Dipartimento di / Department of
Matematica e Applicazioni

Dottorato di ricerca in / PhD program: Mathematics
Ciclo / Cycle: XXXVI
Curriculum in Algebra

The Zassenhaus p-restricted Lie algebra functor

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ANNO ACCADEMICO / ACADEMIC YEAR 2022/2023

Abstract

In 1939, H. Zassenhaus introduced the dimension subgroups $D_n(G)$ over a field of characteristic p, a specific type of composition series for a group G connected to the lower central series, along with the associated p-restricted graded Lie algebra $\mathrm{L}(G)$. Jennings and Lazard later demonstrated that for any group G, this series coincides with the fastest descending series starting at G and closed under commutators and p-powers.

M. Lazard significantly contributed to elucidating the connection between pro-p groups and p-restricted Lie algebras. He established that a free pro-p group G, is finitely generated and free if and only if $\mathrm{L}(G)$ shares these properties. Furthermore, he proved that G is p-adic analytic if and only if $\mathrm{L}(G)$ is nilpotent.

In 1966, J. P. May introduced a trigraded spectral sequence linking the homology of a locally finite, connected, filtered algebra to that of its associated graded algebra. A subsequent contribution came in 1980 when A. I. Lichtman demonstrated that the Zassenhaus p-restricted Lie algebra functor preserves free products.

This thesis investigates the potent tool represented by this functor, consolidating existing knowledge and introducing new results. May's technique is employed to construct a spectral sequence that establishes a relationship between the cohomology ring of $\mathrm{L}(G)$ for a finitely generated pro-p group G and that of G. This construction defines a class of pro-p groups whose spectral sequence collapses on the first page. We characterize such groups in terms of the lifting of minimal graded free resolutions of $\mathbb{F}$ as a $\mathrm{gr}(\mathbb{F}[[G]])$-module to minimal filtered free resolutions of $\mathbb{F}$ as an $\mathbb{F}[[G]]$, providing examples such as uniform or mild pro-p groups and proving closure under direct products and free products.

The main result extends the pro-p version of Lichtman's Theorem to the amalgamated free pro-p product of finitely generated pro-p groups with strongly embedded amalgam. This tool is applied to partially answer a question posed by S. Blumer, C. Quadrelli, and T. Weigel regarding the Zassenhaus p-restricted Lie algebra associated with an oriented right-angled Artin pro-p group.

Abstract

Nel 1939, H. Zassenhaus introdusse i sottogruppi dimensionali $D_n(G)$ su un campo di caratteristica p, un tipo specifico di serie di composizione per un gruppo G legato alla serie centrale inferiore, e insieme l'algebra di Lie p-ristretta graduata associata $\mathrm{L}(G)$. In seguito, Jennings e Lazard dimostrarono che per qualsiasi gruppo G, questa serie coincide con la serie discendente più veloce chi inizia da G e chiusa per commutatori e potenze di p.

Lazard contribuì significativamente a chiarire la connessione tra gruppi pro-p e algebre di Lie p-ristrette. Stabilì che un gruppo pro-p libero G è finitamente generato e libero se e solo se $\mathrm{L}(G)$ condivide queste proprietà. Inoltre, dimostrò che G è p-adicamente analitico se e solo se $\mathrm{L}(G)$ è nilpotente.

Nel 1966, J. P. May introdusse una successione spettrale trigradata che collega l'omologia di un'algebra localmente finita, connessa e filtrata a quella della sua algebra graduata associata. Un contributo successivo avvenne nel 1980 quando A. I. Lichtman dimostrò che il funtore algebra di Lie p-ristretta di Zassenhaus preserva i prodotti liberi.

Questa tesi investiga lo strumento potente rappresentato da questo funtore, consolidando le conoscenze esistenti e introducendo nuovi risultati. La tecnica di May viene utilizzata per costruire una successione spettrale che stabilisce una relazione tra l'anello di coomologia di $\mathrm{L}(G)$ per un gruppo pro-p finitamente generato G e quello di G. Questa costruzione definisce una classe di gruppi pro-p la cui successione spettrale collassa alla prima pagina. Caratterizziamo tali gruppi in termini di sollevamento di risoluzioni libere gradate minimali di $\mathbb{F}$ come modulo $\mathrm{gr}(\mathbb{F}[[G]])$ a risoluzioni libere filtrate minimali di $\mathbb{F}$ come $\mathbb{F}[[G]]$, fornendo esempi come gruppi pro-p uniformi o miti e dimostrando la chiusura sotto prodotti diretti e prodotti liberi.

Il risultato principale estende la versione pro-p del Teorema di Lichtman al prodotto libero pro-p amalgamato di gruppi pro-p finitamente generati con un amalgama strettamente incluso. Questo strumento è utilizzato per fornire una risposta, parziale ma positiva, a una domanda posta da S. Blumer, C. Quadrelli e T. Weigel riguardo all'algebra di Lie p-ristretta associata a un gruppo pro-p di Artin ad angolo retto orientato.

Introduction

Lie algebras from groups

Lie algebras, introduced by Marius Sophus Lie in the 1870s to explore infinitesimal transformations, have played a pivotal role in understanding the interplay between algebraic structures and groups. Initially termed "infinitesimal groups," Lie algebras revealed their significance by providing a more manageable framework compared to groups.

This intrinsic property led to a natural interest in studying Lie algebras derived from groups. One purely algebraic construction, studied by W. Magnus in the 1930s and further developed by E. Witt and P. Hall, was the $\mathbb{N}_0$-graded Lie algebra $\mathrm{L}^\gamma(G) = \bigoplus_n \gamma_n(G)/\gamma_{n+1}(G) \otimes \mathbb{Q}$, whose homogeneous components are quotients of successive terms of the lower central series $\{\gamma_n(G)\}_n$ of G, and whose Lie bracket is induced from the group commutator. Among its applications, in the context of rational homotopy theory, it was used by A. I. Suciu and H. Wang to characterize 1-formal groups, i.e. finitely generated groups whose classifying space K(G, 1) is 1-formal in the sense of D. Sullivan.

Theorem 1 (Suciu, Wang, [20],[22])**.** *A finitely generated group G is 1-formal (over $\mathbb{Q}$) if and only if the following hold.*

1. *G is graded-formal, i.e. $\mathrm{L}^\gamma(G)$ is quadratic.*
2. *G is filtered-formal, i.e. the completion of the universal enveloping algebra of $\mathrm{L}^\gamma(G)$ is isomorphic to the completion of the group algebra $\mathbb{Q}[G]$, which induces the identity on their graded objects.*

Meanwhile, their usefulness in many fields gained them the right to a study on their own. By mimicking the behavior of derivations of algebras, N. Jacobson in 1937 introduced the concept of p-restricted Lie algebras. Not long after, this twisted version found a use in the hands of H. Zassenhaus [1], who, using the dimension subgroups $D_n(G)$ over a field of characteristic p, a particular type of composition series over a group G related to the lower central series, introduced the p-restricted graded Lie algebra $\mathrm{L}(G)$, which is the subject of this thesis. Later Jennings provided a characterization of Zassenhaus' series.

Theorem 2 (Jennings, [2])**.** *Let G be a finite p-group. The dimension subgroup series $\{D_n(G)\}_n$ is the fastest descending series satisfying*

1. *$D_1(G) = G$,*
2. *$[D_n(G), D_m(G)] \leq D_{n+m}(G)$ for every n, m.*
3. *$D_n(G)^p \leq D_{pn}(G)$ for every n.*

But it was M. Lazard who showed the power of this connection. He extended Jennings' result to any group, and applied in particular the construction of Zassenhaus to pro-p groups, establishing several results connecting a pro-p group G to and its Zassenhaus p-restricted Lie algebra $\mathrm{L}(G)$.

Theorem 3 (Lazard, [3])**.** *G is a free pro-p group over n generators, if and only if $\mathrm{L}(G)$ is the graded free p-restricted Lie algebra over n generators.*

Theorem 4 (Lazard, [5])**.** *Let G be a finitely generated pro-p group, the following are equivalent.*

1. *G is p-adic analytic.*
2. *$\mathrm{L}(G)$ is nilpotent.*

In an effort to compute the cohomology of the Steenrod algebra, which appears as the E_2 page of the Adams Spectral Sequence to calculate the mod-p stable homotopy groups of spheres, J. P. May constructed a trigraded spectral sequence that links the homology of a locally finite, connected, filtered algebra, to that of its associated graded algebra. The same construction can be used to relate the cohomology ring of $\mathrm{L}(G)$ for a finitely generated pro-p group G with that of G.

Some progress has been made on investigating the Zassenhaus functor $G \mapsto \mathrm{L}(G)$. For example, A. I. Lichtman proved in 1980 that apart from direct product it preserves also free products.

Theorem 5 (Lichtman, [8])**.** *Let G be a free product of groups G_1 and G_2. Then $\mathrm{L}(G)$ is isomorphic to the free product of p-restricted Lie algebras $\mathrm{L}(G_1)$ and $\mathrm{L}(G_2)$.*

For an arbitrary pro-p group G, it is quite difficult to determine the isomophism type of $\mathrm{L}(G)$. For example, the following question, asked by A. Shalev in a discussion with T. Weigel in 1994 in Istanbul is, to the author's knowledge, still open.

Question 6. *Let L be a 1-generated $\mathbb{N}_0$-graded finite dimensional p-restricted Lie algebra L. Is it always possible to find a finite p-group G for which $\mathrm{L}(G)$ is isomorphic to L?*

Structure of the thesis and main results

Chapter 1 deals with filtered, profinite, augmented algebras, and with modules and minimal resolutions, outlining a suitable algebraic framework to deal with filtrations over finitely generated pro-p groups.

Chapter 2 introduces the tools from the theory of spectral sequences necessary to restate May's result in more suitable terms.

In Chapter 3 we define the concept of strongly collapsing pro-p groups, provide several examples and prove that such class is closed under direct products and free products.

In Chapter 4 we extend the pro-p version of Lichtman's result to amalgamated free pro-p product of groups, under the minimal additional requirement that the amalgam is *strictly embedded* in each factor, i.e. the inclusion of the amalgam in each factor induces an inclusion of the corresponding restricted Lie algebras.

Theorem 7. *Let H, G_1 and G_2 be finitely generated pro-p groups such that H is strictly embedded in G_1 and G_2, and let $G = G_1 \amalg_H G_2$ be their amalgamated free pro-p product. Then the natural morphism of p-restricted Lie algebras*

$$\mathrm{L}(G_1) \amalg_{\mathrm{L}(H)} \mathrm{L}(G_2) \to \mathrm{L}(G).$$

is an isomorphism.

In Chapter 5 we restrict our attention to a particular subclass of strongly collapsing pro-p groups which, in accordance with the terminology of Suciu and Wang will be called filtered-formal. We will provide several examples, and use our previous result to prove that this subclass is closed under free product with amalgamation with strongly embedded amalgam.

In order to investigate on Shalev's question it will be useful to introduce the concept of genus of a finitely generated pro-p group G, as

$$\mathrm{gen}\,(G) = \{G' \mid \mathrm{L}(G') \simeq \mathrm{L}(G)\}\,.$$

Finally, using again Theorem 7, we give a partial answer to a question posed by Blumer, Quadrelli and Weigel ([25]) about oriented right-angled Artin pro-p groups.

Question 8 (Blumer, Quadrelli, Weigel)**.** *Let Γ be a specially oriented graph, let $\lambda : \mathbb{Z}_p \to \mathbb{Z}_p^\times$ be a continuous homomorphism and let $G_{\Gamma,\lambda}$ be the oriented pro-p right-angled Artin group associated to Γ and λ. Is it true that $\mathrm{L}(G_{\Gamma,\lambda})$ is quadratic?*

We prove that $\mathrm{L}(G_{\Gamma,\lambda}) \simeq \mathrm{L}(G_{\ddot{\Gamma}})$, where $G_{\ddot{\Gamma}}$ denotes the right-angled Artin pro-p group associated $G_{\Gamma,\lambda}$.

An affermative answer to Question 8 shows that in general $\mathrm{gen}\,(G_{\Gamma,\lambda})$ contains more than one isomorphism class of pro-p groups when λ is not trivial, i.e. $G_{\ddot{\Gamma}}$. Note however that $G_{\Gamma,\lambda}$ is not filtered formal. We are then lead to asking the following question.

Question 9. *Let G be a finitely generated pro-p group G. Is it true that* $\mathrm{gen}\,(G)$ *contains at most one isomorphism class of filtered formal pro-p groups?*

Chapter 1

Algebras and modules

In this chapter, we will develop the concept of complete profinite augmented algebras and modules that are entirely free over such algebras. The context created allows us, on the one hand, to employ tools such as Pontryagin duality, connected to the realm of profinite groups, and Serre's Lemma (Proposition 1.3.10), linked to that of complete filtrations. On the other hand, it enables us to defer the use of pro-p groups to another chapter (specifically, the third one).

1.1 Filtered modules

1.1.1 Filtered abelian groups

Definition 1.1.1. *We say that an abelian group E is filtered if it has a (decreasing) filtration, that is a function $\nu_E : E \to \overline{\mathbf{Z}} := \mathbb{Z} \cup \{\pm\infty\}$ such $\nu_E(0) = +\infty$ and such that, for any $x, y \in E$,*

$$\nu_E(x - y) \geq \inf(\nu_E(x), \nu_E(y)).$$

Example 1.1.2. *The constant map $\nu_E(x) = 0$ defines the trivial filtration on E.*

If E is filtered, for any n consider the set

$$F^n E := \{x \in E \mid \nu_E(x) \geq n\}.$$

It is nonempty, as it contains 0. If x and y are elements of $F^n E$, so is $x - y$, therefore $F^n E$ is a subgroup of E, and clearly $F^{n+1}E \leq F^n E$. Viceversa, if $\{F^n E\}_n$ is a decreasing series of subgroups, then $\nu_E(x) := \sup\{n \in \mathbb{Z} \mid x \in F^n E\}$ is a filtration.

Definition 1.1.3. *Given filtered abelian groups E and E', we say that a morphism $\phi : E \to E'$ is*

- *filtered, if and only if for all $x \in E$,*

$$\nu_{E'}(\phi(x)) \geq \nu_E(x),$$

- *strictly filtered, or strict, if for all $x \in E$, there is an element $y \in E$ with $\phi(y) = \phi(x)$ such that*

$$\nu_{E'}(\phi(x)) = \nu_E(y).$$

Remark 1.1.4. *In particular, if $\phi : E \to E'$ is injective, it is strict if and only if for all $x \in E$,*

$$\nu_{E'}(\phi(x)) = \nu_E(x).$$

Lemma 1.1.5. *If $\phi : E \to E'$ and $\psi : E' \to E''$ are strict morphisms of filtered abelian groups and either ϕ is surjective or ψ is injective, then $\psi \circ \phi$ is strict.*

Proof. Take $x \in E$. By strictness of ψ, there is an element $y' \in E'$ such that $\psi(\phi(x)) = \psi(y')$ and $\nu_{E''}(\psi(\phi(x))) = \nu_{E'}(y')$.

If ϕ is surjective, there is an element $y \in E$ such that $\phi(y) = y'$. By strictness of ϕ, there is an element $z \in E$ such that $\phi(y) = \phi(z)$ and $\nu_{E'}(\phi(z)) = \nu_E(z)$. Therefore,

$$\psi(\phi(x)) = \psi(y') = \psi(\phi(y)) = \psi(\phi(z),$$
$$\nu_{E''}(\psi(\phi(x))) = \nu_{E'}(y') = \nu_{E'}(\phi(y)) = \nu_{E'}(\phi(z)) = \nu_E(z).$$

If ψ is injective, $\phi(x) = y'$. By strictness of ϕ, there is an element $y \in E$ such that $\phi(x) = \phi(y)$ and $\nu_{E'}(\phi(x)) = \nu_E(y)$. Therefore

$$\psi(\phi(x)) = \psi(\phi(y),$$
$$\nu_{E''}(\psi(\phi(x))) = \nu_{E'}(y') = \nu_{E'}(\phi(x)) = \nu_E(y).$$

□

Definition 1.1.6. *We say that a filtered abelian group E is*

- *bounded from above, if $\nu_E(x) \geq n$ for all $x \in E$ for some $n \in \mathbb{Z}$;*
- *exhaustive, if $\nu_E^{-1}(-\infty) = \emptyset$;*
- *separated, if $\nu_E^{-1}(+\infty) = \{0\}$.*

Example 1.1.7. *Let E be a filtered abelian group and $\phi : E \to E'$ be a surjective morphism. Then we can induce a filtration on E' with respect to which ϕ is strict, as*

$$\nu_{E'}(y) = \sup_{y=\phi(x)} (\nu_E(x)) \quad \forall y \in E'.$$

If ν is another filtration on E' for which ϕ is filtered, then $\nu_{E'}(y) \geq \nu(y)$ for every $y \in E$.

Example 1.1.8. *Let $(\{E_i\}_i, \{\phi_{ij}\}_{i\geq j})$ be an inverse system of filtered abelian groups whose morphisms $\phi_{ij} : E_j \to E_i$ are filtered. Then their inverse limit $\left\{\hat{E}, \phi_i\right\}$ is filtered, with filtration*

$$\nu_{\hat{E}}(x) \coloneqq \inf_i (\nu_{E_i}(\phi_i(x)).$$

Let E be a filtered abelian group. The quotients $\pi_n : E \to E/F^n E$ form an inverse system of filtered abelian groups and the natural morphism $\alpha : E \to \hat{E}$ induced that sends an element $x \in E$ into $(\pi_n(x))_n \in \hat{E}$ is strict.

Definition 1.1.9. *We say that E is complete if α is an isomorphism.*

Remark 1.1.10. *Let E be a filtered abelian group.*

- *If E is complete, it is also separated.*
- *E can be endowed with a topological group structure, with the subgroups F^iE forming a local basis of neighborhoods of 0. As a topological group, $\hat{E}$ is complete.*

For any filtered morphism of abelian groups $\phi : E \to E'$, there is a unique filtered morphism $\phi' : \hat{E} \to \hat{E}'$ such that the following diagram commutes.

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\alpha"] & E' \arrow[d, "\alpha'"] \\ \hat{E} \arrow[r, "\phi'"] & \hat{E}' \end{tikzcd}$$

1.1.2 Filtered modules

Definition 1.1.11. *We say that a ring R is a filtered ring if it has a filtration such that*

1. $\nu_R(rr') \geq \nu_R(r) + \nu_R(r') \quad \forall r, r' \in R$
2. $\nu_R(1) = 0$.

We say that an R-module M is a filtered R-module if it has filtration such that

$$\nu_M(rm) \geq \nu_R(r) + \nu_R(m) \quad \forall r \in R,\ \forall m \in M.$$

If $\mathbb{F}$ is a field, we say that an $\mathbb{F}$-algebra is a filtered algebra if it has filtration such that it is a filtered ring and a filtered $\mathbb{F}$-module, where we consider $\mathbb{F}$ as a filtered ring with trivial filtration.

As before, a filtered ring R is a ring which is filtered as an abelian group and $(F^nR)(F^mR) \subseteq F^{n+m}R$ for all $n, m \in \mathbb{Z}$.

Similarly, a filtered R-module is an R-module which is is filtered as an abelian group and $(F^nR)(F^mM) \subseteq F^{n+m}M$ for all $n, m \in \mathbb{Z}$.

Definition 1.1.12. *Let R be a filtered ring and M an R-module. Then M can be endowed with a filtered structure by $F^iM = (F^iR)M$. We call it the filtration induced by R.*

Example 1.1.13. *If R is a ring and I an ideal of R, then I induces a filtration on R by*

$$\nu_R(r) = \sup \{n \mid r \in I^n\}.$$

Such filtration is bounded from above and it is separated if and only if $\bigcap_i I^i$ equals 0.

Example 1.1.14. *If M and M' are a right and a left filtered R-module respectively, then we can define a filtration on their tensor product as*

$$F^n(M \otimes_R M') := \sum_{i+j=n} F^iM \otimes_R F^jM'.$$

Filtered free modules

Definition 1.1.15. *Let R be a separated filtered ring and M a separated filtered R-module. We say that a set of elements $(x_i)_{i\in I}$ of M is a filtered R-basis if for any family $(r_i)_{i\in I}$ of elements of R where almost all r_i's are zero,*

$$\nu_M\left(\sum_{i\in I} r_i x_i\right) = \inf(\nu_R(r_i) + \nu_M(x_i)).$$

A module that is generated by a filtered basis is called filtered free.

Remark 1.1.16. *A filtered basis is linearly independent. Indeed,*

$$\sum_{i\in I} r_i x_i = 0 \implies \nu\left(\sum_{i\in I} r_i x_i\right) = +\infty \implies \nu(r_i) + \nu(x_i) = +\infty \quad \forall i \in I.$$

Since R is separated, if $r_i \neq 0$ for some i then $\nu(r_i) < +\infty$. It follows that $\nu(x_i) = +\infty$. But as M is also separated, $x_i = 0$.

Remark 1.1.17. *If R is a separated filtered ring and M is a free R-module with basis $(x_i)_{i\in I}$, we may endow M with a filtered R-module structure by assigning to $\nu_M(x_i)$ a nonnegative integer for every $i \in I$, and defining for any $m \in M$, with $m = \sum_i r_i x_i$,*

$$\nu_M(m) \coloneqq \inf(\nu_R(r_i) + \nu_M(x_i)).$$

With such filtration, $(x_i)_{i\in I}$ is a filtered R-basis for M and M is separated.

If additionally $M = A$ is an algebra and, for any i and j,

$$\nu_A(x_i x_j) \geq \nu_A(x_i) + \nu_A(x_j)$$

then A is a filtered algebra.

Remark 1.1.18. *A morphism of filtered algebras $\phi : C \to A$ induces a filtered C-module structure on A by $a \cdot c \coloneqq a\phi(c)$.*

$$\nu_A(a \cdot c) = \nu_A(a\phi(c)) \geq \nu_A(a) + \nu_A(\phi(c)) \geq \nu_A(a) + \nu_A(c).$$

Definition 1.1.19. *We say that morphism of filtered algebras $\phi : C \to A$ is filtered free if A is filtered free as a filtered C-module.*

Remark 1.1.20. *If $\phi : C \to A$ is filtered free, it is injective and strict.*

- *ϕ is injective. Let $c \in C$ such that $\phi(c) = 0$ and take any element a_i of the basis. In particular $\nu_A(a_i) < +\infty$.*

  $$+\infty = \nu(a_i\phi(c)) = \nu(a_i \cdot c) = \nu_A(a_i) + \nu_C(c),$$

  so $\nu_C(c) = +\infty$ and therefore $c = 0$.

- *ϕ is strict. Let $c \in C$ and take any element a_i of the basis. By comparing*

  $$\nu_A(a_i \cdot c) = \nu_A(a_i) + \nu_C(c),$$
  $$\nu_A(a_i \cdot c) = \nu_A(a_i\phi(c)) \geq \nu_A(a_i) + \nu_A(\phi(c)),$$

  it follows that $\nu_A(\phi(c)) \leq \nu_C(c)$.

Completely free modules

Definition 1.1.21. *Let R be a completely filtered ring and M a separated filtered module. We say that a family of elements of M is a topological basis if it is a filtered basis and if M is the completion of the submodule generated by such family.*

A module that admits a topological basis is called completely free.

Lemma 1.1.22. *Let R be a completely filtered ring and I a set, and N the filtered R-module*

$$N \coloneqq \coprod_{i \in I} R.$$

Then the completion $\hat{N}$ of N is contained in the product $\prod_{i \in I} R$ and consists of the elements $(r_i)i \in I$ such that

$$\nu_R(r_i) \to -\infty$$

with respect to the cofinite topology.

Proof. By definition of completion, an element of N is $m = (m_n + F^n N)_{n \in \mathbb{Z}}$ such that for every n, the element $m_n + F^{n+1} N = m_{n+1} + F^n N$ and m_n is a element $(r_{n,i})_{i \in I}$ of N. Therefore,

1. for every n and i we have $r_{n,i} + F^{n+1} R = r_{n,i} + F^{n+1} R$;

2. for every n, the elements $r_{n,i}$ are zero for almost all i.

Since R is complete, by 1) it follows that for every i there is a unique r_i in R such that $r_i + F^n R = r_{n,i} + F^n R$ for every n. So we may write m as $m = (r_i)_{i \in I}$. Also, for any n, the element $r_i + F^n R = r_{n,i} + F^n R$ is zero for almost all i, i.e. $r_i \in F^n R$ for almost all i or, equivalently, $\nu_R(r_i) \leq n$ for almost all i. So $(\nu(r_i))_{i \in I}$ converges to $-\infty$ in the cofinite topology.

Viceversa, let m be an element $(r_i)_{i \in I}$ of $\prod_{i \in I} R$ such that $r_i \in F^n R$ for almost all i. Then m can be written as

$$m = (m_n + F^n N)_{n \in \mathbb{Z}},$$

where $m_n \coloneqq (r_i)_{i \in I}$. □

Proposition 1.1.23. *Let R be a complete filtered ring and M a completely free R-module with topological basis $(x_i)_{i \in I}$. Then the elements of M can be uniquely written as*

$$\sum_{i \in I} r_i x_i$$

where the family $(r_i)_{i \in I}$ is such that

$$\nu_R(r_i) + \nu_{M_i}(x_i) \to -\infty$$

with respect to the cofinite topology.

Proof. Let N be the the filtered free R-module generated by $(x_i)_{i \in I}$ and apply the Lemma 1.1.22 using the fact that $\nu_{M_i}(r_i x_i) = \nu_R(r_i) + \nu_{M_i}(x_i)$. □

1.2 Graded modules

1.2.1 Graded abelian groups

Definition 1.2.1. *We say that an abelian group E is graded if it can be written as a direct sum over the integers of abelian groups*

$$E = \bigoplus_{n \in \mathbb{Z}} E_n.$$

If $x \in E_n$, we say that x is a homogeneous element of degree n, and we write $\deg_E(x) = n$.

Example 1.2.2. *Any group E may be endowed with the trivial grading $E = E_0$.*

Definition 1.2.3. *Given graded abelian groups $E = \bigoplus_n E_n$ and $E' = \bigoplus_n E'_n$ we say that a family of morphisms $\phi_n : E_n \to E'_n$ is morphism of graded abelian groups $\phi : E \to E'$.*

Definition 1.2.4. *We say that a graded abelian group $E = \bigoplus_n E_n$ is*

- *bounded from above, if there is some k for which $E_n = 0$ for all $k \geq n$;*
- *bounded from below, if there is some k for which $E_n = 0$ for all $k \leq n$.*

1.2.2 Graded modules

Definition 1.2.5. *A graded ring $R = \bigoplus_n R_n$ is a ring which is graded as an abelian group, $1 \in R_0$, and $R^n R^m \subseteq R^{n+m}$ for any $n, m \in \mathbb{Z}$.*

A graded R-module $M = \bigoplus_n M_n$ is an R-module which is graded as an abelian group and $R^n M^m \subseteq M^{n+m}$ for any

If $\mathbb{F}$ is a field, we say that an $\mathbb{F}$-algebra is a graded algebra if is graded as a filtered ring and as a graded $\mathbb{F}$-module, where we consider $\mathbb{F}$ as a graded ring with trivial graded structure.

Definition 1.2.6. *If R is a graded ring and M a graded R-module, we say that M is graded free over R if it admits an R-basis of homogeneous elements.*

Definition 1.2.7. *We say that morphism of graded algebras $\phi : C \to A$ is graded free if A is graded free as a graded C-module.*

1.3 The functor gr

Definition 1.3.1. *Let E be a filtered abelian group. For every $i \in \mathbb{N}_0$ define the abelian group*

$$\mathrm{gr}_i E := \frac{F^i E}{F^{i+1} E}$$

We denote by $\mathrm{gr}\, E := \bigoplus_i \mathrm{gr}_i E$ the graded abelian group induced by the filtered abelian group E.

If $\phi : E \to E'$ is a filtered morphism of filtered abelian groups, we denote by $\mathrm{gr}\, \phi : \mathrm{gr}\, E \to \mathrm{gr}\, E'$ the morphism of graded abelian groups induced by ϕ.

If x is an element of E, we denote by $\overline{x}$ the corresponding element in $\operatorname{gr} E$.

Note, that since E is separated, $x = 0$ if and only $\overline{x} = 0$, and that if x is a non zero element, $\deg(\overline{x}) = \nu_E(x)$.

Remark 1.3.2. *If R is a filtered ring, $\operatorname{gr} R$ is a graded ring.*

If M is a filtered R-module, $\operatorname{gr} M$ is a graded $\operatorname{gr} R$-module.

If A is a filtered algebra, $\operatorname{gr} A$ is a graded algebra.

Lemma 1.3.3. *Let be R is a filtered ring, M a filtered R-module and X a set of elements of M. Then X is a filtered R-basis for M if and only if $\overline{X}$ is a homogeneous $\operatorname{gr} R$-basis for $\operatorname{gr} M$.*

As a consequence, a morphism $\phi : C \to A$ of filtered algebras is filtered free if and only if $\operatorname{gr} \phi : \operatorname{gr} C \to \operatorname{gr} A$ is graded free.

Proof. Suppose that $X = (x_i)_{i\in I}$ is a filtered R-basis for M. Let $(r_i)_{i\in I}$ be a family of elements of R where almost all r_i's are zero, such that

$$\sum_i \overline{r_i} \cdot \overline{x_i} = \sum_i (r_i + F^{n_i} R)(x_i + F^{m_i} M) = \sum_i (r_i x_i + F^{n_i+m_i+1} M) = 0$$

We may assume that $\overline{r_i} \cdot \overline{x_i}$ are homogeneous elements of the same degree k. Then

$$\sum_i \overline{r_i} \cdot \overline{x_i} = 0 \iff \sum_i r_i x_i \in F^{k+1} M \iff \nu_M\left(\sum_i r_i x_i\right) \geq k+1$$

But since $(x_i)_{i\in I}$ is a filtered basis, for every $i \in I$

$$\nu_M\left(\sum_i r_i x_i\right) = \nu_R(r_i) + \nu_M(x_i) = \deg(\overline{r_i}) + \deg(\overline{x_i}) = \deg \overline{r_i} \cdot \overline{x_i} = k,$$

against the hypothesis. Viceversa, suppose that $(\overline{x_i})_{i\in I}$ is a homogeneous $\operatorname{gr} R$-basis for $\operatorname{gr} M$. Let $(r_i)_{i\in I}$ be a family of elements of R where almost all r_i's are zero, such that

$$\nu_M\left(\sum_i r_i x_i\right) > \min_i \{\nu_R(r_i) + \nu_M(x_i)\} =: k.$$

Let $J \subseteq I$ the subset of indexes i such that $\nu_R(r_i) + \nu_M(x_i) = \deg(r)k$, i.e. the set of indexes corresponding for which the elements $\overline{r_i} \cdot \overline{x_i}$ are homogeneous of degree k.

$$\left(\sum_i \overline{r_i} \cdot \overline{x_i}\right)_k = \sum_{i\in J} \overline{r_i} \cdot \overline{x_i} = \sum_{i\in J} r_i x_i + F^{k+1} M = 0,$$

against the hypothesis. ∎

1.3.1 The Lifting Lemma

In the following we prove a Lemma to lift graded free resolutions to completely free resolutions. The idea is due to Lazard, who attributes it to Serre ([3], Lemma 2.1.1), but the proof follows P. Day's approach ([10], Theorem 1).

Definition 1.3.4. *We say that an exact sequence of groups*

$$0 \to E \xrightarrow{\phi} E' \xrightarrow{\psi} E'' \to 0.$$

is strict exact if the maps ϕ and ψ are strict.

The following result can be easily verified.

Lemma 1.3.5. *If the sequence*

$$0 \to E \xrightarrow{\phi} E' \xrightarrow{\psi} E'' \to 0.$$

is strict exact, then for every n the sequence

$$0 \to F^n E \xrightarrow{\phi} F^n E' \xrightarrow{\psi} F^n E'' \to 0.$$

is exact. If the filtrations are exhaustive, the converse also holds.

Lemma 1.3.6. *If*

$$0 \to E \xrightarrow{\phi} E' \xrightarrow{\psi} E'' \to 0$$

is a strict exact sequence of filtered abelian groups, then

$$0 \to \operatorname{gr} E \xrightarrow{\operatorname{gr}\phi} \operatorname{gr} E' \xrightarrow{\operatorname{gr}\psi} \operatorname{gr} E'' \to 0$$

is an exact sequence of graded abelian groups.

Proof. By Lemma 1.3.5, the first two rows of the following diagram are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F^n E \arrow[r] \arrow[d] & F^n E' \arrow[r] \arrow[d] & F^n E'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^{n+1} E \arrow[r] \arrow[d] & F^{n+1} E' \arrow[r] \arrow[d] & F^{n+1} E'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{gr}_n E \arrow[r] \arrow[d] & \operatorname{gr}_n E' \arrow[r] \arrow[d] & \operatorname{gr}_n E'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The columns are exact by definition. By the 3×3-lemma, it follows that the bottom row is exact too. □

Lemma 1.3.7. *Let $\phi : E \to E'$ and $\psi : E' \to E''$ be morphisms of filtered abelian groups, and assume that the sequence*

$$0 \to \operatorname{gr} E \xrightarrow{\operatorname{gr}\phi} \operatorname{gr} E' \xrightarrow{\operatorname{gr}\psi} \operatorname{gr} E'' \to 0$$

is exact.

- *If E is separated, then $\phi : F^n E \to F^n E'$ is a monomorphism;*
- *If E' is separated and E'' is complete, then $\psi : F^n E \to F^n E'$ is an epimorphism.*
- *If E, E' and E'' are complete and exhaustive, then the sequence*

$$0 \to E \xrightarrow{\phi} E' \xrightarrow{\psi} E'' \to 0$$

  is strict exact.

Proof. By Lemma 1.3.6 the sequence

$$0 \to \operatorname{gr}_n E \xrightarrow{\operatorname{gr}_n \phi} \operatorname{gr}_n E' \xrightarrow{\operatorname{gr}_n \psi} \operatorname{gr}_n E'' \to 0$$

is exact for every n. We show that, for any $j \geq 1$, the sequence

$$0 \to \frac{F^n E}{F^{n+j} E} \xrightarrow{\phi_j} \frac{F^n E'}{F^{n+j} E'} \xrightarrow{\psi_j} \frac{F^n E''}{F^{n+j} E''} \to 0$$

is exact. We proceed by induction. The case $j = 1$ is true by hypothesis.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{gr}_{n+j-1} E' \arrow[r] \arrow[d, "\operatorname{gr}_{n+j-1} \phi"] & \frac{F^n E'}{F^{n+j} E'} \arrow[r] \arrow[d, "\phi_j"] & \frac{F^n E'}{F^{n+j-1} E'} \arrow[r] \arrow[d, "\phi_{j-1}"] & 0 \\
0 \arrow[r] & \operatorname{gr}_{n+j-1} E \arrow[r] \arrow[d, "\operatorname{gr}_{n+j-1} \psi"] & \frac{F^n E}{F^{n+j} E} \arrow[r] \arrow[d, "\psi_j"] & \frac{F^n E}{F^{n+j-1} E} \arrow[r] \arrow[d, "\psi_{j-1}"] & 0 \\
0 \arrow[r] & \operatorname{gr}_{n+j-1} E'' \arrow[r] \arrow[d] & \frac{F^n E''}{F^{n+j} E''} \arrow[r] \arrow[d] & \frac{F^n E''}{F^{n+j-1} E''} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If $j > 1$, in the above diagram

- the rows are exact;
- the first column is exact by hypothesis
- the the third column is exact by inductive hypothesis.

By the 3×3-lemma, it follows that the second column is exact too. Passing to the limit, we obtain that the following sequence is exact

$$0 \to F^n \hat{E} \to F^n \hat{E}' \to F^n \hat{E}'' \to 0$$

Consider the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigcap_j F^{n+j}E \arrow[r] \arrow[d] & F^nE \arrow[r, "\alpha_n"] \arrow[d, "\phi"] & F^n\hat{E} \arrow[r] \arrow[d] & \operatorname{coker}(\alpha_n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigcap_j F^{n+j}E' \arrow[r] \arrow[d] & F^nE' \arrow[r, "\alpha'_n"] \arrow[d, "\psi"] & F^n\hat{E}' \arrow[r] \arrow[d] & \operatorname{coker}(\alpha'_n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigcap_j F^{n+j}E'' \arrow[r] \arrow[d] & F^nE'' \arrow[r, "\alpha''_n"] \arrow[d] & F^n\hat{E}' \arrow[r] \arrow[d] & \operatorname{coker}(\alpha''_n) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

If the first and fourth column are exact, so is the second.

The result follows from the 4-lemma and the second part of Lemma 1.3.6.

□

Lemma 1.3.8. *Let S be bounded from below graded free $\operatorname{gr} A$-module. Then there is a bounded from above, completely free A-module M such that $\operatorname{gr} M = S$.*

Proof. Let $(w_i)_{i\in I}$ be a homogeneous $\operatorname{gr} A$-basis for S. Consider the formal A-module M' over free a basis $(x_i)_{i\in I}$ for $i \in I$. We can endow M with a filtered module structure by defining

$$\nu_{M'}(x_i) \coloneqq \deg(w_i).$$

Clearly M' is bounded from below. Since $\operatorname{gr} M' = S$ is graded free, M' is filtered free by Lemma 1.3.3. We conclude by taking M as the completion of M'. □

Lemma 1.3.9. *Let M be a completely free A-module, N a completely filtered A-module and $\overline{\phi} : \operatorname{gr} M \to \operatorname{gr} N$ a graded morphism. Then there is a filtered morphism $\phi : M \to N$ such that $\operatorname{gr}\phi = \overline{\phi}$.*

Proof. Let $(x_i)_{i\in I}$ be a topological basis for M, so that $(\overline{x_i})_{i\in I}$ is a homogeneous basis for $\operatorname{gr} M$ by Lemma 1.3.3. For every $i \in I$, with $\nu_M(x_i) = n$,

$$\overline{\phi}(\overline{x_i}) = y_i + F^{n+1}N$$

for some y_i in N with $\nu_N(y_i) = n$. Define $\phi : M \to N$ by

$$\phi(x_i) \coloneqq y_i.$$

□

Proposition 1.3.10. *Let M be a complete filtered A-module and $(Q_\bullet, \overline{\partial})$ a bounded from below graded free $\operatorname{gr} A$-resolution for $\operatorname{gr} M$. Then there exists a bounded from above completely free strict A-resolution $(P_\bullet, \partial)$ for M such that $\operatorname{gr} P_\bullet = Q_\bullet$ and $\operatorname{gr}\partial = \overline{\partial}$.*

Proof. For every n, Lemma 1.3.8 provides a bounded from above, completely free A-module P_n such that $\operatorname{gr} P_n = Q_n$. By Lemma 1.3.9, each $\overline{\partial}_n$ lifts to some morphism $\partial_n : P_n \to P_{n-1}$. Since $\partial_0 : P_0 \to M$ induces an epimorphism $\partial_0 : \operatorname{gr}(Q_0) \to gr(M)$, it must be strict by Lemma 1.3.7. Therefore the sequence

$$0 \to \ker \partial_0 \to P_0 \xrightarrow{\partial_0} M \to 0$$

is strict exact. Hence, by Lemma 1.3.6,

$$0 \to \ker \overline{\partial}_0 \to Q_0 \xrightarrow{\overline{\partial}_0} \operatorname{gr} M \to 0$$

is exact, and we may identify $\ker(\overline{\partial}_0)$ with $\operatorname{gr}(\ker \partial_0)$. Iterating, we see that P must be strict and exact. □

1.4 Complete profinite augmented algebras

1.4.1 Profinite augmented algebras

Let A be a profinite $\mathbb{F}$-algebra. Then there is a direct set I and an inverse system $(\{A_i\}_{i\in I}, \{\phi_{ij}\}_{i\leq j\in I})$ an inverse system of finite dimensional algebras such that

$$A = \varprojlim_i A_i.$$

Proposition 1.4.1. *The following are equivalent.*

1. *There is a continuous map $\varepsilon : A \to \mathbb{F}$.*
2. *For any i the algebra A_i is augmented and for any $i \leq j$ the morphism $\phi_{ij} : A_j \to A_i$ is augmented.*

Definition 1.4.2. *In either case we call A a profinite augmented algebra.*

Proof.

$1 \Rightarrow 2$. If $\varepsilon : A \to \mathbb{F}$ is a continuous morphism and $\mathbb{F}$ is finite, there is $k \in I$ such that $\varepsilon = \varepsilon_k \circ \phi_k$. In particular A_k is augmented. If $k \leq j$, the algebra A_j is also augmented, with augmentation $\varepsilon_j = \varepsilon_k \circ \phi_{kj}$. Since the subset $\{j \in I \mid k \leq j\}$ is cofinal in I,

$$A = \varprojlim_{j\leq k} A_j.$$

$2 \Rightarrow 1$. If $i \in I$ and $\varepsilon_i : A_i \to \mathbb{F}$ is the augmentation map of A_i, define $\varepsilon : A \to \mathbb{F}$ as $\varepsilon := \varepsilon_i \circ \phi_i$. Such morphism doesn't depend on the choice of i. □

Consider the filtration on A induced by the augmentation ideal A^+. Since $\mathbb{F}$ is discrete, A^+ is open in A.

Definition 1.4.3. *If the topology on A is induced by such filtration induced by A^+ we say that A is a complete profinite augmented algebra.*

Proposition 1.4.4. *Let A be a profinite augmented algebra. Assume the following.*

1. *$(A^+)^n$ is open for every n.*

2. A_i^+ is nilpotent for every $i \in I$.

Then A is a complete profinite augmented algebra.

Proof. We need to prove that the powers of the augmentation ideal form a basis for A. This is true if and only if for any $i \in I$ there is a natural number n for which $(A^+)^n \subseteq \ker \phi_i$, where $\phi_i : A \to A_i$ is a natural projection. But A_i^+ is nilpotent if and only if there exist some n for which $(A_i^+)^n = 0$, i.e. for which precisely $(A^+)^n \subseteq \ker \phi_i$. □

Example 1.4.5. *If G is a finitely generated pro-p group, its complete group algebra $\mathbb{F}[[G]]$ is a complete profinite augmented algebra.*

By definition

$$\mathbb{F}[[G]] = \varprojlim_{N \trianglelefteq_o G} \mathbb{F}\left[G/N\right].$$

The augmentation ideal of the group algebra of a finite p-group over a field over characteristic p is nilpotent.

Lemma 1.4.6. *Let A be a complete profinite augmented algebra and M a completely filtered A-module. Let X be a subset of elements of M. The following are equivalent.*

1. *M is a free profinite A-module, with profinite basis X.*

2. *M is a completely free A-module, with topological basis X.*

Proof. We only need to prove that if it is completely free, then it is free profinite. By definition of complete module, we can write

$$M = \varprojlim_k \frac{M}{F^k M},$$

and it is enough to show that, for any k, the A-module M/F^kM is finitely generated. But M/F^kM is generated by those elements of $X = \{x_i\}_{i \in I}$ for which $x_i + F^k \neq 0$. By definition of topological basis, $\nu_M(x_i)$ tends to $-\infty$ in the cofinite topology of I, i.e. the set

$$\{i \in I \mid \nu_M(x_i) > k\} = \left\{i \in I \mid x_i + F^kM \neq 0\right\}$$

is finite. □

1.5 Minimal resolutions

The filtered case

Definition 1.5.1. *Let A be a completely filtered algebra. A surjective morphism of complete augmented A-modules $\phi : P \to M$, with P completely free (graded free) is minimal if the induced map*

$$\frac{P}{A^+P} \to \frac{M}{A^+M}$$

is bijective. A completely free A-resolution $(P_\bullet, \partial)$ for M is minimal if the morphisms

$$P_p \to K_p \coloneqq \operatorname{Im} \partial_p$$

are minimal for all p.

Lemma 1.5.2. *If $\mathbb{F}$ considered as an A-module, then $P_0 \simeq A$. To see this, write $P_0 = V \widehat{\otimes} A$, for some completely filtered $\mathbb{F}$-vector space V. Then there is a chain of isomorphisms*

$$V \simeq \frac{P_0}{A^+ P_0} \simeq \frac{M}{A^+ M} \simeq \mathbb{F}.$$

Lemma 1.5.3. *Let A be a complete augmented algebra. Then $\mathbb{F}$ admits a minimal completely free A-resolution if and only if A^+ does.*

Proof. Let $(P_\bullet, \partial)$ be a minimal completely free A-resolution for $\mathbb{F}$. Then $P_0 = A$ and $P_\bullet \twoheadrightarrow \ker d_0 = A^+$.

Viceversa, let $(P'_\bullet, \partial')$ be a minimal completely free A-resolution for A^+. Combine the two exact sequences

$$\begin{aligned} \ldots \to P'_0 \to A^+ \to 0, \\ 0 \to A^+ \to A \to \mathbb{F} \to 0. \end{aligned}$$

to get a completely free A-resolution for $\mathbb{F}$.

$$\cdots \to P'_0 \to A \to \mathbb{F} \to 0.$$

To see that it is minimal, it is enough to note that the map

$$P'_0 \otimes_A \mathbb{F} \to A \otimes_A \mathbb{F}$$

splits through $A^+ \otimes_A \mathbb{F} = 0$, so it is zero. □

Lemma 1.5.4. *Let A be an augmented algebra, M a complete A-module and $(P_\bullet, \partial)$ a completely free A-resolution for M. The following are equivalent.*

1. *$(P_\bullet, \partial)$ is minimal.*
2. *For every p, $\partial(P_p) \subseteq A^+ P_{p-1}$.*
3. *For every p, the induced map $\partial \otimes_A 1 : P_p \otimes_A \mathbb{F} \to P_{p-1} \otimes_A \mathbb{F}$ is zero.*

Proof.

(1) ⇒ (2): For every p, the composition

$$P_p \xrightarrow{\partial} P_{p-1} \xrightarrow{\partial} K_{p-1}$$

is zero, and so is

$$\frac{P_p}{A^+ P_p} \to \frac{P_{p-1}}{A^+ P_{p-1}} \to \frac{K_{p-1}}{A^+ K_{p-1}}.$$

As the second map is an isomorphism, the first one is zero, so $\partial(P_p) \subseteq A^+ P_{p-1}$.

(2) ⇒ (1): We need to show that, for every p, the map

$$\frac{P_p}{A^+ P_p} \to \frac{K_p}{A^+ K_p}$$

is injective. Let $x \in P_p$ and assume that $\partial(x) \in A^+K_p$. We need to prove that $x \in A^+P_p$. We can write

$$\partial(x) = \sum_i a_i \partial(y_i) = \sum_i \partial(a_i y_i),$$

for some $a_i \in A^+$ and $y_i \in P_p$. Since, by hypothesis, $K_{p+1} \subseteq A^+P_p$,

$$x - \sum_i a_i y_i \in K_{p+1} \subseteq A^+P_p,$$

we conclude that

$$x = (x - \sum_i a_i y_i) + \sum_i a_i y_i \in A^+P_p.$$

(2) $\iff$ (3): For any p, in the commuting diagram

$$\begin{tikzcd} \frac{P_p}{A^+P_p} \arrow[r] \arrow[d] & \frac{P_{p-1}}{A^+P_{p-1}} \arrow[d] \\ P_p \otimes_A \mathbb{F} \arrow[r, "\partial\otimes 1"] & P_{p-1} \otimes_A \mathbb{F} \end{tikzcd}$$

the left and right ones are strict isomorphisms. So the top map is zero if and only if the bottom one is zero. □

Lemma 1.5.5. *Let $(P'_\bullet, \partial')$ be a minimal completely free A-resolution for M' and C a complete augmented algebra which is also a completely free A-module. Then $(P_\bullet, \partial)$ is a minimal completely free C-resolution for M, where*

$$P_\bullet := P'_\bullet \mathbin{\widehat{\otimes}}_A C,$$
$$M := M' \mathbin{\widehat{\otimes}}_A C,$$
$$\partial := \partial' \mathbin{\widehat{\otimes}} 1.$$

Proof.

- $(P_\bullet, \partial)$ is a C-resolution for M. It follows from the flatness of C.
- $(P_\bullet, \partial)$ is completely free. For all n, we can write $P'_p \simeq V_p \mathbin{\widehat{\otimes}} A$. Therefore,

  $$P_p \simeq V_p \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}}_A C \simeq V_p \mathbin{\widehat{\otimes}} C,$$

  which is a completely free C-module.
- $(P_\bullet, \partial)$ is minimal. For any p, in the commuting diagram

  $$\begin{tikzcd} P_p \otimes_C \mathbb{F} \arrow[r] \arrow[d, "\partial\otimes 1"] & P'_p \otimes_A \mathbb{F} \arrow[d, "\partial'\otimes 1"'] \\ P_{p-1} \otimes_C \mathbb{F} \arrow[r] & P'_{p-1} \otimes_A \mathbb{F}, \end{tikzcd}$$

  the top and bottom maps are strict isomorphisms, while the right one is zero. It follows that $\partial \otimes 1 : P_p \mathbin{\widehat{\otimes}}_C \mathbb{F} \to P_{n-1} \mathbin{\widehat{\otimes}}_C \mathbb{F}$ is zero as well.

$\square$

In an analogous way we can prove the following.

Lemma 1.5.6. *For $i = 1, 2$, let $(P^i_\bullet, \partial^i)$ be a completely free minimal A-resolution for M_i. Then $(P\bullet, \partial)$ is a a completely free minimal A-resolution for M, where*

$$P_\bullet := P^1_\bullet \oplus P^2_\bullet,$$
$$M := M_1 \oplus M_2,$$
$$\partial := \partial^1 + \partial^2.$$

Lemma 1.5.7. *Let A be a profinite augmented algebra, M a completely filtered profinite A-module and $(P_\bullet, \partial)$ a completely free A-resolution for M. The following additional condition is equivalent to the others of Lemma 1.5.4.*

4. For every p, the induced map $d : \mathrm{Hom}_A(P_p, \mathbb{F}) \to \mathrm{Hom}_A(P_{p+1}, \mathbb{F})$ is zero.

Proof. By Lemma 1.4.6, for every p the A-module P_p is free profinite. The induced map $d : \mathrm{Hom}_A(P_p, \mathbb{F}) \to \mathrm{Hom}_A(P_{p+1}, \mathbb{F})$ is the compositon of the natural morphisms

$$\mathrm{Hom}_A(P_p, \mathbb{F}) \simeq \mathrm{Hom}_{\mathbb{F}}(P_p \otimes_A \mathbb{F}, \mathbb{F}) \xrightarrow{(\partial\otimes 1)^\wedge} \mathrm{Hom}_{\mathbb{F}}(P_{p+1} \otimes_A \mathbb{F}, \mathbb{F}) \simeq \mathrm{Hom}_A(P_{p+1}, \mathbb{F}),$$

so by Pontryagin duality ([12], Theorem 2.9.6) it is zero if and only if the map

$$(\partial \otimes 1)^\wedge : (P_p \otimes_A \mathbb{F})^\wedge \to (P_{p+1} \otimes_A \mathbb{F})^\wedge$$

is zero. Since $P_p \otimes_A \mathbb{F}$ is profinite abelian group, it is equivalent to the induced map

$$(\partial \otimes 1) : (P_p \otimes_A \mathbb{F}) \to (P_{p+1} \otimes_A \mathbb{F})$$

being zero. $\square$

The graded case

Definition 1.5.8. *Let A be a graded algebra. A surjective morphism of graded augmented A-modules $\phi : P \to M$, with P graded free is minimal if the induced map*

$$\frac{P}{A^+P} \to \frac{M}{A^+M}$$

is bijective. A completely free A-resolution $(P_\bullet, \partial)$ for M is minimal if the morphisms

$$P_p \to K_p := \mathrm{Im}\, \partial_p$$

are minimal for all p.

We state, without proof, the analogous results for the graded case.

Lemma 1.5.9. *Let A be an graded augmented algebra. Then $\mathbb{F}$ admits a minimal graded free A-resolution if and only if A^+ does.*

Lemma 1.5.10. *Let A be an augmented graded algebra, M a graded A-module and $(P_\bullet, \partial)$ a graded free A-resolution for M. The following are equivalent.*

1. $(P_\bullet, \partial)$ *is minimal.*

2. *For every* p, $K_p \subseteq A^+ P_{p-1}$.

3. *For every* p, *the induced map* $\partial \otimes_A 1 : P_p \otimes_A \mathbb{F} \to P_{p-1} \otimes_A \mathbb{F}$ *is zero.*

Lemma 1.5.11. *Let* A *be a locally finite graded augmented algebra,* M *a locally finite graded* A*-module and* $(P_\bullet, \partial)$ *a locally finite graded free augmented* A*-resolution for* M*. The following additional condition is equivalent to the others.*

4. *For every* p, *the induced map* $d : \mathrm{Hom}_A(P_p, \mathbb{F}) \to \mathrm{Hom}_A(P_{p+1}, \mathbb{F})$ *is zero.*

Lemma 1.5.12. *Let* $(P'_\bullet, \partial')$ *be a minimal graded free* A*-resolution for* M' *and* C *an algebra which is also a graded free* A*-module. Then* $(P_\bullet, \partial)$ *is a minimal graded free* C*-resolution for* M*, where*

$$\begin{aligned} P_\bullet &\coloneqq P'_\bullet \mathbin{\widehat{\otimes}_A} C, \\ M &\coloneqq M' \mathbin{\widehat{\otimes}_A} C, \\ \partial &\coloneqq \partial' \mathbin{\widehat{\otimes}} 1. \end{aligned}$$

Lemma 1.5.13. *For* $i = 1, 2$, *let* $(P^i_\bullet, \partial^i)$ *be a minimal graded free* A*-resolution for* M_i*. Then* $(P_\bullet, \partial)$ *is a minimal graded free* A*-resolution for* M*, where*

$$\begin{aligned} P_\bullet &\coloneqq P^1_\bullet \oplus P^2_\bullet, \\ M &\coloneqq M_1 \oplus M_2, \\ \partial &\coloneqq \partial^1 + \partial^2. \end{aligned}$$

Chapter 2

May-type spectral sequences

2.1 Basic definitions

Definition 2.1.1. *A May-type spectral sequence (E_r, d_r) consists of*

1. *A family of abelian groups $\{E_r^{pq}\}$, with $p, q, r \in \mathbb{Z}$ and $r \geq 0$,*
2. *For all p, q, r, morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p+1,q-r}$ such that $d_r^{pq} \circ d_r^{p-1,q+r} = 0$,*
3. *For all p, q, r, an isomorphism*
$$\alpha_r^{pq} : \frac{Z_{r+1}^{pq}}{B_{r+1}^{pq}} \to E_{r+1}^{pq},$$
where $Z_{r+1}^{pq} = \ker(d_r^{pq})$ and $B_{r+1}^{pq} = \mathrm{Im}(d_r^{p-1,q+r})$.

Definition 2.1.2. *A morphism of May-type spectral sequences $f : (E_r, d_r) \to (E'_r, d'_r)$ is a family of morphisms $f_r^{pq} : E_r^{pq} \to E'^{pq}_r$ such that, for any p, q and r, the following diagrams commute*

$$\begin{tikzcd}
E_r^{pq} \arrow[r, "d_r^{pq}"] \arrow[d, "f_r^{pq}"] & E_r^{p+1,q-r} \arrow[d, "f_r^{p+1,q-r}"] \\
E'^{pq}_r \arrow[r, "d'^{pq}_r"] & E'^{p+1,q-r}_r,
\end{tikzcd}
\qquad
\begin{tikzcd}
E_{r+1}^{pq} \arrow[r, "f_{r+1}^{pq}"] \arrow[d, "\alpha_r^{pq}"] & E'^{pq}_{r+1} \arrow[d, "\alpha'^{pq}_r"] \\
\frac{Z_{r+1}^{pq}(E)}{B_{r+1}^{pq}(E)} \arrow[r, "\overline{f_r^{pq}}"] & \frac{Z_{r+1}^{pq}(E')}{B_{r+1}^{pq}(E')}.
\end{tikzcd}$$

2.1.1 Convergence

Definition 2.1.3. *We say that a May-type spectral sequence is regular if, fixed p and q, there exist a t such that $d_r^{pq} = 0$ for $r \geq t$.*

Under such hypothesis, the morphism
$$\overline{\alpha_r^{pq}} : E_r^{pq} \to \frac{E_r^{pq}}{B_{r+1}^{pq}} = \frac{Z_{r+1}^{pq}}{B_{r+1}^{pq}} \simeq E_{r+1}^{pq}.$$
is surjective. It follows that
$$(\{E_r^{pq}\}_{r\geq t}, \{\overline{\alpha_r^{pq}}\}_{r\geq t})$$
is a direct system of abelian groups. We define
$$E_\infty^{pq} := \varinjlim_{r\geq t} E_r^{pq}$$

Definition 2.1.4. *Let $H^\bullet = \{H^p\}_{p\in\mathbb{Z}}$ be a family of filtered groups. We say that a regular May-type spectral sequence (E_r, d_r) converges to $H^\bullet$ if, for every p, q, there are isomorphisms*

$$\beta^{pq} : \mathrm{gr}_q(H^p) \to E_\infty^{pq}$$

and the filtration of H^p is exhaustive and complete for all p.

2.2 May-type spectral sequence associated to an exact couple

Definition 2.2.1. *An exact couple $\mathcal{E} = (D, E, i, j, k)$ is a couple of abelian groups (D, E) with morphisms $i : D \to D$, $j : D \to E$ and $k : E \to D$ such that the following diagram is exact.*

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Proposition 2.2.2. *Given an exact couple $\mathcal{E} = (D, E, i, j, k)$, then $\mathcal{E}' = (D', E', i', j', k')$ is an exact couple,*

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

where

$$D' = i(D), \quad E' = H(E) = \frac{\ker(j \circ k)}{\mathrm{Im}(j \circ k)},$$

and

$$i' = i|_{D'}, \quad j'(i(d)) = [j(d)], \quad k'([e]) = k(e).$$

Proof. We first note that, if we set $d := j \circ k$, then

$$d \circ d = j \circ (k \circ j) \circ k = 0$$

and E' is well defined.

Also $k(j(d)) = 0$, so $d \in \ker(j \circ k)$. In addition, if $i(d) = i(d')$, then $d - d' = k(e)$ for some $e \in E$, and $[j(d)] = [j(d')]$, so j' is well defined.

To show that k' is well defined too, consider $e' \in \ker(j \circ k)$ such that $[e] = [e']$, i.e. $e' = e + j(k(f)$ for some $f \in E$. Then $k'(e') = k(e + j(k(f)) = k(e)$.

Now we will prove that the diagram is exact.

$$\begin{aligned}
\ker i' &= \ker i \cap \operatorname{Im} i = \operatorname{Im} k \cap \ker j \\
&= k(E) \cap \ker j = k(k^{-1}(\ker j)) = k(\ker(j \circ k)) \\
&= \operatorname{Im} k', \\
\ker k' &= \{[e] \in E' \mid e \in \ker k\} = \{[e] \in E' \mid e \in \operatorname{Im} j\} = \{[j(d)] \in E' \mid d \in D\} \\
&= \operatorname{Im} j', \\
\ker j' &= \{i(d) \in D \mid j(d) \in \operatorname{Im}(j \circ k)\} \\
&= \{i(d) \in D \mid d - k(e) \in \ker j = \operatorname{Im} i\} \\
&= \{i(d) \in D \mid d \in \operatorname{Im} i\} \\
&= \operatorname{Im} i'.
\end{aligned}$$

□

We may iterate this process.

Definition 2.2.3. *Given an exact couple $\mathcal{E} = (D, E, i, j, k)$, we define the r-th derived couple $\mathcal{E}_r = (D_r, E_r, i_r, j_r, k_r)$ as*

$$\begin{aligned}
\mathcal{E}_1 &= \mathcal{E}, \\
\mathcal{E}_{r+1} &= \mathcal{E}_r{}'.
\end{aligned}$$

Lemma 2.2.4. *For $r \geq 2$, we have $E_r \simeq Z_r/B_r$, where $Z_r, B_r \subseteq E$ are defined as*

$$Z_r = k^{-1}(i^{r-1}(D)), \quad B_r = j(\ker i^{r-1})$$

Proof. If $r = 2$,

$$\begin{aligned}
Z_2 &= k^{-1}(i(D)) = k^{-1}(\operatorname{Im} i) = k^{-1}(\ker j) \\
&= \ker(j \circ k) = \ker d, \\
B_2 &= j(\ker i) = j(\operatorname{Im} k) = \operatorname{Im}(j \circ k) = \operatorname{Im} d.
\end{aligned}$$

Therefore

$$E_2 = H(E) = \frac{\ker d}{\operatorname{Im} d} = \frac{Z_2}{B_2}.$$

If $r \geq 2$, by induction we obtain

$$E_{r+1} = \frac{\ker\,(d')^r}{\operatorname{Im}\,(d')^r} \simeq \frac{Z'_r}{B'_r},$$

where

$$\begin{aligned}
Z'_r &= (k')^{-1}(\operatorname{Im} i'^{r-1}) = (k')^{-1}(\operatorname{Im} i^r) \\
&= \{[e] \in E' \mid \exists d \in D \text{ s. t. } k(e) = i^r(d)\} \\
&= Z_{r+1}/\operatorname{Im} d, \\
B'_r &= j'(\ker i'^{r-1}) = j'\{i(d) \in i(D) \mid i^r(d) = 0\} \\
&= \{[j(d)] \in E' \mid i^r(d) = 0\} \\
&= B_{r+1}/\operatorname{Im} d.
\end{aligned}$$

We conclude that

$$E_{r+1} \simeq \frac{Z_{r+1}}{B_{r+1}}.$$

□

Notice that

$$\begin{aligned} Z_{r+1} &= k^{-1}(i^r(D)) \subseteq k^{-1}(i^{r-1}(D)) = Z_r, \\ B_r &= j(\ker i^{r-1}) \subseteq j(\ker i^r) = B_r, \\ B_r &= j(\ker i^{r-1}) \subseteq \operatorname{Im} j = \ker k \subseteq k^{-1}(i^{r-1}(D)) = Z_r. \end{aligned}$$

Lemma 2.2.5. *Given an exact couple $\mathcal{E} = (D, E, i, j, k)$ in which $D = D^{pq}$ and $E = E^{pq}$ are bigraded abelian groups, and i, j and k have bidegrees (a, a'), (b, b') and (c, c'), the induced maps i', j' and k' have bidegrees (a, a'), $(b - a, b' - a')$ and (c, c').*

Proof. Let $d' \in D'^{pq}$, i.e. $d' = i(d)$, for some $d \in D^{p-a,q-a'}$. Since inclusions have bidegree $(0, 0)$, it follows that $i'(d') = i(i(d)) \in D'^{p+a,q+a'}$, so i' has bidegree (a, a').

Let $d' \in D'^{pq}$, with $d' = i(d)$ as before, then $j'(d') = [j(d)]$, and $j(d) \in E^{p+b-a,q+b'-a'}$. Since quotients have bidegree $(0, 0)$, then j' has bidegree $(b - a, b' - a')$.

Let $[e] = e + \operatorname{Im}(j \circ k) \in E'^{pq}$, where $j(k(e)) = 0$, for some $e \in E^{pq}$. Then $k'([e]) = k(e) \in D'^{p-c,q-c'} \subseteq D^{p-c,q-c'}$, and k' has degree (c, c'). □

Definition 2.2.6. *A morphism of exact couples $(\phi, \psi) : \mathcal{E} \to \mathcal{F}$ is a couple of morphisms of abelian groups $\phi : D \to D$ and $\psi : E \to E$ that make the following diagram commute.*

$$\begin{tikzcd}
 & E \arrow[dl, "k"'] \arrow[ddd, "\psi"] & \\
D \arrow[rr, "i"] \arrow[d, "\phi"] & & D \arrow[ul, "j"'] \arrow[d, "\phi"] \\
D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\
 & E' \arrow[ul, "k'"] &
\end{tikzcd}$$

Proposition 2.2.7. *An exact couple $\mathcal{E} = (D, E, i, j, k)$ in which $D = D^{pq}$ and $E = E^{pq}$ are bigraded, and i, j and k have bidegrees $(0, 1)$, $(0, 0)$ and $(1, -1)$ determines a May-type spectral sequence $\{E_r^{pq}\}$ starting with $E_1^{pq} = E^{pq}$. A morphism of exact couples induces a morphism of May-type spectral sequences.*

Proof. By the previous lemma and induction, we see that i_r, j_r and k_r have bidegree $(0, 1)$, $(0, -(r - 1))$ and $(1, -1)$, so the map $d_r : E_r \to E_r$ defined as $d_r := j_r \circ k_r$ has bidegree $(0, -r + 1) + (1, -1) = (1, -r)$.

Let $(\phi, \psi) : \mathcal{E} \to \mathcal{F}$ be a morphism of exact couples. Then

$$\begin{aligned} \psi^{p+1,q} \circ d_1^{pq} &= \psi^{p+1,q} \circ j^{p+1,q} \circ k^{pq} = j'^{p+1,q} \circ \phi^{p+1,q} \circ k^{pq} = j'^{p+1,q} \circ k'^{pq} \circ \psi^{pq} \\ &= d_1'^{pq} \circ \psi^{pq}, \end{aligned}$$

therefore there exist a map $\psi_2 : E_2 \to E_2$ such that $\psi_2 \circ d_2 = d_2' \circ \psi_2$. Iterating we get a morphism $(\psi_r) : (E_r, d_r) \to (E_r', d_r')$ of spectral sequences. □

In the graded case, for $r \geq 2$, $E_r^{pq} \simeq Z_r^{pq}/B_r^{pq}$, where $Z_r^{pq}, B_r^{pq} \subseteq E^{pq}$ are, according to their grading,

$$\begin{aligned} Z_r^{pq} &= (k^{pq})^{-1}(\mathrm{Im}(i^{p+1,q-2} \circ \cdots \circ i^{p+1,q-r})) \\ B_r^{pq} &= j^{pq}(\ker(i^{p,q+r-2} \circ \cdots \circ i^{pq})) \end{aligned}$$

2.3 May-type spectral sequence associated to a filtered complex

Proposition 2.3.1. *Given an ascending filtration on a complex $(A^\bullet, \partial)$, there is an exact couple $\mathcal{E} = (D, E, i, j, k)$ whose maps i, j and k have bidegrees $(0,1)$, $(0,0)$ and $(1,-1)$.*

Proof. The short exact sequence

$$0 \to F^{q-1}A^\bullet \overset{m}{\to} F^qA^\bullet \overset{e}{\to} \mathrm{gr}_q(A^\bullet) \to 0$$

induces a long exact sequence

$$\cdots \to H^p(F^{q-1}A^\bullet) \overset{i}{\to} H^p(F^qA^\bullet) \overset{j}{\to} H^p(\mathrm{gr}_q(A^\bullet)) \overset{k}{\to} H^{p+1}(F^{q-1}A^\bullet) \to \dots$$

where $i = H(m)$, $j = H(e)$ and k is the connecting morphism. Define the bigraded objects D and E as

$$\begin{aligned} D^{pq} &= H^p(F^qA^\bullet), \\ E^{pq} &= H^p(F^qA^\bullet/F^{q-1}A^\bullet). \end{aligned}$$

We then have the exact sequence

$$\cdots \to D^{p,q-1} \xrightarrow[(0,1)]{i^{p,q-1}} D^{pq} \xrightarrow[(0,0)]{j^{pq}} E^{pq} \xrightarrow[(1,-1)]{k^{pq}} D^{p+1,q-1} \to \dots$$

$\square$

We can write each term of our spectral sequence E_r^{pq} in a more explitic way (cfr. [11] Section 3.2).

In this case, $i^{p+1,q-2} \circ \cdots \circ i^{p+1,q-r}$ is the map $H^{p+1}(F^{q-r}A^\bullet) \to H^{p+1}(F^{q-1}A^\bullet)$ induced by the short exact sequence

$$0 \to F^{q-r}A^\bullet \to F^{q-1}A^\bullet \to F^{q-1}A^\bullet/F^{q-r}A^\bullet \to 0.$$

It follows that

$$\begin{aligned} Z_r^{pq} &= (k^{pq})^{-1}(\mathrm{Im}(H^{p+1}(F^{q-r}A^\bullet) \to H^{p+1}(F^{q-1}A^\bullet))) \\ &= (k^{pq})^{-1}(\ker(H^{p+1}(F^{q-1}A^\bullet) \to H^{p+1}(F^{q-1}A^\bullet/F^{q-r}A^\bullet))) \\ &= \ker(H^p(F^qA^\bullet/F^{q-1}A^\bullet) \xrightarrow{k^{pq}} H^{p+1}(F^{q-1}A^\bullet) \to H^{p+1}(F^{q-1}A^\bullet/F^{q-r}A^\bullet)) \\ &= \ker(H^p(F^qA^\bullet/F^{q-1}A^\bullet) \to H^{p+1}(F^{q-1}A^\bullet/F^{q-r}A^\bullet)) \\ &= \mathrm{Im}(H^p(F^qA^\bullet/F^{q-r}A^\bullet) \to H^p(F^qA^\bullet/F^{q-1}A^\bullet)), \end{aligned}$$

where the last two maps are induced by the exact sequence

$$0 \to F^{q-1}/F^{q-r} \to F^q/F^{q-r} \to F^q/F^{q-1} \to 0.$$

In an analogous way, $i^{p,q+r-2} \circ \cdots \circ i^{pq}$ is the map $H^p(F^qA^\bullet) \to H^p(F^{q+r-1}A^\bullet)$, so

$$\begin{aligned}
B_r^{pq} &= j^{pq}(\ker(H^p(F^qA^\bullet) \to H^p(F^{q+r-1}A^\bullet))) \\
&= j^{pq}(\mathrm{Im}(H^{p-1}(F^{q+r-1}A^\bullet/F^qA^\bullet) \to H^p(F^qA^\bullet))) \\
&= \mathrm{Im}(H^{p-1}(F^{q+r-1}A^\bullet/F^qA^\bullet) \to H^p(F^qA^\bullet) \xrightarrow{j^{pq}} H^p(F^qA^\bullet/F^{q-1}A^\bullet)) \\
&= \mathrm{Im}(H^{p-1}(F^{q+r-1}A^\bullet/F^qA^\bullet) \to H^p(F^qA^\bullet/F^{q-1}A^\bullet)) \\
&= \ker(H^p(F^qA^\bullet/F^{q-1}A^\bullet) \to H^p(F^{q+r-1}A^\bullet/F^{q-1}A^\bullet)).
\end{aligned}$$

The following lemma is an immediate application of the first theorem of isomomorphism of groups.

Lemma 2.3.2. *If the first line of the commutative diagram*

$$\begin{tikzcd}
 & D \arrow[d, "\gamma"] \arrow[dr] & \\
A \arrow[ur] \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C
\end{tikzcd}$$

is exact, then β induces an isomorphism

$$\begin{tikzcd}
\mathrm{Im}\,\gamma \arrow[r, "\beta"] \arrow[d] & \mathrm{Im}(\beta \circ \gamma) \\
\dfrac{\mathrm{Im}\,\gamma}{\mathrm{Im}\,\alpha} \arrow[ur, dashed, "\hat{\beta}"'] &
\end{tikzcd}$$

It follows that

Proposition 2.3.3.

$$E_r^{pq} \simeq \mathrm{Im}(H^p(F^qA^\bullet/F^{q-r}A^\bullet) \to H^p(F^{q+r-1}A^\bullet/F^{q-1}A^\bullet)).$$

Proof. It follows from the previous lemma applied to the diagram

$$\begin{tikzcd}
 & H^p(F^qA^\bullet/F^{q-r}A^\bullet) \arrow[d] \arrow[dr] & \\
H^{p-1}(F^{q+r-1}A^\bullet/F^qA^\bullet) \arrow[ur] \arrow[r] & H^p(F^qA^\bullet/F^{q-1}A^\bullet) \arrow[r] & H^p(F^{q+r-1}A^\bullet/F^{q-1}A^\bullet).
\end{tikzcd}$$

□

2.3.1 Convergence

Remark 2.3.4. *If $F^sA^\bullet = 0$ for some s, then $d_r^{pq} = 0$ for all $r \geq q - s - 1$, and the induced spectral sequence is regular.*

Proposition 2.3.5. *The filtration over a chain complex $(A^\bullet, \partial)$, induces, for every p, a filtration over its p-th cohomology group $H^p(A^\bullet, \partial)$ as*

$$F^q H^p(A^\bullet, \partial) = (F^q \ker \partial^p + \operatorname{Im} \partial^{p-1}) / \operatorname{Im} \partial^{p-1} = \operatorname{Im}(H^p(F^q A^\bullet) \to H^p(A^\bullet)).$$

Moreover, such filtration is exhaustive if that on $A^\bullet$ is, and it is bounded from below if that on $A^\bullet$ is.

Proof. The inclusions $\ker(\partial^p : F^q A^p \to F^q A^{p+1}) \subseteq \ker \partial^p$ and $\operatorname{Im}(F^q A^{p-1} \to F^q A^p) \subseteq \operatorname{Im} \partial^{p-1}$ induce a well defined map

$$H^p(F^q A^\bullet) = \frac{\ker(\partial^p : F^q A^p \to F^q A^{p+1})}{\operatorname{Im}(\partial^{p-1} : F^q A^{p-1} \to F^q A^p)} \to \frac{\ker \partial^p}{\operatorname{Im} \partial^{p-1}} = H^p(A^\bullet),$$

whose image is $(F^q \ker \partial^p + \operatorname{Im} \partial^{q+1}) / \operatorname{Im} \partial^{p+1}$. If the filtration over A^p is exhaustive,

$$\bigcup_q F^q H^p(A^\bullet) = \frac{\bigcup_q (F^q \ker \partial^p) + \operatorname{Im} \partial^{p-1}}{\operatorname{Im} \partial^{p-1}} = H^p(A^\bullet).$$

Finally, if $F^q A^\bullet = 0$ for some q,

$$F^q H^p(A^\bullet) = \frac{(F^q \ker \partial^p + \operatorname{Im} \partial^{p-1})}{\operatorname{Im} \partial^{p-1}} = 0.$$

□

Proposition 2.3.6. *Let $(A^\bullet, d)$ be a filtered complex, with exhaustive and bounded from below filtration. Then its associated spectral sequence converges to $H^\bullet(A^\bullet)$.*

Proof.

$$\begin{aligned}
E_\infty^{pq} &= \varinjlim_r E_r^{pq} \\
&= \varinjlim_r (\operatorname{Im}(H^p(F^q A^\bullet / F^{q-r} A^\bullet) \to H^p(F^{q+r-1} A^\bullet / F^{q-1} A^\bullet))) \\
&= \operatorname{Im}(\varinjlim_r H^p(F^q A^\bullet / F^{q-r} A^\bullet) \to \varinjlim_r H^p(F^{q+r-1} A^\bullet / F^{q-1} A^\bullet)) \\
&= \operatorname{Im}(H^p(F^q A^\bullet) \to H^p(A^\bullet / F^{q-1} A^\bullet)).
\end{aligned}$$

By applying the preceding lemma to the diagram

$$\begin{tikzcd}
 & H^p(F^q A^\bullet) \arrow[d] \arrow[dr] & \\
H^p(F^{q-1} A^\bullet) \arrow[r] \arrow[ur] & H^p(A^\bullet) \arrow[r] & H^p(A^\bullet / F^{q-1} A^\bullet),
\end{tikzcd}$$

we obtain the isomorphism

$$E_\infty^{pq} \simeq \frac{F^q H^p(A^\bullet)}{F^{q-1} H^p(A^\bullet)}.$$

We proved that the associated spectral sequence weakly converges to $H^\bullet(A^\bullet)$, which by Proposition 2.3.5 is exhaustive and bounded from below, so it converges to $H^\bullet(A^\bullet)$. □

Remark 2.3.7. *Note that $E_\infty^{pq} \simeq Z_\infty^{pq} / B_\infty^{pq}$, where*

$$Z_\infty^{pq} = \varinjlim_r Z_r^{pq} = \operatorname{Im}\left(H^p\left(F^q A^\bullet\right) \to H^p\left(\frac{F^q A^\bullet}{F^{q-1} A^\bullet}\right)\right),$$

$$B_\infty^{pq} = \varinjlim_r B_r^{pq} = \ker\left(H^p\left(\frac{F^q A^\bullet}{F^{q-1} A^\bullet}\right) \to H^p\left(\frac{A^\bullet}{F^{q-1} A^\bullet}\right)\right).$$

2.4 May-type spectral sequence associated to a differential graded algebra

2.4.1 Differential algebras

Definition 2.4.1. *A differential algebra (E, m, d, ϵ) is an algebra (E, m) together with maps $d, \epsilon : E \to E$ such that*

1. $d^2 = 0$ *(derivation)*
2. $\epsilon^2 = \mathbb{1}_E$ *(involution);*
3. $d(m(x \otimes y)) = m(d(x) \otimes y) + m(\epsilon(x) \otimes d(y))$ *(Leibniz rule).*

Lemma 2.4.2. *Let (E, d, m, ϵ) be a differential algebra. Then $H(A) = \ker d / \operatorname{Im} d$ is an algebra, with multiplication induced by m.*

Proof. Application of the Leibniz rule. □

Definition 2.4.3. *A differential graded algebra $A^\bullet = (A^\bullet, \partial^\bullet, m_{\bullet,\bullet})$, or DGA, is a graded complex equipped with a homogeneous map $m_{\bullet,\bullet} : A^\bullet \otimes A^\bullet \to A^\bullet$ such that, for any $a \in A^n$ and $b \in A^{n'}$,*

$$\partial_{n+n'}(m_{n,n'}(a \otimes b)) = m_{n+1,n'}(\partial_n(a) \otimes b) + (-1)^n m_{n,n'+1}(a \otimes \partial_{n'}(b)).$$

2.4.2 Involutions of exact couples

Typically, multiplicative structures existing within exact couples may not be extended to their derived couples. W. S. Massey, in [4], introduced a condition that serves as a criterion for determining when such extensions are possible.

Definition 2.4.4. *An involution (δ, ϵ) of an exact couple $\mathcal{E} = (D, E, i, j, k)$ is a couple of maps $\delta : D \to D$ and $\epsilon : E \to E$ such that $\delta^2 = \mathbb{1}_D$, $\epsilon^2 = \mathbb{1}_E$ and*

1. $i \circ \delta = \delta \circ i$;
2. $j \circ \delta = \epsilon \circ j$;
3. $k \circ \epsilon = -\delta \circ k$.

Lemma 2.4.5. *If an exact couple $\mathcal{E} = (D, E, i, j, k)$ has an involution, so does its derived couple $\mathcal{E}' = (D', E', i', j', k')$*

Proof. Let (δ, ϵ) be an involution for $\mathcal{E} = (D, E, i, j, k)$. Consider the restriction $\delta : i(D) \to D$. As a consequence of 1), the image of such map is contained in $i(D) = D'$. We may define a map $\delta' : D' \to D'$.

Similarly the image of the restriction of ϵ to $\ker d$ is contained in $\ker d$ by 2) and 3), and therefore it induces a map $\ker d \to E'$. Again by 2), this in turn induces a map $\epsilon' : E' \to E'$. Explicitly, $\epsilon'([e]) = [\epsilon(e)]$.

1′. $[i' \circ \delta'](i(d)) = i(\delta(i(d))) = [\delta' \circ i'](i(d))$.

2′. $[j' \circ \delta'](i(d)) = j'(i(\delta(d))) = [j(\delta(d))] = [\epsilon(j(d))] = \epsilon'([j(d)]) = [\epsilon' \circ j'](i(d))$.

3'. $[k' \circ \epsilon']([e]) = k'([\epsilon(e)]) = k(\epsilon(e)) = -\delta(k(e)) = [-\delta' \circ k]([e])$.

$\square$

Definition 2.4.6. *Consider an exact couple $\mathcal{E} = (D, E, i, j, k)$ with involution (δ, ϵ), together with a map $m : E \otimes E \to E$. We say that $\mathcal{E}$ satisfies μ_n if, for any $x \in E$ such that $k(x) = i^n(a)$ for some $a \in D$ and $y \in E$ such that $k(y) = i^n(b)$ for some $b \in D$, there exists $c \in D$ such that*

1. *$k(m(x \otimes y)) = i^n(c)$;*
2. *$j(c) = m(j(a) \otimes y) + m(\epsilon(x) \otimes j(b))$.*

We say that $\mathcal{E}$ satisfies μ if it satisfies μ_n for every $n \geq 0$.

Lemma 2.4.7. *In the previous notation, $\mathcal{E}$ satisfies μ_0 if and only if (E, d, m, ϵ) is a differential algebra.*

Proof. It follows by noticing that $c = i^0(c) = k(m(x \otimes y))$ and the following two equations.

$$\begin{aligned} d(m(x \otimes y)) &= j(k(m(x \otimes y)), \\ m(d(x) \otimes y) + m(\epsilon(x) \otimes d(y)) &= m(j(k(x)) \otimes y) + m(\epsilon(x) \otimes j(k(y))). \end{aligned}$$

$\square$

Lemma 2.4.8. *Let $\mathcal{E} = (D, E, i, j, k)$ an exact couple with involution (δ, ϵ) such that (E, d, m, ϵ) is a differential algebra. Then $\mathcal{E}$ satisfies μ_n if and only if $\mathcal{E}'$ satisfies μ_{n-1}.*

Proof. Suppose $\mathcal{E}$ satisfies μ_n. Take elements $x', y' \in E'$ and $a', b' \in D'$ such that $k'(x) = (i')^{n-1}(a')$ and $k'(y) = (i')^{n-1}(b')$. It means that there are $x, y \in \ker d$ and $a, b \in D$ such that $x' = [x]$, $y' = [y]$, $a' = i(a)$ and $b' = i(b)$. Therefore

$$\begin{aligned} k(x) &= k'([x]) = (i')^{n-1}(a') = i^n(a), \\ k(y) &= k'([y]) = (i')^{n-1}(b') = i^n(b). \end{aligned}$$

By assumption, there is $c \in D$ such that $k(m(x \otimes y)) = i^n(c)$ and $j(c) = m(j(a) \otimes y) + m(\epsilon(x) \otimes b)$. Define $c' = i(c)$. By construction

$$\begin{aligned} k'(m(x' \otimes y')) &= k'(m([x] \otimes [y])) = k'([m(x \otimes y)]) = k(m(x \otimes y)) \\ &= i^n(c)(i')^{n-1}(c'), \\ j'(c') &= [j(c)] = [m(j(a) \otimes y)] + [m(\epsilon(x) \otimes j(b))] \\ &= m([j(a)] \otimes [y]) + m([\epsilon(x)] \otimes [j(a)]) \\ &= m(j'(a') \otimes y') + m(\epsilon'(x') \otimes j'(a')). \end{aligned}$$

The converse follows reversing the equalities, noticing that if $k(x) = i^n(a)$, then $x \in \ker d$. $\square$

Corollary 2.4.9. *If $\mathcal{E}$ satisfies μ, then $\mathcal{E}^n$ is a differential graded algebra for every $n \geq 0$.*

2.4.3 May-type spectral sequences of algebras

Definition 2.4.10. *A differential graded algebra (E, m, d) is a differential algebra which is a (bi-)graded object, and whose involution map $\epsilon : E \to E$ is $\epsilon(e) = (-1)^{|e|}e$.*

Definition 2.4.11. *A May-type spectral sequence of algebras is a May-type spectral sequence $\{E_r^{pq}, d_r^{pq}\}$ together with graded maps $m_r : E_r^{pq} \otimes E_r^{p'q'} \to E_r^{p+p',q+q'}$ such that, for every $r \geq 0$, we have that (E_r, m, d) is a differential graded algebra and m_{r+1} is induced by m_r as*

$$E_{r+1}^{p,q} \otimes E_{r+1}^{p',q'} = H(E_r^{p,q} \otimes E_r^{p',q'}) \xrightarrow{H(m_r)} H(E_r^{p+p',q+q'}) = E_{r+1}^{p+p',q+q'}$$

where $H(E_r^{p,q} \otimes E_r^{p',q'}) = H(E_r^{p,q}) \otimes H(E_r^{p',q'})$.

Definition 2.4.12. *A we say that a differential graded algebra $A^\bullet = (A^\bullet, \partial, m)$ is filtered if it is endowed with a filtration, as a filtered complex $\{F^qA^\bullet\}$, that agrees with the differential and the multiplication, i.e. for any p, q*

- $\partial^p(F^qA^p) \subseteq F^qA^{p+1}$.
- $m(F^qA^p \otimes F^{q'}A^{p'}) \subseteq F^{q+q'}A^{p+p'}$.

Theorem 2.4.13. *Let $A^\bullet = (A^\bullet, \partial, m)$ be a filtered differential graded algebra. Then its associated spectral sequence is a spectral sequence of algebras.*

Proof. Recall the following.

$$D^{pq} = H^p(F^qA^\bullet), \qquad E^{pq} = H^p\left(\frac{F^qA^\bullet}{F^{q-1}A^\bullet}\right),$$

and $j : D^{pq} \to E^{pq}$ is $j([x]) = [\overline{x}]$, for $x \in F^qA^p$, while $k^{pq} : E^{pq} \to D^{p+1,q-1}$ is $k([\bar{x}]) = [\partial x]$. Also $d^{pq} = k^{p+1,q-1} \circ j^{pq}$. With the previous notation, the following diagram commutes,

$$\begin{tikzcd}
E^{p,q} \otimes E^{p'q'} \arrow[dr, "^E m"] & \\
D^{p+1,q-1} \otimes E^{p'q'} \arrow[u, "j^{p+1,q-1}\otimes 1"] \arrow[r, "^D m"] & E^{p+p'+1,q+q'-1},
\end{tikzcd} \tag{2.1}$$

where the maps "^Dm" and "^Em" are induced by m.

Consider the following diagram.

$$\begin{tikzcd}
 & & E^{p+1,q-1} \otimes E^{p'q'} \arrow[dd, "^E m"] \\
 & D^{p+1,q-1} \otimes E^{p'q'} \arrow[ur, "j^{p+1,q-1}\otimes 1"] \arrow[dr, "^D m"] & \\
E^{p,q} \otimes E^{p'q'} \arrow[ur, "k^{pq}\otimes 1"] \arrow[r, "m"] \arrow[dr, "1\otimes k^{p'q'}"'] & E^{p+p',q+q'} \arrow[r, "d^{p+p',q+q'}"] & E^{p+p'+1,q+q'-1} \\
 & E^{p,q} \otimes D^{p'+1,q'-1} \arrow[ur, "m^D"] \arrow[dr, "1\otimes j^{p'+1,q'-1}"'] & \\
 & & E^{p,q} \otimes E^{p'+1,q'-1} \arrow[uu, "m^E"']
\end{tikzcd}$$

Let $x \in F^q A^p$, $y \in F^{q'} A^{p'}$. Then

$$\begin{aligned}(d^{p+p',q+q'} \circ m)([\overline{x}] \otimes [\overline{y}]) &= d^{p+p',q+q'}([\overline{m(x \otimes y)})] \\ &= j^{p+p'+1,q+q'-1}(k^{p+p',q+q'}([\overline{m(x \otimes y)})])) \\ &= j^{p+p'+1,q+q'-1}([\partial(m(x \otimes y))]) \\ &= j^{p+p'+1,q+q'-1}([m(\partial x \otimes y)] + (-1)^p[m(x \otimes \partial y)]) \\ &= [\overline{(m(\partial x \otimes y))}] + (-1)^p([\overline{m(x \otimes \partial y)}]).\end{aligned}$$

On the other hand,

$$\begin{aligned}({}^E m \circ d^{pq} \otimes 1)([\overline{x}] \otimes [\overline{y}]) &= ({}^E m \circ (j^{p+1,q-1} \otimes 1) \circ (k^{pq} \otimes 1))([\overline{x}] \otimes [\overline{y}]) \\ &= ({}^D m \circ (k^{pq} \otimes 1))([\overline{x} \otimes \overline{y}]) \\ &= {}^D m([\partial x] \otimes [\overline{y}]) = [\overline{m(\partial x \otimes y)}].\end{aligned}$$

Similarly, $(m^E \circ (1 \otimes d^{p'q'}))([\overline{x}] \otimes [\overline{y}]) = m^D([\overline{x}] \otimes [\partial y]) = [\overline{m(x \otimes \partial y)}]$.

We proved that $m : E^{pq} \otimes E^{p'q'} \to E^{p+p',q+q'}$ satisfies the Leibniz rule. We need to show that the induced exact couple satisfies μ.

Let $x \in H^p(F^q A^\bullet / F^{q-1} A^\bullet)$, $y \in H^{p'}(F^{q'} A^\bullet / F^{q'-1} A^\bullet)$, $a \in H^{p+1}(F^{q-n-1} A^\bullet)$ and $b \in H^{p'+1}(F^{q'-n-1} A^\bullet)$ such that $k(x) = i^n(a)$ and $k(y) = i^n(b)$. Let $H(\hat{i}) = i$ and $H(\hat{j}) = j$. Choose representatives $\hat{x} \in F^q A^p$, $\hat{y} \in F^{q'} A^{p'}$, $\hat{a} \in F^{q-n-1} A^{p+1}$ and $\hat{b} \in F^{q'-n-1} A^{p'+1}$. Then $\partial x \in F^{q-1} A^{p+1}$, $\partial y \in F^{q'-1} A^{p'+1}$, $\partial \hat{a} = 0$ and $\partial \hat{b} = 0$. Moreover,

$$[\partial \hat{x}] = k(x) = i^n(a) = [\hat{i}^n(\hat{a})].$$

So there is $z \in F^{q-1} A^p$ such that $\partial \hat{x} = \hat{i}^n(\hat{a}) + \partial z$. Similarly, $\partial \hat{y} = \hat{i}^n(\hat{b}) + \partial w$ for some $w \in F^{q'-1} A^{p'}$. Consider

$$k(m(x \otimes y)) = k(m([\hat{x}] \otimes [\hat{y}])) = k([m(\hat{x} \otimes \hat{y})]) = [\partial m(\hat{x} \otimes \hat{y})].$$

Define

$$\hat{c} = m(\hat{a} \otimes \hat{y}) - (-1)^p m(z \otimes \hat{b}) + (-1)^p m(\hat{x} \otimes \hat{b}) - m(\hat{a} \otimes w).$$

Then

$$\begin{aligned}\partial m(\hat{x} \otimes \hat{y}) =& m(\partial \hat{x} \otimes \hat{y}) + (-1)^p m(\hat{x} \otimes \partial \hat{y}) \\ =& m(\hat{i}^n(\hat{a}) \otimes \hat{y}) + m(\partial z \otimes \hat{y}) + (-1)^p m(\hat{x} \otimes \hat{i}^n(\hat{b})) + (-1)^p m(\hat{x} \otimes \partial w) \\ =& m(\hat{i}^n(\hat{a}) \otimes \hat{y}) + \partial m(z \otimes \hat{y}) - (-1)^p m(z \otimes \partial \hat{y}) \\ &+ (-1)^p m(\hat{x} \otimes \hat{i}^n(\hat{b})) + \partial m(\hat{x} \otimes w) - m(\partial \hat{x} \otimes w) \\ =& m(\hat{i}^n(\hat{a}) \otimes \hat{y}) + \partial m(z \otimes \hat{y}) - (-1)^p m(z \otimes \hat{i}^n(\hat{b})) - (-1)^p m(z \otimes \partial w) \\ &+ (-1)^p m(\hat{x} \otimes \hat{i}^n(\hat{b})) + \partial m(\hat{x} \otimes w) - m(\hat{i}^n(\hat{a}) \otimes w) - m(\partial z \otimes w) \\ &= m(\hat{i}^n(\hat{a}) \otimes \hat{y}) + \partial m(z \otimes \hat{y}) - (-1)^p m(z \otimes \hat{i}^n(\hat{b})) - \partial m(z \otimes w) \\ &+ (-1)^p m(\hat{x} \otimes \hat{i}^n(\hat{b})) + \partial m(\hat{x} \otimes w) - m(\hat{i}^n(\hat{b}) \otimes w) \\ &= \hat{i}^n(\hat{c}) + \partial(m(z \otimes \hat{y}) - m(z \otimes w)),\end{aligned}$$

where, in the last passage, we used the commutativity of the diagram

$$
\begin{tikzcd}
F^{q'+1}A^{p'} \otimes F^{q'}A^{p'} \arrow[drr, "m"] & & \\
F^qA^p \otimes F^{q'}A^{p'} \arrow[u, "\hat{\imath}\otimes 1"] \arrow[d, "1\otimes\hat{\imath}"'] \arrow[r] & F^{q+q'}A^{p+p'} \arrow[r, "\hat{\imath}"] & F^{q+q'+1}A^{p+p'} \\
F^{q'}A^{p'} \otimes F^{q'+1}A^{p'} \arrow[urr, "m"'] & &
\end{tikzcd}
$$

Define $c = [\hat{c}]$. Then

$$k(m(x \otimes y)) = [\hat{\imath}^n(\hat{c})] = i^n([\hat{c}]) = i^n(c).$$

By the commutativity of the diagram

$$
\begin{tikzcd}
\dfrac{F^qA^p}{F^{q-1}A^p} \otimes F^{q'}A^{p'} \arrow[drr, "m"] & & \\
F^qA^p \otimes F^{q'}A^{p'} \arrow[u, "\hat{\jmath}\otimes 1"] \arrow[d, "1\otimes\hat{\jmath}"'] \arrow[r] & F^{q+q'}A^{p+p'} \arrow[r, "\hat{\jmath}"] & \dfrac{F^{q+q'}A^{p'}}{F^{q+q'-1}A^{p+p'}} \\
F^{q'}A^{p'} \otimes \dfrac{F^{q'}A^{p'}}{F^{q'-1}A^{p'}} \arrow[urr, "m"'] & &
\end{tikzcd}
$$

we have

$$
\begin{aligned}
\hat{\jmath}(m(\hat{a} \otimes \hat{y})) &= m(\hat{\jmath}(\hat{a}) \otimes \hat{y}), \\
\hat{\jmath}(m(\hat{x} \otimes \hat{b})) &= m(\hat{x} \otimes \hat{\jmath}(b)).
\end{aligned}
$$

Since $m(z \otimes \hat{b})$ and $m(\hat{a} \otimes w)$ lie in $F^{q+q'-n-2}A^{p+p'+1}$, then

$$\hat{\jmath}(m(z \otimes \hat{b})) = \hat{\jmath}(m(\hat{a} \otimes w)) = 0.$$

We conclude that

$$j(c) = [\hat{\jmath}(\hat{c})] = m(j(a) \otimes y) + (-1)^p m(x \otimes j(b)),$$

and we may apply Corollary 2.4.9.

$\square$

2.4.4 Convergence

If a May-type spectral sequence of algebras (E_r, d_r) is regular, for any p, q

$$E_\infty^{pq} \otimes E_\infty^{p'q'} = \varinjlim_{r} \left(E_r^{pq} \otimes E_r^{p'q'} \right).$$

It follows that there is a unique morphism $m_\infty : E_\infty^{pq} \otimes E_\infty^{p'q'} \to E_\infty^{p+p',q+q'}$ that makes, for all r, the following diagram commute

$$\begin{tikzcd}
E_\infty^{pq} \otimes E_\infty^{p'q'} \arrow[r, "m"] \arrow[d, "\beta^{pq}\otimes\beta^{p'q'}"] & E_\infty^{p+p',q+q'} \arrow[d] \\
E_r^{pq} \otimes E_r^{p'q'} \arrow[r, "m_r"] & E_r^{p+p',q+q'}.
\end{tikzcd}$$

Definition 2.4.14. *Let $H^\bullet = \{H^p\}_{p\in\mathbb{Z}}$ be a family of filtered groups. We say that a regular May-type spectral sequence of algebras (E_r, d_r) converges to $H^\bullet$ as an algebra if it converges to $H^\bullet$ and*

$$\begin{tikzcd}
\mathrm{gr}_q(H^p) \otimes \mathrm{gr}_{q'}(H^{p'}) \arrow[r, "\hat{m}"] \arrow[d, "\beta^{pq}\otimes\beta^{p'q'}"] & \mathrm{gr}_{q+q'}(H^{p+p'}(A^\bullet)) \arrow[d] \\
E_\infty^{pq} \otimes E_\infty^{p'q'} \arrow[r, "m_\infty"] & E_\infty^{p+p',q+q'}.
\end{tikzcd}$$

Lemma 2.4.15. *For $i = 1, 2, 3$, consider the commuting diagrams*

$$\begin{tikzcd}
 & D_i \arrow[d, "\gamma_i"] \arrow[dr] & \\
A_i \arrow[ur] \arrow[r, "\alpha_i"] & B_i \arrow[r, "\beta_i"] & C_i
\end{tikzcd}$$

Suppose the following hold.

1. *The sequences*

$$A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i$$

   are exact for $i = 1, 2, 3$.

2. *there are maps $m_B : B_1 \otimes B_2 \to B_3$, $m_D : D_1 \otimes D_2 \to D_3$ and $m_C : C_1 \otimes C_2 \to C_3$ that make the following diagram commute.*

$$\begin{tikzcd}
D_1 \otimes D_2 \arrow[r, "m_D"] \arrow[d, "\gamma_1\otimes\gamma_2"] & D_3 \arrow[d, "\gamma_3"] \\
B_1 \otimes B_2 \arrow[r, "m_B"] \arrow[d, "\beta_1\otimes\beta_2"] & B_3 \arrow[d, "\beta_3"] \\
C_1 \otimes C_2 \arrow[r, "m_C"] & C_3.
\end{tikzcd}$$

   Then, there is a morphism

$$\hat{m} : \frac{\operatorname{Im}\gamma_1}{\operatorname{Im}\alpha_1} \otimes \frac{\operatorname{Im}\gamma_2}{\operatorname{Im}\alpha_2} \to \frac{\operatorname{Im}\gamma_3}{\operatorname{Im}\alpha_3}$$

that makes the following diagram commute.

$$\begin{tikzcd}
\frac{\operatorname{Im}\gamma_1}{\operatorname{Im}\alpha_1} \otimes \frac{\operatorname{Im}\gamma_2}{\operatorname{Im}\alpha_2} \arrow[r, "\hat{m}"] \arrow[d, "\hat{\beta}_1\otimes\hat{\beta}_2"] & \frac{\operatorname{Im}\gamma_3}{\operatorname{Im}\alpha_3} \arrow[d, "\hat{\beta}_3"] \\
\operatorname{Im}(\beta_1\circ\gamma_1) \otimes \operatorname{Im}(\beta_2\circ\gamma_2) \arrow[r, "m_C"] & \operatorname{Im}(\beta_3\circ\gamma_3)
\end{tikzcd}$$

where, for $i = 1, 2, 3$, the isomorphism $\hat{\beta}_i$ is the unique map that makes the following diagram commute.

$$\begin{tikzcd} \operatorname{Im}\gamma_i \arrow[r, "\beta"] \arrow[d, "\pi_i"] & \operatorname{Im}(\beta_i \circ \gamma_i) \\ \dfrac{\operatorname{Im}\gamma_i}{\operatorname{Im}\alpha_i} \arrow[ur, dashed, "\hat{\beta}_i"'] & \end{tikzcd}.$$

Proof. For $i = 1, 2, 3$, the morphisms $\hat{\beta}_i$ are well defined as the maps that satisfy the following equality.

$$\hat{\beta}_i \circ \pi_i = \beta_i. \tag{$\hat{\beta}_i$}$$

Condition (2) reads

$$\gamma_3 \circ m_D = m_B \circ (\gamma_1 \otimes \gamma_2), \tag{γ}$$
$$\beta_3 \circ m_B = m_C \circ (\beta_1 \otimes \beta_2). \tag{β}$$

The image of the the restriction of m_B to

$$\operatorname{Im}\gamma_1 \otimes \operatorname{Im}\gamma_2 \to B_3$$

is in $\operatorname{Im}\gamma_3$ by (γ). We will denote again by m_B such restriction. We have to prove that $m_B(\operatorname{Im}\alpha_1 \otimes \operatorname{Im}\gamma_2 + \operatorname{Im}\gamma_1 \otimes \operatorname{Im}\alpha_2) \subseteq \operatorname{Im}\alpha_3 = \ker\beta_3$.

Since $\beta_1 \circ \alpha_1 = 0$,

$$\begin{aligned} \beta_3(m_B(\operatorname{Im}\alpha_1 \otimes \operatorname{Im}\gamma_2)) &= \operatorname{Im}(\beta_3 \circ m_B \circ (\alpha_1 \otimes \gamma_2)) \\ &\overset{(\beta)}{=} \operatorname{Im}(m_C \circ (\beta_1 \otimes \beta_2) \circ (\alpha_1 \otimes \gamma_2)) \\ &= \operatorname{Im}(m_C \circ ((\beta_1 \circ \alpha_1) \otimes (\beta_2 \circ \gamma_2)) = 0, \end{aligned}$$

Analogously, $\beta_3(m_B(\operatorname{Im}\gamma_1 \otimes \operatorname{Im}\alpha_2)) = 0$.

So there is a unique map $\hat{m}$ that makes the following diagram commute.

$$\begin{tikzcd} \operatorname{Im}\gamma_1 \otimes \operatorname{Im}\gamma_2 \arrow[r, "m_B"] \arrow[d, "\pi_1 \otimes \pi_2"] & \operatorname{Im}\gamma_3 \arrow[d, "\pi_3"] \\ \dfrac{\operatorname{Im}\gamma_1}{\operatorname{Im}\alpha_1} \otimes \dfrac{\operatorname{Im}\gamma_2}{\operatorname{Im}\alpha_2} \arrow[r, "\hat{m}"] & \dfrac{\operatorname{Im}\gamma_3}{\operatorname{Im}\alpha_3} \end{tikzcd}. \tag{2.2}$$

We need to prove that $m_C \circ (\hat{\beta}_1 \otimes \hat{\beta}_2) = \hat{\beta}_3 \circ \hat{m}$.

$$\begin{aligned} [m_C \circ (\hat{\beta}_1 \otimes \hat{\beta}_2)] \circ (\pi_1 \otimes \pi_2) &\overset{(\hat{\beta}_1, \hat{\beta}_2)}{=} m_C \circ (\beta_1 \otimes \beta_2) \overset{(\beta)}{=} \beta_3 \circ m_B \\ &\overset{(\hat{\beta}_3)}{=} \hat{\beta}_3 \circ \pi_3 \circ m_C \overset{(2.2)}{=} [\hat{\beta}_3 \circ \hat{m}] \circ (\pi_1 \otimes \pi_2). \end{aligned}$$

Since $\pi_1 \otimes \pi_2$ is an epimorphism, we are done. □

Proposition 2.4.16. *Let $(A^\bullet, d)$ be a filtered differential algebra, with exhaustive and bounded from below filtration. Then its associated May-type spectral sequence converges to $H^\bullet(A^\bullet)$ as an algebra.*

Proof. Apply Lemma 2.4.15 to following commuting diagrams.

$$\begin{tikzcd}
 & H^p(F^qA^\bullet) \arrow[d] \arrow[dr] & \\
H^p(F^{q-1}A^\bullet) \arrow[r] \arrow[ur] & H^p(A^\bullet) \arrow[r] & H^p(A^\bullet/F^{q-1}A^\bullet),
\end{tikzcd}$$

$$\begin{tikzcd}
 & H^{p'}(F^{q'}A^\bullet) \arrow[d] \arrow[dr] & \\
H^{p'}(F^{q'-1}A^\bullet) \arrow[r] \arrow[ur] & H^{p'}(A^\bullet) \arrow[r] & H^{p'}(A^\bullet/F^{q'-1}A^\bullet),
\end{tikzcd}$$

$$\begin{tikzcd}
 & H^{p+p'}(F^{q+q'}A^\bullet) \arrow[d] \arrow[dr] & \\
H^{p+p'}(F^{q+q'-1}A^\bullet) \arrow[r] \arrow[ur] & H^{p+p'}(A^\bullet) \arrow[r] & H^{p+p'}(A^\bullet/F^{q+q'-1}A^\bullet),
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^p(F^qA^\bullet) \otimes H^{p'}(F^{q'}A^\bullet) \arrow[r] \arrow[d] & H^{p+p'}(F^{q+q'}A^\bullet) \arrow[d] \\
H^p(A^\bullet) \otimes H^{p'}(A^\bullet) \arrow[r] \arrow[d] & H^{p+p'}(A^\bullet) \arrow[d] \\
H^p(A^\bullet/F^{q-1}A^\bullet) \otimes H^{p'}(A^\bullet/F^{q'-1}A^\bullet) \arrow[r] & H^{p+p'}(A^\bullet/F^{q+q'-1}A^\bullet).
\end{tikzcd}$$

Then Lemma 2.4.15 produces a map

$$\hat{m} : \mathrm{gr}_q(H^p(A^\bullet)) \otimes \mathrm{gr}_{q'}(H^{p'}(A^\bullet)) \to \mathrm{gr}_{q+q'}(H^{p+p'}(A^\bullet))$$

that makes the following diagram commute.

$$\begin{tikzcd}
\mathrm{gr}_q(H^p(A^\bullet)) \otimes \mathrm{gr}_{q'}(H^{p'}(A^\bullet)) \arrow[r, "\hat{m}"] \arrow[d] & \mathrm{gr}_{q+q'}(H^{p+p'}(A^\bullet)) \arrow[d] \\
E_\infty^{pq} \otimes E_\infty^{p'q'} \arrow[r, "m"] & E_\infty^{p+p',q+q'}.
\end{tikzcd}$$

$\square$

Chapter 3

Strongly collapsing pro-p groups

3.1 Strongly collapsing modules

Let A be a complete ring and M a complete A-module. Let $(P_\bullet, \partial)$ be a completely free bounded from above strict A-resolution for M.

For $n \geq 0$ we will denote the n-th group of continuous cohomology ([12], Chapter 6) by

$$\mathrm{Ext}^n_A(M, \mathbb{F}) = H^n(\mathrm{Hom}(P_\bullet, \mathbb{F})).$$

Define the (increasing) discrete bounded from below filtered $\mathbb{F}$-complex $(C^\bullet, d)$ as

$$C^\bullet := \mathrm{Hom}_A(P_\bullet, \mathbb{F}),$$

with the filtration

$$F^q C^p = F^q \mathrm{Hom}_A(P_p, \mathbb{F}) = \left\{ \phi \in \mathrm{Hom}_A(P_p, \mathbb{F}) \mid \phi(F^{q+1} P_p) = 0 \right\}.$$

The following Lemma provides the requirements of Theorem 2.3.6.

Lemma 3.1.1. *For every p the filtration on C^p is bounded from below and exhaustive, and $\mathrm{gr}\, C^p$ is isomorphic to $\mathrm{Hom}_{\mathrm{gr}\, A}(Q_p, \mathbb{F})$.*

Proof. For any p, the filtration on C^p is bounded from below, as that on P_p is bounded from above. To show that it is exhaustive, take a morphism of A-modules $f : P_p \to \mathbb{F}$. Since f is continuous and $\mathbb{F}$ is finite, f factorizes through an element of the basis $F^{q+1} P_p$, i.e. there is a morphism

$$\hat{f} : \frac{P_p}{F^{q+1} P_p} \to \mathbb{F}$$

that makes the following diagram commute

$$\begin{tikzcd} P_p \arrow[r, "f"] \arrow[d, two heads] & \mathbb{F} \\ \frac{P_p}{F^{q+1} P_p} \arrow[ur, dashed, "\hat{f}"'] & \end{tikzcd} .$$

Since $\hat{f} \in F^q C^p$, we established an isomorphism

$$C^p \longrightarrow \bigcup_q F^q C^p,$$
$$f \longmapsto \hat{f}.$$

Consider now a morphism $f : P_p \to \mathbb{F}$ whose image $F^{q+1}P_p$ is zero. It induces a morphism of $\operatorname{gr} A$-modules from $\operatorname{gr}_q C^p$ to $\mathbb{F}$ by

$$\frac{F^q P_p}{F^{q+1}P_p} \to \frac{P^p}{F^{q+1}P^p} \xrightarrow{\hat{f}} \mathbb{F}.$$

All elements of $\operatorname{Hom}^q_{\operatorname{gr} A}(Q_p, \mathbb{F})$ may be obtained this way, so we defined a surjective morphism of $\mathbb{F}$-vector spaces whose kernel consists of those morphisms of A-modules that send $F^q C^p$ to zero, i.e. those that belong to $F^{q-1}C^p$. □

By Proposition 2.3.6 we obtain the following.

Proposition 3.1.2. *Let A be a complete profinite augmented algebra over $\mathbb{F}$, and M be a complete A-module. Then there is a May-type spectral sequence (E_r^{pq}, d_r^{pq}) starting with*

$$E_1^{pq} = \operatorname{Ext}^p_{\operatorname{gr} A}(\operatorname{gr}_q M, \mathbb{F})$$

that converges to $\operatorname{Ext}^p_A(M, \mathbb{F})$.

Definition 3.1.3. *We say that M is a strongly collapsing A-module if the maps*

$$d_1^{pq} : E_1^{pq} \to E_1^{p+1,q-1}$$

are zero for all p, q.

Remark 3.1.4. *M is a strongly collapsing A-module if and only if $\operatorname{Ext}^p_{\operatorname{gr} A}(\operatorname{gr}_q M, \mathbb{F})$ is isomorphic to $\operatorname{Ext}^p_A(M, \mathbb{F})$ as $\mathbb{F}$-vector spaces.*

Proposition 3.1.5. *Let A be a complete profinite augmented algebra over $\mathbb{F}$, and M be a complete A-module. The following are equivalent.*

1. *M is strongly collapsing.*
2. *If $(Q_\bullet, \overline{\partial})$ is a bounded from below locally finite graded free minimal $\operatorname{gr} A$-resolution for $\operatorname{gr} M$, there is a bounded from above completely free minimal A-resolution $(P_\bullet, \partial)$ for M such that $(\operatorname{gr} P_\bullet, \operatorname{gr} \partial) = (Q_\bullet, \overline{\partial})$.*
3. *If $(P_\bullet, \partial)$ is a bounded from above completely free minimal A-resolution for M, the complex $(\operatorname{gr} P_\bullet, \operatorname{gr} \partial)$ is a bounded from below locally finite graded free minimal $\operatorname{gr} A$-resolution for $\operatorname{gr} M$.*

Proof. $3 \Rightarrow 1$. Consider the complex $(C^\bullet, d)$. If we assume that $(P_\bullet, \partial)$ is minimal, so is $(\operatorname{gr} P_\bullet, \operatorname{gr} \partial)$ by hypothesis. By Lemma 1.5.4, that is equivalent to d and $\operatorname{gr} d$ being zero. Therefore $H^{p+1}(F^{q-1}C^\bullet) = 0$ for $p > 0$. It follows that $k_1^{pq} = 0$ for all p, q and so

$$d_1^{pq} = j_1^{p+1,q-1} \circ k_1^{pq} = 0.$$

$1 \Rightarrow 2$. Let $(Q_\bullet, \overline{\partial})$ be a resolution as above. By Proposition 1.3.10 there exists a bounded from above completely free A-resolution $(P_\bullet, \partial)$ for M such that

$$(\operatorname{gr} P_\bullet, \operatorname{gr} \partial) = (Q_\bullet, \overline{\partial}).$$

We need to prove that $(P_\bullet, \partial)$ is minimal. Denote by $(C^\bullet, d)$ the filtered complex $\operatorname{Hom}_A(P_\bullet, \mathbb{F})$, so that $\operatorname{gr} C^\bullet = \operatorname{Hom}_{\operatorname{gr} A}(Q_\bullet, \mathbb{F})$. We need to prove that $d = 0$. Consider the exact couple

$$\begin{tikzcd} D_1 \arrow[rr, "i_1"] & & D_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd}$$

that induced the spectral sequence. By hypothesis, $d_1 = j_1 \circ k_1 = 0$. For all n, it follows that $d_n = j_n \circ k_n = 0$ in the n-th derived exact couple

$$\begin{tikzcd} D_n \arrow[rr, "i_n"] & & D_n \arrow[dl, "j_n"] \\ & E_n \arrow[ul, "k_n"] & \end{tikzcd}$$

By exactness,

$$j_n \circ k_n = 0 \iff \operatorname{Im} k_n \subseteq \ker j_n \iff \ker i_n \subseteq \operatorname{Im} i_n.$$

Since the map $i_1 : H^p(F^q C^\bullet) \to H^p(F^{q+1} C^\bullet)$ is induced by the inclusion $F^q C^\bullet \hookrightarrow F^{q+1} C^\bullet$, the map $i_n : H^p(F^q C^\bullet) \to H^p(F^{q+n} C^\bullet)$ is induced by the inclusion $F^q C^\bullet \hookrightarrow F^{q+n} C^\bullet$.

In the diagram below,

$$\begin{tikzcd} F^{q-n}C^{p-1} \arrow[r, "d"] & F^{q-n}C^p \arrow[r, "d"] \arrow[d, hook] & F^{q-n}C^{p+1} \\ F^qC^{p-1} \arrow[r, "d"] & F^qC^p \arrow[r, "d"] \arrow[d, hook] & F^qC^{p+1} \\ F^{q+n}C^{p-1} \arrow[r, "d"] & F^{q+n}C^p \arrow[r, "d"] & F^{q+n}C^{p+1} \end{tikzcd}$$

take $x \in F^q C^p$ with $dx = 0$, so that $[x] \in H^p(F^q C^\bullet)$. Then

- $[x] \in \ker i_n$ if $\exists y \in F^{q+n} C^{p-1}$ such that $dy = x$,
- $[x] \in \operatorname{Im} i_n$ if $x \in F^{q-n} C^p$.

From the inclusion $\ker i_n \subseteq \operatorname{Im} i_n$, for any n and for any p, q it follows that

$$y \in F^{q+n} C^{p-1} \text{ and } dy \in F^q C^p \implies dy \in F^{q-n} C^p,$$

or, equivalently, that

$$y \in F^q C^{p-1} \text{ and } dy \in F^{q-n} C^p \implies dy \in F^{q-2n} C^p.$$

By iterating, we obtain that, for any p, q

$$y \in F^q C^{p-1} \text{ and } dy \in F^{q-1} C^p \implies dy \in \bigcap_{m \geq 1} F^{q-2^m} C^p,$$

and since the filtration is separated, $dy = 0$. But as the map

$$\operatorname{gr} C^p = \operatorname{Hom}_{\operatorname{gr} A}(Q_p, \mathbb{F}) \to \operatorname{Hom}_{\operatorname{gr} A}(Q_{p-1}, \mathbb{F}) = \operatorname{gr} C^{p-1}$$

is zero, by minimality of $(Q_\bullet, \overline{\partial})$, we obtain that $d(F^q C) = 0$ for any q. By exhaustivity of the filtration, $d = 0$.

$2 \Rightarrow 3$. Let $(P_\bullet, \partial)$ be a bounded from above completely free minimal A-resolution for M. Consider a bounded from below locally finite graded free minimal $\operatorname{gr} A$-resolution $(Q_\bullet, \overline{\partial})$ for $\operatorname{gr} M$. By hypothesis there is a bounded from above completely free minimal A-resolution $(P'_\bullet, \partial')$ for M such that $(\operatorname{gr} P_\bullet, \operatorname{gr} \partial) = (Q_\bullet, \overline{\partial})$. By minimality, $P_\bullet$ and $P'_\bullet$ are isomorphic, and so are $\operatorname{gr} P_\bullet$ and $\operatorname{gr} P'_\bullet$. Since $\operatorname{gr} P'$ is minimal, $\operatorname{gr} P_\bullet$ is minimal as well.

□

Lemma 3.1.6. *Let A_1, A_2 and C be complete profinite augmented algebras such that C, for $i = 1, 2$, is a completely free A_i-module.*

For $i = 1, 2$, let M_i be a strongly collapsing A_i-module.

Then $M \coloneqq M_1 \widehat{\otimes}_{A_1} C \oplus M_2 \widehat{\otimes}_{A_2} C$ is strongly collapsing as a C-module.

Proof. For $i = 1, 2$, let $(Q^i_\bullet, \overline{\partial^i})$ be bounded from below locally finite graded free minimal $\operatorname{gr} A_i$-resolution $\operatorname{gr} M_i$. For $i = 1, 2$, $\operatorname{gr} C$ is a bounded from below locally finite graded free $\operatorname{gr} A_i$-module. By Lemma 1.5.12, the complex

$$(Q^i_\bullet \otimes_{\operatorname{gr} A_i} \operatorname{gr} C, \overline{\partial^i} \otimes 1)$$

is bounded from below locally finite graded free minimal $\operatorname{gr} C$-resolution for $\operatorname{gr} M_i \otimes_{\operatorname{gr} A_i} \operatorname{gr} C$. Set

$$Q_\bullet \coloneqq Q^1_\bullet \otimes_{\operatorname{gr} A_1} \operatorname{gr} C \oplus Q^2_\bullet \otimes_{\operatorname{gr} A_2} \operatorname{gr} C,$$

and $\overline{\partial} = \overline{\partial^1} \otimes 1 + \overline{\partial^2} \otimes 1$. By Lemma 1.5.13, $(Q_\bullet, \overline{\partial})$ is a minimal graded free $\operatorname{gr} C$-resolution for $\operatorname{gr} M$.

For $i = 1, 2$, by hypothesis, there exists a bounded from above completely free minimal A_i-resolution $(P^i_\bullet, \partial^i)$ for M_i such that $(\operatorname{gr} P^i_\bullet, \operatorname{gr} \partial^i) = (Q^i_\bullet, \overline{\partial}_i)$. Therefore, by Lemma 1.5.5, the complex $(P^i_\bullet \otimes_{A_i} C, \partial^i \otimes 1)$ is a bounded from above completely free minimal C-resolution $M_i \otimes_{A_i} C$. Define

$$P_\bullet = P^1_\bullet \otimes_{A_1} C \oplus P^2_\bullet \otimes_{A_2} C.$$

and $\partial = \partial^1 \otimes 1 + \partial^2 \otimes 1$. By Lemma 1.5.6, $(P_\bullet, \partial)$ is a bounded from above completely free minimal C-resolution M. We conclude noticing that $\operatorname{gr} P_\bullet = Q_\bullet$ and $\operatorname{gr} \partial = \overline{\partial}$. □

Lemma 3.1.7. *$\mathbb{F}$ is a strongly collapsing A-module if and only if A^+ is.*

Proof. It is the immediate consequence of Lemma 1.5.3 and Lemma 1.5.9. □

3.1.1 Multiplicative structure

For a completely filtered $\mathbb{F}$-algebra A, we denote by $(\mathrm{Bar}_\bullet(A, M), \partial_\bullet)$ the complete unreduced bar complex

$$\mathrm{Bar}_p(A, M) = A \mathbin{\widehat{\otimes}} \underbrace{A \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} A}_{p \text{ times}} \mathbin{\widehat{\otimes}} M$$

with differentials $d_p : \mathrm{Bar}_p(A) \to \mathrm{Bar}_{p-1}(A)$ defined as

$$\begin{aligned} d_p(a_0 \otimes a_1 \otimes \cdots \otimes a_p \otimes m) = {} & a_0 a_1 \otimes \cdots \otimes a_p \otimes m + \\ & \sum_{i=1}^{p} (-1)^{i+1} a_0 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_p + \\ & (-1)^p a_0 \otimes a_1 \otimes \cdots \otimes a_p m. \end{aligned}$$

It is a completely free bounded from above A-resolution for M, and

$$C^\bullet = \mathrm{Hom}_A(\mathrm{Bar}_\bullet(A, \mathbb{F}), \mathbb{F}))$$

is a filtered differential graded algebra.

Therefore as an immediate application of Theorem 2.4.16 we obtain the following.

Proposition 3.1.8. *Let A be a complete profinite augmented algebra over $\mathbb{F}$. Then there is a May-type spectral sequence of algebras (E_r^{pq}, d_r^{pq}) starting with*

$$E_1^{pq} = \mathrm{Ext}^p_{\mathrm{gr}\, A}(\mathbb{F}, \mathbb{F})$$

that converges, as an algebra, to $\mathrm{Ext}^p_A(\mathbb{F}, \mathbb{F})$.

We obtain the following immediate corollary.

Corollary 3.1.9. *The following are equivalent.*

1. *$\mathbb{F}$ is a strongly collapsing A-module.*

2. *$\mathrm{Ext}^p_{\mathrm{gr}\, A}(\mathbb{F}, \mathbb{F})$ is isomorphic to $\mathrm{Ext}^p_A(\mathbb{F}, \mathbb{F})$ as an algebra.*

3.2 Strongly collapsing pro-p groups

Definition 3.2.1. *We say that a finitely generated pro-p group G is strongly collapsing if $\mathbb{F}$ is strongly collapsing as an $\mathbb{F}[[G]]$-module.*

The previous discussions shows that a finitely generated pro-p group G is strongly collapsing if and only if $H^\bullet(G)$ is isomorphic as an algebra to $H^\bullet(\mathrm{L}(G))$.

Definition 3.2.2. *A graded algebra A is called Koszul if*

$$\mathrm{Ext}^{pq}_A(\mathbb{F}, \mathbb{F}) = 0 \quad \textit{for } p, q \geq 1 \textit{ and } p \neq q.$$

Proposition 3.2.3. *Let G be a finitely generated pro-p group. If $\mathrm{gr}(\mathbb{F}[[G]])$ is Koszul, then G is strongly collapsing.*

Proof. For all $p, q \geq 1$, either $\mathrm{Ext}^{pq}_{\mathrm{gr}(\mathbb{F}[[G]])}(\mathbb{F}, \mathbb{F})$ or $\mathrm{Ext}^{p+1,q-1}_{\mathrm{gr}(\mathbb{F}[[G]])}(\mathbb{F}, \mathbb{F})$ is zero. In either case, the map

$$d_1^{pq} : \mathrm{Ext}^{pq}_{\mathrm{gr}(\mathbb{F}[[G]])}(\mathbb{F}, \mathbb{F}) \to \mathrm{Ext}^{p+1,q-1}_{\mathrm{gr}(\mathbb{F}[[G]])}(\mathbb{F}, \mathbb{F})$$

is zero. $\square$

3.3 Examples

3.3.1 Uniform pro-p groups

Definition 3.3.1. *A finite p-group G is powerful if*

- $G' \leq G^4$, *if* $p = 2$;
- $G' \leq G^p$, *if* $p \geq 3$.

A subgroup $N \leq G$ powerfully embeds in G, or simply p.e. G, if

- $[N, G] \leq N^4$, *if* $p = 2$;
- $[N, G] \leq N^p$, *if* $p \geq 3$.

Remark 3.3.2. *G is powerful iff it p.e. itself.*

Proposition 3.3.3. *If N p.e. G, so does N^p.*

Proof. See [16], Proposition 2.3. □

Corollary 3.3.4. *If G is powerful, then G^{p^k} p.e. G for any $k \geq 0$.*

Proposition 3.3.5. *If G is powerful,*

$$D_n(G) = D_{p^k}(G) = G^{p^k}, \quad p^{k-1} < n \leq p^k.$$

Proof. We assume that p is odd. The case $p = 2$ is analogous.

We first show that $D_{p^k}(G) = G^{p^k}$. Since $D_n = D^p_{\lceil \frac{n}{p} \rceil}[D_{n-1}, G]$, clearly $G^{p^k} \leq D_{p^k}$. For the converse, we proceed by induction on k.

- $k = 0$. Trivial.
- $k > 0$.

$$\begin{aligned} D_{p^{k+1}} &= D^p_{p^k}[D_{p^{k+1}-1}, G] = G^{p^{k+1}}[D_{p^{k+1}-1}, G] \leq G^{p^{k+1}}[D_{p^k}, G] \\ &= G^{p^{k+1}}[G^{p^k}, G] \leq G^{p^{k+1}}, \end{aligned}$$

where in the last passage we used the fact that $G^{p^{k+1}}$ p.e. G.

For $p^k < n \leq p^{k+1}$, we have $D_n(G) \leq D_{p^{k+1}}(G) = G^{p^{k+1}}$. Viceversa, by induction,

$$G^{p^{k+1}} = (G^{p^k})^p = (D_{p^{k-1}+1}(G))^p \leq D_{p^k+p}(G) \leq D_{p^{k+1}}(G).$$

□

Definition 3.3.6. *A pro-p group G is powerful if*

- $G' \leq \overline{G^4}$, *if* $p = 2$;
- $G' \leq \overline{G^p}$, *if* $p \geq 3$.

A subgroup $N \leq_c G$ is powerfully embedded in G if

- $[N, G] \leq \overline{N^4}$, *if* $p = 2$*;*
- $[N, G] \leq \overline{N^p}$, *if* $p \geq 3$.

Remark 3.3.7.

- *G is powerful if and only if it p.e. itself.*
- *$N \leq_c G$ p.e. in G iff NU/U p.e. in G/U for every $U \trianglelefteq_o G$.*

Lemma 3.3.8. *Let G be a pro-p group. For any $n \in \mathbb{N}$,*

$$D_n(G) = \varprojlim_{U \trianglelefteq_o G} D_n\left(\frac{G}{U}\right).$$

Proof. It follows from the fact that, for any $N \leq_c G$ and any $n \in \mathbb{N}$,

$$\overline{N^n} = \varprojlim_{U \trianglelefteq_o G} \left\{\frac{NU}{U}\right\}^n$$

and that, for any $N, H \leq_c G$,

$$[N, H] = \varprojlim_{U \trianglelefteq_o G} \left[\frac{NU}{U}, \frac{HU}{U}\right].$$

□

As a consequence,

Proposition 3.3.9. *If G is powerful pro-p group,*

$$D_n(G) = D_{p^k}(G) = G^{p^k}, \quad p^{k-1} < n \leq p^k.$$

Remark 3.3.10. *The restricted $\mathbb{F}$-Lie algebra associated to G has shape*

$$\mathrm{L}(G) = \coprod_{k \geq 0} \frac{G^{p^k}}{G^{p^{k+1}}}.$$

In particular, it is abelian.

Proposition 3.3.11. *For any k, i, the map $x \mapsto x^{p^k}$ induces an epimorphism*

$$\frac{D_{p^k}(G)}{D_{p^{k+1}}(G)} \to \frac{D_{p^{k+i}}(G)}{D_{p^{k+i+1}}(G)}.$$

Definition 3.3.12. *A powerful pro-p group is uniform if such map is an isomorphism.*

Remark 3.3.13. *As a consequence, for a uniform pro-p group G, the p-restricted Lie algebra power map $-^{[p]} : \mathrm{L}_i(G) \to \mathrm{L}_{pi}(G)$ is an isomorfism.*

It follows that

Proposition 3.3.14. *The graded algebra* $\mathrm{gr}(\mathbb{F}[[G]])$ *associated to a uniform pro-p group G is graded free and commutative.*

Since any graded free and commutative algebra is Koszul, we achieve the following.

Corollary 3.3.15. *If G is a uniform pro-p group, it is strongly collapsing.*

3.3.2 Mild pro-p groups

Let V be an $\mathbb{F}$-vector space of finite dimension and consider the tensor algebra $A = T(V)$. It is a locally finite graded algebra. Let $\mathcal{R} \subseteq A$ be a homogeneous ideal.

Lemma 3.3.16. *The following is an exact sequence of locally finite graded $A/\mathcal{R}$-modules.*

$$0 \to \frac{\mathcal{R}}{\mathcal{R}A^+} \to \frac{A^+}{\mathcal{R}A^+} \to \frac{A}{\mathcal{R}} \to \mathbb{F} \to 0. \tag{3.1}$$

Proof. Consider the short exact sequence of A-modules $0 \to A^+ \to A \to \mathbb{F} \to 0$ and apply the functor $- \otimes_A A/\mathcal{R}$. We obtain the long exact sequence

$$\mathrm{Tor}_1^A(A, A/\mathcal{R}) \to \mathrm{Tor}_1^A(\mathbb{F}, A/\mathcal{R}) \to A^+ \otimes_A A/\mathcal{R} \to A \otimes_A A/\mathcal{R} \to \mathbb{F} \otimes_A A/\mathcal{R} \to 0,$$

for which the following hold:

- $\mathrm{Tor}_1^A(A, A/\mathcal{R}) = 0$, by definition;
- $\mathbb{F} \otimes_A A/\mathcal{R} = \mathbb{F}$;
- $A \otimes_A A/\mathcal{R} = A/\mathcal{R}$;
- $A^+ \otimes_A A/\mathcal{R} = A^+/A^+\mathcal{R}$:
- $\mathrm{Tor}_1^A(A, A/\mathcal{R}) = \ker(A^+/A^+\mathcal{R} \to A/\mathcal{R}) = \mathcal{R}/A^+\mathcal{R}$.

□

Definition 3.3.17. *We say that $\mathcal{R}$ is strongly freely generated by A if $\mathcal{R}/A^+\mathcal{R}$ is a graded free-$A/\mathcal{R}$ module.*

Remark 3.3.18. *If $\mathcal{R}$ is strongly freely generated by A, then* (3.1) *is an $A/\mathcal{R}$-free resolution for $\mathbb{F}$.*

Let G be a finitely generated pro-p group with minimal presentation

$$1 \to R \to F \to G \to 1,$$

i.e. such that $R \leq \Phi(F)$. Consider the filtration $R_n = R \cap D_n(F)$ on R, and define $\mathrm{L}'(R)$ as the locally finite graded restricted Lie algebra induced by the filtration. It induces a short exact sequence of locally finite graded restricted Lie algebras

$$0 \to \mathrm{L}'(R) \to \mathrm{L}(F) \to \mathrm{L}(G) \to 0.$$

Applying the universal enveloping algebra functor, we get a a short exact sequence of locally finite graded algebras

$$0 \to \mathcal{R} \to \mathrm{gr}(\mathbb{F}[[F]]) \to \mathrm{gr}(\mathbb{F}[[G]]) \to 0,$$

where $\mathcal{R}$ is the ideal generated by the image of $\mathrm{L}'(R)$ in $\mathfrak{u}(\mathrm{L}(F)) = \mathrm{gr}(\mathbb{F}_p[[F]]) =: A$. Since A is 1-generated and $R \leq \Phi(F) = D_2(F)$, it follows that $\mathcal{R}$ is 2-generated as an A-module.

Definition 3.3.19. *We say that G is mild if $\mathcal{R}$ is strongly freely generated by A.*

Proposition 3.3.20. *If G is mild, it is strongly collapsing.*

Proof. Since A^+ is 1-generated as an A-module, so is $A^+/A^+\mathcal{R}$ as an $A/\mathcal{R}$-module. Analogously, $\mathcal{R}/A^+\mathcal{R}$ is 2-generated. In addition, $A^+/A^+\mathcal{R}$ is graded free over $A/\mathcal{R}$, for A^+ is free over A, while $\mathcal{R}/A^+\mathcal{R}$ is graded free over $A/\mathcal{R}$ by hypothesis. Therefore (3.1) is an $A/\mathcal{R}$-graded free resolution for $\mathbb{F}$, and we can use it to compute the cohomology groups of the $A/\mathcal{R}$-module $\mathbb{F}$, where $A/\mathcal{R} = \mathrm{gr}(\mathbb{F}[[G]])$. Since the resolution has length 2,

$$\mathrm{Ext}^{pq}_{\mathrm{gr}(\mathbb{F}[[G]])}(\mathbb{F},\mathbb{F}) = 0 \text{ for } p \geq 3.$$

Since $\mathrm{L}(G)$ is 1-generated,

$$\mathrm{Ext}^{1,q}_{\mathrm{gr}(\mathbb{F}[[G]])}(\mathbb{F},\mathbb{F}) = 0 \text{ for } q \geq 2.$$

It follows that for all p, q

$$d_1^{pq} : \mathrm{Ext}^{pq}_{\mathrm{gr}(\mathbb{F}[[G]])}(\mathbb{F},\mathbb{F}) \to \mathrm{Ext}^{p+1,q-1}_{\mathrm{gr}(\mathbb{F}[[G]])}(\mathbb{F},\mathbb{F})$$

is zero. □

3.3.3 Powerful p-central p-groups

In the following, p is an odd prime.

Definition 3.3.21. *A group G is p-central if $G^p \leq Z(G)$.*

Definition 3.3.22. *We say that a p-central p-group has the Ω-extension property, or ΩEP, if there exists a p-central group K such that $K/K^p = G$.*

Remark 3.3.23. *Clearly an abelian p-group G is both powerful and p-central. Moreover, if G is p-elementary abelian, it has the ΩEP, with $K = G$.*

Lemma 3.3.24. *Let G be a powerful p-group, with rank n and exponent s. Then $\mathrm{L}(G)$ has a presentation*

$$\mathrm{L}(G) = \langle x_1, \ldots, x_n \mid x_i^{[p]^{s_i}},\ \ [x_i, x_j] \text{ for } i, j = 1, \ldots, n \rangle.$$

for some $s_i \leq s$.

Proof. By definition,

$$D_r(G) = (D_{\lceil \frac{r}{p} \rceil}(G))^p [G, D_{r-1}(G)].$$

Since G is powerful, $D_{p^k+1}(G) = G^{p^{k+1}} = \cdots = D_{p^k}(G)$, therefore

$$\mathrm{L}_r(G) = \begin{cases} \frac{G^{p^k}}{G^{p^{k+1}}} & r = p^k, \\ 0 & \text{otherwise.} \end{cases}$$

Since p is odd, it is abelian. □

Lemma 3.3.25. *Consider the $\mathbb{F}_p$-algebra*

$$A = \frac{\mathbb{F}_p[x_1, \dots, x_n]}{(x_1^{p^{s_1}}, \dots, x_n^{p^{s_n}})}.$$

Then

$$\mathrm{Ext}_A(\mathbb{F}_p, \mathbb{F}_p) = \Lambda[x_1, \dots, x_n] \otimes \mathbb{F}_p[y_1, \dots, y_n],$$

where the x_i's have degree 1 and the y_i's have degree 2.

Proof. It follows by noticing that A corresponds to the $\mathbb{F}_p$-group algebra of the group

$$G = \mathbb{Z}/p^{s_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{s_n}\mathbb{Z},$$

which has the required cohomology. $\square$

Proposition 3.3.26. *Let G be a powerful p-group. Then*

$$H^\bullet(\mathrm{L}(G)) = \Lambda[x_1, \dots, x_n] \otimes \mathbb{F}_p[y_1, \dots, y_n],$$

where the x_i's have degree 1 and the y_i's have degree 2.

Proof. By Lemma 3.3.24,

$$\mathfrak{u}(L(G)) = \frac{\mathbb{F}_p[x_1, \dots, x_n]}{(x_1^{p^{s_1}}, \dots, x_n^{p^{s_n}})}.$$

Therefore, we conclude applying Lemma 3.3.25. $\square$

T. Weigel provided of classification of finite groups with mod p cohomology isomorphic to that of an abelian group ([14], Theorem A), from which, we can deduce the following result.

Corollary 3.3.27. *Let G be a p-group with $p > 2$, and $n = \dim(\Omega_1(G))$, where*

$$\Omega_1(G) = \langle g \in G \mid g^p = 1 \rangle.$$

The following are equivalent.

1. *$H^\bullet(G) = \Lambda[x_1, \dots, x_n] \otimes \mathbb{F}_p[y_1, \dots, y_n]$, where the x_i's have degree 1 and the y_i's have degree 2.*

2. *G has the ΩEP.*

Since G is strongly collapsing if and only if $H^\bullet(G) \simeq H^\bullet(\mathrm{L}(G))$, we get the following.

Corollary 3.3.28. *Let G be a powerful p-group. The following are equivalent.*

1. *G has the ΩEP.*

2. *G is strongly collapsing.*

3.4 Constructions

Proposition 3.4.1. *If G_1 and G_2 are are strongly collapsing pro-p groups, then $G_1 \times G_2$ and $G_1 \amalg G_2$ are strongly collapsing pro-p groups.*

Proof. By definition of strongly collapsing pro-p group, $\mathbb{F}$ is strongly collapsing as an $\mathbb{F}[[G_1]]$-module and as an $\mathbb{F}[[G_2]]$-module. By Lemma 3.1.7, that is equivalent to saying that $I(\!(G_1)\!)$ is strongly collapsing as an $\mathbb{F}[[G_1]]$-module and $I(\!(G_2)\!)$ is strongly collapsing as an $\mathbb{F}[[G_2]]$-module.

If $G = G_1 \amalg G_2$, its group algebra is $\mathbb{F}[[G]] = \mathbb{F}[[G_1]] \amalg \mathbb{F}[[G_2]]$. Its augmentation ideal is

$$I(\!(G)\!) = I(\!(G_1)\!) \mathbin{\widehat{\otimes}}_{\mathbb{F}[[G_1]]} \mathbb{F}[[G]] \oplus I(\!(G_2)\!) \mathbin{\widehat{\otimes}}_{\mathbb{F}[[G_2]]} \mathbb{F}[[G]].$$

If $G = G_1 \times G_2$, its group algebra is $\mathbb{F}[[G]] = \mathbb{F}[[G_1]] \mathbin{\widehat{\otimes}} \mathbb{F}[[G_2]]$ and its augmentation ideal is

$$\begin{aligned} I(\!(G)\!) &= I(\!(G_1)\!) \mathbin{\widehat{\otimes}} \mathbb{F}[[G_2]] \oplus \mathbb{F}[[G_1]] \mathbin{\widehat{\otimes}} I(\!(G_2)\!) \\ &= I(\!(G_1)\!) \mathbin{\widehat{\otimes}}_{\mathbb{F}[[G_1]]} \mathbb{F}[[G]] \oplus I(\!(G_2)\!) \mathbin{\widehat{\otimes}}_{\mathbb{F}[[G_2]]} \mathbb{F}[[G]]. \end{aligned}$$

In either case, $\mathbb{F}[[G]]$ is completely free as a $\mathbb{F}[[G_1]]$-module and as a $\mathbb{F}[[G_2]]$-module, and we fall under the hypotheses of by Lemma 3.1.6 $I(\!(G)\!)$ is strongly collapsing as a $\mathbb{F}[[G]]$-module. We conclude applying again Lemma 3.1.7. □

Chapter 4

A pro-p generalization of a result by Lichtman

If G_1 and G_2 are pro-p groups, and H is a common (closed) subgroup, their *amalgamated free pro-p product with amalgamation* $G = G_1 \amalg_H^p G_2$ is the universal object that fits the commuting diagram

$$\begin{array}{ccc} H & \longrightarrow & G_1 \\ \downarrow & & \downarrow \\ G_2 & \longrightarrow & G. \end{array}$$

It corresponds to the pro-p completion of their amalgamated free product as abstract groups. Unlike with abstract groups, the induced maps $G_1 \to G$ and $G_2 \to G$ may not be injective, as was shown by L. Ribes ([7]). We say that G is a proper amalgamated free product otherwise.

When H is the trivial subgroup, G is called the *free pro-p product* of G_1 and G_2. Using the fact that for a finitely generated pro-p group G the graded algebras $\mathrm{gr}(\mathbb{F}[[G]])$ and $\mathrm{gr}(\mathbb{F}[G])$ are isomorphic (cfr. [16], chapter 12), Theorem 5 may be immediately adapted to the category of pro-p groups.

Theorem 4.0.1. *Let G_1 and G_2 be finitely generated pro-p groups and let G be free product in the category of pro-p groups. Then $\mathrm{L}(G)$ is isomorphic to the free product of p-restricted Lie algebras $\mathrm{L}(G_1)$ and $\mathrm{L}(G_2)$.*

In this chapter we extend this result to amalgamated free pro-p product under the mild and reasonable condition that the inclusion of the filtration on the amalgam fits those on the factors.

To state this assumption, we introduce the notion of strict embedding, both for abstract groups and for pro-p groups

Definition 4.0.2. *A (closed) subgroup H of G is strictly embedded in G if, for all n,*

$$D_n(H) = D_n(G) \cap H.$$

For finitely generated pro-p groups, that is equivalent to requiring that the inclusion of H in G induces an inclusion of $\mathrm{L}(H)$ in $\mathrm{L}(G)$.

Theorem 4.0.3. *Let H, G_1 and G_2 be finitely generated pro-p groups such that H is strictly embedded in G_1 and G_2, and let $G = G_1 \amalg_H^p G_2$ be their amalgamated free pro-p product. Then the natural morphism of p-restricted Lie algebras*

$$\mathrm{L}(G_1) \amalg_{\mathrm{L}(H)} \mathrm{L}(G_2) \to \mathrm{L}(G).$$

is an isomorphism.

An immediate consequence is that amalgamated free pro-p product is proper.

4.1 Amalgamated free products of algebras

Definition 4.1.1. *Let A_1, A_2 and C be $\mathbb{F}$-algebras, with morphisms $\lambda_1 : C \to A_1$ and $\lambda_2 : C \to A_2$. Consider the algebra*

$$T(A_1, A_2) = \bigoplus_n T_n(A_1, A_2),$$

where

$$T_n(A_1, A_2) = \text{sum of tensor products of } n \text{ terms over } A_1 \text{ and } A_2,$$

and multiplication is linearly defined by juxtaposition of terms.

Let $I \subseteq T(A_1, A_2)$ be the ideal generated by the elements

$$\{a \otimes b - ab,\ \ \lambda_1(c) - \lambda_2(c) \mid a, b \in A_i,\ \ i = 1, 2,\ \ c \in C\}$$

We define the amalgamated free product of A_1, A_2 with respect to C as the $\mathbb{F}$-algebra

$$A_1 \amalg_C A_2 \coloneqq T(A_1, A_2)/I.$$

If we denote by $\psi_1 : A_1 \to A_1 \amalg_C A_2$ and $\psi_2 : A_2 \to A_1 \amalg_C A_2$ the natural morphisms

$$A_1 \hookrightarrow T(A_1, A_2) \twoheadrightarrow A_1 \amalg_C A_2,$$
$$A_2 \hookrightarrow T(A_1, A_2) \twoheadrightarrow A_1 \amalg_C A_2,$$

there is a commuting diagram

$$\begin{tikzcd} C \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"] & A_1 \arrow[d, "\psi_1"] \\ A_2 \arrow[r, "\psi_2"] & A_1 \amalg_C A_2. \end{tikzcd}$$

As an algebra, $A = A_1 \amalg_C A_2$ is characterized by the following universal property. If D is an F-algebra and $\phi_1 : A_1 \to D$ and $\phi_2 : A_2 \to D$ are morphisms such that $\phi_1 \circ i_1 = \phi_2 \circ i_2$, there exists a unique morphism $\phi : A \to D$ such that $\phi_1 = \phi \circ \psi_1$ and $\phi_2 = \phi \circ \psi_2$.

In general, the sole condition that C is a subalgebra of A_1 and A_2 does not guarantee that their amalgamated free product is proper, i.e. that both A_1 and A_2 are subalgebras of A, as the following example shows.

Example 4.1.2. *Consider the algebras C, A_1 and A_2, with $C \subseteq A_1$ and $C \subseteq A_2$, defined by the following presentations.*

$$\begin{aligned} C &\coloneqq \langle c \rangle, \\ A_1 &\coloneqq \langle c, x \mid xc = cx = 0 \rangle, \\ A_2 &\coloneqq \langle c, y \mid yc = cy = 1 \rangle. \end{aligned}$$

Their amalgamated free product

$$A_1 \amalg_C A_2 = \langle c, x, y \mid xc = cx = 0,\ yc = cy = 1 \rangle.$$

is isomorphic to A_2, because

$$x = x(cy) = (xc)y = 0.$$

In particular, A_1 is not a subalgebra of $A_1 \amalg_C A_2$.

However, that is true when both A_1 and A_2 are free over C, as we can establish a normal form-type result.

Proposition 4.1.3. *Let C, A_1 and A_2 be $\mathbb{F}$-algebras, and $\lambda_1 : C \to A_1$ and $\lambda_2 : C \to A_2$ injective morphisms. Suppose, for $i = 1, 2$, that A_i is free as a right C-module, with basis $\mathcal{B}_i$, with $1 \in \mathcal{B}_i$. Consider the tuples*

$$(v_1, \ldots, v_n)$$

where $n \in \mathbf{N}$ and $v_l \in \mathcal{B}_{i_l}$, with $i_l \in \{1, 2\}$ and $v_l \neq 1$, for $l = 1, \ldots, n$. We say that the tuple $(v_1, \ldots, v_n)$ is reduced if $i_l \neq i_{l+1}$ for $l = 1, \ldots, n-1$. We denote by $\mathcal{W}$ the set of reduced tuples together with 1 and define a map $\Psi : \mathcal{W} \to A_1 \amalg_C A_2$ by

$$\Psi(v_1, \ldots, v_n) = \psi_{i_1}(v_1) \ldots \psi_{i_n}(v_n).$$

Then $A_1 \amalg_C A_2$ is a free C-right module, and the image of Ψ forms a C-basis for $A_1 \amalg_C A_2$.

Proof of the Proposition. If we denote by W the free right C-module generated by the reduced words $\mathcal{W}$, we may extend Ψ to a monomorphism of right C-modules $\Psi : W \to A_1 \amalg_C A_2$. We need to prove that such morphism is bijective.

First note that, by definition,

$$\Psi((v_1, \ldots, v_n)c) = \psi_{i_1}(v_1) \ldots \psi_{i_n}(v_n)\psi_1(\lambda_1(c)).$$

By induction on n it can be shown that $\Psi : W \to A_1 \amalg_C A_2$ is surjective, so we need only to prove that it is injective.

We define

$$\mathcal{W}_1 = \{(v_1, \ldots, v_n) \in \mathcal{W} \mid i_n = 2\}$$

and $W_1 \coloneqq \operatorname{span}_C \{\mathcal{W}_1\}$. Consider the map $\theta_1 : \mathcal{W}_1 \otimes \mathcal{B}_1 \to \mathcal{W}$ defined as

$$\theta_1((v_1, \ldots, v_{n-1}) \otimes v) \coloneqq \begin{cases} (v_1, \ldots, v_{n-1}, v) & v \neq 1, \\ (v_1, \ldots, v_{n-1}) & v = 1. \end{cases}$$

It is invertible, with inverse $\psi_1 : \mathcal{W} \to \mathcal{W}_1 \otimes \mathcal{B}_1$ defined as

$$\psi_1(v_1, \dots, v_n) = \begin{cases} (v_1, \dots, v_{n-1}) \otimes v_n & i_n = 1, \\ (v_1, \dots, v_n) \otimes 1 & i_n = 2. \end{cases}$$

Then θ_1 extends to an invertible map of right C-modules $\theta_1 : W_1 \otimes A_1 \to W$. For any $a \in A_1$ we may then construct a linear map $f_{1,a} : W \to W$ in the following way: any element reduced word w has shape $w = \theta_1(w' \otimes v)$, for some $w' \in \mathcal{W}_1$ and $v \in \mathcal{B}_1$. We then define

$$f_{1,a}(wc) \coloneqq \theta_1(w' \otimes v\psi_1(c)a).$$

Also, for any $c \in C$ we define $f_c : W \to W$ as $f_c(wc') = wc'c$.

Such maps satisfy the following properties.

1. $f_{1,a}(\theta_i(w' \otimes a')) = \theta_1(w' \otimes a'a)$. It is a consequence of A_1 being a free right C-module and of the linearity of θ_1.
2. $f_{1,\lambda_1(c)} = f_c$. If $w = \theta_1(w' \otimes v)$,
$$f_{1,\psi_1(c)}(wc') = \theta_1(w' \otimes v\lambda_1(c'c)) = \theta(w' \otimes v)c'c = wc'c = f_c(wc')$$
3. $f_{i,1} = 1_W$. Obvious.
4. $f_{1,aa'} = f_{1,a'} \circ f_{1,a}$. It follows from 1).
5. $f_{1,ra+ra'} = rf_{1,a} + r'f_{1,a'}$. It follows again by linearity of θ_1.

Similarly, for any $a \in A_2$ we may define maps $f_{2,a} : W \to W$ enjoying the same properties. Consider $B = \mathrm{End}_R(W)$. It is an $\mathbb{F}$-algebra, with operations

$$(r_1 f_1 + r_2 f_2)(w) \coloneqq r_1 f_1(w) + r_2 f_2(w),$$
$$f_1 \cdot f_2 \coloneqq f_2 \circ f_1.$$

For given i and $a \in A_i$ we may define $\theta_i : A_i \to B$ as $\theta_i(a) = f_{i,a}$. By 3), 4) and 5) every θ_i is a $\mathbb{F}$-algebra homomorphism. By 2) the following diagram commutes,

$$\begin{array}{ccc} C & \xrightarrow{\lambda_1} & A_1 \\ \big\downarrow{\scriptstyle \lambda_2} & & \big\downarrow{\scriptstyle \theta_1} \\ A_2 & \xrightarrow{\theta_2} & B. \end{array}$$

So by the universal property of the pushout, there is a unique morphism of $\mathbb{F}$-algebras $\rho : A_1 \amalg_C A_2 \to B$ such that $\theta_i = \rho \circ \psi_i$ for $i = 1, 2$. Take a reduced word $w = (v_1, \dots, v_n)$. For any $c \in C$

$$\begin{aligned} wc = (v_1, \dots, v_n)c = f_c(v_1, \dots, v_n) &= f_c(\theta_{i_n}((v_1, \dots, v_{n-1}) \otimes v_n) \\ &= f_c(f_{i_n,v_n}(v_1, \dots, v_{n-1})) = \dots = (f_c \circ f_{i_n,v_n} \circ \dots \circ f_{i_1,v_1})(1) \\ &= (f_{i_1,v_1} \cdot \dots f_{i_n,v_n} \cdot f_c)(1) = \rho(\psi_{i_1}(v_1) \dots \psi_{i_n}(v_n)\psi_1(\lambda_1(c)))(1) \\ &= \rho(\Psi(w)c)(1). \end{aligned}$$

Take now $x \in W$ such that $\Psi(x) = 0$. Write $x = \sum_{w \in \mathcal{W}} wc_w$.

$$0 = \rho(\Psi(x))(1) = \sum_{w \in \mathcal{W}} \rho(\Psi(w)c_w)(1) = \sum_{w \in \mathcal{W}} wc_w = x,$$

and Ψ is injective. $\square$

Corollary 4.1.4. *Under the previous hypotheses, the maps ψ_1 and ψ_2 are injective.*

Proof. Let $i = 1$. Take $a \in A_1$ such that $\psi_1(a) = 0$. If $a = \sum_{v \in \mathcal{B}_1} v c_v$,

$$0 = \psi_1(a) = \sum_{v \in \mathcal{B}_1} \psi_1(v)\psi_1(\lambda_1(c_v)) = \sum_{v \in \mathcal{B}_1} \Psi(v) c_v.$$

Therefore $c_v = 0$ for any $v \in \mathcal{B}_1$, and so $a = 0$. □

As another immediate corollary, we retrieve a known result by Shirshov ([18]).

Corollary 4.1.5. *In the pushout of p-restricted Lie algebras*

$$\begin{tikzcd} L_0 \arrow[r, hook, "\lambda_2"] \arrow[d, hook, "\lambda_1"] & L_1 \arrow[d, "\psi_2"] \\ L_2 \arrow[r, "\psi_1"] & L_1 \amalg_{L_0} L_2, \end{tikzcd}$$

the morphisms ψ_1 and ψ_2 are injective.

Proof. By the p-restricted PBW Theorem,

1. a Lie algebra L is contained in its universal enveloping algebra $\mathfrak{u}(L)$,

2. for any subalgebra $L' \subseteq L$, the algebra $\mathfrak{u}(L)$ is free as a $\mathfrak{u}(L')$-module.

The functor $L \to \mathfrak{u}(L)$ has a right adjoint, therefore it preserves pushouts, and by 1) it is faithful, so it reflects monomorphisms. By 2) we may apply Corollary 4.1.4 to the diagram of algebras

$$\begin{tikzcd} \mathfrak{u}(L_0) \arrow[r, "\mathfrak{u}(\lambda_2)"] \arrow[d, "\mathfrak{u}(\lambda_1)"] & \mathfrak{u}(L_1) \arrow[d, "\mathfrak{u}(\psi_2)"] \\ \mathfrak{u}(L_2) \arrow[r, "\mathfrak{u}(\psi_1)"] & \mathfrak{u}(L_1) \amalg_{\mathfrak{u}(L_0)} \mathfrak{u}(L_2), \end{tikzcd}$$

to obtain that $\mathfrak{u}(\psi_1)$ and $\mathfrak{u}(\psi_2)$ are injective, and so are ψ_1 and ψ_2. □

4.1.1 Amalgamated free product of filtered algebras

Let C, A_1 and A_2 be filtered algebras, and $\lambda_1 : C \to A_1$ and $\lambda_2 : C \to A_2$ filtered free morphisms, with filtered basis $\mathcal{B}_i$, with $1 \in \mathcal{B}_i$. By Proposition 4.1.3 the image of Ψ form a basis for $A_1 \amalg_C A_2$. By Lemma 1.1.17, if we set

$$\nu(\psi_{i_1}(v_1) \dots \psi_{i_n}(v_n)) := \nu_{A_{i_1}}(v_1) + \dots + \nu_{A_{i_n}}(v_n).$$

we can endow $A_1 \amalg_C A_2$ with an algebra filtration in such a way that $\psi_1 \circ \lambda_1 = \psi_2 \circ \lambda_2 : C \to A_1 \amalg_C A_2$ is filtered free, with filtered basis formed by the image of Ψ.

We call ν the standard filtration on $A_1 \amalg_C A_2$.

Lemma 4.1.6. *With the standard filtration, ψ_1 and ψ_2 are strict*

Proof. For $i = 1, 2$, take an element $a \in A_i$, with $a = \sum_{v \in \mathcal{B}_1} v c_v$. Then

$$\begin{aligned} \nu(\psi_i(a)) &= \nu\left(\sum_{v \in \mathcal{B}_i} \psi_i(v)\psi_i(\lambda_i(c_v))\right) = \nu\left(\sum_{v \in \mathcal{B}_i} \Psi(v) c_v\right) \\ &= \inf(\nu_{A_i}(v) + \nu_C(c_v)) = \nu_{A_i}(a). \end{aligned}$$

□

4.1.2 Amalgamated free product of graded algebras

Let C, A_1 and A_2 be graded algebras, and $\lambda_1 : C \to A_1$ and $\lambda_2 : C \to A_2$ graded free morphisms, with homogeneous basis $\mathcal{B}_i$, with $1 \in \mathcal{B}_i$. By Proposition 4.1.3 the image of Ψ form a basis for $A_1 \amalg_C A_2$. We can endow it with a grading

$$\deg(\psi_{i_1}(v_1)\dots\psi_{i_n}(v_n)) \coloneqq \deg_{A_{i_1}}(v_1) + \dots + \deg_{A_{i_n}}(v_n).$$

so that $\psi_1 \circ \lambda_1 = \psi_2 \circ \lambda_2 : C \to A_1 \amalg_C A_2$ is graded free, with homogeneous basis formed by the image of Ψ.

4.2 Proof of Theorem 7

Proposition 4.2.1. *Let C, A_1 and A_2 be filtered algebras, and $\lambda_1 : C \to A_1$ and $\lambda_2 : C \to A_2$ filtered free morphisms. If $B \coloneqq A_1 \amalg_C A_2$ is filtered with the standard filtration, the natural map $\overline{\tau} : \operatorname{gr} A_1 \amalg_{\operatorname{gr} C} \operatorname{gr} A_2 \to \operatorname{gr} B$ is an isomorphism of graded algebras.*

Proof. Consider, for $i = 1, 2$, the set $\mathcal{B}_i$ is a filtered C-basis for λ_i.

Proposition 4.1.3 produces a filtered C-basis $\mathcal{V}$ for $\psi_1 \circ \lambda_1 = \psi_2 \circ \lambda_2$, where B is filtered by the standard filtration.

On the other hand, by Lemma 1.3.3 the set $\overline{\mathcal{B}_i}$ is a graded $\operatorname{gr} C$-basis for $\operatorname{gr} \lambda_i$, and Proposition 4.1.3 produces a graded $\operatorname{gr} C$-basis for $\operatorname{gr} \lambda_1 \circ \psi_1' = \operatorname{gr} \lambda_2 \circ \psi_2'$ $\mathcal{W}$. Since $\psi_1 \circ \lambda_1 = \psi_2 \circ \lambda_2$ is strict, $\operatorname{gr} C$-basis of $\operatorname{gr} \psi_1 \circ \operatorname{gr} \lambda_1 = \operatorname{gr} \psi_2 \circ \operatorname{gr} \lambda_2$ is graded free, by Lemma 1.3.3, with graded $\operatorname{gr} C$-basis $\overline{\mathcal{V}}$. Since $\overline{\tau}$ induces a bijection between $\mathcal{W}$ and $\overline{\mathcal{V}}$, it is an isomorphism. □

Lemma 4.2.2. *Let $\phi : C \to A$ be a morphism of augmented algebras. If ϕ is surjective, it is strict.*

Proof. Since $\varepsilon_C = \varepsilon_A \circ \phi$,

$$C^+ = \ker \varepsilon_C = \ker(\varepsilon_A \circ \phi) = \phi^{-1}(\ker \varepsilon_A) = \phi^{-1}(A^+) \implies \phi(C^+) = A^+.$$

Since ϕ is surjective, $\phi(C^+) = \phi(\phi^{-1}(A^+)) = A^+$ and, for every n,

$$\phi(F^n C) = \phi((C^+)^n) = \phi(C^+)^n = (A^+)^n = F^n A.$$

□

Proposition 4.2.3. *Consider $\lambda_1 : C \to A_1$ and $\lambda_2 : C \to A_2$ be morphisms of augmented algebras and $B \coloneqq A_1 \amalg_C A_2$. Then B is an augmented algebra and ψ_1 and ψ_2 are maps of augmented algebras. If, in addition,*

1. *λ_1 and λ_2 are filtered free,*
2. *$\psi_1(A_1^+)$ and $\psi_2(A_2^+)$ generate B^+,*

then the augmented filtration on B coincides with the standard filtration.

Proof. Since $\varepsilon_{A_1} \circ \lambda_1 = \varepsilon_C = \varepsilon_{A_2} \circ \lambda_2$, by the universal property of the pushout there is a unique map of algebras $\varepsilon : B \to R$ such that $\varepsilon \circ \psi_1 = \varepsilon_{A_1}$ and $\varepsilon \circ \psi_2 = \varepsilon_{A_2}$. So B is augmented and ψ_1 and ψ_2 are augmented. Denote by ν' the filtration induced by the augmentation ideal.

Let us assume now that 1) holds. Denote by ν' the filtration induced by the augmentation ideal and by $\{F^n B\}_{n \geq 0}$ the subalgebras induced by the standard filtration ν, so that, for any $b \in B$

- $\nu(b) \geq n \iff b \in F^n B$, and
- $\nu'(b) \geq n \iff b \in (B^+)^n$.

Since ψ_1 and and ψ_2 are augmented, by Lemma 4.2.2 they are filtered with respect to ν'. Moreover, by Lemma 4.1.6 they are strict with respect to ν. It follows so both $\psi_1(A_1^+)$ and $\psi_2(A_2^+)$ are contained in $F^1 B$. By 2) then $B^+ \subseteq F^1 B$, and

$$(B^+)^n \subseteq (F^1 B)^n \subseteq F^n B,$$

from which it follows that $\nu(b) \geq \nu'(b)$ for any $b \in B$. Combining this fact with

$$\begin{aligned} \nu(\psi_{i_1}(v_1) \dots \psi_{i_n}(v_n)) &= \nu_{A_{i_1}}(v_1) + \cdots + \nu_{A_{i_n}}(v_n), \\ \nu'(\psi_{i_1}(v_1) \dots \psi_{i_n}(v_n)) &\geq \nu'(\psi_{i_1}(v_1))) + \dots \nu'(\psi_{i_n}(v_n))) \\ &\geq \nu_{A_{i_1}}(v_1) + \cdots + \nu_{A_{i_n}}(v_n). \end{aligned}$$

Chaining the inequalities, we obtain that

$$\nu'(\psi_{i_1}(v_1) \dots \psi_{i_n}(v_n)) = \nu_{A_{i_1}}(v_1) + \cdots + \nu_{A_{i_n}}(v_n),$$

therefore $\nu = \nu'$. ☐

Proposition 4.2.4. *Let G be a residually p group and H a subgroup of G. The following are equivalent.*

1. *H is strictly embedded in G.*
2. *The inclusion map $\mathbb{F}[H] \to \mathbb{F}[G]$ is filtered free.*

Proof. 1) holds if and only if the induced map $\mathrm{L}(H) \to \mathrm{L}(G)$ is injective. By the p-restricted Poincaré-Birchoff-Witt Theorem ([9], Theorem 2.5.5.1) and Quillen's Theorem ([6], Theorem 1), that is equivalent to the map $\mathrm{gr}(\mathbb{F}[H]) \to \mathrm{gr}(\mathbb{F}[G])$ being graded free, which by Lemma 1.3.3 is in turn equivalent to 2). ☐

Proof of Theorem 7. Let G' be the abstract amalgamated free product of G_1 and G_2 with respect to H, so that G is the pro-p completion of G', and $\mathrm{gr}(\mathbb{F}[G'])$ is isomorphic to $\mathrm{gr}(\mathbb{F}[[G]])$. Set

$$C = \mathbb{F}[H], \quad A_1 = \mathbb{F}[G_1], \quad A_2 = \mathbb{F}[G_2], \quad B = \mathbb{F}[G'].$$

and let $\lambda_1 : C \to A_1$ and $\lambda_2 : C \to A_2$ be the inclusions, which are filtered free by Proposition 4.2.4, while B is the pushout of A_1 and A_2 with respect to C.

For any element $g \in G'$, by the normal form theorem, $g = g_1 \dots g_k$ for some $g_l \in G_{i_l}$, where $i_l \neq i_{l+1}$ for $l = 1, \dots, k-1$, and possibly $g_k \in H$. Therefore

$$(1 - g) = (1 - g_1) + (1 - g_2 \dots g_k) - (1 - g_1)(1 - g_2 \dots g_k)$$

By induction it follows that $B^+ = I(G')$ is generated by $I(G_1) = A_1^+$ and $I(G_2) = A_2^+$. By Proposition 4.2.3 the augmented filtration on $\mathbb{F}[G]$ coincides with the standard filtration, and by applying Proposition 4.2.1 we establish that the natural map of graded algebras

$$\mathrm{gr}(\mathbb{F}[G_1] \amalg_{\mathrm{gr}(\mathbb{F}[H])} \mathrm{gr}(\mathbb{F}[G_2]) \to \mathrm{gr}(\mathbb{F}[G']).$$

is an isomorphism. It follows that also that the map

$$\mathrm{gr}(\mathbb{F}[[G_1]] \amalg_{\mathrm{gr}(\mathbb{F}[[H]])} \mathrm{gr}(\mathbb{F}[[G_2]]) \to \mathrm{gr}(\mathbb{F}[[G]]).$$

is an isomorphism. ☐

Chapter 5

Filtered formality and genus of a pro-p group

5.1 Filtered formal pro-p groups

Strongly collapsing pro-p groups are characterized by having cohomology isomorphic to that of their p-restricted Lie algebra $\mathrm{L}(G)$. Perhaps the simplest case in which this happens is represented by the following class of pro-p groups.

Definition 5.1.1. *A finitely generated pro-p group G is said to be filtered formal if it admits a strict isomorphism of filtered algebras $\phi : \mathbb{F}[[G]] \to \widehat{\mathrm{gr}}(\mathbb{F}[[G]])$.*

Filtered formal groups were introduced by A. I. Suciu and H. Wang to describe a property of 1-formal groups related to its Malcev Lie algebra (details in [20]).

Proposition 5.1.2. *Let H, G_1 and G_2 be filtered formal pro-p groups such that H is strictly embedded in G_1 and G_2, and let $G = G_1 \amalg_H G_2$ be their amalgamated free product. Then G is filtered-formal.*

Lemma 5.1.3. *Consider, in a category, the following commuting diagram*

$$\begin{tikzcd}
 & B_1 \arrow[r, "b"] \arrow[dl, "h_1"'] & B_2 \arrow[dr, "h_2"] & \\
D_1 & A_1 \arrow[u, "f_1"] \arrow[r, "a"] \arrow[d, "g_1"'] & A_2 \arrow[u, "f_2"'] \arrow[d, "g_2"] & D_2. \\
 & C_1 \arrow[r, "c"] \arrow[ul, "l_1"] & C_2 \arrow[ur, "l_2"'] &
\end{tikzcd}$$

If the left side is a pushout, then there is a unique map $d : D_1 \to D_2$ that makes the diagram commute.

If, in addition, the right side is a pushout and a, b and c are isomorphisms, then d is an isomorphism.

Proof. Diagram chasing. □

Proof of Proposition 5.1.2. The functor that sends a filtered algebra to its completion, being left adjoint to the forgetful functor, preserves pushouts. Therefore the previous morphism induces an isomorphism of $\mathbb{F}$-algebras

$$\mathbb{F}[[G_1]] \amalg_{\mathbb{F}[[H]]} \mathbb{F}[[G_2]] \to \mathbb{F}[[G]].$$

Similarly, 7 induces an isomorphism of complete algebras

$$\widehat{\mathrm{gr}}(\mathbb{F}[G_1]) \amalg_{\widehat{\mathrm{gr}}(\mathbb{F}[H])} \widehat{\mathrm{gr}}(\mathbb{F}[G_2]) \to \widehat{\mathrm{gr}}(\mathbb{F}[G]).$$

By hypothesis, there are also isomorphisms

$$\phi_1 : \mathbb{F}[[G_1]] \to \widehat{\mathrm{gr}}(\mathbb{F}[G_1]),$$
$$\phi_2 : \mathbb{F}[[G_2]] \to \widehat{\mathrm{gr}}(\mathbb{F}[G_2]),$$
$$\psi : \mathbb{F}[[H]] \to \widehat{\mathrm{gr}}(\mathbb{F}[H]).$$

We conclude applying Lemma 5.1.3 to the diagram

$$\begin{tikzcd}
& \mathbb{F}[[G_1]] \arrow[dl] \arrow[r, "\phi_1"] & \widehat{\mathrm{gr}}(\mathbb{F}[G_1]) \arrow[dr] & \\
\mathbb{F}[[G]] & \mathbb{F}[[H]] \arrow[u] \arrow[d] \arrow[r, "\psi"] & \widehat{\mathrm{gr}}(\mathbb{F}[H]) \arrow[u] \arrow[d] & \widehat{\mathrm{gr}}(\mathbb{F}[G]). \\
& \mathbb{F}[[G_2]] \arrow[ul] \arrow[r, "\phi_2"] & \widehat{\mathrm{gr}}(\mathbb{F}[G_2]) \arrow[ur] &
\end{tikzcd}$$

And noticing that the isomorphism it produces $\mathbb{F}[[G]] \to \hat{\mathrm{gr}}(\mathbb{F}[[G]])$ induces the identity on $\mathrm{gr}(\mathbb{F}[[G]])$ by construction. □

5.1.1 Free pro-p groups

The simplest class of filtered formal pro-p groups is formed by finitely free pro-p groups.

Proposition 5.1.4. *Finitely generated free pro-p group are filtered formal.*

Lemma 5.1.5 (([10], Lemma 25))**.** *Let A be an algebra $\mathbb{F}\langle X_1, \dots, X_n\rangle / J \neq 0$, where J is generated by homogeneous polynomial in the x_i's, and $I = (X_1, \dots, X_n)/J$. With respect to the filtration induced by I, the map $x \mapsto \overline{x}$ defines a strict isomorphism of filtered algebras.*

$$A \to \mathrm{gr}\, A.$$

Proof. of the Proposition Let F be a finitely generated pro-p group and $\mathrm{rk}(F) = n$. Then F is the pro-p completion of the free group F' generated over n elements.

Clearly $\mathrm{gr}(\mathbb{F}[[F]]) \simeq \mathrm{gr}(\mathbb{F}[F']) \simeq \mathbb{F}\langle X_1, \dots, X_n\rangle$, and Lemma 5.1.5 for $J = 0$ shows that $\mathrm{gr}(\mathbb{F}[F']) \simeq \mathbb{F}[F']$. Passing to the completions we obtain that $\mathbb{F}[[F]]$ is isomorphic to $\hat{\mathrm{gr}}(\mathbb{F}[[F]])$. □

5.1.2 Uniform abelian pro-p groups

Proposition 5.1.6. *Let G be a finitely generated pro-p group. Then G is uniform and filtered formal if and only if it is abelian and torsion free.*

Proof. If G is abelian and torsion free, $G \simeq \mathbb{Z}_p^n$ where $n = \partial(G)$, and

$$\mathbb{F}[[G]] \simeq \mathbb{F}[[X_1, \dots, X_n]],$$

And clearly $\mathbb{F}[[G]] \simeq \hat{\mathrm{gr}}(\mathbb{F}[[G]])$. Viceversa, if G is uniform, from 3.3.14 it follows that

$$\hat{\mathrm{gr}}(\mathbb{F}[[G]]) \simeq \mathbb{F}[[X_1, \dots, X_n]].$$

Since G is filtered free, $\mathbb{F}[[G]]$ is isomorphic to $\hat{\mathrm{gr}}(\mathbb{F}[[G]])$, therefore $\mathbb{F}[[G]]$ is commutative and torsion free, and so is G. □

5.1.3 Right-angled Artin pro-p groups

Undirected graphs

A *undirected* graph $\ddot{\Gamma} = (V, \ddot{E})$ consists of a non-empty set of *vertices* V and a set of *edges* $\ddot{E} \subseteq \mathcal{P}_2(V)$, where $\mathcal{P}_2(V)$ denotes the set of subsets of V of cardinality 2. If $\ddot{E} = \mathcal{P}_2(V)$ we say that $\ddot{\Gamma}$ is *complete*.

Subgraphs are defined in the obvious way, and we say that a subgraph $\ddot{\Gamma}' = (V', \ddot{E}')$ of an undirected graph $\ddot{\Gamma} = (V, \ddot{E})$ is *induced* if

$$\ddot{E}' = \ddot{E} \cap \mathcal{P}_2(V').$$

We say that such a subgraph is *proper* if $V' \subsetneq V$.

A finite complete, and therefore induced, subgraph with n vertices is called an n-*clique*.

Let $\ddot{\Gamma}_1 = (V_1, \ddot{E}_1)$ and $\ddot{\Gamma}_2 = (V_2, \ddot{E}_2)$ be two undirected graphs, with a common induced proper subgraph $\Lambda = (V', \ddot{E}')$. The patching of $\ddot{\Gamma}_1$ and $\ddot{\Gamma}_2$ along Λ is the graph $\ddot{\Gamma} = (V, \ddot{E})$ with

$$V = V_1 \cup V_2,$$
$$\ddot{E} = \ddot{E}_1 \cup \ddot{E}_2,$$

where we identify the vertices lying in $V_1 \cap V'$ and in $V_2 \cap V'$, and the edges lying in $\ddot{E}_1 \cap \ddot{E}'$ and in $\ddot{E}_2 \cap \ddot{E}'$.

A finite undirected graph $\ddot{\Gamma} = (V, \ddot{E})$ is *chordal* if there are no induced subgraphs of $\ddot{\Gamma}$ which are circuits of length at least 4.

Chordal graphs are characterized by the following property ([17], Prop. 5.5.1).

Proposition 5.1.7. *A finite undirected graph $\ddot{\Gamma}$ is chordal if and only if one of the two occurs.*

- $\ddot{\Gamma}$ *is complete*
- $\ddot{\Gamma}$ *decomposes as patching of two induced proper subgraphs $\ddot{\Gamma}_1$ and $\ddot{\Gamma}_2$, along with a common clique Δ.*

We will always assume graphs to be finite.

Pro-p right-angled Artin groups

Associated to an undirected graph $\ddot{\Gamma} = (V, \ddot{E})$ we can define the following objects:

- the right-angled Artin group $G_{\ddot{\Gamma}}$

$$G_{\ddot{\Gamma}} = \langle V \mid [v, w],\ \ \{v, w\} \in \ddot{E} \rangle,$$

- the right-angled Artin pro-p group $\hat{G}_{\ddot{\Gamma}}$, as the pro-p completion of $G_{\ddot{\Gamma}}$,
- the right-angled Artin p-restricted Lie algebra $L_{\ddot{\Gamma}}$

$$\mathrm{L}_{\ddot{\Gamma}} = \langle V \mid [v, w],\ \ \{v, w\} \in \ddot{E} \rangle,$$

- the right-angled Artin associative algebra $A_{\ddot\Gamma}$

$$A_{\ddot\Gamma} = \left\{ V \mid vw = wv, \ \ \{v, w\} \in \ddot{E} \right\}.$$

Inclusions of induced subgraphs induce split monomorphisms in each category.

We quote some results from [21].

Theorem 5.1.8 (Theorem 1.4i)**.** *$A_{\ddot\Gamma}$ is a Hopf algebra, and the identity $V \to V$ induces a natural isomorphism of Hopf algebras $A_{\ddot\Gamma} \to \mathfrak{u}(\mathrm{L}_{\ddot\Gamma})$.*

The powers of the augmentation ideal define a filtration on $A_{\ddot\Gamma}$.

Theorem 5.1.9 (Theorem 1.3ii)**.** *The map $v \mapsto 1+v$ defines a strict morphism of filtered algebras $\mathbb{F}[[G_{\ddot\Gamma}]] \to \hat{A}_{\ddot\Gamma}$.*

Theorem 5.1.10 (Theorem 1.4ii)**.** *The identity $V \to V$ induces a natural morphism of p-restricted Lie algebras $\mathrm{L}_{\ddot\Gamma} \to \mathrm{L}(G_{\ddot\Gamma})$.*

Proposition 5.1.11. *If $\ddot\Gamma = (V, \ddot{E})$ is an undirected graph, the pro-p group $\hat{G}_{\ddot\Gamma}$ is filtered formal.*

Proof. Theorem 5.1.8 and Theorem 5.1.10 tell us that $A_{\ddot\Gamma}$ id the p-restricted enveloping algebra of $\mathrm{L}(G_{\ddot\Gamma})$. Quillen's isomorphism $\mathfrak{u}(\mathrm{L}(G_{\ddot\Gamma})) \to \mathrm{gr}(\mathbb{F}[[G_{\ddot\Gamma}]])$ provides a strict isomorfism of filtered algebras $\mathrm{gr}(\mathbb{F}[[G_{\ddot\Gamma}]]) \to A_{\ddot\Gamma}$, which induced an isomorphism on their completions

$$\hat{A}_{\ddot\Gamma} \to \hat{\mathrm{gr}}(\mathbb{F}[[G_{\ddot\Gamma}]]).$$

Combining such isomorphism with that of Theorem 5.1.9, we obtain a strict isomorphism of filtered algebras

$$\mathbb{F}[[G]] \to \widehat{\mathrm{gr}}(\mathbb{F}[[G]])$$

$\square$

5.2 Genus of a pro-p group

Definition 5.2.1. *Given an $\mathbf{N}_0$-graded locally finite p-restricted Lie algebra L we define its genus as the set*

$$\mathrm{gen}\,(L) = \{G \ \textit{pro-p group} \mid \mathrm{L}(G) \ \textit{is isomorphic to} \ L\}.$$

Given a finitely generated pro-p group G, we define its genus as the set of pro-p groups $\mathrm{gen}\,(G) = \mathrm{gen}\,(\mathrm{L}(G))$.

Question 6 can be rephrased in the following way.

Question 5.2.2. *Is it true that for any $\mathbf{N}_0$-graded finite dimensional p-restricted Lie algebra L the set $\mathrm{gen}\,(L)$ is non-empty?*

Definition 5.2.3. *A pro-p group G is said to be rigid when $|\,\mathrm{gen}\,(G)\,| = 1$.*

Proposition 5.2.4. *Free pro-p groups are rigid.*

Proof. Let F be the free pro-p group over n generators. It was shown by Lazard that $\mathrm{L}(F)$ is a free p-restricted Lie algebra on n generators (details in [19], Theorem 1.3.8). Viceversa, let G be a finitely generated pro-p group whose associated p-restricted Lie algebra $\mathrm{L}(G)$ is free. In particular, $\mathrm{L}(G)$, so G is strongly collapsing and G has cohomological dimension 1. By a characterization of free pro-p groups ([15], Theorem 4.12), G is free. $\square$

Rigidity is a strong property of a pro-p group, as generally some information is lost in the linearization $G \mapsto \mathrm{L}(G)$. In other cases, we are still able to characterize elements of the genus of a pro-p group in in terms of other properties, such as cohomology.

Proposition 5.2.5. *Let G be a uniform pro-p group. Then*

$$\begin{aligned}\operatorname{gen}(G) &\overset{a)}{=} \{G' \mid G' \text{ is uniform and } \mathrm{d}(G) = \mathrm{d}(G')\}\\ &\overset{b)}{=} \{G' \mid G' \text{ is uniform and } H^\bullet(G) \simeq H^\bullet(G')\}.\end{aligned}$$

Proof. We first prove a). Let G' be a uniform pro-p group with

$$\dim(\mathrm{L}_1(G)) = \mathrm{d}(G) = \mathrm{d}(G') = \dim(\mathrm{L}_1(G')).$$

By Proposition 3.3.11 it follows that $\mathrm{L}(G')$ is isomorphic to $\mathrm{L}(G)$. Viceversa, let G' be a pro-p group and $\alpha : \mathrm{L}(G') \to \mathrm{L}(G)$ an isomorphism of graded p-restricted Lie algebras. For every $i \geq 0$ the following diagram

$$\begin{tikzcd} \mathrm{L}_i(G) \arrow[r, "\alpha_i"] \arrow[d, "-^{[p]}"] & \mathrm{L}_i(G') \arrow[d, "-^{[p]}"] \\ \mathrm{L}_{ip}(G) \arrow[r, "\alpha_{ip}"] & \mathrm{L}_{ip}(G') \end{tikzcd}$$

commutes, and the power map $-^{[p]} : \mathrm{L}_i(G') \to \mathrm{L}_{ip}(G')$ is an isomorphism. We conclude that G' is uniform.

To prove b), we recall that G' is uniform if and only if $H^\bullet(G') = \bigwedge_\bullet H^1(G')$ ([13], Theorem 5.1.5), and that $\dim H^1(G') = \mathrm{d}(G')$, and use part a). $\square$

Corollary 5.2.6. *Let G be a uniform pro-p group and let $G_1, G_2 \in \operatorname{gen}(G)$. If G_1 and G_2 are filtered formal, they are isomorphic.*

Proof. By Proposition 5.1.6, we know that G_1 and G_2 are uniform and $\partial(G_1) = \partial(G_2) = n = \partial(G) =: n$. Since they are filtered formal, by Proposition 5.1.6 they are abelian and torsion free, so they are isomorphic. $\square$

Note that Corollary 5.2.6 gives a positive answer to Question 9 when G is a uniform pro-p group.

Proposition 5.2.7. *Let G be a mild pro-p group with quadratic cohomology. Then*

$$\operatorname{gen}(G) = \{G' \mid G' \text{ is mild and } H^\bullet(G) \simeq H^\bullet(G')\}.$$

Proof. If G is mild and $H^\bullet(G)$ is quadratic, $H^\bullet(G)$ is Koszul and $\mathfrak{u}(\mathrm{L}(G))$ is isomorphic to the quadratic dual of $H^\bullet(G)$ ([23], Theorem 1.3). Since G' is a finitely generated pro-p group of cohomological dimension 2 and $H^\bullet(G')$ is Koszul, G' is mild ([23], Proposition 1.4). Therefore $\mathfrak{u}(\mathrm{L}(G'))$, being isomorphic to quadratic dual of $H^\bullet(G') \simeq H^\bullet(G)$, is isomorphic to $\mathfrak{u}(\mathrm{L}(G))$. $\square$

5.3 Oriented right-angled Artin pro-p groups

5.3.1 Oriented graphs

An *oriented* graph $\Gamma = (V, E)$ consists of a non-empty set of vertices V, which again we assume finite, partitioned as a disjoint union $V = V_s \sqcup V_o$ and a set of edges $E \subseteq V \times V \setminus \Delta(V)$, where

$$\Delta(V) = \{(v, v) \in V \times V \mid v \in V\}.$$

The projections onto the first and the second coordinate define two maps, the *origin* $o : E \to V$ and the *terminus* $t : E \to V$. The set E can be partitioned as $E = E_s \sqcup E_o$, where the set

$$E_s = \{e \in E \mid (t(e), o(e)) \notin E\}$$

is called the set of *special* edges, while E_o is the set of *ordinary* edges. The partition on the vertices has to satisfy the following condition: if $e \in E$, then $o(e) \in V_o$.

We say that an oriented graph Γ is *special* if the terminus of every special edge is a special vertex.

For every oriented graph $\Gamma = (E, V)$ there is an undirected graph $\ddot{\Gamma} = (V, \ddot{E})$, where

$$\ddot{E} = \{\{o(e), v(e)\} \mid e \in E\}.$$

An oriented graph Γ is said to be *chordal* if $\ddot{\Gamma}$ is chordal.

5.3.2 Oriented pro-p groups

Definition 5.3.1. *An oriented pro-p group is a couple (G, θ), where G is a pro-p group and $\theta : G \to \mathbb{Z}_p^\times$ is a morphism, called the orientation of G.*

A morphism of oriented pro-p groups $\phi : (G, \theta) \to (G', \theta')$ is a morphism of pro-p groups such that $\theta = \theta' \circ \phi$.

Amalgamated free pro-p product of oriented pro-p groups

Let (G_1, θ_1) and (G_2, θ_2) be oriented pro-p groups with a common subgroup H on which the orientations agree. The amalgamated free pro-p product $G_1 \amalg_H G_2$ can be endowed with the structure of a pro-p oriented group, whose orientation is induced by the universal property.

$$\begin{tikzcd}
 & H \arrow[dl, hook'] \arrow[dr, hook] & \\
G_1 \arrow[r] \arrow[dr, "\theta_1"] & G_1 \amalg_H G_2 \arrow[d, dashed, "\theta"] & G_2 \arrow[l] \arrow[dl, "\theta_2"'] \\
 & \mathbb{Z}_p^\times &
\end{tikzcd}$$

We say that

$$(G, \theta) = (G_1, \theta_1) \amalg_H (G_2, \theta_2)$$

is the amalgamated free pro-p product of (G_1, θ_1) and (G_2, θ_2) with respect to H.

5.3.3 Oriented pro-p right-angled Artin groups

Given an oriented graph $\Gamma = (E, V)$ and a continuous morphism $\lambda : \mathbb{Z}_p \to \mathbb{Z}_p^\times$, called *linear orientation*, we call the pro-p defined by the pro-p presentation

$$G_{\Gamma,\lambda} = \langle v \in V \mid \ [v, w] = \begin{cases} 1 & \text{if } e \in E_0, \\ v^{1-\lambda(1)} & \text{if } e \in E_s \end{cases} \quad \forall e = (v, w) \in E \rangle.$$

the oriented right-angled Arting pro-p group associated to Γ and λ (cfr. [24]). It naturally carries the structure of oriented pro-p group, with orientation $\theta_{\Gamma,\lambda} : G_{\Gamma,\lambda} \to \mathbb{Z}_p^\times$ given by

$$\theta_{\Gamma,\lambda}(v) = \begin{cases} 1 & \text{if } v \in V_o, \\ \lambda(1) & \text{if } v \in V_s. \end{cases}$$

Proposition 5.3.2 ([25], Proposition 4.11)**.** *Let $\Gamma = (V, E)$ be a specially oriented graph and $\lambda : \mathbb{Z}_p \to \mathbb{Z}_p^\times$ a linear orientation. If $\Delta = (V(\Delta), E(\Delta))$ is a clique of Γ, then the inclusion $V(\Delta) \to V$ induces a monomorphism of oriented pro-p groups*

$$(G_{\Delta,\lambda}, \theta_{\Delta,\lambda}) \to (G_{\Gamma,\lambda}, \theta_{\Gamma,\lambda}).$$

Lemma 5.3.3 ([25], Fact 4.1)**.** *If Γ is the patching of two induced subgraphs Γ_1, Γ_2 along a common induced subgraph Δ and $\lambda : \mathbb{Z}_p \to \mathbb{Z}_p^\times$ any linear orienteation, the associated oriented pro-p group $G_{\Gamma,\lambda}$ decomposes as an oriented free product of with amalgamation of pro-p groups*

$$(G_{\Gamma,\lambda}, \theta_{\Gamma,\lambda}) = (G_{\Gamma_1,\lambda}, \theta_{\Gamma_1,\lambda}) \amalg_{G_{\Delta,\lambda}} (G_{\Gamma_2,\lambda}, \theta_{\Gamma_2,\lambda}).$$

Proposition 5.3.4. *If a special oriented graph Γ is chordal, for any linear orientation λ,*

$$G_{\Gamma,\lambda} \in \operatorname{gen}(G_{\ddot{\Gamma}})$$

Proof. We consider the two cases of Proposition 5.1.7. If Γ is complete, it has at most one special vertex, say v. For any vertex $w \in V$,

$$[v, w] = v^{1-\lambda(1)} \in D^p(G_{\Gamma_\lambda})$$

as $\mathrm{Im}(\lambda) \subseteq 1 + p\mathbb{Z}_p$ (or $\mathrm{Im}(\lambda) \subseteq 1 + 4\mathbb{Z}_p$ if $p = 2$). Then the image of $[v, w]$ in $\mathrm{L}(G_{\Gamma_\lambda})$ is zero, and $\mathrm{L}(G_{\Gamma_\lambda}) \simeq \mathrm{L}_{\ddot{\Gamma}}$.

Otherwise the graph Γ decomposes as the patching of two induced proper subgraphs Γ_1 and Γ_2, along a common clique Δ. By Lemma 5.3.3

$$(G_{\Gamma,\lambda}, \theta_{\Gamma,\lambda}) = (G_{\Gamma_1,\lambda}, \theta_{\Gamma_1,\lambda}) \amalg_{G_{\Delta,\lambda}} (G_{\Gamma_2,\lambda}, \theta_{\Gamma_2,\lambda}).$$

By Proposition 5.3.2 the inclusions $\Delta \hookrightarrow \Gamma_1$ and $\Delta \hookrightarrow \Gamma_2$ induce monomorphisms of oriented pro-p groups $\mu_1 : G_{\Delta,\lambda} \to G_{\Gamma_1,\lambda}$ and $\mu_2 : G_{\Delta,\lambda} \to G_{\Gamma_2,\lambda}$. By inductive hypothesis

$$\mathrm{L}(G_{\Gamma_1,\lambda}) \simeq \mathrm{L}_{\ddot{\Gamma}_1},$$
$$\mathrm{L}(G_{\Gamma_2,\lambda}) \simeq \mathrm{L}_{\ddot{\Gamma}_2},$$
$$\mathrm{L}(G_{\Delta,\lambda}) \simeq \mathrm{L}_{\ddot{\Delta}}.$$

As a consequence, $\mathrm{L}(m_1)$ and $\mathrm{L}(m_2)$ are injective, and $G_{\Delta,\lambda}$ is strictly embedded in $G_{\Gamma_1,\lambda}$ and in $G_{\Gamma_2,\lambda}$.

By theorem 7,

$$\mathrm{L}(G_\lambda) \simeq \mathrm{L}(G_{\Gamma_1,\lambda}) \amalg_{\mathrm{L}(G_{\Delta,\lambda})} \mathrm{L}(G_{\Gamma_2,\lambda}) \simeq \mathrm{L}_{\ddot{\Gamma}_1} \amalg_{\mathrm{L}_{\ddot{\Delta}}} \mathrm{L}_{\ddot{\Gamma}_2} \simeq \mathrm{L}_{\ddot{\Gamma}} \simeq \mathrm{L}(G_{\ddot{\Gamma}}).$$

□

We are now able to provide a partial answer to Question 8.

Corollary 5.3.5. *Let* Γ *be a specially oriented chordal graph, let* $\lambda : \mathbb{Z}_p \to \mathbb{Z}_p^\times$ *be a continuous homomorphism and let* $G_{\Gamma,\lambda}$ *be the oriented pro-p right-angled Artin group associated to* Γ *and* λ*. Then* $\mathrm{L}(G_{\Gamma,\lambda})$ *is quadratic.*

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