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PAPER

Compatible pairs of Hamiltonian operators of the first and third orders

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Abstract

We compute general compatibility conditions between a weakly nonlocal homogeneous Hamiltonian operator and a third-order homogeneous Hamiltonian operator. Such operators determine a bi-Hamiltonian structure for many integrable PDEs (Korteweg–De Vries, Camassa–Holm, dispersive water waves, Dym, WDVV and others). Remarkably, the full set of conditions is purely algebraic and the first-order operator is completely determined by commuting systems of conservation laws that are Hamiltonian with respect to the third-order operator. We illustrate the above results with several examples, some of which, concerning WDVV equations, are new.

1. Introduction: formulation of the problem and main results

One of the most important ways to define integrable hierarchies of PDEs is through a compatible pair[6] of Hamiltonian operators A_1, A_2. Indeed, by Magri's theorem [37] an integrable hierarchy of PDEs in n dependent variables $u^i = u^i(t,x)$, $i = 1,\dots,n$, of the form

$$u^i_t = A_1^{ij}\left(\frac{\delta H_n}{\delta u^j}\right)$$

is defined by recursively solving the equation

$$A_1\left(\frac{\delta H_{n+1}}{\delta u^i}\right) = A_2\left(\frac{\delta H_n}{\delta u^i}\right)$$

for H_{n+1}. That implies that all PDEs in the hierarchy are endowed with an infinite sequence of conserved densities $\{H_n\}$ that commute with respect to the Poisson brackets defined by both Hamiltonian operators as

$$\{F,G\}_i = \int \frac{\delta F}{\delta u^h} A_i^{hk} \frac{\delta G}{\delta u^k}\mathrm{d}x, \qquad i = 1,2.$$

That is the essence of the celebrated bi-Hamiltonian formalism for PDEs. In practice, the applicability of the method, i.e. the well-posedness of the recursion relations, relies on some additional conditions like the vanishing of certain cohomology groups ensuring the existence of solutions at each step of the iteration.

[6] We recall that two Hamiltonian operators A_1, A_2 are compatible if and only if any linear combination $\lambda A_1 + \mu A_2$ of them ('pencil') is a Hamiltonian operator.

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If we have a closer look at bi-Hamiltonian equations that were found at the beginning of the research work on integrable PDEs, we notice that many of them are determined by Hamiltonian operators that are homogeneous with respect to the grading given by the count of x-derivatives in their expressions. The study of such operators was initiated by Dubrovin and Novikov [14, 15]. The list of the above equations includes, but is not exhausted by, the KdV equation, the AKNS Equation [16, 31], the Kaup–Broer system [30], the Dispersive Water Waves system [3, 30], the coupled Dym system [2], the Monge–Ampère system[7], the WDVV first-order systems and others (see [33, 45, 51] for a more complete discussion and an extensive list of references). Basing on remarks in [44], we introduced two types of bi-Hamiltonian equations determined by the following types of bi-Hamiltonian pairs:

- bi-Hamiltonian pairs (A_1, A_2) of KdV-type [33, 36]:

$$A_1 = P_1 + R_k, \qquad A_2 = Q_1$$

  where P_1, Q_1 are homogeneous first-order Hamiltonian operators, R_k is a homogeneous k-th order Hamiltonian operator, and all operators are mutually compatible. Equations of this type include the KdV, AKNS, Kaup–Broer, Dispersive Water Waves and Dym equations;

- bi-Hamiltonian pairs (A_1, A_2) of WDVV-type [45]:

$$A_1 = R_k, \qquad A_2 = Q_1$$

  where Q_1 and R_k are compatible operators as above. Equations of this type include the WDVV and Monge–Ampère equations, after rewriting them as first-order systems of conservation laws.

Conditions under which homogeneous operators of the first, second and third orders are Hamiltonian are well known, see e.g. the review by Mokhov [39]. On the other hand, the compatibility of two operators is a way more complicated problem.

Compatibility of two homogeneous first-order operators, local and nonlocal, has been extensively studied (see, for example, the review Paper [41] and [40] for the nonlocal case) as such pairs of operators define the dispersionless limit of dispersive integrable hierarchies [13]. Computing the conditions is not extremely difficult, as they boil down to the Hamiltonian property for a first-order operator, which depends linearly on one parameter.

The compatibility of homogeneous operators with different degrees of homogeneity is a relatively simple matter only at low degrees of homogeneity. For instance, compatible operators of order 1 and of order 0 can be found in [39]; they were used in [10] in order to study bi-Hamiltonian pairs of non-homogeneous operators. Compatible operators of the first-order modified by means of a non-local term defined by isometries were considered in [47].

However, conditions of compatibility of homogeneous operators with different degrees of homogeneity become much more difficult to compute as the order increases. In general, the only possibility is the direct calculation of the Schouten bracket $[P, R_k]$. This is a natural operation on variational multivectors (see [11, 25, 38, 42] for an introductory exposition), which involves an increasing amount of calculations for higher degrees of homogeneity. For more details on various possible approaches to compute the Schouten bracket in case of weakly nonlocal operators, see [8, 9, 34].

Some initial attempts to the calculation of compatibility conditions between a first-order homogeneous operator P and a higher-order operator R_k of the simplest type can be found in [6, 28, 35] (where $k = 3$ and $R_3^{ij} = f^{ij}\partial_x^3$, (f^{ij}) is a constant non-degenerate symmetric matrix) and [36] (where $k = 2$ and $R_2^{ij} = \omega^{ij}\partial_x^2$, (ω^{ij}) is a constant non-degenerate skew-symmetric matrix).

The main difficulty in the above calculations is to be able to determine a 'minimal' set of conditions that are necessary and sufficient to the vanishing of the Schouten bracket. Indeed, most of the coefficients are differential consequences of conditions that are 'hidden' inside a few selected coefficients.

In this paper, motivated by the study of bi-Hamiltonian pairs of KdV (see for instance [2, 3, 30]) and WDVV type (see [19, 46, 48, 51]), we compute a (possibly minimal) set of compatibility conditions for a first-order weakly nonlocal homogeneous Hamiltonian operator and a third-order homogeneous Hamiltonian operator.

It is well known that homogeneous operators are invariant with respect to the group of diffeomorphisms of the dependent variables. This suggests one to reduce one of the two considered operators to

[7] It is a Monge–Ampère-type equation rewritten as a first-order quasilinear system of PDEs, see [39, 43].

a canonical form with respect to this group. Such forms are known both for local first-order operators (that can be reduced to constant form thanks to a classical result of Dubrovin and Novikov [15]) and for general third-order operators. These can be reduced to the so-called canonical Doyle–Potëmin form [12, 49, 50]:

$$R_3^{ij} = \mathrm{D}_x \left(f^{ij} \mathrm{D}_x + c_s^{ij} u_x^s \right) \mathrm{D}_x.$$

As a consequence of the Hamiltonian property, f_{ij} is a Monge metric of the form

$$f_{ij} = \phi_{\alpha\beta} \psi_i^\alpha \psi_j^\beta \tag{1}$$

where ϕ is a constant non-degenerate symmetric square matrix of order n, and $\psi_k^\gamma = \psi_{ks}^\gamma u^s + \omega_k^\gamma$, with $\psi_{ij}^\gamma = -\psi_{ji}^\gamma$, is a non-degenerate square matrix of dimension n, with the constants $\phi_{\alpha\beta}$, ψ_{ij}^γ and ω_k^γ satisfying the relations

$$\phi_{\beta\gamma} \left(\psi_{il}^\beta \psi_{jk}^\gamma + \psi_{jl}^\beta \psi_{ki}^\gamma + \psi_{kl}^\beta \psi_{ij}^\gamma \right) = 0, \tag{2a}$$

$$\phi_{\beta\gamma} \left(\omega_i^\beta \psi_{jk}^\gamma + \omega_j^\beta \psi_{ki}^\gamma + \omega_k^\beta \psi_{ij}^\gamma \right) = 0. \tag{2b}$$

The residual group that preserves the canonical forms of P and R_3 is the group of affine transformations of the dependent variables. However, since the conditions that imply the Hamiltonian property of third-order operators are known only in the Doyle–Potëmin form it is natural to assume R_3 to be of such form. This choice simplifies a lot the computations, even if one has to pay the price that the differential geometry of the compatibility conditions is lost.

On the other hand, it is known that Doyle–Potëmin form is preserved by an extended group, that of the *reciprocal projective transformations* [21, 22], which are projective transformations of the dependent variables coupled with a t-preserving reciprocal transformation of the independent variables:

$$\tilde{u}^i = \frac{T_j^i u^j + T_0^i}{\Delta}, \quad \mathrm{d}\tilde{x} = \Delta \, \mathrm{d}x, \quad \mathrm{d}\tilde{t} = \mathrm{d}t \quad \text{where} \quad \Delta = T_j^0 u^j + T_0^0 \neq 0.$$

Since reciprocal projective transformations do not preserve locality we will consider first-order operators containing a weak nonlocality,

$$P^{ij} = g^{ij} \mathrm{D}_x + \Gamma_s^{ij} u_x^s + c^{\alpha\beta} w_{\alpha k}^i u_x^k \mathrm{D}_x^{-1} w_{\beta l}^j u_x^l. \tag{3}$$

This choice, besides being natural from the point of view of the invariance group of the problem, is also justified by the existence of several examples of systems whose bi-Hamiltonian pair contains one of such commuting pairs (e.g. in [51]).

The main results of the paper are listed below.

1. We calculated the full set of compatibility conditions for the operators P and R_3 (this is a nontrivial generalization of the results of [35]), theorems 2 and 5. In particular, we prove that the nonlocal 'tail' coefficients $w_{\alpha j}^i u_x^j$ of the first-order operator are Hamiltonian systems of conservation laws of the third-order operator for every α:

$$u_t^i = \left(w_\alpha^i \right)_x = w_{\alpha k}^i u_x^k = R_3^{ik} \left(\frac{\delta H}{\delta u^k} \right), \qquad i,k = 1, \ldots, n. \tag{4}$$

Such systems are completely described in [23] (see also [53]): they have the form $w_\alpha^i = \psi_\gamma^i Z_\alpha^\gamma$ where (ψ_γ^i) is the inverse matrix of (ψ_i^γ) in (1) and $Z_\alpha^\gamma = \eta_{\alpha k}^\gamma u^k + \xi_\alpha^\gamma$ are linear functions such that the following linear algebraic equations hold:

$$\phi_{\beta\gamma} \left[\psi_{ij}^\beta \eta_{\alpha k}^\gamma + \psi_{jk}^\beta \eta_{\alpha i}^\gamma + \psi_{ki}^\beta \eta_{\alpha j}^\gamma \right] = 0, \tag{5a}$$

$$\phi_{\beta\gamma} \left[\psi_{ik}^\beta \xi_\alpha^\gamma + \omega_k^\beta \eta_{\alpha i}^\gamma - \omega_i^\beta \eta_{\alpha k}^\gamma \right] = 0. \tag{5b}$$

So, the space of the above Hamiltonian systems is a finite-dimensional vector space.

2. We integrated part of the remaining conditions of compatibility, and reduced all other conditions to algebraic equations, theorem 14. In that process, we found the Structure Formula (corollary 12)

$$g^{ij} = \psi^i_\gamma Z^{\gamma j} + \psi^j_\gamma Z^{\gamma i} - c^{\alpha\beta} w^i_\alpha w^j_\beta$$

for the components of the first-order operator (the metric and the nonlocal 'tail'). The summands $\psi^i_\gamma Z^{\gamma j}$ are again Hamiltonian systems of conservation laws of the third-order operator for any $j = 1, \dots, n$, with $Z^{\gamma j} = \eta^{\gamma j}_k u^k + \xi^{\gamma j}$ where the coefficients $\eta^{\gamma j}_k$, $\xi^{\gamma j}$ fulfil the same system of algebraic equations of (5).

3. Finally, in theorem 17 we proved that the Hamiltonian property of the first-order operator implies that the above Hamiltonian systems commute: $V^i_{,k} W^k_{,j} = W^i_{,k} V^k_{,j}$ where '$,k$' stands for the derivative with respect to u^k and V^i, W^i are any of the above Hamiltonian systems $\psi^i_\gamma Z^{\gamma j}$, $w^i_\alpha = \psi^i_\gamma Z^\gamma_\alpha$.

4. We describe the impact of our results on several examples, two of which are new: the first-order Hamiltonian operators for first-order WDVV systems in dimensions $N = 4$ and $N = 5$.

We remark that, years ago, it was found that a third-order homogeneous Hamiltonian operator R_3 is completely determined by a quadratic line complex, which is an algebraic variety in the Plücker embedding of the projective space with affine coordinates (u^i) [21, 22]. Moreover, it was found [23] that systems of conservation laws that are Hamiltonian with respect to R_3 can be identified with linear line congruences, which are also a projective variety in the Plücker embedding of a suitable projective space.

After a reciprocal transformation of projective type, all the above algebraic varieties are preserved. We still do not have a direct proof that first-order operators P that are compatible with third-order operators R_3 are form-invariant with respect to such transformations, but it is very reasonable to conjecture it (see also the discussion in [46]).

The paper is organized as follows. In section 2 we recall the main facts about first- and third-order homogeneous Hamiltonian operators. These facts are used in section 3, that is devoted to the computation of compatibility conditions and to their consequences. Section 4 contains examples of bi-Hamiltonian structures of KdV- and WDVV-type. An appendix is devoted to specific, nontrivial parts of the computation of compatibility conditions.

2. Hamiltonian structures and their compatibility

In this paper, the main problem is to determine the compatibility conditions between two Hamiltonian operators belonging to two distinct classes: a first-order weakly nonlinear Hamiltonian operator P of hydrodynamic type (see, e.g. [17]) and a third-order homogeneous Hamiltonian operator $R := R_3$ [14]. In this Section we will summarize the main properties of operators in these two classes.

2.1. Weakly nonlinear Hamiltonian operator of hydrodynamic type

Weakly nonlinear Hamiltonian operators of hydrodynamic type $P = (P^{ij})$:

$$P^{ij} = g^{ij} \mathrm{D}_x + \Gamma^{ij}_s u^s_x + c^{\alpha\beta} w^i_{\alpha k} u^k_x \mathrm{D}_x^{-1} w^j_{\beta l} u^l_x$$

are a direct nonlocal generalization of Dubrovin–Novikov homogeneous Hamiltonian operators of the first order [15], and were first introduced by Ferapontov (see the review papers [17, 18]). Here g^{ij}, Γ^{ij}_k and $w^i_{\alpha k}$ are functions of the dependent variables $u^i = u^i(t,x)$, and indices i, j, k, l are in the range $\{1, \dots, n\}$. The matrix $(c^{\alpha\beta})$ is square and constant; in principle, its dimension m is not related with n. The operator P reduces to the Dubrovin–Novikov case when $c^{\alpha\beta} = 0$. Throughout this paper, we assume that the leading coefficient matrix (g^{ij}) is non-degenerate: $\det(g^{ij}) \neq 0$. We set $(g_{ij}) = (g^{ij})^{-1}$.

We recall that, under the action of a transformation of the dependent variables $\tilde{u}^i = U^i(u^j)$, $i = 1, \dots, n$, g^{ij} transform as the components of a contravariant 2-tensor, and the quantities $\Gamma^i_{jk} = -g_{jp}\Gamma^{pi}_k$ transform as the Christoffel symbols of a linear connection. Moreover, the quantities $W_\alpha = (w^i_{\alpha j})$ transform as the components of a $(1,1)$ tensor, and when contracted with velocities as in $v^i_\alpha = w^i_{\alpha j} u^j_x$ they transform as the components of a vector field $v_\alpha = w^i_{\alpha j} u^j_x \partial/\partial u^i$.

It is well-known that the operator P is Hamiltonian if and only if its associated bracket is Poisson, i.e., it is skew-symmetric and satisfies the Jacobi identity. This is equivalent to the requirements that P must be formally skew-adjoint and the Schouten bracket $[P,P]$ vanishes. It was proved [18] (see also [45]) that such requirements are equivalent to the following conditions:

1. $g=(g_{ij})$ is a metric, and Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of g; the latter property is equivalent to the conditions

$$g^{ij}_k=\Gamma^{ij}_k+\Gamma^{ji}_k, \tag{6}$$
$$g^{is}\Gamma^{jk}_s=g^{js}\Gamma^{ik}_s \tag{7}$$

where $g^{ij}_{,k}=\partial g^{ij}/\partial u^k$;

2. the $(1,1)$ tensors $W_\alpha=(w^i_{\alpha j})$ are symmetric with respect to the metric g_{ij}:

$$g^{is}w^j_{\alpha s}=g^{js}w^i_{\alpha s};$$

3. the tensors W_α fulfil the further property

$$\nabla_k w^i_{\alpha j}=\nabla_j w^i_{\alpha k}; \tag{8}$$

4. the vector fields $v_\alpha=w^i_{\alpha k}u^k_x\frac{\partial}{\partial u^i}$ are a commuting family:

$$[v_\alpha,v_\beta]=0$$

where $[\cdot,\cdot]$ is the Jacobi bracket [29]. When the components of v_α and v_β are total x-derivatives: $w^i_{\alpha j}u^j_x=(w^i_\alpha)_x$ (and similarly for v_β), the above condition reduces to the commutativity of the matrix product: $W_\alpha W_\beta=W_\beta W_\alpha$, or explicitly

$$w^i_{\alpha k}w^k_{\beta j}=w^i_{\beta k}w^k_{\alpha j}.$$

When $w^i_{\alpha j}u^j_x=(w^i_\alpha)_x$, we will use the notation $w^i_{\alpha j}=w^i_{\alpha,j}=\partial w^i_\alpha/\partial u^j$;

5. the Riemannian curvature[8] of the Levi-Civita connection of g_{ij} fulfils

$$R^{ij}_{kl}=c^{\alpha\beta}\left(w^i_{\alpha l}w^j_{\beta k}-w^i_{\alpha k}w^j_{\beta l}\right); \tag{9}$$

6. the matrix $(c^{\alpha\beta})$ is symmetric and nondegenerate.

The above conditions have a nice geometric interpretation that is given in [17, 18].

2.2. Third-order Hamiltonian operator in canonical form

The most general form of a third-order homogeneous operator $R=(R^{ij})$, in accordance with the definition given in [14], is

$$R^{ij}=f^{ij}\mathrm{D}_x^3+b^{ij}_s u^s_x\mathrm{D}_x^2+\left[c^{ij}_s u^s_{xx}+c^{ij}_{st}u^s_x u^t_x\right]\mathrm{D}_x+d^{ij}_s u^s_{xxx}+d^{ij}_{st}u^s_x u^t_{xx}+d^{ij}_{srt}u^s_x u^r_x u^t_x,$$

but in this generality a set of conditions that ensure the Hamiltonian property of R is unknown. (As usual we assume that (f^{ij}) is a non-degenerate matrix, and we set $(f_{ij})=(f^{-1})^{ij}$.) What is known though is that, as in the first-order case, f_{ij} transform as the components of a covariant metric. Moreover, the quantities $-\frac{1}{3}f_{jk}d^{ki}_s$ transform as the Christoffel symbols of a linear connection, which is symmetric and flat as proved independently in [12] and [49, 50]. This result has also been generalized to higher-order Hamiltonian operators [7].

On the other hand, there always exists a change of dependent variables that brings a Hamiltonian operator of the above form into what we call *Doyle–Potëmin canonical form*,

$$R^{ij}=\mathrm{D}_x\left(f^{ij}\mathrm{D}_x+c^{ij}_s u^s_x\right)\mathrm{D}_x.$$

This form has considerably less coefficients, and the Hamiltonian property is equivalent to

$$f^{ij}_k=c^{ij}_k+c^{ji}_k, \tag{10a}$$

$$c^{ij}_k f^{kp}=-c^{pj}_k f^{ki}, \tag{10b}$$

[8] We follow the Kobayashi–Nomizu convention for the Riemannian curvature; see [45] for more details.

$$c^{ij}_k f^{kp} + c^{jp}_k f^{ki} + c^{pi}_k f^{kj} = 0, \tag{10c}$$

$$c^{ij}_{k,l} f^{kp} = c^{ip}_k c^{kj}_l - c^{pi}_k c^{kj}_l - c^{pj}_k c^{ik}_l - c^{pj}_k c^{ki}_l. \tag{10d}$$

It was proved [21] that, introducing the quantities $c_{ijk} = f_{iq} f_{jp} c^{pq}_k$ the above system simplifies into

$$c_{ijk} = \frac{1}{3}\left(f_{ik,j} - f_{ij,k}\right), \tag{11a}$$

$$f_{ij,k} + f_{jk,i} + f_{ki,j} = 0, \tag{11b}$$

$$c_{ijk,l} = -f^{pq} c_{pil} c_{qjk}. \tag{11c}$$

A covariant 2-tensor f satisfying equations (11b) must be a Monge metric of a quadratic line complex (see [21] and reference therein). This is an algebraic variety that is a section of the Plücker variety by means of a quadric. Here, the Plücker embedding that we are considering is that of the projective space whose affine chart is (u^i). In particular, being Monge, f_{ij} is a quadratic polynomial in the field variables u. It was further proved in [4, 12, 49, 50] that the tensor f_{ij} can be factorized as

$$f_{ij} = \phi_{\alpha\beta}\psi^\alpha_i \psi^\beta_j \quad \left(\text{or, in a matrix form,} \quad f = \Psi\Phi\Psi^\top\right)$$

where ϕ is a constant non-degenerate symmetric matrix of order n, and $\psi^\gamma_k = \psi^\gamma_{ks} u^s + \omega^\gamma_k$, with $\psi^\gamma_{ij} = -\psi^\gamma_{ji}$, is a non-degenerate square matrix of dimension n, with the constants $\phi_{\beta\gamma}$, ψ^γ_{ij} and ω^γ_k satisfying the relations (2).

It is known [23] that a third-order Hamiltonian operator R is the Hamiltonian operator of a first-order system of conservation laws of the form $u^i_t = (V^i)_x$, $i = 1, \dots, n$ (see (4)) if

$$f_{im} V^m_{,j} = f_{jm} V^m_{,i}, \tag{12a}$$

$$c_{mkl} V^m_{,i} + c_{mik} V^m_{,l} + c_{mli} V^m_k = 0, \tag{12b}$$

$$f_{ks} V^k_{,ij} = c_{smj} V^m_{,i} + c_{smi} V^m_{,j}, \tag{12c}$$

where $V^i_{,j} = \partial V^i / \partial u^j$. Nevertheless, it turns out that this result can be improved.

Theorem 1. *In the system (12) the conditions (12b) are a consequence of (12a) and (12c).*

Proof. Let us take the difference of the condition (12c) with the same condition with the indices s and j exchanged:

$$f_{ks} V^k_{,ij} - f_{kj} V^k_{,is} - c_{skj} V^k_{,i} - c_{ski} V^k_{,j} + c_{jks} V^k_{,i} + c_{jki} V^k_{,s} = 0. \tag{13}$$

We have

$$\begin{aligned} f_{ks} V^k_{,ij} - f_{kj} V^k_{,is} &= \left(f_{ks} V^k_{,j} - f_{kj} V^k_{,s}\right)_{,i} - \left(f_{ks,i} V^k_{,j} - f_{kj,i} V^k_{,s}\right) \\ &= \left(c_{ksi} + c_{ski}\right) V^k_{,j} - \left(c_{kji} + c_{jki}\right) V^k_{,s}, \end{aligned}$$

where we used (12a). Then, (13) becomes

$$\begin{aligned} &- c_{skj} V^k_{,i} - c_{ski} V^k_{,j} + c_{jks} V^k_{,i} + c_{jki} V^k_{,s} + \left(c_{ksi} + c_{ski}\right) V^k_{,j} - \left(c_{kji} + c_{jki}\right) V^k_{,s} \\ &\quad = -c_{skj} V^k_{,i} + c_{jks} V^k_{,i} + c_{ksi} V^k_{,j} - c_{kji} V^k_{,s} \\ &\quad = \left(c_{jsk} + c_{kjs}\right) V^k_{,i} + c_{jks} V^k_{,i} + c_{ksi} V^k_{,j} - c_{kji} V^k_{,s} \\ &\quad = c_{kjs} V^k_{,i} + c_{ksi} V^k_{,j} + c_{kij} V^k_{,s} = 0. \end{aligned}$$

□

The interested reader can find in [23] the general expression of the Hamiltonian system of conservation laws of the above type.

3. Conditions of compatibility

3.1. Variational Schouten bracket of operators

In what follows we will assume that ψ denotes a covector-valued density: $\psi = \psi_i \mathrm{d}u^i \otimes (\mathrm{d}t \wedge \mathrm{d}x)$ and $\phi = \phi^i \partial/\partial u^i$ denotes a (generalized) vector field. Their coefficients depend on field variables (u^i) and their x-derivatives up to a finite (although unspecified) order. Such elements are dual in the geometric theory of the calculus of variations; conservation laws are uniquely represented by their characteristic vectors, which are covector-valued densities, and symmetries are generalized vector fields [5, 42].

A Hamiltonian operator A can be regarded as a variational bivector, i.e. a bi-differential operator taking two covectors as arguments, with values into densities: $A(\psi^1, \psi^2)$ is a density, and it is defined up to total x-derivatives. The variational Schouten bracket of two Hamiltonian operators $[A_1, A_2]$ is a variational three-vector. We will use the following formula for the variational Schouten bracket (this formula is used by Dubrovin and Zhang [13]):

$$[A_1, A_2]\left(\psi^1, \psi^2, \psi^3\right) = \left[\ell_{A_1,\psi^1}\left(A_2\left(\psi^2\right)\right)\left(\psi^3\right) + \ell_{A_2,\psi^1}\left(A_1\left(\psi^2\right)\right)\left(\psi^3\right) + \text{cyclic}\left(\psi^1, \psi^2, \psi^3\right)\right].$$

See [5, 11, 24, 25, 38, 42] for a comprehensive theory. The square brackets in the right-hand side of the above expression mean that the result is an equivalence class with respect to total x-derivatives. In what follows we will drop the square brackets for ease of reading, although retaining the fact that computations hold up to total x-derivatives.

We recall the formula for the linearization of an operator A and its adjoint. If $A(\psi) = a^{ij\sigma}\mathrm{D}_\sigma\psi_j$ then

$$\ell_{A,\psi}(\phi) = \frac{\partial a^{ij\sigma}}{\partial u^k_\tau}\mathrm{D}_\sigma\psi_j\mathrm{D}_\tau\phi^k, \quad \ell^*_{A,\psi^1}\left(\psi^2\right) = (-1)^{|\tau|}\,\mathrm{D}_\tau\left(\psi^2_i\frac{\partial a^{ij\sigma}}{\partial u^k_\tau}\mathrm{D}_\sigma\psi^1_j\right).$$

Hereafter, σ is a multiindex that denotes with respect to which arguments the differentiation takes place. We recall that if Δ, $\square$ are two operators in total derivatives then $(\Delta\circ\square)^* = \square^*\circ\Delta^*$. Note that $(\mathrm{D}^{-1})^* = -\mathrm{D}^{-1}$. Moreover, we recall that *linearization commutes with total derivatives also in case of negative exponent*. There is a linearization formula for the composition of operators:

$$\ell_{A\circ B,\psi}(\varphi) = \ell_{A,B\psi}(\varphi) + A\circ\ell_{B,\psi}(\varphi).$$

In view of next computations we introduce the following notation:

$$R = \mathrm{D}_x\circ F\circ\mathrm{D}_x \quad \text{where} \quad F = f^{ij}\mathrm{D}_x + c^{ij}_k u^k_x,$$
$$P = L + c^{\alpha\beta}N_{\alpha\beta} \quad \text{where} \quad L = g^{ij}\mathrm{D}_x + \Gamma^{ij}_k u^k_x, \quad N_{\alpha\beta} = w^j_{\alpha k}u^k_x\mathrm{D}_x{}^{-1}w^j_{\beta h}u^h_x$$

Then we have

$$[P, R] = [L, R] + c^{\alpha\beta}[N_{\alpha\beta}, R]$$

where

$$[L, R]\left(\psi^1, \psi^2, \psi^3\right) = \ell_{L,\psi^1}\left(R\psi^2\right)\psi^3 + \mathrm{D}_x\left(\ell_{F,\mathrm{D}_x\psi^1}\left(L\psi^2\right)\right)\psi^3 + \text{cyclic}\left(\psi^1, \psi^2, \psi^3\right)$$

and

$$[N_{\alpha\beta}, R]\left(\psi^1, \psi^2, \psi^3\right) = \ell_{N,\psi^1}\left(R\psi^2\right)\psi^3 + \mathrm{D}_x\left(\ell_{F,\mathrm{D}_x\psi^1}\left(N_{\alpha\beta}\psi^2\right)\right)\psi^3 + \text{cyclic}\left(\psi^1, \psi^2, \psi^3\right).$$

The building blocks of our computation are the linearizations of L, F and $N_{\alpha\beta}$,

$$\ell_{L,\psi^1}(\varphi) = g^{ij}_k\mathrm{D}_x\psi^1_j\varphi^k + \left(\Gamma^{ij}_{h,k}u^h_x\varphi^k + \Gamma^{ij}_h\mathrm{D}_x\varphi^h\right)\psi^1_j,$$
$$\ell_{F,\psi^1}(\varphi) = f^{ij}_k\mathrm{D}_x\psi^1_j\varphi^k + \left(c^{ij}_{h,k}u^h_x\varphi^k + c^{ij}_h\mathrm{D}_x\varphi^h\right)\psi^1_j,$$
$$\ell_{N_{\alpha\beta},\psi^1}(\varphi) = \left(w^i_{\alpha k,l}u^k_x\varphi^l + w^i_{\alpha k}\mathrm{D}_x\varphi^k\right)\mathrm{D}_x{}^{-1}\left(w^j_{\beta h}u^h_x\psi^1_j\right) + w^i_{\alpha h}u^h_x\mathrm{D}_x{}^{-1}\left(w^j_{\beta h,l}u^h_x\varphi^l + w^j_{\beta h}\mathrm{D}_x\varphi^h\right)\psi^1_j.$$

Introducing the notation

$$\tilde\psi^i_\alpha = \mathrm{D}_x{}^{-1}\left(w^j_{\alpha k}u^k_x\psi^i_j\right),$$

we can rewrite the nonlocal part of the linearization (up to total x-derivatives, or 'integrating by parts') as

$$\ell_{N_{\alpha\beta},\psi^1}(\varphi) = \left(w^i_{\alpha k,l}u^k_x\varphi^l + w^i_{\alpha k}\mathrm{D}_x\varphi^k\right)\tilde{\psi}^1_\beta - \left(w^j_{\beta h,l}u^h_x\varphi^l + w^j_{\beta h}\mathrm{D}_x\varphi^h\right)\tilde{\psi}^1_\alpha.$$

In order to compute the components of the three-vector $[P,R]$, we will use the algorithm developed in [9] (see also [8]). Then, we will gradually simplify the coefficients of the three-vector using the Hamiltonian properties of P and R. It will leave us with a set of conditions whose vanishing is equivalent to the compatibility of P and R.

Since the expressions of the coefficients of the three-vector are huge we limit ourselves to giving a precise description on how to obtain them, describing the main points and issues of the simplification process and presenting the conditions under which all coefficients of the three-vector vanish.

3.2. The algorithm

The three-vector $[P,R]$ has coefficients that do not determine it uniquely, in view of the fact that $[P,R]$ is defined up to total x-derivatives. So, our aim is to obtain a unique, total x-derivative free representative of $[P,R]$. The algorithm in [9] is divided in two parts.

First step: computing the nonlocal part. We bring all summands that are linear in $\tilde{\psi}^i_\alpha$'s to the form

$$c^{jp}\tilde{\psi}^a_\alpha \mathrm{D}_x{}^n\psi^b_j\psi^c_p \tag{14}$$

where c^{jp} are coefficients depending on (u^i,u^i_σ) and $(a,b,c)\in\{(1,2,3),(3,1,2),(2,3,1)\}$. We will do this by 'integration by parts'; for example,

$$c^{jp}\tilde{\psi}^1_\alpha\psi^2_j\mathrm{D}_x{}^n\psi^3_p = (-1)^n\mathrm{D}_x{}^n\left(c^{jp}\tilde{\psi}^1_\alpha\psi^2_j\right)\psi^3_p + \text{total } x-\text{derivative terms},$$

so that any summand which is not of the type of (14) is brought to that type up to total x-derivatives.

In the process, we will create new local terms that will be added to the local part of the three-vector. Note that $[P,R]$ does not contain, in this particular case, nonlocal terms that are quadratic in $\tilde{\psi}^i_\alpha$'s.

Second step: computing the local part. We operate on the local part of the three-vector as usual. More precisely, we bring the three-vector to a form that is of zero order with respect to the argument ψ^3 using 'integration by parts'; this form is free from total x-derivatives. For example,

$$c^{ijp}\mathrm{D}_x{}^{m_1}\psi^1_i\mathrm{D}_x{}^{m_2}\psi^2_j\mathrm{D}_x{}^{m_3}\psi^3_p = (-1)^{m_3}\mathrm{D}_x{}^{m_3}\left(c^{jp}\mathrm{D}_x{}^{m_1}\psi^1_i\mathrm{D}_x{}^{m_2}\psi^2_j\right)\psi^3_p + \text{total } x-\text{derivative terms}.$$

3.3. Preparatory calculations

First of all, we need to compute the four summands in $[P,R]$, after which we will apply a cyclic permutation:

$$\begin{aligned}
\ell_{L,\psi^1}(R(\psi^2))(\psi^3) &= g^{pi}_k\partial_x\psi^1_i\left(f^{kj}\partial^3_x\psi^2_j + ((f^{kj}_{,h}+c^{kj}_h)u^h_x)\partial^2_x\psi^2_j + (c^{kj}_{h,l}u^l_xu^h_x + c^{kj}_hu^h_{2x})\partial_x\psi^2_j\right)\psi^3_p\\
&+ \Big(\Gamma^{pi}_{h,k}u^h_x\left(f^{kj}\partial^3_x\psi^2_j + ((f^{kj}_{,h}+c^{kj}_h)u^h_x)\partial^2_x\psi^2_j + (c^{kj}_{h,l}u^l_xu^h_x + c^{kj}_hu^h_{2x})\partial_x\psi^2_j\right)\\
&+ \Gamma^{pi}_k\partial_x\left(f^{kj}\partial^3_x\psi^2_j + ((f^{kj}_{,h}+c^{kj}_h)u^h_x)\partial^2_x\psi^2_j + (c^{kj}_{h,l}u^l_xu^h_x + c^{kj}_hu^h_{2x})\partial_x\psi^2_j\right)\Big)\psi^1_i\psi^3_p,\\
\partial_x(\ell_{F,\partial_x(\psi^1)}(L(\psi^2)))(\psi^3) &= \partial_x\Big(f^{pi}_k\partial^2_x\psi^1_i\left(g^{kj}\partial_x(\psi^2_j) + \Gamma^{kj}_hu^h_x\psi^2_j\right)\\
&+ \left(c^{pi}_{h,k}u^h_x\left(g^{kj}\partial_x(\psi^2_j) + \Gamma^{kj}_hu^h_x\psi^2_j\right) + c^{pi}_h\partial_x\left(g^{hj}\partial_x(\psi^2_j) + \Gamma^{hj}_lu^l_x\psi^2_j\right)\right)\partial_x(\psi^1_i)\Big)\psi^3_p,\\
\ell_{N_{\alpha\beta},\psi^1}(R(\psi^2))(\psi^3) &= \Big(w^p_{\alpha k,l}u^k_x\left(f^{lj}\partial^3_x\psi^2_j + ((f^{lj}_{,h}+c^{lj}_h)u^h_x)\partial^2_x\psi^2_j + (c^{lj}_{h,k}u^k_xu^h_x + c^{lj}_hu^h_{2x})\partial_x\psi^2_j\right)\\
&+ w^p_{\alpha k}\partial_x\left(f^{kj}\partial^3_x\psi^2_j + ((f^{kj}_{,h}+c^{kj}_h)u^h_x)\partial^2_x\psi^2_j + (c^{kj}_{h,l}u^l_xu^h_x + c^{kj}_hu^h_{2x})\partial_x\psi^2_j\right)\Big)\tilde{\psi}^1_\beta\psi^3_p\\
&- \Big(w^i_{\beta h,l}u^h_x\left(f^{lj}\partial^3_x\psi^2_j + ((f^{lj}_k+c^{lj}_k)u^k_x)\partial^2_x\psi^2_j + (c^{lj}_{h,k}u^k_xu^h_x + c^{lj}_hu^h_{2x})\partial_x\psi^2_j\right)\\
&+ w^i_{\beta h}\partial_x\left(f^{hj}\partial^3_x\psi^2_j + ((f^{hj}_k+c^{hj}_k)u^k_x)\partial^2_x\psi^2_j + (c^{hj}_{l,k}u^k_xu^l_x + c^{hj}_ku^k_{2x})\partial_x\psi^2_j\right)\Big)\psi^1_i\tilde{\psi}^3_\alpha,\\
\partial_x(\ell_{F,\partial_x(\psi^1)}(N_{\alpha\beta}(\psi^2)))(\psi^3) &= \partial_x\left(f^{pi}_k\partial^2_x\psi^1_iw^k_{\alpha l}u^l_x\tilde{\psi}^2_\beta + \left(c^{pi}_{h,k}u^h_xw^k_{\alpha l}u^l_x\tilde{\psi}^2_\beta + c^{pi}_h\partial_x\left(w^h_{\alpha l}u^l_x\tilde{\psi}^2_\beta\right)\right)\partial_x\psi^1_i\right)\psi^3_p.
\end{aligned}$$

Let us introduce the following notation:

$$T_{LR} = [L,R]\left(\psi^1,\psi^2,\psi^3\right), \qquad T_{N_{\alpha\beta}} = [N_{\alpha\beta},R]\left(\psi^1,\psi^2,\psi^3\right), \qquad T_{N_{\alpha\beta}} = \mathfrak{N}_{\alpha\beta} + \mathfrak{L}_{\alpha\beta}$$

where $\mathfrak{N}_{\alpha\beta}$ contains only nonlocal terms and $\mathfrak{L}_{\alpha\beta}$ contains only local terms. We further split the above expressions into the following summands:

$$\begin{aligned} T_{LR} &= T_{LR}^{(123)} + T_{LR}^{(312)} + T_{LR}^{(231)},\\ \mathfrak{N}_{\alpha\beta} &= \mathfrak{N}_{\alpha\beta}^{(123)} + \mathfrak{N}_{\alpha\beta}^{(312)} + \mathfrak{N}_{\alpha\beta}^{(231)},\\ \mathfrak{L}_{\alpha\beta} &= \mathfrak{L}_{\alpha\beta}^{(123)} + \mathfrak{L}_{\alpha\beta}^{(312)} + \mathfrak{L}_{\alpha\beta}^{(231)}, \end{aligned}$$

where the superscript indicate the order of the arguments ψ^1, ψ^2, ψ^3 in the above four summands with respect to the cyclic permutation.

3.4. Computing the nonlocal part

Applying the first step of the algorithm to $c^{\alpha\beta}\mathfrak{N}_{\alpha\beta}$ yields the nonlocal part T_n of the three-vector $[P,R]$ in a canonical form. We will obtain all three of its components with respect to the three different orderings of the arguments ψ^1, ψ^2, ψ^3 that we chose in (14), and we write $T_n = T_n^{(123)} + T_n^{(312)} + T_n^{(231)}$. We list only the $(3,1,2)$ sector:

$$\begin{aligned} T_n^{(312)} &= c^{\alpha\beta}\left(w^q_{\beta k}f^{kj} - w^j_{\beta k}f^{kq}\right)\tilde\psi^3_\alpha \mathrm{D}_x{}^4\psi^1_j\psi^2_q\\ &+ 2c^{\alpha\beta}\left(-c^{jq}_k w^k_{\beta p} + w^q_{\beta k}c^{jk}_p + c^{qj}_k w^k_{\beta p} - w^j_{\beta k}c^{qk}_p\right)u^p_x\tilde\psi^3_\alpha \mathrm{D}_x{}^3\psi^1_j\psi^2_q\\ &+ c^{\alpha\beta}\left(\mathrm{D}_x\left(\left(-w^j_{\beta h}c^{qh}_p + c^{qj}_k w^k_{\beta p} - c^{jq}_k w^k_{\beta p} + w^q_{\beta h}c^{jh}_p\right)u^p_x\right) + S^{jq}_{\beta lp}u^l_x u^p_x\right)\tilde\psi^3_\alpha \mathrm{D}_x{}^2\psi^1_j\psi^2_q\\ &+ c^{\alpha\beta}\left(\mathrm{D}_x\left(S^{jq}_{\beta lp}u^l_x u^p_x\right) + c^{\alpha\beta}\left(-c^{qj}_k w^k_{\alpha p} + w^q_{\alpha k}c^{kj}_p - w^j_{\beta p,k}f^{kq}\right)u^p_{xxx}\right)\tilde\psi^3_\alpha \mathrm{D}_x\psi^1_j\psi^2_q\\ &+ c^{\alpha\beta}\left(w^j_{\beta p,k} - w^j_{\beta k,p}\right)u^p_{xxxx}f^{kq}\tilde\psi^3_\alpha\psi^1_j\psi^2_q \end{aligned}$$

where

$$S^{jq}_{\beta lp} := w^j_{\beta l,k}c^{kq}_p - c^{jq}_{k,l}w^k_{\beta p} + c^{jq}_{l,k}w^k_{\beta p} - w^q_{\beta k,l}c^{kj}_p.$$

Theorem 2. *The nonlocal part T_n of the three-vector $[P,R]$ in a canonical form vanishes if and only if the following conditions are satisfied:*

$$w^j_{\beta p,k} = w^j_{\beta k,p}, \tag{15a}$$

$$w^q_{\beta k}f^{kj} = w^j_{\beta k}f^{kq}, \tag{15b}$$

$$w^j_{\alpha p,k}f^{kq} = w^q_{\alpha k}c^{kj}_p - c^{qj}_k w^k_{\alpha p}. \tag{15c}$$

Proof. Indeed, $S^{jq}_{\beta lp} + S^{jq}_{\beta pl}$ vanishes identically in view of the known conditions, see lemma 18 in the appendix. In turn, the identity

$$-c^{jq}_k w^k_{\beta p} + w^q_{\beta k}c^{jk}_p + c^{qj}_k w^k_{\beta p} - w^j_{\beta k}c^{qk}_p = 0 \tag{16}$$

is a simple differential consequence of (15*b*). □

The condition (15*a*) implies that there exist w^i_α such that $w^i_{\alpha k} = w^i_{\alpha,k} = \partial w^i_\alpha/\partial u^k$.

Corollary 3. *The potentials w^i_α of the coefficients of the nonlocal part of P can be interpreted, for every index α, as fluxes of systems of first-order conservation laws that are Hamiltonian with respect to R:*

$$u^i_t = \left(w^i_\alpha\right)_x = w^i_{\alpha,k}u^k_x = R^{ih}\left(\frac{\delta H}{\delta u^h}\right).$$

Moreover, we have $w^i_\alpha = \psi^i_\gamma Z^\gamma_\alpha$ where $(\psi^i_\gamma) = (\psi^\gamma_i)^{-1}$ and ψ^γ_i is a part of the decomposition of the Monge metric $f_{ij} = \phi_{\alpha\beta}\psi^\alpha_i\psi^\beta_j$ (see section 2.2), and $Z^\gamma_\alpha = \eta^\gamma_{\alpha k}u^k + \xi^\gamma_\alpha$ are linear functions that solve the linear algebraic system

$$\begin{aligned} \phi_{\beta\gamma}\left[\psi^\beta_{ij}\eta^\gamma_{\alpha k} + \psi^\beta_{jk}\eta^\gamma_{\alpha i} + \psi^\beta_{ki}\eta^\gamma_{\alpha j}\right] &= 0,\\ \phi_{\beta\gamma}\left[\psi^\beta_{ik}\xi^\gamma_\alpha + \omega^\beta_k\eta^\gamma_{\alpha i} - \omega^\beta_i\eta^\gamma_{\alpha k}\right] &= 0. \end{aligned}$$

Proof. The conditions (15*b*) and (15*c*) are exactly the conditions found in [23] for Hamiltonian systems of R (see also section 2.2). □

Remark 4. Note that we proved in theorem 1 that one of the conditions in [23] is redundant, namely (12*b*); that is the same as (16) after lowering the indices.

3.5. Computing the local part

We shall compute a canonical form of the local part of the three-vector $[P, R]$, which is $T_{LR} + c^{\alpha\beta}\mathfrak{L}_{\alpha\beta}$. Its expression can be easily obtained from $T^{(123)}_{LR} + c^{\alpha\beta}\mathfrak{L}^{(123)}_{\alpha\beta}$ by means of a cyclic permutation. A long computation yields the following expressions:

$$
\begin{aligned}
T^{(123)}_{LR} = {} & g^{pi}_k f^{kj} \partial_x \psi^1_i \partial^3_x \psi^2_j \psi^3_p + g^{pi}_k \left(\left(f^{kj}_{,h} + c^{kj}_h \right) u^h_x \right) \partial_x \psi^1_i \partial^2_x \psi^2_j \psi^3_p + g^{pi}_k \left(c^{kj}_{h,l} u^l_x u^h_x + c^{kj}_h u^h_{2x} \right) \partial_x \psi^1_i \partial_x \psi^2_j \psi^3_p \\
& + \left(\Gamma^{pi}_{h,k} u^h_x f^{kj} + \Gamma^{pi}_k \partial_x \left(f^{kj} \right) + \Gamma^{pi}_k \left(\left(f^{kj}_{,h} + c^{kj}_h \right) u^h_x \right) \right) \psi^1_i \partial^3_x \psi^2_j \psi^3_p \\
& + \left(\Gamma^{pi}_{h,k} u^h_x \left(\left(f^{kj}_{,m} + c^{kj}_m \right) u^m_x \right) + \Gamma^{pi}_k \partial_x \left(\left(\left(f^{kj}_{,h} + c^{kj}_h \right) u^h_x \right) \right) + \Gamma^{pi}_k \left(c^{kj}_{h,l} u^l_x u^h_x + c^{kj}_h u^h_{2x} \right) \right) \psi^1_i \partial^2_x \psi^2_j \psi^3_p \\
& + \Gamma^{pi}_k f^{kj} \psi^1_i \partial^4_x \psi^2_j \psi^3_p + \left(\Gamma^{pi}_{h,k} u^h_x \left(c^{kj}_{m,l} u^l_x u^m_x + c^{kj}_m u^m_{2x} \right) + \Gamma^{pi}_k \partial_x \left(c^{kj}_{h,l} u^l_x u^h_x + c^{kj}_h u^h_{2x} \right) \right) \psi^1_i \partial_x \psi^2_j \psi^3_p \\
& + \left(\partial_x \left(f^{pi}_k \right) g^{kj} + f^{pi}_k \partial_x \left(g^{kj} \right) + f^{pi}_k \Gamma^{kj}_h u^h_x + c^{pi}_{h,k} u^h_x g^{kj} + c^{pi}_h \partial_x \left(g^{hj} \right) + c^{pi}_h \Gamma^{hj}_l u^l_x \right) \partial^2_x \psi^1_i \, \partial_x \psi^2_j \psi^3_p \\
& + \left(\partial_x \left(f^{pi}_k \right) \Gamma^{kj}_h u^h_x + f^{pi}_k \partial_x (\Gamma^{kj}_h u^h_x) + c^{pi}_{h,k} u^h_x \Gamma^{kj}_l u^l_x + c^{pi}_h \partial_x (\Gamma^{hj}_l u^l_x) \right) \partial^2_x \psi^1_i \, \psi^2_j \psi^3_p \\
& + f^{pi}_k g^{kj} \partial^3_x \psi^1_i \partial_x (\psi^2_j) \psi^3_p + f^{pi}_k \Gamma^{kj}_h u^h_x \partial^3_x \psi^1_i \psi^2_j \psi^3_p + \left(f^{pi}_k g^{kj} + c^{pi}_h g^{hj} \right) \partial^2_x \psi^1_i \partial^2_x \psi^2_j \psi^3_p \\
& + \left(\partial_x (c^{pi}_{h,k} u^h_x g^{kj}) + c^{pi}_{h,k} u^h_x \Gamma^{kj}_l u^l_x + \partial_x \left(c^{pi}_h \partial_x (g^{hj}) \right) + c^{pi}_h \partial_x (\Gamma^{hj}_l u^l_x) + \partial_x (c^{pi}_h \Gamma^{hj}_l u^l_x) \right) \partial_x (\psi^1_i) \partial_x \psi^2_j \psi^3_p \\
& + \left(c^{pi}_{h,k} u^h_x g^{kj} + c^{pi}_h \partial_x (g^{hj}) + \partial_x (c^{pi}_h g^{hj}) + c^{pi}_h \Gamma^{hj}_l u^l_x \right) \partial_x \psi^1_i \partial^2_x \psi^2_j \psi^3_p \\
& + \left(\partial_x (c^{pi}_{h,k} u^h_x \Gamma^{kj}_l u^l_x) + \partial_x (c^{pi}_h \partial_x (\Gamma^{hj}_l u^l_x)) \right) \partial_x (\psi^1_i) \psi^2_j \psi^3_p + c^{pi}_h g^{hj} \partial_x \psi^1_i \partial^3_x \psi^2_j \psi^3_p
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{L}^{(123)}_{\alpha\beta} = {} & t_{000} \psi^1_i \psi^2_j \psi^3_p + t_{001} \psi^1_i \psi^2_j \mathrm{D}_x \psi^3_p + t_{002} \psi^1_i \psi^2_j \mathrm{D}^2_x \psi^3_p + t_{003} \psi^1_i \psi^2_j \mathrm{D}^3_x \psi^3_p + t_{010} \psi^1_i \mathrm{D}_x \psi^2_j \psi^3_p \\
& + t_{020} \psi^1_i \mathrm{D}^2_x \psi^2_j \psi^3_p + t_{011} \psi^1_i \mathrm{D}_x \psi^2_j \mathrm{D}_x \psi^3_p + t_{100} \mathrm{D}_x \psi^1_i \psi^2_j \psi^3_p + t_{101} \mathrm{D}_x \psi^1_i \psi^2_j \mathrm{D}_x \psi^3_p \\
& + t_{110} \mathrm{D}_x \psi^1_i \mathrm{D}_x \psi^2_j \psi^3_p + t_{102} \mathrm{D}_x \psi^1_i \psi^2_j \mathrm{D}^2_x \psi^3_p + t_{200} \mathrm{D}^2_x \psi^1_i \psi^2_j \psi^3_p + t_{201} \mathrm{D}^2_x \psi^1_i \psi^2_j \mathrm{D}_x \psi^3_p + t_{300} \mathrm{D}^3_x \psi^1_i \psi^2_j \psi^3_p
\end{aligned}
$$

where the coefficients are:

$$
\begin{aligned}
t_{000} = {} & -w^i_{\beta h} f^{hj} \partial^3_x \left(w^p_{\alpha k} u^k_x \right) - 4 \partial^3_x \left(w^i_{\beta h} f^{hj} \right) w^p_{\alpha k} u^k_x - 4 \partial_x \left(w^i_{\beta h} f^{hj} \right) \partial^2_x \left(w^p_{\alpha k} u^k_x \right) - 6 \partial^2_x \left(w^i_{\beta h} f^{hj} \right) \partial_x \left(w^p_{\alpha k} u^k_x \right) \\
& + \left(w^i_{\beta h,l} u^h_x f^{lj} + w^i_{\beta h} \partial_x \left(f^{hj} \right) + w^i_{\beta h} \left(\left(f^{hj}_k + c^{hj}_k \right) u^k_x \right) \right) \partial^2_x \left(w^p_{\alpha m} u^m_x \right) \\
& + 3 \partial^2_x \left(w^i_{\beta h,l} u^h_x f^{lj} + w^i_{\beta h} \partial_x \left(f^{hj} \right) + w^i_{\beta h} \left(\left(f^{hj}_k + c^{hj}_k \right) u^k_x \right) \right) w^p_{\alpha m} u^m_x \\
& + 3 \partial_x \left(w^i_{\beta h,l} u^h_x f^{lj} + w^i_{\beta h} \partial_x \left(f^{hj} \right) + w^i_{\beta h} \left(f^{hj}_k + c^{hj}_k \right) u^k_x \right) \partial_x \left(w^p_{\alpha m} u^m_x \right) \\
& - \left(w^i_{\beta h,l} u^h_x \left(\left(f^{lj}_k + c^{lj}_k \right) u^k_x \right) + w^i_{\beta h} \partial_x \left(\left(f^{hj}_k + c^{hj}_k \right) u^k_x \right) + w^i_{\beta h} (c^{hj}_{l,k} u^k_x u^l_x + c^{hj}_k u^k_{2x}) \right) \partial_x \left(w^p_{\alpha m} u^m_x \right) \\
& - 2 \partial_x \left(\left(w^i_{\beta h,l} u^h_x ((f^{lj}_k + c^{lj}_k) u^k_x) + w^i_{\beta h} \partial_x ((f^{hj}_k + c^{hj}_k) u^k_x) + w^i_{\beta h} (c^{hj}_{l,k} u^k_x u^l_x + c^{hj}_k u^k_{2x}) \right) \right) w^p_{\alpha m} u^m_x \\
& + \left(w^i_{\beta q,l} u^q_x (c^{lj}_{h,k} u^k_x u^h_x + c^{lj}_h u^h_{2x}) + w^i_{\beta h} \partial_x (c^{hj}_{l,k} u^k_x u^l_x + c^{hj}_k u^k_{2x}) \right) w^p_{\alpha m} u^m_x \\
& + \left(\partial_x (f^{pi}_k w^k_{\alpha l} u^l_x) + c^{pi}_{h,k} u^h_x w^k_{\alpha l} u^l_x + c^{pi}_h \partial_x (w^h_{\alpha l} u^l_x) \right) \partial_x \left(w^j_{\beta m} u^m_x \right) \\
& + 2 \partial_x \left(\partial_x (f^{pi}_k w^k_{\alpha l} u^l_x) + c^{pi}_{h,k} u^h_x w^k_{\alpha l} u^l_x + c^{pi}_h \partial_x (w^h_{\alpha l} u^l_x) \right) w^j_{\beta m} u^m_x \\
& - f^{pi}_{,h} w^h_{\alpha l} u^l_x \partial^2_x \left(w^j_{\beta k} u^k_x \right) - 3 \partial^2_x \left(f^{pi}_{,h} w^h_{\alpha l} u^l_x \right) w^j_{\beta k} u^k_x \\
& - 3 \partial_x \left(f^{pi}_{,h} w^h_{\alpha l} u^l_x \right) \partial_x \left(w^j_{\beta k} u^k_x \right) - \partial_x \left(c^{pi}_{h,k} u^h_x w^k_{\alpha l} u^l_x + c^{pi}_h \partial_x (w^h_{\alpha l} u^l_x) \right) w^j_{\beta m} u^m_x, \\
t_{001} = {} & -3 w^i_{\beta h} f^{hj} \partial^2_x \left(w^p_{\alpha k} u^k_x \right) - 8 \partial_x \left(w^i_{\beta h} f^{hj} \right) \partial_x (w^p_{\alpha k} u^k_x) - 6 \partial^2_x \left(w^i_{\beta h} f^{hj} \right) w^p_{\alpha k} u^k_x \\
& + 2 \left(w^i_{\beta h,l} u^h_x f^{lj} + w^i_{\beta h} \partial_x (f^{hj}) + w^i_{\beta h} ((f^{hj}_k + c^{hj}_k) u^k_x) \right) \partial_x (w^p_{\alpha m} u^m_x) \\
& + 3 \partial_x \left(w^i_{\beta h,l} u^h_x f^{lj} + w^i_{\beta h} \partial_x (f^{hj}) + w^i_{\beta h} (f^{hj}_k + c^{hj}_k) u^k_x \right) w^p_{\alpha m} u^m_x \\
& - \left(w^i_{\beta h,l} u^h_x ((f^{lj}_k + c^{lj}_k) u^k_x) + w^i_{\beta h} \partial_x ((f^{hj}_k + c^{hj}_k) u^k_x) + w^i_{\beta h} (c^{hj}_{l,k} u^k_x u^l_x + c^{hj}_k u^k_{2x}) \right) w^p_{\alpha m} u^m_x
\end{aligned}
$$

$$
\begin{aligned}
&+2\big(\partial_x(f^{pi}_k w^k_{\alpha l}u^l_x)+c^{pi}_{h,k}u^h_x w^k_{\alpha l}u^l_x+c^{pi}_h\partial_x(w^h_{\alpha l}u^l_x)\big)w^j_{\beta m}u^m_x\\
&-3f^{pi}_{,h}w^h_{\alpha l}u^l_x\partial_x\big(w^j_{\beta k}u^k_x\big)-6\partial_x\big(f^{pi}_k w^k_{\alpha l}u^l_x\big)w^j_{\beta k}u^k_x,\\
t_{002}=&-3w^i_{\beta h}f^{hj}\partial_x\big(w^p_{\alpha k}u^k_x\big)-4\partial_x\big(w^i_{\beta h}f^{hj}\big)w^p_{\alpha k}u^k_x\\
&+\big(w^i_{\beta h,l}u^h_x f^{lj}+w^i_{\beta h}\partial_x(f^{hj})+w^i_{\beta h}((f^{hj}_k+c^{hj}_k)u^k_x)\big)w^p_{\alpha m}u^m_x-3\big(f^{pi}_{,h}w^h_{\alpha l}u^l_x\big)w^j_{\beta k}u^k_x,\\
t_{003}=&-w^i_{\beta h}f^{hj}w^p_{\alpha k}u^k_x,\\
t_{010}=&\big(\partial_x(f^{pi}_k w^k_{\alpha l}u^l_x)+c^{pi}_{h,k}u^h_x w^k_{\alpha l}u^l_x+c^{pi}_h\partial_x(w^h_{\alpha l}u^l_x)\big)w^j_{\beta m}u^m_x\\
&-2f^{pi}_{,h}w^h_{\alpha l}u^l_x\partial_x\big(w^j_{\beta k}u^k_x\big)-3\partial_x\big(f^{pi}_{,h}w^h_{\alpha l}u^l_x\big)w^j_{\beta k}u^k_x,\\
t_{020}=&-f^{pi}_{,h}w^h_{\alpha l}u^l_x w^j_{\beta k}u^k_x,\\
t_{011}=&-3f^{pi}_{,h}w^h_{\alpha l}u^l_x w^j_{\beta k}u^k_x,\\
t_{100}=&-4w^i_{\beta h}f^{hj}\partial^2_x\big(w^p_{\alpha k}u^k_x\big)-12\partial^2_x\big(w^i_{\beta h}f^{hj}\big)w^p_{\alpha k}u^k_x-12\partial_x\big(w^i_{\beta h}f^{hj}\big)\partial_x\big(w^p_{\alpha k}u^k_x\big)\\
&+3\big(w^i_{\beta h,l}u^h_x f^{lj}+w^i_{\beta h}\partial_x(f^{hj})+w^i_{\beta h}(f^{hj}_k+c^{hj}_k)u^k_x\big)\partial_x\big(w^p_{\alpha m}u^m_x\big)\\
&+6\partial_x\big(w^i_{\beta h,l}u^h_x f^{lj}+w^i_{\beta h}\partial_x(f^{hj})+w^i_{\beta h}(f^{hj}_k+c^{hj}_k)u^k_x\big)w^p_{\alpha m}u^m_x\\
&-2\big(w^i_{\beta h,l}u^h_x((f^{lj}_k+c^{lj}_k)u^k_x)+w^i_{\beta h}\partial_x((f^{hj}_k+c^{hj}_k)u^k_x)+w^i_{\beta h}(c^{hj}_{l,k}u^k_x u^l_x+c^{hj}_k u^k_{2x})\big)w^p_{\alpha m}u^m_x\\
&+c^{pi}_{h,k}u^h_x w^k_{\alpha l}u^l_x w^j_{\beta m}u^m_x+c^{pi}_h\partial_x(w^h_{\alpha l}u^l_x)w^j_{\beta m}u^m_x+\partial_x(c^{pi}_h w^h_{\alpha l}u^l_x w^j_{\beta m}u^m_x),\\
t_{101}=&-8w^i_{\beta h}f^{hj}\partial_x\big(w^p_{\alpha k}u^k_x\big)-12\partial_x\big(w^i_{\beta h}f^{hj}\big)w^p_{\alpha k}u^k_x\\
&+3\big(w^i_{\beta h,l}u^h_x f^{lj}+w^i_{\beta h}\partial_x(f^{hj})+w^i_{\beta h}(f^{hj}_k+c^{hj}_k)u^k_x\big)w^p_{\alpha m}u^m_x,\\
t_{102}=&-4w^i_{\beta h}f^{hj}w^p_{\alpha k}u^k_x,\\
t_{110}=&c^{pi}_h w^h_{\alpha l}u^l_x w^j_{\beta m}u^m_x,\\
t_{200}=&-6w^i_{\beta h}f^{hj}\partial_x\big(w^p_{\alpha k}u^k_x\big)-12\partial_x\big(w^i_{\beta h}f^{hj}\big)w^p_{\alpha k}u^k_x\\
&+3\big(w^i_{\beta h,l}u^h_x f^{lj}+w^i_{\beta h}\partial_x(f^{hj})+w^i_{\beta h}(f^{hj}_k+c^{hj}_k)u^k_x\big)w^p_{\alpha m}u^m_x+\big(f^{pi}_k w^k_{\alpha l}u^l_x w^j_{\beta k}u^k_x+c^{pi}_h w^h_{\alpha l}u^l_x w^j_{\beta m}u^m_x\big),\\
t_{201}=&-6w^i_{\beta h}f^{hj}w^p_{\alpha k}u^k_x,\\
t_{300}=&-4w^i_{\beta h}f^{hj}w^p_{\alpha k}u^k_x.
\end{aligned}
$$

Theorem 5. *The Hamiltonian operators P and R are compatible if and only if*

- *for every α the coefficients $w^i_{\alpha j}u^j_x$ are total x-derivatives of functions w^i_α, which can be interpreted as fluxes of systems of first-order conservation laws*

$$u^i_t=\big(w^i_\alpha\big)_x=w^i_{\alpha,k}u^k_x$$

  that are Hamiltonian with respect to R;

- *the following conditions hold:*

$$\Gamma^{ij}_k f^{kp}=\Gamma^{pj}_k f^{ki}, \tag{17a}$$

$$\Big(\Gamma^{ij}_k-\Gamma^{ji}_k\Big)f^{kp}+\Big(\Gamma^{pj}_k-\Gamma^{jp}_k\Big)f^{ki}+\Big(c^{pi}_k g^{kj}-c^{pj}_k g^{ki}\Big)+\Big(c^{ip}_k g^{kj}-c^{ij}_k g^{kp}\Big)=0, \tag{17b}$$

$$\Big(\Gamma^{ij}_{k,l}+c^{\alpha\beta}w^i_{\alpha k}w^j_{\beta l}\Big)f^{kp}=\Gamma^{pj}_k c^{ki}_l-\Gamma^{kj}_l c^{pi}_k, \tag{17c}$$

$$
\begin{aligned}
&-c^{ij}_{k,l}g^{kp}-c^{pi}_{k,l}g^{kj}-c^{ji}_{k,l}g^{kp}+c^{pi}_{l,k}g^{kj}+c^{ij}_{l,k}g^{kp}+c^{jp}_{l,k}g^{ki}\\
&+c^{ki}_l\Gamma^{pj}_k-c^{kp}_l\Gamma^{ij}_k-2c^{ki}_l\Gamma^{jp}_k+2c^{kp}_l\Gamma^{ji}_k-c^{jk}_l\Gamma^{ip}_k+c^{jk}_l\Gamma^{pi}_k\\
&-c^{ji}_k\Gamma^{pk}_l+c^{jp}_k\Gamma^{ik}_l+c^{\alpha\beta}\Big(w^i_{\alpha k}f^{kj}w^p_{\beta l}-w^j_{\alpha k}f^{kp}w^i_{\beta l}\Big)=0.
\end{aligned}
\tag{17d}
$$

Proof. We recall that the second step consists in the 'integration by parts' with respect to the argument ψ^3 in the local three-vector $T_{LR}+c^{\alpha\beta}\mathfrak{L}_{\alpha\beta}$ in order to obtain a local three-vector T_l, which is of zero order with respect to that argument. That amounts to quite a long computation, which yields the expression

$$T_l=\sum_{0\leqslant k+l\leqslant 4} a_{kl}\mathrm{D}_x{}^k\psi_i^1\mathrm{D}_x{}^l\psi_j^2\psi_p^3,$$

where most of the coefficients are too cumbersome to display. The simplest coefficients are a_{40}, a_{22} and a_{31}. The vanishing of a_{40} (as well as of a_{04}) immediately results in

$$\Gamma_k^{ij}f^{kp}=\Gamma_k^{pj}f^{ki}, \tag{18}$$

vanishing of a_{22} is equivalent to the condition

$$3\left(\Gamma_k^{ij}-\Gamma_k^{ji}\right)f^{kp}=-2\left(c_k^{pi}g^{kj}-c_k^{pj}g^{ki}\right)-c_k^{ip}g^{kj}+c_k^{jp}g^{ki}-c_k^{ji}g^{kp}+c_k^{ij}g^{kp}. \tag{19}$$

The vanishing of a_{31} is equivalent to the condition

$$\left(\Gamma_k^{ij}-\Gamma_k^{ji}\right)f^{kp}+\left(\Gamma_k^{pj}-\Gamma_k^{jp}\right)f^{ki}+\left(c_k^{pi}g^{kj}-c_k^{pj}g^{ki}\right)+\left(c_k^{ip}g^{kj}-c_k^{ij}g^{kp}\right)=0. \tag{20}$$

Equation (19) implies equation (20): it is enough to replace the differences of Christoffel symbols in (20) with the help of (19) to realize that. On the other hand, equation (20) implies (19): subtracting from (20) the same equation with i,j interchanged we get (19). So, the two conditions are equivalent and each of them yields the vanishing of both a_{22} and a_{31}. Moreover, the coefficient a_{13} vanishes in view of these two conditions as well.

Other coefficients are too cumbersome but we can deduce more from looking at their highest terms. Thus, vanishing of the coefficient of u^l_{xxx} in a_{10} results in

$$\left(\Gamma_{k,l}^{ij}+c^{\alpha\beta}w^i_{\alpha k}w^j_{\beta l}\right)f^{kp}=\Gamma_k^{pj}c_l^{ki}-\Gamma_l^{kj}c_k^{pi}, \tag{21}$$

and that of u^k_{xxxx} in a_{00} reduces to

$$\Gamma_{k,l}^{ij}-\Gamma_{l,k}^{ij}=c^{\alpha\beta}\left(w^j_{\alpha k}w^i_{\beta l}-w^i_{\alpha k}w^j_{\beta l}\right). \tag{22}$$

The latter condition is satisfied in view of (21), (18) and (10*b*). Indeed,

$$\begin{aligned}
\left(\Gamma_{k,l}^{ij}+c^{\alpha\beta}w^i_{\alpha k}w^j_{\beta l}\right)f^{kp}f_{ps}&=\left(\Gamma_k^{pj}c_l^{ki}-\Gamma_l^{kj}c_k^{pi}\right)f_{ps}\\
\Gamma_{k,l}^{ij}+c^{\alpha\beta}w^i_{\alpha k}w^j_{\beta l}&=\left(\Gamma_s^{rj}c_l^{si}-\Gamma_l^{sj}c_s^{ri}\right)f_{rk}\\
\Gamma_{k,l}^{ij}+c^{\alpha\beta}w^i_{\alpha k}w^j_{\beta l}-\left(\Gamma_{l,k}^{ij}+c^{\alpha\beta}w^i_{\alpha l}w^j_{\beta k}\right)&=\left(\Gamma_s^{rj}c_l^{si}-\Gamma_l^{sj}c_s^{ri}\right)f_{rk}-\left(\Gamma_s^{rj}c_k^{si}-\Gamma_k^{sj}c_s^{ri}\right)f_{rl}\\
&=\Gamma_s^{rj}c_l^{si}f_{rk}-\Gamma_l^{sj}c_s^{ri}f_{rk}+\Gamma_l^{sj}c_s^{ri}f_{kr}-\Gamma_s^{rj}c_l^{si}f_{kr}\equiv 0.
\end{aligned}$$

Differentiating (18) and killing the derivatives of Christoffel symbols there with the help of (21) yields a useful identity

$$\Gamma_k^{ij}c_l^{pk}-\Gamma_k^{pj}c_l^{ik}+\Gamma_l^{kj}c_k^{ip}-\Gamma_l^{kj}c_k^{pi}=0, \tag{23}$$

which is equivalent to the vanishing of both a_{30} and a_{03}. With the above identities at our disposal we can proceed to simplifying the remaining coefficients. Thus, the coefficient a_{21} readily gives rise to the condition

$$\begin{aligned}
&-c_{k,l}^{ij}g^{kp}-c_{k,l}^{pi}g^{kj}-c_{k,l}^{ji}g^{kp}+c_{l,k}^{pi}g^{kj}+c_{l,k}^{ij}g^{kp}+c_{l,k}^{jp}g^{ki}+c_l^{ki}\Gamma_k^{pj}-c_l^{kp}\Gamma_k^{ij}-2c_l^{ki}\Gamma_k^{jp}+2c_l^{kp}\Gamma_k^{ji}\\
&-c_l^{jk}\Gamma_k^{ip}+c_l^{jk}\Gamma_k^{pi}-c_k^{ji}\Gamma_l^{pk}+c_k^{jp}\Gamma_l^{ik}+c^{\alpha\beta}\left(w^i_{\alpha k}f^{kj}w^p_{\beta l}-w^j_{\alpha k}f^{kp}w^i_{\beta l}\right)=0,
\end{aligned} \tag{24}$$

the coefficient a_{20} is

$$\begin{aligned}
&\left(-\Gamma_l^{kj}c_k^{pi}-\Gamma_k^{pj}c_l^{ik}+c_k^{ip}\Gamma_l^{kj}+\Gamma_k^{ij}c_l^{pk}\right)u^l_{xx}+\left(\Gamma_{k,m}^{pj}c_l^{ki}-\Gamma_{k,l}^{ij}c_m^{kp}-\Gamma_l^{kj}c_{k,m}^{pi}+\Gamma_l^{kj}c_{m,k}^{pi}\right.\\
&\left.+c^{\alpha\beta}\left(w^p_{\alpha k}c_l^{ki}w^j_{\beta m}-w^i_{\alpha k}c_l^{kp}w^j_{\beta m}\right)\right)u^l_xu^m_x,
\end{aligned}$$

where the first term vanishes in view of (23), and the second one through symmetrization gives rise to the condition

$$\begin{aligned}\Gamma^{pj}_{k,m}c^{ki}_{l}-\Gamma^{ij}_{k,l}c^{kp}_{m}-\Gamma^{kj}_{l}c^{pi}_{k,m}+\Gamma^{kj}_{l}c^{pi}_{m,k}+\Gamma^{pj}_{k,l}c^{ki}_{m}-\Gamma^{ij}_{k,m}c^{kp}_{l}-\Gamma^{kj}_{m}c^{pi}_{k,l}+\Gamma^{kj}_{m}c^{pi}_{l,k}\\ +c^{\alpha\beta}\left(w^{p}_{\alpha k}c^{ki}_{l}w^{j}_{\beta m}-w^{i}_{\alpha k}c^{kp}_{l}w^{j}_{\beta m}-w^{i}_{\alpha k}c^{kp}_{m}w^{j}_{\beta l}+w^{p}_{\alpha k}c^{ki}_{m}w^{j}_{\beta l}\right)=0.\end{aligned}\tag{25}$$

The coefficient a_{11} identically vanishes in view of the above conditions, while the coefficient a_{02} is equivalent to the condition (25). The coefficient a_{10} is still quite lengthy after preliminary simplifications, so we opt to present it part-wise. First of all, the highest coefficient thereof (of u^l_{xxx}) vanishes identically; the coefficient of $u^l_{xx}u^m_x$ is not symmetrical,

$$\begin{aligned}&-4\Gamma^{ij}_{k,l}c^{pk}_{m}-5\Gamma^{ij}_{k,l}c^{kp}_{m}+3\Gamma^{ij}_{k,m}c^{kp}_{l}+4\Gamma^{ij}_{k,m}c^{pk}_{l}+5\Gamma^{pj}_{k,m}c^{ki}_{l}-3\Gamma^{pj}_{k,l}c^{ki}_{m}\\ &+4\Gamma^{ij}_{k}c^{kp}_{l,m}+4\Gamma^{ij}_{k}c^{pk}_{l,m}-4\Gamma^{ij}_{k}c^{pk}_{m,l}-4\Gamma^{ij}_{k}c^{kp}_{m,l}-5\Gamma^{kj}_{l}c^{pi}_{k,m}\\ &+3\Gamma^{kj}_{m}c^{pi}_{k,l}+\Gamma^{kj}_{m}c^{pi}_{l,k}+\Gamma^{kj}_{l}c^{pi}_{m,k}+4\Gamma^{pj}_{k}c^{ki}_{l,m}-4\Gamma^{pj}_{k}c^{ki}_{m,l}\\ &+c^{\alpha\beta}\Big(5c^{ki}_{l}w^{p}_{\alpha k}w^{j}_{\beta m}-3c^{ki}_{m}w^{p}_{\alpha k}w^{j}_{\beta l}+3c^{kp}_{l}w^{i}_{\alpha k}w^{j}_{\beta m}-5c^{kp}_{m}w^{i}_{\alpha k}w^{j}_{\beta l}\\ &-4c^{pk}_{m}w^{i}_{\beta k}w^{j}_{\alpha l}-4c^{ip}_{k}w^{k}_{\alpha l}w^{j}_{\beta m}+4c^{pi}_{k}w^{j}_{\alpha m}w^{k}_{\beta l}+4c^{ik}_{l}w^{p}_{\alpha k}w^{j}_{\beta m}\Big),\end{aligned}$$

and therefore it is better to write down its symmetrization and anti-symmetrization with respect to l, m. The symmetrical part gives rise to the condition (25), and its antisymmetrical counterpart to

$$\begin{aligned}&-\Gamma^{ij}_{k,l}c^{pk}_{m}+\Gamma^{ij}_{k,m}c^{kp}_{l}+\Gamma^{ij}_{k,m}c^{pk}_{l}-\Gamma^{pj}_{k,l}c^{ki}_{m}-\Gamma^{ij}_{k,l}c^{kp}_{m}+\Gamma^{pj}_{k,m}c^{ki}_{l}\\ &+\Gamma^{ij}_{k}c^{kp}_{l,m}+\Gamma^{ij}_{k}c^{pk}_{l,m}-\Gamma^{ij}_{k}c^{pk}_{m,l}-\Gamma^{ij}_{k}c^{kp}_{m,l}-\Gamma^{kj}_{l}c^{pi}_{k,m}+\Gamma^{pj}_{k}c^{ki}_{l,m}-\Gamma^{pj}_{k}c^{ki}_{m,l}+\Gamma^{kj}_{m}c^{pi}_{k,l}\\ &+c^{\alpha\beta}\left(c^{kp}_{l}w^{i}_{\alpha k}w^{j}_{\beta m}-c^{ki}_{m}w^{p}_{\alpha k}w^{j}_{\beta l}+c^{pk}_{l}w^{i}_{\alpha k}w^{j}_{\beta m}-c^{pk}_{m}w^{i}_{\alpha k}w^{j}_{\beta l}+c^{ki}_{l}w^{p}_{\alpha k}w^{j}_{\beta m}-c^{kp}_{m}w^{i}_{\alpha k}w^{j}_{\beta l}\right)=0,\end{aligned}$$

which vanishes in view of lemma 21. The coefficient of $u^l_x u^m_x u^n_x$ in a_{10} is

$$\frac{\partial}{\partial u^n}\left(-\Gamma^{ij}_{k,l}c^{kp}_{m}+\Gamma^{pj}_{k,m}c^{ki}_{l}-\Gamma^{kj}_{l}c^{pi}_{k,m}+\Gamma^{kj}_{m}c^{pi}_{l,k}+c^{\alpha\beta}\left(c^{ki}_{m}w^{p}_{\alpha k}w^{j}_{\beta l}-c^{kp}_{l}w^{i}_{\alpha k}w^{j}_{\beta m}\right)\right)+c^{\alpha\beta}\left(S^{ip}_{\alpha nl}w^{j}_{\beta m}+S^{ij}_{\alpha nl}w^{p}_{\beta m}\right).$$

The first part thereof vanishes in view of (25), while the latter vanishes due to lemma 18. Finally, the most lengthy coefficient a_{00} simplifies marvelously to just

$$c^{\alpha\beta}\partial_x\left(S^{ip}_{\alpha nl}w^{j}_{\beta m}u^l_x u^m_x u^n_x\right),$$

which is identically zero due to lemma 18 as well. Finally, we can show that the coefficient a_{01} simplifies to (22), and thus vanishes identically as well. □

An interesting corollary provides a new structure arising from the first-order operator P.

Corollary 6. *The cotangent space of the space of dependent variables (u^i) is endowed by the Christoffel symbols Γ^{ij}_k with a structure of associative algebra (without unity):*

$$\Gamma^{ij}_{s}\Gamma^{sk}_{l}-\Gamma^{ik}_{s}\Gamma^{sj}_{l}=0.\tag{26}$$

Proof. Indeed, the equation (22) implies that the equation (9) is equivalently rewritten as (26). □

3.6. The structure of g

We prove a Structure Formula for g with a series of lemmas. Our initial aim is to integrate the conditions (21) and (22) out of all compatibility conditions of P and R.

Lemma 7. *There exists a matrix of functions r^{ij} such that*

$$\Gamma^{ij}_{k}+c^{\alpha\beta}w^{i}_{\alpha k}w^{j}_{\beta}=r^{ij}_{k}.\tag{27}$$

Proof. Indeed,

$$\left(\Gamma^{ij}_{k}+c^{\alpha\beta}w^{i}_{\alpha k}w^{j}_{\beta}\right)_{,l}-\left(\Gamma^{ij}_{l}+c^{\alpha\beta}w^{i}_{\alpha l}w^{j}_{\beta}\right)_{,k}=0$$

in view of the condition of compatibility (22). □

Hence, by introducing a matrix of functions r^{ij} as above we integrate condition (22).

Lemma 8. *We have the identity*

$$g^{ij} + c^{\alpha\beta} w^i_\alpha w^j_\beta = r^{ij} + r^{ji}. \tag{28}$$

Proof. It is enough to symmetrize the equality (27) with respect to the indices i, j. □

The above lemma shows that the formula (28) integrates the equation (6).

Lemma 9. *The system of equations (17a) and (17c) is equivalent to the system:*

$$f^{lk} r^{ij}_{,k} = f^{ik} r^{lj}_{,k}, \tag{29a}$$

$$f^{kp} r^{ij}_{,kl} = r^{pj}_{,k} c^{ki}_l - r^{kj}_{,l} c^{pi}_k. \tag{29b}$$

Proof. Indeed, the first equation follows from the identities

$$f^{lk} \Gamma^{ij}_k = f^{ik} \Gamma^{lj}_k, \quad f^{lk} w^i_{\alpha k} = f^{ik} w^l_{\alpha k}.$$

The second equation is obtained by rewriting the condition (21) using r^{ij} as a variable:

$$\left(r^{ij}_{,kl} - c^{\alpha\beta} w^i_{\alpha k,l} w^j_\beta \right) f^{kp} = \left(r^{pj}_k - c^{\alpha\beta} w^p_{\alpha k} w^j_\beta \right) c^{ki}_l - \left(r^{kj}_{,l} - c^{\alpha\beta} w^k_{\alpha l} w^j_\beta \right) c^{pi}_k.$$

Since

$$c^{\alpha\beta} \left(w^i_{\alpha k,l} f^{kp} + w^k_{\alpha l} c^{pi}_k - w^p_{\alpha k} c^{ki}_l \right) w^j_\beta = 0,$$

and the arguments can be reversed, we have the result. □

Note that the above potential function r^{ij} is not a gradient with respect to any of the involved pseudo-Riemannian metrics f_{ij}, g_{ij}. However, there is the following interesting relation.

Proposition 10. *We have the identity*

$$\left(f_{bk} r^{kj} \right)_{,a} - \left(f_{ak} r^{kj} \right)_{,b} = \left(f_{kb,a} - f_{ka,b} \right) r^{kj} = 3 r^{kj} c_{kab}.$$

Proof. Indeed, the lower indices version of (29a) is $f_{bi} r^{ij}_{,a} = f_{ai} r^{ij}_{,b}$, from which the result follows. □

Let us introduce the parametrization $r^{ij} = \psi^i_\gamma Z^{\gamma j}$. The equation (29b) transforms into the equation

$$Z^{\gamma j}_{,hk} = 0,$$

and thus $Z^{\gamma r} = \eta^{\gamma r}_k u^k + \xi^{\gamma r}$ for some constants $\eta^{\gamma r}_k$ and $\xi^{\gamma r}$.

Proposition 11. *The equation (29a) is equivalent to the two linear algebraic equations*

$$\phi_{\beta\gamma} \left[\psi^\beta_{ij} \eta^{\gamma r}_k + \psi^\beta_{jk} \eta^{\gamma r}_i + \psi^\beta_{ki} \eta^{\gamma r}_j \right] = 0,$$

$$\phi_{\beta\gamma} \left[\psi^\beta_{ik} \xi^{\gamma r} + \omega^\beta_k \eta^{\gamma r}_i - \omega^\beta_i \eta^{\gamma r}_k \right] = 0.$$

Proof. In [23] a closed form solution of the system

$$f^{lk} w^i_k = f^{ik} w^l_k, \tag{30}$$

$$w^i_{kl} f^{kp} + w^k_l c^{pi}_k - w^p_k c^{ki}_l = 0 \tag{31}$$

is presented by means of the parametrization $w^i = \psi^i_\gamma Z^\gamma$. In particular, it is proved that the last equation (31) is equivalent to $Z^\gamma_{,ij} = 0$, thus proving that $Z^\gamma = \eta^\gamma_k u^k + \xi^\gamma$ is a vector of linear functions. The equation (30) is proved to be equivalent to the (linear) algebraic system

$$\phi_{\beta\gamma} \left[\psi^\beta_{ij} \eta^\gamma_k + \psi^\beta_{jk} \eta^\gamma_i + \psi^\beta_{ki} \eta^\gamma_j \right] = 0,$$

$$\phi_{\beta\gamma} \left[\psi^\beta_{ik} \xi^\gamma + \omega^\beta_k \eta^\gamma_i - \omega^\beta_i \eta^\gamma_k \right] = 0.$$

Here, repeating the above argument using the parametrization $r^{ij} = \psi^i_\gamma Z^{\gamma j}$ we obtain the result. □

In other words, the matrix (r^{ij}) can be interpreted as a family of n systems of first-order conservation laws, indexed by j, which is Hamiltonian with respect to the operator R:

$$u^i_t = \left(r^{ij}\right)_x = R^{ik}\left(\frac{\delta H^j}{\delta u^k}\right),$$

where H^j is the Hamiltonian density (see [20] for more details); the fluxes have the form $r^{ij} = \psi^i_\alpha Z^{\alpha j}$ described in the proof of the above theorem.

The following Structure Formula for the metric of the first-order operator P is a straightforward consequence of (28).

Corollary 12 (Structure Formula). *We have*

$$g^{ij} = \psi^i_\gamma Z^{\gamma j} + \psi^j_\gamma Z^{\gamma i} - c^{\alpha\beta} w^i_\alpha w^j_\beta$$

where w^i_α are potentials of the coefficients of the nonlocal part in the expression (3): $w^i_{\alpha k} = w^i_{\alpha,k}$.

The above results allow us to prove a conjecture that was made in [45].

Corollary 13. *The metric g^{ij} is of the form*

$$g^{ij} = \psi^i_\alpha Q^{\alpha\beta} \psi^j_\beta,$$

where $Q^{\alpha\beta}$ is the matrix of quadratic functions defined by

$$Q^{\alpha\beta} = Z^{\alpha j}\psi^\beta_j + \psi^\alpha_i Z^{\beta i} - c^{\gamma\delta} Z^\alpha_\gamma Z^\beta_\delta \quad \text{with} \quad w^i_\alpha = \psi^i_\gamma Z^\gamma_\alpha.$$

Moreover, taking into account the results of this section the main result can be formulated as follows.

Theorem 14. *Given the two Hamiltonian operators P, R as in sections 2.1 and 2.2, the compatibility conditions $[P, R] = 0$ are equivalent to:*

1. *For every α the vector function (w^i_α) is the vector of fluxes of a system of conservation laws, which are Hamiltonian with respect to R.*
2. $$g^{ij} = \psi^i_\gamma Z^{\gamma j} + \psi^j_\gamma Z^{\gamma i} - c^{\alpha\beta} w^i_\alpha w^j_\beta$$

   where $Z^{\gamma j} = \eta^{\gamma j}_k u^k + \xi^{\gamma j}$ are matrices of linear functions whose coefficients fulfill the (linear) algebraic system

$$\phi_{\beta\gamma}\left[\psi^\beta_{ij}\eta^{\gamma j}_k + \psi^\beta_{jk}\eta^{\gamma j}_i + \psi^\beta_{ki}\eta^{\gamma j}_j\right] = 0,$$
$$\phi_{\beta\gamma}\left[\psi^\beta_{ik}\xi^{\gamma j} + \omega^\beta_k \eta^{\gamma j}_i - \omega^\beta_i \eta^{\gamma j}_k\right] = 0.$$

3. $$\left(\Gamma^{ij}_k - \Gamma^{ji}_k\right) f^{kp} + \left(\Gamma^{pj}_k - \Gamma^{jp}_k\right) f^{ki} + \left(c^{pi}_k g^{kj} - c^{pj}_k g^{ki}\right) + \left(c^{ip}_k g^{kj} - c^{ij}_k g^{kp}\right) = 0.$$

4. $$\begin{aligned} &- c^{ij}_{k,l} g^{kp} - c^{pi}_{k,l} g^{kj} - c^{ji}_{k,l} g^{kp} + c^{pi}_{l,k} g^{kj} + c^{ij}_{l,k} g^{kp} + c^{jp}_{l,k} g^{ki} \\ &+ c^{ki}_l \Gamma^{pj}_k - c^{kp}_l \Gamma^{ij}_k - 2c^{ki}_l \Gamma^{jp}_k + 2c^{kp}_l \Gamma^{ji}_k - c^{jk}_l \Gamma^{ip}_k + c^{jk}_l \Gamma^{pi}_k \\ &- c^{ji}_k \Gamma^{pk}_l + c^{jp}_k \Gamma^{ik}_l + c^{\alpha\beta}\left(w^i_{\alpha k} f^{kj} w^p_{\beta l} - w^j_{\alpha k} f^{kp} w^i_{\beta l}\right) = 0. \end{aligned}$$

After having simplified the compatibility equations, we can now ask ourselves if it is possible to provide an interpretation of the Hamiltonian properties of P written down in section 2.1 in terms of the above Hamiltonian systems of R. We have the following results.

Proposition 15. *The equation (8) is equivalent to the requirement that the vector of fluxes $\{w^i_\alpha \mid i = 1, \ldots, n\}$ and $\{r^{hk} \mid h = 1, \ldots, n\}$ commute for every choice of α and k.*

Proof. The equation (8) is equivalent to

$$\Gamma^{kj}_{p} w^{q}_{\alpha k} = \Gamma^{qj}_{k} w^{k}_{\alpha p}.$$

Then, in view of formula (27) and the commutativity of w^i_α with every other w^j_β, we obtain the result. □

Proposition 16. *The equation (9) is equivalent to the requirement that the vector of fluxes $\{r^{hk} \mid h = 1, \dots, n\}$ and $\{r^{hj} \mid h = 1, \dots, n\}$ commute for every choice of k and j.*

Proof. As we have seen (corollary 6), the equation (22) implies the equation (26). By the formula (27), the previous Proposition and the commutativity of the fluxes w^i_α we obtain the result. □

In view of our previous results, the structure possessed by P due to its compatibility with R allows to reinterpret the Hamiltonian property of P itself, using the Hamiltonian systems of conservation laws of R whose fluxes are $\{w^i_\alpha \mid i = 1, \dots, n\}$ and $\{r^{hk} \mid h = 1, \dots, n\}$.

Theorem 17 (Hamiltonian property). *Let P, R be a pair of Hamiltonian operators described in sections 2.1 and 2.2, respectively. If P and R are compatible, then the Hamiltonian property of P is equivalent to the following requirements:*

1. $g^{ik} w^j_{\alpha k} = g^{jk} w^i_{\alpha k}$;
2. $g^{ik} r^{jl}_k = g^{jk} r^{il}_k$;
3. *The two families of vector of fluxes $\{w^i_\alpha \mid i = 1, \dots, n\}$ and $\{r^{hk} \mid h = 1, \dots, n\}$ are commuting, both between elements of the same family and between elements of the two distinct families:*

$$r^{ik}_{,p} r^{ph}_{,j} = r^{ih}_{,p} r^{pk}_{,j}, \quad r^{ik}_{,p} w^p_{\alpha,j} = w^i_{\alpha,p} r^{pk}_{,j}, \quad w^i_{\alpha,p} w^p_{\beta,j} = w^i_{\beta,p} w^p_{\alpha,j}.$$

4. Examples

In this section, we consider various examples of compatible pairs P, R and determine the structure of P, according to the Structure Formula (corollary 12) by providing $Z^{\gamma i}$, $c^{\alpha\beta}$ and w^i_α with respect to a chosen decomposition $f_{ij} = \phi_{\alpha\beta} \psi^\alpha_i \psi^\beta_j$.

Some of the examples can be found in the literature. In the past, compatibility was checked by pen and paper (see, e.g. [2, 3, 19, 26, 27]); recently, new theoretical advances [9] and computer algebra packages [8] allowed large compatibility calculations, also when P is nonlocal [45, 48, 51, 52].

However, the results of this paper enable us to find the 'fine structure' of the metric of P, and to check the compatibility by means of the simplified, purely algebraic equations to which we reduced the problem. Indeed, we could not manage to compute compatibility for the new WDVV examples below using the above computer algebra packages for Schouten bracket. On the other hand, the compatibility conditions that we found in this paper are easy to program and enable us to compute in much higher dimensions.

We will provide two categories of examples.

The dispersive examples come from the classification in [33], and are of KdV-type. Here we just confirm the Structure Formula by providing the expression of $Z^{\alpha i}$ for some examples in the classification.

The dispersionless examples are endowed by bi-Hamiltonian structures of WDVV-type [45]. Three examples come from that paper, again we provide the expression of $Z^{\gamma i}$. The expression of Z^γ_α was already known; what was not known is the fact that such systems enter in the Structure Formula and contribute to the definition of the metric of the first-order Hamiltonian operator.

Two further dispersionless examples are completely new. They arise in WDVV Equations in dimensions four and five, after rewriting them as commuting quasilinear systems of first-order conservation laws. It was recently proved in [46] that such systems are endowed with a third-order Hamiltonian structure, and it was conjectured that they are endowed also with a first-order Hamiltonian structure. Our results below confirm the conjecture.

In all examples, we checked that commutativity mentioned in theorem 17 holds.

4.1. Bi-Hamiltonian structures of KdV-type

Dispersive water waves. The system

$$u^1_t = \frac{1}{4} u^2_{xxx} + \frac{1}{2} u^2 u^1_x + u^1 u^2_x,$$

$$u^2_t = u^1_x + \frac{3}{2}u^2u^2_x$$

is equivalent to dispersive water waves system up to a Miura transformation [3]. Among others, it is bi-Hamiltonian with respect to operators $R+P_1$ and P_2 where R is a third-order homogeneous operator defined by the Monge metric with

$$\psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \phi = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and P_1 and P_2 are first-order local homogeneous Hamiltonian operators defined by the metrics through the Structure Formula (corollary 12) with

$$Z_1 = \begin{pmatrix} -\frac{1}{4}u^2 & 1-\mu \\ \mu & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0 & \frac{1}{4}u^1 - \mu \\ \frac{1}{4}u^1 + \mu & \frac{1}{4}u^2 \end{pmatrix},$$

respectively. Here, μ is an arbitrary parameter. Generally, a matrix Z is defined up to such inessential constants in this construction.

Case $R^{(2)}$ from [33]: **pencil** $g_{\lambda,11}$. The pair of Hamiltonian operators $R+P_1$ and P_2 where R is a third-order homogeneous operator defined by the Monge metric with

$$\psi = \begin{pmatrix} 1 & 0 \\ u^2 & -u^1 \end{pmatrix}, \quad \phi = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and P_1 and P_2 are first-order homogeneous Hamiltonian operators defined by the metrics through the Structure Formula with

$$Z_1 = \begin{pmatrix} -u^1 & u^2+\mu \\ -\mu u^1 & -\mu u^2 - 1 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0 & -u^1+\mu \\ -\mu u^1 & -\mu u^2 \end{pmatrix},$$

define the bi-Hamiltonian hierarchy

$$u_{t_i} = (P_1+R)\,\delta C_i, \quad i = 1,2,$$

where $C_1 = \int_{S^1} u^1 \mathrm{dx}$ and $C_2 = \int_{S^1} \frac{u^2}{u^1}\mathrm{dx}$ are Casimirs of P_2.

Case $R^{(2)}$ from [33]: **pencil** $g_{\lambda,13}$. The pair of Hamiltonian operators $R+P_1$ and P_2 where R is a third-order homogeneous operator defined the Monge metric with

$$\psi = \begin{pmatrix} 1 & 0 \\ u^2 & -u^1 \end{pmatrix}, \quad \phi = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and P_1 and P_2 are first-order homogeneous local Hamiltonian operators defined by the metrics through the Structure Formula with

$$Z_1 = \begin{pmatrix} u^2 & u^1+\mu \\ -\mu u^1 & -\mu u^2 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0 & \mu \\ -\mu u^1 - 1 & -\mu u^2 \end{pmatrix},$$

define the bi-Hamiltonian hierarchy

$$u_{t_i} = (P_1+R)\,\delta C_i, \quad i = 1,2,$$

where $C_1 = \int_{S^1} \frac{1}{2}(u^1)^2 \mathrm{dx}$ and $C_2 = \int_{S^1} u^2 \mathrm{dx}$ are Casimirs of P_2.

Case $R^{(3)}$ from [33]: **pencil** $g_{\lambda,12}$. The pair of Hamiltonian operators $R+P_1$ and P_2 where R is a third-order homogeneous operator defined the Monge metric with

$$\psi = \begin{pmatrix} 1 & 0 \\ u^2 & -u^1 \end{pmatrix}, \quad \phi = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and P_1 and P_2 are first-order homogeneous local Hamiltonian operators defined by the metrics through the Structure Formula with metrics with

$$Z_1 = \begin{pmatrix} u^1-u^2 & -u^2+\mu \\ -\mu u^1 + 1 & 1-\mu u^2 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} -1 & \mu \\ -\mu u^1 - u^2 & -(\mu+2)\,u^2 \end{pmatrix},$$

define the bi-Hamiltonian hierarchy

$$u_{t_i} = (P_1 + R)\,\delta C_i, \quad i = 1, 2,$$

where $C_1 = \int_{S^1}(u^1 - u^2)\mathrm{dx}$ and $C_2 = \int_{S^1}\sqrt{(u^2)^2 - 2u^1u^2}\,\mathrm{dx}$ are Casimirs of P_2.

4.2. Bi-Hamiltonian structures of WDVV-type

Hyperbolic Monge–Ampère system. The hyperbolic Monge–Ampère equation $u_{tt}u_{xx} - u_{xt}^2 = -1$ is a widely studied differential equation. We adopt the viewpoint in [39] where the equation is rewritten as the hydrodynamic-type system

$$u^1_t = u^2_x, \quad u^2_t = \left(\frac{\left(u^2\right)^2 - 1}{u^1}\right)_x$$

via the change of variables $u^1 = u_{xx}$, $u^2 = u_{xt}$ (see also [45]). It admits a third-order homogeneous Hamiltonian operator R determined by a Monge metric whose factors are

$$\psi = \begin{pmatrix} 1 & 0 \\ -u^2 & u^1 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It also admits four compatible local first-order Hamiltonian operators that are defined by

$$Z_1 = \begin{pmatrix} 0 & u^2 \\ 0 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} u^1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z_3 = \begin{pmatrix} -u^2 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z_4 = \begin{pmatrix} 0 & u^1 \\ 0 & 0 \end{pmatrix}.$$

A four-component example from[1]. It is believed that there is a unique integrable case within the class of systems of conservation laws that admits a Hamiltonian formulation through a third-order homogeneous Hamiltonian operator (see the discussion at the end of [20]),

$$\begin{aligned} u^1_t &= u^3_x, \\ u^2_t &= u^4_x, \\ u^3_t &= \left(\frac{u^1u^2u^4 + u^3\left(\left(u^3\right)^2 + \left(u^4\right)^2 - \left(u^2\right)^2 - 1\right)}{u^1u^3 + u^2u^4}\right)_x, \\ u^4_t &= \left(\frac{u^1u^2u^3 + u^4\left(\left(u^3\right)^2 + \left(u^4\right)^2 - \left(u^1\right)^2 - 1\right)}{u^1u^3 + u^2u^4}\right)_x. \end{aligned} \tag{32}$$

The above system was first derived in [1] by means of projective-geometric considerations. In [23] a Lax pair and a third-order homogeneous Hamiltonian operator R in canonical form were found for the system. In [45] a compatible first-order homogeneous Hamiltonian operator P was also found. The operator R is parametrized by a Monge metric $f_{ij} = \sum_k \psi^k_i\psi^k_j$ (ϕ is the unit matrix),

$$\psi = \begin{pmatrix} -u^2 & u^1 & -u^4 & u^3 \\ -u^3 & -u^4 & u^1 & u^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The metric defining its first-order Hamiltonian operator of Ferapontov type is parametrized by the two-by-two unit matrix c and $Z = -\frac{1}{2}\psi$. From now on the omit inessential parameters in Z in view of their number to avoid unnecessarily cumbersome expressions.

We recall that WDVV in arbitrary dimension, when rewritten as a first-order quasilinear system of conservation laws, is endowed with a third-order homogeneous Hamiltonian operator R [46]. In the series of examples below we will construct what seems to be a reasonable candidate for a first-order homogeneous operator of Ferapontov type for WDVV systems in arbitrary dimension.

WDVV in dimension three. The WDVV system that is associated with a Dubrovin normal form $\eta = \eta^{(2)}$ in dimension three,

$$\eta^{(2)} = \begin{pmatrix} \mu & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

takes the form

$$\mu f_{ttt}f_{xxt} - f_{ttt} + f_{xxt}^2 - f_{xxx}f_{xtt} - \mu f_{xxt}^2 = 0,$$

which in the new dependent variables $u^1 = f_{xxx}$, $u^2 = f_{xxt}$, $u^3 = f_{xtt}$ can be rewritten as the hydrodynamic-type system

$$\left(u^i\right)_t = \left(V^i\right)_x, \quad i = 1, \dots 3,$$

where $V^i = \psi^i_\alpha w_1^\alpha$,

$$w_1 = \begin{pmatrix} 0 & u^2 & u^3 \end{pmatrix},$$

and

$$\psi = (\psi_i^\alpha) = \begin{pmatrix} -u^2 & \mu u^3 + u^1 & -\mu u^2 + 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

A third-order Hamiltonian operator is determined by the Monge metric $f_{ij} = \psi_i^\alpha \phi_{\alpha\beta} \psi_j^\alpha$ with

$$(\phi_{ij}) = \begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the first-order operator P of Ferapontov type is defined through the Structure Formula (corollary 12) by $Z^{\gamma j} = \eta_k^{\gamma j} u^k + \xi^{\gamma j}$,

$$\left(Z^{\gamma j}\right) = \begin{pmatrix} \frac{3}{2}u^2 & u^3 & 0 \\ -\frac{3}{2} - \mu u^2 & u^1 & 2u^2 \\ -\mu u^3 & u^2 & 2u^3 \end{pmatrix},$$

and the matrix c,

$$c = \mu \begin{pmatrix} 1 & 0 \\ 0 & -\mu \end{pmatrix}.$$

Note that P becomes local when $\mu = 0$, and it is equal to the operators obtained in [19] and [51].

WDVV in dimension four: new example. The WDVV system that is associated with the Dubrovin normal form $\eta = \eta^{(2)}$ in dimension four takes the form

$$\begin{aligned}
&\mu\left(f_{yyz}f_{zzz} - f_{yzz}^2\right) + 2f_{yyz}f_{xyz} - f_{yyy}f_{xzz} - f_{xyy}f_{yzz} = 0,\\
&\mu\left(f_{xzz}f_{yzz} - f_{zzz}f_{xyz}\right) + f_{xxy}f_{yzz} - f_{xxz}f_{yyz} + f_{zzz} + f_{xyy}f_{xzz} - f_{xyz}^2 = 0,\\
&\mu\left(f_{yyz}f_{xzz} - f_{xyz}f_{yzz}\right) + f_{xxy}f_{yyz} - f_{xxz}f_{yyy} + f_{yzz} = 0,\\
&\mu\left(f_{xzz}^2 - f_{xxz}f_{zzz}\right) + f_{xxy}f_{xzz} - 2f_{xxz}f_{xyz} + f_{xxx}f_{yzz} = 0,\\
&\mu\left(f_{xxz}f_{yzz} - f_{xzz}f_{xyz}\right) + f_{xxz}f_{xyy} - f_{yyz}f_{xxx} + f_{xzz} = 0,\\
&\mu\left(f_{xxz}f_{yyz} - f_{xyz}^2\right) + f_{xxy}f_{xyy} - f_{xxx}f_{yyy} + 2f_{xyz} = 0
\end{aligned}$$

(only 4 equations here are functionally independent though, see [46]), which in the new dependent variables $u^1 = f_{xxx}$, $u^2 = f_{xxy}$, $u^3 = f_{xxz}$, $u^4 = f_{xyy}$, $u^5 = f_{xyz}$, $u^6 = f_{xzz}$ can be rewritten as the system of two hydrodynamic-type systems

$$\left(u^i\right)_y = \left(V^i\right)_x, \quad \left(u^i\right)_z = \left(W^i\right)_x \quad i = 1, \dots 6,$$

where $V^i = \psi^i_\alpha w_1^\alpha$, $W^i = \psi^i_\alpha w_2^\alpha$, w's are determined to be

$$w_1 = \begin{pmatrix} -2u^5 & -u^6 & u^2 & 0 & u^4 & u^5 \end{pmatrix}, \quad w_2 = \begin{pmatrix} -u^6 & 0 & u^3 & 0 & u^5 & u^6 \end{pmatrix},$$

and

$$\psi = (\psi_i^\alpha) = \begin{pmatrix} u^4 & 0 & -\mu u^5 & -u^1 & \mu u^3 & 0 \\ u^5 & u^3 & -u^2 - \mu u^6 & 0 & -u^1 & \mu u^3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -u^5 & u^4 & -u^3 & \mu u^6 + u^2 & -\mu u^5 + 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Both these systems admit two compatible Hamiltonian operators of the aforementioned form. In particular, the two systems admits a third-order homogeneous Hamiltonian operator R, which was found in [51], defined by the Monge metric $f_{ij} = \psi_i^\alpha \phi_{\alpha\beta} \psi_j^\alpha$ with

$$(\phi_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\mu & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\mu & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

and the new first-order Hamiltonian operator of Ferapontov type (of the system $(u^i)_y = (V^i)_x$ only, for brevity) defined by

$$(Z^{ij}) = \begin{pmatrix} 2u^1 & \mu u^6 & 0 & 0 & -2u^5 & -2u^6 \\ 0 & -u^3 & 0 & 0 & -u^6 & 0 \\ 0 & -\mu u^3 & u^1 & 0 & u^2 & 2u^3 \\ 2u^3 & u^5 & u^6 & 0 & 0 & 0 \\ 0 & -\mu u^5 - 1 & u^2 & 0 & u^4 & 2u^5 \\ 0 & -\mu u^6 & u^3 & 0 & u^5 & 2u^6 \end{pmatrix}$$

and the matrix c,

$$c = \mu \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\mu \end{pmatrix},$$

where the columns are associated with the unit matrix w_0, and the vectors w_1 and w_2, respectively.

WDVV in dimension five: new example. The WDVV system that is associated with a Dubrovin normal form $\eta = \eta^{(2)}$ in dimension five can be rewritten as the system of three hydrodynamic-type systems; one of them is

$$\left(u^i\right)_3 = \left(V^i\right)_2, \quad i = 1, \dots 10,$$

where $V^i = \psi_\alpha^i w_2^\alpha$, the dependent variables (here we switch to the notation (t^2, t^3, t^4, t^5) for the independent variables) are

$$u^1 = f_{222},\ u^2 = f_{223},\ u^3 = f_{233},\ u^4 = f_{224},\ u^5 = f_{234},\ u^6 = f_{244},\ u^7 = f_{225},\ u^8 = f_{235},\ u^9 = f_{245},\ u^{10} = f_{255},$$

w's are determined to be

$$\begin{aligned} w_1 &= \begin{pmatrix} u^1 & u^2 & 0 & u^4 & -u^7 & -u^8 & u^7 & 0 & 0 & u^{10} \end{pmatrix} \\ w_2 &= \begin{pmatrix} u^2 & u^3 & -u^7 & u^5 & 0 & u^9 & u^8 & 0 & u^{10} & 0 \end{pmatrix} \\ w_3 &= \begin{pmatrix} u^4 & u^5 & u^8 & u^6 & u^9 & 0 & u^9 & u^{10} & 0 & 0 \end{pmatrix} \\ w_4 &= \begin{pmatrix} u^7 & u^8 & 0 & u^9 & 0 & 0 & u^{10} & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$\psi = (\psi_i^\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u^5 & -u^3+u^4 & u^2 & -u^2 & u^1 & 0 & \mu u^8 & -\mu u^7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -u^6 & -u^5 & 0 & 0 & u^2 & u^1 & \mu u^9-1 & 0 & -\mu u^7 & 0 \\ 0 & -u^6 & -u^5 & u^5 & u^3-u^4 & u^2 & 0 & \mu u^9-1 & -\mu u^8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -u^9 & -u^8 & 0 & -u^7 & 0 & 0 & \mu u^{10}+u^4 & u^2 & u^1 & -\mu u^7 \\ 0 & -u^9 & -u^8 & 0 & -u^7 & 0 & u^5 & \mu u^{10}+u^3 & u^2 & -\mu u^8 \\ 0 & 0 & 0 & -u^9 & -u^8 & -u^7 & u^6 & u^5 & \mu u^{10}+u^4 & -\mu u^9+1 \end{pmatrix}.$$

This system admits two compatible Hamiltonian operators of the aforementioned form. In particular, the third-order Hamiltonian operator R (found in [51]) is defined by the Monge metric $f_{ij} = \psi_i^\alpha \phi_{\alpha\beta} \psi_j^\alpha$ with

$$(\phi_{\alpha\beta}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -\mu \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \end{pmatrix}$$

and a first-order Hamiltonian operator of Ferapontov type, which is new and is defined by the matrix

$$(Z^{ij}) = \begin{pmatrix} 0 & 0 & -\mu u^7 & -\mu u^7 & 0 & 0 & u^1 & u^2 & u^4 & 2u^7 \\ 0 & 0 & -\mu u^8 & -\mu u^8 & 0 & 0 & u^2 & u^3 & u^5 & 2u^8 \\ 0 & -u^1 & -\frac{3}{2}u^2 & 0 & 0 & 0 & 0 & -u^7 & u^8 & 0 \\ 0 & 0 & -\mu u^9 & -\mu u^9-1 & 0 & 0 & u^4 & u^5 & u^6 & 2u^9 \\ -2u^1 & -u^2 & 0 & 0 & 0 & 0 & -u^7 & 0 & u^9 & 0 \\ -2u^2 & -u^3+u^4 & \frac{3}{2}u^5 & 0 & 0 & 0 & -u^8 & u^9 & 0 & 0 \\ 0 & 0 & -\mu u^{10} & -\mu u^{10} & 0 & 0 & u^7 & u^8 & u^9 & 2u^{10} \\ 0 & 0 & 0 & u^7 & 0 & 0 & 0 & 0 & u^{10} & 0 \\ 0 & u^7 & \frac{3}{2}u^8 & u^8 & 0 & 0 & 0 & u^{10} & 0 & 0 \\ 2u^7 & u^8 & 0 & u^9 & 0 & 0 & u^{10} & 0 & 0 & 0 \end{pmatrix},$$

and the matrix c,

$$c = \mu \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu \end{pmatrix}.$$

5. Discussion

So far, we have been able to derive a set of conditions whose vanishing is equivalent to the vanishing of the Schouten bracket $[P, R_3]$. The strength of our results lies in the fact that the Structure Formula in corollary 12 allows to reduce the system $[P, R_3] = 0$ to algebraic equations. The equations that determine a third-order homogeneous operator in Doyle–Potëmin canonical form (see section 2.2) are indeed algebraic, and by the Structure Formula, a compatible first-order homogeneous operator is determined by a finite number of constants.

If we consider a third-order operator as given, the equations are partly linear (the equations (5) that determine Hamiltonian systems), and partly algebraic of second degree (the symmetry of the Jacobian

matrices of the vectors of fluxes of the systems with respect to the metric g and the commutativity relations).

In our computational experiments, the additional conditions (17*b*) and (17*d*) seem not to play any role in the search for first-order operators P that are compatible with a given R_3, after having used all other conditions. There exists the possibility that they vanish identically; we will address this problem in future research work.

We are currently working on compatibility between second-order and first-order homogeneous Hamiltonian operators [32]; the calculations that we did until now allow us to confirm that many of the results of this Paper hold in a strikingly similar way also in that case.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix

Here we give the details of the proof of some results that we use to simplify the coefficients of $[P,R]$.

Lemma 18. $S^{jq}_{\beta lp}+S^{jq}_{\beta pl}=0$.

Proof. In order to get rid of derivatives with the help of (10*d*) and (15*c*), we multiply $S^{jq}_{\beta lp}+S^{jq}_{\beta pl}$ by $f^{kl}f^{ps}$,

$$
\begin{aligned}
&\left(w^j_{\beta k,h}c^{hq}_p-c^{jq}_{h,k}w^h_{\beta p}+c^{jq}_{k,h}w^h_{\beta p}-w^q_{\beta h,k}c^{hj}_p\right)f^{kl}f^{ps}\\
&\quad+\left(w^j_{\beta p,h}c^{hq}_k-c^{jq}_{h,p}w^h_{\beta k}+c^{jq}_{p,h}w^h_{\beta k}-w^q_{\beta h,p}c^{hj}_k\right)f^{kl}f^{ps}\\
&=w^j_{\beta h,k}f^{kl}c^{hq}_p f^{ps}-c^{jq}_{h,k}f^{hp}w^s_{\beta p}f^{kl}+c^{jq}_{k,h}f^{kl}w^h_{\beta p}f^{ps}-w^q_{\beta h,k}f^{kl}c^{hj}_p f^{ps}\\
&\quad+w^j_{\beta h,p}f^{ps}c^{hq}_k f^{kl}-c^{jq}_{h,p}f^{hk}w^l_{\beta k}f^{ps}+c^{jq}_{p,h}f^{ps}w^h_{\beta k}f^{kl}-w^q_{\beta h,p}f^{ps}c^{hj}_k f^{kl}\\
&=\left(-c^{lj}_k w^k_{\beta h}+w^l_{\beta k}c^{kj}_h\right)c^{hq}_p f^{ps}-\left(c^{jp}_h c^{hq}_k-c^{pj}_h c^{hq}_k-c^{pq}_h\left(c^{hj}_k+c^{jh}_k\right)\right)w^s_{\beta p}f^{kl}\\
&\quad+\left(c^{jl}_k c^{kq}_h-c^{lj}_k c^{kq}_h-c^{lq}_k\left(c^{kj}_h+c^{jk}_h\right)\right)w^h_{\beta p}f^{ps}-\left(-c^{lq}_k w^k_{\beta h}+w^l_{\beta k}c^{kq}_h\right)c^{hj}_p f^{ps}\\
&\quad+\left(-c^{sj}_p w^p_{\beta h}+w^s_{\beta p}c^{pj}_h\right)c^{hq}_k f^{kl}-\left(c^{jk}_h c^{hq}_p-c^{kj}_h c^{hq}_p-c^{kq}_h\left(c^{hj}_p+c^{jh}_p\right)\right)w^l_{\beta k}f^{ps}\\
&\quad+\left(c^{js}_p c^{pq}_h-c^{sj}_p c^{pq}_h-c^{sq}_p\left(c^{pj}_h+c^{jp}_h\right)\right)w^h_{\beta k}f^{kl}-\left(-c^{sq}_p w^p_{\beta h}+w^s_{\beta p}c^{pq}_h\right)c^{hj}_k f^{kl}.
\end{aligned}
$$

Now, we collect the coefficients of $w^k_{\beta h}$, $w^s_{\beta h}$ and $w^l_{\beta h}$ (k is a dummy index, and s and l are fixed),

$$
\begin{aligned}
&\left(c^{hq}_p\left(c^{sj}_k f^{pl}+c^{lj}_k f^{sp}\right)-c^{hj}_p\left(c^{lq}_k f^{sp}+c^{sq}_k f^{pl}\right)\right)w^k_{\beta h}\\
&\quad+\left(\left(c^{jl}_p c^{pq}_k-c^{lj}_p c^{pq}_k-c^{lq}_p c^{pj}_k-c^{lq}_p c^{jp}_k\right)f^{hk}-\left(2c^{lq}_k c^{hj}_p-c^{jh}_p c^{lq}_k\right)f^{kp}-c^{hq}_p c^{lp}_k f^{kj}\right)w^s_{\beta h}\\
&\quad+\left(\left(c^{js}_p c^{pq}_k-c^{sj}_p c^{pq}_k-c^{sq}_p c^{pj}_k-c^{sq}_p c^{jp}_k\right)f^{hk}-\left(2c^{hj}_p c^{sq}_k-c^{jh}_p c^{sq}_k\right)f^{kp}-c^{hq}_p c^{sp}_k f^{kj}\right)w^l_{\beta h}.
\end{aligned}
$$

All three coefficients vanish. Indeed, in the coefficient of $w^k_{\beta h}$ we use (10*b*) and rename the dummy indices to get

$$\begin{aligned} &c^{hq}_p\left(c^{sj}_k f^{pl}+c^{lj}_k f^{sp}\right)-c^{hj}_p\left(c^{lq}_k f^{sp}+c^{sq}_k f^{pl}\right)\\ &\quad=c^{hq}_p c^{sj}_k f^{pl}+c^{hq}_p c^{lj}_k f^{sp}-c^{hj}_p c^{lq}_k f^{sp}-c^{hj}_p c^{sq}_k f^{pl}\\ &\quad=-\left(c^{lq}_p c^{sj}_k+c^{sq}_p c^{lj}_k-c^{sj}_p c^{lq}_k-c^{lj}_p c^{sq}_k\right)f^{ph}\\ &\quad=\left(-c^{lq}_k c^{sj}_p-c^{sq}_k c^{lj}_p+c^{sj}_p c^{lq}_k+c^{lj}_p c^{sq}_k\right)f^{ph}=0. \end{aligned}$$

The coefficients of $w^s_{\beta h}$ and $w^l_{\beta h}$ are the same up to permutation, so we consider only one of them. As a preparatory step, we use again (10*b*) and redenote the dummy indices to yield,

$$\begin{aligned} &\left(c^{js}_p c^{pq}_k-c^{sj}_p c^{pq}_k-c^{sq}_p c^{pj}_k-c^{sq}_p c^{jp}_k\right)f^{hk}-\left(2c^{hj}_p c^{sq}_k-c^{jh}_p c^{sq}_k\right)f^{kp}-c^{hq}_p c^{sp}_k f^{kj}\\ &\quad=\left(c^{js}_p c^{pq}_k-c^{sj}_p c^{pq}_k\right)f^{hk}-c^{sq}_k\left(c^{jk}_p f^{ph}+c^{hj}_p f^{pk}+c^{kh}_p f^{pj}\right)-c^{hq}_p c^{sp}_k f^{kj}, \end{aligned}$$

where the middle summand vanishes due to (10*c*), thus leaving us with

$$\begin{aligned} &\left(c^{js}_p c^{pq}_k-c^{sj}_p c^{pq}_k\right)f^{hk}-c^{hq}_p c^{sp}_k f^{kj}\\ &\quad=-c^{hq}_k\left(c^{js}_p f^{pk}+c^{kj}_p f^{ps}+c^{sk}_p f^{pj}\right)=0 \end{aligned}$$

due to (10*c*) again. □

Lemma 19. $R^{ijp}_{lm}+R^{ijp}_{ml}=0$ *where*

$$R^{ijp}_{lm}:=-\Gamma^{ij}_{k,l}c^{kp}_m+\Gamma^{pj}_{k,l}c^{ki}_m-\Gamma^{kj}_l c^{pi}_{k,m}+\Gamma^{kj}_l c^{pi}_{m,k}+c^{\alpha\beta}\left(c^{ki}_l w^p_{\alpha k}w^j_{\beta m}-c^{kp}_l w^i_{\alpha k}w^j_{\beta m}\right).$$

Proof. First of all, we rewrite (21) as

$$\Gamma^{ij}_{k,l}+c^{\alpha\beta}w^i_{\alpha k}w^j_{\beta l}=\left(\Gamma^{tj}_r c^{ri}_l-\Gamma^{rj}_l c^{ti}_r\right)f_{tk},$$

and simplify $R^{ijp}_{lm}+R^{ijp}_{ml}$ to

$$\begin{aligned} &-\left(\Gamma^{tj}_r c^{ri}_l-\Gamma^{rj}_l c^{ti}_r\right)f_{tk}c^{kp}_m+\left(\Gamma^{tj}_r c^{rp}_m-\Gamma^{rj}_m c^{tp}_r\right)f_{tk}c^{ki}_l-\left(\Gamma^{tj}_r c^{ri}_m-\Gamma^{rj}_m c^{ti}_r\right)f_{tk}c^{kp}_l+\left(\Gamma^{tj}_r c^{rp}_l-\Gamma^{rj}_l c^{tp}_r\right)f_{tk}c^{ki}_m\\ &-\Gamma^{kj}_l c^{pi}_{k,m}+\Gamma^{kj}_m c^{pi}_{l,k}-\Gamma^{kj}_m c^{pi}_{k,l}+\Gamma^{kj}_l c^{pi}_{m,k}. \end{aligned}$$

In order to get rid of the remaining derivatives with the help of (10*d*), we multiply the above expression by $f^{lx}f^{my}$,

$$\begin{aligned} &\Big((\Gamma^{tj}_r c^{ri}_l-\Gamma^{rj}_l c^{ti}_r)f_{mk}c^{kp}_t-(\Gamma^{tj}_r c^{rp}_m-\Gamma^{rj}_m c^{tp}_r)f_{lk}c^{ki}_t+(\Gamma^{tj}_r c^{ri}_m-\Gamma^{rj}_m c^{ti}_r)f_{lk}c^{kp}_t-(\Gamma^{tj}_r c^{rp}_l-\Gamma^{rj}_l c^{tp}_r)f_{mk}c^{ki}_t\\ &\quad-\Gamma^{kj}_l c^{pi}_{k,m}+\Gamma^{kj}_m c^{pi}_{l,k}-\Gamma^{kj}_m c^{pi}_{k,l}+\Gamma^{kj}_l c^{pi}_{m,k}\Big)f^{lx}f^{my}\\ &=(\Gamma^{tj}_r c^{ri}_l-\Gamma^{rj}_l c^{ti}_r)f_{mk}c^{kp}_t f^{lx}f^{my}-(\Gamma^{tj}_r c^{rp}_m-\Gamma^{rj}_m c^{tp}_r)f_{lk}c^{ki}_t f^{lx}f^{my}\\ &\quad+(\Gamma^{tj}_r c^{ri}_m-\Gamma^{rj}_m c^{ti}_r)f_{lk}c^{kp}_t f^{lx}f^{my}-(\Gamma^{tj}_r c^{rp}_l-\Gamma^{rj}_l c^{tp}_r)f_{mk}c^{ki}_t f^{lx}f^{my}\\ &\quad-(c^{pl}_k c^{ki}_m-c^{lp}_k c^{ki}_m-c^{li}_k c^{kp}_m-c^{li}_k c^{pk}_m)\Gamma^{xj}_l f^{my}+(c^{px}_l c^{li}_k-c^{xp}_l c^{li}_k-c^{xi}_l c^{lp}_k-c^{xi}_l c^{pl}_k)\Gamma^{kj}_m f^{my}\\ &\quad-(c^{pm}_k c^{ki}_l-c^{mp}_k c^{ki}_l-c^{mi}_k c^{kp}_l-c^{mi}_k c^{pk}_l)\Gamma^{yj}_m f^{lx}+(c^{py}_m c^{mi}_k-c^{yp}_m c^{mi}_k-c^{yi}_m c^{mp}_k-c^{yi}_m c^{pm}_k)\Gamma^{kj}_l f^{lx}, \end{aligned}$$

where we use (10*b*) and (18) to reduce the expression to the form with collectable coefficients of Γ^{xj}_l and Γ^{yj}_m,

$$\begin{aligned} &\left(-c^{li}_k c^{kp}_m f^{my}-c^{pl}_k c^{ki}_m f^{my}-c^{yi}_m c^{pm}_k f^{lk}+c^{lp}_k c^{ki}_m f^{my}+c^{li}_k c^{pk}_m f^{my}+c^{py}_m c^{mi}_k f^{lk}\right)\Gamma^{xj}_l\\ &\quad+\left(-c^{mi}_k c^{kp}_l f^{lx}-c^{pm}_k c^{ki}_l f^{lx}+c^{mp}_k c^{ki}_l f^{lx}+c^{mi}_k c^{pk}_l f^{lx}+c^{px}_m c^{li}_k f^{mk}-c^{xi}_m c^{pl}_k f^{mk}\right)\Gamma^{yj}_m. \end{aligned}$$

Both coefficients are the same up to permutation, and thus, we need to show that only one of them vanishes. Again, we use (10*b*) to reduce the expression to the form, where the cyclic condition (10*c*) is evident,

$$\begin{aligned}
&-c_k^{li}c_m^{kp}f^{my}-c_k^{pl}c_m^{ki}f^{my}-c_k^{yi}c_m^{pk}f^{lm}-c_m^{lp}c_k^{yi}f^{mk}-c_k^{li}c_m^{yk}f^{mp}+c_k^{li}c_m^{ky}f^{mp}\\
&=-c_k^{mi}c_m^{yp}f^{kl}-c_k^{yi}c_m^{kl}f^{mp}-c_k^{yi}c_m^{pk}f^{ml}-c_m^{lp}c_k^{yi}f^{mk}-c_k^{li}c_m^{yk}f^{mp}+c_k^{li}c_m^{ky}f^{mp}\\
&=-c_k^{mi}c_m^{yp}f^{kl}+c_k^{yi}c_m^{lp}f^{mk}-c_m^{lp}c_k^{yi}f^{mk}-c_k^{li}c_m^{yk}f^{mp}+c_k^{li}c_m^{ky}f^{mp}\\
&=c_k^{li}\left(c_m^{yp}f^{mk}+c_m^{pk}f^{my}+c_m^{ky}f^{mp}\right)=0.
\end{aligned}$$

□

Lemma 20. *The condition (23) can be rewritten in lower indices as*

$$c_{mkl}\Gamma^{mj}_{\alpha i}+c_{mik}\Gamma^{mj}_{\alpha l}+c_{mli}\Gamma^{mj}_{\alpha k}=0. \tag{33}$$

Lemma 21.

$$\begin{aligned}
&-\Gamma^{ij}_{k,l}c_m^{pk}+\Gamma^{ij}_{k,m}c_l^{kp}+\Gamma^{ij}_{k,m}c_l^{pk}-\Gamma^{pj}_{k,l}c_m^{ki}-\Gamma^{ij}_{k,l}c_m^{kp}+\Gamma^{pj}_{k,m}c_l^{ki}\\
&+\Gamma^{ij}_{k}c^{kp}_{l,m}+\Gamma^{ij}_{k}c^{pk}_{l,m}-\Gamma^{ij}_{k}c^{pk}_{m,l}-\Gamma^{ij}_{k}c^{kp}_{m,l}-\Gamma^{kj}_{l}c^{pi}_{k,m}+\Gamma^{pj}_{k}c^{ki}_{l,m}-\Gamma^{pj}_{k}c^{ki}_{m,l}+\Gamma^{kj}_{m}c^{pi}_{k,l}\\
&-c_m^{ki}w_k^pw_l^j+c_l^{kp}w_k^iw_m^j+c_l^{pk}w_k^iw_m^j-c_m^{pk}w_k^iw_l^j+c_l^{ki}w_k^pw_m^j-c_m^{kp}w_k^iw_l^j=0.
\end{aligned}$$

Proof. We start by removing all derivatives from the LHS by using identities (10*d*) and (17*c*),

$$\begin{aligned}
&-c_m^{pk}c_l^{ri}\Gamma_k^{sj}f_{sr}+c_l^{kp}c_m^{ri}\Gamma_k^{sj}f_{sr}+c_l^{pk}c_m^{ri}\Gamma_k^{sj}f_{sr}-c_m^{kp}c_l^{ri}\Gamma_k^{sj}f_{sr}-c_m^{ki}c_l^{rp}\Gamma_k^{sj}f_{sr}+c_l^{ki}c_m^{rp}\Gamma_k^{sj}f_{sr}\\
&-c_m^{pk}c_k^{ri}\Gamma_l^{sj}f_{sr}+c_l^{kp}c_k^{ri}\Gamma_m^{sj}f_{sr}+c_l^{pk}c_k^{ri}\Gamma_m^{sj}f_{sr}-c_m^{kp}c_k^{ri}\Gamma_l^{sj}f_{sr}-c_m^{ki}c_k^{rp}\Gamma_l^{sj}f_{sr}+c_l^{ki}c_k^{rp}\Gamma_m^{sj}f_{sr}\\
&+\left(c_k^{ps}c_l^{ri}+c_k^{si}c_l^{pr}+c_k^{si}c_l^{rp}\right)f_{rs}\Gamma_m^{kj}-\left(c_k^{ps}c_m^{ri}+c_k^{si}c_m^{pr}+c_k^{si}c_m^{rp}\right)f_{rs}\Gamma_l^{kj}+\left(c_l^{si}c_m^{rk}-c_m^{ri}c_l^{sk}\right)f_{rs}\Gamma_k^{pj}
\end{aligned}$$

where we used the condition (10*d*) in the form

$$c^{ij}_{p,l}=\left(c_p^{is}c_l^{rj}+c_p^{sj}c_l^{ir}+c_p^{sj}c_l^{ri}\right)f_{rs},$$

which is achieved by using the conditions (10*b*) and (10*c*) in (10*d*). Now using the condition (10*b*) and renaming dummy indices we can rewrite Γ_l^{sj} and Γ_m^{sj} in the second line as Γ_l^{kj} and Γ_m^{kj} and sum up these terms with the third line to get

$$c_r^{pk}c_l^{ri}\Gamma_m^{sj}f_{ks}-c_m^{ri}c_r^{pk}\Gamma_l^{sj}f_{sk}+\left(c_l^{si}c_m^{rk}-c_m^{ri}c_l^{sk}\right)f_{rs}\Gamma_k^{pj}$$

thus reducing the whole expression to

$$-c_l^{ri}\left(c_m^{pk}\Gamma_k^{sj}f_{sr}-c_r^{pk}\Gamma_k^{sj}f_{sm}-c_m^{sk}f_{rs}\Gamma_k^{pj}\right)-c_m^{ri}\left(c_r^{pk}\Gamma_k^{sj}f_{sl}+c_l^{pk}\Gamma_k^{sj}f_{rs}-c_l^{sk}f_{rs}\Gamma_k^{pj}\right). \tag{34}$$

Lowering indices in (34) by $f_{ai}f_{bj}f_{cp}$ we get

$$-\left(c_{dcm}\Gamma_k^{dj}-c_{dck}\Gamma_m^{dj}-c_{dkm}\Gamma_c^{dj}\right)c_{asl}f^{sk}+\left(c_{dcl}\Gamma_k^{dj}-c_{dck}\Gamma_l^{dj}-c_{asm}\Gamma_c^{dj}\right)c_{asm}f^{sk},$$

which vanishes in view of (33). □

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