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Oriented pro- ℓ groups with the Bogomolov–Positselski property

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Abstract

For a prime number ℓ we say that an oriented pro- ℓ group (G, θ) has the Bogomolov–Positselski property if the kernel of the canonical projection on its maximal θ -abelian quotient $\pi_{G, \theta}^{\text{ab}}: G \rightarrow G(\theta)$ is a free pro- ℓ group contained in the Frattini subgroup of G . We show that oriented pro- ℓ groups of elementary type have the Bogomolov–Positselski property (cf. Theorem 1.2). This shows that Efrat’s Elementary Type Conjecture implies a positive answer to Positselski’s version of Bogomolov’s Conjecture on maximal pro- ℓ Galois groups of a field \mathbb{K} in case that $\mathbb{K}^\times/(\mathbb{K}^\times)^\ell$ is finite. Secondly, it is shown that for an H^\bullet -quadratic oriented pro- ℓ group (G, θ) the Bogomolov–Positselski property can be expressed by the injectivity of the transgression map $d_2^{2,1}$ in the Hochschild–Serre spectral sequence (cf. Theorem 1.4).

Keywords: Maximal pro- ℓ Galois groups, Bogomolov’s Conjecture, Oriented pro- ℓ groups, Kummerian oriented pro- ℓ groups

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1 Introduction

By an ℓ -oriented profinite group for a prime number ℓ we understand a profinite group G together with a continuous homomorphism of profinite groups $\theta: G \rightarrow \mathbb{Z}_\ell^\times$, where \mathbb{Z}_ℓ^\times denotes the group of units of the ring of ℓ -adic integers \mathbb{Z}_ℓ . An ℓ -oriented pro- ℓ group (G, θ) will be simply called an oriented pro- ℓ group. For a field \mathbb{K} , we denote by $G_{\mathbb{K}} = \text{Gal}(\mathbb{K}^{\text{sep}}/\mathbb{K})$ its absolute Galois group, where \mathbb{K}^{sep} denotes a separable closure of \mathbb{K} . For any prime number ℓ , $G_{\mathbb{K}}$ carries naturally the cyclotomic ℓ -orientation $\tilde{\theta}_{\mathbb{K}, \ell}: G_{\mathbb{K}} \rightarrow \mathbb{Z}_\ell^\times$ (cf. Example 2.1 and [26, (1.3)]). The following conjecture formulated by L. Positselski in [20, Conjecture 2] was motivated by an earlier conjecture of F. Bogomolov (cf. [2] and [20, Conjecture 1], see also Remark 3.3).

Conjecture 1.1 *Let \mathbb{K} be a field containing a primitive ℓ th-root of unity, and also $\sqrt{-1}$ if $\ell = 2$, and set*

$${}^{\ell\infty}\sqrt{\mathbb{K}} = \mathbb{K}(\sqrt[n]{a}, a \in \mathbb{K}, n \geq 1).$$

Then the maximal pro- ℓ Galois group of ${}^{\ell\infty}\sqrt{\mathbb{K}}$ is a free pro- ℓ group.

A profinite group G admits a maximal pro- ℓ quotient $G(\ell) = G/O^\ell(G)$, where $O^\ell(G)$ is the closed normal subgroup of G being generated by all pro- q Sylow subgroups for all prime numbers $q \neq \ell$. Apart from $\ker(\theta)$, an oriented pro- ℓ group (G, θ) contains the distinguished closed subgroups

$$K_\theta(G) = \text{cl} \left(\left\langle h^{-\theta(g)}ghg^{-1} \mid g \in G, h \in \ker(\theta) \right\rangle \right) \tag{1.1}$$

and

$$I_\theta(G) = \text{cl} \left(\left\langle h \in \ker(\theta) \mid \exists k \in \mathbb{N}_0 : h^{\ell^k} \in K_\theta(G) \right\rangle \right) \tag{1.2}$$

—the former introduced in [10]—i.e., $I_\theta(G)$ is the closure of the isolator (cf. §66 [12]) of $K_\theta(G)$ in $\ker(\theta)$. An oriented pro- ℓ group (G, θ) is said to be θ -abelian, if the subgroup $K_\theta(G)$ is trivial and if $\ker(\theta)$ is a free abelian pro- ℓ group (in this case G is a free abelian-by-cyclic pro- ℓ group for $\ell \neq 2$, cf. Remark 2.2). By definition $K_\theta(G)$ is a closed normal subgroup of G contained in the Frattini subgroup $\Phi(G) = \text{cl}(G^\ell \cdot [G, G])$ of G . Note that

$$[\ker(\theta), \ker(\theta)] \subseteq K_\theta(G) \subseteq \ker(\theta), \tag{1.3}$$

so that the quotient $\ker(\theta)/K_\theta(G)$ is an abelian pro- ℓ group, and $I_\theta(G)/K_\theta(G)$ is its torsion subgroup. In particular, if $\theta : G \rightarrow \mathbb{Z}_\ell^\times$ is trivial (i.e., θ is identically equal to 1), then $K_\theta(G)$ coincides with the closure of the commutator subgroup of G .

Every oriented pro- ℓ group (G, θ) admits a maximal θ -abelian quotient $(G(\theta), \bar{\theta})$, where $G(\theta) = G/I_\theta(G)$ and $\bar{\theta} : G(\theta) \rightarrow \mathbb{Z}_\ell^\times$ is the homomorphism induced by θ . Namely, $(G(\theta), \bar{\theta})$ is $\bar{\theta}$ -abelian and one has a canonical surjective homomorphism

$$\pi_{G,\theta}^{\text{ab}} : (G, \theta) \longrightarrow (G(\theta), \bar{\theta})$$

of oriented pro- ℓ groups satisfying the following: for every morphism $\psi : (G, \theta) \rightarrow (A, \theta^\circ)$ of oriented pro- ℓ groups onto a θ° -abelian pro- ℓ group (A, θ°) there exists a unique morphism of oriented pro- ℓ groups $\psi_\theta^{\text{ab}} : (G(\theta), \bar{\theta}) \rightarrow (A, \theta^\circ)$ such that $\psi = \psi_\theta^{\text{ab}} \circ \pi_{G,\theta}^{\text{ab}}$ (cf. Proposition 2.3).

The hypothesis of Conjecture 1.1 on the primitive ℓ th-roots lying in \mathbb{K} implies that the maximal pro- ℓ quotient $G_{\mathbb{K}}(\ell)$ of the absolute Galois group $G_{\mathbb{K}}$ carries naturally an ℓ -orientation

$$\tilde{\theta}_{\mathbb{K},\ell} : G_{\mathbb{K}}(\ell) \longrightarrow \mathbb{Z}_\ell^\times. \tag{1.4}$$

So, Conjecture 1.1 predicts that $I_{\tilde{\theta}_{\mathbb{K},\ell}}(G_{\mathbb{K}}(\ell))$ is a free pro- ℓ group contained in the Frattini subgroup $\Phi(G_{\mathbb{K}}(\ell))$ of $G_{\mathbb{K}}(\ell)$ (cf. Proposition 2.6 and § 3.1). At this point it should be mentioned that in fact one has to deal with two properties of oriented pro- ℓ groups. The oriented pro- ℓ group (G, θ) is said to be Kummerian, if $I_\theta(G) = \ker(\pi_{G,\theta}^{\text{ab}})$ is contained in the Frattini subgroup $\Phi(G)$ of G . This property can be reformulated in several different ways (cf. Proposition 2.6). Bearing this fact in mind we say that the Kummerian (cf. §2.3) oriented pro- ℓ group (G, θ) has the Bogomolov–Positselski property, if $I_\theta(G) = \ker(\pi_{G,\theta}^{\text{ab}})$ is a free pro- ℓ group. For example, the oriented pro- ℓ group $(G, \mathbf{1})$, where $\mathbf{1}$ is the trivial ℓ -orientation on G , is Kummerian if, and only if, the maximal abelian pro- ℓ quotient $G^{\text{ab}} = G/G'$ is a free abelian pro- ℓ group, and has the Bogomolov–Positselski property if,

and only if, it is Kummerian and the closure of the commutator subgroup of G is a free pro- ℓ group.

The class of oriented pro- ℓ groups \mathbf{ET}_ℓ of elementary type is the smallest class of oriented pro- ℓ groups containing \mathbb{Z}_ℓ with all its ℓ -orientations, all Demushkin pro- ℓ groups with their natural ℓ -orientation (cf. [26, Proposition 5.2]) and which is closed with respect to free products in the category of oriented pro- ℓ groups and fibre products (cf. Sect. 5.3). The Elementary Type Conjecture formulated by Efrat [8] predicts that for every field \mathbb{K} containing an ℓ th-root of unity (and also $\sqrt{-1}$ if $\ell = 2$) satisfying $|\mathbb{K}^\times/(\mathbb{K}^\times)^\ell| < \infty$ the oriented pro- ℓ group $(G_{\mathbb{K}}(\ell), \tilde{\theta}_{\mathbb{K}, \ell})$ must be of elementary type. The first main purpose of this paper is to establish the following theorem relating the Elementary Type Conjecture with Conjecture 1.1.

Theorem 1.2 *Every oriented pro- ℓ group of elementary type has the Bogomolov–Positselski property.*

From Theorem 1.2 one concludes the following (cf. Proposition 5.13):

Corollary 1.3 *Let \mathbb{K} be a field containing a primitive ℓ th-root of 1 (and also $\sqrt{-1}$ if $\ell = 2$), such that the quotient $\mathbb{K}^\times/(\mathbb{K}^\times)^\ell$ is finite. Then Conjecture 1.1 holds true in the following cases:*

- (a) \mathbb{K} is finite;
- (b) \mathbb{K} is a pseudo algebraically closed (PAC) field, or an extension of relative transcendence degree 1 of a PAC field;
- (c) \mathbb{K} is an extension of transcendence degree 1 of a local field;
- (d) \mathbb{K} is ℓ -rigid (for the definition of ℓ -rigid field see [32, p. 722]);
- (e) \mathbb{K} is an algebraic extension of a global field of characteristic not ℓ .

By the Norm Residue Theorem (cf. [11, 31, 33, 34]), the mod ℓ -Milnor K -ring $K_\bullet^M(\mathbb{K})/\ell$ of a field \mathbb{K} is isomorphic to the cohomology algebra $H^\bullet(G_{\mathbb{K}}(\ell), \mathbb{F}_\ell)$ provided $\ell \neq \text{char}(\mathbb{K})$ and \mathbb{K} contains a primitive ℓ th-root of unity. Moreover, L. Positselski showed in [20, Theorem 1.4] that Conjecture 1.1 is a consequence of a strong Koszulity property of the cohomology algebra $H^\bullet(G_{\mathbb{K}}(\ell), \mathbb{F}_\ell)$.

Our second objective is to establish the following criterion ensuring the Bogomolov–Positselski property of an abstract oriented pro- ℓ group (G, θ) . Surprisingly, it only depends on low-dimensional group cohomology, but in a sophisticated way (cf. Theorem 4.5).

Theorem 1.4 *Let (G, θ) be a Kummerian oriented pro- ℓ group with a quadratic \mathbb{F}_ℓ -cohomology algebra $H^\bullet(G, \mathbb{F}_\ell)$, and let*

$$s: \quad \{1\} \longrightarrow I_\theta(G) \longrightarrow G \longrightarrow G(\theta) \longrightarrow \{1\} \tag{1.5}$$

be the canonical extension of pro- ℓ groups. Then G has the Bogomolov–Positselski property if, and only if, the transgression map

$$d_2^{2,1}: H^2(G(\theta), H^1(I_\theta(G), \mathbb{F}_\ell)) \longrightarrow H^4(G(\theta), \mathbb{F}_\ell) \tag{1.6}$$

is injective.

Remark 1.5 As \mathfrak{s} is a Frattini pro- ℓ cover (i.e., $I_\theta(G)$ is contained in the Frattini subgroup of G , cf. Sect. 3.2), inflation yields an isomorphism $j^1: H^1(G(\theta), \mathbb{F}_\ell) \rightarrow H^1(G, \mathbb{F}_\ell)$. Since $H^\bullet(G, \mathbb{F}_\ell)$ is quadratic, inflation may also be considered as a surjective homomorphism of graded \mathbb{F}_ℓ -algebras

$$j^\bullet: H^\bullet(G(\theta), \mathbb{F}_\ell) \longrightarrow H^\bullet(G, \mathbb{F}_\ell), \tag{1.7}$$

where the left-side term of (1.7) is the exterior algebra generated by $H^1(G(\theta), \mathbb{F}_\ell)$ (cf. Sect. 4.1). By [20, Theorem 1.4], (G, θ) has the Bogomolov–Positselski property provided $H^\bullet(G, \mathbb{F}_\ell)$ is a Koszul \mathbb{F}_ℓ -algebra and $\ker(j^\bullet)$ is a Koszul $H^\bullet(G, \mathbb{F}_\ell)$ -module (cf. [20, §3.3]). Hence the natural question arising in this context is, whether one can express $\ker(d_2^{2,1})$ in terms of $\text{Ext}_{H^\bullet(G, \mathbb{F}_\ell)}^{s,t}(\mathbb{F}_\ell, \ker(j^\bullet))$, $s \neq t$.

2 Oriented pro- ℓ groups

For a pro- ℓ group G and a positive integer n , G^n will denote the closed subgroup of G generated by the n -th powers of all elements of G . Moreover, for two elements $g, h \in G$, we set

$${}^g h = ghg^{-1}, \quad \text{and} \quad [g, h] = {}^g h \cdot h^{-1},$$

and for two subgroups H_1, H_2 of G , $[H_1, H_2]$ will denote the closed subgroup of G generated by all commutators $[g, h]$ with $g \in H_1$ and $h \in H_2$. In particular, G' will denote the closure of the commutator subgroup of G .

2.1 ℓ -Orientations of profinite groups

Let \mathbb{Z}_ℓ denote the ring of ℓ -adic integers, and let \mathbb{Z}_ℓ^\times denote its group of units. Note that \mathbb{Z}_ℓ^\times is a virtual pro- ℓ group, in more detail:

- (a) if $\ell \neq 2$ then the Sylow pro- ℓ subgroup of \mathbb{Z}_ℓ^\times is $1 + \ell\mathbb{Z}_\ell = \{1 + \ell\lambda \mid \lambda \in \mathbb{Z}_\ell\}$, which is free pro- ℓ cyclic;
- (b) if $\ell = 2$ then $\mathbb{Z}_2^\times = 1 + 2\mathbb{Z}_2 \simeq \mathbb{Z}/2 \times (1 + 4\mathbb{Z}_2)$, and the factor $1 + 4\mathbb{Z}_2$ is isomorphic to \mathbb{Z}_2 .

An oriented pro- ℓ group (G, θ) is a pro- ℓ group G together with a continuous group homomorphism $\theta: G \rightarrow \mathbb{Z}_\ell^\times$. Moreover, (G, θ) is said to be torsion-free if $\ell \neq 2$, or if $\ell = 2$ and $\text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$ —observe that in a torsion-free oriented pro- ℓ group (G, θ) , G need not be a torsion free pro- ℓ group, e.g., $(\mathbb{Z}/\ell, \mathbf{1})$ is a torsion-free oriented pro- ℓ as $\text{im}(\mathbf{1}) = \{1\}$.

Oriented pro- ℓ groups were introduced by Efrat [8] under the name “cyclotomic pro- ℓ pairs”. For an oriented pro- ℓ group (G, θ) , $\mathbb{Z}_\ell(1)$ will denote the continuous left $\mathbb{Z}_\ell[[G]]$ -module which is isomorphic to \mathbb{Z}_ℓ as an abelian pro- ℓ group, such that $g \cdot v = \theta(g) \cdot v$ for every $g \in G$ and $v \in \mathbb{Z}_\ell(1)$ (cf. [26, § 1]). Conversely, if a pro- ℓ group G comes endowed with a continuous left $\mathbb{Z}_\ell[[G]]$ -module M which is isomorphic to \mathbb{Z}_ℓ as an abelian pro- ℓ group, then M induces an orientation $\theta: G \rightarrow \mathbb{Z}_\ell^\times$ by $\theta(g) \cdot v = g \cdot v$ for every $g \in G$ and $v \in M$, such that $M \simeq \mathbb{Z}_\ell(1)$.

The fundamental examples of oriented pro- ℓ groups arise in Galois theory (cf. [10, § 4]).

Example 2.1 For a field \mathbb{K} , let $\bar{\mathbb{K}}^{\text{sep}}$ denote a separable closure of \mathbb{K} , and let μ_{ℓ^∞} denote the group of roots of 1 of ℓ -power order lying in $\bar{\mathbb{K}}^{\text{sep}}$. If \mathbb{K} contains a primitive ℓ th-root of

unity, then μ_{ℓ^∞} is contained in the maximal pro- ℓ extension $\mathbb{K}(\ell)$ of \mathbb{K} . As $\mu_{\ell^\infty} \simeq \mathbb{Z}[\frac{1}{\ell}]/\mathbb{Z}$ and $\text{Aut}(\mathbb{Z}[\frac{1}{\ell}]/\mathbb{Z})$ is isomorphic to \mathbb{Z}_ℓ^\times , the action of the maximal pro- ℓ Galois group $G_{\mathbb{K}}(\ell) = \text{Gal}(\mathbb{K}(\ell)/\mathbb{K})$ of \mathbb{K} on μ_{ℓ^∞} fixes the primitive ℓ th-roots of unity, and induces the ℓ -cyclotomic character

$$\tilde{\theta}_{\mathbb{K},\ell}: G_{\mathbb{K}}(\ell) \longrightarrow \mathbb{Z}_\ell^\times.$$

In particular,

$$\sigma(\zeta) = \zeta^{\tilde{\theta}_{\mathbb{K},\ell}(\sigma)} \quad \text{for all } \sigma \in G_{\mathbb{K}}(\ell), \zeta \in \mu_{\ell^\infty}.$$

Furthermore, one has $\text{im}(\tilde{\theta}_{\mathbb{K},\ell}) = 1 + \ell^f \mathbb{Z}_\ell$ —where f is the positive integer satisfying $|\mu_{\ell^\infty} \cap \mathbb{K}^\times| = \ell^f$ —in case $\mu_{\ell^\infty} \cap \mathbb{K}^\times$ is non-empty and finite, and $\text{im}(\tilde{\theta}_{\mathbb{K},\ell}) = \{1\}$ if $\mu_{\ell^\infty} \subseteq \mathbb{K}^\times$. The continuous $G_{\mathbb{K}}(\ell)$ -module $\mathbb{Z}_\ell(1)$ induced by the cyclotomic character is called the 1st Tate twist of \mathbb{Z}_ℓ (cf. [19, Definition 7.3.6]), and for every $n \geq 1$, $\mathbb{Z}_\ell(1)/\ell^n$ is isomorphic to the $G_{\mathbb{K}}(\ell)$ -module of the ℓ^n th roots of 1.

Note that oriented pro- ℓ groups form a category \mathbf{Or}_ℓ , i.e., for $(G, \theta), (H, \theta') \in \text{ob}(\mathbf{Or}_\ell)$ a morphism of oriented pro- ℓ groups $\phi: (G, \theta) \rightarrow (H, \theta')$ is a continuous group homomorphism $\phi: G \rightarrow H$ of pro- ℓ groups satisfying $\theta' \circ \phi = \theta$.

For an oriented pro- ℓ group (G, θ) one has the following constructions.

- (a) Let N be a normal subgroup of G such that $N \subseteq \ker(\theta)$. Then one has an oriented pro- ℓ group

$$(G, \theta)/N := (G/N, \tilde{\theta}),$$

where $\tilde{\theta}: G/N \rightarrow \mathbb{Z}_\ell^\times$ is the orientation induced by θ .

- (b) Let A be an abelian pro- ℓ group. Then one has an oriented pro- ℓ group

$$A \rtimes (G, \theta) := (A \rtimes G, \tilde{\theta}),$$

where $gag^{-1} = a^{\theta(g)}$ for all $g \in G$ and $a \in A$, and $\tilde{\theta} = \theta \circ \pi$, where $\pi: A \rtimes G \rightarrow G$ is the canonical projection.

2.2 The maximal θ -abelian quotient of an oriented pro- ℓ group

Let (G, θ) be a torsion-free oriented pro- ℓ group. Then $G/\ker(\theta) \simeq \text{im}(\theta)$ is torsion-free, and thus either trivial or isomorphic to \mathbb{Z}_ℓ . Therefore, the epimorphism $G \rightarrow G/\ker(\theta)$ splits, and since $ghg^{-1} \equiv h^{\theta(g)} \pmod{K_\theta(G)}$ for every $g \in G$ and $h \in \ker(\theta)$, one concludes that

$$(G, \theta)/K_\theta(G) \simeq \frac{\ker(\theta)}{K_\theta(G)} \rtimes (\text{im}(\theta), \text{Id}_{\text{im}(\theta)}). \tag{2.1}$$

Remark 2.2 By (2.1), if (G, θ) is a torsion-free θ -abelian oriented pro- ℓ group, then it is isomorphic to the oriented pro- ℓ group $\ker(\theta) \rtimes (\text{im}(\theta), \text{Id}_{\text{im}(\theta)})$. Conversely, if A is a free abelian pro- ℓ group, and $(\tilde{G}, \tilde{\theta})$ is an oriented pro- ℓ group satisfying $\ker(\tilde{\theta}) = \{1\}$, then the oriented pro- ℓ group $(G, \tilde{\theta}) = A \rtimes (\tilde{G}, \tilde{\theta})$ is $\tilde{\theta}$ -abelian, since $\ker(\tilde{\theta}) = A$ is a free abelian pro- ℓ group, and as $ghg^{-1} = h^{\tilde{\theta}(g)}$ for every $g \in \tilde{G}$ and $h \in A$ and thus $K_{\tilde{\theta}}(G) = \{1\}$.

Let (G, θ) be an oriented pro- ℓ group. Put $\tilde{G} = G/I_\theta(G)$ and let $\tilde{\theta}: \tilde{G} \rightarrow \mathbb{Z}_\ell^\times$ denote the induced orientation. Since the quotient $\ker(\theta)/I_\theta(G)$ is torsion-free (cf. Sect. 1), the ori-

ented pro- ℓ group $(G(\theta), \bar{\theta})$ is $\bar{\theta}$ -abelian. This group together with the canonical projection

$$\pi_{G,\theta}^{\text{ab}} : G \longrightarrow G(\theta) \tag{2.2}$$

has the following universal property.

Proposition 2.3 *Let (G, θ) be an oriented pro- ℓ group, let (A, θ°) be an oriented θ° -abelian pro- ℓ group, and let $\psi : (G, \theta) \rightarrow (A, \theta^\circ)$ be a continuous homomorphism of oriented pro- ℓ groups. Then ψ factors through $\pi_{G,\theta}^{\text{ab}}$, i.e., there exists a (unique) continuous group homomorphism*

$$\psi_{G,\theta}^{\text{ab}} : (G(\theta), \bar{\theta}) \longrightarrow (A, \theta^\circ)$$

satisfying $\psi = \psi_{G,\theta}^{\text{ab}} \circ \pi_{G,\theta}^{\text{ab}}$.

Proof As ψ is a homomorphism of oriented pro- ℓ groups, and as (A, θ°) is θ° -abelian, one has

$$\psi(\ker(\theta)) \subseteq \ker(\theta^\circ) \quad \text{and} \quad \psi(K_\theta(G)) \subseteq K_{\theta^\circ}(A) = \{1\}. \tag{2.3}$$

As $\ker(\theta^\circ)$ is torsion-free, this implies that $\psi(I_\theta(G)) = \{1\}$. Hence the induced homomorphism $\psi_{G,\theta}^{\text{ab}} : G(\theta) \rightarrow A$ of oriented pro- ℓ groups has the desired properties. \square

Remark 2.4 Let $(G, \theta) \simeq A \rtimes ((G, \theta)/\ker(\theta))$ be a torsion-free θ -abelian oriented pro- ℓ group. Then for every subgroup H of G one has

$$H \simeq (H \cap A) \rtimes (H/\ker(\theta|_H)),$$

and thus the oriented pro- ℓ group $(H, \theta|_H)$ is split $\theta|_H$ -abelian (cf. [26, Remark 3.12]).

2.3 Kummerian oriented pro- ℓ groups

Let (G, θ) be an oriented torsion-free pro- ℓ group. Since $\text{im}(\theta) \subseteq 1 + \ell\mathbb{Z}_\ell$, the action of G on the quotient $\mathbb{Z}_\ell(1)/\ell$ of the continuous G -module $\mathbb{Z}_\ell(1)$ is trivial, i.e., $\mathbb{Z}_\ell(1)/\ell \simeq \mathbb{F}_\ell$ as a trivial left $\mathbb{Z}_\ell[[G]]$ -module. In the proof of the subsequent proposition we will make use of the following

Fact 2.5 *Let A be an abelian pro- ℓ group, and let B be a closed subgroup of A which is a direct summand of A satisfying $B \subseteq A^\ell$. Then $B = \{0\}$.*

Proof Let $A = B \oplus C$. Then $A^\ell = B^\ell \oplus C^\ell$, and as $B^\ell \subseteq B$, and $B \cap C = \{0\}$ one concludes that $B \subseteq B^\ell$, i.e., $B = B^\ell = \Phi(B)$. Hence $B = \{0\}$. \square

A torsion-free oriented pro- ℓ group (G, θ) is said to be Kummerian if the following equivalent properties are satisfied.

Proposition 2.6 *Let (G, θ) be a torsion-free oriented pro- ℓ group. Then the following are equivalent:*

- (i) *the map $H^1(G, \mathbb{Z}_\ell(1)/\ell^n) \rightarrow H^1(G, \mathbb{F}_\ell)$ induced by the epimorphism of discrete left G -modules $\mathbb{Z}_\ell(1)/\ell^n \rightarrow \mathbb{Z}_\ell(1)/\ell \simeq \mathbb{F}_\ell$, is surjective for every $n \geq 1$ (cf. [10]).*
- (ii) *The quotient $\ker(\theta)/K_\theta(G)$ is a free abelian pro- ℓ group.*

- (iii) *The oriented pro- ℓ group $(G, \theta)/K_\theta(G) = (G/K_\theta(G), \bar{\theta})$ is $\bar{\theta}$ -abelian.*
- (iv) *$K_\theta(G)$ is isolated in $\ker(\theta)$, i.e., $I_\theta(G) = K_\theta(G)$.*
- (v) *The group $H_{\text{cts}}^2(G, \mathbb{Z}_\ell(1))$ is a torsion-free \mathbb{Z}_ℓ -module.*
- (vi) *$I_\theta(G) \subseteq \Phi(G)$.*

(Here H_{cts}^* denotes continuous cochain cohomology as defined by Tate [30]).

Proof For G finitely generated the equivalence between (i) and (ii) was shown in [10, Theorem 5.6], and the equivalence between (ii) and (iii) follows from Remark 2.2. For general G the equivalences were shown in [24, Theorem 1.2]. The equivalence between (i) and (v) is shown in [26, Proposition 2.1], and (iii) \Leftrightarrow (iv) is a direct consequence of (2.1) and Remark 2.2. Hence (i)–(v) are equivalent. As $K_\theta(G) \subseteq \Phi(G)$ one has (iv) \Rightarrow (vi). Thus it remains to show that (vi) \Rightarrow (iv). Let $\pi : G \rightarrow G/\Phi(G)$ denote the canonical projection, and let

$$\pi_* : \ker(\theta)/K_\theta(G) \cdot \ker(\theta)^\ell \longrightarrow G/\Phi(G) \tag{2.4}$$

denote the induced map—note that $K_\theta(G) \ker(\theta)^\ell = \ker(\theta)^\ell [G, \ker(\theta)]$, by (1.1). As $\text{im}(\theta)$ —which is isomorphic to either \mathbb{Z}_ℓ or $\{1\}$ —is projective, the 5-term exact sequence associated to the Hochschild–Serre spectral sequence yields an exact sequence

$$H^1(G, \mathbb{F}_\ell) \xrightarrow{\pi_*^\vee} H^1(\ker(\theta), \mathbb{F}_\ell)^G \longrightarrow \{0\} \tag{2.5}$$

Thus, by Pontrjagin duality, π_* is injective. Note that

$$\text{tor}(\ker(\theta)/K_\theta(G)) = I_\theta(G)/K_\theta(G) \tag{2.6}$$

is a direct summand of the abelian pro- ℓ group $\ker(\theta)/K_\theta(G)$ (cf. Sect. 1). Since $\pi(I_\theta(G)) = \{1\}$ by (vi), and since π_* is injective, one concludes that $I_\theta(G) \subseteq K_\theta(G) \cdot \ker(\theta)^\ell$. Hence $I_\theta(G)/K_\theta(G) = \{1\}$ by Fact 2.5. \square

Example 2.7 (a) If (G, θ) is a torsion-free θ -abelian pro- ℓ group, then, by Proposition 2.6-(ii), (G, θ) is Kummerian, as $K_\theta(G) = \{1\}$ and $\ker(\theta)$ is free abelian by definition.

- (b) If G is a free pro- ℓ group, then by Proposition 2.6-(v) the oriented pro- ℓ group (G, θ) is Kummerian for any orientation $\theta : G \rightarrow \mathbb{Z}_\ell^\times$, as $\text{cd}(G) = 1$ (cf. [19, Proposition 3.5.17]).
- (c) If (G, θ) is an oriented pro- ℓ group with trivial orientation $\theta \equiv \mathbf{1}$, then (G, θ) is Kummerian if, and only if, the abelianization G^{ab} is a free abelian pro- ℓ group (cf. [10, Example 3.5-(1)]).

The following result is a consequence of Kummer theory (cf. [10, Theorem 4.2]).

Theorem 2.8 *Let \mathbb{K} be a field containing a primitive ℓ th-root of 1 (and also $\sqrt{-1}$ if $\ell = 2$). Then $(G_{\mathbb{K}}(\ell), \theta_{\mathbb{K}, \ell})$ is a torsion-free Kummerian oriented pro- ℓ group.*

From Proposition 2.3 and Proposition 2.6-(iv), one concludes the following fact.

Corollary 2.9 *Let (G, θ) be a Kummerian torsion-free oriented pro- ℓ group. Then $(G/K_\theta(G), \bar{\theta})$ is the maximal θ -abelian quotient of G .*

3 The Bogomolov–Positselski property

3.1 Bogomolov’s conjecture

Let \mathbb{K} be a field containing a primitive ℓ th-root of 1 (and also $\sqrt{-1}$ if $\ell = 2$), and let $\mathbb{L} = {}^{\ell\infty}\sqrt{\mathbb{K}}$ denote the compositum of all radical extensions $\mathbb{K}(\sqrt[n]{a})$, with $a \in \mathbb{K}^\times$ and $n \geq 1$, i.e.,

$$\mathbb{L} = {}^{\ell\infty}\sqrt{\mathbb{K}} = \mathbb{K}(\sqrt[n]{a} \mid a \in \mathbb{K}^\times, n \geq 1). \tag{3.1}$$

The maximal pro- ℓ Galois group $G_{\mathbb{L}}(\ell)$ of the field \mathbb{L} is equal to the pro- ℓ group $K_{\tilde{\theta}_{\mathbb{K},\ell}}(G_{\mathbb{K}}(\ell))$ associated to the oriented pro- ℓ group $(G_{\mathbb{K}}(\ell), \tilde{\theta}_{\mathbb{K},\ell})$ (cf. [10, Theorem 4.2]). Observe that the ℓ -cyclotomic character associated to the maximal pro- ℓ Galois group of \mathbb{L} is the trivial ℓ -orientation $\mathbf{1}: K_{\tilde{\theta}_{\mathbb{K},\ell}}(G_{\mathbb{K}}(\ell)) \rightarrow \{1\} \subseteq \mathbb{Z}_\ell^\times$.

Motivated by a conjecture formulated by Bogomolov [2]—see Remark 3.3, L. Positselski stated the following conjecture on the pro- ℓ group $G_{\mathbb{L}}(\ell) = K_{\tilde{\theta}_{\mathbb{K},\ell}}(G_{\mathbb{K}}(\ell))$ (cf. [20, Conjecture 1.2]).

Conjecture 3.1 *Let \mathbb{K} be a field containing a primitive ℓ th-root of 1, and also $\sqrt{-1}$ if $\ell = 2$. Then the maximal pro- ℓ Galois group $G_{\mathbb{L}}(\ell)$ of $\mathbb{L} = {}^{\ell\infty}\sqrt{\mathbb{K}}$ is a free pro- ℓ group.*

Conjecture 3.1 is the motivation for the following definition.

Definition 3.2 A Kummerian oriented pro- ℓ group (G, θ) is said to have the Bogomolov–Positselski property if the subgroup $K_\theta(G)$ is a free pro- ℓ group.

Hence, Conjecture 3.1 may be restated as follows: if \mathbb{K} is a field containing a primitive ℓ th-root of 1 (and also $\sqrt{-1}$ if $\ell = 2$), then the oriented pro- ℓ group $(G_{\mathbb{K}}(\ell), \tilde{\theta}_{\mathbb{K},\ell})$ has the Bogomolov-Positselski property.

Remark 3.3 The original formulation of Bogomolov’s conjecture states that if \mathbb{K} is a field containing an algebraically closed field then the (closure of the) commutator subgroup of the Sylow pro- ℓ subgroup of the absolute Galois group $G_{\mathbb{K}}$ of \mathbb{K} is a free pro- ℓ group. Furthermore, the (closure of the) commutator subgroup of the maximal pro- ℓ Galois group $G_{\mathbb{K}}(\ell)$ should be a free pro- ℓ group as well (see also [3, Conjecture 6.2] and [18, § 3.1.2], where the conjecture is stated for function fields).

In [20], Positselski observed that the only essential part of the condition about the algebraically closed subfield of \mathbb{K} is that \mathbb{K} should contain all the roots of 1 of ℓ -power order. Consequently, he formulated the following conjecture (cf. [20, Conjecture 1.1]): the pro- ℓ Sylow subgroup of the absolute Galois group $G_{\mathbb{L}}$, with $\mathbb{L} = {}^{\ell\infty}\sqrt{\mathbb{K}}$ and \mathbb{K} an arbitrary field, is a free pro- ℓ group, i.e., $cd_\ell(G_{\mathbb{L}}) \leq 1$ (or, equivalently, $G_{\mathbb{L}}$ is ℓ -projective, cf. [29, §I.3.4, Proposition 16]). Note that this conjecture is stronger than Conjecture 3.1, and likely hard to approach, while—as stated by Positselski himself, cf. [20, § 1.3]—the latter is closer to Bogomolov’s original conjecture.

Example 3.4 (a) Let (G, θ) be a torsion-free θ -abelian oriented pro- ℓ group. Then (G, θ) is Kummerian (cf. Example 2.7-(a)), and by Proposition 2.6-(iv) one has

$$I_\theta(G) = K_\theta(G) = \{1\}.$$

So, (G, θ) has the Bogomolov–Positselski property.

- (b) Let (G, θ) be an oriented pro- ℓ group with G being a free pro- ℓ group. Then (G, θ) is Kummerian by Example 2.7-(b), and it has the Bogomolov–Positselski property as every closed subgroup of G is again a free pro- ℓ group.
- (c) Let

$$G = \langle x, y, z \mid [x, y] = z, [x, z] = [y, z] = 1 \rangle$$

$$= \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z}_\ell \right\}$$

be the Heisenberg group over \mathbb{Z}_ℓ , and set $(G, \mathbf{1})$, where $\mathbf{1}: G \rightarrow \mathbb{Z}_\ell^\times$ is the trivial orientation. Then $K_\theta(G) = G' \simeq \mathbb{Z}_\ell$ is the cyclic pro- ℓ subgroup generated by z , and $G^{\text{ab}} \simeq \mathbb{Z}_\ell^2$. Hence (G, θ) is Kummerian by Example 2.7-(c), and it has the Bogomolov–Positselski property. Nevertheless, G does not occur as the maximal pro- ℓ Galois group of any field containing μ_{ℓ^∞} (cf. [23, Example 5.4]).

3.2 Self-isolated pro- ℓ groups and Frattini pro- ℓ covers

Let G be a pro- ℓ group, and let $H \subseteq G$ be a subgroup. The isolator of H is the subgroup

$$\text{Iso}(H) = \text{cl}(\langle g \in G \mid g^n \in H \text{ for some } n \geq 1 \rangle)$$

(cf. [12, § 66]). We say that H is self-isolated if $\text{Iso}(H) = H$. In particular, if N is a normal subgroup of G , then G is self-isolated if, and only if, the quotient G/N is a torsion-free pro- ℓ group. The following fact is almost straightforward.

Fact 3.5 *Let (G, θ) be a torsion-free θ -abelian oriented pro- ℓ group. Let N be a normal subgroup of G contained in both $\ker(\theta)$ and $\Phi(G)$. If N is self-isolated, then $N = \{1\}$.*

Proof By Remark 2.2, $\Phi(G) \cap \ker(\theta) = \ker(\theta)^\ell$. As $N \subseteq \ker(\theta)$ is an isolated subgroup, it is a direct summand of $\ker(\theta)$. Thus by Fact 2.5, N is trivial. \square

Fact 3.5 has the following consequence.

Proposition 3.6 *Let (G, θ) be a torsion-free Kummerian oriented pro- ℓ group. Let $N \trianglelefteq G$ be a closed normal, self-isolated, subgroup of G contained in $\ker(\theta)$ satisfying*

$$K_\theta(G) \subseteq N \subseteq \Phi(G).$$

Then $N = K_\theta(G)$.

A Frattini pro- ℓ cover of pro- ℓ groups is a short exact sequence of pro- ℓ groups

$$\{1\} \longrightarrow N \longrightarrow G \xrightarrow{\tau} \tilde{G} \longrightarrow \{1\} \tag{3.2}$$

satisfying $N \subseteq \Phi(G)$. One also says that $\tau: G \rightarrow \tilde{G}$ is a Frattini pro- ℓ cover of \tilde{G} . One may characterize those pro- ℓ groups which may be completed into Kummerian oriented pro- ℓ groups with the Bogomolov–Positselski property as follows.

Theorem 3.7 *A pro- ℓ group G may be completed into a Kummerian oriented pro- ℓ group (G, θ) with the Bogomolov–Positselski property if, and only if, G is a Frattini pro- ℓ cover (3.2) of \tilde{G} , where $(\tilde{G}, \tilde{\theta})$ is a $\tilde{\theta}$ -abelian oriented pro- ℓ group and N is a free pro- ℓ group.*

Proof If (G, θ) is Kummerian with the Bogomolov–Positselski property, then, by Proposition 2.6, $(G/K_\theta(G), \bar{\theta}) = (G, \theta)/K_\theta(G)$ is $\bar{\theta}$ -abelian and $N = K_\theta(G)$ is a free pro- ℓ group by Definition 3.2. This shows one implication.

Conversely, if $(\tilde{G}, \bar{\theta})$ is $\bar{\theta}$ -abelian, then the epimorphism of oriented pro- ℓ groups $(G, \theta) \rightarrow (\tilde{G}, \bar{\theta})$ factors through $(G, \theta)/I_\theta(G)$ by Proposition 2.3. Hence $I_\theta(G) \subseteq N$, while $N \subseteq \Phi(G)$ by hypothesis, thus (G, θ) is Kummerian by Proposition 2.6:(vi). Thus, $I_\theta(G) = K_\theta(G)$ by Proposition 2.6:(iv), and since

$$K_\theta(G) = I_\theta(G) \subseteq N \subseteq \Phi(G),$$

Proposition 3.6 yields $N = K_\theta(G)$, i.e., (G, θ) has the Bogomolov–Positselski property. \square

4 The Bogomolov–Positselski property and cohomology

4.1 Quadratic cohomology and the Norm Residue Theorem

Let G be a pro- ℓ group. The cohomology groups $H^n(G, \mathbb{F}_\ell)$, $n \geq 1$, where \mathbb{F}_ℓ is the trivial G -module isomorphic—as abelian group—to $\mathbb{F}_\ell = \mathbb{Z}/\ell\mathbb{Z}$, come endowed with the bilinear cup-product

$$H^s(G, \mathbb{F}_\ell) \times H^t(G, \mathbb{F}_\ell) \xrightarrow{\cup} H^{s+t}(G, \mathbb{F}_\ell), \quad s, t \geq 0,$$

which is associative and graded-commutative, i.e., $\beta \cup \alpha = (-1)^{st} \alpha \cup \beta$ for $\alpha \in H^s(G, \mathbb{F}_\ell)$ and $\beta \in H^t(G, \mathbb{F}_\ell)$ (cf. [19, Chap. I, § 4]). Thus,

$$H^\bullet(G, \mathbb{F}_\ell) = \coprod_{n \geq 0} H^n(G, \mathbb{F}_\ell)$$

is a connected \mathbb{N}_0 -graded, graded-commutative, associative \mathbb{F}_ℓ -algebra.

For an \mathbb{F}_ℓ -vector space V , let $\mathbf{T}^\bullet V$ denote the \mathbb{F}_ℓ -tensor algebra, i.e.,

$$\mathbf{T}^\bullet V = \coprod_{n \in \mathbb{N}_0} \mathbf{T}^n V \quad \text{where} \quad \mathbf{T}^n V = V^{\otimes n}. \tag{4.1}$$

The \mathbb{N}_0 -graded associative \mathbb{F}_ℓ -algebra \mathbf{A}_\bullet is said to be generated in degree 1, if the canonical homomorphism $\phi_\bullet: \mathbf{T}^\bullet \mathbf{A}_1 \rightarrow \mathbf{A}_\bullet$ of \mathbb{N}_0 -graded associative \mathbb{F}_ℓ -algebras is surjective. Moreover, \mathbf{A}_\bullet is said to be quadratic, if it is 1-generated and $\ker(\phi_\bullet) = \langle \ker(\phi_2) \rangle$, i.e., the ideal $\ker(\phi_\bullet)$ is generated in degree 2.

Definition 4.1 A pro- ℓ group G is said to be H^\bullet -quadratic if $H^\bullet(G, \mathbb{F}_\ell)$ is a quadratic algebra.

For an \mathbb{F}_ℓ -vector space V , let $\mathbf{\Lambda}^\bullet V = \mathbf{T}^\bullet V / \langle v \otimes v \mid v \in V \rangle$ denote the exterior \mathbb{F}_ℓ -algebra spanned by V , and $\mathbf{S}^\bullet V = \mathbf{T}^\bullet V / \langle v \otimes w - w \otimes v \mid v, w \in V \rangle$ denote the symmetric \mathbb{F}_ℓ -algebra spanned by V . Then G is H^\bullet -quadratic if the cup-product induces an isomorphism of graded \mathbb{F}_ℓ -algebras

$$\mathbf{\Xi}^\bullet H^1(G, \mathbb{F}_\ell) / \langle W \rangle \xrightarrow{\sim} H^\bullet(G, \mathbb{F}_\ell), \tag{4.2}$$

where $\mathbf{\Xi}^\bullet = \mathbf{\Lambda}^\bullet$ if ℓ is odd, and $\mathbf{\Xi}^\bullet = \mathbf{S}^\bullet$ if $\ell = 2$. Moreover,

$$W = \ker \left(\mathbf{\Xi}^2(H^1(G, \mathbb{F}_\ell)) \xrightarrow{\cup} H^2(G, \mathbb{F}_\ell) \right). \tag{4.3}$$

By the Norm Residue Theorem, if the field \mathbb{K} contains a primitive ℓ th-root of unity, then the maximal pro- ℓ Galois group $G_{\mathbb{K}}(\ell)$ is H^\bullet -quadratic (cf. [21] or [26]).

Remark 4.2 Let $\ell = 2$ and let G be a pro-2 group. Then one has $\alpha \cup \alpha = 0$ for every $\alpha \in H^1(G, \mathbb{F}_2)$ if, and only if, the map

$$H^1(G, \mathbb{Z}/4) \longrightarrow H^1(G, \mathbb{F}_2),$$

induced by the epimorphism of trivial G -modules $\mathbb{Z}/4 \rightarrow \mathbb{F}_2$, is surjective (cf. [26, Fact 7.1]). In particular, if (G, θ) is a torsion-free Kummerian oriented pro-2 group, one concludes that $\alpha \cup \alpha = 0$ for all $\alpha \in H^1(G, \mathbb{F}_2)$. This is the case for $(G_{\mathbb{K}}(2), \tilde{\theta}_{\mathbb{K},2})$, with \mathbb{K} a field containing $\sqrt{-1}$, i.e., $H^\bullet(G_{\mathbb{K}}(2), \mathbb{F}_2)$ is quadratic and also a quotient of the exterior algebra $\Lambda^\bullet H^1(G_{\mathbb{K}}(2), \mathbb{F}_2)$.

Example 4.3 Let (G, θ) be torsion-free θ -abelian oriented pro- ℓ group. Then G is a torsion free powerful pro- ℓ group (cf. [5, Chap. 4, § 1]), and

$$G \simeq \varprojlim_{i \in I} A_i \rtimes \text{im}(\theta)$$

for finitely generated free abelian pro- ℓ groups A_i . Thus by M. Lazard’s theorem (cf. [14]) one has $\Lambda_\bullet H^1(G, \mathbb{F}_\ell)$ (see, e.g., [26, Theorem 3.13]), and hence G is H^\bullet -quadratic.

4.2 Quadratic cohomology and the Bogomolov–Positselski property

Let (G, θ) be a torsion-free Kummerian oriented pro- ℓ group. The short exact sequence of pro- ℓ groups

$$\{1\} \longrightarrow I_\theta(G) \longrightarrow G \longrightarrow G(\theta) \longrightarrow \{1\} \tag{4.4}$$

induces the 5-terms exact sequence in cohomology

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^1(G(\theta), \mathbb{F}_\ell) & \xrightarrow{\text{inf}_{G(\theta),G}^1} & H^1(G, \mathbb{F}_\ell) & \xrightarrow{\text{res}_{G, I_\theta(G)}^1} & H^1(I_\theta(G), \mathbb{F}_\ell)^{G(\theta)} \\
 & & & & & & \searrow d_2^{0,1} \\
 & & & & H^2(G(\theta), \mathbb{F}_\ell) & \xrightarrow{\text{inf}_{G(\theta),G}^2} & H^2(G, \mathbb{F}_\ell)
 \end{array} \tag{4.5}$$

(cf. [19, Proposition 1.6.7]). As (G, θ) is Kummerian, one has $I_\theta(G) = K_\theta(G) \subseteq \Phi(G)$ (cf. Proposition 2.6(iv)). Hence $\text{inf}_{G(\theta),G}^1$ is an isomorphism and $\text{res}_{G, I_\theta(G)}^1$ is the 0-map. As $(G(\theta), \tilde{\theta}) = (G, \theta)/I_\theta(G)$ is $\tilde{\theta}$ -abelian, one has

$$H^\bullet(G(\theta), \mathbb{F}_\ell) \simeq \Lambda^\bullet H^1(G, \mathbb{F}_\ell) \tag{4.6}$$

(cf. Example 4.3). If in addition G is H^\bullet -quadratic, then $H^\bullet(G, \mathbb{F}_\ell)$ is a quotient of $\Lambda^\bullet H^1(G, \mathbb{F}_\ell)$ (cf. Remark 4.2). In particular, the inflation map $\psi^\bullet = \text{inf}_{G(\theta),G}^\bullet$ induces a surjective homomorphism of \mathbb{N}_0 -graded \mathbb{F}_ℓ -algebras

$$H^\bullet(G(\theta), \mathbb{F}_\ell) \simeq \Lambda^\bullet H^1(G, \mathbb{F}_\ell) \xrightarrow{\psi^\bullet} H^\bullet(G, \mathbb{F}_\ell) \tag{4.7}$$

satisfying

$$\ker(\psi_n) \simeq \ker(\psi_2) \wedge (\Lambda^{n-2}H^1(G, \mathbb{F}_\ell)) \quad \text{for all } n \geq 2. \tag{4.8}$$

Since $\text{res}_{G, K_\theta(G)}^1$ is trivial, one concludes from 4.5 that $d_2^{0,1}$ is injective, $\text{im}(d_2^{0,1}) = \ker(\psi_2)$, and $H^2(G, \mathbb{F}_\ell) \simeq H^2(G(\theta), \mathbb{F}_\ell)/\text{im}(d_2^{0,1})$. Thus, as $H^\bullet(G, \mathbb{F}_\ell)$ is quadratic, one has

$$H^\bullet(G, \mathbb{F}_\ell) \simeq H^\bullet(G(\theta), \mathbb{F}_\ell)/\langle \text{im}(d_2^{0,1}) \rangle. \tag{4.9}$$

4.3 A cohomological criterion

Let (G, θ) be a Kummerian torsion-free oriented pro- ℓ group which is H^\bullet -quadratic. Let $(E_r^{s,t}, d_r^{s,t})$ denote the Hochschild–Serre spectral sequence with coefficients in \mathbb{F}_ℓ associated to the short exact sequence (4.4), i.e.,

$$E_2^{s,t} = H^s(G(\theta), H^t(I_\theta(G), \mathbb{F}_\ell)) \implies E_\infty^{s,t}, \quad s, t \geq 0, \tag{4.10}$$

with differentials $d_r^{s,t}: E_r^{s,t} \rightarrow E_r^{s+r,t-r+1}$ satisfying $d_r \circ d_r = 0$ (cf. [19, Chap. II, § 4]). In particular, by (4.7) one has $E_2^{s,0} \simeq \Lambda^\bullet H^1(G, \mathbb{F}_\ell)$. For $\alpha \in E_2^{s,0} = H^s(G(\theta), \mathbb{F}_\ell)$, $s \geq 0$, and $\beta \in E_2^{0,1} = H^1(I_\theta(G), \mathbb{F}_\ell)^{G(\theta)}$, one has

$$\begin{aligned} \alpha \cup \beta &\in H^s(G(\theta), \mathbb{F}_\ell \otimes H^1(I_\theta(G), \mathbb{F}_\ell)) = E_2^{s,1}, \\ d_2^{s,1}(\alpha \cup \beta) &= (-1)^{s+1} \alpha \cup d_2^{0,1}(\beta) \in E_2^{s+2,0} \end{aligned} \tag{4.11}$$

(cf. [19, Chap. II, Example 4.5]).

Proposition 4.4 *Let (G, θ) be a torsion-free Kummerian oriented pro- ℓ group with G being H^\bullet -quadratic. Then*

- (i) $E_\infty^{s,t}$ is concentrated on the 0th line, i.e., $E_\infty^{s,t} = 0$ for every $s \geq 0$ and $t \geq 1$;
- (ii) $E_3^{s,0} \simeq E_\infty^{s,0} \simeq H^s(G, \mathbb{F}_\ell)$ for every $s \geq 0$.

Proof Since (G, θ) is Kummerian, by (4.6) one has $E_2^{s,0} \simeq \Lambda^\bullet H^1(G, \mathbb{F}_\ell)$. For every $t \geq 0$ there exists a descending separating filtration $(F^k H^t(G, \mathbb{F}_\ell))_{0 \leq k \leq t}$ satisfying $F^0 H^t(G, \mathbb{F}_\ell) = H^t(G, \mathbb{F}_\ell)$ and

$$F^s H^{s+t}(G, \mathbb{F}_\ell)/F^{s+1} H^{s+t}(G, \mathbb{F}_\ell) \simeq E_\infty^{s,t} \tag{4.12}$$

, where $F^{s+t+1} H^{s+t}(G, \mathbb{F}_\ell) = \{0\}$ (cf. [1, p. 99]). By [19, Chap. II, § 4, Example 1], the composite of the maps

$$E_2^{s,0} = H^s(G(\theta), \mathbb{F}_p) \twoheadrightarrow E_3^{s,0} \twoheadrightarrow \dots \twoheadrightarrow E_\infty^{s,0} \twoheadrightarrow H^s(G, \mathbb{F}_p)$$

is the s -th left edge morphism (cf. [19, p. 99]) and hence coincides with the inflation map $\text{inf}_{G(\theta), G}^s$, which is surjective by (4.7). Thus $F^0 H^t(G, \mathbb{F}_\ell) = H^t(G, \mathbb{F}_\ell)$ for all $t \geq 0$, i.e., $E_\infty^{s,0} \simeq H^\bullet(G, \mathbb{F}_p)$, and consequently $E_\infty^{k,t} = 0$ for every $1 \leq k \leq t$. This shows (i).

By (4.11), one has canonical homomorphisms of \mathbb{N}_0 -graded \mathbb{F}_ℓ -algebras

$$\begin{aligned} \sigma^\bullet: H^\bullet(G(\theta), \mathbb{F}_\ell)/\langle \text{im}(d_2^{0,1}) \rangle &\longrightarrow E_3^{\bullet,0}, \\ \tau^\bullet: E_3^{\bullet,0} &\longrightarrow H^\bullet(G, \mathbb{F}_\ell). \end{aligned} \tag{4.13}$$

Moreover, σ^\bullet and τ^\bullet are surjective, σ^k and τ^k are isomorphisms for $k \in \{0, 1, 2\}$, and their composition is an isomorphism of quadratic \mathbb{F}_ℓ -algebras by (4.9). Thus σ^\bullet and τ^\bullet are isomorphisms which shows (ii). \square

Let (G, θ) be a Kummerian torsion-free oriented pro- ℓ group, and put $K_\theta(G)^{\text{ab}} = K_\theta(G)/K_\theta(G)'$. Recall that if \mathbb{K} is a field containing a primitive ℓ th-root of 1 and $(G, \theta) = (G_{\mathbb{K}}(\ell), \bar{\theta}_{\mathbb{K}, \ell})$, then $K_\theta(G)^{\text{ab}}$ is a free abelian pro- ℓ group, as the oriented pro- ℓ group $(K_\theta(G), \theta|_{K_\theta(G)})$ is again Kummerian, and since $\theta|_{K_\theta(G)}$ is trivial. The short exact sequence of pro- ℓ groups

$$\{1\} \longrightarrow K_\theta(G)^{\text{ab}} \xrightarrow{\iota} G/K_\theta(G)' \xrightarrow{\pi} G(\theta) \longrightarrow \{1\}, \tag{4.14}$$

where $G(\theta) = G/K_\theta(G)$, defines a cohomology class $u \in H_{\text{cts}}^2(G(\theta), K_\theta(G)^{\text{ab}})$ (cf. [19, p. 143]), where $K_\theta(G)^{\text{ab}}$ is considered as a topological left $\mathbb{Z}_\ell[[G(\theta)]]$ -module and H_{cts}^* denotes continuous cochain cohomology (cf. [19, Chap. II, § 7]). Since $[G, K_\theta(G)] \subseteq \Phi(G)$, one has

$$\text{Hom}(K_\theta(G), \mathbb{F}_\ell) = \text{Hom}(K_\theta(G), \mathbb{F}_\ell)^{G(\theta)} = E_2^{0,1}.$$

Thus, the pairing

$$\begin{aligned} K_\theta(G)^{\text{ab}} \times E_2^{0,1} &\longrightarrow \mathbb{F}_\ell, \\ (hK_\theta(G)', \beta) &\longmapsto \beta(h), \quad \text{for } h \in K_\theta(G), \end{aligned}$$

induces a map

$$\phi_u: E_2^{2,1} = H^2(G(\theta), \text{Hom}(K_\theta(G), \mathbb{F}_\ell)) \longrightarrow E_2^{4,0} = H^4(G(\theta), \mathbb{F}_\ell) \tag{4.15}$$

given by $\phi_u(\alpha) = u \cup \alpha$ (cf. [19, p. 114]).

Theorem 4.5 *Let (G, θ) be a Kummerian torsion-free oriented pro- ℓ group with G an H^\bullet -quadratic pro- ℓ group. Then the following are equivalent.*

- (i) (G, θ) has the Bogomolov–Positselski property;
- (ii) the differential map $d_2^{2,1}: E_2^{2,1} \rightarrow E_2^{4,0} \simeq \Lambda^4 H^1(G(\theta), \mathbb{F}_\ell)$ is injective;
- (iii) the map ϕ_u is injective, i.e., $u \cup \alpha \neq 0$ for every non-trivial $\alpha \in E_2^{2,1}$.

If these conditions hold, then the spectral sequence $E_2^{s,t} \Rightarrow E_\infty^{s,t}$ collapses at the E_3 -page, i.e., $E_3 = E_\infty$.

Proof By Proposition 4.4(ii), for every $s \geq 0$ one has $E_3^{s,0} \simeq E_4^{s,0} \simeq \dots \simeq E_\infty^{s,0}$. Since, by definition, $E_4^{s,0} = E_3^{s,0}/\text{im}(d_3^{s-3,2})$, one concludes that the maps

$$d_3^{s-3,2}: E_3^{s-3,2} \longrightarrow E_3^{s,0} \simeq H^s(G, \mathbb{F}_\ell)$$

must be the 0-maps for every $s \geq 3$. In particular, $E_4^{0,2} = \ker(d_3^{0,2})$ is equal to $E_3^{0,2}$, which is $\ker(d_2^{0,2})$ by definition. As $E_r^{s,t}$ is a first-quadrant spectral sequence, one has $E_{r+1}^{0,2} = \ker(d_r^{0,2})$ and the map $d_r^{0,2}: E_r^{0,2} \rightarrow E_r^{r,3-r} = 0$ is the 0-map for every $r \geq 4$. This implies that $E_3^{0,2} = E_4^{0,2} = \dots = E_\infty^{0,2}$. Thus, applying Proposition 4.4(i), yields

$$0 = E_\infty^{0,2} = E_3^{0,2} = \ker(d_2^{0,2}), \tag{4.16}$$

i.e., $d_2^{0,2}: E_2^{0,2} \rightarrow E_2^{2,1}$ is injective.

Moreover, one has $E_3^{2,1} = E_4^{2,1} = E_\infty^{2,1}$, as $E_{r+1}^{2,1} = \ker(d_r^{2,1})/\text{im}(d_r^{2-r,r})$ and both maps

$$d_r^{2,1}: E_r^{2,1} \longrightarrow E_r^{2+r,2-r} = 0 \quad \text{and} \quad d_r^{2-r,r}: E_r^{2-r,r} = 0 \longrightarrow E_r^{2,1}$$

are the 0-maps for every $r \geq 3$. Applying Proposition 4.4(i) again yields

$$0 = E_\infty^{2,1} = E_3^{2,1} = \ker(d_2^{2,1})/\text{im}(d_2^{0,2}), \tag{4.17}$$

i.e., $\ker(d_2^{2,1}) = \text{im}(d_2^{0,2})$.

Thus, if (G, θ) has the Bogomolov–Positselski property, then $I_\theta(G) = K_\theta(G)$ is a free pro- ℓ group. Then $H^t(K_\theta(G), \mathbb{F}_\ell) = 0$ for every $t \geq 2$ (cf. [19, Proposition 3.5.17]), and thus $E_r^{0,t} = 0$ for all $r \geq 2$ and $t \geq 2$. In particular, the map $d_2^{0,2}: H^2(I_\theta(G), \mathbb{F}_\ell)^{G(\theta)} \rightarrow E_2^{2,1}$ is trivial, and hence by (4.17), one has $\ker(d_2^{2,1}) = 0$. This proves the implication (i) \Rightarrow (ii).

Conversely, if $d_2^{2,1}$ is injective, then, by (4.17), one has $\text{im}(d_2^{0,2}) = \ker(d_2^{2,1}) = 0$. Since $d_2^{0,2}$ is injective by (4.16), this implies that $E_2^{0,2} = H^2(I_\theta(G), \mathbb{F}_\ell)^{G(\theta)} = 0$. Since G is a pro- ℓ group, the equality $H^2(I_\theta(G), \mathbb{F}_\ell)^{G(\theta)} = 0$ implies that $H^2(I_\theta(G), \mathbb{F}_\ell) = 0$, and thus $I_\theta(G)$ is free by [19, Proposition 3.5.17]. This proves the implication (ii) \Rightarrow (i). The equivalence between (ii) and (iii) follows from [19, Theorem 2.4.4].

Finally, if $I_\theta(G)$ is a free pro- ℓ group, one has $E_r^{s,t} = 0$ for all $s \geq 0, t \geq 2$, and $r \geq 2$. Hence, all maps $d_3^{s,t}$ are trivial, for all $s, t \geq 0$, so that $E_3^{s,t} = E_\infty^{s,t}$. \square

Question 4.6 Let (G, θ) be a Kummerian torsion-free pro- ℓ group with G being an H^\bullet -quadratic pro- ℓ group, and let $(E_r^{s,t}, d_r^{s,t})$ be the Hochschild–Serre spectral sequence associated to (4.4). By Proposition 4.4, for every $s \geq 0$ one has $E_3^{s,0} \simeq E_\infty^{s,0}$, and $E_\infty^{s,t} = 0$ for $s \geq 0$ and $t \geq 1$. Moreover, by Theorem 4.5, if (G, θ) has the Bogomolov–Positselski property, then

$$E_3^{s,t} \simeq E_\infty^{s,t} \quad \text{for every } s, t \geq 0, \tag{4.18}$$

i.e., $E_r^{s,t}$ collapses at the E_3 -page. It would be interesting to understand whether (4.18) implies the Bogomolov–Positselski property for (G, θ) . We suspect that the answer should be affirmative. However, we could not find any evidence for this speculation.

Remark 4.7 Let \mathbb{K} be a field containing a primitive ℓ th-root of unity (and also $\sqrt{-1}$ if $\ell = 2$), put $\mathbb{L} = {}^\ell\sqrt{\mathbb{K}}$ and consider the torsion-free Kummerian oriented pro- ℓ group $(G_{\mathbb{K}}(\ell), \hat{\theta}_{\mathbb{K},\ell})$. The oriented pro- ℓ group $(I_{\hat{\theta}_{\mathbb{K},\ell}}(G_{\mathbb{K}}(\ell)), \mathbf{1})$ is again Kummerian and torsion free, and thus one has

$$I_{\hat{\theta}_{\mathbb{K},\ell}}(G_{\mathbb{K}}(\ell))' = K_1(G_{\mathbb{L}}(\ell)) = G_{{}^\ell\sqrt{\mathbb{L}}}(\ell), \tag{4.19}$$

$$K_{\hat{\theta}_{\mathbb{K},\ell}}(G_{\mathbb{K}})^{\text{ab}} = G_{\mathbb{L}}(\ell)^{\text{ab}} = \text{Gal}({}^\ell\sqrt{\mathbb{L}}/\mathbb{L}), \tag{4.20}$$

where the latter is a free abelian pro- ℓ group [cf. Example 2.7-(c)]. Hence, the short exact sequence (4.14) translates into

$$\{1\} \longrightarrow \text{Gal}({}^\ell\sqrt{\mathbb{L}}/\mathbb{L}) \xrightarrow{\iota} \text{Gal}({}^\ell\sqrt{\mathbb{L}}/\mathbb{K}) \xrightarrow{\pi} \text{Gal}(\mathbb{L}/\mathbb{K}) \longrightarrow \{1\}. \tag{4.21}$$

Recall that by Kummer theory one has an isomorphism of (discrete) ℓ -elementary abelian groups $H^1(\text{Gal}({}^\ell\sqrt{\mathbb{L}}/\mathbb{L}), \mathbb{F}_\ell) \simeq \mathbb{L}^\times/(\mathbb{L}^\times)^\ell$, where $\mathbb{L}^\times = \mathbb{L} \setminus \{0\}$ denotes the

multiplicative group of the field \mathbb{L} . Then by Theorem 4.5 the cohomology element $u \in H^2_{\text{cts}}(\text{Gal}(\mathbb{L}/\mathbb{K}), \text{Gal}(\sqrt[p^\infty]{\mathbb{L}}/\mathbb{L}))$ associated to the extension of pro- ℓ groups (4.21) induces a homomorphism

$$\phi_{u,\mathbb{L}}: H^2\left(\text{Gal}(\mathbb{L}/\mathbb{K}), \mathbb{L}^\times/(\mathbb{L}^\times)^\ell\right) \longrightarrow H^4(\text{Gal}(\mathbb{L}/\mathbb{K}), \mathbb{F}_\ell)$$

which is injective if, and only if, \mathbb{L} satisfies Conjecture 3.1. In view of Theorem 4.5, the knowledge of the structure of $\mathbb{L}^\times/(\mathbb{L}^\times)^\ell$ as continuous $\text{Gal}(\mathbb{L}/\mathbb{K})$ -module, or an arithmetic interpretation of the map $\phi_{u,\mathbb{L}}$, may contribute to the solution of Conjecture 3.1.

5 Oriented pro- ℓ groups of elementary type

5.1 Demushkin groups and one-relator pro- ℓ groups

A Demushkin group is a Poincaré duality pro- ℓ group of dimension 2, namely, a pro- ℓ group G whose \mathbb{F}_ℓ -cohomology satisfies the following conditions:

- (i) $\dim(H^1(G, \mathbb{F}_\ell)) < \infty$;
- (ii) $H^2(G, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell$;
- (iii) cup-product induces a perfect pairing $H^1(G, \mathbb{F}_\ell) \times H^1(G, \mathbb{F}_\ell) \rightarrow H^2(G, \mathbb{F}_\ell)$;

(cf. [19, Definition 3.9.9]). Note that by condition (ii) such a pro- ℓ group G has a single defining relation, namely, G may be defined as the quotient F/N of a free pro- ℓ group F over a normal subgroup $N \subseteq F$ generated as a normal subgroup of F by a single element contained in $\Phi(F)$ (cf., e.g., [19, pp. 231–232]).

A Demushkin group comes equipped with a distinguished orientation $\partial_G: G \rightarrow \mathbb{Z}_\ell^\times$, induced by the action of G on its dualizing module, described in [13, Theorem 4]. The orientation $\partial_G: G \rightarrow \mathbb{Z}_\ell^\times$ is the only orientation which completes G into a Kummerian oriented pro- ℓ group (G, ∂_G) (cf. [26, Proposition 5.2]). The oriented pro- ℓ group (G, ∂_G) enjoys also the Bogomolov–Positselski property.

Theorem 5.1 *Let G be a Demushkin group, endowed with the canonical orientation $\partial_G: G \rightarrow \mathbb{Z}_\ell^\times$, and suppose that $\text{im}(\partial_G) \subseteq 1 + 4\mathbb{Z}_2$ if $\ell = 2$. Then the oriented pro- ℓ group (G, ∂_G) has the Bogomolov–Positselski property.*

Proof Since (G, ∂_G) is Kummerian, by [26, Proposition 5.2], Proposition 2.6(iii) and Remark 2.2, one has $G/I_{\partial_G}(G) \simeq \mathbb{Z}_\ell^{d-1} \times \mathbb{Z}_\ell$, with $d = \dim(H^1(G, \mathbb{F}_\ell))$. Therefore, $I_{\partial_G}(G) = K_{\partial_G}(G)$ is a subgroup of G of infinite index, and thus it is a free pro- ℓ group by [29, § I.4.5, Exercise 5(b)]. \square

As mentioned above, Demushkin groups have a single defining relation. One may prove the Bogomolov–Positselski property also for 1-relator pro- ℓ groups G with quadratic \mathbb{F}_ℓ -cohomology which can be completed into a Kummerian oriented pro- ℓ group $(G, \mathbf{1})$ with a trivial orientation.

Proposition 5.2 *Let G be a finitely generated pro- ℓ group with a single defining relation such that*

- (i) $H^\bullet(G, \mathbb{F}_\ell)$ is a quadratic algebra;
- (ii) $(G, \mathbf{1})$ is Kummerian.

Then $(G, \mathbf{1})$ has the Bogomolov–Positselski property.

Proof Since $(G, \mathbf{1})$ is Kummerian, the quotient G^{ab} is a free abelian pro- ℓ group (cf. Example 2.7(c)). We need to show that $G' = K_1(G) = I_1(G)$ is a free pro- ℓ group.

Since G has a single defining relation, $H^2(G, \mathbb{F}_\ell) \simeq \mathbb{F}_\ell$ (cf. [19, Corollary 3.9.5]). Moreover, since $H^\bullet(G, \mathbb{F}_\ell)$ is quadratic, $H^2(G, \mathbb{F}_\ell)$ is generated by cup products $\chi \cup \psi$ with $\chi, \psi \in H^1(G, \mathbb{F}_\ell)$, so that the cup product from $H^1(G, \mathbb{F}_\ell)$ to $H^2(G, \mathbb{F}_\ell)$ is not trivial (see also [22, Proposition 4.2]). Consequently, [35, Corollary 2] yields a short exact sequence of pro- ℓ groups

$$\{1\} \longrightarrow N \longrightarrow G \longrightarrow \tilde{G} \longrightarrow \{1\}$$

which satisfies the following three properties: N is a free pro- ℓ group; \tilde{G} is a Demushkin group; and for every subgroup S of G containing N , the inflation map

$$\text{inf}_{S,N}^2: H^2(S/N, \mathbb{F}_\ell) \longrightarrow H^2(S, \mathbb{F}_\ell) \tag{5.1}$$

is an isomorphism (this last property is shown to hold in the proof of [35, Corollary 2]).

Since G is finitely generated, also \tilde{G} is finitely generated. Moreover, by (5.1) the inflation map $H^2(\tilde{G}, \mathbb{F}_\ell) \rightarrow H^2(G, \mathbb{F}_\ell)$ is an isomorphism, and thus by the five-terms exact sequence (cf. [19, Proposition 1.6.7]) the restriction map

$$\text{res}_{G,N}^1: H^1(G, \mathbb{F}_\ell) \longrightarrow H^1(N, \mathbb{F}_\ell)^G$$

is surjective. Since $(G, \mathbf{1})$ is Kummerian, and since $\text{res}_{G,N}^1$ is surjective, [24, Theorem 1.2] implies that also the oriented pro- ℓ group $(\tilde{G}, \mathbf{1}) = (G, \mathbf{1})/N$ is Kummerian. Hence, the canonical orientation $\tilde{\partial}_{\tilde{G}}: \tilde{G} \rightarrow \mathbb{Z}_\ell^\times$ must coincide with the trivial orientation $\mathbf{1}$ (cf. [26, Proposition 5.2]). By Theorem 5.1, the oriented pro- ℓ group $(\tilde{G}, \mathbf{1})$ has the Bogomolov–Positselski property, and thus $K_1(\tilde{G})$ —which coincides with \tilde{G}' —is a free pro- ℓ group.

Let S be the normal subgroup of G containing N such that $S/N \simeq \tilde{G}'$. Thus, $G/S \simeq \tilde{G}/\tilde{G}'$ is abelian, and therefore $S \supseteq G'$. By (5.1), one has $H^2(S/N, \mathbb{F}_\ell) \simeq H^2(S, \mathbb{F}_\ell)$, and the term on the left-hand side is trivial as S/N is a free pro- ℓ group. Hence, also $H^2(S, \mathbb{F}_\ell) = 0$, and S is a free pro- ℓ group (cf. [19, Proposition 3.5.17]). Since $G' \subseteq S$, and $\text{cd}_\ell(G') \leq \text{cd}_\ell(S) = 1$, G' must be free (cf. [29, § 3.3, Proposition 14]). \square

Remark 5.3 Let F be a finitely generated free pro- ℓ group, let r be an element of $\Phi(F)$ and let R denote the normal subgroup of F generated by r . Suppose that $\ell \neq 2$. By [22, Proposition 4.2] and Example 2.7(c), the pro- ℓ group $G = F/R$ satisfies the conditions (i)–(ii) in Proposition 5.2 if, and only if, $r \in F'$ and $r \notin F^p \cdot [F', F]$.

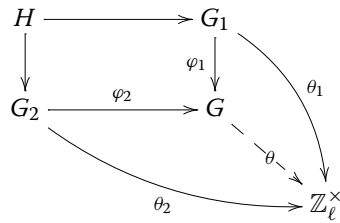
5.2 Free constructions

By [8, § 3], the free product of two oriented pro- ℓ groups (G_1, θ_1) and (G_2, θ_2) is the oriented pro- ℓ group (G, θ) where G is the free pro- ℓ product of G_1, G_2 , and $\theta: G \rightarrow \mathbb{Z}_\ell^\times$ is the orientation induced by θ_1, θ_2 via the universal property of G (see also [26, § 3.4]).

One may extend the above definition to free amalgamated pro- ℓ products of oriented pro- ℓ groups (we refer to [28, § 9.2] for the definition of free amalgamated pro- ℓ products).

Definition 5.4 Let (G_1, θ_1) and (G_2, θ_2) be two oriented pro- ℓ groups such that G_1 and G_2 have a common subgroup $H \subseteq G_1, G_2$ satisfying $\theta_1|_H = \theta_2|_H$. The amalgamated pro- ℓ product of oriented pro- ℓ groups of (G_1, θ_1) and (G_2, θ) with amalgamation in H is the oriented pro- ℓ group $(G, \theta) = (G_1, \theta_1) \amalg_H^\ell (G_2, \theta_2)$, where $G = G_1 \amalg_H^\ell G_2$ is the free

amalgamated pro- ℓ product of G_1 and G_2 over H , and $\theta: G \rightarrow \mathbb{Z}_\ell^\times$ is the orientation which makes the diagram



commute.

Note that the morphisms φ_1 and φ_2 may not be injective (cf. [28, p. 369]). If they are, the free amalgamated pro- ℓ product is said to be proper.

If $H = \{1\}$, then $(G_1, \theta_1) \amalg_H^\ell (G_2, \theta_2)$ coincides with the free product of oriented pro- ℓ groups. In this case we simply write $(G_1, \theta_1) \amalg^\ell (G_2, \theta_2)$, instead of $(G_1, \theta_1) \amalg_{\{1\}}^\ell (G_2, \theta_2)$. Free products of oriented pro- ℓ groups preserve Kummerianity (cf. [10, Proposition 7.5]).

Proposition 5.5 *Let (G_1, θ_1) and (G_2, θ_2) be two Kummerian oriented pro- ℓ groups. Then the free product $(G_1, \theta_1) \amalg^\ell (G_2, \theta_2)$ is again Kummerian.*

We prove that—under certain conditions—if the free amalgamated pro- ℓ product of two Kummerian oriented pro- ℓ groups with the Bogomolov–Positselski property is again Kummerian, then it has also the Bogomolov–Positselski property.

Theorem 5.6 *Let (G_1, θ_1) and (G_2, θ_2) be torsion free Kummerian oriented pro- ℓ groups with the Bogomolov–Positselski property, with common finitely generated subgroup $U = G_1 \cap G_2$ such that $\theta_1|_U = \theta_2|_U$ and that (U, θ_U) is θ_U -abelian, where $\theta_U = \theta_i|_U$ for $i = 1, 2$. Suppose that*

- (i) *the amalgamated pro- ℓ product $(G, \theta) = (G_1, \theta_1) \amalg_U^\ell (G_2, \theta_2)$ is Kummerian;*
- (ii) *the restriction maps*

$$\text{res}_{G, G_i}^1: H^1(G, \mathbb{F}_\ell) \rightarrow H^1(G_i, \mathbb{F}_\ell) \quad \text{and} \quad \text{res}_{G_i, U}^1: H^1(G_i, \mathbb{F}_\ell) \rightarrow H^1(U, \mathbb{F}_\ell)$$

are surjective for both $i = 1, 2$.

Then (G, θ) has the Bogomolov–Positselski property.

Remark 5.7 (a) If U in the statement of Theorem 5.6 is the trivial group, then (G, θ) is the usual free product of oriented pro- ℓ groups, and the two conditions are satisfied by (G, θ) . For condition (i), see Proposition 5.5, and condition (ii) is trivially satisfied. Hence, the Bogomolov–Positselski property is preserved by free products of oriented pro- ℓ groups.

- (b) By duality, for $i \in \{1, 2\}$ the map res_{G, G_i}^1 , respectively the map $\text{res}_{G_i, U}^1$, is surjective if, and only if, the map $\bar{\iota}_i: G_i/\Phi(G_i) \rightarrow G/\Phi(G)$ induced by the inclusion $\iota_i: U \hookrightarrow G_i$, respectively the map $\bar{\iota}_{U, i}: U/\Phi(U) \rightarrow G_i/\Phi(G_i)$ induced by the inclusion $\iota_{U, i}: U \hookrightarrow G_i$, is injective.

Proof By [21, Theorem A], U is a uniformly powerful pro- ℓ group, and therefore [25, Proposition 5.22] implies that $G = G_1 \amalg_U^\ell G_2$ is a proper amalgam. Moreover, by hypoth-

esis one has the monomorphisms of ℓ -elementary abelian groups \bar{i}_i and $\bar{i}_{U,i}$, with $i = 1, 2$ [cf. Remark 5.7(b)]. Hence, also $\bar{i}_U = \bar{i}_i \circ \bar{i}_{U,i}: U/\Phi(U) \rightarrow G/\Phi(G)$ is injective for both $i = 1, 2$.

Let $\iota_U: U \hookrightarrow G$ be the inclusion of U in G , and for $i = 1, 2$, set

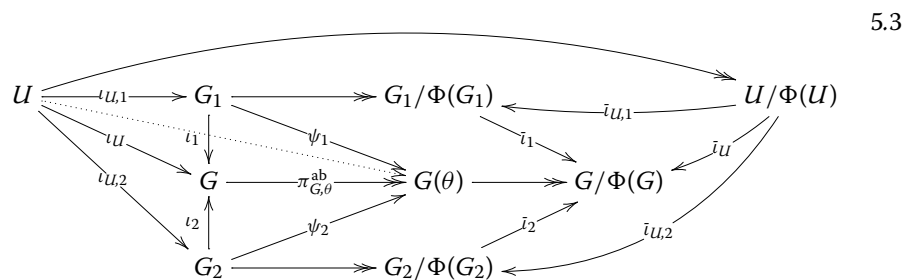
$$\psi_U = \pi_{G,\theta}^{\text{ab}} \circ \iota_U: U \longrightarrow G(\theta) = G/K_\theta(G),$$

$$\psi_i = \pi_{G,\theta}^{\text{ab}} \circ \iota_i: G_i \longrightarrow G(\theta) = G/K_\theta(G).$$

Then

$$\ker(\psi_U) = U \cap K_\theta(G) \quad \text{and} \quad \ker(\psi_i) = G_i \cap K_\theta(G). \tag{5.2}$$

Now consider the commutative diagram



where the dotted arrow from U to $G(\theta)$ is ψ_U . By Remark 2.4, the oriented pro- ℓ groups $(\text{im}(\psi_U), \theta|_{\text{im}(\psi_U)})$ and $(\text{im}(\psi_i), \theta|_{\text{im}(\psi_i)})$ are $\theta|_{\text{im}(\psi_U)}$ - and $\theta|_{\text{im}(\psi_i)}$ -abelian, respectively. In particular,

$$\ker(\psi_i) \supseteq I_{\theta_i}(G_i) = K_{\theta_i}(G_i), \tag{5.4}$$

where the left-hand side inclusion follows by Proposition 2.3, and the right-side equality follows by Proposition 2.6(iv), as (G_i, θ_i) is Kummerian for $i \in \{1, 2\}$ by hypothesis. Consequently, the pro- ℓ groups $\text{im}(\psi_U)$ and $\text{im}(\psi_i)$ are torsion-free, so that $\ker(\psi_U)$ and $\ker(\psi_i)$ are self-isolated subgroups of U and G_i respectively. On the other hand, by duality one has $\ker(\psi_U) \subseteq \Phi(U)$ and $\ker(\psi_i) \subseteq \Phi(G_i)$, as the maps \bar{i}_U and \bar{i}_i are injective. Altogether, by (5.2) and (5.4) one has

$$K_{\theta|_U}(U) = \{1\} \subseteq U \cap K_\theta(G) \subseteq \Phi(U) \quad \text{and} \quad K_{\theta_i}(G_i) \subseteq G_i \cap K_\theta(G) \subseteq \Phi(G_i),$$

and thus $\{1\} = U \cap K_\theta(G)$ and $K_{\theta_i}(G_i) = G_i \cap K_\theta(G)$ by Proposition 3.6.

Now, let $\mathcal{T} = (\mathcal{V}(\mathcal{T}), \mathcal{E}(\mathcal{T}))$ be the pro- ℓ tree whose vertices and edges are given by

$$\mathcal{V}(\mathcal{T}) = \{gG_1, gG_2 \mid g \in G\} \quad \text{and} \quad \mathcal{E}(\mathcal{T}) = \{gU, g\bar{U} \mid g \in G\},$$

respectively. In particular, every edge $gU \in \mathcal{E}(\mathcal{T})$ defines an origin, the G_1 -coset gG_1 and a terminus, the G_2 -coset gG_2 . For $g\bar{U} \in \mathcal{E}(\mathcal{T})$ the roles of the terminus and origin are interchanged. Then \mathcal{T} is a second countable pro- ℓ tree, with a natural G -action (cf. [27, Example 6.2.3]). For $\nu = gG_i \in \mathcal{V}(\mathcal{T})$ and $\mathbf{e} = hU \in \mathcal{E}(\mathcal{T})$, with $g, h \in G$ and $i \in \{1, 2\}$, let K_ν and $K_{\mathbf{e}}$ denote the stabilizers of ν and \mathbf{e} in $K_\theta(G)$, respectively. Hence

$$K_\nu = \{x \in K_\theta(G) \mid xg \in gG_i\} = K_\theta(G) \cap gG_i g^{-1},$$

$$K_{\mathbf{e}} = \{x \in K_\theta(G) \mid xh \in hU\} = K_\theta(G) \cap hUh^{-1}.$$

Since $K_\theta(G)$ is a normal subgroup of G , for every $v = gG_i \in \mathcal{V}(T)$ the subgroup K_v is isomorphic to $K_\theta(G) \cap G_i = K_{\theta_i}(G_i)$, which is free by hypothesis; while for every $e = hU \in \mathcal{E}(T)$ the subgroup K_e is equal to $\{1\}$, and hence no non-trivial element of $K_\theta(G)$ stabilizes an edge. Therefore, by [16, Theorem 5.6], $K_\theta(G)$ has the following decomposition as free pro- ℓ product:

$$K_\theta(G) = \left(\coprod_{v \in \mathcal{V}'} K_v \right) \amalg F, \tag{5.5}$$

for some subset \mathcal{V}' of $\mathcal{V}(T)$, where F is a free pro- ℓ group. Hence $K_\theta(G)$ is the free pro- ℓ product of free pro- ℓ groups, and thus it is a free pro- ℓ group as well. \square

Example 5.8 Let (G_1, θ_1) and (G_2, θ_2) be the oriented pro- ℓ groups with

$$G_1 = \left\langle x, y_1, y_3 \mid [y_1, y_3] = 1, {}^x y_j = y_j^{1+\ell}, \forall j \in \{1, 3\} \right\rangle \simeq \mathbb{Z}_\ell^2 \rtimes \mathbb{Z}_\ell,$$

$$G_2 = \left\langle x, y_2, y_3 \mid [y_2, y_3] = 1, {}^x y_j = y_j^{1+\ell}, \forall j \in \{2, 3\} \right\rangle \simeq \mathbb{Z}_\ell^2 \rtimes \mathbb{Z}_\ell,$$

and such that $\theta_i(x) = 1 + \ell$ and $\theta_i(y_i) = \theta_i(y_3) = 1$ for both $i = 1, 2$. By Remark 2.2, these two oriented pro- ℓ groups are respectively θ_1 - and θ_2 -abelian. Set $U = G_1 \cap G_2$ —i.e. U is the subgroup generated by x, y_3 . Clearly, $\theta_1|_U = \theta_2|_U$, and

$$(U, \theta_i|_U) = \langle y_3 \rangle \rtimes (\langle x \rangle, \theta_i|_{\langle x \rangle}) \quad \text{for both } i = 1, 2,$$

which is $\theta_i|_U$ -abelian by Remark 2.4. Moreover, it is straightforward to see that the maps $\bar{i}_{U,i}: U/\Phi(U) \rightarrow G_i/\Phi(G_i)$ are injective for both $i = 1, 2$. Now let (G, θ) be the oriented pro- ℓ group $(G_1, \theta_1) \amalg_U^\ell (G_2, \theta_2)$. Then

$$G = \left\langle x, y_1, y_2, y_3 \mid [y_1, y_3] = [y_2, y_3] = 1, {}^x y_i = y_i^{1+\ell}, \forall i \in \{1, 2, 3\} \right\rangle$$

and $\theta(x) = 1 + \ell, \theta(y_j) = 1$ for $j = 1, 2, 3$. Moreover, one has an epimorphism of oriented pro- ℓ groups $\tau: (G, \theta) \rightarrow (\bar{G}, \bar{\theta})$, where

$$\bar{G} = \left\langle \bar{x}, \bar{y}_1, \bar{y}_2, \bar{y}_3 \mid [\bar{y}_j, \bar{y}_{j'}] = 1, \bar{x} \bar{y}_j = \bar{y}_j^{1+\ell}, \forall j, j' \in \{1, 2, 3\} \right\rangle \simeq \mathbb{Z}_\ell^3 \rtimes \mathbb{Z}_\ell,$$

and $\bar{x} = \tau(x), \bar{y}_j = \tau(y_j)$ for $j = 1, 2, 3$. By Remark 2.2, $(\bar{G}, \bar{\theta})$ is $\bar{\theta}$ -abelian, and thus $\ker(\tau) \supseteq I_\theta(G)$ by Proposition 2.3. On the other hand, it is straightforward to see that $\Phi(G) \supseteq \ker(\tau)$, and hence (G, θ) is Kummerian by Proposition 2.6-(vi). Since (G_1, θ_1) and (G_2, θ_2) have the Bogomolov–Positselski property by Example 3.4-(a), Theorem 5.6 implies that also (G, θ) has the Bogomolov–Positselski property. Observe that G is H^\bullet -quadratic (cf. [25, Remark 5.25-(c)]).

5.3 Pro- ℓ groups of elementary type

Let (G, θ) be an oriented pro- ℓ group, and let A be a free abelian pro- ℓ group. Recall that the semidirect product $A \rtimes (G, \theta) = (A \rtimes G, \theta \circ \pi)$ is the oriented pro- ℓ group where $gag^{-1} = a^{\theta(g)}$ for all $a \in A$ and $g \in G$, and $\pi: A \rtimes G \rightarrow G$ is the canonical projection (cf. [8, § 3]).

The following is straightforward (cf., e.g., [10, Proposition 3.6]).

Proposition 5.9 *Given an oriented pro- ℓ group (G, θ) and a free abelian pro- ℓ group A , one has $K_{\theta \circ \pi}(A \rtimes G) = K_\theta(G)$. In particular, $A \rtimes (G, \theta)$ is Kummerian if, and only if, (G, θ)*

is Kummerian; and $A \rtimes (G, \theta)$ has the Bogomolov–Positselski property if, and only if, (G, θ) has the Bogomolov–Positselski property.

The family \mathbf{ET}_ℓ of oriented pro- ℓ groups of elementary type is the smallest class of finitely generated oriented pro- ℓ groups satisfying (cf. [8, § 3])

- (a) the oriented pro- ℓ group $(G, \bar{\theta}_G)$, with G a Demushkin group, is of elementary type;
- (b) the oriented pro- ℓ group $(\mathbb{Z}_\ell, \theta)$, with $\theta: \mathbb{Z}_\ell \rightarrow \mathbb{Z}_\ell^\times$ arbitrary, is of elementary type;
- (c) if the oriented pro- ℓ group (G, θ) is of elementary type and A is a finitely generated free abelian pro- ℓ group, then also the semidirect product $A \rtimes (G, \theta)$ is of elementary type;
- (d) if (G_1, θ_1) and (G_2, θ_2) are oriented pro- ℓ groups of elementary type then also the free pro- ℓ product $(G_1, \theta_1) \amalg^\ell (G_2, \theta_2)$ is of elementary type.

Remark 5.10 (a) In the original definition of oriented pro-2 groups of elementary type one has that also the cyclic group C_2 of order 2, endowed with the non-trivial orientation $\theta_{C_2}: C_2 \twoheadrightarrow \{\pm 1\} \subset \mathbb{Z}_2^\times$, is a pro-2 group of elementary type (cf. [8, p. 242]). Since our results always assume oriented pro- ℓ groups to be torsion-free, we may safely exclude (C_2, θ_{C_2}) from the above definition of oriented pro- ℓ groups of elementary type.

- (b) From the results in [26, § 3.3–3.4], one may deduce that a finitely generated subgroup H of an oriented pro- ℓ groups of elementary type (G, θ) gives rise to a pro- ℓ groups of elementary type $(H, \theta|_H)$.
- (c) If (F, θ) is a torsion-free oriented pro- ℓ group with F a finitely generated free pro- ℓ group and $\theta: F \rightarrow \mathbb{Z}_\ell^\times$ any orientation, then (F, θ) is of elementary type. Indeed, if $\theta = \mathbf{1}$, then (F, θ) is isomorphic to the free pro- ℓ product of d copies of the oriented pro- ℓ group $(\mathbb{Z}_\ell, \mathbf{1})$, where d is the minimal number of generators of F . Otherwise, $\text{im}(\theta) \simeq \mathbb{Z}_\ell$, and the short exact sequence of pro- ℓ groups

$$\{1\} \longrightarrow \ker(\theta) \longrightarrow F \longrightarrow \text{im}(\theta) \longrightarrow \{1\}$$

splits. In this case, let $\{x_1, \dots, x_d\}$ be a minimal generating set where $\theta(x_1) \neq 1$ and $\theta(x_i) = 1$ for $i \geq 2$, and let H be the subgroup of F generated by $\{x_2, \dots, x_d\}$, which is free. Then, $(F, \theta) \simeq (H, \mathbf{1}) \amalg^\ell (\text{im}(\theta), \text{id}_{\text{im}(\theta)})$, where both factors are oriented pro- ℓ groups of elementary type.

From Example 2.7-(b), Sect. 5.1, and Propositions 5.5 and 5.9, one concludes that oriented pro- ℓ groups of elementary type are Kummerian. I. Efrat’s Elementary Type Conjecture states that if \mathbb{K} is a field containing a primitive ℓ th-root of 1 (and also $\sqrt{-1}$ if $\ell = 2$) and if the maximal pro- ℓ Galois group $G_{\mathbb{K}}(\ell)$ is finitely generated, then $(G_{\mathbb{K}}(\ell), \bar{\theta}_{\mathbb{K}, \ell})$ is of elementary type (cf. [6, 7], see also [15, § 10] and [26, § 7.5]).

Example 5.11 The oriented pro- ℓ group (G, θ) as in Example 5.8 is not of elementary type. Indeed, the subgroup of G generated by $\{x, y_1, y_2\}$ contains a finitely generated subgroup which does not complete into a Kummerian oriented pro- ℓ group (cf. [24, Example 5.3])—in particular, G does not occur as the maximal pro- ℓ Galois group of a field containing a primitive ℓ th-root of unity (and also $\sqrt{-1}$ if $\ell = 2$). Therefore, (G, θ) is not of elementary type by Remark 5.10-(b).

Theorem 5.12 *Let (G, θ) be an oriented pro- ℓ group of elementary type. Then (G, θ) has the Bogomolov–Positselski property.*

Proof If G is a free pro- ℓ group, then (G, θ) has the Bogomolov–Positselski property by Example 3.4-(a). If G is a Demushkin group and $\theta = \bar{\theta}_G$, then $(G, \bar{\theta}_G)$ has the Bogomolov–Positselski property by Theorem 5.1.

By Proposition 5.9, if $(G, \theta) = A \rtimes (G_0, \theta|_{G_0})$ where A is a free abelian pro- ℓ group and the right side factor is an oriented pro- ℓ group of elementary type, then (G, θ) has the Bogomolov–Positselski property—provided that $(G_0, \theta|_{G_0})$ has the Bogomolov–Positselski property.

Finally, by Theorem 5.6, if $(G, \theta) = (G_1, \theta_1) \amalg^\ell (G_2, \theta_2)$ and both (G_1, θ_1) and (G_2, θ_2) have the Bogomolov–Positselski property, then also (G, θ) has the Bogomolov–Positselski property. \square

Let \mathbb{K} be a field containing a primitive ℓ th-root of unity, and set $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$. Since Kummer theory yields an isomorphism of (discrete) ℓ -elementary abelian pro- ℓ groups $H^1(G_{\mathbb{K}}(\ell), \mathbb{F}_\ell)^\vee \simeq \mathbb{K}^\times / (\mathbb{K}^\times)^\ell$, the pro- ℓ group $G_{\mathbb{K}}(\ell)$ is finitely generated if, and only if, the quotient $\mathbb{K}^\times / (\mathbb{K}^\times)^\ell$ is finite. One has the following (see [17, Theorem D], and [9] for item (f)).

Proposition 5.13 *Let \mathbb{K} be a field containing a primitive ℓ th-root of 1 (and also $\sqrt{-1}$ if $\ell = 2$), such that the quotient $\mathbb{K}^\times / (\mathbb{K}^\times)^\ell$ is finite. Then the oriented pro- ℓ group $(G_{\mathbb{K}}(\ell), \theta_{\mathbb{K}, \ell})$ is of elementary type in the following cases:*

- (a) \mathbb{K} is finite;
- (b) \mathbb{K} is a pseudo algebraically closed (PAC) field, or an extension of relative transcendence degree 1 of a PAC field;
- (c) \mathbb{K} is an extension of transcendence degree 1 of a local field;
- (d) \mathbb{K} is ℓ -rigid (cf. [32, p. 722], see also [4, § 3]);
- (e) \mathbb{K} is algebraic extension of a global field of characteristic not ℓ ;
- (f) $\mathbb{K} = \mathbb{k}((T))$, where $(G_{\mathbb{k}}(\ell), \theta_{\mathbb{k}, \ell})$ is of elementary type.

Corollary 1.3 follows from Theorem 5.12 and Proposition 5.13.

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