotes the space of classical symbols on  $T^*\tilde{U}$  with values in  $\tilde{E}$  of order  $m$ . Let  $L_{\rho,\delta}^m(\tilde{U}, \tilde{E} \boxtimes \tilde{E}^*)$  (respectively  $L_{cl}^m(\tilde{U}, \tilde{E} \boxtimes \tilde{E}^*)$ ) be the space of pseudodifferential operators on  $\tilde{U}$  of order  $m$  from sections of  $E$  to sections of  $E$  with full symbol  $a \in S_{\rho,\delta}^m(T^*\tilde{U}, \tilde{E})$  (respectively  $a \in S_{cl}^m(T^*\tilde{U}, \tilde{E})$ ). Eventually, we write  $H \in L_{\rho,\delta}^m(X, E \otimes E^*)$  (respectively  $H \in L_{cl}^m(X, E \otimes E^*)$ ) if for any orbifold chart  $(\tilde{U}, G_U) \rightarrow U$ , there is a  $\tilde{H}_U \in L_{\rho,\delta}^m(\tilde{U}, \tilde{E} \boxtimes \tilde{E}^*)$  (respectively  $\tilde{H}_U \in L_{cl}^m(\tilde{U}, \tilde{E} \boxtimes \tilde{E}^*)$ ) such that

$$H(x, y) \equiv \sum_{g \in G_U} \tilde{H}_U(\tilde{x}, g \cdot \tilde{y})$$

for  $(x, y) \in U \times U$ , where  $\pi(\tilde{x}) = x$ ,  $\pi(\tilde{y}) = y$ ,  $\pi : \tilde{U} \rightarrow U$  is the natural projection and  $\tilde{E}$  is the lifting of  $E$ .

### 2.5. Microlocal Szegő kernel

On each open set  $U \subseteq X$ , we define operators  $S_U^{(q)}$  having the same microlocal structure described in our main results. We call this operators *approximate Szegő kernels*. We shall review some results in [11] concerning the existence of a microlocal Hodge decomposition of the Kohn Laplacian on an open set of a CR manifold where the Levi form is non-degenerate. Since the proofs are local we can carry out the same analysis on the chart  $(G_U, \tilde{U})$  for a given open set  $U$ . For the following theorems we refer to Chapter 6, Chapter 7 and Chapter 8 of Part I in [11].

**Theorem 2.1.** *We assume that the  $G_U$ -invariant Levi form is non-degenerate of constant signature  $(n_-, n_+)$  at each point of  $\tilde{U}$ . Let  $q \neq n_-, n_+$ . Then, there is a properly supported operator  $A_{\tilde{U}} \in L_{\frac{1}{2}, \frac{1}{2}}^{-1}(\tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$  such that  $\square_b^{(q)} A_{\tilde{U}} = I + F_{\tilde{U}}$  on  $\tilde{U}$ ,  $F_{\tilde{U}}$  is a smoothing operator on  $\tilde{U}$ .*

Until further notice, we assume that the Levi form is non-degenerate of constant signature  $(n_-, n_+)$  on  $X$ . Let  $p_0(x, \xi) \in C^\infty(T^*X)$  be the principal symbol of  $\square_b^{(q)}$ . Note that  $p_0(x, \xi)$  is a  $G_U$ -invariant polynomial of degree 2 in  $\xi$  when written locally on  $\tilde{U}$ . Recall that the characteristic manifold of  $\square_b^{(q)}$  is given by  $\Sigma = \Sigma^+ \cup \Sigma^-$ , where  $\Sigma^+$  and  $\Sigma^-$  are given by

$$\Sigma^+ = \{(x, \lambda \omega_0(x)) \in T^*X; \lambda > 0\}, \quad \Sigma^- = \{(x, \lambda \omega_0(x)) \in T^*X; \lambda < 0\}, \tag{7}$$

where  $\omega_0 \in C^\infty(X, T^*X)$  is the uniquely determined global contact 1-form.

**Theorem 2.2.** *With the same assumptions as in the previous theorem, let  $q = n_-$  or  $n_+$ . Then, by adopting notation as in Section 2.4, there exist properly supported continuous operators  $A \in L_{\frac{1}{2}, \frac{1}{2}}^{-1}(\tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ , and operators  $S_-, S_+ \in L_{\frac{1}{2}, \frac{1}{2}}^0(\tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ , such that*

$$\begin{aligned} \square_b^{(q)} A + S_- + S_+ &\equiv I \text{ on } \tilde{U}, \\ \square_b^{(q)} S_- &\equiv 0 \text{ on } \tilde{U}, \quad \square_b^{(q)} S_+ \equiv 0 \text{ on } \tilde{U}, \\ A &\equiv A^* \text{ on } \tilde{U}, \quad S_- A \equiv 0 \text{ on } \tilde{U}, \quad S_+ A \equiv 0 \text{ on } \tilde{U}, \\ S_- &\equiv S_-^* \equiv S_-^2 \text{ on } \tilde{U}, \\ S_+ &\equiv S_+^* \equiv S_+^2 \text{ on } \tilde{U}, \\ S_- S_+ &\equiv S_+ S_- \equiv 0 \text{ on } \tilde{U}, \end{aligned} \tag{8}$$

where  $A^*, S_-^*$  and  $S_+^*$  are the formal adjoints of  $A, S_-$  and  $S_+$  with respect to  $(\cdot | \cdot)$  respectively and  $S_-(\tilde{x}, \tilde{y})$  satisfies

$$S_-(\tilde{x}, \tilde{y}) \equiv \int_0^\infty e^{i\varphi_-(\tilde{x}, \tilde{y})t} s_-(\tilde{x}, \tilde{y}, t) dt \text{ on } \tilde{U} \tag{9}$$

with a symbol  $s_- \in S_{cl}^n(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$  such that

$$\begin{aligned} s_-(\tilde{x}, \tilde{y}, t) &\sim \sum_{j=0}^\infty s_-^j(\tilde{x}, \tilde{y}) t^{n-j} \quad \text{in } S_{1,0}^n(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \\ s_-^j &\in C^\infty(\tilde{U} \times \tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j \in \mathbb{N}_0, \end{aligned} \tag{10}$$

and phase function  $\varphi_-$  such that  $\varphi = \varphi_-$  satisfies

$$\begin{aligned} \varphi &\in C^\infty(\tilde{U} \times \tilde{U}), \quad \text{Im } \varphi(\tilde{x}, \tilde{y}) \geq 0, \\ \varphi(\tilde{x}, \tilde{x}) &= 0, \quad \varphi(\tilde{x}, \tilde{y}) \neq 0 \text{ if } \tilde{x} \neq \tilde{y}, \\ d_x \varphi(\tilde{x}, \tilde{y})|_{\tilde{x}=\tilde{y}} &= -\omega_0(\tilde{x}), \quad d_y \varphi(\tilde{x}, \tilde{y})|_{\tilde{x}=\tilde{y}} = \omega_0(\tilde{x}), \\ \varphi(\tilde{x}, \tilde{y}) &= -\bar{\varphi}(\tilde{y}, \tilde{x}). \end{aligned} \tag{11}$$

Moreover, there is a function  $f \in C^\infty(\tilde{U} \times \tilde{U})$  such that

$$p_0(\tilde{x}, \varphi_x^L(\tilde{x}, \tilde{y})) - f(\tilde{x}, \tilde{y})\varphi(\tilde{x}, \tilde{y}) \tag{12}$$

vanishes to infinite order at  $\tilde{x} = \tilde{y}$ . Similarly,

$$S_+(\tilde{x}, \tilde{y}) \equiv \int_0^\infty e^{i\varphi_+(\tilde{x}, \tilde{y})t} s_+(\tilde{x}, \tilde{y}, t) dt \text{ on } \tilde{U} \tag{13}$$

with  $s_+ \in S_{cl}^n(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$  satisfying

$$s_+(\tilde{x}, \tilde{y}, t) \sim \sum_{j=0}^\infty s_+^j(\tilde{x}, \tilde{y}) t^{n-j} \tag{14}$$

in  $S_{1,0}^n(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ ,

$$s_+^j \in C^\infty(\tilde{U} \times \tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j \in \mathbb{N}_0,$$

and  $-\bar{\varphi}_+(\tilde{x}, \tilde{y})$  satisfies (11) and (12). Moreover, if  $q \neq n_+$ , then  $s_+(\tilde{x}, \tilde{x}, t)$  vanishes to infinite order at  $\tilde{x} = \tilde{y}$ . If  $q \neq n_-$ , then  $s_-(\tilde{x}, \tilde{y}, t)$  vanishes to infinite order at  $\tilde{x} = \tilde{y}$ .

The operators  $S_+, S_-$  are called *approximate Szegő kernels*.

**Remark 2.1.** With the notations and assumptions used in Theorem 2.2, furthermore suppose that  $q = n_- \neq n_+$ . Since  $s_+(\tilde{x}, \tilde{y}, t)$  vanishes to infinite order at  $\tilde{x} = \tilde{y}$ , we have  $S_+ \equiv 0$  on  $\tilde{U}$ . Similarly, if  $q = n_+ \neq n_-$ , then  $S_- \equiv 0$  on  $\tilde{U}$ .

We pause and introduce some notations. For a given point  $p \in \tilde{U}$ , let  $\{W_j\}_{j=1}^n$  be an orthonormal frame of  $(T^{1,0}\tilde{U}, \langle \cdot | \cdot \rangle)$  near  $p$ , for which the Levi form is diagonal at  $p$ . Put

$$\mathcal{L}_p(W_j, \overline{W}_\ell) = \mu_j(p) \delta_{j,\ell}, \quad j, \ell = 1, \dots, n.$$

We will denote by

$$\det \mathcal{L}_p = \prod_{j=1}^n \mu_j(p). \tag{15}$$

Let  $\{e_j\}_{j=1}^n$  denote the basis of  $T^{*0,1}\tilde{U}$ , dual to  $\{\overline{W}_j\}_{j=1}^n$ . We assume that  $\mu_j(p) < 0$  if  $1 \leq j \leq n_-$  and  $\mu_j(p) > 0$  if  $n_- + 1 \leq j \leq n$ . Put

$$\begin{aligned} \mathcal{N}(p, n_-) &:= \{c e_1(p) \wedge \dots \wedge e_{n_-}(p); c \in \mathbb{C}\}, \\ \mathcal{N}(p, n_+) &:= \{c e_{n_-+1}(p) \wedge \dots \wedge e_n(p); c \in \mathbb{C}\} \end{aligned}$$

and let

$$\tau_{p,n_-} : T_p^{*0,q}\tilde{U} \rightarrow \mathcal{N}(p, n_-), \quad \tau_{p,n_+} : T_p^{*0,n-q}\tilde{U} \rightarrow \mathcal{N}(p, n_+), \tag{16}$$

be the orthogonal projections onto  $\mathcal{N}(p, n_-)$  and  $\mathcal{N}(p, n_+)$  with respect to  $\langle \cdot | \cdot \rangle$ , respectively. For  $J = (j_1, \dots, j_q)$ ,  $1 \leq j_1 < \dots < j_q \leq n$ , let  $e_J := e_{j_1} \wedge \dots \wedge e_{j_q}$ . For  $|I| = |J| = q$ ,  $I, J$  are strictly increasing, let  $e_I \otimes (e_J)^*$  be the linear transformation from  $T^{*0,q}X$  to  $T^{*0,q}X$  given by

$$(e_I \otimes (e_J)^*)(e_K) = \delta_{J,K} e_I,$$

for every  $|K| = q$ ,  $K$  is strictly increasing, where  $\delta_{J,K} = 1$  if  $J = K$ ,  $\delta_{J,K} = 0$  if  $J \neq K$ . For any  $f \in T^{*0,q}X \boxtimes (T^{*0,q}X)^*$ , we have

$$f = \sum'_{|I|=|J|=q} c_{I,J} e_I \otimes (e_J)^*,$$

$c_{I,J} \in \mathbb{C}$ , for all  $|I| = |J| = q$ ,  $I, J$  are strictly increasing, where  $\sum'$  means that the summation is performed only over strictly increasing multi-indices. We call  $c_{I,J} e_I \otimes (e_J)^*$  the component of  $f$  in the direction  $e_I \otimes (e_J)^*$ . Let  $I_0 = (1, 2, \dots, q)$ . We can check that

$$\tau_{p,n_-} = e_{I_0}(p) \otimes (e_{I_0}(p))^*.$$

The following formula for the leading term  $s_-^0$  on the diagonal follows from [11, §9]. The formula for the leading term  $s_+^0$  on the diagonal follows similarly.

**Theorem 2.3.** Let  $q = n_-$ . For the leading term  $s_-^0$  of the expansion (10) of  $s_-$ , we have

$$s_-^0(\tilde{x}_0, \tilde{x}_0) = \frac{1}{2} \pi^{-n-1} |\det \mathcal{L}_{\tilde{x}_0}| \tau_{\tilde{x}_0, n_-}, \quad \tilde{x}_0 \in \tilde{U}.$$

Let  $q = n_+$ . For the leading term  $s_+^0$  of the expansion (14) of  $s_+$ , we have

$$s_+^0(\tilde{x}_0, \tilde{x}_0) = \frac{1}{2} \pi^{-n-1} |\det \mathcal{L}_{\tilde{x}_0}| \tau_{\tilde{x}_0, n_+}, \quad \tilde{x}_0 \in \tilde{U}.$$

**Definition 2.2.** With the assumptions and notations used in Theorem 2.2, let  $\varphi_1, \varphi_2 \in \mathcal{C}^\infty(\tilde{U} \times \tilde{U})$ . We assume that  $\varphi_1$  and  $\varphi_2$  satisfy (11). We say that  $\varphi_1$  and  $\varphi_2$  are equivalent on  $\tilde{U}$  if for any  $b_1(x, y, t) \in S_{\text{cl}}^n(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$  we can find  $b_2(x, y, t) \in S_{\text{cl}}^n(\tilde{U} \times \tilde{U} \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$  such that

$$\int_0^\infty e^{i\varphi_1(x,y)t} b_1(x, y, t) dt \equiv \int_0^\infty e^{i\varphi_2(x,y)t} b_2(x, y, t) dt \quad \text{on } \tilde{U}$$

and vice versa.

**Theorem 2.4.** *With the assumptions and notations used in Theorem 2.2, let  $q = n_-$ . There is a phase  $\hat{\varphi}_-(\tilde{x}, \tilde{y}) \in C^\infty(\tilde{U} \times \tilde{U})$  satisfying (11) such that  $\hat{\varphi}_-(g \cdot \tilde{x}, g \cdot \tilde{y}) = \hat{\varphi}_-(\tilde{x}, \tilde{y})$ , for every  $g \in G_U$  and  $\hat{\varphi}_-$  is equivalent to  $\varphi_-$  on  $\tilde{U}$ .*

**Proof.** Fix  $g \in G_U, g \neq I$ . Let

$$S_{-,g}, S_{+,g} : \Omega_c^{0,q}(\tilde{U}) \rightarrow \Omega_c^{0,q}(\tilde{U})$$

be the continuous operators with distribution kernels  $S_-(g \cdot \tilde{x}, g \cdot \tilde{y}), S_+(g \cdot \tilde{x}, g \cdot \tilde{y})$  respectively, where  $S_-, S_+$  are as in Theorem 2.2. Let  $S_g := S_{-,g} + S_{+,g}, S := S_- + S_+$ . From the first equation in (8) and note that

$$S_g \square_b^{(q)} \equiv 0 \text{ on } \tilde{U},$$

we have

$$S_g \circ S \equiv S_g \text{ on } \tilde{U}. \tag{17}$$

Now,

$$\begin{aligned} & (S_g \circ S)(\tilde{x}, \tilde{y}) \\ &= \int S(g \cdot \tilde{x}, g \cdot \tilde{u}) S(\tilde{u}, \tilde{y}) dV_X(\tilde{u}) \\ &= \int S(g \cdot \tilde{x}, \tilde{u}) S(g^{-1} \cdot \tilde{u}, \tilde{y}) dV_X(\tilde{u}) \equiv S(\tilde{x}, \tilde{y}) \text{ on } \tilde{U}. \end{aligned} \tag{18}$$

From (17) and (18), we get

$$S_g = S_{-,g} + S_{+,g} \equiv S_- + S_+. \tag{19}$$

Since  $WF'(S_{-,g}) = WF'(S_-) = \Sigma^- \cap T^*\tilde{U}, WF'(S_{+,g}) = WF'(S_+) = \Sigma^+ \cap T^*\tilde{U}$ , we have  $S_{-,g} \equiv S_-, S_{+,g} \equiv S_+$ . From  $S_{-,g} \equiv S_-$ , we can repeat the proof of [15, Theorem 5.4] and deduce that there is a  $f(\tilde{x}, \tilde{y}) \in C^\infty(\tilde{U} \times \tilde{U}), f(\tilde{x}, \tilde{x}) = 1$ , such that

$$\varphi_-(\tilde{x}, \tilde{y}) - f(\tilde{x}, \tilde{y})\varphi_-(g \cdot \tilde{x}, g \cdot \tilde{y})$$

vanishes to infinite order at  $\tilde{x} = \tilde{y}$ . From global theory of complex Fourier integral operators of Melin-Sjöstrand [20],  $\varphi_-(\tilde{x}, \tilde{y})$  is equivalent to  $\varphi_-(g \cdot \tilde{x}, g \cdot \tilde{y})$  on  $\tilde{U}$ . Let  $\hat{\varphi}_-(\tilde{x}, \tilde{y}) := \frac{1}{|G_U|} \sum_{g \in G_U} \varphi_-(g \cdot \tilde{x}, g \cdot \tilde{y})$ . Then,  $\hat{\varphi}_-(\tilde{x}, \tilde{y})$  satisfies (11) such that  $\hat{\varphi}_-(g \cdot \tilde{x}, g \cdot \tilde{y}) = \hat{\varphi}_-(\tilde{x}, \tilde{y})$ , for every  $g \in G_U$ , and  $\hat{\varphi}_-$  is equivalent to  $\varphi_-$  on  $\tilde{U}$ .  $\square$

**Theorem 2.5.** *With the assumptions and notations used in Theorem 2.2, let  $q = n_-$ . We can take the phase  $\varphi_-$  and symbol  $s_-$  in (9) so that  $\varphi_-$  and  $s_-$  are  $G_U$ -invariant.*

*Similarly, let  $q = n_+$ . We can take the phase  $\varphi_+$  and symbol  $s_+$  in (13) so that  $\varphi_+$  and  $s_+$  are  $G_U$ -invariant.*

**Proof.** Let  $q = n_-$ . From Theorem 2.5, we see that we can take  $\varphi_-$  so that  $\varphi_-$  is  $G_U$ -invariant. From the proof of Theorem 2.5, we see that  $S_- \equiv S_{-,g}$ , for every  $g \in G_U$ . Hence, we can replace  $s_-(\tilde{x}, \tilde{y})$  by  $\frac{1}{|G_U|} s_-(g \cdot \tilde{x}, g \cdot \tilde{y})$ . When  $q = n_+$ , the proofs for  $\varphi_+$  and  $s_+$  are similar.  $\square$

### 2.6. Geometric conditions for the closed range property

We recall that, given  $q \in \{0, \dots, n\}$ , the Levi form is said to satisfy condition  $Y(q)$  at  $p \in X$ , if  $\mathcal{L}_p$  has at least either  $\min(q + 1, n - q)$  pairs of eigenvalues with opposite signs or  $\max(q + 1, n - q)$  eigenvalues of the same sign.

**Remark 2.2.** Assume that the Levi form is non degenerate of constant signature  $(n_-, n_+)$ . Given  $q \in \{0, \dots, n\}$ , then  $Y(q)$  holds if  $q \notin \{n_-, n_+\}$ . Furthermore, notice that if  $q = n_- = n_+, q - 1 \notin \{n_-, n_+\}$  and  $q + 1 \notin \{n_-, n_+\}$ , then  $Y(q - 1)$  and  $Y(q + 1)$  hold. Eventually, we remark that if  $q = n_-$  and  $|n_- - n_+| > 1$ , then  $Y(q - 1)$  and  $Y(q + 1)$  hold. In particular if  $X$  is a strongly pseudoconvex CR orbifold of dimension greater than 5, then  $Y(1)$  holds.

We can repeat the same proof as in manifold case with small modification and get the following:

**Theorem 2.6.** *Let  $X$  be a compact CR orbifold and suppose that  $Y(q)$  holds. Then,  $\square_b^{(q)}$  has closed range.*

*Furthermore, suppose that  $Y(q)$  fails but  $Y(q - 1)$  and  $Y(q + 1)$  hold, then  $\square_b^{(q)}$  has closed range.*

In the following, we will give a proof of the first part of Theorem 2.6 under the assumption that the Levi form is non-degenerate of constant signature  $(n_-, n_+)$ . Since the proof is related to the construction of parametrices for Kohn Laplacians on orbifolds, therefore we think it is worth to give a proof under the assumption that the Levi form is non-degenerate of constant signature  $(n_-, n_+)$ . We suppose that  $Y(q)$  holds. Then,  $q \notin \{n_-, n_+\}$ . We shall prove:

$$\exists C > 0 \text{ such that } \|\square_b^{(q)} u\| \geq C \|u\| \quad \forall u \perp \text{Ker} \square_b^{(q)}; \tag{20}$$

which is equivalent to prove that  $\square_b^{(q)}$  has closed range, see Lemma C.1.1 in [18]. We need the following lemma.

**Lemma 2.1.** *There exists a continuous operator*

$$A : H^s(X, T^{*0,q}X) \rightarrow H^{s+1}(X, T^{*0,q}X), \quad \forall s \in \mathbb{R},$$

such that  $\square_b^{(q)} A = I + F$  where  $F$  is smoothing on  $X$ .

**Proof of Lemma 2.1.** Fix an orbifold chart  $(G_U, \tilde{U}) \rightarrow U$ . By Theorem 2.1, we can define on  $\tilde{U}$ ,

$$\hat{A}_U(x, y) = \frac{1}{|G_U|} \sum_{h, g \in G_U} A_{\tilde{U}}(h \cdot \tilde{x}, g \cdot \tilde{y}) \quad \text{and} \quad \hat{F}_U(x, y) = \frac{1}{|G_U|} \sum_{h, g \in G_U} F_{\tilde{U}}(h \cdot \tilde{x}, g \cdot \tilde{y}),$$

so that  $\hat{A}_U : H_{\text{comp}}^s(\tilde{U}, T^{*0,q}X) \rightarrow H_{\text{loc}}^{s+1}(\tilde{U}, T^{*0,q}X)$  is continuous, for all  $s \in \mathbb{R}$ ,  $\hat{F}_U$  is smoothing and they satisfy the following

$$\square_b^{(q)} \hat{A}_U = I + \hat{F}_U \quad \text{on} \quad U,$$

where  $\hat{A}_U$  and  $\hat{F}_U$  are as in Theorem 2.1. Now, since  $X$  is compact we can consider an open cover  $\{U_j\}_{j \in J}$ ,  $(\tilde{U}_j, G_{U_j}) \rightarrow U_j$  orbifold chart, for every  $j$ , and a partition of unity  $\chi_j \in C_c^\infty(U_j)$  so that we have

$$\square_b^{(q)} \hat{A}_{U_j} \chi_j = \chi_j + \hat{F}_{U_j} \chi_j \quad \text{on} \quad U_j.$$

Eventually, set  $\hat{A} = \sum_j \hat{A}_j \chi_j$  and  $\hat{F} = \sum_j \hat{F}_j \chi_j$ , we can check

$$\square_b^{(q)} \hat{A} = I + \hat{F},$$

globally on  $X$  and we get the lemma.  $\square$

We are ready to prove (20). Suppose by the contrary that (20) does not hold. Then there exists a sequence of  $(0, q)$  forms such that  $u_j \in L^2_{(0,q)}(X)$ ,  $j = 1, 2, \dots$ , with  $\|u_j\| = 1$  and  $u_j \perp \text{Ker} \square_b^{(q)}$  such that  $\|\square_b^{(q)} u_j\| < \frac{1}{j} \|u_j\|$  for each  $j = 1, 2, \dots$ . By Lemma 2.1, we have

$$\hat{A}^* \square_b^{(q)} u_j = u_j + \hat{F}^* u_j, \quad j = 1, 2, \dots,$$

where  $\hat{A}^*$  and  $\hat{F}^*$  are adjoints of  $\hat{A}$  and  $\hat{F}$  respectively. Since  $\hat{A}^* : L^2_{(0,q)}(X) \rightarrow H^1(X, T^{*0,1}X)$  is continuous, we get

$$\|u_j\|_1 - \|\hat{F}^* u_j\|_1 \leq \|\hat{A}^* \square_b^{(q)} u_j\|_1 \leq C \|\square_b^{(q)} u_j\| \leq \frac{C}{j} \|u_j\|,$$

for all  $j = 1, 2, \dots$ , where  $\|\cdot\|_1$  denotes the Sobolev norm of order one and  $C > 0$  is a constant independent of  $u_j$ . Since  $\hat{F} : L^2_{(0,q)}(X) \rightarrow H^1(X, T^{*0,q}X)$  is continuous, we get

$$\|u_j\|_1 \leq C', \quad \text{for all } j = 1, 2, \dots,$$

where  $C' > 0$  is a constant independent of  $j$ . By Rellich's Lemma, there is a subsequence  $1 \leq j_1 < j_2 < \dots$  such that  $u_{j_s} \rightarrow u$  in  $L^2_{(0,q)}(X)$  as  $s \rightarrow +\infty$ . Since  $u_j \perp \text{Ker} \square_b^{(q)}$  for each  $j$ , then  $u \perp \text{Ker} \square_b^{(q)}$ ; but  $\square_b^{(q)} u = \lim_s \square_b^{(q)} u_{j_s} = 0$  and thus we get a contradiction.

We can repeat the proof of [15, Theorem 6.24] with minor change and get the following

**Theorem 2.7.** *Assume that the Levi form is non-degenerate of constant signature  $(n_-, n_+)$  on  $X$ . Suppose that  $X$  admits a transversal and CR  $\mathbb{R}$ -action. For any  $q \in \{0, 1, \dots, n\}$ ,*

$$\square_b^{(q)} : \text{Dom} \square_b^{(q)} \subset L^2_{(0,q)}(X) \rightarrow L^2_{(0,q)}(X)$$

has closed range.

### 3. Proof of Theorem 1.1

Let

$$\Pi^{(q)} : L^2_{(0,q)}(X) \rightarrow \text{Ker } \square_b^{(q)}$$

be the orthogonal projection, the so called Szegő projection. Let us recall the following known global result.

**Theorem 3.1.** *Suppose that the Kohn Laplacian  $\square_b^{(q)} : \text{Dom } \square_b^{(q)} \subset L^2_{(0,q)}(X) \rightarrow L^2_{(0,q)}(X)$  has closed range, then there exists a bounded operator  $N : L^2_{(0,q)}(X) \rightarrow \text{Dom } \square_b^{(q)}$  such that*

$$N \square_b^{(q)} + \Pi^{(q)} = I \quad \text{on } \text{Dom } \square_b^{(q)} \quad \text{and} \quad \square_b^{(q)} N + \Pi^{(q)} = I \quad \text{on } L^2_{(0,q)}(X).$$

Consider an orbifold chart  $(G_U, \tilde{U}) \rightarrow U$ , let

$$S_-, S_+, A : \Omega_c^{0,q}(\tilde{U}) \rightarrow \Omega_c^{0,q}(\tilde{U})$$

be as in Theorem 2.2. Recall that  $S_-, S_+$  and  $A$  are properly supported on  $\tilde{U}$ . Define on  $\tilde{U}$  the following kernels

$$\hat{S}_-(x, y) = \sum_{g \in G_U} S_-(\tilde{x}, g \cdot \tilde{y}), \quad \hat{S}_+(x, y) = \sum_{g \in G_U} S_+(\tilde{x}, g \cdot \tilde{y}),$$

and

$$\hat{A}(x, y) = \frac{1}{|G_U|} \sum_{g, h \in G_U} A(h \cdot \tilde{x}, g \cdot \tilde{y}).$$

It is not difficult to see that  $\hat{S}_-, \hat{S}_+$  and  $A$  are properly supported. For every  $u \in \Omega_c^{0,q}(U)$ , we have

$$\hat{A}u = \frac{1}{|G_U|} \sum_{g, h \in G_U} \int_{\tilde{U}} A(h \cdot \tilde{x}, g \cdot \tilde{y}) u(\tilde{y}) dV_{\tilde{U}}(\tilde{y}) = \frac{1}{|G_U|} \sum_{h, g \in G_U} \int_{\tilde{U}} A(h \cdot \tilde{x}, \tilde{y}) u(g^{-1} \cdot \tilde{y}) dV_{\tilde{U}}(\tilde{y}),$$

which is in  $\Omega_c^{0,q}(U)$  because  $A$  is proper and  $u \circ g^{-1}$  is compactly supported. Similarly one can deal with  $\hat{S}_-$  and  $\hat{S}_+$ . Thus, we have  $\hat{S}_-, \hat{S}_+, \hat{A} : \Omega_c^{0,q}(U) \rightarrow \Omega_c^{0,q}(U)$ . Moreover, from Theorem 2.5, we see that

$$\hat{S}_-(\tilde{x}, \tilde{y}) \equiv \frac{1}{|G_U|} \sum_{g, h \in G_U} \int_{\tilde{U}} S_-(h \cdot \tilde{x}, g \cdot \tilde{y}) \text{ on } \tilde{U},$$

$$\hat{S}_+(\tilde{x}, \tilde{y}) \equiv \frac{1}{|G_U|} \sum_{g, h \in G_U} \int_{\tilde{U}} S_+(h \cdot \tilde{x}, g \cdot \tilde{y}) \text{ on } \tilde{U}.$$

By the properties of  $S_-, S_+$  and  $A$ , we have

$$\square_b^{(q)} \hat{A} + \hat{S}_- + \hat{S}_+ = I \quad \text{on } \tilde{U} \tag{21}$$

and

$$\square_b^{(q)} \hat{S}_- \equiv 0 \quad \text{on } \tilde{U}. \tag{22}$$

Now, apply  $\Pi^{(q)}$  to (21), for each  $u \in \Omega_c^{0,q}(U)$  we have

$$\Pi^{(q)}(\square_b^{(q)} \hat{A} + \hat{S}_- + \hat{S}_+)u = \Pi^{(q)}u, \tag{23}$$

where  $\Pi^{(q)}u \in L^2_{(0,q)}(X)$  and the left hand side is well-defined since we have already checked that  $\hat{S}_-, \hat{S}_+, \hat{A}$  are properly supported. Now, notice that for each  $u, v \in \Omega_c^{0,q}(U)$  we have  $(\Pi^{(q)} \square_b^{(q)} \hat{A}u | v) = (\hat{A}u | \square_b^{(q)} \Pi v) = 0$  which implies  $\Pi^{(q)} \square_b^{(q)} \hat{A} = 0$  on  $\Omega_c^{0,q}(U)$ . Thus, by (23), we also get the following desired property

$$\Pi^{(q)}(\hat{S}_- + \hat{S}_+) = \Pi^{(q)} \quad \text{on } \Omega_c^{0,q}(U). \tag{24}$$

Recall from Theorem 3.1 the existence of the operator  $N$ ; since  $\hat{S}_-$  and  $\hat{S}_+$  are properly supported we have

$$N \square_b^{(q)}(\hat{S}_- + \hat{S}_+) + \Pi^{(q)}(\hat{S}_- + \hat{S}_+) = \hat{S}_- + \hat{S}_+ \quad \text{on } \Omega_c^{0,q}(U).$$

Now, we get that  $\square_b^{(q)}(\hat{S}_- + \hat{S}_+)$  is a smoothing operator on  $U$ , which we will denote by  $F$ . Thus, by (24) we write

$$NF + \Pi^{(q)} = \hat{S}_- + \hat{S}_+ =: \hat{S} \quad \text{on } \Omega_c^{0,q}(U). \tag{25}$$

We recall that since  $F$  is smoothing and properly supported on  $U$ , it maps  $\mathcal{E}'(U, T^{*0,q}X)$  to compactly supported smooth sections, so  $F : \mathcal{E}'(U, T^{*0,q}X) \rightarrow \Omega_c^{0,q}(U)$ . Thus, by composition we get in (25) an operator  $NF : \mathcal{E}'(U, T^{*0,q}X) \rightarrow L^2_{(0,q)}(X)$ . Taking the adjoint in (25), we get  $\hat{S}^* - \Pi^{(q)} = F^*N$  on  $\Omega_c^{0,q}(U)$  and thus by making use again of (25) we obtain

$$(\hat{S}^* - \Pi^{(q)})(\hat{S} - \Pi^{(q)}) = F^*N^2F \quad \text{on } \Omega_c^{0,q}(U) \tag{26}$$

where it is easy to see that the right hand side is smoothing. Thus, the left hand side of (26) is also smoothing and we have

$$\hat{S}^*\hat{S} - \hat{S}^*\Pi^{(q)} - \Pi^{(q)}\hat{S} + \Pi^2 \equiv 0.$$

Eventually by (24) we have

$$\Pi^{(q)} \equiv \hat{S}^*\hat{S}. \tag{27}$$

From (21) and  $\hat{S}^*\square_b^{(q)} \equiv 0$ , we get

$$\hat{S}^*\hat{S} \equiv \hat{S}. \tag{28}$$

From (27) and (28), we get

$$\Pi^{(q)} \equiv \hat{S}_- + \hat{S}_+. \tag{29}$$

From Theorem 2.5 and (29), Theorem 1.1 follows.

### 3.1. Quantization commutes with reduction

In this section, we assume that  $X$  admits a CR compact Lie group action  $G$ ,  $\dim G = d$ . We will work Assumption 2.1 and suppose that  $X$  is strongly pseudoconvex. We will use the same notations as in Section 2.3. The moment map associated to  $\omega_0$  is the map given by  $\mu : X \rightarrow \mathfrak{g}^*$  such that for all  $x \in X$  and  $\xi \in \mathfrak{g}$ , we have

$$\langle \mu(x), \xi \rangle = \omega_0(x)(\xi_X). \tag{30}$$

Let us recall the following theorem which is well-known in the setting of symplectic manifolds. Throughout this section we will always assume that 0 is a regular value, and thus  $\mu^{-1}(0)$  is a suborbifold of  $X$ .

**Proposition 3.1.** *Suppose that 0 is a regular value, then the action of  $G$  on  $\mu^{-1}(0)$  is locally free.*

**Proof.** Since 0 is a regular value then for each  $p \in \mu^{-1}(0)$ ,  $d_p\mu : T_pX \rightarrow \mathfrak{g}^*$  is surjective. Let us identify  $\mathfrak{g} \cong \mathfrak{g}^*$ , then for every  $\xi \in \mathfrak{g} \setminus \{0\}$  there exists  $V \in T_pX$  such that  $d_p\mu(V) = \xi$ . Thus, we have

$$0 \neq \langle d_p\mu(V), \xi \rangle = d_p\omega_0(\xi_X(p), V) = -b_p(\xi_X(p), J_pV)$$

and we can conclude that  $\xi_X(p) \neq 0$ .  $\square$

**Proposition 3.2.** *Suppose 0 is a regular value, then  $X_G = \mu^{-1}(0)/G$  is an orbifold.*

**Proof.** By the previous proposition the action of  $G$  on  $\mu^{-1}(0)$  is locally free. From the proof of Corollary B.31 in [10] we can conclude.  $\square$

Corollary B.31 in [10] is a consequence of the Slice Theorem which states that there exists a  $G$ -equivariant diffeomorphism  $\phi$  between  $G \times_{G_x} D$  and a tubular neighborhood of the orbit  $G \cdot x$ , we refer to Appendix B in [10] for precise definitions. More precisely,  $\phi$  is induced by the exponential map  $E_p : U_0 \rightarrow U \subseteq M$ , where  $U_0$  is an open neighborhood of 0 in  $T_xM$ . We have  $G_x$  equivariant decomposition

$$T_xX = T_x(G \cdot x) \oplus W$$

where  $W$  is the normal bundle to  $G \cdot x$ . As a consequence of this, it is easy to see the CR orbifold structure on  $X$  descends naturally to a CR orbifold structure on  $X$  and so the other  $G$ -invariant structures defined on  $X$ . Moreover, we can repeat the proof of [17, Theorem 2.5] and deduce that  $X_G$  is a strongly pseudoconvex CR orbifold.

We take the Hermitian metric  $\langle \cdot | \cdot \rangle$  on  $\mathbb{C}TX$  so that  $\langle \cdot | \cdot \rangle$  is  $G$ -invariant. Let  $\langle \cdot | \cdot \rangle_{X_G}$  be the Hermitian metric on  $\mathbb{C}TX_G$  induced by  $\langle \cdot | \cdot \rangle$  and let  $(\cdot | \cdot)_{X_G}$  be the inner product on  $L^2(X_G)$  induced by  $\langle \cdot | \cdot \rangle_{X_G}$ . Let  $\square_{b, X_G}^{(0)}$  be the Kohn Laplacian on the orbifold  $X_G$  with respect to  $(\cdot | \cdot)_{X_G}$ . Let  $\Pi_{X_G} : L^2(X_G) \rightarrow \square_{b, X_G}^{(0)}$  be the Szegő projection. Fix  $p \in \mu^{-1}(0)$ . Let  $(\tilde{U}, G_U) \rightarrow U$  be an orbifold chart of  $X$ ,  $p \in U$ . By the previous discussion we have that  $G \cdot p = [p] \in X_G$  and a sufficiently small neighborhood  $U_{[p]}$  of  $[p]$  in  $X_G$  has a orbifold local chart  $(G_p \times G_U, \tilde{W}) \rightarrow U_{[p]}$ , where  $\tilde{W}/G_p = (\tilde{U} \cap \mu^{-1}(0))/G_p$  and  $G_p := \{g \in G; gp = p\}$ . Thus, from Theorem 1.1, the Szegő kernel for the orbifold  $X_G$  has the following expression in the local chart  $(G_p \times G_U, \tilde{W}) \rightarrow U_{[p]}$

$$\Pi_{X_G}(x, y) \equiv \sum_{g \in G_p} \int_0^{+\infty} e^{it\varphi_-^{X_G}(\tilde{x}, g \cdot \tilde{y})} s_-^{X_G}(\tilde{x}, g \cdot \tilde{y}, t) dt \tag{31}$$

where  $x, y \in U_{[p]}$  and  $\varphi_-^{X_G}$  and  $s_-^{X_G}$  are respectively the complex phase function and the symbol of  $S_{X_G}$  as described in Theorem 1.1 for the CR orbifold  $X_G$ .

Let us recall that

$$(\text{Ker } \square_b^{(0)})^G := \left\{ u \in \text{Ker } \square_b^{(0)}; g^*u = u, \forall g \in G \right\}.$$

Let  $\Pi_G : L^2(X) \rightarrow (\text{Ker } \square_b^{(0)})^G$  be the orthogonal projection ( $G$ -invariant Szegő projection). In [13] and [8], we studied the  $G$ -invariant Szegő kernel.

We pause and introduce some notations. Fix  $x \in \mu^{-1}(0)$ , consider the map

$$\begin{aligned} R_x : \mathfrak{g}^* &\rightarrow \mathfrak{g}^*, \\ u &\rightarrow R_x u, \langle R_x u | v \rangle = \langle d\omega_0(x), Ju \wedge v \rangle, \quad v \in \mathfrak{g}^*, \end{aligned}$$

where  $J$  is the natural complex structure map on  $HX := \text{Re } T^{1,0}X$ . Let

$$\det R_x = \lambda_1(x) \cdots \lambda_d(x),$$

$\lambda_j(x)$ ,  $j = 1, \dots, d$ , are the eigenvalues of  $R_x$  with respect to  $\langle \cdot | \cdot \rangle$ . Put

$$Y_x := \{gx \in \mu^{-1}(0); g \in G\}$$

and let  $dV_{Y_x}$  be the volume form on  $Y_x$  induced by the given Hermitian metric  $\langle \cdot | \cdot \rangle$ . Put

$$V_{\text{eff}} := \int_{Y_x} dV_{Y_x}.$$

We now come back to our situation. From Theorem 1.1, we can repeat the procedure in [8] and deduce the following: Fix  $p \in \mu^{-1}(0)$ . Let  $(\tilde{U}, G_U) \rightarrow U$  be an orbifold chart of  $X$ ,  $p \in U$ . Then, the distributional kernel of  $\Pi_G$  satisfies

$$\Pi_G(x, y) \equiv \sum_{g \in G_p} \int_0^{+\infty} e^{it\Phi_-(\tilde{x}, g \cdot \tilde{y})} a_-(\tilde{x}, g \cdot \tilde{y}, t) dt \quad \text{on } U \times U, \tag{32}$$

where

$$a_-(\tilde{x}, \tilde{y}, t) \sim \sum_{j=0}^{+\infty} a_-^j(\tilde{x}, \tilde{y}) t^{n-d/2-j} \in S_{1,0}^{n-d/2}(\tilde{U} \times \tilde{U} \times \mathbb{R}),$$

$a_-^j \in \mathcal{C}^\infty(\tilde{U} \times \tilde{U})$ ,  $j \in \mathbb{N}_0$ , and for every  $\tilde{x} \in \widetilde{\mu^{-1}(0)}$ ,  $\widetilde{\mu^{-1}(0)} \subset \tilde{U}$  is the lifting of  $\mu^{-1}(0)$  on  $\tilde{U}$ , we have

$$a_-^0(\tilde{x}, \tilde{x}) = 2^{d-1} \frac{1}{V_{\text{eff}}(x)|G_x|} \pi^{-n-1+\frac{d}{2}} |\det R_{\tilde{x}}|^{-\frac{1}{2}} |\det \mathcal{L}_{\tilde{x}}|. \tag{33}$$

We refer to [13] or [8] for a precise description of the complex phase function  $\Phi_-$  and the symbol  $a_-$ . Notice that  $x$  and  $y$  can be assumed to be  $x = \exp_p(v)$  and  $y = \exp_p(w)$  for some  $v, w \in W$  since  $S_-^G$  is  $G$ -invariant.

Now we can introduce explicitly a map  $\sigma : (\text{Ker } \square_b^{(0)})^G \rightarrow \text{Ker } \square_{b, X_G}^{(0)}$ . Let  $\mathcal{C}^\infty(\mu^{-1}(0))^G$  denote the set of all  $G$ -invariant smooth functions on  $\mu^{-1}(0)$ . Let

$$\iota_G : \mathcal{C}^\infty(\mu^{-1}(0))^G \rightarrow \mathcal{C}^\infty(X_G)$$

be the natural identification. Let  $\iota : \mu^{-1}(0) \rightarrow X$  be the natural inclusion and let  $\iota^* : \mathcal{C}^\infty(X) \rightarrow \mathcal{C}^\infty(\mu^{-1}(0))$  be the pull-back of  $\iota$ . Let

$$f(x) = \sqrt{V_{\text{eff}}(x) |G_x|}.$$

Eventually, let  $E \in L_{\text{cl}}^{-\frac{d}{4}}(X_G)$  be an elliptic pseudodifferential operator with principal symbol  $\sigma_E^0(x, \xi) = |\xi|^{-\frac{d}{4}}$ , we define

$$\begin{aligned} \sigma : \mathcal{C}^\infty(X) &\rightarrow \text{Ker } \square_{b, X_G}^{(0)}, \\ u &\mapsto \Pi_{X_G} \circ E \circ \iota_G \circ f \circ \iota^* \circ \Pi_G u. \end{aligned} \tag{34}$$

Note that  $\sigma$  is taken from [17, (1.20)]. Let  $\sigma^* : \mathcal{C}^\infty(X_G) \rightarrow \mathcal{D}'(X)$  be the formal adjoint of  $\sigma$ . From (31) and (32), we can use the same method developed in [17] with some modification and obtain

**Theorem 3.2.** *Under the hypothesis of this section, suppose that  $\square_{b, X_G}^{(0)}$  has closed range. Then,  $\sigma^*$  maps  $\mathcal{C}^\infty(X_G)$  continuously to  $\mathcal{C}^\infty(X)$ . Hence,  $\sigma^* \sigma : \mathcal{C}^\infty(X) \rightarrow \mathcal{C}^\infty(X)$  is a well-defined continuous operator whose distribution kernel satisfies*

$$\sigma^* \sigma = \Pi_G + \Gamma, \tag{35}$$

where  $\Gamma : \mathcal{C}^\infty(X) \rightarrow \mathcal{C}^\infty(X)$  is a continuous map and satisfies the following: there exists  $\varepsilon > 0$  such that  $\Gamma : H^s(X) \rightarrow H^{s+\varepsilon}(X)$  is continuous for every  $s \in \mathbb{R}$ ,  $H^s(X)$  is the Sobolev space of functions over  $X$  order  $s$ .

Moreover,

$$\sigma \sigma^* = \Pi_{X_G} + \hat{\Gamma}, \tag{36}$$

where  $\hat{\Gamma} : \mathcal{C}^\infty(X_G) \rightarrow \mathcal{C}^\infty(X_G)$  is a continuous map and satisfies the following: there exists  $\delta > 0$  such that  $\hat{\Gamma} : H^s(X_G) \rightarrow H^{s+\delta}(X_G)$  is continuous for every  $s \in \mathbb{R}$ ,  $H^s(X_G)$  is the Sobolev space of functions over  $X_G$  order  $s$ .

It should be mentioned that Theorem 3.2 is the orbifold version of [17, Theorem 1.5].

From Theorem 3.2, we see that  $\sigma$  can be extended by density to

$$\sigma : L^2(X) \rightarrow \text{Ker } \square_{b, X_G}^{(0)}. \tag{37}$$

**Corollary 3.1.** *Under the preceding assumptions and notations, the map  $\sigma$  given by (34) and (37) has the following properties:*

- (i)  $\text{Ker } \sigma \cap (\text{Ker } \square_b^{(0)})^G$  is a finite dimensional subspace of  $\mathcal{C}^\infty(X)$ .
- (ii)  $\text{Coker } \sigma \cap \text{Ker } \square_{b, X_G}^{(0)}$  is a finite dimensional subspace of  $\mathcal{C}^\infty(X_G)$ .

Moreover, we have  $\sigma : (\text{Ker } \square_b^{(0)})^G \rightarrow \text{Ker } \square_{b, X_G}^{(0)}$  is a Fredholm map.

**Proof.** Let  $u \in \text{Ker } \sigma \cap (\text{Ker } \square_b^{(0)})^G$ . From (35), we have

$$0 = \sigma^* \sigma u = (\Pi_G + \Gamma)u = u + \Gamma u. \tag{38}$$

From the previous theorem and (38), we deduce that  $u \in H^\varepsilon(X)$ . By applying again the previous theorem and (38), we get  $u \in H^{2\varepsilon}(X)$ . Continuing in this way, we conclude that  $u \in \mathcal{C}^\infty(X)$ .

Suppose that  $\text{Ker } \sigma \cap (\text{Ker } \square_b^{(0)})^G$  is not a finite dimensional subspace of  $\mathcal{C}^\infty(X)$ . We can find  $u_j \in \text{Ker } \sigma \cap (\text{Ker } \square_b^{(0)})^G$ ,  $j = 1, 2, \dots$ ,  $(u_j | u_k) = \delta_{j,k}$ , for every  $j, k = 1, \dots$ . From the previous theorem and (38), we notice that  $\{u_j\}_{j \in \mathbb{N}}$  is a bounded set in  $H^\varepsilon(X)$ . By Rellich's lemma, there is a subsequence  $1 \leq j_1 < j_2 < \dots$ ,  $\lim_{s \rightarrow +\infty} j_s = +\infty$ , such that  $u_{j_s} \rightarrow u$  in  $L^2(X)$  as  $s \rightarrow +\infty$ , for some  $u \in L^2(X)$ . Since  $(u_j | u_k) = \delta_{j,k}$ , for every  $j, k = 1, \dots$ , we have

$$\|u_{j_s} - u_{k_s}\|_{L^2} = \sqrt{2}$$

and we get a contradiction since  $u_{j_s}$  and  $u_{k_s}$  converge to  $u$  as  $k_s, j_s$  goes to infinity. Thus,  $\text{Ker } \sigma \cap (\text{Ker } \square_b^{(0)})^G$  is a finite dimensional subspace of  $\mathcal{C}^\infty(X)$ .

We can repeat the procedure above with minor change and deduce that  $\text{Coker } \sigma \cap \text{Ker } \square_{b, X_G}^{(0)}$  is a finite dimensional subspace of  $\mathcal{C}^\infty(X_G)$ .  $\square$

Recall that when  $X$  is a circle orbifold, we have a natural circle action which commutes with the action of  $G$  on  $X$  and it is transversal to the CR structure. This induces a decomposition of  $(\text{Ker } \square_b^{(0)})^G$  into Fourier components  $(\text{Ker } \square_b^{(0)})_k^G$  and similarly  $\text{Ker } \square_{b, X_G}^{(0)}$  splits as a direct sum  $(\text{Ker } \square_{b, X_G}^{(0)})_k$ ,  $k \in \mathbb{Z}$ . Now, we can take the limits as  $k$  goes to infinity in the previous corollary and we get that quantization commutes with reduction for  $k$  large,  $X$  and  $X_G$  may be orbifolds.

### 4. Applications to complex geometry

Now, assume that  $X$  admits a transversal and CR locally free  $S^1$  action  $e^{i\theta}$ . We take the Reeb vector field  $R$  to be the vector field on  $X$  induced by the  $S^1$  action. For every  $\ell \in \mathbb{N}$ , put

$$X_\ell := \left\{ x \in X; e^{i\theta} x \neq x, \theta \in \left( 0, \frac{2\pi}{\ell} \right), e^{i\frac{2\pi}{\ell}} x = x \right\}. \tag{39}$$

For every  $k \in \mathbb{Z}$ , set

$$\text{Ker } \square_{b,k}^{(q)} := \left\{ u \in \text{Ker } \square_b^{(q)}; (e^{i\theta})^* u = e^{ik\theta} u \right\}.$$

Let

$$\Pi_k^{(q)} : L_{(0,q)}^2(X) \rightarrow \text{Ker } \square_{b,k}^{(q)}$$

be the orthogonal projection and let  $\Pi_k^{(q)}(x, y) \in C^\infty(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$  be the distribution kernel of  $\Pi_k^{(q)}$ . From Theorem 1.1, we can repeat the procedure in [8] and get

**Theorem 4.1.** *With the assumptions and notations used above, assume that the Levi form is non-degenerate of constant signature  $(n_-, n_+)$  on  $X$  and let  $q = n_-$ . Fix  $p \in X$  and assume that  $p \in X_\ell$ , for some  $\ell \in \mathbb{N}$ . Consider an open set  $U \subset X$ ,  $p \in U$ , and an orbifold chart  $(\tilde{U}, G_U) \rightarrow U$ . We have as  $k \rightarrow +\infty$ ,*

$$\Pi_k^{(q)}(x, y) = \sum_{g \in G_U} \sum_{j=0}^{\ell-1} e^{\frac{2\pi kj}{\ell}} e^{ik\Psi(\tilde{x}, e^{i\frac{2\pi j}{\ell}} \cdot g \cdot \tilde{y})} b(\tilde{x}, e^{i\frac{2\pi j}{\ell}} \cdot g \cdot \tilde{y}, k) + \sum_{g \in G_U} \sum_{j=0}^{\ell-1} F_k(\tilde{x}, e^{i\frac{2\pi j}{\ell}} \cdot g \cdot \tilde{y}) \tag{40}$$

where

$$\begin{aligned} &\Psi \in C^\infty(\tilde{U} \times \tilde{U}), \\ &\Psi(\tilde{x}, \tilde{x}) = 0, \text{ for all } \tilde{x} \in \tilde{U}, \\ &\frac{1}{C} \inf_{e^{i\theta} \in S^1} \left\{ \text{dist}^2(\tilde{x}, e^{i\theta} \tilde{y}) \right\} \leq \text{Im } \Psi(\tilde{x}, \tilde{y}) \leq C \inf_{e^{i\theta} \in S^1} \left\{ \text{dist}^2(\tilde{x}, e^{i\theta} \tilde{y}) \right\}, \\ &\forall (\tilde{x}, \tilde{y}) \in \tilde{U} \times \tilde{U}, C > 1 \text{ is a constant,} \end{aligned} \tag{41}$$

and

$$\begin{aligned} &b(\tilde{x}, \tilde{y}, k) \sim \sum_{j=0}^{+\infty} b_j(\tilde{x}, \tilde{y}) k^{n-j} \text{ in } S^n(1; \tilde{U} \times \tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \\ &b_j(\tilde{x}, \tilde{y}) \in C^\infty(\tilde{U} \times \tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j = 0, 1, \dots, \\ &b_0(\tilde{x}, \tilde{x}) = \frac{1}{2} \pi^{-n-1} |\det \mathcal{L}_{\tilde{x}_0}| \tau_{\tilde{x}_0, n_-}, \tilde{x}_0 \in \tilde{U}, \end{aligned} \tag{42}$$

$\Psi(\tilde{x}, \tilde{y})$ ,  $b(\tilde{x}, \tilde{y}, k)$  are  $G_U$ -invariant. Furthermore,  $F_k \in C^\infty(\tilde{U} \times \tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ ,  $F_k(\tilde{x}, \tilde{y})$  is  $G_U$ -invariant and satisfies the following: for every  $N_1, N \in \mathbb{N}$ , there is a constant  $C_{N_1, N}$  independent of  $k$  such that  $\|F_k(\tilde{x}, \tilde{y})\|_{C^{N_1}(\tilde{U} \times \tilde{U})} \leq C_{N_1, N} k^{-N}$ , for every  $k \gg 1$ .

We refer the reader to [15, Definition 8.1] for the meaning of semi-classical symbol spaces  $S^n(1; \tilde{U} \times \tilde{U}, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ .

**Remark 4.1.** Here, we compare our result with [18, Theorem 5.4.11]. Let  $(M, \omega)$  be an Hodge orbifold. Now, we will identify  $H^0(M, L^k)$  with  $\text{Ker } \square_{b,k}^{(0)}$  and we write  $X$  for the circle bundle in  $L^*$ . Now,  $X$  is a strictly pseudoconvex domain, for simplicity we take  $\langle \cdot | \cdot \rangle$  so that the eigenvalues of the Levi form are equal to one. For simplicity, we assume that the  $S^1$  action on the fiber of  $L^*$  is free and hence we do not have the decomposition  $X_\ell$  given by formula (39). Fix  $p \in X$  and we work locally on an orbifold chart  $(\tilde{U}, G_U) \rightarrow U$  of  $X$ ,  $U$  is a small open set of  $p$  in  $X$  and we assume that  $g \cdot \tilde{p} = \tilde{p}$  for each  $g \in G_U$ , where  $\pi(\tilde{p}) = p$ ,  $\pi : \tilde{U} \rightarrow U$  is the natural projection. By our formula, we have as  $k \rightarrow +\infty$ ,

$$\Pi_k^{(0)}(x, x) = \sum_{g \in G_U} e^{ik\Psi(\tilde{x}, g \cdot \tilde{x})} b(\tilde{x}, g \cdot \tilde{x}, k) + O(k^{-\infty}). \tag{43}$$

By [14, Theorem 3.8], we can take local coordinates  $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{2n+1})$ ,  $\tilde{z}_j = \tilde{x}_{2j-1} + i\tilde{x}_{2j}$ ,  $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_{2n+1})$ ,  $\tilde{w}_j = \tilde{y}_{2j-1} + i\tilde{y}_{2j}$   $j = 1, \dots, n$ , defined on an open set  $\tilde{U} = \hat{U} \times (-\epsilon, \epsilon)$ ,  $\epsilon > 0$ , near  $p$ , where  $\hat{U}$  is an open set of  $0 \in \mathbb{C}^n$  in  $\mathbb{C}^n$ , so that  $p \equiv 0$ ,  $R = \partial/\partial\tilde{x}_{2n+1}$  is the Reeb vector field and near  $p$ , we have

$$\begin{aligned} \Psi(\tilde{x}, \tilde{y}) &= \tilde{x}_{2n+1} - \tilde{y}_{2n+1} + \Psi_0(\tilde{z}, \tilde{w}) + \Psi_1(\tilde{z}, \tilde{w}), \\ \Psi_0(\tilde{z}, \tilde{w}) &:= i|\tilde{z}|^2 + i|\tilde{w}|^2 - 2i(\tilde{z}|\tilde{w}), \end{aligned} \tag{44}$$

where

$$\Psi_1(\tilde{z}, \tilde{w}) = O(|\tilde{z} - \tilde{w}|^3). \tag{45}$$

Here, we identify  $\tilde{z}$  and  $\tilde{w}$  with

$$\sum_{j=1}^n \tilde{z}_j \frac{\partial}{\partial \tilde{z}_j} \quad \text{and} \quad \sum_{j=1}^n \tilde{w}_j \frac{\partial}{\partial \tilde{w}_j} \quad \text{in } \mathbb{C}T_{\tilde{x}}X, \quad \tilde{x} = (\tilde{z}, 0).$$

Let  $\epsilon_0 \ll \epsilon, \epsilon > 0$ , sufficiently small and let  $\tilde{U}_0 = \hat{U} \times (-\epsilon_0, \epsilon_0)$ . We write

$$g \cdot \tilde{x} = (g \cdot \tilde{z}, \theta_g + \tilde{x}_{2n+1}), \quad \theta_g \in \mathbb{R},$$

where  $\theta_g$  is small if  $\tilde{x}_{2n+1}$  is small. Note that  $\theta_g$  is the same as in the discussion before [18, Theorem 5.4.11].

Now, we need to choose  $\tilde{y} = g \cdot \tilde{x}$  in equation (44) to compare with [18, Theorem 5.4.11]. Furthermore, as in [18, pg. 241], let  $\hat{U}_g = \{\tilde{z} \in \hat{U}; g \cdot \tilde{z} = \tilde{z}\}$  be the fixed point-set of  $g \in G_U$ . Inspired by the discussion before [18, Theorem 5.4.11], let  $\hat{N}_g$  be the normal bundle of  $\hat{U}_g$  in  $\hat{U}$  and we identify  $\hat{W}_g = \{Y \in \hat{N}_g; |Y| < \hat{\epsilon}\}$ , where  $\hat{\epsilon} > 0$  is small, with a neighborhood of  $\hat{U}_g$  in  $\hat{U}$  by the exponential map. For  $\tilde{z} \in \hat{W}_g$  we write  $\tilde{z} = (\tilde{z}_{1,g}, \tilde{z}_{2,g})$  where  $\tilde{z}_{1,g} \in \hat{U}_g$  and  $\tilde{z}_{2,g} \in \hat{N}_{g,\tilde{z}_{1,g}}$ , and we identify  $\tilde{z}_{1,g} \equiv (\tilde{z}_{1,g}, 0), \tilde{z}_{2,g} \equiv (0, \tilde{z}_{2,g})$ . It is not difficult to see that

$$\begin{aligned} g \cdot \tilde{z} &= \tilde{z}_{1,g} + g \cdot \tilde{z}_{2,g} + O(|\tilde{z}_{2,g}|^2), \\ |\tilde{z} - g \cdot \tilde{z}|^2 &= O(|\tilde{z}_{2,g}|^2), \\ \langle \tilde{z} | (I - g)\tilde{z} \rangle &= \langle \tilde{z}_{2,g} | (I - g)\tilde{z}_{2,g} \rangle + O(|\tilde{z}_{2,g}|^3), \end{aligned} \tag{46}$$

where  $I$  is the identity matrix. Now, by substituting (46) in (44), we get for (43)

$$\begin{aligned} \Pi_k^{(0)}(x, x) &= b(\tilde{x}, g \cdot \tilde{x}, k) + \sum_{I \neq g \in G_U} e^{-ik\theta_g - 2k\langle \tilde{z}_{2,g} | (I - g)\tilde{z}_{2,g} \rangle + ik\Psi_1(\tilde{x}, g \cdot \tilde{x})} b(\tilde{x}, g \cdot \tilde{x}, k). \end{aligned} \tag{47}$$

From (45) and (46), we can check that

$$k\Psi_1(\tilde{x}, g \cdot \tilde{x}) = \frac{1}{\sqrt{k}} O(|\sqrt{k}\tilde{z}_{2,g}|^3). \tag{48}$$

From (48) and Taylor expansion,  $e^{ik\Psi_1(\tilde{x}, g \cdot \tilde{x})}$  can be written as a polynomial in  $\sqrt{k}\tilde{z}_{2,g}$ . From this observation and (47), we can deduce similar expansion as [18, Theorem 5.4.1].

Now, we take local coordinates  $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{2n+1}), \tilde{z}_j = \tilde{x}_{2j-1} + i\tilde{x}_{2j}, j = 1, \dots, n$ , so that  $\langle \frac{\partial}{\partial \tilde{z}_\ell} | \frac{\partial}{\partial \tilde{z}_\ell} \rangle = \delta_{j,\ell} + O(|\tilde{z}|)$ ,  $j, \ell = 1, \dots, n$ . For every  $g \in G_U$ , since  $g \cdot p = p$ , we have

$$\langle \tilde{z}_{2,g} | (I - g)\tilde{z}_{2,g} \rangle = \langle \tilde{z}_{2,g} | (I - \hat{g})\tilde{z}_{2,g} \rangle_{\mathbb{C}^n} + O(|\tilde{z}_{2,g}|^2), \tag{49}$$

where  $\langle \cdot | \cdot \rangle_{\mathbb{C}^n}$  denotes the flat Hermitian metric on  $\mathbb{C}^n$ , that is  $\langle z | w \rangle_{\mathbb{C}^n} = \sum_{j=1}^n z_j \overline{w_j}, z = (z_1, \dots, z_n) \in \mathbb{C}^n, w = (w_1, \dots, w_n) \in \mathbb{C}^n$ , and  $\hat{g} \in U(n)$  acting  $\mathbb{C}$ -linearly and isometric on  $\mathbb{C}^n$  with respect to  $\langle \cdot | \cdot \rangle_{\mathbb{C}^n}$ . Put  $\hat{G}_U := \{\hat{g}; g \in G_U\}$ . Then,  $\hat{G}_U \subset U(n)$  is a finite subgroup acting  $\mathbb{C}$ -linearly and isometric on  $\mathbb{C}^n$  with respect to  $\langle \cdot | \cdot \rangle_{\mathbb{C}^n}$ . From (47), (48) and (49), we get

$$\begin{aligned} \Pi_k^{(0)}(x, x) &= b(\tilde{x}, g \cdot \tilde{x}, k) + \sum_{I \neq g \in \hat{G}_U} e^{-ik\theta_g - 2k\langle \tilde{z}_{2,g} | (I - \hat{g})\tilde{z}_{2,g} \rangle_{\mathbb{C}^n} + \frac{1}{\sqrt{k}} O(|\sqrt{k}\tilde{z}_{2,g}|^3)} b(\tilde{x}, g \cdot \tilde{x}, k). \end{aligned} \tag{50}$$

It should be noticed that all the calculations above work well for  $\Pi_k^{(q)}(x, x), q > 0$ .

It is good to mention that all the calculation in Remark 4.1 works well for the Szegő kernel  $\Pi_k^{(q)}(x, x)$  of CR manifolds endowed with a transversal locally free circle action.

As an easy consequence of the asymptotic expansion of the Szegő kernel along the diagonal we also have the following theorem, that can be compared with Corollary 1.2 in [22]. In fact, in [22], a compact Hodge manifold  $M$  with quantizing

circle bundle  $X$  is considered;  $X$  is a compact CR manifold with a natural built-in circle action. R. Paoletti considers a new locally free Hamiltonian and holomorphic circle action on  $M$  which lifts to a new circle action on  $X$  satisfying the hypothesis we are assuming in Theorem 4.2. The asymptotic of the dimensions of the  $k$ -th Fourier component  $H_k(X) := \text{Ker } \square_{b,k}^{(0)}$  of the Hardy space of  $X$  with respect to the new circle action is then given in terms of the geometric data of  $M$ . Moreover, let us mention that the holomorphic Morse inequalities for orbifolds were obtained by M. Puchol in [23].

**Theorem 4.2.** *Under the same assumptions and notations as in Theorem 4.1 above, assume that  $X = X_{\ell_0} \cup X_{\ell_1} \cup \dots \cup X_{\ell_u}$ ,  $1 \leq \ell_0 < \ell_1 < \dots < \ell_u$ ,  $\ell_j \in \mathbb{N}$ ,  $j = 0, 1, \dots, u$ ,  $u \in \mathbb{N}$ . Denote by  $p$  the least common multiple of  $l_0, l_1, \dots, l_u$ . We have*

$$\lim_{k \rightarrow +\infty} \frac{1}{(kp)^n} \dim \text{Ker } \square_{b,kp}^{(q)} = \frac{\ell_0}{2} \pi^{-n-1} \int_X |\det \mathcal{L}_X| dV_X.$$

We work with the same assumptions and notations as in Theorem 4.1 above and assume that  $X$  is strongly pseudoconvex. Let  $M$  be the Hodge orbifold obtained by  $M = X/S^1$ . There exists a complex proper line orbibundle  $(L, h^L)$  such that  $X \rightarrow M$  is the circle orbibundle  $X \subset L^*$ . For  $k \in \mathbb{N}$ , let  $H^0(M, L^k)$  be the space of global holomorphic sections of  $M$  with values in  $L^k$ . The space  $\text{Ker } \square_{b,k}^{(0)}$  is isomorphic to  $H^0(M, L^k)$ . Let  $\{f_j\}_{j=1}^{d_k} \subset H^0(M, L^k)$  be an orthonormal basis. The Kodaira-Bailey map is given by

$$\begin{aligned} \Phi_k : M &\rightarrow \mathbb{C}P^{d_k-1}, \\ x &\rightarrow [f_1(x), \dots, f_{d_k}(x)]. \end{aligned} \tag{51}$$

As an application of Theorem 4.1, we will give an analytic proof of the following version of Kodaira-Bailey embedding theorem. The original proof Kodaira-Bailey embedding theorem was given by Baily in [2]. It should be mentioned that Ma and Marinescu, see [18, Theorem 5.4.20], gave a proof of Kodaira-Bailey embedding theorem and they also gave the isomorphism of the local ring structure. Let  $M_{\text{sing}}$  be the set of singular points of  $M$ .

**Theorem 4.3.** *With the same assumptions and notations as in Theorem 4.1 and suppose that  $X$  is strongly pseudoconvex. There is a  $N_0 \in \mathbb{N}$  with  $\ell_j | N_0$ , for every  $j = 1, \dots, u$ , where  $\ell_j$  is as in Theorem 4.1, and  $k_0 \in \mathbb{N}$ , such that for every  $k \geq k_0$ ,*

- (i) *the Kodaira-Bailey map  $\Phi_{kN_0} : M \hookrightarrow \mathbb{C}P^{d_{kN_0}-1}$  is injective,*
- (ii) *we can find orbifold charts  $(\tilde{U}_j, G_{U_j})$ ,  $j = 1, \dots, \ell$ , such that*

$$M_{\text{sing}} \subset \bigcup_{j=1}^N U_j,$$

$U_j = \tilde{U}_j/G_{U_j}$ ,  $j = 1, \dots, \ell$ , and for every  $j$ , there is a map  $\hat{\Phi}_{k,j} : \tilde{U}_j \rightarrow \mathbb{C}P^{d_{kN_0}-1}$  such that

$$\Phi_{kN_0}(x) = \sum_{g \in G_{U_j}} \hat{\Phi}_{k,j}(g \cdot x)$$

and  $d_x \hat{\Phi}_{k,j}$  is injective at every point of  $\tilde{U}_j$ ,

- (iii)  $d_x \Phi_{kN_0}$  is injective at every point of  $x \in M$  and

$$x \notin \bigcup_{j=1}^N U_j.$$

#### 4.1. Proof of Theorem 4.2

We have the following formula

$$\dim \text{Ker } \square_{b,kp}^{(q)} = \int_X \text{Tr } \Pi_{kp}^{(q)}(x, x) dV_X(x).$$

As explained in the introduction,  $X$  can be written as a disjoint union based on the orbit type induced by the circle action and  $X_{\ell_0}$  is an open and dense subset of  $X$ . Moreover, from (40), we see that there is a constant  $C > 0$  independent of  $k$  such that

$$\text{Tr } \Pi_{kp}^{(q)}(x, x) \leq Ck^n, \tag{52}$$

for all  $x \in X$ . From (52), we can apply Lebesgue's dominated convergence theorem and get

$$\lim_{k \rightarrow +\infty} k^{-n} \dim \text{Ker} \square_{b,k}^{(q)} = \int_{X_{\ell_0}} \lim_{k \rightarrow +\infty} k^{-n} \text{Tr} \Pi_{kp}^{(q)}(x, x) dV_X(x). \tag{53}$$

From (40) and (53), Theorem 4.2 follows.

#### 4.2. Proof of Theorem 4.3

In this section, we will identify  $H^0(M, L^k)$  with  $\text{Ker} \square_{b,k}^{(0)}$  and we write  $X$  for the circle bundle in  $L^*$ . We take  $\langle \cdot | \cdot \rangle$  so that the eigenvalues of the Levi form are equal to one and hence

$$b_0(\tilde{x}, \tilde{x}) = \frac{1}{2} \pi^{-n-1}, \quad \tilde{x}_0 \in \tilde{U}, \tag{54}$$

where  $b_0$  is as in (42). For simplicity, we assume that the  $S^1$  action on the fiber of  $L^*$  is free, hence we do not have decomposition  $X_\ell$  given by (39). Fix  $p \in X$  and we work locally on an orbifold chart  $(\tilde{U}, G_U) \rightarrow U$ ,  $U$  is a small open set of  $p$ . By (40), we have as  $k \rightarrow +\infty$ ,

$$\Pi_k^{(0)}(x, y) = \sum_{g \in G_U} e^{ik \Psi(\tilde{x}, g \cdot \tilde{y})} b(\tilde{x}, g \cdot \tilde{y}, k) + O(k^{-\infty}).$$

From now on, we replace  $L$  by  $L^m$ ,  $m \in \mathbb{N}$ ,  $m \gg 1$ , so that  $\theta_g = 0$  in (50) and we get

$$\begin{aligned} \Pi_k^{(0)}(x, x) &= b(\tilde{x}, g \cdot \tilde{x}, k) + \sum_{l \neq g \in \hat{G}_U} e^{-2k(\tilde{z}_{2,g} | (l-\hat{g})\tilde{z}_{2,g})_{\mathbb{C}^n} + \frac{1}{\sqrt{k}} O(|\sqrt{k}\tilde{z}_{2,g}|^3)} b(\tilde{x}, g \cdot \tilde{x}, k), \end{aligned} \tag{55}$$

where  $\langle \cdot | \cdot \rangle_{\mathbb{C}^n}$  denotes the flat Hermitian metric on  $\mathbb{C}^n$ ,  $\hat{G}_U \subset U(n)$  is a finite subgroup acting  $\mathbb{C}$ -linearly and isometric on  $\mathbb{C}^n$  with respect to  $\langle \cdot | \cdot \rangle_{\mathbb{C}^n}$ . From (54) and (55), we deduce

$$\Pi_k^{(0)}(x, x) = \frac{1}{2} \pi^{-n-1} \left( 1 + \sum_{l \neq g \in \hat{G}_U} e^{-2k(\tilde{z}_{2,g} | (l-\hat{g})\tilde{z}_{2,g})_{\mathbb{C}^n}} \right) k^n + O(k^{n-\frac{1}{2}}). \tag{56}$$

In order to show that the Kodaira map is well-defined, in [18, Theorem 5.4.18] the authors prove [18, Lemma 5.4.14], which ensures the uniform positivity of the Bergman kernel near the singularity. In fact by (56) and [18, Lemma 5.4.14] we get:

**Lemma 4.1.** *With the assumptions and notations above, let  $p$  be a singular point and let  $(\tilde{U}, G_U)$  be a local chart of  $X$  with  $g \cdot \tilde{p} = \tilde{p}$  for each  $g \in G_U$ , where  $\pi(\tilde{p}) = p$ ,  $\pi : \tilde{U} \rightarrow U$  is the natural projection. There exists  $C_{G,n} > 0$  such that*

$$\Pi_k^{(0)}(x, x) \geq C_{G,n} k^n, \quad \text{for every } x \in U.$$

If  $p$  is a regular point, it is not difficult to see that Lemma 4.1 holds. The Kodaira map is well-defined if

$$Bl_k := \{x \in M; s(x) = 0 \text{ for all } s \in H^0(M, L^k)\} = \emptyset.$$

Now, since the manifold is compact, from Lemma 4.1, the Kodaira map is well-defined if  $k \gg 1$ .

Here, the main goal is to give a pure analytic proof of the Kodaira embedding theorem for orbifolds by using Theorem 1.1. Recall that we replace  $L$  by  $L^m$ ,  $m \in \mathbb{N}$ ,  $m \gg 1$ , so that  $\theta_g = 0$  in (50). From Lemma 4.1, there is a  $k_0 \in \mathbb{N}$  such that for every  $k \geq k_0$ , for every  $x_0 \in X$ , there is a  $u_k \in H^0(M, L^k)$  such that  $u_k(x_0) \neq 0$ . From now on, we replace  $L$  by  $L^{k_0}$ . Hence, for every  $x_0 \in X$ , there is a  $u \in H^0(M, L)$  such that  $u(x_0) \neq 0$ .

Now, we are going to study the differential of the map  $\Phi_{kN_0}$  is injective, if  $k \gg 1$ , for some  $N_0 \in \mathbb{N}$ . We will identify  $x \in M$  with  $(x, 0) \in X$ . Consider an open set  $U$  around a point  $x_0 \in X$  and an orbifold chart  $(\tilde{U}, G_U) \rightarrow U$  such that  $\tilde{x}_0$  is a fix point of  $G_U$ . Let  $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{2n+1})$  be local coordinates of  $\tilde{U}$  with  $\tilde{x}(\tilde{x}_0) = 0$  and  $R = \partial/\partial\tilde{x}_{2n+1}$ . Let  $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_n) \in \mathbb{C}^n$ ,  $z_j = \tilde{x}_{2j-1} + i\tilde{x}_{2j}$ ,  $j = 1, \dots, n$ . Let  $\chi \in C_c^\infty(\tilde{U}, \overline{\mathbb{R}_+})$  with  $\int \chi = 1$  and  $\chi(\tilde{z}, \tilde{x}_{2n+1}) = \tau(|\tilde{z}|, \tilde{x}_{2n+1})$  for some  $\tau \in C_c^\infty(\mathbb{R}^2, \overline{\mathbb{R}_+})$ . For  $k \in \mathbb{N}$ , put

$$\chi_k(\tilde{x}) = \chi(k^{\frac{1}{2}}\tilde{z}, k^{1+\varepsilon}\tilde{x}_{2n+1}) \quad \text{and} \quad \hat{\chi}_k(\tilde{x}) := \sum_{g \in G_U} \chi_k(g \cdot \tilde{x}),$$

where  $\varepsilon > 0$  is a small constant. For each  $j = 1, \dots, n$ , let

$$\begin{aligned} \hat{f}_j : \tilde{U} &\rightarrow \mathbb{C}, & \tilde{z} &\mapsto f_j(\tilde{z}) = \tilde{z}_j, \\ \tilde{f}_j : \tilde{U} &\rightarrow \mathbb{C}, & \tilde{z} &\mapsto \tilde{f}_j(\tilde{z}) = \sum_{g \in G_U} g \cdot \tilde{z}_j. \end{aligned}$$

Note that  $\tilde{f}_j$  is  $G_U$ -invariant and smooth and thus descends naturally to a continuous function  $f_j : U \rightarrow \mathbb{C}$  on  $U$ . From Theorem 4.1, we see that

$$\Pi_k^{(0)}(x, y) = \sum_{g \in G_U} \hat{\Pi}_k^{(0)}(\tilde{x}, g \cdot \tilde{y}) \tag{57}$$

on  $\tilde{U} \times \tilde{U}$ , where  $\hat{\Pi}_k^{(0)}(\tilde{x}, \tilde{y}) = e^{ik\Psi(\tilde{x}, \tilde{y})} b(\tilde{x}, \tilde{y}, k) + O(k^{-\infty})$ ,  $\Psi(\tilde{x}, \tilde{y})$  and  $b(\tilde{x}, \tilde{y}, k)$  are as in Theorem 4.1. Let us set

$$\begin{aligned} u_k^{(j)} &= k^{1+\varepsilon+2n} \Pi_k^{(0)}(f_j \hat{\chi}_k) \in \text{Ker } \square_{b,k}^{(0)}, \\ \hat{u}_k^{(j)} &= k^{1+\varepsilon+2n} \hat{\Pi}_k^{(0)}(\hat{f}_j \chi_k) \in C^\infty(\tilde{U}, L^k). \end{aligned}$$

Note that  $u_k^{(j)}$  is a global section but  $\hat{u}_k^{(j)}$  is a local section. From (57), we have on  $\tilde{U}$ ,

$$u_k^{(j)}(x) = \sum_{g \in G_U} \hat{u}_k^{(j)}(g \cdot \tilde{x}), \quad j = 1, \dots, n. \tag{58}$$

From  $\Psi(x, x) = 0$ , (55), and Lemma 4.1 it is straightforward to check that

$$\begin{aligned} \lim_{k \rightarrow +\infty} \left( \frac{\partial}{\partial z_q} \hat{u}_k^{(j)} \right) (\tilde{x}_0) &= \delta_{j,q} C_j, \\ \lim_{k \rightarrow +\infty} k \hat{u}_k^{(j)} (\tilde{x}_0) &= 0, \end{aligned} \tag{59}$$

for every  $j, q = 1, \dots, n$ , where  $C_j > 0$  is a constant,  $j = 1, \dots, q$ . Recall that we replace  $L$  by  $L^{k_0}$ . Hence, we can find  $u \in H^0(M, L)$  such that  $u(x_0) \neq 0$ . From (59), it is not difficult to see that the differential of the map

$$\tilde{x} \in \tilde{U} \rightarrow \left( \frac{\hat{u}_k^{(1)}(\tilde{x})}{u^k(\tilde{x})}, \frac{\hat{u}_k^{(2)}(\tilde{x})}{u^k(\tilde{x})}, \dots, \frac{\hat{u}_k^{(n)}(\tilde{x})}{u^k(\tilde{x})} \right)$$

is injective at  $\tilde{x}_0$  if  $k \gg 1$ . Hence, there is a  $m_0 = m_0(x_0) \in \mathbb{N}$  and an open set  $\tilde{W}_{\tilde{x}_0} \subset \tilde{U}$  of  $\tilde{x}_0$  such that  $u(\tilde{x}) \neq 0$ , for every  $\tilde{x} \in \tilde{W}_{\tilde{x}_0}$ ,

the differential of the map

$$\tilde{U} \ni \tilde{x} \mapsto \left( \frac{\hat{u}_{m_0}^{(1)}(\tilde{x})}{u^{m_0}(\tilde{x})}, \frac{\hat{u}_{m_0}^{(2)}(\tilde{x})}{u^{m_0}(\tilde{x})}, \dots, \frac{\hat{u}_{m_0}^{(n)}(\tilde{x})}{u^{m_0}(\tilde{x})} \right) \text{ is injective at every point of } \tilde{W}_{\tilde{x}_0} \tag{60}$$

and  $u^{m_0}(x), u_{m_0}^{(1)}(x), \dots, u_{m_0}^{(n)}(x)$  are linearly independent. For every  $k \in \mathbb{N}$ , consider sections  $g_1, \dots, g_{d_{(k+1)m_0} - n - 1} \in H^0(M, L^{(k+1)m_0})$  such that

$$\{u^{(k+1)m_0}, u^{km_0} u_{m_0}^{(1)}, \dots, u^{km_0} u_{m_0}^{(n)}, g_1, \dots, g_{d_{(k+1)m_0} - n - 1}\}$$

is a basis of  $H^0(M, L^{(k+1)m_0})$ . As a consequence of (60), it is easy to see that the differential of the map

$$\tilde{x} \mapsto \left( \frac{u^{km_0} \hat{u}_{m_0}^{(1)}}{u^{(k+1)m_0}}(\tilde{x}), \dots, \frac{u^{km_0} \hat{u}_{m_0}^{(n)}}{u^{(k+1)m_0}}(\tilde{x}), \frac{g_1}{u^{(k+1)m_0}}(\tilde{x}), \dots, \frac{g_{d_{(k+1)m_0} - n - 1}}{u^{(k+1)m_0}}(\tilde{x}) \right) \tag{61}$$

is injective on  $\tilde{W}_{\tilde{x}_0}$ , for every  $k \in \mathbb{N}$ . Now, assume that  $M_{\text{sing}} \subset W_{x_0} \cup W_{x_1} \cup \dots \cup W_{x_r}$ ,  $r \in \mathbb{N}$ ,  $W_{x_j} = \tilde{W}_{\tilde{x}_j} / G_{U_j}$ ,  $(\tilde{U}_j, G_{U_j})$  is an orbifold chart defined near  $x_j \in M_{\text{sing}}$ ,  $\tilde{W}_{\tilde{x}_j} \subset \tilde{U}_j$ ,  $x_j \in \tilde{W}_{\tilde{x}_j}$ , is as above,  $j = 0, 1, \dots, r$ . Let  $m_j = m(x_j) \in \mathbb{N}$  be as above,  $j = 0, 1, \dots, r$ . Let  $N_0 := [m_1, \dots, m_r]$ . From (61), it is straightforward to see that for  $k \gg 1$ , the Kodaira map  $\Phi_{kN_0}$  satisfies (ii) in Theorem 4.3. From now on, we replace  $L$  by  $L^{N_0}$ . Thus, for  $k \gg 1$ , the Kodaira map  $\Phi_k$  satisfies (ii) in Theorem 4.3. The proof of (iii) in Theorem 4.3 is similar as smooth case.

We now need to prove that  $\Phi_k$  is globally injective. By absurd, up to passing to a subsequence, suppose that there are  $x_k, y_k \in M$  with  $x_k \neq y_k$  such that  $\Phi_k(x_k) = \Phi_k(y_k)$ , for each  $k$ . We prove the theorem by showing that we get a contradiction.

Assume that  $\lim_{k \rightarrow +\infty} x_k = x_0$ ,  $\lim_{k \rightarrow +\infty} y_k = y_0$ . If  $x_0 \neq y_0$ , we can repeat the procedure in [12] and find  $u_k, v_k \in H^0(M, L^k)$  such that  $|u_k(x_k)|_{h^k}^2 \geq Ck^n$ ,  $|v_k(x_k)|_{h^k}^2 \leq \frac{C}{2}k^n$ ,  $|u_k(y_k)|_{h^k}^2 \leq \frac{C}{2}k^n$ ,  $|v_k(y_k)|_{h^k}^2 \geq Ck^n$ , for all  $k \gg 1$ , where  $C > 0$  is a constant independent of  $k$ . Thus,  $\Phi_k(x_k) \neq \Phi_k(y_k)$ , for  $k$  large. We get a contradiction and thus we must have  $x_0 = y_0$ .

For every  $k \in \mathbb{N}$ , put

$$A_k := \{(x, y) \in M \times M; \Phi_k(x) = \Phi_k(y)\}.$$

We need

**Lemma 4.2.** *There is a constant  $k_0$  such that for every  $j \in \mathbb{N}$  and every  $k \in \mathbb{N}$  with  $k \geq k_0$ , we have*

$$A_{j+k} \subset A_j.$$

**Proof.** From Lemma 4.1, we see that there is a  $k_0 \in \mathbb{N}$  such that for every  $x \in X$  and every  $k \in \mathbb{N}$ ,  $k \geq k_0$ , we can find  $u_k \in \text{Ker } \square_{b,k}^{(0)}$  such that  $|u_k(x)| \geq \frac{1}{2}$ . Fix  $j \in \mathbb{N}$ , let  $k \in \mathbb{N}$  with  $k \geq k_0$  and let  $(x, y) \notin A_j$ . We claim that  $(x, y) \notin A_{j+k}$ . Since  $(x, y) \notin A_j$ , we can find  $v_j \in \text{Ker } \square_{b,j}^{(0)}$  so that  $v_j(x) = 1$  and  $v_j(y) = 0$ . Let  $u_k \in \text{Ker } \square_{b,k}^{(0)}$  such that  $|u_k(x)| \geq \frac{1}{2}$ . Then,  $u_k v_j \in \text{Ker } \square_{b,j+k}^{(0)}$ ,  $|(u_k v_j)(x)| \geq \frac{1}{2}$ ,  $|(u_k v_j)(y)| = 0$ . Thus,  $(x, y) \notin A_{j+k}$ . The lemma follows.  $\square$

We now have  $\lim_{k \rightarrow +\infty} x_k = \lim_{k \rightarrow +\infty} y_k = 0$ . Consider an open set  $U$  around a given point  $x_0 \in X_\ell$  and an orbifold chart  $(\tilde{U}, G_U) \rightarrow U$  such that  $x_0$  is a fix point of  $G_U$ . Let  $x = (x_1, \dots, x_{2n+1})$  be local coordinates of  $\tilde{U}$  with  $x(x_0) = 0$ . For every  $j \in \mathbb{N}$ , take  $k_j \in \mathbb{N}$  with  $k_j \geq k_0$  such that

$$|x_{j+k_j}| \leq \frac{1}{j^3} \quad \text{and} \quad |y_{j+k_j}| \leq \frac{1}{j^3}. \tag{62}$$

From Lemma 4.2, we see that  $(x_{j+k_j}, y_{j+k_j}) \subset A_j$ . Thus, for every  $j$ , we replace  $(x_j, y_j)$  by  $(x_{j+k_j}, y_{j+k_j})$  and conclude that there are  $x_k, y_k \in M$ ,  $x_k \neq y_k$  such that  $\Phi_k(x_k) = \Phi_k(y_k)$ , for each  $k$  and

$$\lim_{k \rightarrow +\infty} |k^2 x_k| = \lim_{k \rightarrow +\infty} |k^2 y_k| = 0. \tag{63}$$

Put

$$f_k(t) = \frac{|\Pi_k^{(0)}(tx_k + (1-t)y_k, y_k)|^2}{\Pi_k^{(0)}(tx_k + (1-t)y_k)\Pi_k^{(0)}(y_k)}$$

where we write  $\Pi_k^{(0)}(x) = \Pi_k^{(0)}(x, x)$ , for any  $x \in X$ . From (63), we can check that

$$f_k(t) = \sum_{g \in G_U} e^{-2k \text{Im} \Psi(t\tilde{x}_k + (1-t)\tilde{y}_k, g \cdot \tilde{y}_k)} \cdot \left[ 1 + \frac{1}{k} \tilde{R}_k(t) + \varepsilon_k(t) \right], \tag{64}$$

where

$$|\partial_t^j \tilde{R}_k(t)| \leq C_j |x_k - y_k|^j \quad \text{and} \quad |\partial_t^j \varepsilon_k(t)| \leq C_{N,j} k^{-N} |x_k - y_k|^j,$$

for all  $j \in \mathbb{N} \cup \{0\}$  and every  $N \in \mathbb{N}$ , where  $C_j, C_{N,j} > 0$  are constants independent of  $k$ . For ease of notation, pose  $F_k(t) := -2k \text{Im} \Psi(t\tilde{x}_k + (1-t)\tilde{y}_k, g \cdot \tilde{y}_k)$  for given  $\tilde{x}_k, \tilde{y}_k$  and  $g \in G_U$ . From (63), we have

$$|F'_k(t)| = | \langle -2kp \text{Im} \Psi'_x(t\tilde{x}_k + (1-t)\tilde{y}_k, g \cdot \tilde{y}_k), \tilde{x}_k - \tilde{y}_k \rangle | \leq \frac{1}{c_0 k} |\tilde{x}_k - \tilde{y}_k|$$

and

$$F''_k(t) = \langle -2kp \text{Im} \Psi''_x(t\tilde{x}_k + (1-t)\tilde{y}_k, g \cdot \tilde{y}_k), \tilde{x}_k - \tilde{y}_k \rangle < -c_0 k |\tilde{x}_k - \tilde{y}_k|^2$$

for the computation of the second derivative of (64), here  $c_0$  is a positive constant. We get

$$\limsup_k \frac{f''(t)}{k |x_k - y_k|^2} < 0 \tag{65}$$

for each  $t$ .

By Cauchy-Schwartz inequality we have  $0 \leq f_k(t) \leq 1$ , for any  $t \in [0, 1]$ . From  $\Phi_k(x_k) = \Phi_k(y_k)$ , for each  $k$ , we can check that  $f_k(0) = f_k(1) = 1$ . Thus, for each  $k$ , there is a  $t_k \in [0, 1]$  such that  $f''(t_k) = 0$ . Hence,

$$\limsup_k \frac{f''(t)}{k |x_k - y_k|^2} \geq 0$$

which contradicts (65).

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## Data availability

No data was used for the research described in the article.

## References

- [1] A. Adem, J. Leida, Y. Ruan, *Orbifolds and Stringy Topology*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2007.
- [2] W. Baily, On the imbedding of V-manifolds in projective space, *Am. J. Math.* 79 (1957) 403–430.
- [3] L. Boutet de Monvel, J. Sjöstrand, Sur la singularité des noyaux de Bergman et de Szegő, *Astérisque* 34–35 (1976) 123–164.
- [4] C.P. Boyer, K. Galicki, *Sasakian Geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.
- [5] L. Chen, Y. Yuan, Hardy spaces and Szegő projection on quotient domains, arXiv:2310.12151, 2023.
- [6] X. Dai, K. Liu, X. Ma, On the asymptotic expansion of Bergman kernel, *J. Differ. Geom.* 72 (1) (2006) 1–41.
- [7] K. Fritsch, H. Herrmann, C.-Y. Hsiao, G-equivariant embedding theorems for CR manifolds of high codimension, *Mich. Math. J. Adv. Publ.* (2021) 1–44, <https://doi.org/10.1307/mmj/20205864>.
- [8] A. Galasso, C.-Y. Hsiao, Toeplitz operators on CR manifolds and group actions, *J. Geom. Anal.* 33 (2023) 21, <https://doi.org/10.1007/s12220-022-01078-9>.
- [9] A. Galasso, C.-Y. Hsiao, Functional calculus and quantization commutes with reduction for Toeplitz operators on CR manifolds, *Math. Z.* 308 (2024) 5, <https://doi.org/10.1007/s00209-024-03561-1>.
- [10] V. Ginzburg, Y. Karshon, V. Guillemin, *Moment Map, Cobordisms and Hamiltonian Group Actions*, *Mathematical Surveys and Monographs* Volume, vol. 98, 2002, 350 pp.
- [11] C.-Y. Hsiao, Projections in several complex variables, *Mém. Soc. Math. Fr.* 123 (2010) 1–136.
- [12] C.-Y. Hsiao, Bergman kernel asymptotics and a pure analytic proof of the Kodaira embedding theorem, in: *Complex Analysis and Geometry*, in: Springer Proceedings in Mathematics and Statistics, 2015.
- [13] C.-Y. Hsiao, R.-T. Huang, G-invariant Szegő kernel asymptotics and CR reduction, *Calc. Var. Partial Differ. Equ.* 60 (1) (2021) 47.
- [14] C.-Y. Hsiao, G. Marinescu, Asymptotics of spectral function of lower energy forms and Bergman kernel of semi-positive and big line bundles, *Commun. Anal. Geom.* 22 (1) (2014) 1–108.
- [15] C.-Y. Hsiao, G. Marinescu, On the singularities of the Szegő projections on lower energy forms, *J. Differ. Geom.* 107 (2017) 83–155.
- [16] C.-Y. Hsiao, W.-C. Shen, On the second coefficient of the asymptotic expansion of Boutet de Monvel–Sjöstrand, *Bull. Inst. Math. Acad. Sin. (N.S.)* 15 (4) (2020) 329–365.
- [17] C.-Y. Hsiao, X. Ma, G. Marinescu, Geometric quantization on CR manifolds, *Commun. Contemp. Math.* 25 (10) (2023) 2250074, <https://doi.org/10.1142/S0219199722500742>.
- [18] G. Marinescu, X. Ma, *Holomorphic Morse Inequalities and Bergman Kernels*, *Progress in Mathematics*, Series, vol. 254, Birkhäuser, Basel, 2007.
- [19] G. Marinescu, X. Ma, Toeplitz operators on symplectic manifolds, *J. Geom. Anal.* 18 (2) (2008) 565–611.
- [20] A. Melin, J. Sjöstrand, *Fourier Integral Operators with Complex Valued Phase Functions*, Springer Lecture Notes in Math, vol. 459, Springer-Verlag, Berlin, 1974, pp. 120–223.
- [21] J.W. Milnor, J.D. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton, New Jersey, 1974.
- [22] R. Paoletti, Asymptotics of Szegő kernels under Hamiltonian torus actions, *Isr. J. Math.* 191 (1) (2012) 363–403.
- [23] M. Puchol, Holomorphic Morse inequalities for orbifolds, *Math. Z.* 289 (3–4) (2018) 1237–1260.
- [24] J. Song, The Szegő kernel on an orbifold circle bundle, arXiv:math/0405071.