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Spectral Theory and Resonances for Point Interactions in Unbounded Domains

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Introduction

In the 1930s physicists were interested in studying nuclear interactions. This kind of interaction is characterized by a very strong intensity and at the same time a very short range of influence. One of the first modelizations of this interaction were the works of Bethe and Peierls [9] and Thomas [47] in which a "zero range potential" was introduced, and of Fermi ([21]) from which the common denominations "Fermi pseudopotential" for point interactions derives. Later studies aimed at the rigorous definition of this concept led to the work of Berezin and Faddeev of 1961 [8], where it was formulated through von Neumann-Kreĭn's theory of self-adjoint extensions.

The mathematical formulation of a "delta-like" potential (from now on we will use freely the term delta interaction or point interaction in dimension two and three) is based on the fact that the corresponding Schrödinger's operator "$-\Delta + \lambda\delta_y$" must coincide with the free Laplacian on $C_0^\infty(\mathbb{R}^n \setminus \{ y \})$. This mimics the idea that the particle does not feel the interaction if it cannot be found at the point y. The theory of self-adjoint extensions, when applied to dimension two and three (the one dimensional case is much easier and will not considered in this thesis if not as a comparison case) shows that there exists a one parameter family of self-adjoint extensions of this symmetric restriction of the free Laplacian in $\mathbb{R}^n$ indexed by a parameter α. This constant appears in the abstract treatment as an index for the individual self-adjoint extension, and it could be recovered as a coupling constant (possibly renormalized, this is especially relevant in dimension three) or a suitable function of the coupling constant, so that it should not be confused with the formal coefficient λ multiplying the δ function in $-\Delta + \lambda\delta_y$ if not in dimension one where the delta interaction can be recovered as a form perturbation of the Laplacian.

The point interaction can be obtained as a local approximation, starting from a Schrödinger operator $-\Delta + V$ scaled suitably, but with a taming of the standard δ scaling $\frac{1}{\epsilon^d}V(\frac{x}{\epsilon})$; moreover, only in dimension three, the Schrödinger operator $-\Delta + V$ needs to have a zero-energy resonance (see Section 1.1 for further details). In dimension one the scaling is the usual delta scaling, and the point interaction obtained can be considered on the same footing of a perturbation of the Laplacian by means of a potential. Instead, in dimension two and three, the scaling is different from the delta scaling and subjected to certain conditions on the approximating potentials and this bears very relevant differences in spectral properties of regular potentials and singular interactions of the delta-type. For example, it is known that for $n = 1, 2$, the operator $-\Delta + V$ with $V \in C_0^\infty(\mathbb{R}^n)$ has an eigenvalues whenever $\int_{\mathbb{R}^n} V(x)\, \mathrm{d}x < 0$, while in $\mathbb{R}^3$, the potential needs to reach a certain intensity to cause the insurgence of eigenvalues. This is a consequence of the Cwikel-Lieb-Rozemblum inequality [34] which bounds the number of negative eigenvalues for rather general potentials, however not including point interactions

$$N_-(V) \leq L_n \int_{\mathbb{R}^3} [V(x)]_-^{\frac{3}{2}} \, \mathrm{d}x.$$

For the point interaction in $\mathbb{R}^3$, despite it being a "one-point support" interaction, instead an eigenvalue exist if and only if $\alpha < 0$, and in $\mathbb{R}^2$ a negative eigenvalue exists for any member of the family of point interactions, irrespectively from the sign and value of α. In general, a systematic theory of N-point (N possibly infinite) interactions in $\mathbb{R}^n$ and of their spectral properties has been carried out (see the treatise [2] and reference references, updated to 2005 and hugely increased in the last twenty years).

The study of point interaction has also been extended to proper subsets of $\mathbb{R}^n$. In [11] the class of point interaction in a bounded domain coupled with Dirichlet boundary conditions is constructed using the technique of boundary value spaces and it is also shown how the introduction of this point interaction alters the spectrum of the free Laplacian in a ball. A similar approach is taken in [20], in which it is also investigated how the position of the point interaction with respect to the boundary affects the principal eigenvalue.

There is not much literature about point interaction in unbounded domain different from $\mathbb{R}^n$. These domains have characteristics which place them somewhat in the middle of the $\mathbb{R}^n$ and bounded case. For instance, the free Laplacian on this kind of domains has typically purely absolutely continuous spectrum $[0, +\infty)$ as in the $\mathbb{R}^n$ case, while for bounded domains its spectrum is purely discrete (this is true with some restrictions, discussed later in the thesis; for example this is the case of exterior domains and conical domains as the half-plane; it is not true anymore for generic quasi-conical domains or for quasi-cylindrical or quasi-bounded sets). On the other hand the presence of a boundary in unbounded domains affects spectral properties as in the bounded case. So, for example it is interesting to determine under which conditions the point-interaction causes the birth of an eigenvalue and how this condition is affected by boundary conditions. This is one of the main tasks dealt with in the thesis.

When one considers non-compact domains, resonances becomes relevant. Resonances of the operator H are the poles of the meromorphic continuation in the lower complex half-plane of the resolvent $(H - z^2)^{-1}$. Loosely speaking, they behave as a sort of complex eigenvalues corresponding to resonance functions playing the role of eigenfunctions, that are only locally square integrable, and when H is the generator of a Schrödinger or wave dynamics, they correspond to solutions decaying in time. Their use is common in Physics but a rigorous and complete definition has many different declinations (sometimes not equivalent) in the mathematical literature ([31], [18] are representative examples of a time dependent and a stationary definition, respectively). As mentioned before, they are of interest because the solution of the wave or Schrödinger equation propagating on non-compact domains can be expressed, asymptotically for large times, as an expansion on resonance functions [1] as an alternative to the spectral theorem based expansion. Also in this case, when point interactions are considered, resonances have been studied mainly in the $\mathbb{R}^n$ case. The distribution of complex ($\operatorname{Im} z < 0$) resonances has been investigated by Albeverio and Karabash [4–6] and by Lipovský and Lotoreichik [35]. Real resonance in dimension 3 are discussed in [37]. Properties of the resolvent in dimension 2 are instead studied in [15]. Also for the resonances little is known when the domain considered is an unbounded subset of $\mathbb{R}^n$.

In this work we aim to give a first attempt at describing spectral properties and resonances of the Laplacian perturbed by point interaction on unbounded domains.

In the first chapter, which is preparatory but still contains not previously published material, we define the N-point interactions in an unbounded domain, with Dirichlet, Neumann or Robin boundary conditions using the theory of self-adjoint extensions. We also give a different treatment through a different road and, following the work [46] by Teta, where he defines quadratic forms corresponding to the point interaction Laplacians in $\mathbb{R}^n$, we construct quadratic forms corresponding to delta-interactions in unbounded domains.

The second chapter is devoted to the study of the spectrum of 1-point interaction in unbounded domains. The point spectrum in these domains is constituted at most of an eigenvalue, due to Kreĭn formula. The existence and behavior of the eigenvalue is determined by two elements: the value of the parameter α and the position of the point interaction with respect to the boundary. General conditions are found for exterior domains in $\mathbb{R}^3$ and $\mathbb{R}^2$ with Dirichlet boundary conditions and for domains Ω, star-shaped with respect to the point in which the "delta intersction" is located, for Neumann boundary conditions, that include the half-space and other domains with unbounded boundary. Then the critical α_c for which an eigenvalue exists is determined in some particular cases in which the Green's function is known explicitly such as the half-space, the half-plane, the exterior of the disk and the exterior of a sphere. The behavior of α_c changes in a noticeable way when compared to the one in $\mathbb{R}^n$ for a single interaction.

In the third chapter we define resonances for point interactions in domains (according to a definition agreeing with the Sjostrand-Zworski theory for regular perturbation of the Laplacian and of their more general "black-box" definition [18, 43, 44]) and study their distribution on the half-space and half-plane for 1-point interaction. In the former case, for Dirichlet and Neumann boundary conditions, we determine that there are infinite resonances (while for $N = 1$ in $\mathbb{R}^3$ there is at most one resonance). Moreover each of this resonances is localized in a precise vertical strip of the complex plane. Also a logarithmic relation between real and imaginary part of the resonances is established. As a byproduct of this, an asymptotic on the number of resonances in a ball of radius R centered at the origin for $R \to +\infty$ is performed. For Robin boundary condition in the half-space and Dirichlet and Neumann ones in the half-plane, we give an estimate on the real part of the resonances contained in a horizontal strip of width β. As a non-trivial application we give the resonance expansion for the wave evolution. Finally, we studied the semiclassical distribution of resonances for the one-point interaction on the half-space. A previous result in the one dimensional case of a delta interaction placed on the half-line with Dirichlet boundary condition at the free end (also called the Winter model in physical literature) is given in the recent work [17] on the half-line. The present results constitute a first generalization of this analysis to a three dimensional case.

The analysis of spectral theory and resonances of point interactions in domains is essentially a virgin soil, and so there are several refinements, extension and developments of the results here presented that can be object of future study. Here we mention

a) the completion of the analysis of the exterior domain case to general (Robin) boundary conditions;

b) the more ambitious analysis of the behavior of point interaction with respect to Glazman classification of unbounded domains: quasi-conical, quasi-cylindrical and quasi-bounded;

c) the extension of the analysis of the distribution of resonances on more general domains than the half-space and the corresponding resonance expansion for time dependent problems.

The characteristic of point interactions on domains is that of being in a sense explicitly solvable models, less than the point interactions $\mathbb{R}^n$ case, but more than the regular potential perturbation of the Laplacian. So the expectation is that their analysis here begun should give on one hand the chance of new results in the restricted field of study of point interaction, and on the other hand a source of inspiration and intuition of possible behavior for the standard regular potential case.

Chapter 1.

Point Interactions in a Domain

1.1. Point Interactions in $\mathbb{R}^3$

In this section we recall some properties of the N centers point interaction in $\mathbb{R}^3$. Refer to the book [2] by Albeverio and Høegh-Krohn for additional details.

We consider the non-negative operator

$$H_Y = -\Delta|_{C_0^\infty(\mathbb{R}^3 \setminus \{y_j\}_{j=1}^N)} .$$

Its closure is the symmetric operator

$$\dot{H}_Y = -\Delta|_{D(\dot{H}_Y)} ,$$

where

$$D(\dot{H}_Y) = \{\varphi \in H^2(\mathbb{R}^3) |\, \varphi(y_j) = 0, y_j \in Y, j = 1, \dots, N\}.$$

$\dot{H}_Y$ is not self-adjoint, but in fact has deficiency indices (N, N) and deficiency subspaces

$$\mathrm{Ran}\,(\dot{H}_Y \pm i)^\perp = \mathrm{span}\left\{G^{0,3}_{\sqrt{\pm i}}(\cdot - y_1), \dots, G^{0,3}_{\sqrt{\pm i}}(\cdot - y_N)\right\}, \quad \mathrm{Im}\,\sqrt{\pm i} > 0$$

(for the definition of Green's functions $G^{0,3}_z$ see (A.3) in Appendix A.1). From the theory of self-adjoint extensions it then follows that there exists an N^2-parameters family of self-adjoint extensions of $\dot{H}_Y$. Imposing locality of the interactions, we define the self-adjoint extension $-\Delta_{\alpha,Y}$ as follows ($\alpha = \{\alpha_j\}_{j=1}^N$). The domain $D(-\Delta_{\alpha,Y})$ is made by all functions ψ of the form

$$\psi(x) = \varphi_z(x) + \sum_{j=1}^N q_j G^{0,3}_z(x - y_j), \quad x \in \mathbb{R}^3 \setminus Y,$$

with $\varphi_z \in H^2(\mathbb{R}^3)$, $q_j \in \mathbb{C}$ and $z^2 \in \rho(-\Delta_{\alpha,Y})$, $\operatorname{Im} z > 0$. And with ψ of this form, the action is given by

$$(-\Delta_{\alpha,Y} - z^2)\psi = (-\Delta - z^2)\varphi_z.$$

Moreover, let $\psi \in D(-\Delta_{\alpha,Y})$ and $\psi = 0$ in an open set $U \subseteq \mathbb{R}^3$. Then $-\Delta_{\alpha,Y}\psi = 0$ in U.

Through Kreĭn's formula (Theorem B.1.1), we can also give the expression for the resolvent

$$(-\Delta_{\alpha,Y} - z^2)^{-1} = (-\Delta - z^2)^{-1} + \sum_{j,l=1}^{N} [\Gamma^{\Omega}_{\alpha,Y}(z)]^{-1}_{jl} (G^{0,3}_z(\cdot - y_l), \cdot) G^{0,3}_z(\cdot - y_j),$$
$$z^2 \in \rho(-\Delta_{\alpha,Y}), \quad \operatorname{Im} z > 0, \quad (1.1)$$

where

$$\Gamma^{\Omega}_{\alpha,Y}(z)_{jl} = \begin{cases} -G^{0,3}_z(y_j - y_l) & j \neq l \\ \alpha_j - \frac{iz}{4\pi} & j = l \end{cases}.$$

Remark. Point interactions can be obtained also through local approximation. Here we recall the result for a one point interaction situated in the origin. The construction is done starting from a Schrödinger operator $-\Delta + V$ and scaling it suitably, but with a taming of the δ scaling (see [2] and [3] for the missing details):

- For $n = 2$ the scaling is of the form $-\Delta + \frac{g(\ln \varepsilon)}{\varepsilon^2} V(\frac{\cdot}{\varepsilon})$ with $g(y) = g_1 y + g_2 y^2 + o(y^2)$ and a relation between g_1, g_2 and V determines the α in $-\Delta_{\alpha,0}$ as $\varepsilon \to 0$.
- For $n = 3$ the scaling is of the form $-\Delta + \frac{1+g\varepsilon}{\varepsilon^2} V(\frac{\cdot}{\varepsilon})$ with $g \in \mathbb{R}$. Again g and V determine α in the norm resolvent limit, but a further crucial condition appear in $3d$: $-\Delta + V$ needs to have a zero energy resonance or otherwise the scaling limit coincides with the free Laplacian.

1.2. Point Interactions in a Domain in $\mathbb{R}^3$ Through Self-Adjoint Extensions

1.2.1. Boundary Conditions

We want to construct Laplacians with point interactions also for domains Ω different than the whole $\mathbb{R}^3$. Let $\varphi, \psi \in C^2(\overline{\Omega})$ and consider

$$\begin{aligned}(-\Delta\varphi, \psi)_{L^2(\Omega)} &= \int_\Omega (-\Delta\varphi)(x)\overline{\psi(x)}\,\mathrm{d}x = \int_\Omega \nabla\varphi(x)\cdot\overline{\nabla\psi(x)}\,\mathrm{d}x - \int_\Omega \nabla\cdot\left(\overline{\psi(x)}\nabla\varphi(x)\right)\mathrm{d}x \\ &= \int_\Omega \nabla\cdot\left(\varphi(x)\overline{\nabla\psi(x)} - \overline{\psi(x)}\nabla\varphi(x)\right)\mathrm{d}x + \int_\Omega \varphi(x)\overline{(-\Delta\psi)(x)}\,\mathrm{d}x \\ &= \int_{\partial\Omega}\left(\varphi(x)\overline{\frac{\partial\psi}{\partial\nu}(x)} - \overline{\psi(x)}\frac{\partial\varphi}{\partial\nu}(x)\right)\mathrm{d}\sigma + (\varphi, -\Delta\psi)_{L^2(\Omega)},\end{aligned}$$

where we repeatedly integrated by parts and used the divergence theorem. We note that for generic $\psi, \varphi \in C^2(\overline{\Omega})$ the Laplacian is not symmetric. In order to have symmetry, the boundary term must vanish. So we look for subsets of $C^2(\overline{\Omega})$ such that the restriction of the Laplacian to those ones actually is symmetric. Sufficient conditions are

- $\varphi|_{x\in\partial\Omega} = 0$ and $\psi|_{x\in\partial\Omega} = 0$;.
- There exists some real valued $\eta \in C(\partial\Omega)$ such that

$$\left.\frac{\partial\varphi}{\partial\nu}\right|_{x\in\partial\Omega} = -\eta\varphi|_{x\in\partial\Omega} \quad \text{and} \quad \left.\frac{\partial\psi}{\partial\nu}\right|_{x\in\partial\Omega} = -\eta\psi|_{x\in\partial\Omega}.$$

  Anyway, we will only consider η being positive real constant in the following.

So to obtain a symmetric Laplacian then boundary conditions must be imposed. We define respectively

- the Dirichlet Laplacian on Ω as the operator with domain

$$D(-\Delta^{\Omega,\mathrm{D}}) = \left\{ \psi \in C^\infty(\overline{\Omega}) \;\middle|\; \psi|_{x\in\partial\Omega} = 0 \right\}.$$

- the Robin Laplacian as the one with domain

$$D(-\Delta^{\Omega,\mathrm{R},\eta}) = \left\{ \psi \in C^\infty(\overline{\Omega}) \;\middle|\; \left.\left(\frac{\partial\psi}{\partial\nu} + \eta\psi\right)\right|_{x\in\partial\Omega} = 0 \right\}.$$

  We also name $-\Delta^{\Omega,\mathrm{R},0} = -\Delta^{\Omega,\mathrm{N}}$, the Neumann Laplacian and it follows that its domain is

$$D(-\Delta^{\Omega,\mathrm{N}}) = \left\{ \psi \in C^\infty(\overline{\Omega}) \;\middle|\; \left.\frac{\partial\psi}{\partial\nu}\right|_{x\in\partial\Omega} = 0 \right\}.$$

For all these operators the action is the usual action of the Laplacian.

These operators are symmetric but not self-adjoint. That's because $C^\infty(\Omega)$ is too small and the domain of the adjoint properly contains it. The "right" space is $H^2(\Omega)$. Functions in H^2 are only defined almost everywhere, but due to Morrey's embedding theorem (see for example Theorem 12.55 in [33]) each of them has a continuous rapresentative. Hence the boundary value can be intended as the classical evaluation of a function. The same does not hold for the notion of the value of their derivative on the boundary. This means that, to also treat Neumann and Robin boundary conditions, the evaluation of the boundary must be intended in a different way, which extends the classical evaluation on continuous functions. The following theorem states that under regularity hypotheses for $\partial\Omega$, this extension exists

Theorem 1.2.1 ([33]). *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$ be an open set whose boundary $\partial\Omega$ is Lipschitz continuous. There exists a unique linear operator*

$$\mathrm{Tr} : H^1(\Omega) \to L^2_{\mathrm{loc}}(\partial\Omega)$$

such that

(i) $\mathrm{Tr}\,(u) = u$ *on* $\partial\Omega$ *for all* $u \in H^1(\Omega) \cap C(\overline{\Omega})$.

(ii) the integration by parts formula

$$\int_\Omega u \frac{\partial \psi}{\partial x_i}\,\mathrm{d}x = -\int_\Omega \psi \frac{\partial u}{\partial x_i}\,\mathrm{d}x + \int_{\partial\Omega} \psi\,\mathrm{Tr}\,(u)\nu_i\,\mathrm{d}\sigma_{n-1}$$

(where ν is the outward normal vector and σ_{n-1} is the $(n-1)$-dimensional Hausdorff measure) holds for all $u \in H^1(\Omega)$, for all $\psi \in C^1_0(\mathbb{R}^n)$ and all $i = 1, \dots, N$.

(iii) for every $R > 0$ there exist two constants $c_R, \varepsilon_R > 0$ depending on R and Ω such that

$$\int_{B(0,R)\cap\partial\Omega} |\mathrm{Tr}\,(u)|^2\,\mathrm{d}\sigma_{n-1} \leq c_R \varepsilon^{-1} \int_{B(0,R)\cap(\Omega\setminus\Omega_\varepsilon)} |u|^2\,\mathrm{d}x + c_R\varepsilon \int_{B(0,R)\cap(\Omega\setminus\Omega_\varepsilon)} \|\nabla u\|^2\,\mathrm{d}x,$$

for every $0 < \varepsilon \leq \varepsilon_R$, where $\Omega_\varepsilon = \{\, x \in \Omega \mid \mathrm{dist}\,(x, \partial\Omega) > \varepsilon \,\}$.

The function $\mathrm{Tr}\,(u)$ is called the *trace* of u on $\partial\Omega$.

We can observe that if $\partial\Omega$ is bounded, Tr is bounded from $H^2(\Omega) \to L^2(\partial\Omega)$. We give the following definition

Definition 1.2.1. Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$ be the half-space $\mathbb{H}^n$ or an open set with $\partial\Omega$ bounded and Lipschitz. We call the operator

$$\gamma_D : H^1(\Omega) \to L^2(\partial\Omega) \quad \gamma_D u = \mathrm{Tr}\,(u)$$

the Dirichlet trace operator.

Definition 1.2.2. Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$ be the half-space $\mathbb{H}^n$ or an open set with $\partial\Omega \in C^{1,1}$ bounded. We call the operator

$$\gamma_N : H^2(\Omega) \to L^2(\partial\Omega) \quad \gamma_D u = \mathrm{Tr}\left(\frac{\partial u}{\partial \nu}\right)$$

the Neumann trace operator.

The following theorems states that for domains of this kind the trace is bounded in $L^2(\Omega)$.

Theorem 1.2.2 ([30])**.** *For the half space $\mathbb{H}^n$:*

- *if $s \geq 1$, then the operator $\gamma_{\mathrm{D}} : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1}) \hookrightarrow L^2(\mathbb{R}^{n-1})$ is bounded;*
- *if $s > 3/2$, then the operator $\gamma_{\mathrm{N}} : H^s(\mathbb{R}^n) \to H^{s-3/2}(\mathbb{R}^{n-1}) \hookrightarrow L^2(\mathbb{R}^{n-1})$ is bounded.*

Theorem 1.2.3 (See Theorem 18.40 in [33])**.** *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an open set whose boundary $\partial\Omega$ is bounded and Lipschitz. Then there exists some constant C such that*

$$\|\gamma_D u\|_{L^2(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)} .$$

Theorem 1.2.4 (See Theorem 18.51 in [33])**.** *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an open set with $\partial\Omega \in C^{1,1}$ bounded. Then*

$$(\gamma_D, \gamma_N) : H^2(\Omega) \to L^2(\partial\Omega)$$

is a bounded operator.

From now on, we will only consider cases covered by these result unless otherwise stated. Hence Ω will be one of the following

- the half-space $\mathbb{H}^n = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0 \}$;
- a domain with $\partial\Omega$ Lipschitz and bounded if the domain of the operator considered is $H^1(\Omega)$;
- a domain with $C^{1,1}$ $\partial\Omega$ and bounded if the domain of the operator considered is $H^2(\Omega)$.

On these kinds of domains, now we can interpret the boundary value even for functions defined almost everywhere and so we can consider finally the self-adjoint form of the symmetric Laplacians defined above (we use the same symbol because the symmetric Laplacians won't be used anymore in the following):

- the Dirichlet Laplacian: $D(-\Delta^{\Omega,\mathrm{D}}) = \left\{ \psi \in H^2(\Omega) \,\middle|\, \gamma_{\mathrm{D}}\psi = 0 \right\} = H^2_0(\Omega)$;
- the Neumann Laplacian: $D(-\Delta^{\Omega,\mathrm{N}}) = \left\{ \psi \in H^2(\Omega) \,\middle|\, \gamma_{\mathrm{N}}\psi = 0 \right\}$;
- the Robin Laplacian: $D(-\Delta^{\Omega,\mathrm{R},\eta}) = \left\{ \psi \in H^2(\Omega) \,\middle|\, (\gamma_{\mathrm{N}} + \eta\gamma_{\mathrm{D}})\psi = 0 \right\}$.

We remark that η will be always supposed to be non-negative. We also use $-\Delta^{\Omega}$ as a generic symbol for one of the three operators above. The action of the Laplacians defined above is

$$-\Delta^{\Omega} : D(-\Delta^{\Omega}) \to L^2(\Omega) \quad -\Delta^{\Omega}\psi = -\Delta\psi$$

We will also refer to the Green's functions of $-\Delta^{\Omega}$ with $G^{\Omega}_{z,y}$ in place of $G^{\Omega,\mathrm{D}}_{z,y}$, $G^{\Omega,\mathrm{N}}_{z,y}$ or $G^{\Omega,\mathrm{R},\eta}_{z,y}$ which are the Green's function with Dirichlet, Neumann and Robin boundary conditions respectively. In the same way, we will call the generic corrector function $h^{\Omega}_{z,y}$ and the ones with specific prescribed boundary conditions $h^{\Omega,\mathrm{D}}_{z,y}$, $h^{\Omega,\mathrm{N}}_{z,y}$ or $h^{\Omega,\mathrm{R},\eta}_{z,y}$ (see Appendix A.1).

1.2.2. Boundary Triples and Point Perturbations

Here we use the theory of boundary value spaces (the main result used are taken from [25] and mentioned in Appendix B.3) to construct self-adjoint extensions associated with point perturbation of the Laplacian in domains Ω of the type described above with Dirichlet, Neumann or Robin boundary conditions. Let $\left\{ y_j \right\}_{j=1}^{N}$ be distinct points in Ω. We consider the closed symmetric operator $\big(H_0, D(H_0)\big)$

$$D(H_0) = \left\{ \psi \in D(-\Delta^{\Omega}) \;\middle|\; \psi(y_j) = 0,\ j = 1, \dots, N \right\} \tag{1.2}$$

$$H_0\psi = -\Delta\psi. \tag{1.3}$$

In order to find the domain of the adjoint through von Neumann decomposition formula, we characterize its deficiency subspaces.

Lemma 1.2.1. *Let $\mathcal{H}_{\pm} = \mathrm{Ran}\,(H_0 \pm i)^{\perp}$ be the deficiency subspaces related to the operators H_0 defined in (1.2) and (1.3); then*

$$\mathcal{H}_{\pm} = \mathrm{span}\left\{ G^{\Omega}_{\sqrt{\mp i},y_j},\ j = 1, \dots, N \right\}.$$

Proof. $\psi \in \mathcal{H}_{\pm}$ implies

$$\begin{cases} (\psi, (-\Delta \pm i)\varphi)_{L^2(\Omega)} = 0 \\ \psi \in L^2(\Omega) \end{cases} , \quad \forall \varphi \in D(H_0).$$

But since, $\varphi(y_j) = 0\ \forall j = 1, \dots, N$, we have that

$$\left(G^{\Omega}_{\sqrt{\mp i},y_j}, (-\Delta \pm i)\varphi\right)_{L^2(\Omega)} = \left((-\Delta \mp i)G^{\Omega}_{\sqrt{\mp i},y_j}, \varphi\right)_{L^2(\Omega)} = \varphi(y_j) = 0,$$

which means that $\operatorname{span}\left\{ G^{\Omega}_{\sqrt{\mp i},y_j}, j = 1,\dots,N \right\} \subseteq \mathcal{H}_\pm$. Let $\psi_\pm \in \operatorname{Ran}(H_0 \pm i)^\perp$ and $\varphi \in D(H_0)$. Then there exist numbers $c_1^\pm,\dots,c_N^\pm$ independent of φ such that

$$(\psi_\pm, (-\Delta \pm i)\varphi)_{L^2(\Omega)} = \sum_{j=1}^N c_j^\pm \varphi(y_j). \tag{1.4}$$

In fact, let

$$\tilde{\varphi} = \varphi - \sum_{j=1}^N \varphi(y_j)\chi_j, \tag{1.5}$$

where $\chi_j \in C_0^\infty(\Omega)$, $\chi(y_j) = 1$ and $\operatorname{supp}\chi_j \cap \operatorname{supp}\chi_l = \emptyset$, $j,l = 1,\dots,N$, $j \neq l$. Then $\tilde{\varphi} \in D(H_0)$. Substituting (1.5) in (1.4) and using this fact we obtain that $c_j^\pm = (\psi_\pm, (-\Delta \pm i)\chi_j)$. But the constants $c_1^\pm,\dots,c_N^\pm$ are uniquely determined by $\psi_\pm$. In fact, let $\tilde{\psi}_\pm$ be such that

$$(\tilde{\psi}_\pm, (-\Delta \pm i)\varphi)_{L^2(\Omega)} = \sum_{j=1}^N c_j^\pm \varphi(y_j).$$

Then $((\psi_\pm - \tilde{\psi}_\pm), (-\Delta \pm i)\varphi)_{L^2(\Omega)} = 0\ \forall \varphi \in D(H_0)$, which implies that $\psi_\pm = \tilde{\psi}_\pm$.

Finally, we observe that

$$\psi_\pm = \sum_{j=1}^N c_j^\pm G^{\Omega}_{\sqrt{\mp i},y_j}$$

satisfies (1.4), thereby proving $\mathcal{H}_\pm \subseteq \operatorname{span}\left\{ G^{\Omega}_{\sqrt{\mp i},y_j}, j = 1,\dots,N \right\}$. □

The next step in the construction of the sought self-adjoint extensions is to find a boundary value space for H_0. In order to do so, we define the operators $\Gamma_1, \Gamma_2 : D(H_0^*) \to \mathbb{C}^N$ as follows

$$(\Gamma_1\psi)_j = \lim_{x \to y_j} 4\pi \left|x - y_j\right| \psi(x), \quad j = 1,\dots,N \tag{1.6}$$

$$(\Gamma_2\psi)_j = \lim_{x \to y_j} \left(\psi(x) - \frac{(\Gamma_1\psi)_j}{4\pi \left|x - y_j\right|} \right), \quad j = 1,\dots,N. \tag{1.7}$$

We claim that

Theorem 1.2.5. *The triple $(\mathbb{C}^N, \Gamma_1, \Gamma_2)$ defined by (1.6) and (1.7) forms a boundary value space for H_0.*

Proof. By the von Neumann formula (Theorem B.2.1), we can write two generic vectors $\psi, \varphi \in D(H_0^*)$ as follows

$$\psi = \psi_0 + \sum_{k=1}^{N}\Big(a_k G^{\Omega}_{\sqrt{i},y_k} + b_k G^{\Omega}_{\sqrt{-i},y_k}\Big), \quad \varphi = \varphi_0 + \sum_{k=1}^{N}\Big(\alpha_k G^{\Omega}_{\sqrt{i},y_k} + \beta_k G^{\Omega}_{\sqrt{-i},y_k}\Big), \tag{1.8}$$

where $\psi_0, \varphi_0 \in D(H_0)$ and $a_k, b_k, \alpha_k, \beta_k \in \mathbb{C}$. We now consider

$$(\psi, H_0^*\varphi)_{L^2(\Omega)} - (H_0^*\psi, \varphi)_{L^2(\Omega)}. \tag{1.9}$$

Using the action of H_0^* given by von Neumann formula and (1.8) we have (we omit the subscript of the inner product for brevity)

$$\begin{aligned}
(\psi, H_0^*\varphi) &= \Big(\psi, H_0^*\Big(\varphi_0 + \sum_{l=1}^{N}\Big(\alpha_l G^{\Omega}_{\sqrt{i},y_l} + \beta_l G^{\Omega}_{\sqrt{-i},y_l}\Big)\Big)\Big) \\
&= (\psi, H_0\varphi_0) + i\sum_{l=1}^{N}\overline{\alpha_l}\Big(\psi, G^{\Omega}_{\sqrt{i},y_l}\Big) - i\sum_{l=1}^{N}\overline{\beta_l}\Big(\psi, G^{\Omega}_{\sqrt{-i},y_l}\Big) \\
&= (\psi_0, H_0\varphi_0) + \sum_{k=1}^{N} a_k\Big(G^{\Omega}_{\sqrt{i},y_k}, H_0\varphi_0\Big) + \sum_{k=1}^{N} b_k\Big(G^{\Omega}_{\sqrt{-i},y_k}, H_0\varphi_0\Big) + i\sum_{l=1}^{N}\overline{\alpha_l}\Big(\psi_0, G^{\Omega}_{\sqrt{i},y_l}\Big) \\
&\quad - i\sum_{l=1}^{N}\overline{\beta_l}\Big(\psi_0, G^{\Omega}_{\sqrt{-i},y_l}\Big) + i\sum_{k,l=1}^{N} a_k\overline{\alpha_l}\Big(G^{\Omega}_{\sqrt{i},y_k}, G^{\Omega}_{\sqrt{i},y_l}\Big) + i\sum_{k,l=1}^{N} b_k\overline{\alpha_l}\Big(G^{\Omega}_{\sqrt{-i},y_k}, G^{\Omega}_{\sqrt{i},y_l}\Big) \\
&\quad - i\sum_{k,l=1}^{N} a_k\overline{\beta_l}\Big(G^{\Omega}_{\sqrt{i},y_k}, G^{\Omega}_{\sqrt{-i},y_l}\Big) - i\sum_{k,l=1}^{N} b_k\overline{\beta_l}\Big(G^{\Omega}_{\sqrt{-i},y_k}, G^{\Omega}_{\sqrt{-i},y_l}\Big).
\end{aligned}$$

Similarly

$$\begin{aligned}
(H_0^*\psi, \varphi) &= (H_0\psi_0, \varphi_0) + \sum_{l=1}^{N}\overline{\alpha_l}\Big(H_0\psi_0, G^{\Omega}_{\sqrt{i},y_l}\Big) + \sum_{l=1}^{N}\overline{\beta_l}\Big(H_0\psi_0, G^{\Omega}_{\sqrt{-i},y_l}\Big) - i\sum_{k=1}^{N} a_k\Big(G^{\Omega}_{\sqrt{i},y_k}, \varphi_0\Big) \\
&\quad + i\sum_{k=1}^{N} b_k\Big(G^{\Omega}_{\sqrt{-i},y_k}, \varphi_0\Big) - i\sum_{k,l=1}^{N} a_k\overline{\alpha_l}\Big(G^{\Omega}_{\sqrt{i},y_k}, G^{\Omega}_{\sqrt{i},y_l}\Big) + i\sum_{k,l=1}^{N} b_k\overline{\alpha_l}\Big(G^{\Omega}_{\sqrt{-i},y_k}, G^{\Omega}_{\sqrt{i},y_l}\Big) \\
&\qquad - i\sum_{k,l=1}^{N} a_k\overline{\beta_l}\Big(G^{\Omega}_{\sqrt{i},y_k}, G^{\Omega}_{\sqrt{-i},y_l}\Big) + i\sum_{k,l=1}^{N} b_k\overline{\beta_l}\Big(G^{\Omega}_{\sqrt{-i},y_k}, G^{\Omega}_{\sqrt{-i},y_l}\Big).
\end{aligned} \tag{1.10}$$

Taking the difference, (1.9) becomes

$$\sum_{k=1}^{N} a_k\Big(G^{\Omega}_{\sqrt{i},y_k},(H_0-i)\varphi_0\Big)+\sum_{k=1}^{N} b_k\Big(G^{\Omega}_{\sqrt{-i},y_k},(H_0+i)\varphi_0\Big)-\sum_{l=1}^{N}\overline{\alpha_l}\Big((H_0-i)\psi_0,G^{\Omega}_{\sqrt{i},y_l}\Big)$$
$$-\sum_{l=1}^{N}\overline{\beta_l}\Big((H_0+i)\psi_0,G^{\Omega}_{\sqrt{-i},y_l}\Big)+2i\sum_{k,l=1}^{N}a_k\overline{\alpha_l}\Big(G^{\Omega}_{\sqrt{i},y_k},G^{\Omega}_{\sqrt{i},y_l}\Big)-2i\sum_{k,l=1}^{N}b_k\overline{\beta_l}\Big(G^{\Omega}_{\sqrt{-i},y_k},G^{\Omega}_{\sqrt{-i},y_l}\Big). \quad (1.11)$$

Now, given the definition of $G^{\Omega}_{\sqrt{\mp i},y_j}$ and that $\psi_0,\varphi_0\in D(H_0)$, each term of the first four sums vanishes. We exchange k and l in the last sum and so

$$(\psi,H_0^*\varphi)-(H_0^*\psi,\varphi)=2i\sum_{k,l=1}^{N}a_k\overline{\alpha_l}\Big(G^{\Omega}_{\sqrt{i},y_k},G^{\Omega}_{\sqrt{i},y_l}\Big)-2i\sum_{k,l=1}^{N}b_l\overline{\beta_k}\Big(G^{\Omega}_{\sqrt{-i},y_l},G^{\Omega}_{\sqrt{-i},y_k}\Big).$$

We observe that $G^{\Omega}_{\sqrt{-i},y_j}=\overline{G^{\Omega}_{\sqrt{i},y_j}}$. This means that

$$\Big(G^{\Omega}_{\sqrt{-i},y_l},G^{\Omega}_{\sqrt{-i},y_k}\Big)=\overline{\Big(G^{\Omega}_{\sqrt{i},y_l},G^{\Omega}_{\sqrt{i},y_k}\Big)}=\Big(G^{\Omega}_{\sqrt{i},y_k},G^{\Omega}_{\sqrt{i},y_l}\Big),$$

which implies

$$(\psi,H_0^*\varphi)-(H_0^*\psi,\varphi)=2i\sum_{k,l=1}^{N}\Big(a_k\overline{\alpha_l}-b_l\overline{\beta_k}\Big)\Big(G^{\Omega}_{\sqrt{i},y_k},G^{\Omega}_{\sqrt{i},y_l}\Big). \quad (1.12)$$

Let's consider the inner product $\Big(G^{\Omega}_{\sqrt{i},y_k},G^{\Omega}_{\sqrt{i},y_l}\Big)$. We start with the case $k\neq l$. By the definition of $G^{\Omega}_{\sqrt{\mp i},y_j}$, we have

$$\begin{aligned}\Big(G^{\Omega}_{\sqrt{i},y_k},G^{\Omega}_{\sqrt{i},y_l}\Big)&=-i\Big(iG^{\Omega}_{\sqrt{i},y_k},G^{\Omega}_{\sqrt{i},y_l}\Big)=-i\Big(H_0G^{\Omega}_{\sqrt{i},y_k},G^{\Omega}_{\sqrt{i},y_l}\Big)+i\overline{G^{\Omega}_{\sqrt{i},y_l}(y_k)}\\&=-i\Big(G^{\Omega}_{\sqrt{i},y_k},H_0G^{\Omega}_{\sqrt{i},y_l}\Big)+iG^{\Omega}_{\sqrt{-i},y_l}(y_k)\\&=-\Big(G^{\Omega}_{\sqrt{i},y_k},G^{\Omega}_{\sqrt{i},y_l}\Big)-iG^{\Omega}_{\sqrt{i},y_k}(y_l)+iG^{\Omega}_{\sqrt{-i},y_l}(y_k),\end{aligned}$$

equivalent to

$$\Big(G^{\Omega}_{\sqrt{i},y_k},G^{\Omega}_{\sqrt{i},y_l}\Big)=-\frac{i}{2}\Big(G^{\Omega}_{\sqrt{i},y_k}(y_l)-G^{\Omega}_{\sqrt{-i},y_l}(y_k)\Big). \quad (1.13)$$

Now, if $k=l$, both terms in the difference in (1.13) diverge, but the quantity has still a finite limit. We rewrite last equation using the definition for $G^{\Omega}_{\sqrt{\mp i},y_j}$

$$\Big(G^{\Omega}_{\sqrt{i},y_k},G^{\Omega}_{\sqrt{i},y_l}\Big)=-\frac{i}{2}\Big(G^{0,3}_{\sqrt{i},y_k}(y_l)-G^{0,3}_{\sqrt{-i},y_l}(y_k)-h^{\Omega}_{\sqrt{i},y_k}(y_l)+h^{\Omega}_{\sqrt{-i},y_l}(y_k)\Big).$$

The last two terms are finite for $k = l$ and we can write

$$h^{\Omega}_{\sqrt{-i},y_k}(y_k) - h^{\Omega}_{\sqrt{i},y_k}(y_k) = h^{\Omega}_{\sqrt{-i},y_k}(y_k) - \overline{h^{\Omega}_{\sqrt{-i},y_k}(y_k)} = 2i\,\mathrm{Im}\,h^{\Omega}_{\sqrt{-i},y_k}(y_k).$$

For the first two terms instead holds

$$G^{0,3}_{\sqrt{i},y_k}(y_l) - G^{0,3}_{\sqrt{-i},y_l}(y_k) = G^{0,3}_{\sqrt{i},y_k}(y_l) - G^{0,3}_{\sqrt{-i},y_k}(y_l).$$

So, to evaluate the expression on the left side, we can take the limit $y_l \to y_k$ of the right one

$$\lim_{y_l\to y_k}\Big(G^{0,3}_{\sqrt{i},y_k}(y_l) - G^{0,3}_{\sqrt{-i},y_k}(y_l)\Big) = \lim_{y_l\to y_k} \frac{e^{-\sqrt{-i}|x_l-x_k|} - e^{-\sqrt{i}|x_l-x_k|}}{4\pi\,|x_l - x_k|} = \frac{\sqrt{i} - \sqrt{-i}}{4\pi}.$$

So we found that

$$\Big(G^{\Omega}_{\sqrt{i},y_k}, G^{\Omega}_{\sqrt{i},y_l}\Big)_{L^2(\Omega)} = \begin{cases} -\frac{i}{2}\Big(G^{\Omega}_{\sqrt{i},y_k}(y_l) - G^{\Omega}_{\sqrt{-i},y_l}(y_k)\Big) & k \neq l \\ \mathrm{Im}\,h^{\Omega}_{\sqrt{-i},y_k}(y_k) + \frac{i}{8\pi}(\sqrt{-i} - \sqrt{i}) & k = l \end{cases}.$$

Then (1.12) can be rewritten as

$$\begin{aligned}(\psi, H_0^*\varphi) - (H_0^*\psi, \varphi) &= \sum_{\substack{k,l=1\\k\neq l}}^{N}\Big(a_k\overline{\alpha_l} - b_l\overline{\beta_k}\Big)\Big(G^{\Omega}_{\sqrt{i},y_k}(y_l) - G^{\Omega}_{\sqrt{-i},y_l}(y_k)\Big)\\ &- \frac{1}{4\pi}\sum_{k=1}^{N}\Big(a_k\overline{\alpha_k} - b_k\overline{\beta_k}\Big)(\sqrt{-i} - \sqrt{i}) - \sum_{k=1}^{N}\Big(a_k\overline{\alpha_k} - b_k\overline{\beta_k}\Big)\Big(h^{\Omega}_{\sqrt{i},y_k}(y_k) - h^{\Omega}_{\sqrt{-i},y_k}(y_k)\Big). \end{aligned} \tag{1.14}$$

We consider now the operators Γ_1 and Γ_2 acting on $D(H_0^*)$

$$(\Gamma_1\psi)_j = a_j + b_j \tag{1.15}$$

$$(\Gamma_2\psi)_j = \sum_{\substack{k=1\\k\neq j}}^{N}\Big(a_k G^{\Omega}_{\sqrt{i},y_k}(y_j) + b_k G^{\Omega}_{\sqrt{-i},y_k}(y_j)\Big) - \frac{1}{4\pi}(a_j\sqrt{-i} + b_j\sqrt{i}) - a_j h^{\Omega}_{\sqrt{i},y_j}(y_j) - b_j h^{\Omega}_{\sqrt{-i},y_j}(y_j). \tag{1.16}$$

Some computations show that

$$(\Gamma_1\psi,\Gamma_2\varphi)_{\mathbb{C}^N}-(\Gamma_2\psi,\Gamma_1\varphi)_{\mathbb{C}^N}=\sum_{\substack{k,l=1\\k\neq l}}^{N}\Big(a_k\overline{\alpha_l}-b_l\overline{\beta_k}\Big)\Big(G^{\Omega}_{\sqrt{i},y_k}(y_l)-G^{\Omega}_{\sqrt{-i},y_l}(y_k)\Big)$$
$$+\sum_{\substack{k,l=1\\k\neq l}}^{N}a_k\overline{\beta_l}\Big(G^{\Omega}_{\sqrt{i},y_k}(y_l)-G^{\Omega}_{\sqrt{i},y_l}(y_k)\Big)+\sum_{\substack{k,l=1\\k\neq l}}^{N}b_k\overline{\alpha_l}\Big(G^{\Omega}_{\sqrt{-i},y_k}(y_l)-G^{\Omega}_{\sqrt{-i},y_l}(y_k)\Big)$$
$$-\frac{1}{4\pi}\sum_{k=1}^{N}\Big(a_k\overline{\alpha_k}-b_k\overline{\beta_k}\Big)(\sqrt{-i}-\sqrt{i})-\sum_{k=1}^{N}\Big(a_k\overline{\alpha_k}-b_k\overline{\beta_k}\Big)\Big(h^{\Omega}_{\sqrt{i},y_k}(y_k)-h^{\Omega}_{\sqrt{-i},y_k}(y_k)\Big).\quad(1.17)$$

If the second and third sums in (1.17) are zero, then (1.14) and (1.17) coincide and the result follows. In fact it holds that $\Big(G^{\Omega}_{\sqrt{\pm i},y_k}(y_l)-G^{\Omega}_{\sqrt{\pm i},y_l}(y_k)\Big)=0\ \forall k,l=1,\dots N\ k\neq l.$ That follows from the following computation which uses the definition of $G^{\Omega}_{\sqrt{\pm i},y_j}$

$$\begin{aligned}\Big(G^{\Omega}_{\sqrt{\pm i},y_k},G^{\Omega}_{\sqrt{\mp i},y_l}\Big)_{L^2(\Omega)}&=\mp i\Big(\pm G^{\Omega}_{\sqrt{\pm i},y_k},G^{\Omega}_{\sqrt{\mp i},y_l}\Big)_{L^2(\Omega)}\\&=\mp i\Big(\pm G^{\Omega}_{\sqrt{\pm i},y_k},H_0G^{\Omega}_{\sqrt{\mp i},y_l}\Big)_{L^2(\Omega)}\pm iG^{\Omega}_{\sqrt{\pm i},y_l}(y_k)\\&=\Big(G^{\Omega}_{\sqrt{\pm i},y_k},G^{\Omega}_{\sqrt{\mp i},y_l}\Big)_{L^2(\Omega)}\mp iG^{\Omega}_{\sqrt{\pm i},y_k}(y_l)\pm iG^{\Omega}_{\sqrt{\pm i},y_l}(y_k).\end{aligned}$$

□

This result allows a general characterization of the self-adjoint extensions of H_0 in terms of operatorial boundary conditions. Theorem B.3.2 tells us that the self-adjoint extensions of H_0 can be parametrized through the element of the set

$$W=\Big\{\,E=\Big(B\,C\Big)\;\Big|\;BC^*=CB^*,\mathrm{Ran}\,E=n\,\Big\}$$

(see Appendix B.3). Let $(A,B)\in W$, then the associated self-adjoint extension H^{AB} of H_0 is defined as follows

$$D\Big(H^{AB}\Big)=\{\,\psi\in D(H_0^*)\;\Big|\;A\Gamma_1\psi=B\Gamma_2\psi\,\},\quad H^{AB}\psi=H_0^*\psi.$$

The following theorem holds

Theorem 1.2.6. *Fix $(A, B) \in W$. Let H^{AB} be the related self-adjoint extension defined above and R_z^{AB} its resolvent. For any $z \in \mathbb{C} \setminus \mathbb{R}$, the following representation holds:*

$$\begin{aligned} D\Big(H^{AB}\Big) = \Big\{\psi \in L^2(\Omega) | \, \psi = \varphi^z + \sum_{k=1}^N q_k G^{\Omega}_{z,y_k}, \; \varphi^z \in D(-\Delta^{\Omega}), q_j = (\Gamma_1\psi)_j, \\ \sum_{j=1}^N B_{kj}\varphi^z(y_j) = \sum_{j=1}^N (B\Gamma(z) + A)_{kj} q_j, \; k = 1, \ldots, N\Big\} \end{aligned} \tag{1.18}$$

$$H^{AB}\psi = -\Delta\varphi^z + z^2 \sum_{k=1}^N q_k G^{\Omega}_{z,y_k} \tag{1.19}$$

$$R_z^{AB}\psi = R_z\psi + \sum_{j,k,l}^N (B\Gamma(z) + A)^{-1}_{jl} B_{jk} R_z\psi(y_k) G^{\Omega}_{z,y_j}, \quad \forall \psi \in L^2(\Omega) \tag{1.20}$$

$$\Gamma(z)_{kj} = \begin{cases} -G^{\Omega}_{z,y_j}(y_k) & j \neq k \\ h^{\Omega}_{z,y_k}(y_k) - \frac{iz}{4\pi} & j = k \end{cases}, \tag{1.21}$$

where $R_z = (-\Delta^{\Omega} - z^2)^{-1}$ is the resolvent operator associated to $-\Delta$ with the corresponding boundary conditions in Ω.

Proof. By the von Neumann decomposition formula, we can write

$$\psi = \psi_0 + \sum_{k=1}^N \Big(a_k G^{\Omega}_{\sqrt{i},y_k} + b_k G^{\Omega}_{\sqrt{-i},y_k}\Big), \quad \psi_0 \in D(H_0). \tag{1.22}$$

We fix $z \in \mathbb{C} \setminus \mathbb{R}$ and set

$$\varphi^z = \psi_0 + \sum_{k=1}^N \Big(a_k G^{\Omega}_{\sqrt{i},y_k} + b_k G^{\Omega}_{\sqrt{-i},y_k}\Big) - \sum_{k=1}^N q_k G^{\Omega}_{z,y_k} \tag{1.23}$$

$$q_j = (\Gamma_1\psi)_j = a_j + b_j. \tag{1.24}$$

We show that $\varphi^z \in D(-\Delta^{\Omega})$. The boundary condition is satisfied because φ^z is a finite sum of terms satisfying it. We only need to prove that $\varphi^z \in H^2(\Omega)$. The decaying at infinity is fast enough for all terms $\psi_0 \in D(H_0)$ and it implies $\psi_0 \in H^2(\Omega)$. So we need to check only possible singularities. These are contained in the set $\{ y_k \}_{k=1}^N$. Around each y_k the behaviour of the function φ^z is controlled by the one of

$$f_k(x) = a_k \frac{e^{-\sqrt{-i}|x-y_k|}}{4\pi\,|x-y_k|} + b_k \frac{e^{-\sqrt{i}|x-y_k|}}{4\pi\,|x-y_k|} - q_k \frac{e^{iz|x-y_k|}}{4\pi\,|x-y_k|}.$$

Both f_k and ∇f_k are finite for $x \to y_k$,

$$\lim_{x\to y_k} f_k(x) = -\frac{1}{4\pi}(a_k \sqrt{-i} + b_k \sqrt{i} - iq_k z) \tag{1.25}$$

$$\begin{aligned}\nabla f_k(x) = q_k(x - x_k)\frac{e^{iz|x-y_k|}(1 - iz\,|x - y_k|)}{4\pi\,|x - y_k|^3} - a_k(x - x_k)\frac{e^{-\sqrt{-i}|x-y_k|}(1 + \sqrt{-i}\,|x - y_k|)}{4\pi\,|x - y_k|^3}\\ - b_k(x - x_k)\frac{e^{-\sqrt{i}|x-y_k|}(1 + \sqrt{i}\,|x - y_k|)}{4\pi\,|x - y_k|^3} \sim \frac{x - y_k}{4\pi\,|x - y_k|}(-ia_k + ib_k + z^2 q_k),\end{aligned} \tag{1.26}$$

which means that $\nabla f_k \in L^2(B_r(y_k))$ for $r > 0$ small enough. Also $\Delta f_k \in L^2(\Omega)$. In fact, using the definition ogìf the Green function, we have

$$\Delta f_k(x) = -ia_k\frac{e^{-\sqrt{-i}|x-y_k|}}{4\pi\,|x - y_k|} + ib_k\frac{e^{-\sqrt{i}|x-y_k|}}{4\pi\,|x - y_k|} - zq_k\frac{e^{-\sqrt{z}|x-y_k|}}{4\pi\,|x - y_k|} + (-a_k - b_k + q_k)\delta(x - y_k),$$

but the distributional term is cancelled due to $q_k = a_k+b_k$. So $\varphi^z \in D(H_0)$ and the representations (1.22) and

$$\psi = \varphi^z + \sum_{k=1}^{N} q_k G^{\Omega}_{z,y_k}, \quad \varphi^z \in D(-\Delta^{\Omega}) \tag{1.27}$$

are equivalent.

The value $\varphi^z(y_k)$ is linked to $(\Gamma_1\psi)_k$ and $(\Gamma_2\psi)_k$. In fact

$$\begin{aligned}\varphi^z(x_k) &= \lim_{x\to y_k}\Big(a_k G^{\Omega}_{\sqrt{i},y_k}(x) + b_k G^{\Omega}_{\sqrt{-i},y_k}(x) - q_k G^{\Omega}_{z,y_k}(x)\Big) + \sum_{\substack{l=1\\ l\neq k}}^{N}\Big(a_l G^{\Omega}_{\sqrt{i},y_l}(y_k) + b_l G^{\Omega}_{\sqrt{-i},y_l}(y_k) - q_l G^{\Omega}_{z,y_l}(y_k)\Big)\\ &= (\Gamma_2\psi)_k - q_k\frac{iz}{4\pi} + q_k h^{\Omega}_{z,y_k}(y_k) - \sum_{\substack{l=1\\ l\neq k}}^{N} q_l G^{\Omega}_{z,y_l}(y_k) = (\Gamma_2\psi)_k + \Gamma(z)_{kl} q_l.\end{aligned}$$

Combining this with the boundary relation $A\Gamma_1\psi = B\Gamma_2\psi$, we obtain

$$\sum_{j=1}^{N} B_{kj}\varphi^z(y_j) = \sum_{j=1}^{N}(B\Gamma(z) + A)_{kj} q_j \tag{1.28}$$

and the representation of $D(H^{AB})$ in (1.18) is proved.

Let $\psi \in H^{AB}$ and $\varphi \in D(H_0)$. In order to prove the formula for the action of H^{AB} we consider the inner product

$$(H^{AB}\psi, \varphi)_{L^2(\Omega)} = \Big(H^{AB}\Big(\varphi^z + \sum_{k=1}^{N} q_k G^{\Omega}_{z,y_k}\Big), \varphi\Big)_{L^2(\Omega)} = (H_0^*\varphi^z, \varphi) + \sum_{k=1}^{N} q_k\Big(H_0^* G^{\Omega}_{z,y_k}, \varphi\Big)$$
$$= (H_0\varphi^z, \varphi) + \sum_{k=1}^{N} q_k z^2 (G^{\Omega}_{z,y_k}, \varphi).$$

This equation being valid $\forall \varphi \in D(H_0)$ implies (1.19).

Using Theorem B.1.1 with H_0 and $-\Delta^{\Omega}$ we can write

$$R_z^{AB}\varphi = R_z\varphi + \sum_{j=1}^{N} q_j(\varphi) G^{\Omega}_{z,y_j}, \tag{1.29}$$

with $q_k(\varphi)$ to be determined. Comparing this expression with (1.27), we have that $R_z\varphi = \varphi^z$. So, recalling (1.28), we have

$$\sum_{j=1}^{N} B_{kj} R_z\varphi(y_j) = \sum_{j=1}^{N} (B\Gamma(z) + A)_{kj} q_j(\varphi),$$

or, in matrix form $BR_z\varphi(z) = (B\Gamma(z) + A)q(\varphi)$. Solving for q we obtain $q(\varphi) = (B\Gamma(z) + A)^{-1}BR_z\varphi(x)$, which corresponds to

$$q_j(\varphi) = \sum_{k,l=1}^{N} (B\Gamma(z) + A)^{-1}_{jk} B_{kl} R_z\varphi(y_l).$$

Substituting this expression into (1.29) gives the desired result. □

Remark. A property of the operator Γ_1 and Γ_2 is that they describe the asymptotic behaviour of any $\psi \in D(H_0)$ as $x \to y_k$:

$$\psi \sim \frac{1}{4\pi |x - y_l|}(\Gamma_1\psi)_k + (\Gamma_2\psi)_k + o(1).$$

This suggests that the boundary condition $A\Gamma_1\psi = B\Gamma_2\psi$ pairs the asymptotic behaviour of the coefficient of the singular part $\Gamma_1\psi$ with the evaluation $\Gamma_2\psi$ of the regular part of ψ. Whenever $\forall k = 1, \dots N$, the coefficient of the singular part $(\Gamma_2\psi)_k$ depends only on the regular part $(\Gamma_1\psi)_k$ in the same point, the interaction described by this self-adjoint extension is said to be local. This condition can be achieved asking both A and B to be diagonal. We can parametrize the most general local interaction of this kind by having $A_{jk} = \alpha_j\delta_{jk}$ and $B_{jk} = \delta_{jk}$, with $\alpha_j \in \mathbb{R}$. In this case the resolvent has the simpler form

$$R_z^{AB}\psi = R_z\psi + \sum_{j,k}^{N} (\Gamma^{\Omega}_{\alpha,Y}(z))^{-1}_{jk} R_z\psi(y_k) G^{\Omega}_{z,y_j}, \tag{1.30}$$

with

$$(\Gamma^{\Omega}_{\alpha,Y}(z))_{jk} = \begin{cases} -G^{\Omega}_{z,y_k}(y_j) & j \neq k \\ \alpha_k - \frac{iz}{4\pi} + h^{\Omega}_{z,y_k}(y_k) & j = k \end{cases}. \tag{1.31}$$

To end this section we explicitly discuss the case, which will be studied in the next chapter, that is the one center interaction. We will use the symbol $-\Delta^{\Omega}_{\alpha,y}$ to describe the one point interaction Laplacians with generic boundary conditions on $\partial\Omega$. Similarly $-\Delta^{\Omega,\mathrm{D}}_{\alpha,y}$, $-\Delta^{\Omega,\mathrm{N}}_{\alpha,y}$ and $-\Delta^{\Omega,\mathrm{R},\eta}_{\alpha,y}$ will be used when a specific boundary condition is considered.

Proposition 1.2.1. *Let $-\Delta^{\Omega}_{\alpha,y}$ be the one parameter family of one center interaction Laplacians in $\Omega \subset \mathbb{R}^3$. For any $z \in \rho(-\Delta^{\Omega}_{\alpha,y}) \cap \rho(-\Delta^{\Omega})$, the following representation holds:*

$$D\Big(-\Delta^{\Omega}_{\alpha,y}\Big) = \Big\{\psi \in L^2(\Omega) |\, \psi = \varphi^z + qG^{\Omega}_{z,y},\ \varphi^z \in D(-\Delta^{\Omega}), q = \Gamma_1\psi,\ \varphi^z(y) = \Gamma^{\Omega}_{\alpha,y} q\Big\} \tag{1.32}$$

$$-\Delta^{\Omega}_{\alpha,y}\psi = -\Delta\varphi^z + z^2 qG^{\Omega}_{z,y} \tag{1.33}$$

$$(-\Delta^{\Omega}_{\alpha,y} - z^2)^{-1}\psi = (-\Delta^{\Omega} - z^2)^{-1}\psi + (\Gamma^{\Omega}_{\alpha,y})^{-1}(-\Delta^{\Omega} - z^2)^{-1}\psi(y) G^{\Omega}_{z,y}, \quad \forall\psi \in L^2(\Omega) \tag{1.34}$$

$$\Gamma^{\Omega}_{\alpha,y}(z) = \alpha - \frac{iz}{4\pi} + h^{\Omega}_{z,y}(y). \tag{1.35}$$

Similarly, letting $Y = \{ y_1, \ldots, y_N \}$, $-\Delta^{\Omega}_{\alpha,Y}$, $-\Delta^{\Omega,\mathrm{D}}_{\alpha,Y} - \Delta^{\Omega,\mathrm{N}}_{\alpha,Y}$ $-\Delta^{\Omega,\mathrm{R},\eta}_{\alpha,Y}$ symbols will indicate the N-points interaction local Laplacians. Their definitions is immediate from Theorem 1.2.6 after the substitutiosn $A_{jk} = \alpha_j\delta_{jk}$ and $B = \delta_{jk}$.

1.3. Point Interactions in a Domain in $\mathbb{R}^2$ Through Self-Adjoint Extensions

We can follow the same construction contained in the previous section, to build N-points interactions as self-adjoint extensions of the Laplacians for domains in $\mathbb{R}^2$. The different form of the Green function for $-\Delta - z^2$ in $\mathbb{R}^2$ (see Appendix A.1), causes a different choice for the boundary triple $(\mathbb{C}^N, \Gamma_1, \Gamma_2)$, that is

$$(\Gamma_1\psi)_j = \lim_{x\to y_j}\left(-\frac{2\pi\psi(x)}{\ln\left|x-y_j\right|}\right) \quad j = 1,\dots,N \tag{1.36}$$

$$(\Gamma_2\psi)_j = \lim_{x\to y_j}\left(\psi(x) + \frac{1}{2\pi}(\Gamma\psi)_j)\ln|x-y|\right) \quad j = 1,\dots,N. \tag{1.37}$$

This triple is proved to be a boundary value space for the operator H_0, which has an analogous definition of the $n = 3$ case, which just the caveat of Ω being a domain in $\mathbb{R}^2$. So, through Theorem B.3.2, a family of self adjoint extensions is defined. If we limit ourselves to the local case we can characterize this family $-\Delta^\Omega_{\alpha,Y}$ them as follows (also compare this with the N-point interaction Laplacian in the whole $\mathbb{R}^2$ constructed throughout chapter II.4 of [2]).

Theorem 1.3.1. *Let $Y = \left\{y_j\right\}_{j=1}^N$ and $\alpha_1,\dots,\alpha_N \in \mathbb{R}$. For any $z \in \mathbb{C}\setminus\mathbb{R}$, the following representation holds:*

$$\begin{aligned} D\left(-\Delta^\Omega_{\alpha,Y}\right) = \Bigg\{\psi \in L^2(\Omega)|\,\psi = \varphi^z + \sum_{k=1}^N q_k G^\Omega_{z,y_k},\ \varphi^z \in D(-\Delta^\Omega), q_j = (\Gamma_1\psi)_j,\\ \varphi^z(y_k) = \sum_{j=1}^N (\Gamma^\Omega_{\alpha,Y}(z))_{kj} q_j,\ k = 1,\dots,N\Bigg\} \end{aligned} \tag{1.38}$$

$$(-\Delta^\Omega_{\alpha,Y})\psi = -\Delta\varphi^z + z^2\sum_{k=1}^N q_k G^\Omega_{z,y_k} \tag{1.39}$$

$$(-\Delta^\Omega_{\alpha,Y} - z^2)^{-1}\psi = (-\Delta - z^2)^{-1}\psi + \sum_{j,k,l}^N (\Gamma^\Omega_{\alpha,Y})^{-1}_{jl}(-\Delta - z^2)^{-1}\psi(y_k)G^\Omega_{z,y_j}, \quad \forall\psi \in L^2(\Omega) \tag{1.40}$$

$$(\Gamma^\Omega_{\alpha,Y}(z))_{kj} = \begin{cases} -G^\Omega_{z,y_j}(y_k) & j \neq k \\ \alpha_k + \frac{\gamma + \ln\frac{z}{2i}}{2\pi} + h^\Omega_{z,y_k}(y_k) & j = k \end{cases}, \tag{1.41}$$

where γ is Euler's constant.

We also give the explicitly the representation for the one center case.

Proposition 1.3.1. *Let $-\Delta^\Omega_{\alpha,y}$ be the one parameter family of one center interaction Laplacians in $\Omega \subset \mathbb{R}^2$. For any $z \in \rho(-\Delta^\Omega_{\alpha,y}) \cap \rho(-\Delta^\Omega)$, the following representation holds:*

$$D\left(-\Delta^\Omega_{\alpha,y}\right) = \left\{\psi \in L^2(\Omega)|\,\psi = \varphi^z + qG^\Omega_{z,y},\ \varphi^z \in D(-\Delta^\Omega), q = \Gamma_1\psi,\ \varphi^z(y) = \Gamma^\Omega_{\alpha,y}(z)q\right\} \tag{1.42}$$

$$-\Delta^\Omega_{\alpha,y}\psi = -\Delta\varphi^z + z^2 qG^\Omega_{z,y} \tag{1.43}$$

$$(-\Delta^\Omega_{\alpha,y} - z^2)^{-1}\psi = (-\Delta^\Omega - z^2)^{-1}\psi + (\Gamma^\Omega_{\alpha,y})^{-1}(-\Delta^\Omega - z^2)^{-1}\psi(y)G^\Omega_{z,y}, \quad \forall\psi \in L^2(\Omega) \tag{1.44}$$

$$\Gamma^\Omega_{\alpha,y}(z) = \alpha + \frac{\gamma + \ln\frac{z}{2i}}{2\pi} + h^\Omega_{z,y}(y). \tag{1.45}$$

1.4. Point Interactions Through Quadratic Forms

Another path to construct point interactions is through quadratic forms. Kato's representation theorem states that, given a symmetric, densely defined, closed and bounded from below quadratic form q, there exists a unique self-adjoint operator Q such that

$$q(\psi,\varphi) = (Q\psi,\varphi) \quad \forall\psi \in D(T) \quad \text{and} \quad \forall\varphi \in D(q).$$

So, we aim to construct the quadratic forms associated with the self-adjoint operators $-\Delta^{\Omega}_{\alpha,Y}$. In order to do so we will follow the same approach taken by Teta in [46] to construct point interaction quadratic form in the whole $\mathbb{R}^n$. Differences will arise in order to take count of the boundaries of Ω.

1.4.1. Dirichlet, Neumann and Robin Quadratic Forms

Here we recall the definitions of the classic Dirichlet, Neumann and Robin quadratic forms:

- the Dirichlet quadratic form D^{Ω} is defined as

$$D(\mathrm{D}^{\Omega}) = H^1_0(\Omega) \quad \mathrm{D}^{\Omega}(\psi,\psi) = \int_{\Omega} |\nabla\psi(x)|^2 \, \mathrm{d}x = \|\psi\|^2_{L^2(\Omega)} \,;$$

- the Neumann quadratic form N^{Ω} is defined as

$$D(\mathrm{N}^{\Omega}) = H^1(\Omega) \quad \mathrm{N}^{\Omega}(\psi,\psi) = \int_{\Omega} |\nabla\psi(x)|^2 \, \mathrm{d}x = \|\psi\|^2_{L^2(\Omega)} \,;$$

- the Robin quadratic form $R^{\Omega,\eta}$ is defined as

$$D(\mathrm{R}^{\Omega,\eta}) = H^1(\Omega) \tag{1.46}$$

$$\mathrm{R}^{\Omega,\eta}(\psi,\psi) = \int_{\Omega} |\nabla\psi(x)|^2 \, \mathrm{d}x + \eta \int_{\partial\Omega} |(\gamma_{\mathrm{D}}\psi)(x)|^2 \, \mathrm{d}\sigma = \|\psi\|^2_{L^2(\Omega)} + \eta \, \|\gamma_{\mathrm{D}}\psi\|^2_{L^2(\partial\Omega)} \,. \tag{1.47}$$

.

F^{Ω} will be used as a generic symbol for the classic quadratic forms.

1.4.2. N-Points Quadratic Forms on a Domain

Our aim is constructing quadratic forms who corresponds to self-adjoint extension of the Laplacian. The elements of the domain of these quadratic forms will have a regular part, which belongs to $D(F^{\Omega})$ and a singular part, being a linear combination of the functions G_{z,y_j}. Moreover we want the new form to agree with F^{Ω} over $D(F^{\Omega})$. With these ideas in mind and using the classical forms expressions, we can define the N-points Dirichlet, Neumann and Robin quadratic forms as follows.

Definition 1.4.1. Let $Y = \left\{ y_j \right\}_{j=1}^N$ be distinct points in $\Omega \subset \mathbb{R}^n$ with $n = 1, 2$ and $\alpha = \left\{ \alpha_j \right\}_{j=1}^N$ with $\alpha_j \in \mathbb{R}$ $\forall i = 1, \ldots, N$. For each imaginary z with $\operatorname{Im} z > 0$ we define

- the N-points interaction Dirichlet quadratic form

$$D(\mathrm{D}^{\Omega}_{\alpha,Y}) = \left\{ \psi \in L^2(\Omega) \;\middle|\; \exists q_1, \ldots, q_n \text{ s.t. } \psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{D}}_{z,y_j} \in H^1_0(\Omega) \right\} \tag{1.48}$$

$$\mathrm{D}^{\Omega}_{\alpha,Y}(\psi,\psi) = \mathcal{D}^{\Omega}_{z,Y}(\psi,\psi) + \sum_{j,k=1}^{N} (\Gamma^{\Omega,\mathrm{D}}_{\alpha,Y}(z))_{kj} \overline{q_k} q_j \tag{1.49}$$

$$\mathcal{D}^{\Omega}_{z,Y}(\psi,\psi) = \left\| \nabla\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{D}}_{z,y_j}\Big) \right\|^2_{L^2(\Omega)} - z^2 \left\| \psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{D}}_{z,y_j} \right\|^2_{L^2(\Omega)} + z^2 \|\psi\|^2_{L^2(\Omega)} \; ; \tag{1.50}$$

- the N-points interaction Neumann quadratic form

$$D(\mathrm{N}^{\Omega}_{\alpha,Y}) = \left\{ \psi \in L^2(\Omega) \;\middle|\; \exists q_1, \ldots, q_n \text{ s.t. } \psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{N}}_{z,y_j} \in H^1(\Omega) \right\} \tag{1.51}$$

$$\mathrm{N}^{\Omega}_{\alpha,Y}(\psi,\psi) = \mathcal{N}^{\Omega}_{z,Y}(\psi,\psi) + \sum_{j,k=1}^{N} (\Gamma^{\Omega,\mathrm{N}}_{\alpha,Y}(z))_{kj} \overline{q_k} q_j \tag{1.52}$$

$$\mathcal{N}^{\Omega}_{z,Y}(\psi,\psi) = \left\| \nabla\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{N}}_{z,y_j}\Big) \right\|^2_{L^2(\Omega)} - z^2 \left\| \psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{N}}_{z,y_j} \right\|^2_{L^2(\Omega)} + z^2 \|\psi\|^2_{L^2(\Omega)} \; ; \tag{1.53}$$

- the N-points interaction Robin quadratic form (η being positive)

$$D(\mathrm{R}^{\Omega,\eta}_{\alpha,Y}) = \left\{ \psi \in L^2(\Omega) \;\middle|\; \exists q_1, \ldots, q_n \text{ s.t. } \psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j} \in H^1(\Omega) \right\} \tag{1.54}$$

$$\mathrm{R}^{\Omega,\eta}_{\alpha,Y}(\psi,\psi) = \mathcal{R}^{\Omega,\eta}_{z,Y}(\psi,\psi) + \sum_{j,k=1}^{N} (\Gamma^{\Omega,\mathrm{R},\eta}_{\alpha,Y}(z))_{kj} \overline{q_k} q_j \tag{1.55}$$

$$\begin{aligned} \mathcal{R}^{\Omega,\eta}_{z,Y}(\psi,\psi) = {} & \left\| \nabla\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big) \right\|^2_{L^2(\Omega)} - z^2 \left\| \psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j} \right\|^2_{L^2(\Omega)} \\ & + z^2 \|\psi\|^2_{L^2(\Omega)} + \eta \left\| \gamma_{\mathrm{D}}\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big) \right\|^2_{L^2(\partial\Omega)} . \end{aligned} \tag{1.56}$$

$\mathrm{F}^{\Omega}_{\alpha,Y}$ and $\mathcal{F}^{\Omega}_{z,Y}$ will be used to denote collectively quadratic forms of this kind.

Proposition 1.4.1. *The quadratic forms* $\mathrm{F}^{\Omega}_{\alpha,Y}$ *are independent of the choice of* $z \in i\mathbb{R}$.

Proof. To start, we observe that $D(\mathrm{F}^{\Omega}_{\alpha,Y})$ is independent of z. In fact, if $\psi - qG^{\Omega}_{z,y} \in H^1(\Omega)$ ($H^1_0(\Omega)$ for $\mathrm{D}^{\Omega}_{\alpha,Y}$), then also $\psi - qG^{\Omega}_{w,y} \in H^1(\Omega)$ ($H^1_0(\Omega)$ respectively). This is true because $\psi - qG^{\Omega}_{w,y} = (\psi - qG^{\Omega}_{z,y}) + (qG^{\Omega}_{z,y} - qG^{\Omega}_{w,y})$, which is a sum of functions in $H^1(\Omega)$ (or $H^1_0(\Omega)$). This also implies that for different value of z we can still use the same q. Extending this to N-points is straightforward.

Let $z, w \in i\mathbb{R}$. We consider $\mathcal{F}^{\Omega}_{z,Y}(\psi,\psi) - \mathcal{F}^{\Omega}_{w,Y}(\psi,\psi)$. We distinguish between the Dirichlet form and the other ones.

- For $\mathrm{D}^{\Omega}_{\alpha,Y}$ we have

$$\begin{aligned}\mathcal{D}^{\Omega}_{z,Y}(\psi,\psi) - \mathcal{D}^{\Omega}_{w,Y}(\psi,\psi) &= \left(\nabla\left(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{D}}_{z,y_j}\right), \nabla\left(\psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{D}}_{z,y_k}\right)\right) + z^2 \|\psi\|^2 \\ &- z^2\left(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{D}}_{z,y_j}, \psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{D}}_{z,y_k}\right) - \left(\nabla\left(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{D}}_{w,y_j}\right), \nabla\left(\psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{D}}_{w,y_k}\right)\right) \\ &- w^2 \|\psi\|^2 + w^2\left(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{D}}_{w,y_j}, \psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{D}}_{w,y_k}\right). \quad (1.57)\end{aligned}$$

Now we manipulate the terms containing gradients

$$\begin{aligned}&\left(\nabla\left(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{D}}_{z,y_j}\right), \nabla\left(\psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{D}}_{z,y_k}\right)\right) - \left(\nabla\left(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{D}}_{w,y_j}\right), \nabla\left(\psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{D}}_{w,y_k}\right)\right) = \\ &= -\left(\nabla\left(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{D}}_{z,y_j}\right), \sum_{k=1}^{N} q_k \nabla\left(G^{\Omega,\mathrm{D}}_{z,y_k} - G^{\Omega,\mathrm{D}}_{w,y_k}\right)\right) - \left(\sum_{j=1}^{N} q_j \nabla\left(G^{\Omega,\mathrm{D}}_{z,y_j} - G^{\Omega,\mathrm{D}}_{w,y_j}\right), \nabla\left(\psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{D}}_{z,y_k}\right)\right) \\ &- \left(\sum_{j=1}^{N} q_j \nabla\left(G^{\Omega,\mathrm{D}}_{z,y_j} - G^{\Omega,\mathrm{D}}_{w,y_j}\right), \sum_{k=1}^{N} q_k \nabla\left(G^{\Omega,\mathrm{D}}_{z,y_k} - G^{\Omega,\mathrm{D}}_{w,y_k}\right)\right) = \left(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{D}}_{z,y_j}, \sum_{k=1}^{N} q_k\left(w^2 G^{\Omega,\mathrm{D}}_{w,y_k} - z^2 G^{\Omega,\mathrm{D}}_{z,y_k}\right)\right) \\ &+ \left(\sum_{j=1}^{N} q_j\left(w^2 G^{\Omega,\mathrm{D}}_{w,y_j} - z^2 G^{\Omega,\mathrm{D}}_{z,y_j}\right), \psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{D}}_{z,y_k}\right) + \left(\sum_{j=1}^{N} q_j\left(G^{\Omega,\mathrm{D}}_{z,y_j} - G^{\Omega,\mathrm{D}}_{w,y_j}\right), \sum_{k=1}^{N} q_k\left(w^2 G^{\Omega,\mathrm{D}}_{w,y_k} - z^2 G^{\Omega,\mathrm{D}}_{z,y_k}\right)\right),\end{aligned} \quad (1.58)$$

where we integrated by parts and used that $\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{D}}_{z,y_j} \in H^1_0(\Omega)$ and the definition of $G^{\Omega,\mathrm{D}}_{\cdot,\cdot}$.

- For $\mathrm{R}^{\Omega,R,\eta}_{\alpha,Y}$ (including the $\eta = 0$ case) we have

$$\mathcal{R}^{\Omega,\eta}_{z,Y}(\psi,\psi) - \mathcal{R}^{\Omega,\eta}_{w,Y}(\psi,\psi) = \left(\nabla\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big), \nabla\Big(\psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big)\right) + z^2\,\|\psi\|^2$$
$$-z^2\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}, \psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big) + \eta\left(\gamma_{\mathrm{D}}\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big), \gamma_{\mathrm{D}}\Big(\psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big)\right)_{L^2(\partial\Omega)}$$
$$-\left(\nabla\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{w,y_j}\Big), \nabla\Big(\psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{R},\eta}_{w,y_k}\Big)\right) - w^2\,\|\psi\|^2 + w^2\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{w,y_j}, \psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{R},\eta}_{w,y_k}\Big)$$
$$-\eta\left(\gamma_{\mathrm{D}}\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{w,y_j}\Big), \gamma_{\mathrm{D}}\Big(\psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{R},\eta}_{w,y_k}\Big)\right)_{L^2(\partial\Omega)}. \tag{1.59}$$

We integrate by parts the terms with gradients

$$\left(\nabla\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big), \nabla\Big(\psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big)\right) - \left(\nabla\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{w,y_j}\Big), \nabla\Big(\psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{R},\eta}_{w,y_k}\Big)\right) =$$
$$= \Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}, \sum_{k=1}^{N} q_k\Big(w^2 G^{\Omega,\mathrm{R},\eta}_{w,y_k} - z^2 G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big)\Big) + \Big(\sum_{j=1}^{N} q_j\Big(w^2 G^{\Omega,\mathrm{R},\eta}_{w,y_j} - z^2 G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big), \psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big)$$
$$+\Big(\sum_{j=1}^{N} q_j\Big(G^{\Omega,\mathrm{R},\eta}_{z,y_j} - G^{\Omega,\mathrm{R},\eta}_{w,y_j}\Big), \sum_{k=1}^{N} q_k\Big(w^2 G^{\Omega,\mathrm{R},\eta}_{w,y_k} - z^2 G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big)\Big) + \eta\Big(\gamma_{\mathrm{D}}\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big), \sum_{k=1}^{N} q_k \gamma_{\mathrm{D}}\Big(G^{\Omega,\mathrm{R},\eta}_{z,y_k} - G^{\Omega,\mathrm{R},\eta}_{w,y_k}\Big)\Big)$$
$$+\eta\Big(\sum_{j=1}^{N} q_j \gamma_{\mathrm{D}}\Big(G^{\Omega,\mathrm{R},\eta}_{z,y_j} - G^{\Omega,\mathrm{R},\eta}_{w,y_j}\Big), \gamma_{\mathrm{D}}\Big(\psi - \sum_{k=1}^{N} q_k G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big)\Big) + \eta\Big(\sum_{j=1}^{N} q_j \gamma_{\mathrm{D}}\Big(G^{\Omega,\mathrm{R},\eta}_{z,y_j} - G^{\Omega,\mathrm{R},\eta}_{w,y_j}\Big), \sum_{k=1}^{N} q_k \gamma_{\mathrm{D}}\Big(G^{\Omega,\mathrm{R},\eta}_{w,y_k} - G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big)\Big), \tag{1.60}$$

where we used the definition of $G^{\Omega,\mathrm{R},\eta}_{\cdot,\cdot}$. If we substitute (1.60) into (1.59), some tedious computations shows that all the boundary terms cancel out.

In both cases the remaining terms are

$$\mathcal{F}^{\Omega}_{z,Y}(\psi,\psi) - \mathcal{F}^{\Omega}_{w,Y}(\psi,\psi) = \Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega}_{z,y_j}, \sum_{k=1}^{N} q_k\Big(w^2 G^{\Omega}_{w,y_k} - z^2 G^{\Omega}_{z,y_k}\Big)\Big)$$
$$+\Big(\sum_{j=1}^{N} q_j\Big(w^2 G^{\Omega}_{w,y_j} - z^2 G^{\Omega}_{z,y_j}\Big), \psi - \sum_{k=1}^{N} q_k G^{\Omega}_{z,y_k}\Big) + \Big(\sum_{j=1}^{N} q_j\Big(G^{\Omega}_{z,y_j} - G^{\Omega}_{w,y_j}\Big), \sum_{k=1}^{N} q_k\Big(w^2 G^{\Omega}_{w,y_k} - z^2 G^{\Omega}_{z,y_k}\Big)\Big)$$
$$+ (z^2 - w^2)\,\|\psi\|^2 - z^2\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega}_{z,y_j}, \psi - \sum_{k=1}^{N} q_k G^{\Omega}_{z,y_k}\Big) + w^2\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega}_{w,y_j}, \psi - \sum_{k=1}^{N} q_k G^{\Omega}_{w,y_k}\Big). \tag{1.61}$$

After eliminating all the cancelling terms, we are left with

$$\begin{aligned}\mathcal{F}^{\Omega}_{z,Y}(\psi,\psi) - \mathcal{F}^{\Omega}_{w,Y}(\psi,\psi) &= (z^2 - w^2)\sum_{j,k=1}^{N} \overline{q_j} q_k \Big(G^{\Omega}_{w,y_j}, G^{\Omega}_{z,y_k}\Big)_{L^2(\Omega)} \\ &= \sum_{j,k=1}^{N} \Big(G^{\Omega}_{z,y_j}(y_k) - G^{\Omega}_{w,y_k}(y_j)\Big)\overline{q_j} q_k,\end{aligned}$$

last step being due to the resolvent identity. In the terms with $j = k$, $G^{\Omega}_{\cdot,y_j}(y_j)$ diverges, but the difference stays finite. we distinguish $n = 2$ and $n = 3$:

- For $n = 2$ it holds that

$$\begin{aligned}G^{\Omega}_{z,y_j}(y_j) - G^{\Omega}_{w,y_j}(y_j) &= \lim_{x\to y_j}\Big(G^{\Omega}_{z,y_j}(x) - G^{\Omega}_{w,x}(y_j)\Big) \\ &= \lim_{x\to y_j} \frac{i}{4}\Big(H_0^{(1)}(z\left|x-y_j\right|) - H_0^{(1)}(w\left|x-y_j\right|)\Big) - h^{\Omega}_{z,y_j}(y_j) + h^{\Omega}_{w,y_j}(y_j).\end{aligned}$$

  We set $z = is$ and $w = it$ with $s, t > 0$. Using (C.18), together with (C.22), we have

$$\lim_{x\to y_j} \frac{i}{4}\Big(H_0^{(1)}(z\left|x-y_j\right|) = \lim_{x\to y_j} \frac{K_0(s\left|x-y_j\right|) - K_0(t\left|x-y_j\right|)}{2\pi} = -\frac{\gamma + \ln s}{2\pi} + \frac{\gamma + \ln t}{2\pi}$$

  and so

$$G^{\Omega}_{z,y_j}(y_j) - G^{\Omega}_{w,y_j}(y_j) = -\alpha_j - \frac{\gamma + \ln\frac{z}{i}}{2\pi} + \alpha_j + \frac{\gamma + \ln\frac{w}{i}}{2\pi}.$$

- For $n = 3$ it holds

$$\begin{aligned}G^{\Omega}_{z,y_j}(y_j) - G^{\Omega}_{w,y_j}(y_j) &= \lim_{x\to y_j}\Big(G^{\Omega}_{z,y_j}(x) - G^{\Omega}_{w,x}(y_j)\Big) \\ &= \lim_{x\to y_j} \frac{e^{iz\left|x-y_j\right|} - e^{iw\left|x-y_j\right|}}{4\pi\left|x-y_j\right|} - h^{\Omega}_{z,y_j}(y_j) + h^{\Omega}_{w,y_j}(y_j) \\ &= -\alpha_j + \frac{iz}{4\pi} - h^{\Omega}_{z,y_j}(y_j) + \alpha_j - \frac{iw}{4\pi} + h^{\Omega}_{w,y_j}(y_j).\end{aligned}$$

In both cases this means that $G^{\Omega}_{z,y_j}(y_k) - G^{\Omega}_{w,y_k}(y_j) = (\Gamma^{\Omega}_{\alpha,Y}(w))_{jk} - (\Gamma^{\Omega}_{\alpha,Y}(z))_{jk}$ and hence

$$\mathcal{F}^{\Omega}_{z,Y}(\psi,\psi) - \mathcal{F}^{\Omega}_{w,Y}(\psi,\psi) = \sum_{j,k=1}^{N}(\Gamma^{\Omega}_{\alpha,Y}(w))_{kj}\overline{q_k}q_j - \sum_{j,k=1}^{N}(\Gamma^{\Omega}_{\alpha,Y}(z))_{kj}\overline{q_k}q_j.$$

□

Proposition 1.4.2. *The quadratic forms* $\mathrm{F}^{\Omega}_{\alpha,Y}$ *are closed and bounded below.*

Proof. z in the definition of $\mathrm{F}^{\Omega}_{\alpha,Y}$ is imaginary. This means that

$$\mathrm{F}^{\Omega}_{\alpha,Y}(\psi,\psi) \geq z^2 \|\psi\|^2_{L^2(\Omega)} + \sum_{j,k=1}^{N} (\Gamma^{\Omega}_{\alpha,Y}(z))_{kj}\overline{q_k}q_j.$$

We decompose $\Gamma^{\Omega}_{\alpha,Y}(z)$ in $\Gamma^{\Omega}_{\alpha,Y}(z) = \Gamma^{0}_{\alpha,Y}(z) + K(z)$ where

$$(\Gamma^{0}_{\alpha,Y}(z))_{jk} = \begin{cases} -G^{0,3}_{z,y_k}(y_j) & j \neq k \\ \alpha_k - \frac{iz}{4\pi} & j = k \end{cases} \qquad (K(z))_{jk} = h_{z,y_k}(y_j).$$

The eigenvalues of $\Gamma^{0}_{\alpha,Y}(z)$ are strictly increasing in $\operatorname{Im} z$ (see [2] page 116). Let $\lambda^0_1 = \lambda^0_1(z,\alpha,Y)$ be the lowest eigenvalue of $\Gamma^{0}_{\alpha,Y}(z)$ and $\lambda_1 = \lambda_1(z,\alpha,Y)$ be the lowest one for $\Gamma^{\Omega}_{\alpha,Y}(z)$. Then

$$\sum_{j,k=1}^{N} (\Gamma^{0}_{\alpha,Y}(z))_{kj}\overline{q_k}q_j \geq \lambda^0_1(z,\alpha,Y) \sum_{j=1}^{N} \left|q_j\right|^2.$$

This means that there exists an imaginary $z_0(\alpha, Y)$ such that $\forall z \in i\mathbb{R}$ with $\operatorname{Im} z > \operatorname{Im} z_0$

$$\sum_{j,k=1}^{N} (\Gamma^{0}_{\alpha,Y}(z))_{kj}\overline{q_k}q_j \geq \lambda^0_1(z,\alpha,Y) \sum_{j=1}^{N} \left|q_j\right|^2 > \lambda^0_1(z_0,\alpha,Y) \sum_{j=1}^{N} \left|q_j\right|^2.$$

Now, since both $\Gamma^{0}_{\alpha,Y}(z)$ and $K(z)$ are Hermitian and $\lim_{\operatorname{Im} z \to +\infty} \|K(z)\|^2_F = 0$, as a consequence of Corollary 6.3.8. in [26], we have that for $\operatorname{Im} z$ large enough

$$\sum_{j,k=1}^{N} (\Gamma^{\Omega}_{\alpha,Y}(z))_{kj}\overline{q_k}q_j \geq \lambda_1(z,\alpha,Y) \sum_{j=1}^{N} \left|q_j\right|^2 > \lambda^0_1(z_0,\alpha,Y) \sum_{j=1}^{N} \left|q_j\right|^2.$$

This choice of z is not restrictive because $\mathrm{F}^{\Omega}_{\alpha,Y}$ is z independent and so the boundedness from below is proved.

In order to prove that the forms are closed it is more convenient to consider

$$\mathrm{F}^{\Omega}_{\alpha,Y,z}(\psi,\psi) = \mathrm{F}^{\Omega}_{\alpha,Y}(\psi,\psi) - z^2 \|\psi\|^2_{L^2(\Omega)}.$$

Let $\{\psi_n\}$ be a sequence in $D(\mathrm{F}^{\Omega}_{\alpha,Y,z})$ such that $\psi_n \to \psi$ in $L^2(\Omega)$ and $\lim_{n,m\to+\infty} \mathrm{F}^{\Omega}_{\alpha,Y,z}(\psi_n - \psi_m, \psi_n - \psi_m) = 0$. This implies

$$\lim_{n,m\to+\infty} \left\| \left(\psi_n - \sum_{j=1}^{N} q^n_j G^{\Omega}_{z,y_j}\right) - \left(\psi_m - \sum_{j=1}^{N} q^m_j G^{\Omega}_{z,y_j}\right) \right\|_{H^1(\Omega)} = \|\varphi_n - \varphi_m\|_{H^1(\Omega)} = 0 \tag{1.62}$$

$$\sum_{j,k=1}^{N} (\Gamma^{\Omega}_{\alpha,Y}(z))_{jk}\left(q^n_j - q^m_j\right)\overline{\left(q^n_k - q^n_k\right)} = 0, \tag{1.63}$$

where $\varphi_n = \psi_n - \sum_{j=1}^N q_j^n G^{\Omega}_{z,y_j}$. And so there exist $\varphi \in H^1(\Omega)$ and $q_1, \dots, q_N \in \mathbb{C}$ such that

$$\|\varphi_n - \varphi\|_{H^1(\Omega)} = 0 \tag{1.64}$$

$$\left\| \sum_{j=1}^N q_j^n G^{\Omega}_{z,y_j} - \sum_{j=1}^N q_j G^{\Omega}_{z,y_j} \right\|_{L^2(\Omega)} = 0. \tag{1.65}$$

This means that $\psi = \varphi + \sum_{j=1}^N q_j G^{\Omega}_{z,y_j}$, i.e. $\psi \in D\left(\mathrm{F}^{\Omega}_{\alpha,Y}\right)$ and hence $\mathrm{F}^{\Omega}_{\alpha,Y}$ is closed. □

At this point we have proved that $\mathrm{F}^{\Omega}_{\alpha,Y}$ are symmetric (it is immediate from their definition), closed and bounded from below. So there's a unique bounded from below and self-adjoint operator associated to $\mathrm{F}^{\Omega}_{\alpha,Y}$ via Theorem B.4.1. We proceed to prove that this operator is in fact $-\Delta^{\Omega}_{\alpha,Y}$.

Proposition 1.4.3. *The quadratic form* $(\mathrm{D}^{\Omega}_{\alpha,Y}, D(\mathrm{D}^{\Omega}_{\alpha,Y}))$ *is associated to the operator* $-\Delta^{\Omega,\mathrm{D}}_{\alpha,Y}$ *so that*

$$\mathrm{D}^{\Omega}_{\alpha,Y}(\psi,\varphi) = (-\Delta^{\Omega,\mathrm{D}}_{\alpha,Y}\psi,\varphi)_{L^2(\Omega)} \quad \forall \psi \in D(-\Delta^{\Omega,\mathrm{D}}_{\alpha,Y}) \quad \forall \varphi \in D(\mathrm{D}^{\Omega}_{\alpha,Y}),$$

where

$$\begin{aligned} D(-\Delta^{\Omega,\mathrm{D}}_{\alpha,Y}) = \Bigg\{ &\psi \in D(\mathrm{D}^{\Omega}_{\alpha,Y}) | \, \psi - \sum_{j=1}^N q_j G^{\Omega,\mathrm{D}}_{z,y_j} \in H^2_0(\Omega), \\ &\left(\psi - \sum_{j=1}^N q_j G^{\Omega,\mathrm{D}}_{z,y_j}\right)(y_k) = \sum_{j=1}^N (\Gamma^{\Omega,\mathrm{D}}_{\alpha,Y}(z))_{kj} q_j, \; k = 1, \dots, N \Bigg\} \end{aligned} \tag{1.66}$$

$$(-\Delta^{\Omega,\mathrm{D}}_{\alpha,Y} - z^2)\psi = (-\Delta - z^2)\left(\psi - \sum_{j=1}^N q_j G^{\Omega,\mathrm{D}}_{z,y_j}\right). \tag{1.67}$$

Proof. We note that if we identify $\psi - \sum_{j=1}^N q_j G^{\Omega,\mathrm{D}}_{z,y_j}$ with φ^z, it is immediate to see that the action and domain of $-\Delta^{\Omega,\mathrm{D}}_{\alpha,Y}$ described here coincide with the ones in Theorem 1.2.6.

So, let $\psi \in D(-\Delta^{\Omega,\mathrm{D}}_{\alpha,Y})$, $\varphi \in D(\mathrm{D}^{\Omega}_{\alpha,Y})$ and be $\phi \in L^2(\Omega)$ such that

$$\mathrm{D}^{\Omega}_{\alpha,Y}(\psi,\varphi) = (\phi,\varphi)_{L^2(\Omega)}$$

or more explicitly

$$\begin{aligned} \left(\nabla\left(\psi - \sum_{j=1}^N q_j^{\psi} G^{\Omega,\mathrm{D}}_{z,y_j}\right), \nabla\left(\varphi - \sum_{k=1}^N q_k^{\varphi} G^{\Omega,\mathrm{D}}_{z,y_k}\right)\right) - z^2\left(\psi - \sum_{j=1}^N q_j^{\psi} G^{\Omega,\mathrm{D}}_{z,y_j}, \varphi - \sum_{k=1}^N q_k^{\varphi} G^{\Omega,\mathrm{D}}_{z,y_k}\right) \\ + \sum_{j,k=1}^N (\Gamma^{\Omega,\mathrm{D}}_{\alpha,Y}(z))_{kj} \overline{q_k^{\varphi}} q_j^{\psi} = (\phi - z^2\psi, \varphi). \end{aligned} \tag{1.68}$$

Since, $\varphi \in D(\mathrm{D}^{\Omega}_{\alpha,Y})$, the boundary term appearing through integration by parts vanishes and so last equation becomes

$$\left((-\Delta - z^2)\Big(\psi - \sum_{j=1}^{N} q_j^{\psi} G^{\Omega,\mathrm{D}}_{z,y_j}\Big), \varphi - \sum_{k=1}^{N} q_k^{\varphi} G^{\Omega,\mathrm{D}}_{z,y_k}\right)_{L^2(\Omega)} + \sum_{j,k=1}^{N} (\Gamma^{\Omega,\mathrm{D}}_{\alpha,Y}(z))_{kj} \overline{q_k^{\varphi}} q_j^{\psi} = (\phi - z^2\psi, \varphi)_{L^2(\Omega)}.$$

This is valid $\forall \varphi \in D(\mathrm{D}^{\Omega}_{\alpha,Y})$ only if

$$(-\Delta - z^2)\Big(\psi - \sum_{j=1}^{N} q_j^{\psi} G^{\Omega,\mathrm{D}}_{z,y_j}\Big) = \phi - z^2\psi = (-\Delta^{\Omega}_{\alpha,Y} - z^2)\psi.$$

The remaining terms are

$$\left((-\Delta - z^2)\Big(\psi - \sum_{j=1}^{N} q_j^{\psi} G^{\Omega,\mathrm{D}}_{z,y_j}\Big), \sum_{k=1}^{N} q_k^{\varphi} G^{\Omega,\mathrm{D}}_{z,y_k}\right)_{L^2(\Omega)} = \sum_{j,k=1}^{N} (\Gamma^{\Omega,\mathrm{D}}_{\alpha,Y}(z))_{kj} \overline{q_k^{\varphi}} q_j^{\psi}.$$

Since $\gamma_D\Big(\psi - \sum_{j=1}^{N} q_j^{\psi} G^{\Omega,\mathrm{D}}_{z,y_j}\Big) = \gamma_D\Big(\varphi - \sum_{j=1}^{N} q_j^{\varphi} G^{\Omega,\mathrm{D}}_{z,y_j}\Big) = 0$, we can integrate by parts twice with no boundary term and obtain

$$\left(\psi - \sum_{j=1}^{N} q_j^{\psi} G^{\Omega,\mathrm{D}}_{z,y_j}, \sum_{k=1}^{N} q_k^{\varphi} (-\Delta - z^2) G^{\Omega,\mathrm{D}}_{z,y_k}\right)_{L^2(\Omega)} = \sum_{j,k=1}^{N} (\Gamma^{\Omega,\mathrm{D}}_{\alpha,Y}(z))_{kj} \overline{q_k^{\varphi}} q_j^{\psi}.$$

By definition $(-\Delta - z^2) G^{\Omega,\mathrm{D}}_{z,y_k} = \delta(x - y_k)$ and hence

$$\sum_{k=1}^{N} \overline{q_k^{\varphi}} \Big(\psi - \sum_{j=1}^{N} q_j^{\psi} G^{\Omega,\mathrm{D}}_{z,y_j}\Big)(y_k) = \sum_{j,k=1}^{N} (\Gamma^{\Omega,\mathrm{D}}_{\alpha,Y}(z))_{kj} \overline{q_k^{\varphi}} q_j^{\psi}.$$

This must be verified $\forall q_k^{\varphi} \in \mathbb{C}$, $k = 1, \ldots, N$, which implies

$$\Big(\psi - \sum_{j=1}^{N} q_j^{\psi} G^{\Omega,\mathrm{D}}_{z,y_j}\Big)(y_k) = \sum_{j=1}^{N} (\Gamma^{\Omega,\mathrm{D}}_{\alpha,Y}(z))_{kj} q_j^{\psi}.$$

□

Proposition 1.4.4. *The quadratic form* $(\mathrm{R}^{\Omega,\eta}_{\alpha,Y}, D(\mathrm{R}^{\Omega}_{\alpha,Y}, \eta))$ *(with* $\eta \geq 0$, $\eta = 0$ *being the Neumann case) is associated to the operator* $-\Delta^{\Omega,\mathrm{R},\eta}_{\alpha,Y}$ *so that*

$$\mathrm{R}^{\Omega,\eta}_{\alpha,Y}(\psi, \varphi) = (-\Delta^{\Omega,\mathrm{R},\eta}_{\alpha,Y}\psi, \varphi)_{L^2(\Omega)} \quad \forall \psi \in D(-\Delta^{\Omega,\mathrm{R},\eta}_{\alpha,Y}) \quad \forall \varphi \in D(\mathrm{R}^{\Omega}_{\alpha,Y}),$$

where

$$D(-\Delta^{\Omega,\mathrm{R},\eta}_{\alpha,Y}) = \Big\{\psi \in D(\mathrm{R}^{\Omega,\eta}_{\alpha,Y}) |\, \psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j} \in H^2(\Omega),\ (\gamma_N + \eta\gamma_D)\psi = 0,$$

$$\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big)(y_k) = \sum_{j=1}^{N} (\Gamma^{\Omega,\mathrm{R},\eta}_{\alpha,Y}(z))_{kj} q_j,\ k = 1,\dots,N\Big\} \tag{1.69}$$

$$(-\Delta^{\Omega,\mathrm{R},\eta}_{\alpha,Y} - z^2)\psi = (-\Delta - z^2)\Big(\psi - \sum_{j=1}^{N} q_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big). \tag{1.70}$$

Proof. Let $\psi \in D(-\Delta^{\Omega,\mathrm{R},\eta}_{\alpha,Y})$, $\varphi \in D(\mathrm{R}^{\Omega,\eta}_{\alpha,Y})$ and be $\phi \in L^2(\Omega)$ such that

$$\mathrm{R}^{\Omega,\eta}_{\alpha,Y}(\psi,\varphi) = (\phi,\varphi)_{L^2(\Omega)}$$

or more explicitly

$$\Big(\nabla\Big(\psi - \sum_{j=1}^{N} q^{\psi}_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big), \nabla\Big(\varphi - \sum_{k=1}^{N} q^{\varphi}_k G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big)\Big) - z^2\Big(\psi - \sum_{j=1}^{N} q^{\psi}_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}, \varphi - \sum_{k=1}^{N} q^{\varphi}_k G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big)$$
$$+\,\eta\Big(\gamma_D\Big(\psi - \sum_{j=1}^{N} q^{\psi}_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big), \gamma_D\Big(\varphi - \sum_{k=1}^{N} q^{\varphi}_k G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big)\Big)_{L^2(\Omega)} + \sum_{j,k=1}^{N} (\Gamma^{\Omega,\mathrm{R},\eta}_{\alpha,Y}(z))_{kj}\overline{q^{\varphi}_k} q^{\psi}_j = (\phi - z^2\psi, \varphi). \tag{1.71}$$

We integrate by parts the first term, recalling also that ψ and $G^{\Omega,\mathrm{R},\eta}_{z,y_k}$, $k = 1,\dots,N$ satisfy the Robin boundary condition. For these reasons the boundary terms cancel out

$$\Big((-\Delta - z^2)\Big(\psi - \sum_{j=1}^{N} q^{\psi}_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big), \varphi - \sum_{k=1}^{N} q^{\varphi}_k G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big) + \sum_{j,k=1}^{N} (\Gamma^{\Omega,\mathrm{R},\eta}_{\alpha,Y}(z))_{kj}\overline{q^{\varphi}_k} q^{\psi}_j = (\phi - z^2\psi, \varphi). \tag{1.72}$$

Imposing the cancellation of all terms with φ, regardless of its choice in $D(\mathrm{R}^{\Omega}_{\alpha,Y}, \eta))$, we obtain (1.70). The remaining terms are

$$\Big((-\Delta - z^2)\Big(\psi - \sum_{j=1}^{N} q^{\psi}_j G^{\Omega,\mathrm{R},\eta}_{z,y_j}\Big), \sum_{k=1}^{N} q^{\varphi}_k G^{\Omega,\mathrm{R},\eta}_{z,y_k}\Big)_{L^2(\Omega)} = \sum_{j,k=1}^{N} (\Gamma^{\Omega,\mathrm{R},\eta}_{\alpha,Y}(z))_{kj}\overline{q^{\varphi}_k} q^{\psi}_j.$$

Following the same path as in the proof for the Dirichlet form leads to (1.69). □

Chapter 2.

Spectral Properties of One Point Interactions in Domains

In this chapter we study the spectrum for the one-point interaction Laplacians on domains like the ones considered in the previous chapter. Before doing this, we characterize the spectrum of the free Laplacian $-\Delta^{\Omega}$ on this kind of domains.

We start with the following definition.

Definition 2.0.1 (Conical Semi-Infinite Domain)**.** A domain $\Omega \subset \mathbb{R}^n$ is said to be Conical Semi-Infinite if there exists some $R > 0$ and a closed curve C on S^{n-1} such that for all $x \in \Omega \setminus B_R(0)$, then $\frac{x}{|x|} \in S^{n-1}$ is in the internal region delimited by C on S^{n-1}.

We sum up Theorems 6-8 and 11 in [27] with the following

Theorem 2.0.1. *Let Ω is a conical semi-infinite domain, then the spectra of $-\Delta^{\Omega,\mathrm{D}}$ and $-\Delta^{\Omega,\mathrm{N}}$ are both purely absolutely continuous and equal to* $[0, +\infty)$.

$\mathbb{H}^n$ is a conical semi-infinite domain because it fulfills the requirements with $R = 1$ and C being the intersection of S^n with the set $\partial\mathbb{H}^n = \{ x \in \mathbb{R}^n \mid x_n = 0 \}$. Hence we can conclude

$$\sigma(-\Delta^{\mathbb{H}^n,\mathrm{D}}) = \sigma_{\mathrm{ac}}(-\Delta^{\mathbb{H}^n,\mathrm{D}}) = [0, +\infty) \quad \sigma(-\Delta^{\mathbb{H}^n,\mathrm{N}}) = \sigma_{\mathrm{ac}}(-\Delta^{\mathbb{H}^n,\mathrm{N}}) = [0, +\infty)$$

For the Robin Laplacian on $\mathbb{H}^n$ we instead refer to the work [16] by Cossetti and Krejčiřik. They state the following result.

Theorem 2.0.2. *Let $n \geq 1$ and assume that $\eta \in W^{1,\infty}(\partial\mathbb{H}^n; \mathbb{R})$ is such that*

$$\eta \geq 0$$

and

$$x \cdot \nabla\eta \leq 0.$$

Then $\sigma_{\mathrm{p}}(-\Delta^{\mathbb{H}^n,\mathrm{R},\eta}) = \emptyset$ and $\sigma_{\mathrm{ac}}(-\Delta^{\mathbb{H}^n,\mathrm{R},\eta}) = [0, +\infty)$.

In this theorem the two conditions combine to guarantee the total absence of eigenvalues. In fact, the first one eliminates the possibility of negative eigenvalues by rendering the operator non-negative. Instead the non-trivial existence of non-negative embedded eigenvalues is excluded by the latter one.

Other conditions ensuring the absence of positive eigenvalues for domains with non-compact boundary can be found in Chapter 7 of [39] (e.g. Theorem 7.2.5. and Theorem 7.4.1.).

For exterior domains it holds the same condition of a purely absolutely continuous spectrum contained in $[0, +\infty)$ (See Theorems 4.9 and 4.11 in [32]).

So we can sum up the above results for the Laplacians on $\mathbb{H}^n$ and on Ω unbounded with bounded $\partial\Omega$ and state that in those cases

$$\sigma(-\Delta^{\Omega}) = \sigma_{\mathrm{ac}}(-\Delta^{\Omega}) = [0, +\infty) \quad \text{and} \quad \sigma_{\mathrm{p}}(-\Delta^{\Omega}) = \emptyset.$$

So the spectrum is purely absolutely continuous and there are no embedded eigenvalues in the continuous spectrum. This fact will be useful to determine the one point interaction Laplacians spectrum.

2.1. Characterization of the Eigenvalues of One Point Interaction Laplacians

If we recall Proposition 1.2.1 and Proposition 1.3.1, we can observe that the resolvent of $-\Delta^{\Omega}$ and of $-\Delta^{\Omega}_{\alpha,y}$ differ for a rank one operator. Theorem 9.3.5 in [10] states that for such operators the continuous spectrum is preserved under this perturbation and so

$$\sigma_{\mathrm{c}}(-\Delta^{\Omega}_{\alpha,y}) = \sigma_{\mathrm{c}}(-\Delta^{\Omega}) = [0, +\infty).$$

Moreover, by Theorem 9.3.8 also in [10]), the point spectrum is made of one point at most and

$$\sigma_{\mathrm{p}}(-\Delta^{\Omega}_{\alpha,y}) \subset (-\infty, 0).$$

Remark. The expression of the resolvent reported in Proposition 1.2.1 and Proposition 1.3.1 also shows that the possible negative eigenvalue is z^2, where z is the imaginary solution with $\mathrm{Im}\, z \geq 0$ of the equation $\Gamma^{\Omega}_{\alpha,y}(z) = 0$. Throughout all this chapter z^2 will indicate eigenvalues, while z will be used as a parametrization of the complex plane, employed to identify the singularities of the resolvent $(-\Delta_{\alpha,y} - z^2)^{-1}$.

Remark. We recall the eigenvalue properties for the one-point interaction in the whole $\mathbb{R}^n$, $n = 2, 3$.

For $n = 3$, the eigenvalue is z^2, where z is the solution of

$$\alpha - \frac{iz}{4\pi} = 0 \quad \operatorname{Im} z \geq 0, \tag{2.1}$$

so $z = -4\pi i\alpha$ ($z^2 = -16\pi^2\alpha^2$). The condition $\operatorname{Im} z \geq 0$ is fulfilled only for $\alpha \leq 0$. Moreover, while for $\alpha = 0$, $z = 0$ is a solution of (2.1), it is not an actual eigenvalue, because the corresponding solution

$$G_{0,y}^{0,3} = \frac{1}{4\pi\,|x - y|}$$

of $-\Delta_{\alpha,y}^{\mathbb{R}^3} = 0$ fails to be in $L^2(\mathbb{R}^3)$.

For $n = 2$, instead, the eigenvalue is z^2, where z is the solution

$$\alpha + \frac{\ln \frac{z}{2i} + \gamma}{2\pi} = 0 \quad \operatorname{Im} z \geq 0$$

(here γ is the Euler constant). Since the possible eigenvalue is negative, the substitution $z = ia$, $a > 0$ returns $z = 2ie^{-2\pi\alpha-\gamma}$ and then $z^2 = -4e^{-4\pi\alpha-2\gamma}$.

We give the following

Definition 2.1.1. The critical value α_c for $-\Delta_{\alpha,y}^{\Omega}$ is

$$\alpha_c = \sup\left\{ \alpha \in \mathbb{R} \;\middle|\; -\Delta_{\alpha,y}^{\Omega} \quad \text{admits eigenvalues} \right\}.$$

For example we have that $\alpha_c = 0$ for $-\Delta_{\alpha,y}^{\mathbb{R}^3}$ and $\alpha_c = +\infty$ for $-\Delta_{\alpha,y}^{\mathbb{R}^2}$.

In the following we investigate the conditions under which the possible negative eigenvalue exists and determine α_c for Dirichlet and Neumann boundary conditions in dimension two and three.

2.1.1. Exterior Domains with Dirichlet Boundary Conditions

Proposition 2.1.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $C^{1,1}$ boundary and consider $y \in \Omega$. The critical value α_c for the operator $-\Delta_{\alpha,y}^{\Omega,\mathrm{D}}$ is $\alpha_c = -h_{0,y}^{\Omega,\mathrm{D}}(y) \leq 0$ ($h_{z,y}^{\Omega,\mathrm{D}}$ defined by (A.5) is the harmonic part of the Green's function $G_{z,y}^{\Omega,\mathrm{D}}$). When the eigenvalue exists, it is non-positive and unique.*

Proof. The eigenvalue is the possible solution of

$$\alpha - \frac{iz}{4\pi} + h_{z,y}^{\Omega,\mathrm{D}}(y) = 0, \quad z \in i\mathbb{R}^+.$$

Setting $z = ia$ with $a \geq 0$, we get the equation

$$\frac{a}{4\pi} = -\alpha - h^{\Omega,\mathrm{D}}_{ia,y}(y), \quad a \geq 0, \tag{2.2}$$

whose solution can be interpreted as the abscissas of the intersections of the curves at each side.

The corrector function $h^{\Omega,\mathrm{D}}_{ia,y}$ solves the boundary value problem

$$\begin{cases} (-\Delta + a^2) h^{\Omega,\mathrm{D}}_{ia,y}(x) = 0 & \text{in}\,\Omega \\ h^{\Omega,\mathrm{D}}_{ia,y}(x) = \frac{e^{-a|x-y|}}{4\pi|x-y|} & \text{in}\,\partial\Omega \end{cases}. \tag{2.3}$$

By the regularity of the boundary data we can claim that the solution is smooth in Ω. This fact can be derived also from the representation of the problem (2.3) in terms of layer potentials related to $\partial\Omega$, see e.g. Theorem 7.15 in [36]. Moreover, the solution sought has to be in $H^2(\Omega)$, so it must hold $\lim_{|x|\to+\infty} h^{\Omega,\mathrm{D}}_{ia,y}(x) = 0$ (we observe that the boundary data also obey to the same condition). Hence, $\forall\varepsilon > 0$ there exists $R > 0$ large enough such that $\left|h^{\Omega,\mathrm{D}}_{ia,y}(x)\right| < \varepsilon$, $\forall x$ such that $|x - y| \geq R$. We define $U = \Omega \cap B_R(y)$. By the maximum principle in the form of Theorem A.2.1, we can claim that $h^{\Omega,\mathrm{D}}_{ia,y}$ is strictly positive in U and that reaches its maximum on the boundary,

$$h^{\Omega,\mathrm{D}}_{ia,y}(x) < \sup_{\partial U} h^{\Omega,\mathrm{D}}_{ia,y}(x) \quad \forall x \in U.$$

Given V, W open subset of $\mathbb{R}^3$ the following inclusion holds

$$\partial(V \cap W) \subset (\overline{V} \cap \partial W) \cup (\partial V \cap \overline{W}). \tag{2.4}$$

This implies

$$\begin{aligned} \sup_{\partial U} h^{\Omega,\mathrm{D}}_{ia,y}(x) &\leq \max\left\{ \sup_{\overline{\Omega}\cap\partial B_R(y)} h^{\Omega,\mathrm{D}}_{ia,y}(x), \sup_{\partial\Omega\cap\overline{B_R(y)}} h^{\Omega,\mathrm{D}}_{ia,y}(x) \right\} \\ &\leq \max\left\{ \varepsilon, \sup_{\partial\Omega\cap\overline{B_R(y)}} \frac{e^{-a|x-y|}}{4\pi\,|x-y|} \right\} \leq \max\left\{ \varepsilon, \sup_{\partial\Omega} \frac{e^{-a|x-y|}}{4\pi\,|x-y|} \right\} \\ &= \sup_{\partial\Omega} \frac{e^{-a|x-y|}}{4\pi\,|x-y|}, \end{aligned}$$

for R large enough. Since $\left|h^{\Omega,\mathrm{D}}_{ia,y}\right| < \varepsilon$ arbitrary, outside $B_R(y)$ for sufficiently large R, we can claim that

$$h^{\Omega,\mathrm{D}}_{ia,y}(x) < \sup_{\partial\Omega} \frac{e^{-a|x-y|}}{4\pi\,|x-y|} \quad \forall x \in \Omega.$$

By the minimum principle Theorem A.2.1, we can also claim that

$$h_{ia,y}^{\Omega,\mathrm{D}}(y) \geq \inf_{\partial U} h_{ia,y}^{\Omega,\mathrm{D}}(x) \geq \min\left\{\inf_{\overline{\Omega}\cap\partial B_R(y)} h_{ia,y}^{\Omega,\mathrm{D}}(x),\ \inf_{\partial\Omega\cap\overline{B_R(y)}} h_{ia,y}^{\Omega,\mathrm{D}}(x)\right\}$$
$$\geq \min\left\{-\varepsilon,\ \min_{\partial\Omega\cap\overline{B_R(y)}} \frac{e^{-a|x-y|}}{4\pi|x-y|}\right\} \geq \min\left\{-\varepsilon, \inf_{\partial\Omega} \frac{e^{-a|x-y|}}{4\pi|x-y|}\right\} \geq -\varepsilon.$$

Since ε can be chosen smaller and smaller for R large enough, we can claim that

$$h_{ia,y}^{\Omega,\mathrm{D}}(y) \geq 0. \tag{2.5}$$

Moreover, the derivative $\frac{\partial h_{ia,y}^{\Omega,\mathrm{D}}}{\partial a}$ solves

$$\begin{cases} (-\Delta + a^2)\frac{\partial h_{ia,y}^{\Omega,\mathrm{D}}}{\partial a} = -2a h_{ia,y}^{\Omega,\mathrm{D}} & \text{in } \Omega \\ \frac{\partial h_{ia,y}^{\Omega,\mathrm{D}}}{\partial a}(x) = -\frac{e^{-a|x-y|}}{4\pi} & \text{in } \partial\Omega \end{cases}.$$

The solution belongs to $C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ thanks to the regularity of the source and boundary value. With the help of the maximum principle for the Dirichlet problem Theorem A.2.2, we can write that

$$\begin{cases} \frac{\partial h_{ia,y}^{\Omega,\mathrm{D}}}{\partial a}(x) \geq \min\left\{\inf_{\partial\Omega}\left(-\frac{e^{-a|x-y|}}{4\pi}\right), \inf_{\Omega}\left(-2a h_{ia,y}^{\Omega,\mathrm{D}}(x)\right)\right\} \\ \frac{\partial h_{ia,y}^{\Omega,\mathrm{D}}}{\partial a}(x) \leq \max\left\{\sup_{\partial\Omega}\left(-\frac{e^{-a|x-y|}}{4\pi}\right), \sup_{\Omega}\left(-2a h_{ia,y}^{\Omega,\mathrm{D}}(x)\right)\right\} \end{cases}.$$

This implies that $\frac{\partial h_{ia,y}^{\Omega,\mathrm{D}}}{\partial a} < 0$ and that

$$\left|\frac{\partial h_{ia,y}^{\Omega,\mathrm{D}}}{\partial a}(x)\right| \leq \max\left\{\sup_{\partial\Omega}\left(\frac{e^{-a|x-y|}}{4\pi}\right), \sup_{\partial\Omega}\left(a\frac{e^{-a|x-y|}}{2\pi|x-y|}\right)\right\}.$$

We now inquiry the value of $h_{ia,y}^{\Omega,\mathrm{D}}(y)$ as $a \to +\infty$. The negativity of the derivative with respect to a leaves two alternatives: the limit is $-\infty$ or it is finite and less than $h_{0,y}^{\Omega,\mathrm{D}}(y)$. Let's consider

$$\left|h_{ia,y}^{\Omega,\mathrm{D}}(x)\right| \leq \int \left|\frac{\partial h_{ia,y}^{\Omega,\mathrm{D}}}{\partial a}\right| \mathrm{d}a \leq \int \max\left\{\sup_{\partial\Omega}\left(\frac{e^{-a|x-y|}}{4\pi}\right), \sup_{\partial\Omega}\left(a\frac{e^{-a|x-y|}}{2\pi|x-y|}\right)\right\} \mathrm{d}a.$$

Both the functions argument of sup in the equation above tends to 0 as $|x| \to +\infty$. This means that $\forall\varepsilon > 0$ there exists a $R > 0$, such that both functions are less than ε whenever $x \notin B_R(y)$. Now let $U' = \partial\Omega \cap B_R(y)$. This set being compact and being both functions continuous on $\partial\Omega$ (y is an internal point) imply that the supremum on this U' is actually attained for both functions. So there exist $x_1, x_2 \in U'$ such that

$$\sup_{U'}\left(\frac{e^{-a|x-y|}}{4\pi}\right) = \frac{e^{-a|x_1-y|}}{4\pi} \quad \text{and} \quad \sup_{U'}\left(a\frac{e^{-a|x-y|}}{2\pi\,|x-y|}\right) = a\frac{e^{-a|x_2-y|}}{2\pi\,|x_2-y|}.$$

Then, we can write that

$$\sup_{\partial\Omega}\left(\frac{e^{-a|x-y|}}{4\pi}\right) \leq \max\left\{\frac{e^{-a|x_1-y|}}{4\pi}, \varepsilon\right\} \quad \text{and} \quad \sup_{\partial\Omega}\left(a\frac{e^{-a|x-y|}}{2\pi\,|x-y|}\right) \leq \max\left\{a\frac{e^{-a|x_2-y|}}{2\pi\,|x_2-y|}, \varepsilon\right\}.$$

The arbitrarity of ε, together with the fact $\max\{A, B\} \leq A + B$ if $A, B \geq 0$ implies

$$\begin{aligned}\left|h^{\Omega,\mathrm{D}}_{ia,y}(x)\right| &\leq \int \frac{e^{-a|x_1-y|}}{4\pi}\,\mathrm{d}a + \int a\frac{e^{-a|x_2-y|}}{2\pi\,|x_2-y|}\,\mathrm{d}a \\ &= -\frac{e^{-a|x_1-y|}}{4\pi\,|x_1-x|} - \frac{e^{-a|x_2-y|}}{2\pi\,|x_2-y|^2}\left(a + \frac{1}{|x_2-y|}\right) + c,\end{aligned}$$

for some $c \in \mathbb{R}^+$. So

$$\lim_{a\to+\infty}\left|h^{\Omega,\mathrm{D}}_{ia,y}(y)\right| \leq c < +\infty,$$

which implies

$$\lim_{a\to+\infty} h^{\Omega,\mathrm{D}}_{ia,y}(y) > -\infty.$$

Let's go back to (2.2). We have just proved that the right side stays finite for $a \to +\infty$, while the left side goes to $+\infty$. Both sides are continuous, so to have at least a solution in $[0, +\infty)$ it must hold that $\alpha \leq -h^{\Omega,\mathrm{D}}_{0,y}(y)$. By recalling (2.5) with $a = 0$ we observe that $\alpha_\mathrm{c} \leq 0$. The eigenvalue if existent is unique because we already proved that the point spectrum is made of a single point at most. □

Proposition 2.1.2. *Let $\Omega \subset \mathbb{R}^2$ be an exterior domain with $C^{1,1}$ boundary and consider $y \in \Omega$. Define the function*

$$f : \Omega \times \Omega \times (0 + \infty] \to \mathbb{R} \tag{2.6}$$

$$f(x, y, a) = h^{\Omega,\mathrm{D}}_{ia,y}(x) + \frac{\ln\left(\frac{a}{2}\right) I_0(a\,|x-y|) + \gamma}{2\pi}, \tag{2.7}$$

where $h^{\Omega,\mathrm{D}}_{z,y}$, defined in (A.5), is the harmonic part of the Green's function and I_0 the modified Bessel function of the first kind of order 0 defined in (C.17). Then the function

$$g : \Omega \times \Omega \to \mathbb{R} \quad g(x, y) = \lim_{a\to 0^+} f(x, y, a)$$

is well-defined and the critical value for the operator $-\Delta^{\Omega,\mathrm{D}}_{\alpha,y}$ is $\alpha_\mathrm{c} = -g(y, y)$.

There is at most one eigenvalue and it is non-positive.

Proof. We know that the possible eigenvalue is unique and non-positive. So it is the eventual solution of

$$\alpha + \frac{\ln \frac{z}{2i} + \gamma}{2\pi} + h^{\Omega,\mathrm{D}}_{z,y}(y) = 0 \quad z \in i\mathbb{R}^+,$$

which can be rewritten as

$$\alpha + \frac{\ln \frac{a}{2} + \gamma}{2\pi} + h^{\Omega,\mathrm{D}}_{ia,y}(y) = 0 \quad a \geq 0. \tag{2.8}$$

We recall that $h^{\Omega,\mathrm{D}}_{ia,y}$ solves

$$\begin{cases} (-\Delta + a^2)h^{\Omega,\mathrm{D}}_{ia,y} = 0 & \text{in}\, \Omega \\ h^{\Omega,\mathrm{D}}_{ia,y}(x) = \frac{1}{2\pi} K_0(a\,|x-y|) & \text{in}\, \partial\Omega \end{cases},$$

where K_0 is the modified Bessel function of the second kind of order 0 (see (C.20) with $n = 0$). So by the regularity of the boundary data, for $a > 0$, the solution is smooth in Ω and continuous on its boundary. As in the proof of Proposition 2.1.1, we observe that the solution sought has to be in $H^2(\Omega)$, so it must hold $\lim_{|x|\to+\infty} h^{\Omega,\mathrm{D}}_{ia,y}(x) = 0$. Hence, $\forall\varepsilon > 0$ there exists a ball B_R large enough such that $\left|h^{\Omega,\mathrm{D}}_{ia,y}\right| < \varepsilon$, $\forall x \in \mathbb{R}^2 \setminus B_R(y)$. We can consider the strong maximum principle over the set $U = \Omega \cap B_R(y)$ and claim that

$$h^{\Omega,\mathrm{D}}_{ia,y}(x) > \inf_{\partial U} h^{\Omega,\mathrm{D}}_{ia,y}(x) \quad \forall x \in U.$$

Recalling (2.4), we can write

$$\begin{aligned} \inf_{\partial U} h^{\Omega,\mathrm{D}}_{ia,y}(x) &> \min\left\{ \inf_{\overline{\Omega}\cap\partial B_R(y)} h^{\Omega,\mathrm{D}}_{ia,y}(x), \inf_{\partial\Omega\cap\overline{B_R(y)}} h^{\Omega,\mathrm{D}}_{ia,y}(x) \right\} \\ &> \min\left\{ -\varepsilon, \inf_{\partial\Omega\cap\overline{B_R(y)}} \frac{K_0(a\,|x-y|)}{2\pi} \right\} \geq \min\left\{ -\varepsilon, \inf_{\partial\Omega} \frac{K_0(a\,|x-y|)}{2\pi} \right\} \\ &= -\varepsilon. \end{aligned}$$

Being $\left|h^{\Omega,\mathrm{D}}_{ia,y}\right|$ bounded by ε outside $B_R(y)$, choosing R sufficiently large we can claim that

$$h^{\Omega,\mathrm{D}}_{ia,y}(x) > -\varepsilon \quad \forall x \in \Omega.$$

This implies that $h^{\Omega,\mathrm{D}}_{ia,y}$ and moreover $h^{\Omega,\mathrm{D}}_{ia,y}(y)$ are bounded from below and so the left side of (2.8) goes to $+\infty$ for $a \to +\infty$.

We recall that the function $J_0(w)$ solves the differential equation (C.6). If we set $u(r) = J_0(iar)$, with $r = |x - y|$, it follows that u solves

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} + a^2 u = 0,$$

which, because off the expression of the Laplacian operator in polar coordinates, is equivalent to $(-\Delta + a^2)u = 0$. Using the definition (C.17) of the modified Bessel function of order zero I_0, we have that $u(r) = I_0(ar) = I_0(a\,|x - y|)$. It follows then that u solves the following boundary value problem

$$\begin{cases} (-\Delta + a^2)u(x, y, a) = 0 \\ u(x, y, ia)|_{x\in\partial\Omega} = I_0(a\,|x - y|)|_{x\in\partial\Omega} \end{cases} . \tag{2.9}$$

Since $h^{\Omega,\mathrm{D}}_{ia,y}$ and $u(\cdot, y, a)$ solve the same linear homogeneous equation, then any linear combination of those functions will solve the same equation. The boundary value will be the corresponding linear combination of the boundary values for the original problems. This means that f solves

$$\begin{cases} (-\Delta + a^2)f(x, y, a) = 0 \\ f(x, y, a)|_{x\in\partial\Omega} = \left.\frac{K_0(a|x-y|)+\ln\frac{a}{2}I_0(a|x-y|)+\gamma}{2\pi}\right|_{x\in\partial\Omega} \end{cases} . \tag{2.10}$$

Both $h^{\Omega,\mathrm{D}}_{ia,y}(x)$ and $I_0(a\,|x - y|)$ are continuous for $x = y$ because they are harmonic in Ω. So it makes sense to evaluate $f(y, y, a)$ and since $I_0(0) = 1$ it holds that

$$f(y, y, a) = h^{\Omega,\mathrm{D}}_{ia,y}(y) + \frac{\ln\frac{a}{2} + \gamma}{2\pi}. \tag{2.11}$$

A comparison between this expression and (2.8) show that we can rewrite (2.8) as

$$\alpha + f(y, y, a) = 0 \tag{2.12}$$

and hence the behaviour of $f(y, y, a)$ for $a \to 0^+$ determines the existence of a solution for (2.8). We prove that the boundary datum of the problem (2.10) is continuous from the right in $a = 0$. In fact, thanks to the expansion (C.22) of K_0 for small argument we have

$$\lim_{a\to 0^+} \frac{K_0(a\,|x - y|) + \ln\frac{a}{2}I_0(a\,|x - y|) + \gamma}{2\pi} = -\frac{\ln|x - y|}{2\pi}.$$

This means that problem (2.10) depends continuously on a and the function $g(x, y) = \lim_{a\to 0^+} f(x, y, a)$ is the solution of

$$\begin{cases} -\Delta g(x, y) = 0 \\ g(x, y)|_{x\in\partial\Omega} = -\left.\frac{\ln|x-y|}{2\pi}\right|_{x\in\partial\Omega} \end{cases} .$$

The boundary data regularity implies that $g(\cdot, y) \in C^\infty(\Omega) \cap C(\overline{\Omega})$. This means that $g(y, y)$ is finite and using the form (2.12) of (2.8) we conclude that a solution (which is the only one) is possible if and only if $\alpha \leq -g(y, y)$. □

Remark. In dimension 3 the critical value α_c is non-positive as a consequence of the maximum principle and of the positivity of the boundary datum for problem (2.3).

In dimension 2 the sign of α_c is not a priori determined, because the function

$$\psi_0(x, y) = -\frac{\ln|x - y|}{2\pi}$$

is not positive in general (or even of definite sign on $\partial\Omega$). Instead, for a fixed Ω, one can get operators $-\Delta^{\Omega,D}_{\alpha,y}$ for which α_c is arbitrarily negative large by locating the point interaction close enough to the boundary.

2.1.2. Star-Shaped Domains with Neumann Boundary Condition

Definition 2.1.2 (Star-Shaped Domain)**.** Let $x_0 \in \Omega \subseteq \mathbb{R}^n$ with $\partial\Omega \in C^{0,1}$. We say that Ω is star-shaped with respect to x_0 if

$$(x - x_0) \cdot n(x) \geq 0 \quad \forall x \in \partial\Omega, \tag{2.13}$$

where $n(x)$ is the unit outer normal to $\partial\Omega$ at x.

Proposition 2.1.3. *Let $\Omega \subset \mathbb{R}^3$ be an unbounded domain with C^2 boundary, star-shaped with respect to $y \in \Omega$ and such that $\sigma(-\Delta^{\Omega,N}) = \sigma_{ac}(-\Delta^{\Omega,N}) = [0, +\infty)$. Then the cririical value α_c for the operator $-\Delta^{\Omega,N}_{\alpha,y}$ is $\alpha_c = -h^{\Omega,N}_{0,y}(y) \geq 0$ ($h^{\Omega,D}_{z,y}$ defined by (A.6) is the harmonic part of the Neumann Green's function $G^{\Omega,N}_{z,y}$). When the eigenvalue λ exists, it is non-positive and unique. Moreover, when $\alpha \leq 0$ it holds that*

$$\lambda < -16\pi^2\alpha^2. \tag{2.14}$$

Proof. We know that the point spectrum of $-\Delta^{\Omega,N}_{\alpha,y}$ is made at most of one non-positive point. This eigenvalue, if existent, is equal to z^2, with z solving

$$\alpha - \frac{iz}{4\pi} + h^{\Omega,N}_{z,y}(y) = 0, \quad z \in i\mathbb{R}^+.$$

Setting $z = ia$ with $a \geq 0$, we get the equation

$$\alpha + \frac{a}{4\pi} + h^{\Omega,N}_{ia,y}(y) = 0, \quad a \geq 0. \tag{2.15}$$

Here, the function $h^{\Omega,\mathrm{N}}_{z,y}$ is the solution of the boundary value problem

$$\begin{cases} (-\Delta + a^2)h^{\Omega,\mathrm{N}}_{ia,y}(x) = 0 \\ \frac{\partial h^{\Omega,\mathrm{N}}_{ia,y}}{\partial n}(x)\Big|_{x\in\partial\Omega} = \frac{\partial}{\partial n}\Big(\frac{e^{-a|x-y|}}{4\pi|x-y|}\Big)\Big|_{x\in\partial\Omega}, \end{cases} \quad y \in \Omega \quad a \geq 0. \tag{2.16}$$

We can rewrite this problem as

$$\begin{cases} (-\Delta + a^2)h^{\Omega,\mathrm{N}}_{ia,y}(x) = 0 \\ \frac{\partial h^{\Omega,\mathrm{N}}_{ia,y}}{\partial n}(x)\Big|_{x\in\partial\Omega} = -\frac{e^{-a|x-y|}(1+a|x-y|)}{4\pi|x-y|^3}(x-y)\cdot n(x)\Big|_{x\in\partial\Omega}, \end{cases} \quad y \in \Omega \quad a \geq 0. \tag{2.17}$$

For $a \to 0^+$, the boundary term in (2.17) converges to

$$-\frac{x-y}{4\pi\,|x-y|^3}.$$

So it makes sense to pass to the limit in the boundary value problem (2.17) and this way it becomes

$$\begin{cases} -\Delta h^{\Omega,\mathrm{N}}_{0^+,y}(x) = 0 \\ \nabla h^{\Omega,\mathrm{N}}_{0^+,y}(x)\cdot n(x)\Big|_{x\in\partial\Omega} = -\frac{x-y}{4\pi|x-y|^3}\cdot n(x)\Big|_{x\in\partial\Omega}, \end{cases} \quad y \in \Omega.$$

By regularity of the boundary and boundary value we have that $h^{\Omega,\mathrm{N}}_{0^+,y} \in C^\infty(\Omega) \cap C^1(\overline{\Omega})$ and this implies that $h^{\Omega,\mathrm{N}}_{0^+,y}(y)$ is finite.

The right hand side of the boundary condition in (2.17) is the product of a scalar negative function and $(x-y)\cdot n(x)$. This observation combined with Ω being star-shaped with respect to y implies that $\frac{\partial h^{\Omega,\mathrm{N}}_{ia,y}}{\partial n} \leq 0$ on $\partial\Omega$. $h^{\Omega,\mathrm{N}}_{ia,y} \leq 0$ is then a consequence of Proposition A.2.1, which can be applied thanks to the assumption of Ω having C^2 boundary.

We now prove that for a large enough, the left hand side of (2.15) is positive. To do so we consider set $a \geq b \geq 0$ and consider the following auxiliar boundary value problem

$$\begin{cases} (-\Delta + b^2)h^{\Omega,\mathrm{N}}_{ib,y}(x) = 0 \\ \frac{\partial h^{\Omega,\mathrm{N}}_{ib,y}}{\partial n}(x)\Big|_{x\in\partial\Omega} = -\frac{e^{-b|x-y|}(1+b|x-y|)}{4\pi|x-y|^3}(x-y)\cdot n(x)\Big|_{x\in\partial\Omega} = f(b)(x-y)\cdot n(x)|_{x\in\partial\Omega} \end{cases}, \quad y \in \Omega. \tag{2.18}$$

Then, we define the boundary value problem, whose equation data are the difference of (2.18) and (2.17)

$$\begin{cases} -\Delta\Big(h^{\Omega,\mathrm{N}}_{ib,y}(x) - h^{\Omega,\mathrm{N}}_{ia,y}(x)\Big) + b^2\Big(h^{\Omega,\mathrm{N}}_{ib,y}(x) - h^{\Omega,\mathrm{N}}_{ia,y}(x)\Big) = (a^2-b^2)h^{\Omega,\mathrm{N}}_{ia,y}(x) \\ \frac{\partial}{\partial n}\Big(h^{\Omega,\mathrm{N}}_{ib,y} - h^{\Omega,\mathrm{N}}_{ia,y}\Big)(x)\Big|_{x\in\partial\Omega} = \Big(f(b) - f(a)\Big)(x-y)\cdot n(x)\Big|_{x\in\partial\Omega} \end{cases}, \quad y \in \Omega. \tag{2.19}$$

The right hand side of the first equation in (2.19) is non positive because $a \geq b \geq 0$ and we have proven that $h^{\Omega,\mathrm{N}}_{ia,y} \leq 0$. The same fact holds for the right hand side of the other equation because the function f is increasing

$$\frac{\mathrm{d}f}{\mathrm{d}a}(a) = \frac{a}{4\pi\,|x-y|}e^{-a|x-y|} \geq 0, \quad \forall a \geq 0. \tag{2.20}$$

This means that we can apply Proposition A.2.1 to the problem (2.19) concluding that, whenever $a \geq b \geq 0$, one has $h^{\Omega,\mathrm{N}}_{ia,y} \geq h^{\Omega,\mathrm{N}}_{ib,y}$. In particular, since it has been proven that $h^{\Omega,\mathrm{N}}_{0^+,y}(y)$ is finite, it holds that also $\forall a \geq 0$ $h^{\Omega,\mathrm{N}}_{ia,y}$ is bounded below. We can then conclude that, for a large enough

$$\alpha + \frac{a}{4\pi} + h^{\Omega,\mathrm{N}}_{ia,y}(y) > 0 \tag{2.21}$$

The functions in (2.15) are all continuous in a and so for a solution to exists it needs to happen that $\alpha \leq h^{\Omega,\mathrm{N}}_{0,y}(y)$.

Next we rewrite (2.15) as

$$a = -4\pi(h^{\Omega,\mathrm{N}}_{ia,y} + \alpha).$$

If $\alpha \leq 0$, then $0 < a < -4\pi\alpha$ and (2.14) follows. □

Proposition 2.1.4. *Let $\Omega \subset \mathbb{R}^2$ be an unbounded domain with C^2 boundary, star-shaped with respect to $y \in \Omega$ and such that $\sigma(-\Delta^{\Omega,\mathrm{N}}) = \sigma_{\mathrm{ac}}(-\Delta^{\Omega,\mathrm{N}}) = [0,+\infty)$. Then the operator $-\Delta^{\Omega,\mathrm{N}}_{\alpha,y}$ has one eigenvalue λ and it is non-positive, regardless of α.*

In addition, the harmonic part $h^{\Omega,\mathrm{N}}_{ia,y}$ of the Neumann Green's function of Ω obeys $h^{\Omega,\mathrm{N}}_{ia,y} \leq 0$ in Ω $\forall a > 0$ and $\lim_{a\to 0^+} h^{\Omega,\mathrm{N}}_{ia,y}(y) > -\infty$

Proof. We know that the point spectrum of $-\Delta^{\Omega,\mathrm{N}}_{\alpha,y}$ is made at most of one non-positive point. This eigenvalue z^2, if existent, is such that z is a solution of

$$\alpha + \frac{\ln\frac{z}{2i} + \gamma}{2\pi} + h^{\Omega,\mathrm{N}}_{z,y}(y) = 0 \quad z \in i\mathbb{R}^+$$

($h^{\Omega,\mathrm{N}}_{z,y}$ is defined in (A.6)), or equivalently

$$\alpha + \frac{\ln\frac{a}{2} + \gamma}{2\pi} + h^{\Omega,\mathrm{N}}_{ia,y}(y) = 0 \quad a \geq 0. \tag{2.22}$$

Here the function $h^{\Omega,\mathrm{N}}_{z,y}$ is the solution of the boundary value problem

$$\begin{cases} (-\Delta + a^2)h^{\Omega,\mathrm{N}}_{ia,y}(x) = 0 \\ \frac{\partial h^{\Omega,\mathrm{N}}_{ia,y}}{\partial n}(x)\Big|_{x\in\partial\Omega} = \frac{\partial}{\partial n}\Big(\frac{1}{2\pi}K_0(a\,|x-y|)\Big)\Big|_{x\in\partial\Omega} \end{cases}, \quad y \in \Omega \quad a \geq 0. \tag{2.23}$$

By definition of normal derivative and since $K_0'(w) = -K_1(w)$ ($K_1(w)$ being the modified Bessel function of the second kind of order 1), we can rewrite this problem as

$$\begin{cases} (-\Delta + a^2)h^{\Omega,\mathrm{N}}_{ia,y}(x) = 0 \\ \nabla h^{\Omega,\mathrm{N}}_{ia,y}(x)\cdot n(x)\Big|_{x\in\partial\Omega} = -\frac{a}{2\pi}\frac{K_1(a|x-y|)}{|x-y|}(x-y)\cdot n(x)\Big|_{x\in\partial\Omega} \end{cases}, \quad y\in\Omega \quad y \geq 0. \tag{2.24}$$

Thanks to (C.23) we have that

$$\lim_{a\to 0^+} . -\frac{a}{2\pi}K_1(a\,|x-y|)\frac{x-y}{|x-y|} = -\frac{x-y}{2\pi\,|x-y|^2},$$

which means that, the boundary term is continuous for $a \to 0^+$. So it makes sense to pass to the limit in the boundary value problem (2.24) and this way it becomes

$$\begin{cases} -\Delta h^{\Omega,\mathrm{N}}_{0^+,y}(x) = 0 \\ \nabla h^{\Omega,\mathrm{N}}_{0^+,y}(x)\cdot n(x)\Big|_{x\in\partial\Omega} = -\frac{x-y}{2\pi|x-y|^2}\cdot n(x)\Big|_{x\in\partial\Omega} \end{cases}, \quad y\in\Omega.$$

By regularity of the boundary and boundary value we have that $h^{\Omega,\mathrm{N}}_{0^+,y} \in C^\infty(\Omega)\cap C^1(\overline{\Omega})$ and this implies that $h^{\Omega,\mathrm{N}}_{0^+,y}(y)$ is finite. So, the left side of (2.22) goes to $-\infty$ as a approaches 0.

The right hand side of the boundary condition in (2.24) is the product of a scalar negative function and $(x-y)\cdot n(x)$. This observation combined with Ω being star-shaped with respect to y implies that $\frac{\partial h^{\Omega,\mathrm{N}}_{ia,y}}{\partial n} \leq 0$ on $\partial\Omega$. The C^2 regularity of Ω allows us to invoke Proposition A.2.1 and conclude that $h^{\Omega,\mathrm{N}}_{ia,y} \leq 0$.

Proving that for a large enough the left side of (2.22) is positive, implies that the existence of a solution. Let $a \geq b \geq 0$ and consider the problem

$$\begin{cases} (-\Delta + b^2)h^{\Omega,\mathrm{N}}_{ib,y}(x) = 0 \\ \frac{\partial h^{\Omega,\mathrm{N}}_{ib,y}}{\partial n}(x)\Big|_{x\in\partial\Omega} = -\frac{b}{2\pi}\frac{K_1(b|x-y|)}{|x-y|}(x-y)\cdot n(x)\Big|_{x\in\partial\Omega} = f(b)(x-y)\cdot n(x)|_{x\in\partial\Omega} \end{cases}, \quad y\in\Omega. \tag{2.25}$$

As done in the three dimensional case, we consider the boundary value problem difference of the ones with b and a

$$\begin{cases} -\Delta\Big(h^{\Omega,\mathrm{N}}_{ib,y}(x) - h^{\Omega,\mathrm{N}}_{ia,y}(x)\Big) + b^2\Big(h^{\Omega,\mathrm{N}}_{ib,y}(x) - h^{\Omega,\mathrm{N}}_{ia,y}(x)\Big) = (a^2-b^2)h^{\Omega,\mathrm{N}}_{ia,y}(x) \\ \frac{\partial}{\partial n}\Big(h^{\Omega,\mathrm{N}}_{ib,y} - h^{\Omega,\mathrm{N}}_{ia,y}\Big)(x)\Big|_{x\in\partial\Omega} = \Big(f(b)-f(a)\Big)(x-y)\cdot n(x)\Big|_{x\in\partial\Omega} \end{cases}, \quad y\in\Omega. \tag{2.26}$$

$a \geq b \geq 0$ and $h^{\Omega,\mathrm{N}}_{ia,y} \leq 0$, imply the non-positivity of the right side of the first equation. It also holds that

$$\frac{\mathrm{d}f}{\mathrm{d}a}(a) = \frac{a}{2\pi}K_0(a\,|x-y|) \geq 0, \tag{2.27}$$

which together with $a \geq b \geq 0$ and the fact that the domain is star shaped with respect to y, implies that also $\frac{\partial}{\partial n}\Big(h^{\Omega,\mathrm{N}}_{ib,y} - h^{\Omega,\mathrm{N}}_{ia,y}\Big)(x)\Big|_{x\in\partial\Omega} \leq 0$. So we can apply Proposition A.2.1 and have that

$$h^{\Omega,\mathrm{N}}_{ia,y}(y) \geq h^{\Omega,\mathrm{N}}_{ib,y}(y) \geq h^{\Omega,\mathrm{N}}_{0^+,y}(y) > -\infty.$$

Hence for large a, the left side of (2.22) is positive (the other terms are either constant or diverge to $+\infty$ as $a \to +\infty$) regardless of α, implying that a solution exists $\forall\alpha$. □

Remark. We conjecture that this the last two propositions hold also in an exterior domain, resembling the results valid for the Dirichlet case. Note that in exterior domains, condition $(x - y) \cdot n(x)$ is never true. In fact what is missing is a way to bound from below $h^{\Omega,\mathrm{N}}_{a,y}(y)$ for large $a > 0$.

2.2. The Half-Space

In this and following sections, we explicitly study the spectrum of $-\Delta^{\Omega}_{\alpha,y}$ for different domains Ω. We start with the half-space $\mathbb{H}^3 = \Big\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \;\Big|\; x_3 > 0 \Big\}$. The point source is located in $y = (y_1, y_2, y_3)$ with $y_3 > 0$.

2.2.1. Dirichlet Boundary Condition

The Green's function for $\mathbb{H}^3$ with Dirichlet boundary condition can be found by a simple reflection argument. We have that $\partial\mathbb{H}^3 = \Big\{ x \in \mathbb{R}^3 \;\Big|\; x_3 = 0 \Big\}$. We define $\overline{y} = (y_1, y_2, -y_3)$. Let $x \in \partial\mathbb{H}^3$, then

$$|x - \overline{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 + y_3)^2} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 + y_3)^2} = |x - y|\,.$$

It follows that

$$h^{\mathbb{H}^3,\mathrm{D}}_{z,y}(x) = \frac{e^{iz|x-\overline{y}|}}{4\pi\,|x - \overline{y}|}$$

is the solution of (A.5). We can now prove the following

Proposition 2.2.1. *The operator* $-\Delta^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}$ *has* $\alpha_{\mathrm{c}} = -\frac{1}{8\pi y_3} < 0$. *When the eigenvalue* λ *exists, it is such that*

$$-16\pi^2\alpha^2 < \lambda \leq -\left(4\pi\alpha + \frac{1}{2y_3}\right)^2. \tag{2.28}$$

The corresponding eigenfunction is

$$\psi_\lambda(x,y) = \frac{e^{-\sqrt{|\lambda|}|x-y|}}{4\pi\,|x-y|} - \frac{e^{-\sqrt{|\lambda|}|x-\overline{y}|}}{4\pi\,|x-\overline{y}|}. \tag{2.29}$$

Proof. The eigenvalues are solutions of $\Gamma^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}(z) = 0$, that is

$$\alpha - \frac{iz}{4\pi} + \frac{e^{iz|y-\overline{y}|}}{4\pi\,|y-\overline{y}|} = 0, \tag{2.30}$$

with $z^2 = \lambda \in \mathbb{R}$ and $\operatorname{Im} z \geq 0$. We already know that the possible eigenvalue is negative, but we will anyway directly prove that no positive eigenvalues can occur.

First we look for positive eigenvalues. To do so we substitute the ansatz $z = a$ with $a \in \mathbb{R}$ in (2.30), obtaining

$$\alpha - \frac{ia}{4\pi} + \frac{e^{2iy_3a}}{8\pi y_3} = 0.$$

By equating real and imaginary part we get the system

$$\begin{cases} \cos(2y_3a) = -8\pi\alpha y_3 \\ \sin(2y_3a) = 2y_3a \end{cases}.$$

The second equation admits solution only if $2y_3a = 0$. If $y_3 = 0$, then the first equation cannot be true, so this candidate solution is discarded. If $a = 0$, the first equation admits a solution if and only if $\alpha = -\frac{1}{8\pi y_3}$.

Whenever zero is a solution of the algebraic equation for the eigenvalues, one has to check whether the corresponding solution is in L^2 or not. In this case

$$\psi_0(x,y) = \frac{1}{4\pi\,|x-y|} - \frac{1}{4\pi\,|x-\overline{y}|}.$$

This function is in $L^2(\mathbb{H}^3)$ because

$$|\psi_0(x,y)| = \frac{|x-\overline{y}| - |x-y|}{4\pi\,|x-y|\,|x-\overline{y}|} \leq \frac{|x-\overline{y}-x+y|}{4\pi\,|x-y|\,|x-\overline{y}|} = \frac{y_3}{2\pi\,|x-y|\,|x-\overline{y}|},$$

with the last side of the inequality being in $L^2(\mathbb{H}^3)$.

To find negative eigenvalues we substitute $z = ia$, with $a > 0$, in (2.30) so it becomes

$$\frac{e^{-2y_3a}}{2y_3} = -4\pi\alpha - a. \tag{2.31}$$

We name the first and second members of (2.31) $f(a)$ and $g(a)$ respectively. It holds that:

- $f(0) = \frac{1}{2y_3}$ and $g(0) = -4\pi\alpha$;
- $0 > f'(a) > g'(a) \quad \forall a > 0$;
- $f(0) \geq g(0)$ for $\alpha \geq -\frac{1}{8\pi y_3}$.

This implies that for $\alpha > -\frac{1}{8\pi y_3}$ no solution can exist and so there are no negative eigenvalues for the operator. If instead $\alpha < -\frac{1}{8\pi y_3}$, given that $0 = \lim_{a\to+\infty} f(a) > \lim_{a\to+\infty} g(a) = -\infty$ and that both functions are smooth in a, it must necessarily exist an $\tilde{a} > 0$ such that it solves the equation. This solution is unique because in the interval $(\tilde{a}, +\infty)$ f decreases slower than g and they are equal in $\tilde{a}$.

A useful fact about f is that for $a \geq 0$, $0 < f(a) \leq 1$. Then the solution, if existing, must satisfy

$$0 < -4\pi\alpha - a \leq \frac{1}{y_3}$$

or equivalently $a \in \left[-4\pi\alpha - \frac{1}{2y_3}, -4\pi\alpha\right)$ and hence (2.28) follows. □

Remark. We can use the quadratic form $\mathrm{D}^{\mathbb{H}^3}_{\alpha,y}$ to check if the result above does make sense. Since, $\sigma_{\mathrm{p}}(-\Delta^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}) \subset (-\infty, 0)$, for the function $G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}$ to even be a plausible eigenfunction, it must hold that $\mathrm{D}^{\mathbb{H}^3}_{\alpha,y}\left(G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}, G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}\right) \leq 0$. This will set some condition on the parameters $\alpha \in \mathbb{R}$ and $y_3 \in \mathbb{R}^+$, which shouldn't be stricter than the ones found directly for $-\Delta^{\Omega,\mathrm{D}}_{\alpha,y}$.

So we compute (using $z^2 = \lambda = -|\lambda|$, (1.49) and (1.50))

$$\begin{aligned}
\mathrm{D}^{\mathbb{H}^3}_{\alpha,y}\left(G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}, G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}\right) &= -|\lambda| \int_{\mathbb{R}^2\times\mathbb{R}^+} \left| \frac{e^{-\sqrt{|\lambda|}|x-y|}}{4\pi|x-y|} - \frac{e^{-\sqrt{|\lambda|}|x-\bar{y}|}}{4\pi|x-\bar{y}|} \right|^2 \mathrm{d}x + \alpha + \frac{\sqrt{|\lambda|}}{4\pi} + \frac{e^{-\sqrt{|\lambda|}|y-\bar{y}|}}{4\pi|y-\bar{y}|} \\
&= -\frac{|\lambda|}{16\pi^2} \int_{\mathbb{R}^2\times\mathbb{R}^+} \left(\frac{e^{-2\sqrt{|\lambda|}|x-y|}}{|x-y|^2} - 2\frac{e^{-\sqrt{|\lambda|}(|x-y|+|x-\bar{y}|)}}{|x-y|\,|x-\bar{y}|} + \frac{e^{-2\sqrt{|\lambda|}|x-\bar{y}|}}{|x-\bar{y}|^2} \right) \mathrm{d}x + \alpha + \frac{\sqrt{|\lambda|}}{4\pi} + \frac{e^{-2\sqrt{|\lambda|}y_3}}{8\pi y_3}.
\end{aligned} \tag{2.32}$$

We have that

$$\int_{\mathbb{R}^2\times\mathbb{R}^+} \frac{e^{-2\sqrt{|\lambda|}|x-y|}}{|x-y|^2}\,\mathrm{d}x \underset{u=x-y}{=} \int_{\mathbb{R}^2\times[-y_3,+\infty)} \frac{e^{-2\sqrt{|\lambda|}|u|}}{|u|^2}\,\mathrm{d}u.$$

Passing to polar coordinates, the integration over u_3 is converted to the one on the region $t = \cos\theta \geq -y_3/r$ ($t \in [-1, 1]$). If $y_3/r \geq 1$, the condition is verified $\forall t \in [-1, 1]$, so we can rewrite the last equation as

$$\begin{aligned}\int_{\mathbb{R}^2\times\mathbb{R}^+} \frac{e^{-2\sqrt{|\lambda|}|x-y|}}{|x-y|^2}\,\mathrm{d}x &= 2\pi\int_0^{y_3}\int_{-1}^{1} e^{-2\sqrt{|\lambda|}r}\,\mathrm{d}t\,\mathrm{d}r + 2\pi\int_{y_3}^{+\infty}\int_{-\frac{y_3}{r}}^{1} e^{-2\sqrt{|\lambda|}r}\,\mathrm{d}t\,\mathrm{d}r \\ &= \frac{\pi}{\sqrt{|\lambda|}}\Big(2 - e^{-2\sqrt{|\lambda|}y_3}\Big) + 2\pi y_3\int_{y_3}^{+\infty}\frac{e^{-2\sqrt{|\lambda|}r}}{r}\,\mathrm{d}r.\end{aligned} \tag{2.33}$$

In a similar fashion we prove that

$$\int_{\mathbb{R}^2\times\mathbb{R}^+} \frac{e^{-2\sqrt{|\lambda|}|x-\bar{y}|}}{|x-\bar{y}|^2}\,\mathrm{d}x = \frac{\pi}{\sqrt{|\lambda|}}e^{-2\sqrt{|\lambda|}y_3} - 2\pi y_3\int_{y_3}^{+\infty}\frac{e^{-2\sqrt{|\lambda|}r}}{r}\,\mathrm{d}r \tag{2.34}$$

and

$$\int_{\mathbb{R}^2\times\mathbb{R}^+} \frac{e^{-\sqrt{|\lambda|}(|x-y|+|x-\bar{y}|)}}{|x-y|\,|x-\bar{y}|} = \frac{\pi}{\sqrt{|\lambda|}}e^{-2\sqrt{|\lambda|}y_3}. \tag{2.35}$$

Substitute the expression for the integrals in the quadratic form we get

$$\mathrm{D}^{\mathbb{H}^3}_{\alpha,y}\Big(G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}, G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}\Big) = \alpha + \frac{\sqrt{|\lambda|}}{8\pi} + \frac{e^{-2\sqrt{|\lambda|}y_3}}{8\pi}\Big(\frac{1}{y_3} + \sqrt{|\lambda|}\Big).$$

We $\sqrt{|\lambda|} = a > 0$ and define

$$d(a) = \alpha + \frac{a}{8\pi} + \frac{e^{-2ay_3}}{8\pi}\Big(\frac{1}{y_3} + a\Big).$$

We look for the minimum of this function for $a \in [0, +\infty)$. We study the sign of the derivative

$$\frac{\mathrm{d}d}{\mathrm{d}a}(a) = \frac{1}{8\pi} - \frac{e^{-2ay_3}}{8\pi}(1 + 2ay_3) \geq 0,$$

which happens for $1 + 2ay_3 \leq e^{2ay_3}$. This condition holds true $\forall a$, so the function is monotone increasing in $[0, +\infty]$, and then the minimum is attained for $a = 0$. But

$$d(0) = \alpha + \frac{1}{8\pi y_3},$$

which is non positive for $\alpha \leq -\frac{1}{8\pi y_3}$. This means that, for such values of α, $\mathrm{D}^{\mathbb{H}^3}_{\alpha,y}\Big(G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}, G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}\Big)$ is negative regardless of the value of $\lambda < 0$ (has a negative maximum). This is in agreement with the study of the associated operator $-\Delta^{\Omega,\mathrm{D}}_{\alpha,y}$.

2.2.2. Neumann Boundary Condition

Also in this case, a reflection argument helps. The normal external derivative on $\partial\mathbb{H}^3$ is nothing else than $-\frac{\partial}{\partial x_3}$. So, a smooth function even in x_3 satisfies the boundary condition in (A.6). Let $f : \mathbb{R} \to \mathbb{C}$ be smooth in a neighborhood of $\partial\mathbb{H}^3$. It is straightforward to verify that

$$g(x, y) = f(|\overline{x} - \overline{y}|) + f(|\overline{x} - y|) = f(|x - y|) + f(|x - \overline{y}|),$$

which means that g is even in x_3. By definition of $G^{0,3}_{z,y}$ (see (A.3)), then

$$G^{\mathbb{H}^3,\mathrm{N}}_{z,y}(x) = \frac{e^{iz|x-y|}}{4\pi\,|x - y|} + \frac{e^{iz|x-\overline{y}|}}{4\pi\,|x - \overline{y}|}$$

and hence

$$h^{\mathbb{H}^3,\mathrm{N}}_{z,y}(x) = -\frac{e^{iz|x-\overline{y}|}}{4\pi\,|x - \overline{y}|}.$$

Proposition 2.2.2. *The operator* $-\Delta^{\mathbb{H}^3,\mathrm{N}}_{\alpha,y}$ *has* $\alpha_{\mathrm{c}} = \frac{1}{8\pi y_3} > 0$. *When the eigenvalue* λ *exists, it is such that*

$$-\left(\frac{1}{2y_3} - 4\pi\alpha\right)^2 \le \lambda < -16\pi^2\alpha^2, \quad \text{for} \quad \alpha < 0 \tag{2.36}$$

and

$$-\left(\frac{1}{2y_3} - 4\pi\alpha\right)^2 \le \lambda < 0, \quad \text{for} \quad 0 \le \alpha \le \frac{1}{8\pi y_3}. \tag{2.37}$$

The corresponding eigenfunction is

$$\psi_\lambda(x, y) = \frac{e^{-\sqrt{|\lambda|}|x-y|}}{4\pi\,|x - y|} + \frac{e^{-\sqrt{|\lambda|}|x-\overline{y}|}}{4\pi\,|x - \overline{y}|}. \tag{2.38}$$

Proof. This time the equation for z is

$$\alpha - \frac{iz}{4\pi} - \frac{e^{2iy_3 z}}{8\pi y_3} = 0. \tag{2.39}$$

We know there are no positive eigenvalues. To find the negative eigenvalues we substitute $z = ia$ with $a > 0$ in (2.39), getting

$$\frac{e^{-2y_3 a}}{2y_3} = a + 4\pi\alpha.$$

Let $f(a)$ be the first member and $g(a)$ the other one. It holds that:

- both functions are smooth, f is decreasing and g is increasing;

- $\lim_{a\to+\infty} f(a) = 0$ and $\lim_{a\to+\infty} g(a) = +\infty$;
- $f(0) \geq g(0)$ when $\alpha \leq \frac{1}{8\pi y_3}$.

It follows that a solution, which is unique, is admitted only for $\alpha \leq \frac{1}{8\pi y_3}$ and furthermore that solution is $a = 0$ for $\alpha = \frac{1}{8\pi y_3}$. Though 0 is not an eigenvalue because the corresponding candidate eigenfunction

$$\psi_0(x, y) = \frac{1}{4\pi\,|x - y|} + \frac{1}{4\pi\,|x - \overline{y}|}$$

is not in $L^2(\mathbb{H}^3)$. The positive sign in $h^{\mathbb{H}^3,\mathrm{N}}_{0,y}$ does not allow the partial cancellation that allowed for sufficiebtly fast decay in the Dirichlet case.

Moreover, since f is decreasing, g is increasing and $f(0) = \frac{1}{2y_3}$, the solution must obey

$$-4\pi\alpha < a \leq \frac{1}{2y_3} - 4\pi\alpha.$$

If $\alpha < 0$ this condition is equivalent to (2.36) for λ. Instead, if $\alpha \geq 0$, since the function $p(x) = -x^2$ has a maximum in 0, then $\lambda = -a^2$ takes values in

$$-\left(\frac{1}{2y_3} - 4\pi\alpha\right)^2 \leq \lambda < 0.$$

□

Remark. As for the Dirichlet case, the negativity of the quadratic form on $G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}$ is a necessary condition for it to be an eigenfunction relative to $\lambda < 0$. We verify that this condition holds. By $z^2 = \lambda = -|\lambda|$, (1.52) and (1.53), in fact we have

$$\mathrm{N}^{\mathbb{H}^3}_{\alpha,y}\left(G^{\mathbb{H}^3,\mathrm{N}}_{\sqrt{\lambda},y}, G^{\mathbb{H}^3,\mathrm{N}}_{\sqrt{\lambda},y}\right) = -|\lambda| \int_{\mathbb{R}^2\times\mathbb{R}^+} \left|\frac{e^{-\sqrt{|\lambda|}|x-y|}}{4\pi\,|x-y|} + \frac{e^{-\sqrt{|\lambda|}|x-\overline{y}|}}{4\pi\,|x-\overline{y}|}\right|^2 \mathrm{d}x + \alpha + \frac{\sqrt{|\lambda|}}{4\pi} - \frac{e^{-\sqrt{|\lambda|}|y-\overline{y}|}}{4\pi\,|y-\overline{y}|}.$$

Equations (2.33), (2.34) and (2.35) and simple computations lead to

$$\mathrm{N}^{\mathbb{H}^3}_{\alpha,y}\left(G^{\mathbb{H}^3,\mathrm{N}}_{\sqrt{\lambda},y}, G^{\mathbb{H}^3,\mathrm{N}}_{\sqrt{\lambda},y}\right) = \alpha + \frac{\sqrt{|\lambda|}}{8\pi} - \frac{e^{-2\sqrt{|\lambda|}y_3}}{8\pi}\left(\frac{1}{y_3} + \sqrt{|\lambda|}\right).$$

We set $\sqrt{|\lambda|} = a > 0$ and define

$$n(a) = \alpha + \frac{a}{8\pi} - \frac{e^{-2ay_3}}{8\pi}\left(\frac{1}{y_3} + a\right),$$

whose derivative is

$$\frac{\mathrm{d}n}{\mathrm{d}a}(a) = \frac{1}{8\pi} + \frac{e^{-2ay_3}}{8\pi}(1 + 2ay_3) \geq 0,$$

which is positive $\forall a \geq 0$. Then, n attain its minimum for $a = 0$. But

$$n(0) = \alpha - \frac{1}{8\pi y_3}$$

and so eigenvalues can exist only when $\alpha \leq \frac{1}{8\pi y_3}$, which agrees with the condition for the existence of eigenvalues for $-\Delta^{\Omega,\mathrm{N}}_{\alpha,y}$.

2.2.3. Robin Boundary Condition

Before the study of the eigenvalue for the Robin condition, we discuss some preliminary facts about the Robin Green's function on $\mathbb{H}^3$.

The Green's function on $\mathbb{H}^3$ for the Robin problem is given by the formula (see the analysis by Keller [29])

$$G^{\mathbb{H}^3,\mathrm{R},\eta}_{z,y}(x) = \frac{e^{iz|x-y|}}{4\pi\,|x-y|} - \frac{e^{iz|x-\overline{y}|}}{4\pi\,|x-\overline{y}|} - 2\frac{\partial}{\partial x_3}\int_0^{+\infty} e^{-\eta s}\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\,\mathrm{d}s. \tag{2.40}$$

Here $\overline{y}$ is the usual reflected of y through the half-plane $x_3 = 0$ and $b = (0, 0, 1)$. In order for this expression to be actually meaningful, we have to ask the integral to be finite. In this case this condition is equivalent to ask that

$$\lim_{s\to+\infty} e^{-\eta s}\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|} = 0, \tag{2.41}$$

and this happens if and only if $\mathrm{Im}\, z > -\eta$. This means that for a fixed η, this representation is only valid for $\mathrm{Im}\, z > -\eta$.

We want to obtain a more useful expression for the function

$$h^{\mathbb{H}^3,\mathrm{R},\eta}_{z,y}(x) = \frac{e^{iz|x-\overline{y}|}}{4\pi\,|x-\overline{y}|} + 2\frac{\partial}{\partial x_3}\int_0^{+\infty} e^{-\eta s}\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\,\mathrm{d}s,$$

because $h^{\mathbb{H}^3,\mathrm{R},\eta}_{z,y}(y)$ appears in $\Gamma^{\mathbb{H}^3,\mathrm{R},\eta}_{\alpha,y}(z)$. The presence of the derivative makes it difficult to evaluate the function in y. In order to get rid of the partial derivative, we would like to bring it under the integral sign. While doing so, since what we are really interested in is just $h^{\mathbb{H}^3,\mathrm{R},\eta}_{z,y}(y)$, we can just assume that $x \in B_r(y)$ for some $0 < r < y_3$. Now we can claim that

- (2.41) guarantees integrability.
- Since $|x - \overline{y} + bs|^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 + y_3 + s)^2$ and $0 < y_3 - r \leq x_3 \leq y_3 + r$, $y_3 > 0$ and $s \geq 0$, the argument of the absolute values functions in the integrand is always greater than zero. This means that the integrand is smooth and so in particular that $\forall s \geq 0$ its partial derivative with respect to x_3 exists and is continuous $\forall x_3 \in (y_3 - r, y_3 + r)$

- We compute

$$\frac{\partial}{\partial x_3}\left(e^{-\eta s}\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\right)=\frac{\Big(iz\,|x-\overline{y}+bs|-1\Big)(x_3+y_3+s)}{4\pi\,|x-\overline{y}+bs|^3}e^{iz|x-\overline{y}+bs|-\eta s}$$

and estimate

$$\left|\frac{\partial}{\partial x_3}\left(e^{-\eta s}\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\right)\right|\leq e^{-\operatorname{Im}z|x-\overline{y}+bs|-\eta s}\Big(|z|\,|x-\overline{y}+bs|^{-1}+|x-\overline{y}+bs|^{-2}\Big).$$

By simple geometrical considerations we have that $\max_{x\in B_r(y)}|x-\overline{y}+bs|=2y_3+r+s$ and $\min_{x\in B_r(y)}|x-\overline{y}+bs|=2y_3-r+s$. Then

$$e^{-\operatorname{Im}z|x-\overline{y}+bs|}\leq\begin{cases}e^{-\operatorname{Im}z(2y_3-r+s)} & \text{for }\operatorname{Im}z\geq 0\\ e^{-\operatorname{Im}z(2y_3+r+s)} & \text{for }\operatorname{Im}z<0\end{cases}=e^{-\operatorname{Im}z(2y_3\mp\operatorname{sgn}(\operatorname{Im}z)r+s)},$$

from which follows

$$\left|\frac{\partial}{\partial x_3}\left(e^{-\eta s}\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\right)\right|\leq e^{-\operatorname{Im}z(2y_3\mp\operatorname{sgn}(\operatorname{Im}z)r)}e^{-(\operatorname{Im}z+\eta)s}\Big(|z|\,(s+2y_3-r)^{-1}+(s+2y_3-r)^{-2}\Big),$$

which is integrable because $\operatorname{Im}z+\eta>0$ and $0<r<y_3$.

All this considered, by dominated convergence theorem we conclude that

$$\frac{\partial}{\partial x_3}\int_0^{+\infty}e^{-\eta s}\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\,\mathrm{d}s=\int_0^{+\infty}e^{-\eta s}\frac{\partial}{\partial x_3}\left(\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\right)\mathrm{d}s. \tag{2.42}$$

We can notice that the function argument of the partial derivative depends on x_3 only through terms of the form x_3+y_3+s. This means that

$$\frac{\partial}{\partial x_3}\left(\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\right)=\frac{\partial}{\partial s}\left(\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\right).$$

So we can use integration by parts to rewrite (2.42) as

$$\frac{\partial}{\partial x_3}\int_0^{+\infty}e^{-\eta s}\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\,\mathrm{d}s=\left[e^{-\eta s}\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\right]_0^{+\infty}+\eta\int_0^{+\infty}e^{-\eta s}\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\,\mathrm{d}s.$$

The condition $\eta+\operatorname{Im}z>0$ implies both that the upper evaluation of the first term is zero and that the second term is finite. $h_{z,y}^{\mathbb{H}^3,\mathrm{R},\eta}$ is then

$$h_{z,y}^{\mathbb{H}^3,\mathrm{R},\eta}(x)=-\frac{e^{iz|x-\overline{y}|}}{4\pi\,|x-\overline{y}|}+2\eta\int_0^{+\infty}e^{-\eta s}\frac{e^{iz|x-\overline{y}+bs|}}{4\pi\,|x-\overline{y}+bs|}\,\mathrm{d}s$$

and now we can evaluate it on y

$$h_{z,y}^{\mathbb{H}^3,\mathrm{R},\eta}(y) = -\frac{e^{2iy_3 z}}{8\pi y_3} + 2\eta \int_0^{+\infty} e^{-\eta s} \frac{e^{iz(2y_3+s)}}{4\pi(2y_3+s)} \,\mathrm{d}s.$$

The substitution $t = 2y_3 s$ in the integral, combined with the last equality in (C.2) implies

$$\int_0^{+\infty} e^{-\eta s} \frac{e^{iz(2y_3+s)}}{4\pi(2y_3+s)} \,\mathrm{d}s = \frac{e^{2iy_3 z}}{4\pi} \int_0^{+\infty} \frac{e^{-2y_3(\eta - iz)t}}{t+1} \,\mathrm{d}t = \frac{e^{2y_3\eta}}{4\pi} E_1(2y_3(\eta - iz))$$

(E_1 is the exponential integral which is defined in (C.1)) and finally we can write $h_{z,y}^{\mathbb{H}^3,\mathrm{R},\eta}(y)$ as

$$h_{z,y}^{\mathbb{H}^3,\mathrm{R},\eta}(y) = -\frac{e^{2iy_3 z}}{8\pi y_3} + \frac{\eta}{2\pi} e^{2y_3\eta} E_1(2y_3(\eta - iz)).$$

Remark. It is interesting to notice that in the limit $\eta \to +\infty$ and $\eta \to 0^+$ we obtain $h_{z,y}^{\mathbb{H}^3,\mathrm{D}}(y)$ and $h_{z,y}^{\mathbb{H}^3,\mathrm{N}}(y)$ respectively. This is evident for $\eta \to 0^+$. Instead for the other case we use (C.3), so that

$$\begin{aligned}
h_{z,y}^{\mathbb{H}^3,\mathrm{R},\eta}(y) &= -\frac{e^{2iy_3 z}}{8\pi y_3} + \frac{\eta}{2\pi} e^{2y_3\eta} \left(\frac{e^{-2y_3(\eta - iz)}}{2y_3(\eta - iz)} + O\left(\frac{e^{-2y_3\eta}}{\eta^2} \right) \right) \\
&= -\frac{e^{2iy_3 z}}{8\pi y_3} + \frac{\eta}{2\pi} e^{2y_3\eta} \left(\frac{e^{-2y_3(\eta - iz)}}{2y_3\eta} \left(1 + \frac{iz}{\eta} + O\left(\frac{1}{\eta^2} \right) \right) + O\left(\frac{e^{-2y_3\eta}}{\eta^2} \right) \right) \\
&= -\frac{e^{2iy_3 z}}{8\pi y_3} + \frac{e^{2iy_3 z}}{4\pi y_3} + O\left(\frac{e^{-2y_3\eta}}{\eta^2} \right) = \frac{e^{2iy_3 z}}{8\pi y_3} + O\left(\frac{e^{-2y_3\eta}}{\eta^2} \right).
\end{aligned}$$

Now we can prove the following.

Proposition 2.2.3. *The operator $-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{R},\eta}$ with $\eta > 0$ has*

$$\alpha_{\mathrm{c}} = \frac{1}{8\pi y_3} - \frac{\eta}{2\pi} e^{2y_3\eta} E_1(2y_3\eta), \tag{2.43}$$

where E_1 is the exponential integral defined in (C.1). $\forall \eta > 0$ the critical value α_{c} lies in $\left(-\frac{1}{8\pi y_3}, \frac{1}{8\pi y_3} \right)$.

There is at most a single eigenvalue λ and it is non-positive. The corresponding eigenfunction is

$$\psi_\lambda(x, y) = \frac{e^{-\sqrt{|\lambda|}|x-y|}}{4\pi|x-y|} - \frac{e^{-\sqrt{|\lambda|}|x-\bar{y}|}}{4\pi|x-\bar{y}|} - 2\frac{\partial}{\partial x_3} \int_0^{+\infty} e^{-\eta s} \frac{e^{-\sqrt{|\lambda|}|x-\bar{y}+bs|}}{4\pi|x-\bar{y}+bs|} \,\mathrm{d}s \tag{2.44}$$

($b = (0, 0, 1)$).

Remark. We notice that the last part of Proposition 2.2.3 implies that $\lambda^{\mathrm{R},\eta} < \lambda^{\mathrm{D}}$, if $\lambda^{\mathrm{R},\eta}$ and λ^{D} are the eigenvalues of $-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{R},\eta}$ and $-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}$ respectively. (c.f. with Theorem 3.2. in [7]).

Proof. The equation for eigenvalues is

$$\alpha - \frac{iz}{4\pi} - \frac{e^{2iy_3 z}}{8\pi y_3} + \frac{\eta}{2\pi} e^{2y_3\eta} E_1(2y_3(\eta - iz)) = 0, \quad \operatorname{Im} z > 0. \tag{2.45}$$

Since we know there exists at most an eigenvalue and that it is negative, we substitute $z = ia$ with $a > 0$. (2.45) becomes

$$f(a) = \alpha + \frac{a}{4\pi} - \frac{e^{-2y_3 a}}{8\pi y_3} + \frac{\eta}{2\pi} e^{2y_3\eta} E_1(2y_3(\eta + a)) = 0.$$

Since there exists at most a solution in $[0, +\infty)$, studying the values of the continuous function f at the extrema of the interval is sufficient to conclude whether or not a solution exists. It holds that $f(0) = \alpha - \frac{1}{8\pi y_3} + \frac{\eta}{2\pi} e^{2y_3\eta} E_1(2y_3\eta)$ and that $\lim_{a\to+\infty} f(a) = \lim_{a\to+\infty} \alpha + \frac{a}{4\pi} = +\infty$ and so an eigenvalue exists only if $\alpha < \frac{1}{8\pi y_3} - \frac{\eta}{2\pi} e^{2y_3\eta} E_1(2y_3\eta)$.

This critical value is always between the corresponding critical values for the Dirichlet and Neumann problems. Let $g(\eta) = \frac{\eta}{2\pi} e^{2y_3\eta} E_1(2y_3\eta)$. $g(0) = 0$ (the corresponding critical value being $\frac{1}{8\pi y_3}$) while

$$\lim_{\eta\to+\infty} g(\eta) = \frac{\eta}{2\pi} e^{2y_3\eta} E_1(2y_3\eta) = \lim_{\eta\to+\infty} \frac{\eta}{2\pi} e^{2y_3\eta} \frac{e^{-2y_3\eta}}{2y_3\eta} = \frac{1}{4\pi y_3}$$

(which corresponds to $-\frac{1}{8\pi y_3}$). If we prove that g is monotone increasing, the claim follows. It holds that

$$g'(\eta) = \frac{e^{2y_3\eta}(1 + 2y_3\eta) E_1(2y_3\eta) - 1}{2\pi}.$$

Setting $\rho = 2y_3\eta$, we want $e^{\rho}(1+\rho)E_1(\rho) > 1$ to hold $\forall \rho > 0$. By recalling the inequalities (C.5), it follows that, if

$$\frac{1}{2} \ln\left(1 + \frac{2}{\rho}\right) > \frac{1}{\rho + 1}, \tag{2.46}$$

a fortiori $g'(\eta) > 0$ $\forall \eta > 0$ holds as well. Let $q(\rho) = \frac{1}{2}\ln\left(1 + \frac{2}{\rho}\right) - \frac{1}{\rho+1}$. It's easy to verify that q is monotone decreasing in $(0, +\infty)$, which implies that there is at most one root for q. Moreover $\lim_{\rho\to0^+} q(\rho) = +\infty$. Expanding both sides of (2.46) for $\rho \to +\infty$, we get

$$\frac{1}{2}\ln\left(1 + \frac{2}{\rho}\right) = \frac{1}{\rho} - \frac{1}{\rho^2} + \frac{4}{3\rho^3} + O\left(\frac{1}{\rho^4}\right) \quad \text{and} \quad \frac{1}{1+\rho} = \frac{1}{\rho} - \frac{1}{\rho^2} + \frac{1}{\rho^3} + O\left(\frac{1}{\rho^4}\right).$$

This implies that $q > 0$ in a neighborhood of infinity, so q has no roots and as a consequence g is monotone increasing.

Remark. The critical value α_c for $-\Delta^{\mathbb{H}^3}_{\alpha,y}$, regardless of the boundary conditions, as $y_3 \to +\infty$ goes to zero, which is the critical value for a point interaction in $\mathbb{R}^3$. This fact confirms the intuitive idea that a point interaction very far from the boundary behaves as if it were in the whole space.

□

2.3. The Half-Plane

Now we consider the half-plane $\mathbb{H}^2 = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; x_2 > 0 \right\}$. The point source is in $y = (y_1, y_2) \in \mathbb{H}^2$.

2.3.1. Dirichlet Boundary Condition

Let $\overline{y} = (y_1, -y_2)$. As in the three dimensional case, a reflection works. So, using (A.1), then

$$h_{z,y}^{\mathbb{H}^2,\mathrm{D}}(x) = \frac{i}{4} H_0^{(1)}\big(z\,|x - \overline{y}|\big)$$

The following holds.

Proposition 2.3.1. *The operator* $-\Delta_{\alpha,y}^{\mathbb{H}^2,\mathrm{D}}$ *has* $\alpha_c = \ln(2y_2)/(2\pi)$. *There is at most a single eigenvalue, which lies in* $[-4e^{-4\pi\alpha - 2\gamma}, 0)$ *and the corresponding eigenfunction is*

$$\psi_\lambda(x, y) = \frac{1}{2\pi} K_0\Big(\sqrt{|\lambda|}\,|x - y|\Big) - \frac{1}{2\pi} K_0\Big(\sqrt{|\lambda|}\,|x - \overline{y}|\Big), \tag{2.47}$$

where K_0 *is the modified Bessel function of order zero.*

Proof. By (1.45), the eigenvalues corresponds to the real solutions of

$$\alpha + \frac{\ln \frac{z}{2i} + \gamma}{2\pi} + \frac{i}{4} H_0^{(1)}(2y_2 z) = 0. \tag{2.48}$$

There are no positive eigenvalues, hence we let $z = ia$, with $a > 0$ and look for negative ones. Using (C.18) with $\nu = 0$, (2.48) becomes

$$K_0(2y_2 a) = -2\pi\alpha - \gamma - \ln \frac{a}{2}. \tag{2.49}$$

We name $f(a)$ and $g(a)$ the first and second member of the equation. $f(a) > 0, \forall a > 0$ and so, if a solution exists it must lie in the region where $g(a) > 0$. This happens for $a < 2e^{-2\pi\alpha-\gamma}$ and $f(2e^{-2\pi\alpha-\gamma}) > g(2e^{-2\pi\alpha-\gamma}) = 0$. Let's consider now the behaviour of both functions for $a \to 0$. Both functions diverge to $+\infty$ going towards 0 and this does not allow us to conclude much. A sufficient condition to claim the existence of a solution would be to show that g is above f in a right neighbourhood of 0. Recalling (C.22) we have

$$f(a) = -\ln a - \ln y_2 - \gamma + O(a^2 \ln a).$$

Since both f and g have the same order of magnitude for $a \to 0$ (note that the only diverging terms cancels out, and so the left hand side of (2.48) is continuous in 0, and then 0 is not a priori exclusded as a possible solution) and the vanishing terms in f are smaller than any finite constant for a sufficiently small, we can impose $g(a) > f(a)$ in a right neighbourhood of 0 by an inequality on the coefficients

$$-2\pi\alpha - \gamma + \ln 2 > -\ln y_2 - \gamma, \tag{2.50}$$

equivalent to $\alpha < \ln(2y_2)/(2\pi)$.

If $\alpha = \ln(2y_2)/(2\pi)$ all the the constant term in (2.50) cancel out. This means that 0 is a solution of the equation for such value of α. We then consider the following term in the expansion at $a = 0$. This term is

$$-y_2^2 \ln(ay_2)a^2$$

that is positive in a right neighborhood of $a = 0$ and then $f > g$ in it and so an even number of solutions (possibly zero) exists in $[0, 2e^{-2\pi\alpha-\gamma}]$.

We now consider the derivatives of both f and g. Since $K_0'(w) = -K_1(w)$, we have that

$$f'(a) = -2y_2 K_1(2y_2 a) \quad \text{and} \quad g'(a) = -\frac{1}{a}.$$

If $f'(a) > g'(a)\ \forall a \in (0, 2e^{-2\pi\alpha-\gamma}]$ the solution, if exists, must be unique. Letting $u = 2y_2 a > 0$, the inequality becomes

$$K_1(u) < \frac{1}{u}. \tag{2.51}$$

(C.26), the positivity of I_n and K_n and the fact that $I_0 \geq 1$ for positive argument, imply that (2.51) is verified $\forall u > 0$. Then $f'(a) > g'(a)\ \forall a > 0$ and so (2.49) has at most a solution in $[0, 2e^{-2\pi\alpha-\gamma}]$. This also mean that for $\alpha = \ln(2y_2)/(2\pi)$ there are no solutions other than $a = 0$.

The eigenfunction is obtained combining $(A.1)$, (2.47), $z^2 = \lambda = -|\lambda|$ and (C.18) with $\nu = 0$.

We state that the candidate eigenfunction

$$\psi_0(x, y) = -\frac{1}{2\pi} \ln\left(\frac{|x - y|}{|x - \overline{y}|}\right)$$

is not in $L^2(\mathbb{H}^2)$. We note that the logarithmic singularity at $x = y$ is square integrable and that $\lim_{|x|\to+\infty} \psi_0(x,y) = 0$. We claim that the decay is too slow. In fact

$$\begin{aligned}\psi_0(x,y) &= -\frac{1}{4\pi}\ln\left(\frac{|x-y|^2}{|x-\overline{y}|^2}\right) = -\frac{1}{4\pi}\ln\left(\frac{(x_1-y_1)^2+(x_2-y_2)^2}{(x_1-y_1)^2+(x_2+y_2)^2}\right)\\ &= -\frac{1}{4\pi}\ln\left(1-\frac{4y_2x_2}{(x_1-y_1)^2+(x_2+y_2)^2}\right).\end{aligned}$$

Using the polar coordinates

$$\begin{cases} x_1 - y_1 = \rho\cos\theta \\ x_2 = \rho\sin\theta \end{cases} \qquad \rho\in[0,+\infty), \quad \theta\in[0,\pi],$$

we rewrite ψ_0 as follows

$$\psi_0(\rho,\theta) = -\frac{1}{4\pi}\ln\left(1-\frac{4y_2\rho\sin\theta}{\rho^2\cos^2\theta+(\rho\sin\theta+y_2)^2}\right).$$

Excluding the cases $\theta = 0$ or $\theta = \pi$, where the function is identically zero (according to the boundary condition imposed), we estimate its behaviour for $\rho \to +\infty$.

$$\psi_0(\rho,\theta) \sim \frac{1}{4\pi}\frac{4y_2\rho\sin\theta}{\rho^2} = \frac{y_2}{\pi}\frac{\sin\theta}{\rho}$$

Then

$$|\psi_0(\rho,\theta)|^2 \sim \frac{y_2^2}{\pi^2}\frac{\sin^2\theta}{\rho^2}, \quad \text{for} \quad \rho\to+\infty,$$

which decays too slowly to be integrable with respect to the measure $\rho\,\mathrm{d}\rho$ in $(R,+\infty)$ for some $R > 0$. □

Remark. We rewrite (2.49) isolating α

$$\alpha = -\frac{1}{2\pi}K_0(2y_2a) - \frac{\ln\frac{a}{2}+\gamma}{2\pi}.$$

We let $\alpha \to -\infty$, and study how the corresponding solution behaves. Since for a approaching zero from the right, the two singularities cancel out and the limit is finite, there are no finite point to which a can go as $\alpha \to -\infty$. Instead, for $a \to +\infty$ the first term goes to $-\infty$, while the latter vanishes. Hence we have that $\lambda \to -\infty$ for $\alpha \to -\infty$.

Remark. Also in this case we check the result by verifying that the quadratic form is negative on the eigenfunction.

$$\mathrm{D}^{\mathbb{H}^3}_{\alpha,y}\Big(G^{\mathbb{H}^2,\mathrm{D}}_{\sqrt{\lambda},y}, G^{\mathbb{H}^2,\mathrm{D}}_{\sqrt{\lambda},y}\Big) < \alpha + \frac{1}{2\pi}\Big(\gamma + \ln\frac{\sqrt{|\lambda|}}{2} + K_0(2y_2\sqrt{|\lambda|})\Big),$$

since by definition the integral part of the quadratic form evaluated on the Green's function of the domain is negative. Let's substitute $\sqrt{|\lambda|} = a > 0$ in the upper bound above and let's name it $d(a)$. If d is negative on some non-zero measure subset of $[0, +\infty)$, then same holds for the quadratic form. Let's examine $d(a)$ for $a \to 0^+$:

$$\lim_{a\to 0^+} d(a) = \lim_{a\to 0^+} \alpha + \frac{1}{2\pi}\Big(\gamma + \ln\frac{a}{2} - \ln(y_2 a) - \gamma\Big) = \alpha - \frac{\ln(2y_2)}{2\pi}$$

((C.22) has been used). So the quadratic form is negative in a right neighbourhood of 0 for $\alpha \leq \frac{\ln(2y_2)}{2\pi}$, which is the same condition found directly looking for the eigenvalues.

2.3.2. Neumann Boundary Condition

The same reasoning of the 3d case, together with (A.1) let us conclude that

$$h^{\mathbb{H}^2,\mathrm{N}}_{z,y}(x) = -\frac{i}{4}H^{(1)}_0\big(z\,|x-\bar{y}|\big)$$

is the solution of (A.6) for $\Omega = \mathbb{H}^2$.

Proposition 2.3.2. *The operator* $-\Delta^{\mathbb{H}^2,\mathrm{N}}_{\alpha,y}$ *has* $\alpha_c = +\infty$*. Regardless of* $\alpha \in \mathbb{R}$ *and* $y_2 > 0$ *there is always a single eigenvalue* λ *and it lies in* $(-\infty, -4e^{-4\pi\alpha-2\gamma})$*. The corresponding eigenfunction is*

$$\psi_\lambda(x,y) = \frac{1}{2\pi}K_0\big(\sqrt{|\lambda|}\,|x-y|\big) + \frac{1}{2\pi}K_0\big(\sqrt{|\lambda|}\,|x-\bar{y}|\big), \tag{2.52}$$

where K_0 *is the modified Bessel function of order zero.*

Proof. The possible negative eigenvalue is z^2, where z is the solution of $\Gamma^{\mathbb{H}^2,\mathrm{N}}_{\alpha,y}(z) = 0$, after the substitution $z = ia$, $a > 0$. $a = 0$ is escluded from possible solutions because $\lim_{a\to 0^+}\Gamma^{\mathbb{H}^2,\mathrm{N}}_{\alpha,y}(ia) = +\infty$ and so the function $\Gamma^{\mathbb{H}^2,\mathrm{N}}_{\alpha,y}$ cannot be defined continuously in zero. The equation is ((C.18) has been used)

$$K_0(2y_2 a) = 2\pi\alpha + \gamma + \ln\frac{a}{2}. \tag{2.53}$$

We name $f(a)$ the former side of the equation and the latter side $g(a)$. We observe that:

- both functions are smooth for $a > 0$;
- $\lim_{a\to 0^+} f(a) = +\infty$ and $\lim_{a\to 0^+} g(a) = -\infty$;
- $\lim_{a\to +\infty} f(a) = 0$ and $\lim_{a\to +\infty} g(a) = +\infty$.

Those facts allows us to claim that there necessarily exists a solution for $a > 0$. Moreover we have that $f(a) > 0\ \forall a > 0$, while $g(a) > 0$ for $a > 2e^{-2\pi\alpha-\gamma}$. The solution must then lie in $(2e^{-2\pi\alpha-\gamma}, +\infty)$.

The solution is unique. To prove it we look at the derivatives of both functions:

$$f'(a) = -2y_2K_1(2y_2a) \qquad g'(a) = \frac{1}{a}.$$

If $f'(a) \leq g'(a)$ $\forall a \in (2e^{-2\pi\alpha-\gamma}, +\infty)$ the claim is proven. The inequality is written as $-2y_2K_1(2y_2a) \leq 1/a$, which, posing $u = 2y_2a$, is equivalent to

$$K_1(u) \geq -\frac{1}{u},$$

verified $\forall u > 0$ (and subsequently $\forall a > 0$) because the left side is positive and the right side is negative.

The eigenfunction is obtained combining $(A.1)$, (2.52), $z^2 = \lambda = -|\lambda|$ and (C.18) with $\nu = 0$. □

Remark. We verify that, regardless of the values of α and y_2, the quadratic form evaluated on $G^{\mathbb{H}^2,\mathrm{N}}_{\sqrt{\lambda},y}$ is negative. In the same way as in the Dirichlet case we have that

$$\mathrm{N}^{\mathbb{H}^3}_{\alpha,y}\Big(G^{\mathbb{H}^2,\mathrm{N}}_{\sqrt{\lambda},y}, G^{\mathbb{H}^2,\mathrm{N}}_{\sqrt{\lambda},y}\Big) < \alpha + \frac{1}{2\pi}\Big(\gamma + \ln\frac{\sqrt{|\lambda|}}{2} - K_0\Big(2y_2\sqrt{|\lambda|}\Big)\Big).$$

The term in parenthesys on the right side diverges to $-\infty$ for $\lambda \to 0^+$ regardless of $y_2 > 0$ (see (C.22)). This implies that the quadratic form is somewhere negative in $[0, +\infty)$ $\forall\alpha$ and $\forall y_2 > 0$.

Remark. Also in the Neumann case, by rewriting (2.53) isolating α as follows

$$\alpha = \frac{1}{2\pi}K_0(2y_2a) - \frac{\ln\frac{a}{2} + \gamma}{2\pi},$$

we can determine how the eigenvalue behaves for α large. In this case also $\alpha \to +\infty$ admits an eigenvalue. Both terms goes to $+\infty$ for $a \to 0^+$ and hence $\lambda \to 0$ for $\alpha \to +\infty$. As in the Dirichlet case, the function tends to $-\infty$ only for $a \to +\infty$ and so for $\alpha \to +\infty$ one has $\lambda \to -\infty$.

2.4. Exterior of a Disk

In this section $\Omega = \{x \in \mathbb{R}^2 | \, |x| > R\}$. The point interaction location $y = (y_1, y_2)$ will be described through the polar coordinates ρ and ϕ, while x by r and θ.

2.4.1. Dirichlet Boundary Condition

The Green's function for $-\Delta - z^2$ on Ω can be proven to be equal to (see for example [45])

$$G^{\Omega,\mathrm{D}}_{z,y}(x) = \frac{i}{4}\sum_{n=-\infty}^{+\infty} e^{in(\theta-\phi)}\Big(J_n(zr_<) - \frac{J_n(zR)}{H^{(1)}_n(zR)}H^{(1)}_n(zr_<)\Big)H^{(1)}_n(zr_>), \tag{2.54}$$

where $r_< = \min(r,\rho)$ and $r_> = \max(r,\rho)$ and $H^{(1)}_n$ is the Hankel function of the first kind of order n. By (C.8) and (C.11) comes that $J_{-n}(u)H^{(1)}_{-n}(v) = J_n(u)H^{(1)}_n(v)$ for $u, v \in \mathbb{C}$. Hence we can rewrite

$$\sum_{n=-\infty}^{+\infty} e^{in(\theta-\phi)}J_n(zr_<)H^{(1)}_n(zr_>) = \sum_{n=-\infty}^{+\infty} \cos(n(\theta-\phi))J_n(zr_<)H^{(1)}_n(zr_>),$$

which is equal to $H^{(1)}_0\Big(z\sqrt{r^2+\rho^2-2r\rho\cos(\theta-\phi)}\Big) = H^{(1)}_0\big(z\,|x-y|\big)$ because of (C.13). So we can rewrite (2.54) in the form

$$\begin{aligned} G^{\Omega,\mathrm{D}}_{z,y}(x) &= G^{0,2}_z(x,y) - \frac{i}{4}\sum_{n=-\infty}^{+\infty} e^{in(\theta-\phi)}\frac{J_n(zR)}{H^{(1)}_n(zR)}H^{(1)}_n(zr_<)H^{(1)}_n(zr_>) \\ &= G^{0,2}_z(x,y) - h^{\Omega,\mathrm{D}}_{z,y}(x). \end{aligned} \tag{2.55}$$

Hence

$$h^{\Omega,\mathrm{D}}_{z,y}(x) = \frac{i}{4}\sum_{n=-\infty}^{+\infty} e^{in(\theta-\phi)}\frac{J_n(zR)}{H^{(1)}_n(zR)}H^{(1)}_n(zr_<)H^{(1)}_n(zr_>),$$

and

$$h^{\Omega,\mathrm{D}}_{z,y}(y) = \frac{i}{4}\sum_{n=-\infty}^{+\infty} \frac{J_n(zR)}{H^{(1)}_n(zR)}\Big(H^{(1)}_n(z\rho)\Big)^2. \tag{2.56}$$

Proposition 2.4.1. *The operator $-\Delta^{\Omega,\mathrm{D}}_{\alpha,y}$ with $|y| = \rho$ and Ω being the exterior of a disk of radius R with Dirichlet boundary conditions has $\alpha_\mathrm{c} = -\frac{1}{2\pi}\ln\left(\frac{R}{\rho^2-R^2}\right)$ (α_c is not attained though). When existent, the eigenvalue $\lambda < 0$ is unique and the relative eigenfunction is*

$$\psi_\lambda(x,y) = \frac{1}{2\pi}K_0\big(\sqrt{|\lambda|}\,|x-y|\big) - \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{in(\theta-\phi)}\frac{I_n(\sqrt{|\lambda|}R)}{K_n(\sqrt{|\lambda|}R)}K_n(\sqrt{|\lambda|}r_<)K_n(\sqrt{|\lambda|}r_>). \tag{2.57}$$

Proof. The eigenvalue z^2, which is unique and non-positive, is such that z is a solution of

$$\alpha + \frac{\gamma + \ln\frac{z}{2i}}{2\pi} + \frac{i}{4}\sum_{n=-\infty}^{+\infty} \frac{J_n(zR)}{H^{(1)}_n(zR)}\Big(H^{(1)}_n(z\rho)\Big)^2 = 0. \tag{2.58}$$

If we substitute $z = ia$, with $a > 0$ in (2.58) and use (C.17) and (C.18), it becomes

$$f(a) = \alpha + \frac{\gamma + \ln\frac{a}{2}}{2\pi} + \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{I_n(aR)}{K_n(aR)}\Big(K_n(a\rho)\Big)^2 = 0. \tag{2.59}$$

We now study the solution of this equation in $(0, +\infty)$. We start looking at the behaviour of the different terms for $a \to 0^+$. The logaritmic term diverges to $-\infty$. To study the sum we rewrite it as

$$\sum_{n=-\infty}^{+\infty}\frac{I_n(aR)}{K_n(aR)}\Big(K_n(a\rho)\Big)^2 = \frac{I_0(aR)}{K_0(aR)}\Big(K_0(a\rho)\Big)^2 + \frac{1}{\pi}\sum_{n=1}^{+\infty}c_n(a),$$

with $c_n(a) = \frac{I_n(aR)}{K_n(aR)}\Big(K_n(a\rho)\Big)^2$.We start considering the zeroth term. By (C.19) we have that $I_0(w) = 1 + O(w^2)$ for $w \to 0$. Then, (C.21), paired with some Landau symbols algebra, allows us to write

$$\frac{I_0(aR)}{K_0(aR)}\Big(K_0(a\rho)\Big)^2 = -\ln a + \ln\frac{2R}{\rho^2} - \gamma + O\left(\frac{1}{\ln a}\right) \quad \text{for} \quad a \to 0.$$

We are left to study the infinite sum in the limit $a \to 0$. By (C.23) follows

$$c_n(a) \sim \frac{1}{2n}\left(\frac{R}{\rho}\right)^{2n} \quad a \to 0 \tag{2.60}$$

that is the general term of a summable sequence ($\rho > R$ by hypotheses). In addition (C.31) and (C.32)allows us to get the bound

$$c_n(a) < \frac{\cosh(aR)e^{aR}}{2n}\left(\frac{R}{\rho}\right)^{2n} = \frac{e^{2aR}+1}{4n}\left(\frac{R}{\rho}\right)^{2n} < \frac{e^{2MR}+1}{4n}\left(\frac{R}{\rho}\right)^{2n},$$

where the last inequality is valid for $a < M$ for some $M > 0$ (which is sufficient, since we are looking for the limit towards 0). Moreover this also implies that the sum is continuous in every interval $(0, M)$ with $M > 0$. So for sufficiently small $a > 0$, $c_n(a)$ is bounded by a summable sequence which does not depend on a. By this and (2.60), by dominated convergence we can claim that

$$\lim_{a\to 0^+}\sum_{n=1}^{+\infty}c_n(a) = \sum_{n=1}^{+\infty}\lim_{a\to 0^+}c_n(a) = \sum_{n=1}^{+\infty}\frac{1}{2n}\left(\frac{R}{\rho}\right)^{2n} = -\frac{1}{2}\ln\left(1-\left(\frac{R}{\rho}\right)^2\right) < +\infty.$$

Putting all this together, we have that

$$\lim_{a\to 0^+} f(a) = \alpha + \frac{1}{2\pi}\ln\left(\frac{R}{\rho^2 - R^2}\right). \tag{2.61}$$

On the other side f goes to $+\infty$ for $a \to +\infty$. That's because:

- $\lim_{a\to+\infty} \ln \frac{a}{2} = +\infty$;
- I_n and K_n are positive on $(0, +\infty)$.

Now, since f is continuous in $(0, M)$, $\forall M > 0$, we can determine whether the solution exists or not just by the nature of the sign at the extrema of the interval. Since there is an M large enough such that $f(M) > 0$, then the equation $f(a) = 0$ has at least one solution in this interval only if

$$\alpha < -\frac{1}{2\pi} \ln\left(\frac{R}{\rho^2 - R^2}\right) = \alpha_{\mathrm{c}}.$$

The fact that this possible solution is unique is a consequence of $-\Delta^{\Omega,\mathrm{D}}_{\alpha,y}$ being a one-rank extension of $-\Delta^{\mathrm{D}}$ combined with the aforementioned result from [10] which guarantess that the point spectrum is at most made of one point.

The eigenfunction is found combining (2.55), $z^2 = -|\lambda|$, (C.17) and (C.18).Because of (2.61), f can be extended continuously to $a = 0$ and so, for $\alpha = \alpha_{\mathrm{c}}$, $a = 0$ is a solution of (2.59).

□

Remark. Also for this kind of geometry it holds that $\alpha_{\mathrm{c}} \to +\infty$ when the point interaction gets away from the boundary (here meaning that $\rho \to +\infty$). It also holds that

$$\lim_{\rho\to R^+} \alpha_c = -\infty.$$

This result suggests that if the point interaction gets closer and closer to the boundary mantaining its intensity α, then under a threshold distance no eigenvalues are possible anymore. We can explain this fact with the incompatibility of the Dirichlet boundary condition with a very strong interaction near the boundary.

2.4.2. Neumann Boundary Condition

The Neumann Green's function of $-\Delta - z^2$ on Ω is given by (see [12])

$$\begin{aligned} G^{\Omega,\mathrm{N}}_{z,y}(x) &= \frac{i}{4}\sum_{n=-\infty}^{+\infty} e^{in(\theta-\phi)}\left(J_n(zr_<) - \frac{J_n'(zR)}{H_n^{(1)\prime}(zR)} H_n^{(1)}(zr_<)\right) H_n^{(1)}(zr_>) \\ &= G^{0,2}_z(x,y) - \frac{i}{4}\sum_{n=-\infty}^{+\infty} e^{in(\theta-\phi)} \frac{J_n'(zR)}{H_n^{(1)\prime}(zR)} H_n^{(1)}(zr_<) H_n^{(1)}(zr_>). \end{aligned} \tag{2.62}$$

Then

$$h^{\Omega,\mathrm{N}}_{z,y}(x) = \frac{i}{4}\sum_{n=-\infty}^{+\infty} e^{in(\theta-\phi)} \frac{J_n'(zR)}{H_n^{(1)\prime}(zR)} H_n^{(1)}(zr_<) H_n^{(1)}(zr_>).$$

Proposition 2.4.2. *The operator* $-\Delta_{\alpha,y}^{\Omega,\mathrm{N}}$ *with* $|y| = \rho$ *and* Ω *being the exterior of a disk of radius* R *with Neumann boundary conditions has* $\alpha_{\mathrm{c}} = +\infty$*. There is always a single eigenvalue* λ *which happens to be negative. Its associate eigenfunction is*

$$\psi_\lambda(x,y) = \frac{1}{2\pi}K_0\Big(\sqrt{|\lambda|}\,|x-y|\Big) - \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{in(\theta-\phi)}\frac{I_n'(\sqrt{|\lambda|}R)}{K_n'(\sqrt{|\lambda|}R)}K_n(\sqrt{|\lambda|}r_<)K_n(\sqrt{|\lambda|}r_>). \tag{2.63}$$

Proof. As always the unique possible eigenvalue is negative. In this case it corresponds to the imaginary solution with positive imaginary part of

$$\alpha + \frac{\gamma + \ln\frac{z}{2i}}{2\pi} + \frac{i}{4}\sum_{n=-\infty}^{+\infty}\frac{J_n'(zR)}{H_n^{(1)\prime}(zR)}\Big(H_n^{(1)}(z\rho)\Big)^2 = 0. \tag{2.64}$$

So we set $z = ia$, with $a > 0$ in the previous equation. Using (C.17) and (C.18), the previous equation becomes

$$f(a) = \alpha + \frac{\gamma + \ln\frac{a}{2}}{2\pi} - \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{I_n'(aR)}{(-K_n'(aR))}\Big(K_n(a\rho)\Big)^2 = 0, \tag{2.65}$$

where the minus sign has been taken out to get a positive terms sum. This observation implies that $f(a) \to -\infty$ for $a \to 0^+$, because the logarithmic term goes to $-\infty$ and the other ones are negative or constant. Since f is singular in zero, it implies that $a = 0$ can't be a solution of (2.65). So, proving that f is contionuous in $(0, +\infty)$ and determining for which values of α f is positive in a neighbourhood plus of infinity, equals determining the ones with which (2.65) has a solution.

We prove that f is continuous in $(0, +\infty)$. Let's consider (C.24) and (C.25). The positivity of both $I_n(w)$ and $K_n(w)$ implies that we can claim that $I_n'(w) < I_{n-1}(w)$ and that $-K_n'(w) > \frac{n}{w}K_n(w)$ for $n = 1, 2, \ldots$. We rewrite the sum in (2.65) as

$$\sum_{n=-\infty}^{+\infty}\frac{I_n'(aR)}{(-K_n'(aR))}\Big(K_n(a\rho)\Big)^2 = \frac{I_0'(aR)}{(-K_0'(aR))}\Big(K_0(a\rho)\Big)^2 + 2\sum_{n=1}^{+\infty}\frac{I_n'(aR)}{(-K_n'(aR))}\Big(K_n(a\rho)\Big)^2. \tag{2.66}$$

We call $c_n(a)$ the general term of the last sum. Through the inequalities above, we can bound c_n with the general term of a converging sum

$$c_n(a) < \frac{aR}{n}\frac{I_{n-1}(aR)}{K_n(aR)}\Big(K_n(a\rho)\Big)^2 \sim \left(\frac{R}{\rho}\right)^{2n} \qquad n \to +\infty \tag{2.67}$$

(to get the asymptotics of the upper bound, (C.33) has been employed), which converges because $R < \rho$. By (2.67) follows that there exists an $\overline{n}$ such that $c_n(a) < 2(R/\rho)^{2n}$, $\forall n > \overline{n}$. Since definitely in n, c_n is bounded by the terms of a convergent series, then uniform convergence holds and hence the series converges to a continuous function (each c_n is continuous).

Combining (2.66) and (2.67) with this bound and recalling that $I_0'(w) = I_1(w)$ and $K_0'(w) = -K_1(w)$, we can write that

$$\sum_{n=-\infty}^{+\infty} \frac{I_n'(aR)}{(-K_n'(aR))}\Big(K_n(a\rho)\Big)^2 < \frac{I_1(aR)}{K_1(aR)}\Big(K_0(a\rho)\Big)^2 + \sum_{n=1}^{\bar{n}} \frac{2aR}{n}\frac{I_{n-1}(aR)}{K_n(aR)}\Big(K_n(a\rho)\Big)^2 + L, \qquad (2.68)$$

for some finite $L > 0$. The right hand side of (2.68) contains a finite quantity of terms depending on a. If all of them stay finite in the limit $a \to +\infty$, we can claim that also the left end side stays finite for such limit. To study the behaviour of these terms we state the asymptotic behaviours for the functions $I_n(w)$ and $K_n(w)$ for $w \to +\infty$

$$I_n(w) \sim \frac{e^w}{\sqrt{2\pi w}} \quad K_n(w) \sim \sqrt{\frac{\pi}{2w}}e^{-w}. \qquad (2.69)$$

So that

$$\frac{I_1(aR)}{K_1(aR)}\Big(K_0(a\rho)\Big)^2 \sim \frac{e^{-2a(\rho-R)}}{2a\rho} \to 0 \quad \text{for} \quad a \to +\infty \qquad (2.70)$$

$$\frac{2aR}{n}\frac{I_{n-1}(aR)}{K_n(aR)}\Big(K_n(a\rho)\Big)^2 \sim \frac{R}{n\rho}e^{-2a(\rho-R)} \to 0 \quad \text{for} \quad a \to +\infty \quad \text{and} \quad n = 1, \dots, \bar{n}. \qquad (2.71)$$

The infinite sum in (2.65) is hence finite for $a \to +\infty$, which implies that $f(a) \to +\infty$ for $a \to +\infty$ independently on the value of α. □

2.5. Exterior of a Sphere

Here we consider $\Omega = \{x \in \mathbb{R}^3 | \, |x| > R\}$. The point interaction is placed in $y = (0, 0, y_3)$. We set $|y| = \rho$. x is instead located by the polar coordinates r and θ, those respectively being the distance from the origin and the angle between x and the x_3 axis.

2.5.1. Dirichlet Boundary Condition

In [12] it is reported that the Green's function for $-\Delta - z^2$ on Ω is given by

$$G_{z,y}^{\Omega,\mathrm{D}}(x) = \frac{iz}{4\pi}\sum_{n=0}^{+\infty}(2n+1)P_n(\cos\theta)\left(j_n(zr_<) - \frac{j_n(zR)}{h_n^{(1)}(zR)}h_n^{(1)}(zr_<)\right)h_n^{(1)}(zr_>). \qquad (2.72)$$

Here P_n is the n-th Legendre polynomial. Using (C.39), and the addition theorems (C.37) and (C.38) we can rewrite the first term of the product in (2.72) as

$$\frac{iz}{4\pi}\sum_{n=0}^{+\infty}(2n+1)P_n(\cos\theta)j_n(zr_<)h_n^{(1)}(zr_>) = \frac{i}{4\pi\,|x-y|}\Big(\sin(z\,|x-y|) - i\cos(z\,|x-y|)\Big)$$
$$= \frac{e^{iz|x-y|}}{4\pi\,|x-y|}.$$

This means that

$$h_{z,y}^{\Omega,\mathrm{D}}(y) = \frac{iz}{4\pi}\sum_{n=0}^{+\infty}(2n+1)\frac{j_n(zR)}{h_n^{(1)}(zR)}\Big(h_n^{(1)}(z\rho)\Big)^2.$$

Proposition 2.5.1. *The operator $-\Delta_{\alpha,y}^{\Omega,\mathrm{D}}$ with $|y| = \rho$ and Ω being the exterior of a sphere of radius R with Dirichlet boundary conditions has critical value $\alpha_{\mathrm{c}} = -\frac{R}{4\pi(\rho^2-R^2)}$ (α_{c} is not attained though). When existent, the eigenvalue $\lambda < 0$ is unique and has the following associated eigenfunction*

$$\psi_\lambda(x,y) = \frac{e^{-\sqrt{|\lambda|}|x-y|}}{4\pi\,|x-y|} + \frac{\sqrt{|\lambda|}}{2\pi^2}\sum_{n=0}^{+\infty}(2n+1)P_n(\cos\theta)\frac{i_n^{(1)}(\sqrt{|\lambda|}R)}{k_n(\sqrt{|\lambda|}R)}k_n(\sqrt{|\lambda|}r_<)k_n(\sqrt{|\lambda|}r_>). \quad (2.73)$$

Proof. The condition $\Gamma_{\alpha,y}^{\Omega,\mathrm{D}}(z) = 0$ is explicitly

$$\alpha - \frac{iz}{4\pi} + \frac{iz}{4\pi}\sum_{n=0}^{+\infty}(2n+1)\frac{j_n^{(1)}(zR)}{h_n^{(1)}(zR)}(h_n^{(1)}(z\rho))^2 = 0. \quad (2.74)$$

The single negative eigenvalue can be studied by substituting $z = ia$ with $a \geq 0$ in (2.74):

$$f(a) = \alpha + \frac{a}{4\pi} - \frac{a}{4\pi}\sum_{n=0}^{+\infty}(2n+1)\frac{j_n^{(1)}(iaR)}{h_n^{(1)}(iaR)}(h_n^{(1)}(ia\rho))^2 = 0. \quad (2.75)$$

These spherical Bessel functions' arguments contain an i factor. This can be removed converting them into the modified spherical Bessel functions $i_n^{(1)}$ and k_n via (C.40) and (C.41) obtaining

$$f(a) = \alpha + \frac{a}{4\pi} + \frac{1}{2\pi^2}\sum_{n=0}^{+\infty}a(2n+1)\frac{i_n^{(1)}(aR)}{k_n(aR)}(k_n(a\rho))^2 = 0.$$

We study the function f in the interval $(0,+\infty)$. We observe that if $c_n(a)$ is the general term of the infinite sum, it holds that

$$c_n(a) \sim \frac{\pi}{2\rho}\left(\frac{R}{\rho}\right)^{2n+1} \quad \text{for}\, n \to +\infty,\ \forall a > 0.$$

This implies that c_n is definitely bounded by a summable sequence. Since each term of the sum is continuous in $(0, +\infty)$ it means that f is continuous there. Then, to assure the existence of a solution it suffices to know its sign at the extrema: discording signs imply the eigenvalue existence due to the intermediate value theorem. We start by examining $f(a)$ as $a \to 0^+$. We can find that

$$c_n(a) \sim \frac{\pi}{2\rho}\left(\frac{R}{\rho}\right)^{2n+1} \qquad \text{for } a \to 0^+,$$

using (C.44). In addition, (C.47), combined with (C.31) and (C.32) return the following bound for $c_n(a)$

$$c_n(a) < \frac{\pi(e^{2aR}+1)}{4\rho}\left(\frac{R}{\rho}\right)^{2n+1} < \frac{\pi(e^{2R}+1)}{4\rho}\left(\frac{R}{\rho}\right)^{2n+1},$$

where the last inequality is valid for $n = 0, 1, 2, \dots$ and $a < 1$ (in particular this is valid for sufficiently small positive a). So the dominated convergence theorem can be applied and we can switch order between sum and limit, so that

$$\lim_{a\to 0^+} \sum_{n=0}^{+\infty} c_n(a) = \sum_{n=0}^{+\infty} \frac{\pi}{2\rho}\left(\frac{R}{\rho}\right)^{2n+1} = \frac{\pi R}{2(\rho^2 - R^2)}.$$

This means that

$$\lim_{a\to 0^+} f(a) = \alpha + \frac{R}{4\pi(\rho^2 - R^2)}.$$

Hence for $\alpha = -\frac{R}{4\pi(\rho^2-R^2)}$, $a = 0$ is a solution of (2.75).

Since each term in the infinite sum is positive and that the linear term diverges to $+\infty$ when $a \to +\infty$, we have that f does the same regardless of the sign of α. This means that in order to have a solution to (2.75) the condition.

$$\alpha \leq -\frac{R}{4\pi(\rho^2 - R^2)}$$

needs to be satisfied. Since by the a priori result in [10] we know that the point spectrum is made of at most one point, the possible solution is then unique.

At last we prove that in fact $\lambda = 0$ is not an eigenvalue. We need to verify that the corresponding $\psi_0 \notin L^2(\Omega)$. To do so we use the solution of the Laplace problem in the whole space (A.4), together with the image method and the Kelvin transform. This transformation maps outside point inside the disk of radius R centered in the origin and viceversa. It also maps the point on the circle of radius R on themselves. Moreover it preserves harmonicity. Let u be an harmonic function on $U \subset \mathbb{R}^n$ with U not containing the origin, then its Kelvin transform

$\mathrm{K}[u](x)$ is defined as

$$\mathrm{K}[u](x) = \frac{R}{|x|}u\left(\frac{R^2}{|x|^2}x\right). \tag{2.76}$$

We note that if $|x| = R$, then

$$|x^* - y|^2 = \frac{R^4}{|x|^4}|x|^2 - 2\frac{R^2}{|x|^2}x \cdot y + |y|^2 = |x|^2 - 2x \cdot y + |y|^2 = |x - y|\,. \tag{2.77}$$

Using the formula of the fundamental solution of Laplace equation in $\mathbb{R}^3$ (A.4), and Kelvin transform (2.76), we can write the expression for ψ_0

$$\psi_0(x, y) = \frac{1}{4\pi\,|x - y|} - \frac{R}{4\,|x|\left|y - \frac{R^2}{|x|^2}x\right|}.$$

We expand both terms for $|x| \to +\infty$, obtaining

$$\frac{1}{4\pi\,|x - y|} = \frac{1}{4\pi\,|x|} + O\left(\frac{1}{|x|^2}\right) \quad \frac{R}{4\,|x|\left|y - \frac{R^2}{|x|^2}x\right|} = \frac{R}{4\pi\rho\,|x|} + O\left(\frac{1}{|x|^2}\right),$$

so that $\psi_0(x, y) = \frac{\rho - R}{4\pi\rho|x|}$ which has a decay not sufficiently fast to be square integrable at infinite. □

Remark. Also in this case, the limit $\lim_{\rho\to+\infty}\alpha_c = 0$, and zero is the critical value for the whole $\mathbb{R}^3$. Sending the point interaction towards the $\partial\Omega$, instead causes $\alpha_c \to -\infty$: the boundary "feels" the point interaction too close and this interferes with the realization of the Dirichlet boundary condition.

2.5.2. Neumann Boundary Condition

In this case the Green's function is given by (see [12])

$$G^{\Omega,\mathrm{N}}_{z,y}(x) = \frac{iz}{4\pi}\sum_{n=0}^{+\infty}(2n+1)P_n(\cos\theta)\left(j_n(zr_<) - \frac{j_n'(zR)}{h_n^{(1)\prime}(zR)}h_n^{(1)}(zr_<)\right)h_n^{(1)}(zr_>), \tag{2.78}$$

here j_n' and $h_n^{(1)\prime}$ denote respectively the derivative of j_n and $h_n^{(1)}$. In the same manner as in the Dirichlet case we can write

$$\begin{aligned}G^{\Omega,\mathrm{N}}_{z,y}(x) &= = \frac{e^{iz|x-y|}}{4\pi\,|x - y|} - \frac{iz}{4\pi}\sum_{n=0}^{+\infty}(2n+1)P_n(\cos\theta)\frac{j_n'(zR)}{h_n^{(1)\prime}(zR)}h_n^{(1)}(zr_<)h_n^{(1)}(zr_>)\\ &= G^{0,3}_z(x, y) - h^{\Omega,\mathrm{N}}_{z,y}(x).\end{aligned}$$

Proposition 2.5.2. *The operator* $-\Delta^{\Omega,\mathrm{N}}_{\alpha,y}$ *with* $|y| = \rho$ *and* Ω *being the exterior of a sphere of radius* R *with Neumann boundary conditions, has critical value*

$$\alpha_c = \frac{R}{4\pi(\rho^2 - R^2)} + \frac{1}{4\pi R}\ln\left(1 - \left(\frac{R}{\rho}\right)^2\right) \tag{2.79}$$

(α_c *is not attained though). When existent, the eigenvalue* $\lambda < 0$ *is unique and has the following associated eigenfunction*

$$\psi_\lambda(x,y) = \frac{e^{-\sqrt{|\lambda|}|x-y|}}{4\pi\,|x-y|} + \frac{\sqrt{|\lambda|}}{2\pi^2}\sum_{n=0}^{+\infty}(2n+1)P_n(\cos\theta)\frac{i_n^{(1)\prime}(\sqrt{|\lambda|}R)}{k_n'(\sqrt{|\lambda|}R)}k_n(\sqrt{|\lambda|}r_<)k_n(\sqrt{|\lambda|}r_>). \tag{2.80}$$

Proof. To study the behaviour of the possible negative eigenvalue (which is unique if existent, by the aforementioned result in [10] on the point spectrum of finite-rank extensions), we set $z = ia$ in $\Gamma^{\Omega,\mathrm{N}}_{\alpha,y}(z) = 0$, obtaining

$$f(a) = \alpha + \frac{a}{4\pi} - \frac{1}{2\pi^2}\sum_{n=0}^{+\infty} a(2n+1)\frac{i_n^{(1)\prime}(aR)}{(-k_n'(aR))}(k_n(a\rho))^2 = 0 \tag{2.81}$$

(the minus sign was taken both in and out the sum to make it a positive term sum, since $k_n'(aR) < 0$ $\forall n$ and $\forall a > 0$).

We prove that f happens to be continuous in $(0,+\infty)$ so that we can discuss the existence of the solution just by determining the sign of f at both ends of the interval. Employing the positivity of both $i_n^{(1)}$ and k_n, together with (C.48) and (C.49) we get the bounds $i_n^{(1)\prime}(w) < i_{n-1}^{(1)}(w)$ and $-k_n'(w) > \frac{n+1}{w}k_n(w)$. These are useful to prove that the infinite sum converges $\forall a > 0$. Let's call $c_n(a)$ the general term of the series. It follows that

$$c_n(a) < a^2 R\frac{2n+1}{n+1}\frac{i_{n-1}^{(1)}(aR)}{k_n(aR)}(k_n(a\rho))^2 \sim \frac{\pi}{2\rho}\frac{2n+1}{n+1}\left(\frac{R}{\rho}\right)^{2n+1}, \tag{2.82}$$

This means that $c_n(a)$ is definitely bound by a summable sequence not depending on a. Moreover $c_n(a)$ is a sequence of continuous functions, hence f is continuous. The above asymptotic of this upper bound is obtained using (C.33) together with (C.47).

Since $c_n(a) \to 0$ for $a \to +\infty$ $\forall n = 0, 1, \ldots$, (2.82) allows to conclude that for dominated convergence, also the sum vanishes in the limit. We can then conclude that f diverges to $+\infty$ for $a \to +\infty$.

Now we move to the behaviour of f towards 0. Substituting (C.44), (C.45) and (C.46) in the definition of $c_n(a)$ we find that

$$c_0(a) \sim \frac{\pi R^3}{6\rho}a^3 \quad c_n(a) \sim \frac{\pi}{2\rho}\frac{n}{n+1}\left(\frac{R}{\rho}\right)^{2n+1} \quad n = 1, 2, \ldots \quad \text{for} \quad a \to 0.$$

Since the zeroth term goes to 0, we can ignore it when considering the limit and start the sum at $n = 1$. Now, to get the value of $f(0^+)$ we would like to bring the limit inside the infinite sum. To do so we employ the dominated convergence theorem and so we need to prove that $c_n(a)$ is bounded by an a-independant summable sequence. Recalling that $i_n^{(1)\prime}(w) < i_{n-1}^{(1)}(w)$ and $-k_n'(w) > \frac{n+1}{w} k_n(w)$, some computations involving the use of (C.47), (C.31) and (C.32) lead to the bound

$$c_n(a) < \frac{\pi}{4\rho} \frac{2n+1}{n+1} (e^{2aR} + 1) \left(\frac{R}{\rho}\right)^{2n+1} < \frac{\pi}{4\rho} \frac{2n+1}{n+1} (e^{2R} + 1) \left(\frac{R}{\rho}\right)^{2n+1}$$

valid for small a and $n = 1, 2, \ldots$. Being this bounding sequence summable, we can apply the dominated convergence theorem and claim

$$\lim_{a\to 0^+} \sum_{n=1}^{+\infty} c_n(a) = \sum_{n=1}^{+\infty} \frac{\pi}{2\rho} \frac{n}{n+1} \left(\frac{R}{\rho}\right)^{2n+1} = \frac{\pi R}{2(\rho^2 - R^2)} + \frac{\pi}{2R} \ln\left(1 - \left(\frac{R}{\rho}\right)^2\right).$$

This implies that

$$\lim_{a\to 0^+} f(a) = \alpha - \frac{R}{4\pi(\rho^2 - R^2)} - \frac{1}{4\pi R} \ln\left(1 - \left(\frac{R}{\rho}\right)^2\right). \tag{2.83}$$

We can then conclude that the solution exists only when $f(0^+) < 0$ and so for

$$\alpha < \frac{R}{4\pi(\rho^2 - R^2)} + \frac{1}{4\pi R} \ln\left(1 - \left(\frac{R}{\rho}\right)^2\right) = \alpha_c$$

Equation (2.83) implies that we can extend continuously f in 0. So if $\alpha = \alpha_c$, $a = 0$ is a solution of $\Gamma^{\Omega,N}_{\alpha,y}(ia) = 0$. Following the same ideas as in the Dirichlet case, we can write

$$\psi_0(x, y) = \frac{1}{4\pi |x - y|} + \frac{R}{4 |x| \left|y - \frac{R^2}{|x|^2} x\right|}.$$

Since both terms are positive, there is no hope for cancellation. $\psi_0(x, y) \sim |x|^{-1}$ for $|x| \to +\infty$ and then it is not square integrable in Ω. □

Ω	Boundary Condition	Eigenfunction	α_c	Is α_c attained
$\mathbb{H}^3$	Dirichlet	(2.29)	$-\frac{1}{8\pi y_3}$	Yes
$\mathbb{H}^3$	Neumann	(2.38)	$\frac{1}{8\pi y_3}$	No
$\mathbb{H}^3$	Robin	(2.44)	(2.43)	Not Determined
$\mathbb{H}^2$	Dirichlet	(2.47)	$\ln\left(\frac{2y_2}{2\pi}\right)$	No
$\mathbb{H}^2$	Neumann	(2.52)	$+\infty$	No
Exterior of a Disk	Dirichlet	(2.57)	$-\frac{1}{2\pi}\ln\left(\frac{R}{\rho^2-R^2}\right)$	No
Exterior of a Disk	Neumann	(2.63)	$+\infty$	No
Exterior of a Sphere	Dirichlet	(2.73)	$-\frac{R}{4\pi(\rho^2-R^2)}$	No
Exterior of a Sphere	Neumann	(2.80)	(2.79)	No

Table 2.1.: A brief recap of the specific domains studied in this chapter with different boundary conditions. The equations defining the eigenfunctions and the critical values for each of those problems are recalled.

Chapter 3.

Resonances of One Point Interactions in $\mathbb{H}^n$

3.1. Resonances of N-Points Interactions in a Domain

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be a domain of the kind already considered, i.e. external domains or $\mathbb{H}^n$. The resolvent

$$(-\Delta^\Omega - z^2)^{-1} : L^2(\Omega) \to L^2(\Omega)$$

acts as follows

$$((-\Delta^\Omega - z^2)^{-1} f)(x) = \int_\Omega G^\Omega_{z,y}(x) f(y)\, \mathrm{d}y = \int_\Omega (G^{0,n}_{z,y}(x) - h^\Omega_{z,y}(x)) f(y)\, \mathrm{d}y.$$

We observe that the map $z \to G^\Omega_{z,y}(x)$ is holomorphic in all $\mathbb{C}$, smooth in x and y. But the kernel $G^\Omega_{z,y}(x)$ does not go to zero when $\operatorname{Im} z \leq 0$. So it ceases to be an L^2 kernel. Though, if we consider some $\rho \in C^\infty_{\mathrm{c}}(\Omega)$, then the map

$$\rho(-\Delta^\Omega - z^2)^{-1}\rho : L^2(\Omega) \to L^2(\Omega) \tag{3.1}$$

$$(\rho(-\Delta^\Omega - z^2)^{-1}\rho f)(x) = \int_\Omega \rho(x) G^\Omega_{z,y}(x) \rho(y) f(y)\, \mathrm{d}y, \tag{3.2}$$

called truncated resolvent, is well defined $\forall z \in \mathbb{C}$. So, if we consider the restriction $(-\Delta^\Omega - z^2)^{-1} : L^2_{\mathrm{comp}}(\Omega) \to L^2_{\mathrm{loc}}(\Omega)$, even if initially defined only for $\operatorname{Im} z > 0$, it can be holomorphically continued to $\mathbb{C}$.

Now we consider the N-points interactions on Ω. We explicitly rewrite here the expression (1.30) for the resolvent

$$(-\Delta^\Omega_{\alpha,Y} - z^2)^{-1}\psi = (-\Delta^\Omega - z^2)^{-1}\psi + \sum_{j,k}^{N} (\Gamma^\Omega_{\alpha,Y}(z))^{-1}_{jk} \Big((-\Delta^\Omega - z^2)^{-1}\psi\Big)(y_k) G^\Omega_{z,y_j},$$

defined for $\operatorname{Im} z > 0$. We want to extend this perturbed resolvent as well. We observe that:

- The first term is simply the free resolvent, which can be holomorphically extended to the whole $\mathbb{C}$ as recalled above.
- The other terms contain the composition of the free resolvent $(-\Delta^\Omega - z^2)^{-1}$ with the projector onto the function G^Ω_{z,y_j}. So also these factors can be extended to all $\mathbb{C}$.
- The factors $(\Gamma^\Omega_{\alpha,Y}(z))^{-1}_{jk}$ are not defined whenever the matrix $\Gamma^\Omega_{\alpha,Y}(z)$ is singular.

Summarizing we can state the following

Proposition 3.1.1. *Let $\rho \in C^\infty_c(\Omega)$. The map*

$$z \mapsto \rho(-\Delta^\Omega_{\alpha,Y} - z^2)^{-1}\rho : L^2(\Omega) \to L^2(\Omega)$$

extends as a meromorphic family of operators to $z \in \mathbb{C}$. Its poles are the solution of

$$\det\left(\Gamma^\Omega_{\alpha,Y}(z)\right) = 0.$$

For each of this poles:

- *when $z \neq 0$ and $\operatorname{Im} z \geq 0$, the value z^2 is an eigenvalue;*
- *when $\operatorname{Im} z < 0$, the corresponding z^2 is called a* resonance*;*
- *if 0 is a pole, it is an eigenvalue if the corresponding solution is in $L^2(\Omega)$, while instead it is called* zero energy resonance *if the associated solution of $-\Delta^\Omega_{\alpha,y} = 0$ is in $L^2_{\mathrm{loc}}(\Omega)$ but not in $L^2(\Omega)$.*

Considering the one particle center, the resonances that are not 0 energy resonances are the solution of

$$\Gamma^\Omega_{\alpha,y}(z) = 0, \quad \operatorname{Im} z < 0.$$

The previous definitions agrees with the treatment based on the black box description of Sjöstrand and Zworski in [44] and [43]. See also the treatise [18] by Dyatlov and Zworski.

In the following sections we study the resonances when $\Omega = \mathbb{H}^n$, $n = 2, 3$ with different boundary conditions.

3.2. Resonances for the Half-Space

3.2.1. Dirichlet Boundary Condition

Proposition 3.2.1. *The resonances z^2 of $-\Delta^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}$ are determined by the values z with the following properties:*

- *$\forall\alpha \in \mathbb{R}$ and $y \in \mathbb{H}^3$ there are exactly two values of z, with opposite real part, in every region of the form $\frac{k\pi}{y_3} < |\mathrm{Re}\, z| < \frac{(2k+1)\pi}{2y_3}$ with $k = 1, 2, \ldots$.*

  Moreover

$$\mathrm{Im}\, z = \frac{1}{2y_3} \ln\left(\frac{\sin(2y_3 \,\mathrm{Re}\, z)}{2y_3 \,\mathrm{Re}\, z}\right). \tag{3.3}$$

- *for $\alpha < -\frac{1}{8\pi y_3}$ there is an additional unique z on the negative imaginary semi-axis, while for $\alpha > -\frac{1}{8\pi y_3}$, there are two additional resonances, one for $\mathrm{Re}\, z \in \left(-\frac{\pi}{2y_3}, 0\right)$ and one for $\mathrm{Re}\, z \in \left(0, \frac{\pi}{2y_3}\right)$.*

- *If $\alpha = \frac{1}{8\pi y_3} \ln\left(\frac{\pi}{2} + k\pi\right)$ for some non-negative even k, then $z = \frac{\pi}{4y_3} + \frac{k\pi}{2y_3} - \frac{i}{2y_3} \ln\left(\frac{\pi}{2} + k\pi\right)$ generates an additional resonance.*

- *If $\alpha = \frac{1}{8\pi y_3} \ln\left(-\frac{\pi}{2} - k\pi\right)$ for some negative odd k, then $z = \frac{\pi}{4y_3} + \frac{k\pi}{2y_3} - \frac{i}{2y_3} \ln\left(-\frac{\pi}{2} - k\pi\right)$ generates an additional resonance.*

Proof. Following the idea in the previous section, the resonances are the solution of $\Gamma^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}(z) = 0$, that is (2.30) with $\mathrm{Im}\, z < 0$. We plug $z = a + ib$ in (2.30), with the prescription $b < 0$. This way we get

$$\alpha + \frac{b - ai}{4\pi} + e^{-2y_3 b} \frac{\cos(2y_3 a) + i \sin(2y_3 a)}{8\pi y_3} = 0.$$

Equating real and imaginary part of both sides, we obtain the system in a and b

$$\begin{cases} -\alpha - \frac{b}{4\pi} = e^{-2y_3 b} \frac{\cos(2y_3 a)}{8\pi y_3} \\ \frac{a}{4\pi} = e^{-2y_3 b} \frac{\sin(2y_3 a)}{8\pi y_3} \end{cases}. \tag{3.4}$$

We observe that if $a = 0$, the second equation is satisfied $\forall b$. The first equation of the system is then

$$f(b) = \alpha + \frac{b}{4\pi} + \frac{e^{-2y_3 b}}{8\pi y_3} = 0, \quad b < 0.$$

Since $\lim_{b \to -\infty} f(b) = +\infty$ and f is a continuous function, a solution in $(-\infty, 0)$ exists only if $f(0) < 0$. This happens when $\alpha < -(8\pi y_3)^{-1}$, otherwise there are no resonances on the negative imaginary semi-axis. The possible solution is in fact unique because $f'(b) = (1 - e^{-2y_3 b})/(4\pi) < 0 \; \forall b < 0$.

Another case of interest in the study of (3.4) is when $a = \frac{k\pi}{2y_3}$, for some $k = \pm 1, \pm 2, \ldots$. In this case it is easy to see that the system does not admit any solution, because the right side of the second equation of the system is zero, while the left one isn't. So there are no resonances on the lines $\mathrm{Re}\, z = \frac{k\pi}{2y_3}$, $k = \pm 1, \pm 2, \ldots$ outside the real axis.

Also, since we already considered the $a = 0$ case, we can rewrite the second equation of the system (3.4) as

$$\frac{\sin(2y_3 a)}{2y_3 a} = e^{2y_3 b}, \quad a \neq 0. \tag{3.5}$$

In this way we can verify by hand that no solution can exist on the upper half complex plane, barring the ones on the real axis and on the positive imaginary semi-axis, this being a consequence of the self-adjointness of the operator. Since the left side is always smaller than 1, a solution is possible only if $b < 0$. In this form it is also easy to see that there are no solution when $\pi + 2k\pi < 2y_3 |a| < 2\pi + 2k\pi$, $k = 0, 1, 2, \ldots$, because in that case the left side would be negative, while the right hand side wouldn't.

We now consider the case $a = \frac{\pi}{4y_3} + \frac{k\pi}{2y_3}$ for some $k \in \mathbb{Z}$. (3.4) is then

$$\begin{cases} b = -4\pi\alpha \\ e^{8\pi\alpha y_3} = (-1)^k \left(\frac{\pi}{2} + k\pi\right) \end{cases}.$$

This system is overdetermined and so can have a solution only for certain values of α and y_3. Let's distinguish the different cases:

- if k is positive odd or negative even, then no solution exists regardless of α and y_3 (a would be in one of the intervals mentioned above for which no solution exists);
- if k is non-negative even, then there is a resonance in $z = \frac{\pi}{4y_3} + \frac{k\pi}{2y_3} - \frac{i}{2y_3} \ln\left(\frac{\pi}{2} + k\pi\right)$ if $\alpha = \frac{1}{8\pi y_3} \ln\left(\frac{\pi}{2} + k\pi\right)$;
- if k is negative odd, then there is a resonance in $z = \frac{\pi}{4y_3} + \frac{k\pi}{2y_3} - \frac{i}{2y_3} \ln\left(-\frac{\pi}{2} - k\pi\right)$ if $\alpha = \frac{1}{8\pi y_3} \ln\left(-\frac{\pi}{2} - k\pi\right)$.

At this point, we divide side by side the equations of the system (3.4). Note that it is legit doing so, since we already examined the cases in which one of the sides vanishes. So, dividing the second one by the first, we get

$$\tan(2y_3 a) = -\frac{a}{4\pi\alpha + b},$$

which, solved for b gives

$$b = -4\pi\alpha - \frac{a}{\tan(2y_3 a)}. \tag{3.6}$$

Substituting b in the second equation of (3.4), we get the following equation in a

$$g(a) = \frac{\sin(2y_3 a)}{2y_3 a} = e^{-8\pi\alpha y_3 - \frac{2y_3 a}{\tan(2y_3 a)}} = h(a). \tag{3.7}$$

The following properties about h holds:

- h is even (as g does as well) and so we can limit ourself to consider $a \geq 0$.
- $\lim_{a\to 0} h(a) = e^{-8\pi\alpha y_3 - 1}$, and $\forall k = 0, 1, 2, \ldots$

$$\lim_{a\to\frac{\pi/2+k\pi}{2y_3}} h(a) = e^{-8\pi\alpha y_3}, \quad \lim_{a\to\frac{\pi^-+k\pi}{2y_3}} h(a) = +\infty \quad \text{and} \quad \lim_{a\to\frac{\pi^++k\pi}{2y_3}} h(a) = 0.$$

- The derivative of h is

$$h'(a) = -2y_3 e^{-8\pi\alpha y_3 - \frac{2y_3 a}{\tan(2y_3)}} \frac{\tan(2y_3 a) - 2y_3 a\left(1 + \tan^2(2y_3 a)\right)}{\tan^2(2y_3 a)}.$$

  We claim that $h'(a) \geq 0$ $\forall a \geq 0$. This is a consequence of the inequality

$$p(t) = \frac{\tan t}{1 + \tan^2 t} \leq t$$

  being satisfied $\forall t \geq 0$. In fact $p(0) = 0$ and $p'(t) = \frac{1-\tan^2 t}{1+\tan^2 t} \leq 1$, which implies that the above inequality is valid $\forall t \geq 0$ with the equality only valid in $t = 0$.

- The continuity of h and its derivative in every interval of the form $\left(\frac{k\pi}{y_3}, \frac{(2k+1)\pi}{2y_3}\right)$ with $k = 0, 1, 2, \ldots$, together with $h'(a) \geq 0$, implies that the h is monotone increasing in each of those intervals.
- The second derivative h'' is given by

$$h''(a) = 4y_3^2 e^{-8\pi\alpha y_3 - \frac{2y_3 a}{\tan(2y_3)}} \Big(\cot^2(2y_3 a) + 2\csc^2(2y_3 a) + 4y_3^2 a^2 \csc^4(2y_3 a) - 8y_3 a \cot(2y_3 a)\csc^2(2y_3 a)\Big). \quad (3.8)$$

  It holds that $h'' > 0$ when

$$t^2 + \sin^2 t \cos^2 t + 2\sin^2 t - 4t \sin t \cos t > 0,$$

  with $t = 2y_3 a$. This expression is greater or equal than $t^2 - 2t$, which is positive for $t > 2$. So $h''(a) > 0$ $\forall a > 1/y_3$ in its domain. Since $1/y_3 < \pi/y_3 \leq k\pi/y_3$ for $k = 1, 2, \ldots$, then $h'' > 0$ in each interval $\left(\frac{k\pi}{y_3}, \frac{(2k+1)\pi}{2y_3}\right)$ with $k = 1, 2, \ldots$ and is convex here.

About g we recall that:

- g is continuous and smooth outside zero.
- In each interval of the form $\left(\frac{k\pi}{y_3}, \frac{(2k+1)\pi}{2y_3}\right)$ with $k = 1, 2, \ldots$, g is positive and has a unique maximum M_k, which is the solution of

$$g'(a) = \frac{2y_3 a \cos(2y_3 a) - \sin(2y_3 a)}{2y_3 a^2} = 0$$

in a said interval. Moreover, since

$$g'\left(\frac{\frac{\pi}{2}+2k\pi}{2y_3}\right) = -\frac{2y_3}{\left(\frac{\pi}{2}+2k\pi\right)^2} < 0, \quad \forall k = 1, 2, \ldots,$$

each maximum point is contained in $\left(\frac{k\pi}{y_3}, \frac{\frac{\pi}{2}+2k\pi}{2y_3}\right)$, and in particular in these intervals both $\sin(2y_3a)$ and $\cos(2y_3)$ are positive.

- We also have that

$$g''(a) = -\frac{\left((2y_3a)^2 - 2\right)\sin(2y_3a) + 4y_2a\cos(2y_2a)}{x^3}.$$

  Then, g is concave whevever $q(a) = \left((2y_3a)^2 - 2\right)\sin(2y_3a) + 4y_2a\cos(2y_2a) > 0$. But, since $q\left(\frac{k\pi}{y_3}\right) = 4k\pi > 0$ and $q'(a) = 8y_3^3a^2\cos(2y_3a)$, it follows that in every interval $\left(\frac{k\pi}{y_3}, \frac{\frac{\pi}{2}+2k\pi}{2y_3}\right)$ g is concave.

- In $[0, \pi)$ g is monotone decreasing and $\lim_{a\to 0} g(a) = 1$ which is the absolute supremum for g.

Now we claim that in each interval of the form $\left(\frac{k\pi}{y_3}, \frac{(2k+1)\pi}{2y_3}\right)$ with $k = 1, 2, \ldots$, (3.7) has exactly one solution.

Let a_k be the smallest solution in the $k-th$ interval. We distinguish the two cases $a_k \geq M_k$ and $a_k < M_k$. If $a_k \geq M_k$, the solution is unique because in $\left[a_k, \frac{(2k+1)\pi}{2y_3}\right)$, h is increasing, while g is decreasing. If $a_k < M_k$, given that $M_k \in \left(\frac{k\pi}{y_3}, \frac{\frac{\pi}{2}+2k\pi}{2y_3}\right)$, the function $r(a) = h(a) - g(a)$ is strictly convex in $\left(\frac{k\pi}{y_3}, M_k\right)$. But this, together with $\lim_{a\to\frac{k\pi^+}{y_3}} r(a) = 0$ implies that at there is at most a zero for r in $\left(\frac{k\pi}{y_3}, M_k\right)$. In fact, if there were two internal zeros $a_k < \tilde{a}_k$, there would be two numbers b_k and $\tilde{b}_k$ such that $\frac{k\pi}{y_3} < b_k < a_k < \tilde{b}_k < M_k$ and $r'(b_k) = r'(\tilde{b}_k) = 0$. This contradicts the function r being convex (r' is strictly increasing), hence only a solution is possible. For each interval, this possible unique solution in fact exists. That's because:

- $h\left(\frac{k\pi^+}{y_3}\right) = g\left(\frac{k\pi}{y_3}\right) = 0$ and $h'\left(\frac{k\pi^+}{y_3}\right) = 0 < g'\left(\frac{k\pi}{y_3}\right) = \frac{y_3}{k\pi}$;
- $h\left(\frac{\pi^-+2k\pi}{2y_3}\right) = +\infty$ and $g\left(\frac{\pi^-+2k\pi}{2y_3}\right) = 0$.

It remains to consider the interval $\left[0, \frac{\pi}{2y_3}\right)$. Here g is monotone decreasing and h is monotone increasing, so, at most there is one solution. The solution actually exists only if $\lim_{a\to 0}(h(a) - g(a)) < 0$, which happens if and only if $\alpha > -\frac{1}{8\pi y_3}$.

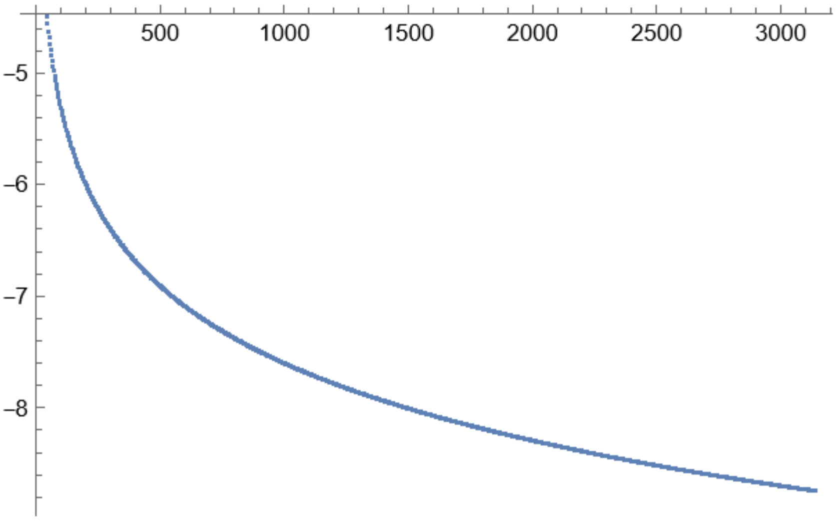

Figure 3.1.: Plot of the real part versus imaginary part of the first 100 values z with $\operatorname{Re} z > \pi$ for which z^2 is a resonance for $\alpha = 0$ and $y_3 = 1$ in the Dirichlet half-space case.

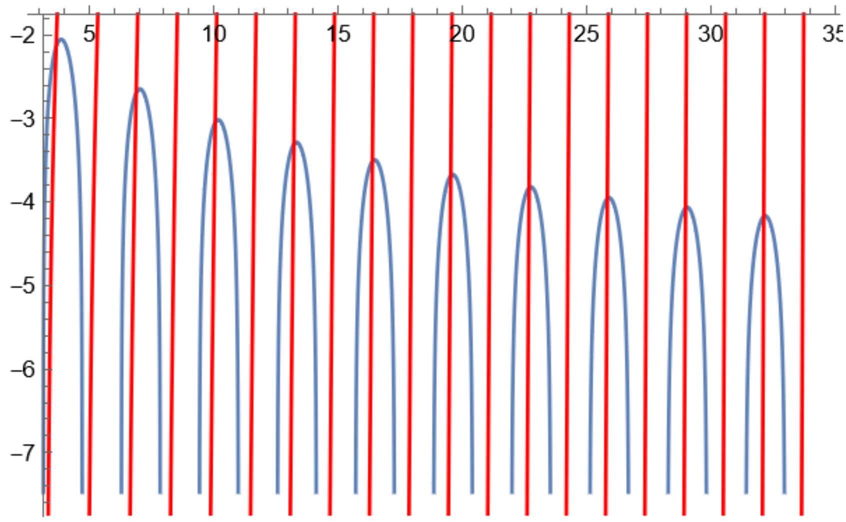

Figure 3.2.: First 10 z with $\operatorname{Re} z > \pi$ such that z^2 is a resonance for $\alpha = 0$ and $y_3 = 1$ as intersections of (3.6) (red) and (3.3) (blue) in the Dirichlet half-space case.

For each solution of (3.7), we can recover the corresponding imaginary part through the relation $b = \frac{1}{2y_3} \ln g(a)$. □

Asymptotics of the Resonance Counting Function

Now we consider the set $\Sigma\Big(-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}\Big)$ of complex z such that z^2 is an eigenvalue or a resonance of $-\Delta_{\alpha,y}^{\Omega,\mathrm{D}}$. Let's define

$$\mathcal{N}_{-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}}(R) = \#\Big\{ z \in \Sigma\Big(-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}\Big) \;\Big|\; |z| < R \Big\}$$

the counting function for the the elements of $\Sigma\Big(-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}\Big)$ counted with the appropriate multiplicity. We now prove that (see [4] for an analogous result for N-point interactions in $\mathbb{R}^n$)

Proposition 3.2.2. *$\forall \alpha \in \mathbb{R}$ and $y \in \mathbb{H}^3$ it holds that*

$$\lim_{R\to+\infty} \frac{\ln\Big(\mathcal{N}_{-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}}(R)\Big)}{\ln R} = 1.$$

Proof. Let a_k^2 be the resonance of $-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}$ such that $a_k \in \left(\frac{k\pi}{y_3}, \frac{(2k+1)\pi}{2y_3}\right)$ and $\{a_k\}_{k=1,2,\dots}$ the associated sequence. We start by proving that

$$\lim_{k\to+\infty} a_k = \frac{(2k+1/2)\pi}{2y_3}. \tag{3.9}$$

We observe that, even if it can occur that $a_1 \in \left(\frac{5\pi}{4y_3}, \frac{3\pi}{2y_3}\right)$, it still happens that, definitely in k, $a_k \in \left(\frac{k\pi}{y_3}, \frac{(2k+1/2)\pi}{2y_3}\right)$. This is true because, recalling (3.7), we have that

$$g\left(\frac{(2k+1/2)\pi}{2y_3}\right) = \frac{1}{(2k+1/2)\pi} \quad \text{and} \quad \lim_{a\to\frac{(2k+1/2)\pi}{2y_3}} h(a) = e^{-8\pi\alpha y_3}$$

and so there exists a $\overline{k}$, such that $\forall k \geq \overline{k}$, $g < h$ when both evaluated at the middle point of the k-th interval, which implies that the unique solution is in the first half of the interval. Let y be in $\left(\frac{\overline{k}\pi}{y_3}, \frac{(2\overline{k}+1/2)\pi}{2y_3}\right)$ and $\{y_l\}_{l=0,1,\dots} = \left\{y + \frac{2l\pi}{2y_3}\right\}_{l=0,1,\dots}$. We have that

$$g(y_l) = \frac{\sin(2y_3 y)}{2y_3 y + 2l\pi} \quad \text{and} \quad h(y_l) = e^{-8\pi\alpha y_3 - \frac{2y_3 y + 2l\pi}{\tan(2y_3 y)}}.$$

Both sequences goes to zero, but $\{h(y_l)\}$ does so faster than $\{g(y_l)\}$, which implies that for sufficiently large l, $g(y_l) > h(y_l)$ and so the solution in the $(\overline{k}+l)$-th interval must be in $\left(y_l, \frac{(2(\overline{k}+l)+1/2)\pi}{2y_3}\right)$. The arbitrarity of y in $\left(\frac{\overline{k}\pi}{y_3}, \frac{(2\overline{k}+1/2)\pi}{2y_3}\right)$ leads to (3.9).

So we have that $a_k = \frac{(2k+1/2)\pi}{2y_3} + o(1)$ for $k \to +\infty$. We use this property to estimate $\mathcal{N}_{-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}}(R)$ as $R \to +\infty$ excluding at most a finite quantity of resonances. Doing so is equivalent to find the highest k for which

$$a_k^2 + \frac{1}{4y_3^2}\ln^2\left(\frac{\sin(2y_3 a_k)}{2y_3 a_k}\right) < R^2.$$

Substituting the asymptotic expression for $k \to +\infty$, we get

$$\left(2k + \frac{1}{2}\right)^2 \pi^2 + \ln^2\left(\left(2k + \frac{1}{2}\right)\pi\right) + o(1) < 4y_3^2 R^2.$$

If we set $s = \left(2k + \frac{1}{2}\right)\pi$, it becomes

$$s^2 + \ln^2(s) + o(1) < 4y_3^2 R^2.$$

We are concerned about large value of s, so, since the left side happens to be increasing for s sufficiently large, the largest s satisfying the inequality will be the one satisfying

$$s^2 + \ln^2(s) + o(1) = 4y_3^2 R^2.$$

We look for the asymptotic for s solving the equation, when $R \to +\infty$. We write

$$s = \sqrt{4y_3^2 R^2 - \ln^2(s)} + o(1). \tag{3.10}$$

We see that for large R, $s > 1$, so that $s < 2y_3 R$ and $s = O(R)$. This implies that $\ln^2(s) = O\left(\ln^2(R)\right)$. Now we substitute this in (3.10), obtaining

$$s = 2y_3 R\sqrt{1 + O\left(\frac{\ln^2(R)}{R^2}\right)} + o(1) = 2y_3 R + o(1).$$

And so for k holds

$$k = \left\lfloor \frac{y_3}{\pi} R - \frac{1}{4} \right\rfloor.$$

But in this way we only counted the resonances with positive real part, so to get the right number, we have to multiply by 2, since for each resonance there is another one, symmetric with respect to the imaginary axis. So

$$\mathcal{N}_{-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}}(R) = 2\left\lfloor \frac{y_3}{\pi} R - \frac{1}{4} \right\rfloor.$$

The sought limit is then

$$\begin{aligned}\lim_{R\to+\infty}\frac{\ln\left(\mathcal{N}_{-\Delta^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}}(R)\right)}{\ln R} &= \lim_{R\to+\infty}\frac{\ln\left(2\left\lfloor\frac{y_3}{\pi}R-\frac{1}{4}\right\rfloor\right)}{\ln R}\\ &= \lim_{R\to+\infty}\frac{\ln\left(2\left(\frac{y_3}{\pi}R-\frac{1}{4}\right)-2\left\{\frac{y_3}{\pi}R-\frac{1}{4}\right\}\right)}{\ln R}\\ &= \lim_{R\to+\infty}\frac{\ln\left(\frac{2y_3}{\pi}R\right)}{\ln R} = \lim_{R\to+\infty}\frac{\ln R}{\ln R} = 1.\end{aligned}$$

□

3.2.2. Neumann Boundary Condition

Proposition 3.2.3. *The resonances z^2 of $-\Delta^{\mathbb{H}^3,\mathrm{N}}_{\alpha,y}$ are determined by the values z with the following properties:*

- *$\forall\alpha\in\mathbb{R}$ and $y\in\mathbb{H}^3$ there are exactly two values z, with opposite real part, in every region of the form $\frac{(2k+1)\pi}{2y_3}<|\operatorname{Re} z|<\frac{(k+1)\pi}{y_3}$ with $k=0,1,\dots$. Moreover*

$$\operatorname{Im} z=\frac{1}{2y_3}\ln\left(-\frac{\sin(2y_3\operatorname{Re} z)}{2y_3\operatorname{Re} z}\right). \tag{3.11}$$

- *For $\alpha>\frac{1}{8\pi y_3}$ there is an additional unique z on the negative imaginary semi-axis.*
- *If $\alpha=\frac{1}{8\pi y_3}\ln\left(\frac{\pi}{2}+k\pi\right)$ for some positive odd k, then $z=\frac{\pi}{4y_3}+\frac{k\pi}{2y_3}-\frac{i}{2y_3}\ln\left(\frac{\pi}{2}+k\pi\right)$ generates an additional resonance.*
- *If $\alpha=\frac{1}{8\pi y_3}\ln\left(-\frac{\pi}{2}-k\pi\right)$ for some negative even k, then $z=\frac{\pi}{4y_3}+\frac{k\pi}{2y_3}-\frac{i}{2y_3}\ln\left(-\frac{\pi}{2}-k\pi\right)$ generates an additional resonance.*

Proof. The resonances z^2 corresponds to the solution z of (2.39) with negative imaginary part. So, in order to study them, we plug $z=a+ib$ with $b<0$ into (2.39), obtaining

$$\alpha+\frac{b-ai}{4\pi}-e^{-2y_3b}\frac{\cos(2y_3a)+i\sin(2y_3a)}{8\pi y_3}=0.$$

By separating real and imaginary part, we get the system

$$\begin{cases}\alpha+\frac{b}{4\pi}=e^{-2y_3b}\frac{\cos(2y_3a)}{8\pi y_3}\\ -\frac{a}{4\pi}=e^{-2y_3b}\frac{\sin(2y_3a)}{8\pi y_3}\end{cases}. \tag{3.12}$$

If z is purely imaginary ($a = 0$), then the second equation is valid $\forall b$. The first one is then written as

$$f(b) = \alpha + \frac{b}{4\pi} - \frac{e^{-2y_3 b}}{8\pi y_3} = 0.$$

$\lim_{b\to-\infty} f(b) = -\infty$, $f(0) = \alpha - \frac{1}{8\pi y_3}$, and the continuity of f imply that a solution exist only if $\alpha > \frac{1}{8\pi y_3}$. $f'(b) = \left(1 + e^{-2y_3 b}\right)/(4\pi)$ ensures unicity.

For the same reason as in the Dirichlet case, there are no solutions for $a = \frac{k\pi}{2y_3}$.

We now rewrite (3.12)' s second equation as

$$\frac{\sin(2y_3 a)}{2y_3 a} = -e^{2y_3 b}, \quad a \neq 0. \tag{3.13}$$

Since the left side is always less than one in absolute value, it is automatically fulfillled that $b < 0$. Moreover, the negativity of the right hand side, implies that no solutions exist for $2k\pi < 2y_3 |a| < \pi + 2k\pi$, $k = 0, 1, 2, \ldots$.

Next we consider $a = \frac{\pi}{4y_3} + \frac{k\pi}{2y_3}$ for some $k \in \mathbb{Z}$. In this case (3.12) becomes

$$\begin{cases} b = -4\pi\alpha \\ e^{8\pi\alpha y_3} = (-1)^{k+1}\left(\frac{\pi}{2} + k\pi\right) \end{cases}.$$

It follows that:

- no solution is possible for both k non-negative even and k negative odd;
- if k is positive odd, then there is a resonance in $z = \frac{\pi}{4y_3} + \frac{k\pi}{2y_3} - \frac{i}{2y_3}\ln\left(\frac{\pi}{2} + k\pi\right)$ if $\alpha = \frac{1}{8\pi y_3}\ln\left(\frac{\pi}{2} + k\pi\right)$;
- if k is negative even, there is a resonance in $z = \frac{\pi}{4y_3} + \frac{k\pi}{2y_3} - \frac{i}{2y_3}\ln\left(-\frac{\pi}{2} - k\pi\right)$ if $\alpha = \frac{1}{8\pi y_3}\ln\left(-\frac{\pi}{2} - k\pi\right)$.

In the same way as in the Dirichlet case, we find the expression (3.6) of b in terms of a. By substituting this expression in (3.13), we get an equation for a

$$g(a) = \frac{\sin(2y_3 a)}{2y_3 a} = -e^{-8\pi\alpha y_3 - \frac{2y_3 a}{\tan(2y_3 a)}} = -h(a), \tag{3.14}$$

where g and h are the same of the previous subsection ones. We use the information we know about those functions to claim that also in this case there is a unique solution for each interval $\left(\frac{\pi+2k\pi}{2y_3}, \frac{\pi+k\pi}{y_3}\right)$.

- $-h\left(\frac{\pi^+ + 2k\pi}{2y_3}\right) = g\left(\frac{\pi+2k\pi}{2y_3}\right) = 0$, $-\frac{2y_3}{(2k+1)\pi} = g'\left(\frac{\pi+2k\pi}{2y_3}\right) < h'\left(\frac{\pi+2k\pi}{2y_3}\right) = 0$ and $h\left(\frac{\pi^- + k\pi}{y_3}\right) = -\infty$ and $g\left(\frac{\pi^- + k\pi}{y_3}\right) = 0$ so in each interval there is at least a solution;

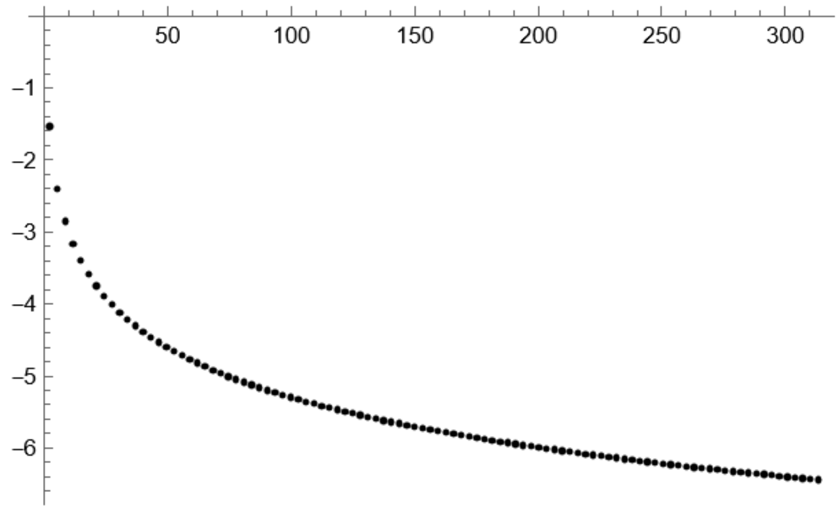

Figure 3.3.: Plot of the real part versus imaginary part of the first 100 values z with $\operatorname{Re} z > \frac{3}{2}\pi$ such that z^2 is a resonance for $\alpha = 0$ and $y_3 = 1$ in the Neumann half-space case.

- in each of those intervals, $-h$ is monotone decreasing and concave;
- in each of those intervals, g is negative and has a unique minimum m_k and since

$$g'\left(\frac{\frac{3}{2}\pi + 2k\pi}{2y_3}\right) = \frac{2y_3}{\left(\frac{3}{2}\pi + 2k\pi\right)^2} > 0, \quad \text{for} \quad k = 0, 1, \dots,$$

then $m_k \in \left(\frac{\pi+2k\pi}{2y_3}, \frac{\frac{3}{2}\pi+2k\pi}{2y_3}\right)$ and here g also happens to be convex.

At this point, an analogous argument to the one used in the Dirichlet case proves unicity. Let a_k be the smallest solution in the k-th interval. If $a_k \geq m_k$ the unicity follows, because in $\left[a_k, \frac{\pi+k\pi}{y_3}\right)$ $-h$ is decreasing, while g is increasing. Otherwise, defining $u(a) = g(a) + h(a)$, the unicity comes from its convexity in $\left(\frac{\pi+2k\pi}{2y_3}, a_k\right]$ and $u\left(\frac{\pi^+ + 2k\pi}{2y_3}\right) = 0$.

We can get the corresponding imaginary part b for each of the solution of (3.14) thanks to the relationship $b = \frac{1}{2y_3} \ln\left(-g(a)\right)$. □

Asymptotics of the Resonance Counting Function

We now prove a property analogous to Proposition 3.2.2 for the Neumann case.

Proposition 3.2.4. *$\forall \alpha \in \mathbb{R}$ and $y \in \mathbb{H}^3$ it holds that*

$$\lim_{R\to+\infty} \frac{\ln\left(\mathcal{N}_{-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{N}}}(R)\right)}{\ln R} = 1.$$

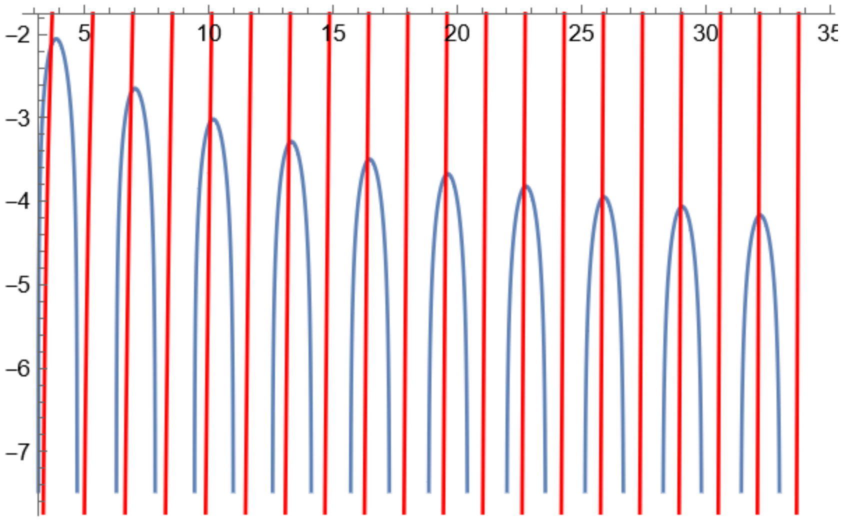

Figure 3.4.: First 10 z with $\operatorname{Re} z > \frac{3}{2}\pi$ such that z^2 is a resonance for $\alpha = 0$ and $y_3 = 1$ as intersections of (3.6) (red) and (3.11) (blue) in the Neumann half space case.

Proof. Let a_k^2 be the resonance of $-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{N}}$ such thst in $a_k \in \left(\frac{(2k+1)\pi}{2y_3}, \frac{(k+1)\pi}{y_3}\right)$ and $\{a_k\}_{k=0,1,\dots}$ the associated sequence. We start by proving that

$$\lim_{k\to+\infty} a_k = \frac{(2k+3/2)\pi}{2y_3}. \tag{3.15}$$

We observe that, even if it can occur that $a_0 \in \left(\frac{(2k+3/2)\pi}{2y_3}, \frac{(k+1)\pi}{y_3}\right)$, it still happens that, definitely in k, $a_k \in \left(\frac{(2k+1)\pi}{2y_3}, \frac{(2k+3/2)\pi}{2y_3}\right)$. This is true because, recalling (3.14), we have that

$$g\left(\frac{(2k+3/2)\pi}{2y_3}\right) = -\frac{1}{(2k+3/2)\pi} \quad \text{and} \quad \lim_{a\to\frac{(2k+3/2)\pi}{2y_3}} -h(a) = -e^{-8\pi\alpha y_3}$$

and so there exists a $\bar{k}$, such that $\forall k \geq \bar{k}$, $g > -h$ when both evaluated at the middle point of the k-th interval, which implies that the unique solution is in the first half of the interval. Let y be in $\left(\frac{(2\bar{k}+1)\pi}{2y_3}, \frac{(2\bar{k}+3/2)\pi}{2y_3}\right)$ and $\{y_l\}_{l=0,1,\dots} = \left\{y + \frac{2l\pi}{2y_3}\right\}_{l=0,1,\dots}$. We have that

$$g(y_l) = \frac{\sin(2y_3 y)}{2y_3 y + 2l\pi} < 0 \quad \text{and} \quad -h(y_l) = -e^{-8\pi\alpha y_3 - \frac{2y_3 y + 2l\pi}{\tan(2y_3 y)}}.$$

Both sequences goes to zero, but $\{h(y_l)\}$ does so faster than $\{g(y_l)\}$, which implies that for sufficiently large l, $g(y_l) < h(y_l)$ and so the solution in the $(\bar{k}+l)$-th interval must be in $\left(y_l, \frac{(2(\bar{k}+l)+3/2)\pi}{2y_3}\right)$. The arbitrarity of y in $\left(\frac{(2\bar{k}+1)\pi}{2y_3}, \frac{(2\bar{k}+3/2)\pi}{2y_3}\right)$ leads to (3.15).

So we have that $a_k = \frac{(2k+3/2)\pi}{2y_3} + o(1)$ for $k \to +\infty$. We use this property to estimate $\mathcal{N}_{-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{N}}}(R)$ as $R \to +\infty$ excluding at most a finite quantity of resonances. Doing so is equivalent to find the highest k for which

$$a_k^2 + \frac{1}{4y_3^2}\ln^2\left(-\frac{\sin(2y_3 a_k)}{2y_3 a_k}\right) < R^2.$$

Substituting the asymptotic expression for $k \to +\infty$, we get

$$\left(2k + \frac{3}{2}\right)^2 \pi^2 + \ln^2\left(\left(2k + \frac{3}{2}\right)\pi\right) + o(1) < 4y_3^2 R^2.$$

Setting $s = \left(2k + \frac{3}{2}\right)\pi$, we get

$$s^2 + \ln^2(s) + o(1) < 4y_3^2 R^2.$$

The rest of the proof mimics almost completely the one for the Dirichlet case. □

3.2.3. Robin Boundary Condition

Recalling that the representation (2.40) holds only in the region $\operatorname{Im} z > -\eta$, in the study of resonances for $-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{R},\eta}$, we will limit ourselves to this region.

Proposition 3.2.5. *The resonances z^2 of $-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{R},\eta}$ with corresponding z in the horizontal strip $-\eta < \operatorname{Im} z \leq 0$, if existent, are a finite quantity and all correspond to z contained in the region*

$$\{\, z \in \mathbb{C} \mid -\tau < \operatorname{Re} z < \tau,\ -\eta < \operatorname{Im} z \leq 0 \,\}$$

and $0 < \tau < \frac{e^{2y_3\eta}}{2y_3} + 3\eta$.

Proof. The resonances z^2 correspond to the solution z of (2.45) with negative imaginary part. So, (2.45), has to be solved in the horizontal strip $-\eta < \operatorname{Im} z \leq 0$. For convenience we substitute $w = \eta - iz$, obtaining the new equation

$$\alpha + \frac{w-\eta}{4\pi} - \frac{e^{-2x_{0,3}(w-\eta)}}{8\pi y_3} + \frac{\eta}{2\pi}e^{2y_3\eta}E_1(2y_3 w) = 0, \quad 0 < \operatorname{Re} w \leq \eta. \tag{3.16}$$

Now we prove that all the z in this strip which generate resonances must be contained in the rectangle $R_{\eta,\tau} = \{w \in \mathbb{C} \,|\, 0 < \operatorname{Re} w \leq \eta,\ -\tau < \operatorname{Im} w < \tau\}$ for some $\tau > 0$. Let $w = a + ib$ with $0 < a \leq \eta$ fixed. Plugging in the previous equation we get

$$\alpha + \frac{a-\eta}{4\pi} + \frac{ib}{4\pi} - \frac{1}{8\pi y_3}e^{2y_3(\eta-a)}e^{-2iy_3 b} + \frac{\eta}{2\pi}e^{2y_3\eta}E_1(2y_3(a+ib)) = 0. \tag{3.17}$$

Let's take the limit for $|b| \to +\infty$ of the left side and consider the absolute value of the summands

- the first two terms are constant;
- the third's modulus diverges to $+\infty$;
- the fourth is constant in absolute value (the variation of b changes only its phase);
- by (C.3), the last term tends to 0 for $|b| \to +\infty$.

This means that in the limit, the left side's modulus goes to $+\infty$ and so the equation cannot have solution in a sufficiently small neighborhood of infinity, or otherwise that it can't have solutions for $|b| \geq \tau$ for some $\tau > 0$ as anticipated.

We now prove that the number of resonances in $R_{\eta,\tau}$ is finite. If they are an infinite quantity then, there must be some number of accumulation points in $\overline{R_{\eta,\tau}}$ of zeros of the left side of (3.16). We divide $R_{\eta,\tau}$ in the three regions $R_1 = \{w \in \mathbb{C} \,|\, 0 < \operatorname{Re} w \leq \eta,\ \varepsilon < \operatorname{Im} w < \tau\}$, $R_2 = \{w \in \mathbb{C} \,|\, 0 < \operatorname{Re} w \leq \eta,\ -\tau < \operatorname{Im} w < -\varepsilon\}$ and $R_3 = \{w \in \mathbb{C} \,|\, 0 < \operatorname{Re} w \leq \eta,\ -\varepsilon < \operatorname{Im} w < \varepsilon\}$. Since the function considered is defined in $\overline{R_1}$, it can be extended in the open set in $\widetilde{R_{\eta,\tau}} = \{w \in \mathbb{C} \,|\, -\varepsilon < \operatorname{Re} w \leq \eta,\ \varepsilon < \operatorname{Im} w < \tau\}$. Here the function is holomorphic and so it having an accumulation point of zeros would imply the function being identically zero. So R_1 does not contain accumulating resonances. The same is valid for R_2. In $\overline{R_3}$ the function is not holomorphic, it not being defined in 0. Then 0 can be an accumulation point of zeros. But (C.4) implies that the last term in (3.16) is the only one diverging in absolute value and so the left side cannot be equal to zero in a small neighbourhood of 0. So there cannot be accumulating resonances in $R_{\eta,\tau}$ and then in this set the resonances are a finite quantity.

Now we give an upper bound for τ. Considering the first equality in (C.2), we can write

$$\begin{aligned}
E_1(2y_3(a+ib)) &= e^{-2y_3(a+ib)} \int_0^{+\infty} \frac{e^{-t}}{t + (2y_3(a+ib))}\,\mathrm{d}t \\
&= e^{-2y_3 a}\Big(\cos(2y_3 b) - i\sin(2y_3 b)\Big)(I_1 - 2iy_3 b I_2) \\
&= e^{-2y_3 a}\Big(\cos(2y_3 b) I_1 - 2y_3 b \sin(2y_3 b) I_2 \\
&\qquad - i\Big(\sin(2y_3 b) I_1 + 2y_3 b \cos(2y_3 b) I_2\Big)\Big),
\end{aligned}$$

where

$$I_1 = \int_0^{+\infty} e^{-t} \frac{t + 2y_3 a}{(t+2y_3 a)^2 + 4y_3^2 b^2}\,\mathrm{d}t \text{and} \quad I_2 = \int_0^{+\infty} \frac{e^{-t}}{(t+2y_3 a)^2 + 4y_3^2 b^2}\,\mathrm{d}t.$$

Plugging this expression in (3.17) and separating real and imaginary part we get the system

$$\begin{cases}
\alpha + \frac{a-\eta}{4\pi} - \frac{1}{8\pi y_3} e^{2y_3(\eta-a)} \cos(2y_3 b) + \frac{\eta}{2\pi} e^{2y_3(\eta-a)}\Big(\cos(2y_3 b) I_1 - 2y_3 b \sin(2y_3 b) I_2\Big) = 0 \\
\frac{b}{4\pi} + \frac{1}{8\pi y_3} e^{2y_3(\eta-a)} \sin(2y_3 b) = \frac{\eta}{2\pi} e^{2y_3(\eta-a)}\Big(\sin(2y_3 b) I_1 + 2y_3 b \cos(2y_3 b) I_2\Big)
\end{cases}. \tag{3.18}$$

Now we consider the second equation and get an upper bound for the absolute value of the right end side. First we consider I_1

$$I_1 = \int_0^{+\infty} e^{-t} \frac{t + 2y_3 a}{(t + 2y_3 a)^2 + 4y_3^2 b^2} \,\mathrm{d}t = \int_0^{+\infty} e^{-t} f(t) \,\mathrm{d}t \leq \max_{0 \leq t < +\infty} f(t).$$

But

$$f'(t) = \frac{4y_3^2 b^2 - (t + 2y_3 a)^2}{\Big((t + 2y_3 a)^2 + 4y_3^2 b^2\Big)^2},$$

which is positive in $-2y_3(|b| + a) \leq t \leq 2y_3(|b| - a)$. Since we are looking for an upper bound on τ, we are considering a region in which $|b| > a$, so the sought maximum is $f\Big(2y_3(|b| - a)\Big) = (4y_3\,|b|)^{-1} \geq I_1$. In the same way

$$I_2 = \int_0^{+\infty} \frac{e^{-t}}{(t + 2y_3 a)^2 + 4y_3^2 b^2} \,\mathrm{d}t = \int_0^{+\infty} e^{-t} g(t) \,\mathrm{d}t \leq g(0) = \frac{1}{4y_3^2(a^2 + b^2)},$$

because g is decreasing on the whole $[0, +\infty)$. So the bound follows

$$\begin{aligned}
\left|\frac{\eta}{2\pi} e^{2y_3\eta}\Big(\sin(2y_3 b) I_1 + 2y_3 b \cos(2y_3 b) I_2\Big)\right| &\leq \frac{\eta}{2\pi} e^{2y_3\eta}(I_1 + 2y_3\,|b|\,I_2) \\
&\leq \frac{\eta}{2\pi} e^{2y_3\eta}\left(\frac{1}{4y_3\,|b|} + \frac{|b|}{2y_3(a^2 + b^2)}\right) \\
&\leq \frac{3}{8\pi y_3} \eta e^{2y_3\eta} \frac{1}{|b|}.
\end{aligned}$$

We also consider a lower bound for the absolute value of the right side of the same equation

$$\left|\frac{b}{4\pi} + \frac{1}{8\pi y_3} e^{2y_3(\eta - a)} \sin(2y_3 b)\right| \geq \frac{|b|}{4\pi} - \frac{1}{8\pi y_3} e^{2y_3(\eta - a)} \left|\sin(2y_3 b)\right|.$$

Here the absolute value on the right side is omitted, because for $|b|$ large enough the expression will be positive. Moreover

$$\left|\frac{b}{4\pi} + \frac{1}{8\pi y_3} e^{2y_3(\eta - a)} \sin(2y_3 b)\right| \geq \frac{|b|}{4\pi} - \frac{e^{2y_3\eta}}{8\pi y_3}.$$

Now we take the inequality

$$\frac{|b|}{4\pi} - \frac{e^{2y_3\eta}}{8\pi y_3} > \frac{3}{8\pi y_3} \eta e^{2y_3\eta} \frac{1}{|b|}.$$

The set of solutions of this inequality cannot contain any solution of the second equation in (3.18) and it is equivalent to $2y_3 |b|^2 - e^{2y_3\eta} |b| - 3\eta e^{2y_3\eta} > 0$. The set of solution is

$$|b| > \frac{e^{2y_3\eta} + \sqrt{e^{4y_3\eta} + 24y_3\eta e^{2y_3\eta}}}{4y_3}$$

and so

$$\tau < \frac{e^{2y_3\eta} + \sqrt{e^{4y_3\eta} + 24y_3\eta e^{2y_3\eta}}}{4y_3} = \frac{e^{2y_3\eta} + e^{2y_3\eta}\sqrt{1 + 24y_3\eta e^{-2y_3\eta}}}{4y_3}.$$

By the identity $\sqrt{1+t} < 1 + t/2$, we have

$$\tau < \frac{e^{2y_3\eta} + e^{2y_3\eta}\left(1 + 12y_3\eta e^{-2y_3\eta}\right)}{4y_3} = \frac{e^{2y_3\eta}}{2y_3} + 3\eta.$$

□

3.3. Resonances for the Half-Plane

3.3.1. Dirichlet Boundary Condition

We study the resonances in a horizontal strip below the real axis.

Proposition 3.3.1. *If* $\alpha \neq \ln(2y_2)/(2\pi)$ *all the resonances* z^2 *of* $-\Delta_{\alpha,y}^{\mathbb{H}^2,\mathrm{D}}$ *with* z *in the strip* $-\chi < \operatorname{Im} z < 0$ *(if existent) have actually* z *contained in the smaller strip* $-\chi < \operatorname{Im} z < -\varepsilon$ *for some* $\varepsilon > 0$*. Moreover there exist a* $\tau > 0$ *such that all these resonances* z^2 *correspond to* z *such that* $|\operatorname{Re} z| < \tau$ *with*

$$\tau < 2e^{\pi e^{2y_2\chi} + \frac{1}{2}\sqrt{\frac{\pi}{y_2\varepsilon}} - 2\pi\alpha - \gamma}.$$

Proof. The resonances are given by those z^2 such that z is a solution of (2.48) with negative imaginary part. We start by looking for possible resonances on the negative imaginary semiaxis. To do so we substitute $z = -ia$ with $a > 0$ in (2.48). It becomes

$$\alpha + \frac{\ln\frac{-a}{2} + \gamma}{2\pi} + \frac{i}{4}H_0^{(1)}(-2iy_2a) = 0.$$

It holds that

$$\ln w = \begin{cases} \ln w - i\pi & \text{if} \quad \operatorname{Arg} w > 0 \\ \ln w + i\pi & \text{if} \quad \operatorname{Arg} w \leq 0 \end{cases}.$$

Employing this, (C.18) with $\nu = 0$, together with (C.27), the previous equation is written as

$$\alpha + \frac{\ln\frac{a}{2} + i\pi + \gamma + K_0(2y_2 a)}{2\pi} - \frac{i}{2} I_0(2y_2 a) = 0.$$

The equation can have a solution only if $I_0(2y_2a) = 1$, which happens only for $a = 0$ and so no resonance can exist on the negative imaginary semiaxis.

Now we study the resonances generated by z in the strip $-\chi \le \operatorname{Im} z < 0$. First we look at possible resonances close to the real axis. Since there are no real z solving (2.48) and the left side of (2.48) is continuous outside 0, for each point of the real axis excepted zero, there is a neighborhood in which there are no solutions of (2.48). This means that for each non-zero real, there is a neighbourhood in which there are no z corresponding to resonances. To determine if this also holds for zero, we study (2.48) near the origin by substituting (C.11) in it. It becomes

$$2\pi\alpha + \ln\frac{z}{2i} + \gamma + \frac{i\pi}{2} - \ln(y_2 z) - \gamma + O(z^2 \ln z) = 0.$$

Given that

$$\ln\frac{z}{2i} = \begin{cases} \ln\frac{z}{2} + \frac{3}{2}\pi i & \text{if} \quad \operatorname{Arg} z \le -\frac{\pi}{2} \\ \ln\frac{z}{2} - \frac{i\pi}{2} & \text{if} \quad \operatorname{Arg} z > -\frac{\pi}{2} \end{cases}, \tag{3.19}$$

we obtain the two expressions

$$\begin{cases} 2\pi\alpha - \ln(2y_2) + 2i\pi + O(z^2\ln z) = 0 & \operatorname{Arg} z \le -\frac{\pi}{2} \\ 2\pi\alpha - \ln(2y_2) + O(z^2\ln z) = 0 & \operatorname{Arg} z > -\frac{\pi}{2} \end{cases}.$$

In the first case, independently on the values of α and y_2, the expression tends to a quantity different from zero (the limit of the imaginary part is 2π). In the second one, instead for $\alpha = \frac{1}{2\pi}\ln(2y_2)$, the expression tends to zero and the possibility of accumulating zeros can't be excluded. Finally we also exclude the possibility of resonances accumulating closer and closer the real axis when $|\operatorname{Re} z| \to +\infty$. To verify this, we take the limit $z \to \pm\infty$ in (2.48). It turns out that, by using (C.14), (C.15) and (C.16), the left hand side of (2.48) does not tend to zero and so the equation is not "asymptotically solved". So we can claim that for $\alpha \ne \frac{1}{2\pi}\ln(2y_2)$, for $\varepsilon > 0$ small enough there are no z corresponding to resonances in the strip $-\varepsilon < \operatorname{Im} z < 0$.

So we can limit our study to the strip $-\chi < \operatorname{Im} z < -\varepsilon$. Recalling (C.9), we can rewrite (2.48) as

$$\alpha + \frac{\ln\frac{z}{2i} + \gamma}{2\pi} + \frac{i}{4} J_0(2y_2 z) - \frac{1}{4} Y_0(2y_2 z) = 0. \tag{3.20}$$

The lower complex half-plane is contained in $-\pi \le \operatorname{Arg} z \le \frac{\pi}{2}$. Hence both (C.28) anf (C.29) hold and we can plug them into (3.20) obtaining

$$2\pi\alpha + \ln\frac{z}{2i} + \gamma = -i\pi J_0(2y_2 z) - K_0(2iy_2 z). \tag{3.21}$$

We look for an upper bound of the modulus of the right hand side. The bound

$$|K_0(2iy_2z)| \leq \int_0^{+\infty} e^{-2y_2|\operatorname{Im} z|\cosh t}\,\mathrm{d}t$$

is a consequence of (C.30). But, recalling that $\cosh t \geq \frac{t^2}{2}$, we have

$$|K_0(2iy_2z)| \leq \int_0^{+\infty} e^{-y_2|\operatorname{Im} z|t^2}\,\mathrm{d}t = \frac{1}{2}\sqrt{\frac{\pi}{y_2\,|\operatorname{Im} z|}} \leq \frac{1}{2}\sqrt{\frac{\pi}{y_2\varepsilon}}.$$

Using this in conjunction with (C.7), the bound

$$|-i\pi J_0(2y_2z) - K_0(2iy_2z)| \leq \pi e^{2y_2\chi} + \frac{1}{2}\sqrt{\frac{\pi}{y_2\varepsilon}}$$

is then immediate. A lower bound for the modulus of the right end of (3.21) is instead given by

$$\left|2\pi\alpha + \ln\frac{z}{2i} + \gamma\right| \geq \left|2\pi\alpha + \ln\left|\frac{z}{2}\right| + \gamma\right| \geq \left|2\pi\alpha + \frac{1}{2}\ln\left(\frac{(\operatorname{Re} z)^2 + \varepsilon^2}{4}\right) + \gamma\right|.$$

The set of solution of the inequality

$$\left|2\pi\alpha + \frac{1}{2}\ln\left(\frac{(\operatorname{Re} z)^2 + \varepsilon^2}{4}\right) + \gamma\right| > e^{2y_2\chi} + \frac{1}{2}\sqrt{\frac{\pi}{y_2\varepsilon}}$$

in $\operatorname{Re} z$ cannot contain any resonance. This set is given by

$$\left(\operatorname{Re} z\right)^2 > 4e^{2\pi e^{2y_2\chi} + \sqrt{\frac{\pi}{y_2\varepsilon}} - 4\pi\alpha - 2\gamma} - \varepsilon^2 \quad \vee \quad \left(\operatorname{Re} z\right)^2 < 4e^{-2\pi e^{2y_2\chi} - \sqrt{\frac{\pi}{y_2\varepsilon}} - 4\pi\alpha - 2\gamma} - \varepsilon^2.$$

By the arbitrarity of ε and the fact that $Ae^{-B\varepsilon^{-1/2}}$ with $A, B > 0$ vanishes faster than ε^2, we have that the second inequality is never verified for ε small enough. So we have proved that exists some $\tau > 0$ such that all the resonances in the strip $-\chi < \operatorname{Im} z < 0$ satisfy $|\operatorname{Re} z| < \tau$ and it holds

$$\tau < \sqrt{4e^{2\pi e^{2y_2\chi} + \sqrt{\frac{\pi}{y_2\varepsilon}} - 4\pi\alpha - 2\gamma} - \varepsilon^2} < 2e^{\pi e^{2y_2\chi} + \frac{1}{2}\sqrt{\frac{\pi}{y_2\varepsilon}} - 2\pi\alpha - \gamma}.$$

□

3.3.2. Neumann Boundary Condition

Proposition 3.3.2. *All the resonances z^2 of $-\Delta_{\alpha,y}^{\mathbb{H}^2,\mathrm{N}}$ with z in the strip $-\chi < \operatorname{Im} z < 0$ (if existent) have z actually contained in the smaller strip $-\chi < \operatorname{Im} z < -\varepsilon$ for some $\varepsilon > 0$. Moreover there exist a $\tau > 0$ such that all these resonances z^2 correspond to z such that $|\operatorname{Re} z| < \tau$ with*

$$\tau < 2e^{\pi e^{2y_2\chi} + \frac{1}{2}\sqrt{\frac{\pi}{y_2\varepsilon}} - 2\pi\alpha - \gamma}$$

Proof. We start by examining the simples case of resonances on the negative imaginary semi-axis. To do so we set $z = -ia$ with $a > 0$ in $\Gamma^{\mathbb{H},\mathrm{N}}_{\alpha,y}(z) = 0$, obtaining

$$\alpha + \frac{\ln \frac{-a}{2} + \gamma}{2\pi} - \frac{i}{4} H_0^{(1)}(-2iy_2 a) = 0.$$

Using the same properties of Bessel function as in the Dirichlet case, this equation is equivalent to

$$\alpha + \frac{\ln \frac{a}{2} + i\pi + \gamma - K_0(2y_2 a)}{2\pi} + \frac{i}{2} I_0(2y_2 a) = 0.$$

Taking the imaginary part of the equation, we see that it can be verified only if $I_0(2y_2 a) = -1$. This equation has no real solutions and so there are no resonances on the negative imaginary semi-axis.

There are no real solution of $\Gamma^{\mathbb{H},\mathrm{N}}_{\alpha,y}(z) = 0$ outside zero. We also prove that there exists a neighbourhood of zero in which there are no solution of that equation. Substituting (C.11) into $\Gamma^{\mathbb{H},\mathrm{N}}_{\alpha,y}(z) = 0$, leads to

$$2\pi\alpha + \ln \frac{z}{2i} + 2\gamma - \frac{i\pi}{2} + \ln(y_2 z) + O(z^2 \ln z) = 0,$$

which, combined with (3.19) implies

$$\begin{cases} 2\ln z + 2\pi\alpha + 2\gamma + \ln(\frac{y_2}{2}) + i\pi + O(z^2 \ln z) = 0 & \operatorname{Arg} z \leq -\frac{\pi}{2} \\ 2\ln z + 2\pi\alpha + 2\gamma + \ln(\frac{y_2}{2}) - i\pi + O(z^2 \ln z) = 0 & \operatorname{Arg} z > -\frac{\pi}{2} \end{cases}.$$

In both cases, for $z \to 0$, the first term's absolute value diverges to $+\infty$, while all the other terms stay finite. This means that the absolute value of both the expressions diverges in the limit and therefore there is a small neighbourhood of zero where the equation has no solutions. In the same way as in the Dirichlet case one also prove that there are no solution of $\Gamma^{\mathbb{H},\mathrm{N}}_{\alpha,y}(z) = 0$ accumulating to the real axis when $|\operatorname{Re} z| \to +\infty$. All these facts together mean that all z that generate resonances must obey $\operatorname{Im} z < -\varepsilon$ for some small $\varepsilon > 0$.

Then we are interested in fixing $\chi > 0$ and studying the resonances in the region $-\chi < \operatorname{Im} z < -\varepsilon$. Since the equation in this case differs from (3.21) only by an opposite side in the left end side, we have that the same precise steps as the one for the Dirichlet case bring us to claim that all solutions in the strip must obey $|\operatorname{Re} z| < \tau$ with the same upper bound for τ. □

3.4. Why Resonances?

After having established the definition of resonances and studied them for $-\Delta^{\mathbb{H}^n}_{\alpha,y}$, one can ask what is their purpose.

Resonances play a role in the description of oscillations and decay of waves propagating in non-compact domains. For example in [18] (pp. 39-45 and 110-111) an asymptotic expansion of the solution of the wave equation for large times is constructed. It is also important that the initial data is compactly supported, so that the slow decay at infinity of the resonance functions is not a concern This expansion is proved when the wave is scattered on a $L^\infty_{\text{comp}}(\mathbb{R}^n)$ potential. Here we prove a similar result for a point interaction on the half-space.

First we give a bound for the resolvent in a region with finite resonances.

Theorem 3.4.1. *Let $\alpha \in \mathbb{R}$ and $y = (y_1, y_2, y_3) \in \mathbb{H}^3$. For any $\rho \in C_0^\infty(\mathbb{H}^3)$ there exist constants A, C and T depending on ρ such that*

$$\left\|\rho(-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}} - z^2)^{-1}\rho\right\|_{L^2(\mathbb{H}^3)\to H^j(\mathbb{H}^3)} \le C\,|z|^{j-1}\,e^{T(\operatorname{Im} z)_-}, \quad j = 0, 1, 2, \tag{3.22}$$

($(x)_- = \max\{-x, 0\}$) for

$$\operatorname{Im} z \ge -A - \delta \ln(1 + |z|), \quad |z| > C_0, \quad \delta < \frac{1}{2y_3}$$

and z not being such that z^2 is a resonance. In particular, there are only finitely many z in the region

$$\{\, z \in \mathbb{C} \mid \operatorname{Im} z \ge -A - \delta \ln(1 + |z|) \,\}, \tag{3.23}$$

for any $A > 0$ that correspond to resonances z^2.

Proof. We start by proving that there are a finite number of resonances in the region (3.23). As shown in the upcoming section, all resonances are on the curve (3.3). This means that the curve

$$\operatorname{Im} z = \frac{1}{2y_3} \ln\left(\frac{1}{2y_3\,|\operatorname{Re} z|}\right),$$

equivalent to

$$|\operatorname{Re} z| = \frac{1}{2y_3} e^{-2y_3 \operatorname{Im} z} \tag{3.24}$$

is above all resonances. The equation of the curve which delimits the region (3.23) can be written in explicit form as follows

$$|\operatorname{Re} z| = \sqrt{\left(e^{-\frac{\operatorname{Im} z + A}{\delta}} - 1\right)^2 - \left(\operatorname{Im} z\right)^2}. \tag{3.25}$$

Both curves are symmetric with respect to the imaginary axis, so we limit ourself to $\operatorname{Re} z > 0$. If it happens that for $\operatorname{Im} z$ large enough, curve (3.25) is above curve (3.24), then for $\operatorname{Im} z$ large enough all resonances are below (3.25) and hence only a finite quantity of them is in the region described by (3.23). This scenario happens for $\frac{1}{\delta} > 2y_3$ regardless of the value of A.

In order to prove (3.22) we use (1.34) and write

$$\begin{aligned}\left\|\rho(-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}-z^2)^{-1}\rho\right\|_{L^2(\mathbb{H}^3)\to H^j(\mathbb{H}^3)} &\le \left\|\rho(-\Delta^{\mathbb{H}^3,\mathrm{D}}-z^2)^{-1}\rho\right\|_{L^2(\mathbb{H}^3)\to H^j(\mathbb{H}^3)}\\ &+\left|\Gamma_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}(z)\right|^{-1}\left\|\rho G_{z,y}^{\mathbb{H}^3,\mathrm{D}}(-\Delta^{\mathbb{H}^3,\mathrm{D}}-z^2)^{-1}\rho\right\|_{L^2(\mathbb{H}^3)\to H^j(\mathbb{H}^3)}. \quad (3.26)\end{aligned}$$

We recall the following bound for the free truncated resolvent ([18] Theorem 3.1)

$$\left\|\rho(-\Delta-z^2)^{-1}\rho\right\|_{L^2(\mathbb{H}^3)\to H^j(\mathbb{H}^3)} \le C(1+|z|)^{j-1}e^{L(\operatorname{Im} z)_-}, \quad j=0,1,$$

where $L>\operatorname{diam}(\operatorname{supp}(\rho))$ and that is also valid if we replace $-\Delta$ with $-\Delta^{\mathbb{H}^3,\mathrm{D}}$. We also observe that if $|z|<C_0$, then $1+|z|<\frac{C_0+1}{C_0}|z|$. So there exists C (we don't change the constants name in bounds) such that

$$\left\|\rho(-\Delta-z^2)^{-1}\rho\right\|_{L^2(\mathbb{H}^3)\to H^j(\mathbb{H}^3)} \le C\,|z|^{j-1}\,e^{L(\operatorname{Im} z)_-}, \quad j=0,1,2.$$

This directly bounds the first term in (3.26). It does the same for the second one if we observe that $G_{z,y}^{\mathbb{H}^3,\mathrm{D}}$ is in $L^2_{\mathrm{loc}}(\mathbb{H}^3)$ and that outside of resonances $\left|\Gamma_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}(z)\right|^{-1}$ is bounded. Hence (3.22) follows. □

The following theorem describe this resonance expansion of the solution of the wave equation.

Theorem 3.4.2. *Let $\alpha\in\mathbb{R}$, $y=(y_1,y_2,y_3)\in\mathbb{H}^3$ and $R(z)=\left(-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}-z^2\right)^{-1}$. Suppose $w(t,x)$ is the solution of*

$$\begin{cases}\left(\frac{\partial^2}{\partial t^2}-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}\right)w(t,x)=0\\ w(0,x)=w_0(x)\in H^1_{\mathrm{comp}}(\mathbb{H}^3)\\ \frac{\partial w}{\partial t}(0,x)=w_1(x)\in L^2_{\mathrm{comp}}(\mathbb{H}^3)\end{cases}. \quad (3.27)$$

Let $\left\{z_j^2\right\}$ be the set of resonances of $\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}$ and λ the possible negative eigenvalue of $\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}$ (see Proposition 2.2.1 for the condition for it to appear).

Then, for any $A>0$,

- *if $\alpha\neq-\frac{1}{8\pi y_3}$*

$$w(t,x)=\sum_{\{\operatorname{Im} z_j>-A\}\cup\left\{i\sqrt{|\lambda|}\right\}} e^{-iz_jt}f_j(x)+\quad E_A(t),$$

where the sum is finite and

$$f_j(x)=-\operatorname{Res}_{z=z_j}\left(iR(z)w_1+zR(z)w_0\right); \quad (3.28)$$

- *if* $\alpha = -\frac{1}{8\pi y_3}$

$$w(t,x) = \sum_{\{\operatorname{Im} z_j > -A\}} e^{-iz_j t} f_j(x) + f_{0,0}(x) + t f_{0,1}(x) + \quad E_A(t),$$

where the sum is finite, f_j *are still defined as in (3.28) and*

$$f_{0,0}(x) + t f_{0,1}(x) = -\operatorname{Res}_{z=0}\Big(iR(z)w_1 + zR(z)w_0\Big).$$

moreover, for any compact, connected set K *such that* $\operatorname{supp}(w_j) \subset K,\ j = 0, 1,$ *there exist constants* $C_{K,A}$ *and* $T_{K,A}$ *such that*

$$\|E_A(t)\|_{H^2(K)} \le C_{K,A} e^{-tA}(\|w_0\|_{H^1(\mathbb{H}^3)} + \|w_1\|_{L^2(\mathbb{H}^3)}), \quad t \ge T_{K,A}.$$

Proof. We start by considering the case $w_0 = 0$ and $\operatorname{supp}(w_1) \subset K$.

By the functional calculus, the propagator of (3.27) can be written as

$$U(t) = \int_0^{+\infty} \frac{\sin tz}{z}\, \mathrm{d}E_z + \frac{\sin t\lambda}{\lambda}\Big(\cdot, G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}\Big) G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}. \tag{3.29}$$

If zero is an eigenvalue, the last term is replaced by

$$t\Big(\cdot, G^{\mathbb{H}^3,\mathrm{D}}_{0,y}\Big) G^{\mathbb{H}^3,\mathrm{D}}_{0,y}.$$

Since for z near $\lambda \neq 0$, $R(z) = (\lambda^2 - z^2)^{-1}\Big(\cdot, G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}\Big) G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y} + Q(z)$, where $Q(z)$ is holomorphic near λ and $\operatorname{Res}_{z=\lambda}(\lambda^2 - z^2) = -(2\lambda)^{-1}$, we have

$$\frac{\sin t\lambda}{\lambda}\Big(\cdot, G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}\Big) G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y} = -\operatorname{Res}_{z=\lambda}(iR(z)e^{-izt}) - \operatorname{Res}_{z=-\lambda}(iR(-z)e^{-izt}) = \Pi_\lambda + \Pi_{-\lambda}. \tag{3.30}$$

If $\lambda = 0$, instead

$$t\Big(\cdot, G^{\mathbb{H}^3,\mathrm{D}}_{0,y}\Big) G^{\mathbb{H}^3,\mathrm{D}}_{0,y} = -t\operatorname{Res}_{z=0}(iR(z)). \tag{3.31}$$

In the following, the term for $\lambda \neq 0$ will be reported whenever it is not needed to distinguish the two cases.

By Stone's formula for the spectral measure $\mathrm{d}E_z$ in terms of $R(z)$, we have

$$\mathrm{d}E_z = \frac{1}{\pi i}(R(z) - R(-z))z\,\mathrm{d}z.$$

Hence

$$\begin{aligned} w(t,x) - \frac{\sin t\lambda}{\lambda}\Big(\cdot, G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y}\Big)G^{\mathbb{H}^3,\mathrm{D}}_{\sqrt{\lambda},y} &= \frac{1}{\pi i}\int_0^{+\infty} \sin tz(R(z)-R(-z))w_1(x)\,\mathrm{d}z \\ &= \frac{1}{2\pi}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} e^{-itz}(R(z)-R(-z))w_1(x)\,\mathrm{d}z \\ &= \frac{1}{2\pi}\int_{\Sigma_{\varepsilon_0}} e^{-itz}(R(z)-R(-z))w_1(x)\,\mathrm{d}z \\ &\quad + \frac{1}{2\pi}\lim_{\varepsilon\to 0^+}\int_{\sigma_\varepsilon} e^{-itz}(R(z)-R(-z))w_1(x)\,\mathrm{d}z, \end{aligned}$$

where ε_0 is such that there are no non-zero poles of $R(z)$ in $D(0,\varepsilon_0)$, Σ_{ε_0} is the union of $\mathbb{R}\setminus(-\varepsilon_0,\varepsilon_0)$ with the semicircle in the upper complex half-plane of radius ε_0 and centered at zero oriented counterclockwise. σ_ε instead is the semicircle in the upper complex half-plane of radius ε, centered at zero and oriented clockwise.

To prove that the integrand decays fast enough for the integral on Σ_{ε_0} to be convergebt we observe that, by the definition of $R(z)$, follows that

$$\Big(R(z)-R(-z)\Big)\Big(-\Delta^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}\Big) = (R(z)-R(-z))\Big(-\Delta^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y} - z^2 + z^2\Big) = z^2\Big(R(z)-R(-z)\Big).$$

From this we can conclude that for $\rho \in C_0^\infty(\mathbb{H}^3)$ equal to one on $\operatorname{supp}(w_1)$,

$$\begin{aligned} \rho\Big(R(z)-R(-z)\Big)\Big(1-\Delta^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}\Big)\rho w_1 &= \rho\Big(R(z)-R(-z)\Big)w_1 + \rho\Big(R(z)-R(-z)\Big)\Big(-\Delta^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}\Big)w_1 \\ &= (1+z^2)\rho\Big(R(z)-R(-z)\Big)w_1 \end{aligned}$$

and so

$$\rho\Big(R(z)-R(-z)\Big)w_1 = \rho\Big(R(z)-R(-z)\Big)(1+z^2)^{-1}\Big(1-\Delta^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}\Big)\rho w_1. \tag{3.32}$$

This, together with (3.22) shows that the integral converges in $L^2_{\mathrm{loc}}(\mathbb{H}^3)$.

The resolvent is holomorphic in a neighborhhod of the origin and hence the integral over σ_ε converges to 0 as $\varepsilon \to 0^+$, unless zero is an eigenvalue for $-\Delta^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}$. In that case, we study the expression $R(z)-R(-z)$ near zero. We recall that 0 is an eigenvalue if and only if $\alpha = -\frac{1}{8\pi y_3}$. In this case

$$R(z)-R(-z) = \frac{e^{iz|x-y|}}{4\pi|x-y|}\frac{e^{iz|x'-y|}}{4\pi|x'-y|}\Big(\Gamma^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}(z)\Big)^{-1} - \frac{e^{-iz|x-y|}}{4\pi|x-y|}\frac{e^{-iz|x'-y|}}{4\pi|x'-y|}\Big(\Gamma^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}(-z)\Big)^{-1}.$$

Some computations show that

$$\mathrm{Res}_{z=0}\Big(R(z)-R(-z)\Big) = \frac{i}{\pi y_3|x-y|\,|x'-y|}\left(\frac{y_3}{3} - \frac{|x-y|+|x'-y|}{2}\right),$$

which means that

$$\frac{1}{2\pi}\lim_{\varepsilon\to 0^+}\int_{\sigma_\varepsilon} e^{-itz}(R(z)-R(-z))w_1(x)\,\mathrm{d}z = \frac{w_1(x)}{2\pi y_3\,|x-y|\,|x'-y|}\left(\frac{y_3}{3}-\frac{|x-y|+|x'-y|}{2}\right)$$
$$= \Pi_0 w_1(x).$$

Let $\rho\in C_0^\infty(\mathbb{H}^3)$ satisfy $\rho=1$ on K. We choose R large enough so that all the resonances z^2 with $\operatorname{Im} z > -A-\delta\ln(1+|\operatorname{Re} z|)$ are contained in the ball $|z|\le R$. We deform the contour of integration in the integral over Σ_{ε_0} using the following contours:

$$\omega_R = \left\{\, s - i\Big(A+\varepsilon+\delta\ln(1+|s|)\Big) \;\Big|\; s\in[-R,R]\,\right\} \tag{3.33}$$

$$\gamma_R^\pm = \{\, \pm R - is \mid 0\le s\le A+\varepsilon+\delta\ln(1+R)\,\}, \quad \gamma_R=\gamma_R^+\cup\gamma_R^- \tag{3.34}$$

$$\gamma_R^\infty = (-\infty,R)\cup(R,+\infty). \tag{3.35}$$

ε is chosen in such a way that ω_R does not cross resonances. We also define

$$\Omega_A = \{\, z \mid \operatorname{Im} z \ge -A-\varepsilon-\delta\ln(1+|\operatorname{Re} z|)\,\}\setminus\{\,0\,\}$$

and define

$$\Pi_A(t) = i\sum_{z_j\in\Omega_A}\operatorname{Res}_{z=z_j}\Big(\rho R(z)\rho e^{-izt}\Big).$$

By combining (3.29), (3.30) (or (3.31) when zero is an eigenvalue) and the residue theorem, we can write that

$$\rho U(t)\rho = \Pi_0\rho + \Pi_A(t) + E_{\omega_R}(t) + E_{\gamma_R}(t) + E_{\gamma_R^\infty}(t) + \Pi_\lambda + \Pi_{-\lambda}, \tag{3.36}$$

where

$$E_\gamma(t) = \frac{1}{2\pi}\int_\gamma e^{-itz}\rho\Big(R(z)-R(-z)\Big)\rho w_1\,\mathrm{d}z.$$

We note that $\Pi_{-\lambda}$ cancels out with the term in Π_A containing the residue at $z=-\lambda$.

Now we prove that the integrals in (3.36) are negligible in the limit $R\to+\infty$ for t large enough. Using (3.32), we bound

$$\begin{aligned}
\left\|E_{\gamma_R^\infty}(t)w_1\right\|_{H^1(\mathbb{H}^3)} &\le C\left\|\int_R^{+\infty} e^{-itz}\rho\Big(R(z)-R(-z)\Big)\rho w_1\,\mathrm{d}z\right\|_{H_1(\mathbb{H}^3)}\\
&\le C\left\|\int_R^{+\infty}\rho\Big(R(z)-R(-z)\Big)(1+z^2)^{-1}\Big(1-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}\Big)w_1\,\mathrm{d}z\right\|_{H_1(\mathbb{H}^3)}\\
&\le C\int_0^{+\infty}(1+z^2)^{-1}\,\|w_1\|_{H^2(\mathbb{H}^3)} = C\arctan\left(\frac{1}{R}\right)\|w_1\|_{H^2(\mathbb{H}^3)} \le \frac{C}{R}\,\|w_1\|_{H^2(\mathbb{H}^3)}\,.
\end{aligned}$$

While, for the integral over γ_R, it holds that

$$\begin{aligned}\left\|E_{\gamma_R}(t)w_1\right\|_{H^1(\mathbb{H}^3)} &= C\left\|\int_0^{A+\varepsilon+\delta\ln(1+R)} e^{-it(R-is)}\rho\Big(R(R-is)-R(-R+is)\Big)\rho w_1(-i)\,\mathrm{d}s\right\|\\ &\le C\int_0^{A+\varepsilon+\delta\ln(1+R)} |s-it|\,e^{-(t-T)s}\,\|w_1\|_{H^2(\mathbb{H}^3)}\,\mathrm{d}s\\ &\le Ce^{-(t-T)\Big(A+\varepsilon+\delta\ln(1+R)\Big)}\,\|w_1\|_{H^2(\mathbb{H}^3)} \le C(1+R)^{-(t-T)\delta}\,\|w_1\|_{H^2(\mathbb{H}^3)}\,,\end{aligned}$$

where we used (3.22). For $t > T$, this bound vanish for $R \to +\infty$.

We now show that for $t \gg 1$,

$$\left\|E_{\omega_R}(t)w_1\right\|_{H^2(\mathbb{H}^3)} \le Ce^{-tA}\,\|w_1\|_{L^2(\mathbb{H}^3)}\,. \tag{3.37}$$

In order to do so, we use (3.22) with $j = 2$, the fact that on ω_R $|z| \ge A+\varepsilon$ and the assumption that there exist a compact set containing ω_R in which there are no poles of $R(z)$. We obtain

$$\begin{aligned}\left\|E_{\omega_R}(t)w_1\right\|_{H^2(\mathbb{H}^3)} &\le C\int_{-\infty}^{+\infty} e^{-it\Big(s-i(A+\varepsilon+\delta\ln(1+|s|))\Big)}\Big(1+|s|\Big)e^{T\delta\ln(1+|s|)}\,\|w_1\|_{L^2(\mathbb{H}^3)}\,\mathrm{d}s\\ &\le Ce^{-tA}\int_{-\infty}^{+\infty} e^{-\delta(t-T)\ln(1+|s|)}\Big(1+|s|\Big)\,\|w_1\|_{L^2(\mathbb{H}^3)}\,\mathrm{d}s\\ &\le Ce^{-tA}\int_{-\infty}^{+\infty}\Big(1+|s|\Big)^{-\delta(t-T)+1}\,\|w_1\|_{L^2(\mathbb{H}^3)}\,\mathrm{d}s.\end{aligned}$$

This last integral is finite if $-\delta(t-T)+1 \le -2$, which is equivalent to $t \ge T + 3/\delta$ and in this case (3.37) is valid.

Since $C_0^\infty(K) \subset H^2(K)$ is dense in $L^2(K)$, the bounds expressed in term of the H^2 norm are still valid if it is replaced by $\|w_1\|_{L^2(\mathbb{H}^3)}$ for $w_1 \in L^2_{\mathrm{comp}}(\mathbb{H}^3)$ and hence the theorem is proven for $w_0 = 0$.

The proof for arbitrary $w_0 \in H^1_{\mathrm{comp}}(\mathbb{H}^3)$ and $w_1 = 0$ follows the same steps, with the replacement $\sin tz/z \longrightarrow \cos tz$ in the formula for $w(t,x)$. □

The same idea of an expansion in term of resonances can be carried out for the Schrödinger operators of the form $H = -\Delta + V$, where V is such that H is self-adjoint (we include, even if not rigorously phrased, the case of V being a "point interaction potential"). By the spectral decomposition, one has that

$$H = \int_{\sigma(H)} \lambda\,\mathrm{d}E(\lambda)$$

and also

$$H_- = \int_{\sigma(H)\cap(-\infty,0]} \lambda\,\mathrm{d}E(\lambda) \quad H_+ = \int_{\sigma(H)\cap(0,+\infty)} \lambda\,\mathrm{d}E(\lambda).$$

If we further assume that $\sigma(H) \cap (-\infty, 0]$ consists of only a finite number of eigenvalues $z_n^2 < 0$ with corresponding eigenfunctions $\psi_n \in L^2(\mathbb{R}^3)$, by the spectral calculus we can claim that, for any $f \in C_0^\infty(\mathbb{R})$

$$f(H_-) = \sum_n f(z_n^2)(\cdot, \psi_n)\psi_n.$$

Can a similar decomposition be carried out for $f(H_+)$? This is possible for particular choices of f. For example, let $f(\lambda) = e^{-it\lambda^2}$, with $\lambda^2, t \in \mathbb{R}^+$. By functional calculus, it follows that

$$e^{-itH_+}(x, x') = -\frac{1}{2\pi i} \int_0^{+\infty} e^{-itz^2} (H - z^2)^{-1}(x, x') 2z \, \mathrm{d}z. \tag{3.38}$$

The following theorem (see [1] for analogous result in $\mathbb{R}^3$) states that an expansion for e^{-itH_+} is possible and it is made in term of resonances.

Theorem 3.4.3. *Let $H = -\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}$ with $\alpha \neq -\frac{1}{8\pi y_3}$ or $H = -\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{N}}$ with $\alpha \neq \frac{1}{8\pi y_3}$. Then there exists some $t_0(x, x', \alpha, y)$ such that for $t > t_0$ the following expansion holds*

$$\begin{aligned} e^{-itH_+}(x, x') = \sum_n 2z_n e^{-itz_n^2} \operatorname{Res}_{z=z_n} \Big(\big(\Gamma_{\alpha,y}^{\mathbb{H}^3}(z)\big)^{-1} \Big) G_{z_n,y}^{\mathbb{H}^3}(x) G_{z_n,y}^{\mathbb{H}^3}(x') \\ + (2\pi i t)^{-\frac{3}{2}} e^{\frac{i}{2t}|x-x'|^2} - \frac{1}{2\pi} \int_0^{+\infty} \Big(\hat{R}\big(e^{-i\frac{\pi}{4}} s\big) - \hat{R}\big(-e^{-i\frac{\pi}{4}} s\big) \Big) e^{-ts} \, \mathrm{d}s, \end{aligned} \tag{3.39}$$

with $\hat{R}(z) = (H - z^2)^{-1} - (-\Delta - z^2)^{-1}$. The sum in (3.39) is taken over all z_n lying in the region $-\frac{\pi}{4} < \operatorname{Arg} k < 0$ such that z_n^2 is a resonance.

Proof. We give the proof for $-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}}$. The Neumann case is proven similarly. By the definition of $\hat{R}$, (3.38) can be rewritten as

$$e^{-itH_+}(x, x') = -\frac{1}{2\pi i} \int_0^{+\infty} e^{-itz^2} \hat{R}(z)(x, x') 2z \, \mathrm{d}z - \frac{1}{2\pi i} \int_0^{+\infty} e^{-itz^2} G_{z,y}^{\mathbb{H}^3,\mathrm{D}}(x, x') 2z \, \mathrm{d}z.$$

It holds that

$$\frac{1}{2\pi i} \int_0^{+\infty} e^{-itz^2} G_{z,y}^{\mathbb{H}^3,\mathrm{D}}(x, x') 2z \, \mathrm{d}z = (2\pi i t)^{-\frac{3}{2}} e^{i|x-x'|^2 2t}.$$

Instead, because of the parity of $\hat{R}(z)$, we can rewrite the other integral as

$$-\frac{1}{2\pi i} \int_0^{+\infty} e^{-itz^2} \big(\hat{R}(z) - \hat{R}(-z) \big)(x, x') 2z \, \mathrm{d}z.$$

Using the residue theorem, this integral can be written as

$$\frac{1}{2\pi i}\int_{C_R} e^{-itz^2}\Big(\hat{R}(z)-\hat{R}(-z)\Big)(x,x')2z\,\mathrm{d}z + \frac{1}{2\pi i}\int_{\gamma_R} e^{-itz^2}\Big(\hat{R}(z)-\hat{R}(-z)\Big)(x,x')2z\,\mathrm{d}z \\ + \sum_n 2z_n e^{-itz_n^2}\,\mathrm{Res}_{z=z_n}\Big(\hat{R}(z)-\hat{R}(-z)\Big)(x,x'), \quad (3.40)$$

where C_R is the arc of circumference of radius R going clockwise from $z = R$ to $z = Re^{-i\pi/4}$, γ_R is the segment going from $z = Re^{-i\pi/4}$ towards the origin and the sum is over the z_n in the region delimited by C_R, γ_R and the real positive semiaxis.

We prove that the integral over C_R vanishes for $R \to +\infty$. We prove this just for the term with $\hat{R}(z)$. Analogous observations prove the same for $\hat{R}(-z)$. By Proposition 3.2.1, there are infinite resonances z_n and $|z_n| \to +\infty$ for $n \to +\infty$. This means that, in order to study the limit $R \to +\infty$, we have to distinguish the cases

- $R \neq |z_n|$, $\forall n \in \mathbb{N}$;
- $R = |z_n|$, for some $n \in \mathbb{N}$.

In the former case, the integral is given by

$$\int_{C_R} e^{-itz^2}\left(\alpha - \frac{iz}{4\pi} + \frac{e^{2izy_3}}{8\pi y_3}\right)^{-1} \frac{e^{iz|x-y|}}{4\pi\,|x-y|}\frac{e^{iz|x'-y|}}{4\pi\,|x'-y|}2z\,\mathrm{d}z. \quad (3.41)$$

We substitute $z = Re^{i\theta}$ and estimate the absolute value of the integrand

$$\left| e^{-itR^2e^{2i\theta}}\left(\alpha - \frac{iRe^{i\theta}}{4\pi} + \frac{e^{2iRe^{i\theta}y_3}}{8\pi y_3}\right)^{-1} \frac{e^{iRe^{i\theta}|x-y|}}{4\pi\,|x-y|}\frac{e^{iRe^{i\theta}|x'-y|}}{4\pi\,|x'-y|}2ie^{i\theta}R^2\right| = |I|\,.$$

The second factor is bounded, because the path does not cross any resonance, hence it holds that

$$|I| \le C(x,x',y)R^2 e^{R\sin\theta(2tR\cos\theta - |x-y| - |x'-y|)}.$$

Since we are in the complex lower half-plane, $\sin\theta < 0$. So if the other factor in the exponent is positive, then in the limit the integral vanishes. By recalling that $-\frac{\pi}{4} < \theta < 0$, we have that

$$2tR\cos\theta - |x-y| - |x'-y| > \sqrt{2}Rt - |x-y| - |x'-y|\,,$$

which is positive for $t > \frac{\sqrt{2}}{2R}\Big(|x-y| + |x'-y|\Big)$ and hence the integral vanishes in the limit.

In the other case, the path of integration has to be modified because the previous path would pass across the pole z_n. So, the integral (3.41) has to be replaced by the following sum of integrals of the same function

$$-\frac{1}{2\pi i}\int_{C_R^1}\dots \mathrm{d}z-\frac{1}{2\pi i}\int_{C_\varepsilon}\dots \mathrm{d}z-\frac{1}{2\pi i}\int_{C_R^2}\dots \mathrm{d}z,$$

where C_ε is the semicircle of radius ε centered at z_n, C_R^1 is the arc of circle of radius R centered at the origin starting at $Re^{-i\frac{\pi}{4}}$ and ending when intersecting C_ε, and C_R^2 is the arc of circle of radius R centered at the origin between the point R and the intersection with C_ε. The path is run clockwise. The integrals on C_R^1 and C_R^2 are proven to vanish in the limit $n\to+\infty$ for t large enough in the same way as in the first case. We consider the remaining integral, it holds that

$$-\frac{1}{2\pi i}\int_{C_\varepsilon}\dots \mathrm{d}z=-\frac{1}{2}\operatorname{Res}_{z=z_n}\left(e^{-itz^2}\left(\alpha-\frac{iz}{4\pi}+\frac{e^{2izy_3}}{8\pi y_3}\right)^{-1}\frac{e^{iz|x-y|}}{4\pi|x-y|}\frac{e^{iz|x'-y|}}{4\pi|x'-y|}2z\right).$$

In order to find the residue, we study the divergent factor

$$\left(\Gamma^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}(z)\right)^{-1}=\left(\alpha-\frac{iz}{4\pi}+\frac{e^{2izy_3}}{8\pi y_3}\right)^{-1}.$$

Using the definition of z_n we have that

$$\begin{aligned}\left(\Gamma^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}(z)\right)&=\alpha-\frac{i}{4\pi}(z-z_n)-\frac{iz_n}{4\pi}+\frac{e^{2iy_3(z-z_n)}}{8\pi y_3}e^{2iy_3z_n}\\&=\frac{e^{2iy_3z_n}}{8\pi y_3}\left(e^{2iy_3(z-z_n)}-1\right)-\frac{i}{4\pi}(z-z_n)\\&=\frac{e^{2iy_3z_n}}{8\pi y_3}\sum_{k=1}^{+\infty}\frac{2^ki^ky_3^k}{k!}(z-z_n)^k-\frac{i}{4\pi}(z-z_n)\\&=\frac{i}{4\pi}\left(e^{2iy_3z_n}-1\right)(z-z_n)+O((z-z_n)^2).\end{aligned}$$

This means that

$$\operatorname{Res}_{z=z_n}\left(\left(\Gamma^{\mathbb{H}^3,\mathrm{D}}_{\alpha,y}(z)\right)^{-1}\right)=\lim_{z\to z_n}\frac{z-z_n}{\alpha-\frac{iz}{4\pi}+\frac{e^{2iy_3z_n}}{8\pi y_3}}=\frac{4\pi i}{1-e^{2iy_3z_n}}.$$

The denominator would vanish only if $z_n=\frac{k\pi}{y_3}$ for some $k\in\mathbb{Z}$. In order for this value to be a resonance, it has to solve

$$\frac{ik}{4y_3}=\alpha+\frac{1}{8\pi y_3},$$

but the the left side is purely imaginary, while the right one is real. So the equation holds only if both sides vanish, which happens if and only if $z_n=0$ and $\alpha=-\frac{1}{8\pi y_3}$. Hence, if we state

$z_n = r_n e^{i\theta_n}$, we have

$$\left| -\frac{1}{2\pi i} \int_{C_\varepsilon} \dots \mathrm{d}z \right| = \left| e^{-itz_n^2} \frac{4\pi i}{e^{2iy_3 z_n} - 1} \frac{e^{iz_n|x-y|}}{4\pi |x-y|} \frac{e^{iz_n|x'-y|}}{4\pi |x'-y|} z_n \right|$$
$$\leq C'(x, x', y) e^{r_n \sin\theta_n (2tr_n \cos\theta_n - |x-y| - |x'-y|)}.$$

Also in this case

$$2tr_n \cos\theta_n - |x-y| - |x'-y| \geq \sqrt{2} tr_n - |x-y| - |x'-y|,$$

with the last expression being positive for $t > \frac{\sqrt{2}}{2r_n}\left(|x-y| + |x'-y|\right)$. So, in the limit $n \to +\infty$ also the integral on C_ε vanishes for t large enough.

Going back to (3.40), we consider the sum in it. First we observe that $\hat{R}(-z)$ has no poles in the region considered, because all its poles are in the upper complex plane. The expression of $(-\Delta_{\alpha,y}^{\mathbb{H}^3,\mathrm{D}} - z^2)^{-1}$ and the fact that for $R \to +\infty$ the region in exam becomes $-\frac{\pi}{4} < \arg z < 0$ let us recollect the expression present in (3.39).

The integral over γ_R can be expressed, with the substitution $z = e^{-i\pi/4}\sqrt{s}$, as follows

$$\frac{1}{2\pi i} \int_0^{+\infty} e^{-ts} \left(\hat{R}\left(e^{-i\frac{\pi}{4}}\sqrt{s}\right) - \hat{R}\left(-e^{-i\frac{\pi}{4}}\sqrt{s}\right) \right)(x, x') 2e^{-i\frac{\pi}{4}}\sqrt{s} e^{-i\frac{\pi}{4}} \frac{\mathrm{d}s}{2\sqrt{s}},$$

which returns the missing term. □

3.5. Semiclassical Asymptotics for Resonances for the Half-Space

In general, as also the previous treatment shows, it is difficult to obtain explicit information on all resonances. In order to understand some properties of them, one can study their asymptotic behaviour on certain regimes. An example of these regimes is the *high energy limit*. To fix ideas we limit ourself to the one point interaction. This limit consist in studying the distribution of the poles of

$$\left(-\Delta_{\alpha,y}^{\Omega} - z^2\right)^{-1}, \quad \text{for } |z| \to +\infty.$$

The high energy limit is an example of *semiclassical limit*. This is obtained rescaling z dividing it by a reference constant $h > 0$ (who mimics Planck's constant) and passing to the limit $h \to 0$. This way, the energy $\left(\frac{z}{h}\right)^2$ goes to infinite, implying that the energies of the phenomena in exam are way higher than the reference energy h. In this case the resolvent is then

$$\left(-\Delta_{\alpha,y}^{\Omega} - \left(\frac{z}{h}\right)^2\right)^{-1} = h^2\left(-h^2\Delta_{\alpha,y}^{\Omega} - z^2\right)^{-1}, \quad \text{for } h \to 0.$$

In applications, typically, the potential which describes the interaction (here the point interaction) is usually assumed to be energy dependent and hence dependent on h in this setting. A common choice for the interaction is of the form $h^{-\beta}$ with $\beta > 0$ (see for example [17] and [23]). We consider both positive and negative sign of the point charge. So the operator considered is $-h^2\Delta^{\Omega}_{\pm h^{-\beta},y}$ and its resonances are the poles in the complex lower half-plane of

$$\left(-\Delta^{\Omega}_{\pm h^{-\beta},y} - \left(\frac{z}{h}\right)^2\right)^{-1} = h^2\Big(-h^2\Delta^{\Omega}_{\pm h^{-\beta},y} - z^2\Big)^{-1}, \quad \text{for } h \to 0.$$

Below we will study the asymptotics of these resonances for $\Omega = \mathbb{H}^3$ with both Dirichlet and Neumann boundary conditions. The approach is borrowed from the work [17] on the half-line by Datchew and Malawo. There the case of the half-line with a delta interaction at a point and Dirichlet boundary conditions at the origin is treated. This is called Winter's model in the literature (see [42] for a recent analysis and reference)

3.5.1. Dirichlet Boundary Condition

We start by studying the semiclassical resonance asymptotics for $-h^2\Delta^{\mathbb{H}^3,\mathrm{D}}_{\pm h^{-\beta},y}$. The definition of the operator, combined with (2.30) implies that the equation for the resonances is

$$\pm h^{-\beta} - \frac{iz}{4\pi h} + \frac{e^{2iy_3\frac{z}{h}}}{8\pi y_3} = 0, \tag{3.42}$$

which can be rewritten as

$$-e^{\pm 8\pi y_3 h^{-\beta}} = \Big(-2iy_3\frac{z}{h} \pm 8\pi y_3 h^{-\beta}\Big)e^{-2iy_3\frac{z}{h}\pm 8\pi y_3 h^{-\beta}}. \tag{3.43}$$

Setting $-w^{\pm} = -e^{\pm 8\pi y_3 h^{-\beta}}$ and $x = -2iy_3\frac{z}{h} \pm 8\pi y_3 h^{-\beta}$, previous equation becomes $-w = xe^x$. The main difference in the two cases is that in the first $w \to +\infty$ as $h \to 0^+$, while in the second $w \to 0$.

The solutions on the complex plane of $-w = xe^x$ are given by $x = W_k(-w)$, where W is the Lambert W function (see Appendix C.1) and k varies over $\mathbb{Z}$. Many facts about the Lambert W function are reported in [14]. Specifically we use that $W_k(w)$ can be expanded in a convergent series (note that this expansion is valid for both w approaching zero and infinity)

$$\begin{aligned} W_k(-w) &= \ln(-w) + 2\pi ik - \ln\Big(\ln(-w) + 2\pi ik\Big) + R_k \\ &= \ln(w) + (2k+1)i\pi - \ln\Big(\ln(w) + (2k+1)i\pi\Big) + R_k, \end{aligned}$$

where

$$R_k = \sum_{j=0}^{+\infty}\sum_{m=1}^{+\infty} c_{j,m}\frac{\ln^m\Big(\ln(w) + (2k+1)i\pi\Big)}{\Big(\ln(w) + (2k+1)i\pi\Big)^{j+m}} \tag{3.44}$$

and

$$c_{j,m} = \frac{(-1)^j}{m!}\begin{bmatrix} j+m \\ j+1 \end{bmatrix}.$$

Here $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ are the Stirling numbers, which are the number of ways to arrange p objects in q cycles. The following Lemma holds.

Lemma 3.5.1. *The series (3.44) is absolutely convergent for w large (or small) enough and $k \in \mathbb{Z}$. More precisely we have the tail estimate*

$$\begin{aligned}\left| R_k - \frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi} \right| &\leq \sum_{j\geq 0, m\geq 1, (j,m)\neq(0,1)} \left| c_{j,m} \frac{\ln^m\left(\ln(w) + (2k+1)i\pi\right)}{\left(\ln(w) + (2k+1)i\pi\right)^{j+m}} \right| \\ &\leq 2\left| \frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi} \right|^2 \end{aligned} \tag{3.45}$$

and

$$\left| \frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi} \right| \leq \frac{1}{2}. \tag{3.46}$$

Proof. We prove the Lemma for both w big and small and distinguish when different observations have to be made. (3.46) is fulfilled for w sufficiently large (or small), because the denominator goes to infinity faster than the numerator. To prove (3.45) we write

$$\begin{aligned}\left| R_k - \frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi} \right| &\leq \sum_{j\geq 0, m\geq 1, (j,m)\neq(0,1)} \left| c_{j,m} \frac{\ln^m\left(\ln(w) + (2k+1)i\pi\right)}{\left(\ln(w) + (2k+1)i\pi\right)^{j+m}} \right| \\ = \sum_{m=2}^{+\infty} \left| c_{0,m} \frac{\ln^m\left(\ln(w) + (2k+1)i\pi\right)}{\left(\ln(w) + (2k+1)i\pi\right)^{m}} \right| &+ \sum_{m=1}^{+\infty}\sum_{j=1}^{+\infty} \left| c_{j,m} \frac{\ln^m\left(\ln(w) + (2k+1)i\pi\right)}{\left(\ln(w) + (2k+1)i\pi\right)^{j+m}} \right| = S_1 + S_2. \end{aligned} \tag{3.47}$$

We recall that $\left[\begin{smallmatrix} m \\ 1 \end{smallmatrix}\right] = (m-1)!$, and so $c_{0,m} = 1/m$. So

$$S_1 \leq \frac{1}{2}\sum_{m=2}^{+\infty} \left| \frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi} \right|^m = \frac{1}{2}\sum_{m=2}^{+\infty} q^m = \frac{q^2}{2(1-q)} \leq \left| \frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi} \right|^2,$$

where the last inequality is valid because $q \leq 1/2$ for w large (or small) enough. We consider the other sum

$$S_2 = \frac{1}{|\ln(w) + (2k+1)i\pi|} \sum_{m=1}^{+\infty} \left| \frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi} \right|^m \sum_{j=1}^{+\infty} \frac{\left|c_{j,m}\right|}{|\ln(w) + (2k+1)i\pi|^{j-1}}. \tag{3.48}$$

Since the double sum is absolutely convergent for large or small w, then it must hold that

$$\left| \frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi} \right|^m \sum_{j=1}^{+\infty} \frac{\left|c_{j,m}\right|}{|\ln(w) + (2k+1)i\pi|^{j-1}} \to 0 \quad \text{for} \quad m \to +\infty.$$

Moreover, for large w, the first multiplicative term is decreasing in w (it is positive and goes to 0) and each term in the sum is decreasing as well. Since all terms are positive, we can conclude that the expression above is decreasing for large w. This means that there is N such that, setting $w = N$, $\forall m \geq 1$ we have

$$\left| \frac{\ln\left(\ln(N) + (2k+1)i\pi\right)}{\ln(N) + (2k+1)i\pi} \right|^m \sum_{j=1}^{+\infty} \frac{\left|c_{j,m}\right|}{|\ln(N) + (2k+1)i\pi|^{j-1}} \leq N.$$

In the same way we have that, for small w, the first multiplicative term is increasing in w (it is positive and tends to 0 for $w \to 0$) and each term in the sum is increasing and positive as well. So the whole expression is increasing for small w and that means that ther is N such that if $w = \frac{1}{N}$, $\forall m \geq 1$

$$\left| \frac{\ln\left(-\ln(N) + (2k+1)i\pi\right)}{-\ln(N) + (2k+1)i\pi} \right|^m \sum_{j=1}^{+\infty} \frac{\left|c_{j,m}\right|}{|-\ln(N) + (2k+1)i\pi|^{j-1}} \leq N$$

holds. To consider together both cases, we can just put $\operatorname{sgn}\left(\ln(w)\right)$ in front of $\ln(N)$. Then, for $|w| \geq N$ large (or $|w| \leq \frac{1}{N}$ respectively), it holds that

$$\begin{aligned}\sum_{j=1}^{+\infty} \frac{\left|c_{j,m}\right|}{|\ln(w) + (2k+1)i\pi|^{j-1}} &\leq \sum_{j=1}^{+\infty} \frac{\left|c_{j,m}\right|}{\left|\operatorname{sgn}\left(\ln(w)\right)\ln(N) + (2k+1)i\pi\right|^{j-1}} \\ &\leq N \left| \frac{\operatorname{sgn}\left(\ln(w)\right)\ln(N) + (2k+1)i\pi}{\ln\left(\operatorname{sgn}\left(\ln(w)\right)\ln(N) + (2k+1)i\pi\right)} \right|^m.\end{aligned}$$

Substituting this in (3.48), we get

$$\begin{aligned}
S_2 &\leq \frac{N}{|\ln(w)+(2k+1)i\pi|}\sum_{m=1}^{+\infty}\left|\frac{\ln\left(\ln(w)+(2k+1)i\pi\right)}{\ln(w)+(2k+1)i\pi}\cdot\frac{\operatorname{sgn}\left(\ln(w)\right)\ln(N)+(2k+1)i\pi}{\ln\left(\operatorname{sgn}\left(\ln(w)\right)\ln(N)+(2k+1)i\pi\right)}\right|^m\\
&= 2N\frac{\left|\ln\left(\ln(w)+(2k+1)i\pi\right)\right|}{|\ln(w)+(2k+1)i\pi|^2}\left|\frac{\operatorname{sgn}\left(\ln(w)\right)\ln(N)+(2k+1)i\pi}{\ln\left(\operatorname{sgn}\left(\ln(w)\right)\ln(N)+(2k+1)i\pi\right)}\right|\frac{1}{2}\sum_{m=0}^{+\infty}p^m,
\end{aligned}$$

where

$$\begin{aligned}
p &= \left|\frac{\ln\left(\ln(w)+(2k+1)i\pi\right)}{\ln(w)+(2k+1)i\pi}\cdot\frac{\operatorname{sgn}\left(\ln(w)\right)\ln(N)+(2k+1)i\pi}{\ln\left(\operatorname{sgn}\left(\ln(w)\right)\ln(N)+(2k+1)i\pi\right)}\right|\\
&= \begin{cases}\frac{s(w)}{s(N)} & w \text{ large}\\ \frac{s(w)}{s\left(\frac{1}{N}\right)} & w \text{ small}\end{cases}.
\end{aligned}$$

Since s is decreasing for large w, $p < 1$. But it also holds that $\lim_{w\to+\infty} s(w) = 0$, so for large w, $p < \frac{1}{2}$. For small w instead s is increasing and $\lim_{w\to 0} s(w) = 0$, which means that for sufficiently small w, $p < \frac{1}{2}$. But,

$$\frac{1}{2}\sum_{m=0}^{+\infty}p^m = \frac{1}{2(1-p)} < 1,$$

for $p < \frac{1}{2}$. This implies

$$S_2 \leq \frac{2N\left|\operatorname{sgn}\left(\ln(w)\right)\ln(N)+(2k+1)i\pi\right|}{\left|\ln\left(\operatorname{sgn}\left(\ln(w)\right)\ln(N)+(2k+1)i\pi\right)\right|}\cdot\frac{\left|\ln\left(\ln(w)+(2k+1)i\pi\right)\right|}{|\ln(w)+(2k+1)i\pi|^2}.$$

Given that $\left|\ln\left(\ln(w)+(2k+1)i\pi\right)\right| \to +\infty$ for both $w\to+\infty$ and $w\to 0$, we have that

$$\frac{2N\left|\operatorname{sgn}\left(\ln(w)\right)\ln(N)+(2k+1)i\pi\right|}{\left|\ln\left(\operatorname{sgn}\left(\ln(w)\right)\ln(N)+(2k+1)i\pi\right)\right|} \leq \left|\ln\left(\ln(w)+(2k+1)i\pi\right)\right|,$$

for w large (respectively small) enough. So,

$$S_2 \leq \left|\frac{\ln\left(\ln(w)+(2k+1)i\pi\right)}{\ln(w)+(2k+1)i\pi}\right|^2$$

and the thesis follows. □

So we can write the resonances $z_k^{\pm}$ in terms of the Lambert W function. For $-h^2\Delta^{\mathbb{H}^3,\mathrm{D}}_{h^{-\beta},y}$ they are given by

$$\begin{aligned} z_k^+ &= \frac{ih}{2x_{0,3}}\Big(W_k(-w) - 8\pi y_3 h^{-\beta}\Big) \\ &= \frac{ih}{2x_{0,3}}\Big(\ln(w) + (2k+1)i\pi - \ln\Big(\ln(w) + (2k+1)i\pi\Big) - 8\pi x_{0,3}h^{-\beta} + R_k\Big) \\ &= \frac{ih}{2x_{0,3}}\Big((2k+1)i\pi - \ln\Big(\ln(w) + (2k+1)i\pi\Big) + R_k\Big), \end{aligned} \tag{3.49}$$

where k varies over $\mathbb{Z}$ and $w = e^{8\pi y_3 h^{-\beta}}$. Instead, for $-h^2\Delta^{\mathbb{H}^3,\mathrm{D}}_{-h^{-\beta},y}$

$$\begin{aligned} z_k^- &= \frac{ih}{2x_{0,3}}\Big(W_k(-w) + 8\pi y_3 h^{-\beta}\Big) \\ &= \frac{ih}{2x_{0,3}}\Big(\ln(w) + (2k+1)i\pi - \ln\Big(\ln(w) + (2k+1)i\pi\Big) + 8\pi x_{0,3}h^{-\beta} + R_k\Big) \\ &= \frac{ih}{2x_{0,3}}\Big((2k+1)i\pi - \ln\Big(\ln(w) + (2k+1)i\pi\Big) + R_k\Big) \end{aligned} \tag{3.50}$$

(here $w = e^{-8\pi y_3 h^{-\beta}}$).

Now we show that under some assumptions over z_k, k is roughly of size h^{-1}.

Lemma 3.5.2. *Let $\varepsilon \in (0,1)$ ve given. Then, for k such that z_k is given by (3.49) or (3.50) and*

$$\varepsilon \le |z_k| \le 1/\varepsilon, \tag{3.51}$$

we have

$$\frac{\varepsilon}{2} \le \frac{|k|\,\pi h}{y_3} \le \frac{2}{\varepsilon}. \tag{3.52}$$

Proof. Using (3.46) and the inverse triangular inequality we have that

$$\left| R_k - \frac{\ln\Big(\ln(w) + (2k+1)i\pi\Big)}{\ln(w) + (2k+1)i\pi} \right| \ge \left| |R_k| - \frac{1}{2} \right|.$$

Combining this with (3.45) and (3.46) we get

$$\left| |R_k| - \frac{1}{2} \right| \le \frac{1}{2},$$

equivalent to $|R_k| \leq 1$. We now prove the first inequality in (3.52). For the sake of contradiction we suppose that $\frac{|k|\pi h}{y_3} < \frac{\varepsilon}{2}$. Combining the first inequality of (3.51), $|R_k| \leq 1$ and (3.49), we get

$$\varepsilon \leq |z_k| \leq \frac{|k|\,\pi h}{y_3} + \frac{h}{2x_{0,3}} \left|\ln\left(\ln(w) + (2k+1)i\pi\right)\right| + \frac{\pi+1}{2y_3}h.$$

Then, using $\frac{|k|\pi h}{y_3} < \frac{\varepsilon}{2}$, we have

$$\varepsilon < \frac{h}{y_3} \left|\ln\left(\ln(w) + (2k+1)i\pi\right)\right| + \frac{\pi+1}{y_3}h.$$

Since the second term in the right side goes to zero faster than the first, it must hold that, for h small enough

$$\frac{\varepsilon}{2} < \frac{h}{y_3} \left|\ln\left(\ln(w) + (2k+1)i\pi\right)\right|,$$

which implies

$$e^{\frac{\varepsilon y_3}{2h}} \leq |\ln(w) + (2k+1)i\pi| \leq 8\pi y_3 h^{-\beta} + 2\,|k|\,\pi + \pi \leq 8\pi y_3 h^{-\beta} + \varepsilon y_3 h^{-1}$$

and this is absurd for h small enough.

To prove the second inequality (3.52) we suppose for the sake of contradiction that $\frac{|k|\pi h}{y_3} > \frac{2}{\varepsilon}$. By $|R_k| \leq 1$, (3.49) and triangular inequality we have that

$$|z_k| \geq \frac{|k|\,\pi h}{y_3} - \frac{h}{2x_{0,3}} |\ln|\ln(w) + (2k+1)i\pi|| - \frac{h}{2y_3} \left|\mathrm{Arg}\left(\ln(w) + (2k+1)i\pi\right)\right| - \frac{\pi+1}{2y_3}h.$$

This, combined with the second inequality of (3.51), implies

$$\begin{aligned}\frac{|k|\,\pi h}{y_3} - \frac{h}{2x_{0,3}} |\ln|\ln(w) + (2k+1)i\pi|| &\leq \frac{1}{\varepsilon} + \frac{h}{2y_3} \left|\mathrm{Arg}\left(\ln(w) + (2k+1)i\pi\right)\right| + \frac{\pi+1}{2y_3}h \\ &\leq \frac{3}{2\varepsilon},\end{aligned}$$

last inequality being valid for h small eoungh. So it holds that

$$2\,|k|\,\pi - \frac{3y_3}{\varepsilon h} \leq |\ln|\ln(w) + (2k+1)i\pi||,$$

hence

$$e^{2|k|\pi - \frac{3y_3}{\varepsilon h}} - 2\,|k|\,\pi \leq |\ln(w)| + \pi.$$

But, since the function $f(x) = e^{x-a} - x$ is increasing in $(a, +\infty)$ and $2|k|\pi > \frac{4y_3}{\varepsilon h}$ we have $f\left(\frac{4y_3}{\varepsilon h}\right) < f\left(2|k|\pi\right) \leq \ln(w) + \pi$ and so

$$e^{\frac{y_3}{\varepsilon h}} \leq 8\pi y_3 h^{-\beta} + \frac{4y_3}{\varepsilon h} + \pi,$$

which is a contradiction for h small enough. □

We now establish asymptotic behaviour for resonances apart from the ones near zero and near infinity. We distinguish between $0 < \beta < 1$ and $\beta > 1$. This different treatment is needed because different values of β changes the relative dominance between $\ln(y)$ and $|2k+1|\pi$ in (3.49). We start with the case $\beta \in (0, 1)$.

Proposition 3.5.1. *Let $\beta \in (0, 1)$ and $\varepsilon \in (0, 1)$ be given. Then there is $h_0 > 0$ such that, when $h \in (0, h_0]$, all solutions of (3.42) satisfying*

$$\varepsilon \leq |z| \leq \frac{1}{\varepsilon},$$

obey

$$0 \leq -\operatorname{Im} z - \frac{h}{2y_3} \ln\left(2y_3 h^{-1} |\operatorname{Re} z|\right) \leq \frac{72\pi^2}{y_3} \varepsilon^{-2} h^{3-2\beta}.$$

Proof. Since $\beta \in (0, 1)$, then the first inequality in (3.52) implies

$$|(2k+1)i\pi| \geq 2|k|\pi - \pi \geq \varepsilon y_3 h^{-1} - \pi > |\ln(w)| = 8\pi y_3 h^{-\beta}, \tag{3.53}$$

for h small enough, so that it is more convenient to rewrite

$$\ln\left(\ln(w) + (2k+1)i\pi\right) = \ln\left((2k+1)i\pi\right) + \ln\left(1 + \frac{\ln(w)}{(2k+1)i\pi}\right)$$

and

$$z_k^{\pm} = \frac{ih}{2y_3}\left((2k+1)i\pi - \ln\left((2k+1)i\pi\right) + R_k'\right), \quad \text{where} \quad R_k' = R_k - \ln\left(1 + \frac{\ln(w)}{(2k+1)i\pi}\right).$$

Using $|R_k| \leq 1$ and (3.53), we get

$$\left|R_k'\right| \leq |R_k| + \left|\ln\left|1 + \frac{\ln(w)}{(2k+1)i\pi}\right|\right| + \left|\operatorname{Arg}\left(1 + \frac{\ln(w)}{(2k+1)i\pi}\right)\right| \leq 1 + \ln 2 + \frac{\pi}{2} \leq 4. \tag{3.54}$$

Hence, by taking the real and imaginary part of $z_k^{\pm}$ we get

$$\begin{cases} \operatorname{Re} z_k^{\pm} = -\frac{(2k+1)\pi h}{2y_3} + \frac{h}{2y_3} \operatorname{Arg}\left((2k+1)i\pi\right) - \frac{h}{2y_3} \operatorname{Im} R_k' \\ \operatorname{Im} z_k^{\pm} = -\frac{h}{2y_3} \ln|(2k+1)\pi| + \frac{h}{2y_3} \operatorname{Re} R_k' \end{cases}. \tag{3.55}$$

We now separate real and imaginary part of (3.42)

$$\begin{cases} e^{-2\frac{x_{0,3}}{h}\operatorname{Im} z_k^{\pm}}\cos\left(2\frac{y_3}{h}\operatorname{Re} z_k^{\pm}\right) = -2\frac{x_{0,3}}{h}\operatorname{Im} z_k^{\pm} \mp 8\pi y_3 h^{-\beta} \\ e^{-2\frac{x_{0,3}}{h}\operatorname{Im} z_k^{\pm}}\sin\left(2\frac{y_3}{h}\operatorname{Re} z_k^{\pm}\right) = 2\frac{x_{0,3}}{h}\operatorname{Re} z_k^{\pm} \end{cases}.$$

Squaring the equations and adding them side by side we have

$$e^{-4\frac{x_{0,3}}{h}\operatorname{Im} z_k^{\pm}} = \frac{4y_3^2}{h^2}\left(\left(\operatorname{Im} z_k^{\pm}\right)^2 + \left(\operatorname{Re} z_k^{\pm}\right)^2\right) \pm 32\pi y_3^2 h^{-\beta-1}\operatorname{Im} z_k^{\pm} + 64\pi^2 y_3^2 h^{-2\beta}$$

or equivalently

$$-4\frac{x_{0,3}}{h}\operatorname{Im} z_k^{\pm} = 2\ln\left(2y_3h^{-1}\left|\operatorname{Re} z_k^{\pm}\right|\right) + \ln(1+t),$$

where

$$t = \frac{4y_3^2\left(\operatorname{Im} z_k^{\pm}\right)^2 \pm 32\pi y_3^2 h^{1-\beta}\operatorname{Im} z_k^{\pm} + 64\pi^2 y_3^2 h^{2-2\beta}}{4y_3^2\left(\operatorname{Re} z_k^{\pm}\right)^2}.$$

We recall that $0 \le \ln(1+t) \le t$ to get

$$0 \le -4\frac{x_{0,3}}{h}\operatorname{Im} z_k^{\pm} - 2\ln\left(2y_3h^{-1}\left|\operatorname{Re} z_k^{\pm}\right|\right) \le t. \tag{3.56}$$

From (3.55) we have, using (3.52), (3.54) and $\operatorname{Arg}\left((2k+1)i\pi\right) = \pi/2$, that, for sufficiently small h

$$\begin{aligned} |\operatorname{Re} z_k| &\ge \frac{|2k+1|\,\pi h}{2y_3} - \left|\frac{h}{2y_3}\operatorname{Im} R_k' - \frac{h}{2y_3}\operatorname{Arg}\left((2k+1)i\pi\right)\right| \\ &\ge \frac{|k|\,\pi h}{y_3} - \frac{\pi h}{2y_3} - \frac{h}{2y_3}\left|\operatorname{Im} R_k'\right| - \frac{h}{2y_3}\left|\operatorname{Arg}\left((2k+1)i\pi\right)\right| \\ &\ge \frac{\varepsilon}{2} - \frac{\pi h}{2y_3} - \frac{2h}{y_3} - \frac{\pi h}{4y_3} \ge \frac{\varepsilon}{3} \end{aligned}$$

and

$$\begin{aligned} \left|\operatorname{Im} z_k^{\pm}\right| &\le \frac{h}{2y_3}\ln\left(2|k|\pi + \pi\right) + \frac{h}{2y_3}\left|\operatorname{Re} R_k'\right| \\ &= \frac{h}{2y_3}\ln\left(2|k|\pi\right) + \frac{h}{2y_3}\ln\left(1 + \frac{1}{2|k|\pi}\right) + \frac{h}{2y_3}\left|\operatorname{Re} R_k'\right| \\ &\le \frac{h}{2y_3}\ln\left(\frac{4y_3}{\varepsilon h}\right) + \frac{\ln 2 + 2}{y_3}h \le \frac{h}{y_3}\ln\left(\frac{4y_3}{\varepsilon h}\right). \end{aligned}$$

For z_k^+ this, together with $\operatorname{Im} z_k^+ < 0$, implies

$$t \leq \frac{4h^2 \ln^2\left(\frac{4y_3}{\varepsilon h}\right) + 64\pi^2 y_3^2 h^{2-2\beta}}{4y_3^2\left(\operatorname{Re} z_k^+\right)^2} \leq \frac{32\pi^2 h^{2-2\beta}}{\left(\operatorname{Re} z_k^+\right)^2} \leq 288\pi^2\varepsilon^{-2}h^{2-2\beta},$$

while for z_k^-

$$t \leq \frac{4h^2 \ln^2\left(\frac{4y_3}{\varepsilon h}\right) + 32\pi x_{0,3} h^{2-\beta} \ln\left(\frac{4y_3}{\varepsilon h}\right) + 64\pi^2 y_3^2 h^{2-2\beta}}{4y_3^2\left(\operatorname{Re} z_k^-\right)^2} \leq \frac{32\pi^2 h^{2-2\beta}}{\left(\operatorname{Re} z_k^-\right)^2} \leq 288\pi^2\varepsilon^{-2}h^{2-2\beta},$$

Plugging this inequality into (3.56) gives the desired inequality. □

A different behaviour happens for $\beta > 1$

Proposition 3.5.2. *Let $\beta > 1$ and $\varepsilon \in (0,1)$ be given. Then there is $h_0 > 0$ such that, when $h \in (0, h_0]$, all solutions to (3.42)$^+$ satisfying*

$$\varepsilon \leq |z| \leq \frac{1}{\varepsilon}$$

obey

$$\left|\operatorname{Im} z + \frac{2y_3 \ln\left(8\pi y_3 h^{-\beta}\right)}{h(2k+1)^2\pi^2}\left(\operatorname{Re} z\right)^2\right| \leq \frac{1+96\varepsilon^{-4}}{4\pi y_3^2} h^{\beta+1} \ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2}h^{2\beta-1}}{2\pi^2 y_3},$$

which, using (3.52), implies

$$\operatorname{Im} z + \frac{\varepsilon^2}{32 y_3} h \ln\left(8\pi y_3 h^{-\beta}\right)\left(\operatorname{Re} z\right)^2 \leq \frac{1+96\varepsilon^{-4}}{4\pi y_3^2} h^{\beta+1} \ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2}h^{2\beta-1}}{2\pi^2 y_3} \tag{3.57}$$

$$\operatorname{Im} z + \frac{8}{\varepsilon^2 y_3} h \ln\left(8\pi y_3 h^{-\beta}\right)\left(\operatorname{Re} z\right)^2 \geq -\frac{1+96\varepsilon^{-4}}{4\pi y_3^2} h^{\beta+1} \ln\left(8\pi y_3 h^{-\beta}\right) - \frac{\varepsilon^{-2}h^{2\beta-1}}{2\pi^2 y_3}. \tag{3.58}$$

Proof. Since $\beta > 1$, by the second inequality in (3.52), we have

$$|2k+1|\,\pi \leq 2\,|k|\,\pi + \pi \leq 4y_3\varepsilon^{-1}h^{-1} + \pi < 8\pi y_3 h^{-\beta} = \ln(w),$$

for h small enough. Recalling (3.49), we write

$$z_k = \frac{ih}{2y_3}\left((2k+1)i\pi - \ln\left(8\pi y_3 h^{-\beta}\right) - \ln\left(1 + \frac{i(2k+1)}{8y_3} h^{\beta}\right) + R_k\right)$$

and so we can ensure that the third term is small for h small. Separating both the real and imaginary part gives

$$\begin{cases} \operatorname{Re} z_k = \frac{h}{2y_3}\left(-(2k+1)\pi + \operatorname{Arg}\left(1 + \frac{i(2k+1)}{8y_3}h^{\beta}\right) - \operatorname{Im} R_k\right) \\ \operatorname{Im} z_k = \frac{h}{2y_3}\left(-\ln\left(8\pi y_3 h^{-\beta}\right) - \ln\left|1 + \frac{i(2k+1)}{8y_3}h^{\beta}\right| + \operatorname{Re} R_k\right) \end{cases}. \tag{3.59}$$

We observe that

$$\ln\left|1 + \frac{i(2k+1)}{8y_3}h^{\beta}\right| = \frac{1}{2}\ln\left(1 + \frac{(2k+1)^2}{64y_3^2}h^{2\beta}\right) = \frac{1}{2}\ln(1+t),$$

with $t = \frac{(2k+1)^2}{64y_3^2}h^{2\beta}$. Considering the imaginary part in (3.59), we get

$$\left|\operatorname{Im} z_k + \frac{h}{2y_3}\ln\left(8\pi y_3 h^{-\beta}\right)\right| = \frac{h}{2y_3}\left|\operatorname{Re} R_k - \ln\left|1 + \frac{i(2k+1)}{8y_3}h^{\beta}\right|\right|. \tag{3.60}$$

To deal with R_k, we recall that $w = 8\pi y_3 h^{-\beta}$ and consider

$$\begin{aligned} \left|\frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi}\right| &\leq \frac{\left|\ln|\ln(w) + (2k+1)i\pi| + i\frac{\pi}{2}\right|}{|8\pi y_3 h^{-\beta} + (2k+1)i\pi|} \\ &= \frac{\ln\left(h^{-\beta}\right)}{h^{-\beta}}\frac{\left|\ln|\ln(w) + (2k+1)i\pi| + i\frac{\pi}{2}\right|}{\ln\left(h^{-\beta}\right)|8\pi y_3 + (2k+1)i\pi h^{\beta}|}, \end{aligned}$$

but

$$\frac{\left|\ln|\ln(w) + (2k+1)i\pi| + i\frac{\pi}{2}\right|}{\ln\left(h^{-\beta}\right)|8\pi y_3 + (2k+1)i\pi h^{\beta}|} \leq \frac{1}{h^{\beta}\ln(h^{-\beta})}\left(\left|\frac{\ln|\ln(w) + (2k+1)i\pi|}{\ln(w) + (2k+1)\pi}\right| + \frac{\frac{\pi}{2}}{|\ln(w) + (2k+1)\pi|}\right).$$

Since

$$\lim_{h\to 0^+}\frac{1}{h^{\beta}\ln(h^{-\beta})}\left|\frac{\ln|\ln(w) + (2k+1)i\pi|}{\ln(w) + (2k+1)\pi}\right| = \frac{1}{8\pi y_3}$$

and

$$\lim_{h\to 0^+}\frac{1}{h^{\beta}\ln(h^{-\beta})}\frac{\frac{\pi}{2}}{|\ln(w) + (2k+1)\pi|} = 0,$$

then

$$\frac{\left|\ln|\ln(w) + (2k+1)i\pi| + i\frac{\pi}{2}\right|}{\ln\left(h^{-\beta}\right)|8\pi y_3 + (2k+1)i\pi h^{\beta}|} \leq \frac{1}{8\pi y_3},$$

for h small enough and finally this means that

$$\left|\frac{\ln\left(\ln(w)+(2k+1)i\pi\right)}{\ln(w)+(2k+1)i\pi}\right| \leq \frac{h^{\beta}\ln\left(h^{-\beta}\right)}{4\pi y_3}. \tag{3.61}$$

By (3.45), we can also write

$$\left|R_k-\frac{\ln\left(\ln(w)+(2k+1)i\pi\right)}{\ln(w)+(2k+1)i\pi}\right| \leq \frac{h^{2\beta}\ln^2\left(h^{-\beta}\right)}{8\pi^2 y_3^2}. \tag{3.62}$$

Turning back to (3.60), we now have

$$\begin{aligned}\left|\operatorname{Im} z_k+\frac{h}{2y_3}\ln\left(8\pi y_3 h^{-\beta}\right)\right| &\leq \frac{h}{2y_3}\left|\operatorname{Re} R_k-\operatorname{Re}\left(\frac{\ln\left(\ln(w)+(2k+1)i\pi\right)}{\ln(w)+(2k+1)i\pi}\right)\right| \\ &+\frac{h}{2y_3}\left|\operatorname{Re}\left(\frac{\ln\left(\ln(w)+(2k+1)i\pi\right)}{\ln(w)+(2k+1)i\pi}\right)-\ln\left|1+\frac{i(2k+1)}{8y_3}h^{\beta}\right|\right| \leq \frac{h^{2\beta+1}\ln^2\left(h^{-\beta}\right)}{16\pi^2 y_3^3}+|A|.\end{aligned} \tag{3.63}$$

We now find a bound for $|A|$. After some manipulations we can write

$$A=\frac{\left(\ln\left(8\pi y_3 h^{-\beta}\right)+\frac{1}{2}\ln(1+t)\right)\frac{h^{\beta+1}}{16\pi y_3^2}+\frac{(2k+1)h^{2\beta+1}}{128\pi y_3^3}\operatorname{Arg}\left(8\pi y_3 h^{-\beta}+(2k+1)i\pi\right)}{1+t}-\frac{h}{4y_3}\ln(1+t).$$

By triangle inequality and $t>0$ follows that

$$\begin{aligned}|A| \leq \frac{h^{\beta+1}}{16\pi y_3^2}\ln\left(8\pi y_3 h^{-\beta}\right)+\frac{|2k+1|\,h^{2\beta+1}}{128\pi y_3^3}\left|\operatorname{Arg}\left(8\pi y_3 h^{-\beta}+(2k+1)i\pi\right)\right| \\ +\frac{h}{4y_3}\ln(1+t)+\frac{h^{\beta+1}}{32\pi y_3^2}\ln(1+t).\end{aligned} \tag{3.64}$$

This expression can be also bounded using $\left|\operatorname{Arg}\left(8\pi y_3 h^{-\beta}+(2k+1)i\pi\right)\right| \leq \frac{\pi}{2}$, $\ln(1+t)\leq t$, the expression for t, $(2k+1)^2 \leq 2(4k^2+1)$ and the second inequality in (3.52) as

$$|A| \leq \frac{h^{\beta+1}}{16\pi y_3^2}\ln\left(8\pi y_3 h^{-\beta}\right)+\frac{h^{3\beta+1}}{1024\pi x_{0,3}^4}+\frac{\varepsilon^{-2}h^{3\beta-1}}{64\pi^3 y_3^2}+\frac{3h^{2\beta+1}}{256 y_3^3}+\frac{\varepsilon^{-1}h^{2\beta}}{64\pi y_3^2}+\frac{\varepsilon^{-2}h^{2\beta-1}}{8\pi^2 y_3}.$$

For $\beta>1$, the dominant term for small h are the first and last term. More specifically the first is dominant for $\beta \geq 2$ and the other for $1<\beta<2$. What said implies that for h small enough

$$|A| \leq \frac{h^{\beta+1}}{8\pi y_3^2}\ln\left(8\pi y_3 h^{-\beta}\right)+\frac{\varepsilon^{-2}h^{2\beta-1}}{4\pi^2 y_3},$$

which implies

$$\left|\operatorname{Im} z_k + \frac{h}{2y_3}\ln\left(8\pi y_3 h^{-\beta}\right)\right| \leq \frac{h^{2\beta+1}\ln^2\left(h^{-\beta}\right)}{16\pi^2 y_3^3} + \frac{h^{\beta+1}}{8\pi y_3^2}\ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2}h^{2\beta-1}}{4\pi^2 y_3}$$
$$\leq \frac{h^{\beta+1}}{4\pi y_3^2}\ln\left(2\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2}h^{2\beta-1}}{2\pi^2 y_3},$$

for h sufficiently small.

Now we look at the real part of (3.59)

$$\left|\operatorname{Re} z_k + \frac{(2k+1)\pi h}{2y_3}\right| = \frac{h}{2y_3}\left|\operatorname{Arg}\left(1 + \frac{i(2k+1)}{8y_3}h^{\beta}\right) - \operatorname{Im} R_k\right|$$
$$\leq \frac{h}{2y_3}\left|\operatorname{Arg}\left(1 + \frac{i(2k+1)}{8y_3}h^{\beta}\right)\right| + \frac{h}{2y_3}\left|\frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi} - R_k\right|$$
$$+ \frac{h}{2y_3}\left|\frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi}\right|. \quad (3.65)$$

Using that $\left|\operatorname{Arg}(1+is)\right| \leq 2|s|$ for small s, the second inequality in (3.52), (3.61) and (3.62) we get the bound

$$\left|\operatorname{Re} z_k + \frac{(2k+1)\pi h}{2y_3}\right| \leq \frac{\varepsilon^{-1}h^{\beta}}{2\pi y_3} + \frac{h^{\beta+1}}{8y_3^2} + \frac{h^{2\beta+1}\ln^2\left(h^{-\beta}\right)}{16\pi^2 y_3^3} + \frac{h^{\beta+1}\ln\left(h^{-\beta}\right)}{8\pi y_3^2} \leq \frac{\varepsilon^{-1}h^{\beta}}{\pi y_3}, \quad (3.66)$$

last inequality being valid for h small enough.

This new inequality can be used to bound the following quantity

$$B = \left|\frac{2y_3}{h(2k+1)^2\pi^2}\ln\left(8\pi y_3 h^{-\beta}\right)\left(\operatorname{Re} z_k\right)^2 - \frac{h}{2y_3}\ln\left(8\pi y_3 h^{-\beta}\right)\right|.$$

By factorizing, (3.66) and by the second inequalities in (3.52) and (3.51) respectively

$$B \leq \frac{2y_3}{h(2k+1)^2\pi^2}\ln\left(8\pi y_3 h^{-\beta}\right)\left(|z_k| + \frac{|2k+1|\pi h}{2y_3}\right)\left|\operatorname{Re} z_k + \frac{(2k+1)\pi h}{2y_3}\right|$$
$$\leq \frac{6\varepsilon^{-2}}{(2k+1)^2\pi^3}h^{\beta-1}\ln\left(8\pi y_3 h^{-\beta}\right).$$

Now we observe that $(2k+1)^2 \geq k^2$ $\forall k \in \mathbb{Z}$, $k \neq 0$ (which is escluded by (3.51)). Using this fact, together with the reciprocal of the first inequality of (3.52), we have

$$B \leq \frac{24\varepsilon^{-4}}{\pi y_3^2}h^{\beta+1}\ln\left(8\pi y_3 h^{-\beta}\right).$$

Finally we can find the bound

$$\left|\operatorname{Im} z_k + \frac{2y_3 \ln\left(8\pi y_3 h^{-\beta}\right)}{h(2k+1)^2\pi^2}\left(\operatorname{Re} z_k\right)^2\right| \leq \left|\operatorname{Im} z_k + \frac{h}{2y_3}\ln\left(8\pi y_3 h^{-\beta}\right)\right|$$
$$+\left|\frac{2y_3 \ln\left(8\pi y_3 h^{-\beta}\right)}{h(2k+1)^2\pi^2}\left(\operatorname{Re} z_k\right)^2 - \frac{h}{2y_3}\ln\left(8\pi y_3 h^{-\beta}\right)\right| \leq$$
$$\leq \frac{1+96\varepsilon^{-4}}{4\pi y_3^2}h^{\beta+1}\ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2}h^{2\beta-1}}{2\pi^2 y_3}. \quad (3.67)$$

To get (3.57) and (3.58) we use respectively that $\forall k \in \mathbb{Z}$, $k \neq 0$, $(2k+1)^2 \leq 16k^2$ and that $(2k+1)^2 \geq k^2$, together with (3.52). □

Now we consider $\beta > 1$ for $(3.42)^-$.

Proposition 3.5.3. *Let $\beta > 1$ and $\varepsilon \in (0,1)$ be given. Then there is $h_0 > 0$ such that, when $h \in (0, h_0]$, all solutions to $(3.42)^-$ satisfying*

$$\varepsilon \leq |z| \leq \frac{1}{\varepsilon} \quad (3.68)$$

obey

$$\left|\operatorname{Im} z + \frac{y_3 \ln\left(8\pi y_3 h^{-\beta}\right)}{2k^2\pi^2 h}\left(\operatorname{Re} z\right)^2\right| \leq \frac{1+24\varepsilon^{-4}}{4\pi y_3^2}h^{\beta+1}\ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2}h^{2\beta-1}}{2\pi^2 y_3},$$

which, using (3.52), implies

$$\operatorname{Im} z + \frac{\varepsilon^2}{8y_3}h\ln\left(8\pi y_3 h^{-\beta}\right)\left(\operatorname{Re} z\right)^2 \leq \frac{1+24\varepsilon^{-4}}{4\pi y_3^2}h^{\beta+1}\ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2}h^{2\beta-1}}{2\pi^2 y_3} \quad (3.69)$$
$$\operatorname{Im} z + \frac{2}{\varepsilon^2 y_3}h\ln\left(8\pi y_3 h^{-\beta}\right)\left(\operatorname{Re} z\right)^2 \geq -\frac{1+24\varepsilon^{-4}}{4\pi y_3^2}h^{\beta+1}\ln\left(8\pi y_3 h^{-\beta}\right) - \frac{\varepsilon^{-2}h^{2\beta-1}}{2\pi^2 y_3}. \quad (3.70)$$

Proof. Since $\beta > 1$, by the second inequality in (3.52), we have

$$|2k+1|\,\pi \leq 2\,|k|\,\pi + \pi \leq 4y_3\varepsilon^{-1}h^{-1} + \pi < 8\pi y_3 h^{-\beta} = |\ln(w)|\,,$$

for h small enough. Recalling (3.50), we write

$$z_k = \frac{ih}{2y_3}\left(2i\pi k - \ln\left(8\pi y_3 h^{-\beta}\right) - \ln\left(1 - \frac{i(2k+1)}{8y_3}h^{\beta}\right) + R_k\right)$$

and so we can ensure that the third term is small for h small. Separating both the real and imaginary part gives

$$\begin{cases} \operatorname{Re} z_k = \frac{h}{2y_3}\left(-2k\pi + \operatorname{Arg}\left(1 - \frac{i(2k+1)}{8y_3}h^\beta\right) - \operatorname{Im} R_k\right) \\ \operatorname{Im} z_k = \frac{h}{2y_3}\left(-\ln\left(8\pi y_3 h^{-\beta}\right) - \ln\left|1 - \frac{i(2k+1)}{8y_3}h^\beta\right| + \operatorname{Re} R_k\right) \end{cases}. \tag{3.71}$$

We observe that

$$\ln\left|1 - \frac{i(2k+1)}{8y_3}h^\beta\right| = \frac{1}{2}\ln\left(1 + \frac{(2k+1)^2}{64y_3^2}h^{2\beta}\right) = \frac{1}{2}\ln(1+t),$$

with $t = \frac{(2k+1)^2}{64y_3^2}h^{2\beta}$.

(3.61) and (3.62) are true and can be proved as done in Proposition 3.5.2. So, by (3.71)

$$\left|\operatorname{Im} z_k + \frac{h}{2y_3}\ln\left(8\pi y_3 h^{-\beta}\right)\right| \le \frac{h}{2y_3}\left|\operatorname{Re} R_k - \operatorname{Re}\left(\frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi}\right)\right|$$
$$+ \frac{h}{2y_3}\left|\operatorname{Re}\left(\frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi}\right) - \ln\left|1 - \frac{i(2k+1)}{8y_3}h^\beta\right|\right| \le \frac{h^{2\beta+1}\ln^2\left(h^{-\beta}\right)}{16\pi^2 y_3^3} + |A|, \tag{3.72}$$

where

$$A = \frac{-\left(\ln\left(8\pi y_3 h^{-\beta}\right) + \frac{1}{2}\ln(1+t)\right)\frac{h^{\beta+1}}{16\pi y_3^2} + \frac{(2k+1)h^{2\beta+1}}{128\pi y_3^3}\operatorname{Arg}\left(-8\pi y_3 h^{-\beta} + (2k+1)i\pi\right)}{1+t} - \frac{h}{4y_3}\ln(1+t).$$

By triangle inequality and $t > 0$ follows that

$$|A| \le \frac{h^{\beta+1}}{16\pi y_3^2}\ln\left(8\pi y_3 h^{-\beta}\right) + \frac{|2k+1|\,h^{2\beta+1}}{128\pi y_3^3}\left|\operatorname{Arg}\left(-8\pi y_3 h^{-\beta} + (2k+1)i\pi\right)\right|$$
$$+ \frac{h}{4y_3}\ln(1+t) + \frac{h^{\beta+1}}{32\pi y_3^2}\ln(1+t). \tag{3.73}$$

This expression can be also bounded using $\left|\operatorname{Arg}\left(8\pi y_3 h^{-\beta} + (2k+1)i\pi\right)\right| \le \pi$, $\ln(1+t) \le t$, the expression for t, $(2k+1)^2 \le 2(4k^2+1)$ and the second inequality in (3.52) as

$$|A| \le \frac{h^{\beta+1}}{16\pi y_3^2}\ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2}h^{3\beta-1}}{64\pi^3 y_3^2} + \frac{h^{3\beta+1}}{1024\pi y_3^4} + \frac{\varepsilon^{-1}h^{2\beta}}{32\pi y_3^2} + \frac{h^{2\beta+1}}{64y_3^3} + \frac{\varepsilon^{-2}h^{2\beta-1}}{8\pi^2 y_3}.$$

For $\beta > 1$, the dominant term for small h are the first and last term. More specifically the first is dominant for $\beta \geq 2$ and the other for $1 < \beta < 2$. What said implies that for h small enough

$$|A| \leq \frac{h^{\beta+1}}{8\pi y_3^2} \ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2} h^{2\beta-1}}{4\pi^2 y_3},$$

which implies

$$\begin{aligned}\left|\operatorname{Im} z_k + \frac{h}{2y_3} \ln\left(8\pi y_3 h^{-\beta}\right)\right| &\leq \frac{h^{2\beta+1} \ln^2\left(h^{-\beta}\right)}{16\pi^2 y_3^3} + \frac{h^{\beta+1}}{8\pi y_3^2} \ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2} h^{2\beta-1}}{4\pi^2 y_3} \\ &\leq \frac{h^{\beta+1}}{4\pi y_3^2} \ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2} h^{2\beta-1}}{2\pi^2 y_3},\end{aligned}$$

for h sufficiently small.

Now we look at the real part of (3.71)

$$\begin{aligned}\left|\operatorname{Re} z_k + \frac{k\pi h}{y_3}\right| &= \frac{h}{2y_3} \left|\operatorname{Arg}\left(1 - \frac{i(2k+1)}{8y_3} h^\beta\right) - \operatorname{Im} R_k\right| \\ &\leq \frac{h}{2y_3} \left|\operatorname{Arg}\left(1 - \frac{i(2k+1)}{8y_3} h^\beta\right)\right| + \frac{h}{2y_3} \left|\frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi} - R_k\right| \\ &\quad + \frac{h}{2y_3} \left|\frac{\ln\left(\ln(w) + (2k+1)i\pi\right)}{\ln(w) + (2k+1)i\pi}\right|. \qquad (3.74)\end{aligned}$$

Using that $\left|\operatorname{Arg}(1 + is)\right| \leq 2|s|$ for small s, the second inequality in (3.52), (3.61) and (3.62) we get the bound

$$\left|\operatorname{Re} z_k + \frac{k\pi h}{y_3}\right| \leq \frac{\varepsilon^{-1} h^\beta}{2\pi y_3} + \frac{h^{\beta+1}}{8y_3^2} + \frac{h^{2\beta+1} \ln^2\left(h^{-\beta}\right)}{16\pi^2 y_3^3} + \frac{h^{\beta+1} \ln\left(h^{-\beta}\right)}{8\pi y_3^2} \leq \frac{\varepsilon^{-1} h^\beta}{\pi y_3}, \qquad (3.75)$$

last inequality being valid for h small enough.

This new inequality can be used to bound the following quantity

$$B = \left|\frac{y_3}{2k^2\pi^2 h} \ln\left(8\pi y_3 h^{-\beta}\right)\left(\operatorname{Re} z_k\right)^2 - \frac{h}{2y_3} \ln\left(8\pi y_3 h^{-\beta}\right)\right|.$$

By factorizing, (3.75) and by the second inequalities in (3.52) and (3.68) respectively

$$\begin{aligned}B &\leq \frac{y_3}{2k^2\pi^2 h} \ln\left(8\pi y_3 h^{-\beta}\right)\left(|z_k| + \frac{|k|\pi h}{y_3}\right)\left|\operatorname{Re} z_k + \frac{k\pi h}{y_3}\right| \\ &\leq \frac{6\varepsilon^{-4}}{\pi y_3^2} h^{\beta+1} \ln\left(8\pi y_3 h^{-\beta}\right).\end{aligned}$$

Finally we can find the desired bound

$$\left|\operatorname{Im} z_k + \frac{y_3 \ln\left(8\pi y_3 h^{-\beta}\right)}{2k^2\pi^2 h}\left(\operatorname{Re} z_k\right)^2\right| \leq \left|\operatorname{Im} z_k + \frac{h}{2y_3}\ln\left(8\pi y_3 h^{-\beta}\right)\right|$$
$$+\left|\frac{2y_3 \ln\left(8\pi y_3 h^{-\beta}\right)}{h(2k+1)^2\pi^2}\left(\operatorname{Re} z_k\right)^2 - \frac{h}{2y_3}\ln\left(8\pi y_3 h^{-\beta}\right)\right| \leq$$
$$\leq \frac{1+24\varepsilon^{-4}}{4\pi y_3^2} h^{\beta+1}\ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2}h^{2\beta-1}}{2\pi^2 y_3}. \quad (3.76)$$

□

3.5.2. Neumann Boundary Condition

By the definition of $-h^2\Delta^{\mathbb{H}^3,\mathrm{N}}_{\pm h^{-\beta},y}$ and (2.39) it follows that the resonances are the solution of

$$\pm h^{-\beta} - \frac{iz}{4\pi h} - \frac{e^{2iy_3\frac{z}{h}}}{8\pi y_3} = 0, \quad (3.77)$$

which can be rewritten as $w_\pm = x_\pm e^{x_\pm}$, where $x_\pm = \pm 8\pi y_3 h^{-\beta} - 2iy_3\frac{z}{h}$ and $w_\pm = e^{\pm 8\pi y_3 h^{-\beta}}$. So the solutions are still expressed in terms of the Lambert W function, with the only difference of it having a positive argument this time. The above mentioned expansion is still valid (one has only to substitute $2k+1$ with $2k$, reason being the change in sign of w)

$$W_k(y) = \ln(w) + 2ik\pi - \ln\left(\ln(w) + 2ik\pi\right) + R_k \quad (3.78)$$
$$R_k = \sum_{j=0}^{+\infty}\sum_{m=1}^{+\infty} c_{j,m}\frac{\ln^m\left(\ln(w)+2ik\pi\right)}{\left(\ln(w)+2ik\pi\right)^{j+m}}. \quad (3.79)$$

The expression for the resonances $z_k^\pm$ in term of the Lambert W function is

$$z_k^\pm = \frac{ih}{2x_{0,3}}\left(2ik\pi - \ln\left(\ln(y_\pm) + 2ik\pi\right) + R_k\right). \quad (3.80)$$

We state corresponding properties of the one of Dirichle boundary condition without an explicit proof. The proof of this facts are just slight modifications of the ones in the previous subsection.

Lemma 3.5.3. *The series (3.79) is absolutely convergent for w large (or small) enough and $k \in \mathbb{Z}$. More precisely we have the tail estimate*

$$\begin{aligned} \left| R_k - \frac{\ln\left(\ln(w) + 2ik\pi\right)}{\ln(w) + 2ik\pi} \right| &\le \sum_{j\ge 0, m\ge 1, (j,m)\neq(0,1)} \left| c_{j,m} \frac{\ln^m\left(\ln(w) + 2ik\pi\right)}{\left(\ln(w) + 2ik\pi\right)^{j+m}} \right| \\ &\le 2 \left| \frac{\ln\left(\ln(w) + 2ik\pi\right)}{\ln(w) + 2ik\pi} \right|^2 \end{aligned} \tag{3.81}$$

and

$$\left| \frac{\ln\left(\ln(w) + 2ik\pi\right)}{\ln(w) + 2ik\pi} \right| \le \frac{1}{2}. \tag{3.82}$$

Lemma 3.5.4. *Let $\varepsilon \in (0,1)$ ve given. Then, for k such that z_k is given by (3.80) and*

$$\varepsilon \le |z_k| \le 1/\varepsilon, \tag{3.83}$$

we have

$$\frac{\varepsilon}{2} \le \frac{|k|\,\pi h}{y_3} \le \frac{2}{\varepsilon}. \tag{3.84}$$

Proposition 3.5.4. *Let $\beta \in (0,1)$ and $\varepsilon \in (0,1)$ be given. Then there is $h_0 > 0$ such that, when $h \in (0, h_0]$, all solutions of (3.77) satisfying*

$$\varepsilon \le |z| \le \frac{1}{\varepsilon},$$

obey

$$0 \le -\operatorname{Im} z - \frac{h}{2y_3} \ln\left(2y_3 h^{-1} |\operatorname{Re} z|\right) \le \frac{72\pi^2}{y_3} \varepsilon^{-2} h^{3-2\beta}.$$

Proposition 3.5.5. *Let $\beta > 1$ and $\varepsilon \in (0,1)$ be given. Then there is $h_0 > 0$ such that, when $h \in (0, h_0]$, all solutions to (3.77)$^+$ satisfying*

$$\varepsilon \le |z| \le \frac{1}{\varepsilon}$$

obey

$$\left| \operatorname{Im} z_k + \frac{2y_3}{hk^2\pi^2} \ln\left(8\pi y_3 h^{-\beta}\right)\left(\operatorname{Re} z_k\right)^2 \right| \le \frac{1 + 96\varepsilon^{-4}}{4\pi y_3^2} h^{\beta+1} \ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2} h^{2\beta-1}}{4\pi^2 y_3},$$

which, using (3.84), implies

$$\operatorname{Im} z + \frac{\varepsilon^2}{2y_3} h \ln\left(8\pi y_3 h^{-\beta}\right)\left(\operatorname{Re} z\right)^2 \leq \frac{1+96\varepsilon^{-4}}{4\pi y_3^2} h^{\beta+1} \ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2} h^{2\beta-1}}{4\pi^2 y_3} \tag{3.85}$$

$$\operatorname{Im} z + \frac{8}{\varepsilon^2 y_3} h \ln\left(8\pi y_3 h^{-\beta}\right)\left(\operatorname{Re} z\right)^2 \geq -\frac{1+96\varepsilon^{-4}}{4\pi y_3^2} h^{\beta+1} \ln\left(8\pi y_3 h^{-\beta}\right) - \frac{\varepsilon^{-2} h^{2\beta-1}}{4\pi^2 y_3}. \tag{3.86}$$

Proposition 3.5.6. *Let $\beta > 1$ and $\varepsilon \in (0, 1)$ be given. Then there is $h_0 > 0$ such that, when $h \in (0, h_0]$, all solutions to (3.77)$^-$ satisfying*

$$\varepsilon \leq |z| \leq \frac{1}{\varepsilon} \tag{3.87}$$

obey

$$\left|\operatorname{Im} z_k + \frac{2y_3 \ln\left(8\pi y_3 h^{-\beta}\right)}{(2k-1)^2\pi^2 h}\left(\operatorname{Re} z_k\right)^2\right| \leq \frac{1+96\varepsilon^{-4}}{4\pi y_3^2} h^{\beta+1} \ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2} h^{2\beta-1}}{4\pi^2 y_3},$$

which, using (3.84), implies

$$\operatorname{Im} z + \frac{\varepsilon^2}{32y_3} h \ln\left(8\pi y_3 h^{-\beta}\right)\left(\operatorname{Re} z\right)^2 \leq \frac{1+24\varepsilon^{-4}}{4\pi y_3^2} h^{\beta+1} \ln\left(8\pi y_3 h^{-\beta}\right) + \frac{\varepsilon^{-2} h^{2\beta-1}}{4\pi^2 y_3} \tag{3.88}$$

$$\operatorname{Im} z + \frac{8}{\varepsilon^2 y_3} h \ln\left(8\pi y_3 h^{-\beta}\right)\left(\operatorname{Re} z\right)^2 \geq -\frac{1+24\varepsilon^{-4}}{4\pi y_3^2} h^{\beta+1} \ln\left(8\pi y_3 h^{-\beta}\right) - \frac{\varepsilon^{-2} h^{2\beta-1}}{4\pi^2 y_3}. \tag{3.89}$$

Appendix A.

Harmonic Functions

A.1. Green's Functions for the Helmholtz Operator

Here $z \in \mathbb{C}$ with $\operatorname{Im} z \geq 0$

A.1.1. Green's Functions in the whole $\mathbb{R}^n$

The plane

Let $y \in \mathbb{R}^2$. The Green's function $G^{0,2}_{z,y}$ for the operator $(-\Delta - z^2) : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is the solution of the equation

$$(-\Delta - z^2) G^{0,2}_{z,y} = \delta(\cdot - y),$$

namely

$$G^{0,2}_{z,y}(x) = \frac{i}{4} H^{(1)}_0\big(z\,|x-y|\big) \tag{A.1}$$

($H^{(1)}_0$ is the Hankel function of the first kind of order 0).

We also recall that the Green's function $G^{0,2}_{0,y}$ for the Laplace operator $-\Delta : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is the solution of the equation

$$-\Delta G^{0,2}_{0,y} = \delta(\cdot - y),$$

namely

$$G^{0,2}_{0,y}(x) = -\frac{1}{2\pi} \ln|x-y|. \tag{A.2}$$

The space

Let $y \in \mathbb{R}^3$. The Green's function $G^{0,3}_{z,y}$ for the operator $(-\Delta - z^2) : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the solution of the equation

$$(-\Delta - z^2)G^{0,3}_{z,y} = \delta(\cdot - y),$$

namely

$$G^{0,3}_{z,y}(x) = \frac{e^{iz|x-y|}}{4\pi\,|x-y|}. \tag{A.3}$$

We also recall that the Green's function $G^{0,3}_{0,y}$ for the operator $\Delta : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the solution of the equation

$$-\Delta G^{0,3}_{0,y} = \delta(\cdot - y),$$

namely

$$G^{0,3}_{0,y}(x) = \frac{1}{4\pi\,|x-y|}. \tag{A.4}$$

A.1.2. Domain with Dirichlet Boundary Conditions

Let Ω be a domain in $\mathbb{R}^n$ with $n = 2$ or $n = 3$ and $y \in \Omega$. The Green's function $G^{\Omega,\mathrm{D}}_{z,y}$ for the operator $(-\Delta^{\Omega,\mathrm{D}} - z^2) : H^2(\Omega) \to L^2(\Omega)$ can be expressed as

$$G^{\Omega,\mathrm{D}}_{z,y}(x) = G^{0,n}_{z,y}(x) - h^{\Omega,\mathrm{D}}_{z,y}(x),$$

where, $G^{0,n}_{z,y}$ are defined in (A.1) and (A.3), and $h^{\Omega,\mathrm{D}}_{z,y}$ is the solution of the boundary value problem

$$\begin{cases} (-\Delta - z^2)h^{\Omega,\mathrm{D}}_{z,y} = 0 & \text{in}\,\Omega \\ h^{\Omega,\mathrm{D}}_{z,y} = G^{0,n}_{z,y} & \text{in}\,\partial\Omega \end{cases}. \tag{A.5}$$

A.1.3. Domain with Neumann Boundary Conditions

Let Ω be a domain in $\mathbb{R}^n$ with $n = 2$ or $n = 3$, $y \in \Omega$ and ν the unit outward normal on $\partial\Omega$. The Green's function $G^{\Omega,\mathrm{N}}_{z,y}$ for the operator $(-\Delta^{\Omega,\mathrm{N}} - z^2) : H^2(\Omega) \to L^2(\Omega)$ can be expressed as

$$G^{\Omega,\mathrm{N}}_{z,y}(x) = G^{0,n}_{z,y}(x) - h^{\Omega,\mathrm{N}}_{z,y}(x),$$

where, $G_{z,y}^{0,n}$ are defined in (A.1) and (A.3), and $h_{z,y}^{\Omega,\mathrm{N}}$ is the solution of the boundary value problem

$$\begin{cases}(-\Delta - z^2)h_{z,y}^{\Omega,\mathrm{N}} = 0 & \text{in}\,\Omega \\ \frac{\partial}{\partial\nu}h_{z,y}^{\Omega,\mathrm{N}} = \frac{\partial}{\partial\nu}G_{z,y}^{0,n} & \text{in}\,\partial\Omega\end{cases}. \tag{A.6}$$

A.1.4. Domain with Robin Boundary Conditions

Let Ω be a domain in $\mathbb{R}^n$ with $n = 2$ or $n = 3$, $y \in \Omega$, ν the unit outward normal on $\partial\Omega$ and η a positive constant. The Green's function $G_{z,y}^{\Omega,\mathrm{R},\eta}$ for the operator $(-\Delta^{\Omega,\mathrm{R},\eta} - z^2) : H^2(\Omega) \to L^2(\Omega)$ can be expressed as

$$G_{z,y}^{\Omega,\mathrm{R},\eta}(x) = G_{z,y}^{0,n}(x) - h_{z,y}^{\Omega,\mathrm{R},\eta}(x),$$

where, $G_{z,y}^{0,n}$ are defined in (A.1) and (A.3), and $h_{z,y}^{\Omega,\mathrm{R},\eta}$ is the solution of the boundary value problem

$$\begin{cases}(-\Delta - z^2)h_{z,y}^{\Omega,\mathrm{R},\eta} = 0 & \text{in}\,\Omega \\ \eta h_{z,y}^{\Omega,\mathrm{R},\eta} + \frac{\partial}{\partial\nu}h_{z,y}^{\Omega,\mathrm{R},\eta} = \eta G_{z,y}^{0,n} + \frac{\partial}{\partial\nu}G_{z,y}^{0,n}, & \text{in}\,\partial\Omega\end{cases}.$$

A.2. Maximum Principles

Theorem A.2.1 ([19])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $c \geq 0$.*

(i) If

$$(-\Delta + c)u \leq 0 \quad \text{in} \quad \Omega,$$

then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+,$$

where $u^+ = \max\{u, 0\}$;

(ii) Likewise, if

$$(-\Delta + c)u \geq 0 \quad \text{in} \quad \Omega,$$

then

$$\max_{\overline{\Omega}} u \geq -\max_{\partial\Omega} u^-,$$

where $u^- = -\min\{u, 0\}$.

Theorem A.2.2 (Maximum Principle for the Dirichlet Problem, [13])**.** *Let* Ω *be an open subset of* $\mathbb{R}^n$*. Assume that*

$$f \in L^2(\Omega) \quad \text{and} \quad u \in H^1(\Omega) \cap C(\overline{\Omega})$$

satisfy

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega u\varphi = \int_\Omega f\varphi \quad \forall \varphi \in H_0^1(\Omega).$$

Then for all $x \in \Omega$

$$\min\left\{ \inf_{\partial\Omega} u, \inf_{\Omega} f \right\} \le u(x) \le \max\left\{ \sup_{\partial\Omega} u, \sup_{\Omega} f \right\}.$$

Definition A.2.1 (Lyapunov Function for the Laplacian)**.** A positive function $\varphi \in C^2(\mathbb{R}^n)$ such that $\lim_{|x|\to+\infty} \varphi(x) = +\infty$ and $\Delta\varphi - \lambda_0\varphi \le 0$ for some $\lambda_0 > 0$ is called a Lyapunov function for the Laplacian.

Proposition A.2.1 ([22])**.** *Let* Ω *be an open set in* $\mathbb{R}^n$ *with* C^2 *boundary. Let* φ *be a Lyapunov function for the Laplacian and in addition suppose that* $\frac{\partial\varphi}{\partial n} \ge 0$ *on* $\partial\Omega$*, where n is the outward unit normal vector to* $\partial\Omega$*. Let* $u \in C_b(\overline{\Omega}) \cap H^2(\Omega \cap B_R)$ *for all* $R > 0$*, such that* $-\Delta u \in C_b(\overline{\Omega})$ *and*

$$\begin{cases} -\Delta u(x) + \lambda u(x) \le 0 & x \in \Omega \\ \frac{\partial u}{\partial n}(x) \le 0 & x \in \partial\Omega \end{cases},$$

for some $\lambda \ge \lambda_0$*. Then* $u \le 0$*.*

Appendix B.

Self-Adjoint Extensions of Symmetric Operators

B.1. Kreĭn's Formula

Let $\dot{A}$ be a densely defined, closed symmetric operator in $\mathscr{H}$ with deficiency indices (N, N), $N \in \mathbb{N}$. Let B and C two self-adjoint extensions of $\dot{A}$ and denote by $\mathring{A}$ the maximal coomon part of B and C. Let (M, M), $0 < M \leq N$ be the deficiency indices of $\mathring{A}$ and $\{\varphi_1(w), \ldots, \varphi_M(w)\}$ span the corresponding deficiency sbspace of $\mathring{A}$

$$\mathring{A}^* \varphi_m(w) = w\varphi_m(w), \quad \varphi_m(w) \in D(\mathring{A}^*), \quad m = 1, \ldots, M, \quad w \in \mathbb{C} \setminus \mathbb{R} \tag{B.1}$$

and $\{\varphi_1(w), \ldots, \varphi_M(w)\}$ linearly independent. Then, the following theorem holds.

Theorem B.1.1 (Kreĭn's Formula, [2])**.** *Let $B, C, \mathring{A}$ and $\dot{A}$ as above. Then*

$$(B - w)^{-1} - (C - w)^{-1} = \sum_{m,n=1}^{M} \lambda_{mn}(w)(\varphi_n(\overline{w}), \cdot)\varphi_m(w). \quad w \in \rho(B) \cap \rho(C),$$

where the matrix $\lambda(w)$ is nonsingular for $w \in \rho(B) \cap \rho(C)$ and $\lambda_{mn}(w)$ and $\varphi_m(w)$, $m, n = 1, \ldots, M$, may be chosen to be analytic in $w \in \rho(B) \cap \rho(C)$. In fact, $\varphi_m(w)$ may be defined as

$$\varphi_m(w) = \varphi_m(w_0) + (w - w_0)(C - w)^{-1}\varphi_m(w_0), \quad m = 1, \ldots, M, \quad w \in \rho(C),$$

where $\varphi_m(w_0)$, $m = 1, \ldots, M$, $w_0 \in \mathbb{C} \setminus \mathbb{R}$, are linear independent solutions of (B.1) for $w = w_0$ and the matrix $\lambda(w)$ satisfies

$$[\lambda(w)]^{-1}_{mn} = [\lambda(w')]^{-1}_{mn} - (w - w')(\varphi_n(\overline{w}), \quad w, w' \in \rho(B) \cap \rho(C).$$

We observe that if $\dot{A} = \mathring{A}$, then in the previous theorem M is replaced by N.

B.2. Von Neumann Decomposition Formula

Theorem B.2.1 ([40]). *Let A be a densely defined, closed symmetric operator in $\mathscr{H}$ with deficiency indices (N, N) and*

$$A^*\varphi_m(\pm i) = \pm i\varphi_m(\pm i), \quad \varphi_m(\pm i) \in D(A^*), \quad m = 1, \dots, N.$$

Then A admits self-adjoint extensions. Their domains are made of functions of the form

$$\psi = \varphi_0 + \sum_{j=1}^{N}\Big(a_m\varphi_m(-i) + b_m\varphi_m(i)\Big),$$

with $\varphi_0 \in D(\dot{A})$. Moreover

$$A^*\psi = \dot{A}\varphi_0 + i\sum_{j=1}^{N}\Big(-a_m\varphi_m(-i) + b_m\varphi_m(i)\Big).$$

B.3. Boundary Value Spaces

Definition B.3.1 (Boundary Value Space, [25]). Let A be a densely defined, closed symmetric operator in $\mathscr{H}$ with equal, finite or infinite, deficiency indices. A triple $(\mathscr{V}, \Gamma_1, \Gamma_2)$, where $\mathscr{V}$ is a Hilbert space and $\Gamma_i : D(A^*) \to \mathscr{V}$ $i = 1, 2$ are bounded linear operators, is called a boundary value space of the operator A if

$$(\psi, A^*\varphi)_{\mathscr{H}} - (A^*\psi, \varphi)_{\mathscr{H}} = (\Gamma_1\psi, \Gamma_2\varphi)_{\mathscr{V}} - (\Gamma_2\psi, \Gamma_1\varphi)_{\mathscr{V}}, \quad \forall \psi, \varphi \in D(A^*) \tag{B.2}$$

$$\text{the map} \quad (\Gamma_1, \Gamma_2) : D(A^*) \to \mathscr{V} \oplus \mathscr{V} \quad \text{is surjective.} \tag{B.3}$$

Theorem B.3.1 ([25]). *For any symmetric operator with deficiency indices (N, N) ($N \le +\infty$) there exists a boundary value space $(\mathscr{V}, \Gamma_1, \Gamma_2)$ with* $\dim \mathscr{V} = N$.

Now let $n \in \mathbb{N}$ and B, C be complex $n \times n$ matrices. Define $E = \begin{pmatrix} B\, C \end{pmatrix}$, where E is the $n \times 2n$ matrix obtained by horizontal juxtaposition of B and C. Let $W = \left\{ E = \begin{pmatrix} B\, C \end{pmatrix} \,\middle|\, BC^* = CB^*, \operatorname{Ran} E = n \right\}$ The following proposition holds.

Theorem B.3.2 ([11]). *Let A be a densely defined, closed symmetric operator in $\mathscr{H}$ with equal deficiency (N, N), $N < +\infty$ and let $(\mathscr{V}, \Gamma_1, \Gamma_2)$ be its boundary value space. There is a bijective correspondence between the self-adjoint extensions of A and the set W defined above. A self-adjoint extension $A^{B,C}$, corresponding to $\begin{pmatrix} B\, C \end{pmatrix} \in W$, is given by the restriction of A^* to those elements $\psi \in D(A^*)$ satisfying the boundary conditions*

$$B\Gamma_1\psi = C\Gamma_2\psi.$$

B.4. Quadratic Forms

Theorem B.4.1 ([28]). *Let $q(\cdot,\cdot)$ be a densely defined, symmetric, closed quadratic form bounded from below in $\mathscr{H}$. There exists a self-adjoint and bounded from below operator Q such that*

(i) *$D(Q) \subset D(q)$ and*

$$q(u, v) = (Qu, v),$$

for every $u \in D(Q)$ and $v \in D(q)$.

(ii) *$D(Q)$ is a core of q.*

(iii) *If $u \in D(q)$, $w \in \mathscr{H}$ and*

$$q(u, v) = (w, v)$$

holds for every v belonging to a core of q, then $u \in D(Q)$ and $Qu = w$. Q and q have the same lower bound.

Appendix C.

Special Functions

Unless otherwise stated, the results within this appendix are taken from [38].

C.1. Lambert W Function

The Lambert W function is defined to be the function satisfying

$$W(w)e^{W(w)} = w.$$

It is a multivalued function with countable many branches. Fig. C.1 shows some of the branches ranges. The principal branch is denoted with W_0 and is separated from the branches W_1 and W_{-1} by the curve $\gamma_0 = \{ -b\cot b + ib, -\pi < b < \pi \}$. W_1 and W_{-1} are separated by the half-line $(-\infty, -1]$. The curves separating the other branches are given by

$$\gamma_{\pm k} = \{ -b\cot b + ib, 2k\pi < \pm b < (2k+1)\pi \} \quad \text{for} \quad k = 1, 2, \ldots$$

C.2. Exponential Integral

Let $w \neq 0$, then the exponential integral $E_1(w)$ is defined by

$$E_1(w) = \int_w^\infty \frac{e^{-u}}{u} \, \mathrm{d}u, \tag{C.1}$$

where the path of integration does not cross the negative real axis or pass through the origin. If $\operatorname{Re} w > 0$, then also the following identities hold

$$E_1(w) = e^{-w} \int_0^\infty \frac{e^{-u}}{u+w} \, \mathrm{d}u = e^{-w} \int_0^{+\infty} \frac{e^{-wu}}{u+1} \, \mathrm{d}u. \tag{C.2}$$

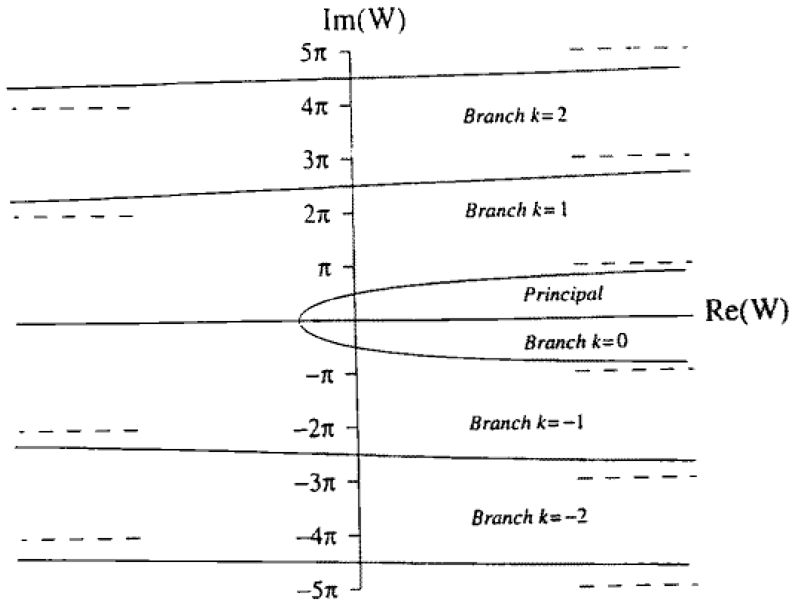

Figure C.1.: The ranges of some branches of $W(w)$, separated by the curves γ_k. The figure is taken from [14]

E_1 has the following properties:

- for $|w| \to +\infty$ it holds that

$$E_1(w) = \frac{e^{-w}}{w} + O\Big(\frac{e^{-w}}{w^2}\Big); \tag{C.3}$$

- for $w \to 0$ it holds that

$$E_1(w) = -\gamma - \ln w + O(w); \tag{C.4}$$

- for $x > 0$ the two following inequalities are verified

$$e^x E_1(x) > \frac{1}{2}\ln\left(1 + \frac{2}{x}\right) \quad xe^x E_1(x) > \frac{x}{x+1}. \tag{C.5}$$

C.3. Bessel and Related Functions

C.3.1. Bessel Functions

Let $\nu \in \mathbb{C}$. Consider the differential equation

$$w^2 \frac{\mathrm{d}^2 f}{\mathrm{d}w^2} + w\frac{\mathrm{d}f}{\mathrm{d}w} + (w^2 - \nu^2)f = 0. \tag{C.6}$$

The two linearly independent solutions of this equations are named $J_\nu(w)$ and $Y_\nu(w)$. These functions are respectively called Bessel functions of the first and second kind of order ν. It holds that, for $n \in \mathbb{Z}$,

$$|J_n(w)| \leq e^{|\mathrm{Im}\, w|} \tag{C.7}$$

$$J_{-n}(w) = (-1)^n J_n(w). \tag{C.8}$$

C.3.2. Hankel Functions

The Hankel function of the first kind of order ν is defined as

$$H_\nu^{(1)}(w) = J_\nu(w) + iY_\nu(w), \tag{C.9}$$

while the Hankel function of the second kind of order ν is defined as

$$H_\nu^{(2)}(w) = J_\nu(w) - iY_\nu(w). \tag{C.10}$$

It holds that

- for small w

$$H_0^{(1)}(w) = 1 + \frac{2i}{\pi}\Big(\ln\frac{w}{2} + \gamma\Big) + O(w^2 \ln w); \tag{C.11}$$

- the Hankel function of the first kind has the following parity property with respect to order

$$H_{-n}^{(1)}(w) = (-1)^n H_n^{(1)}(w); \tag{C.12}$$

- if $u, v, w \in \mathbb{R}^n$ with $w = u - v$, $|v| < |u|$ and let α be the angle between u and v, then

$$H_0^{(1)}\big(|w|\big) = \sum_{n=-\infty}^{+\infty} H_n^{(1)}\big(|u|\big)J_n\big(|v|\big)\cos(n\alpha); \tag{C.13}$$

- Hankel functions have the following behaviour for large argument

$$H_n^{(1)}(w) \sim \sqrt{\frac{2}{\pi w}} e^{i\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)} \quad \text{for} \quad |w| \to +\infty \quad \text{and} \quad -\pi + \delta \le \operatorname{Arg} z \le 2\pi - \delta \tag{C.14}$$

$$H_n^{(2)}(w) \sim \sqrt{\frac{2}{\pi w}} e^{-i\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)} \quad \text{for} \quad |w| \to +\infty \quad \text{and} \quad -2\pi + \delta \le \operatorname{Arg} z \le \pi - \delta; \tag{C.15}$$

- the Hankel function of the first kind has the following parity property with respect to argument

$$H_n^{(1)}(-w) = (-1)^{n+1} H_n^{(2)}(w). \tag{C.16}$$

C.3.3. Modified Bessel Functions

We define the modified Bessel functions as follows

$$I_\nu(w) = e^{\mp \nu i\pi/2} J_\nu\left(e^{\pm i\pi/2} w\right), \quad -\pi \le \pm \operatorname{Arg} w \le \frac{\pi}{2} \tag{C.17}$$

$$K_\nu(w) = \frac{\pi}{2} i e^{\nu i\pi/2} H_\nu^{(1)}\left(e^{i\pi/2} w\right), \quad -\pi \le \operatorname{Arg} w \le \frac{\pi}{2}, \tag{C.18}$$

where I_ν and K_ν denote respectively the modified Bessel function of the first and of the second kind of order ν. We recall some properties of these functions:

- For n non-negative integer the following power series representations hold

$$I_n(w) = \left(\frac{w}{2}\right)^n \sum_{k=0}^{+\infty} \frac{\left(\frac{w^2}{4}\right)^k}{k!(k+n)!} \tag{C.19}$$

and

$$K_n(w) = \frac{1}{2}\left(\frac{w}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(-\frac{w^2}{4}\right)^k + (-1)^{n+1} \ln \frac{w}{2} I_n(w)$$
$$+ (-1)^n \frac{1}{2}\left(\frac{w}{2}\right)^n \sum_{k=0}^{+\infty} \frac{\Big(\psi(k+1) + \psi(n+k+1)\Big)\left(\frac{w^2}{4}\right)^k}{k!(n+k)!}, \tag{C.20}$$

where ψ is the digamma function. Using the definition of the ψ, for $n = 0$, last equation can be written as

$$K_0(w) = -\left(\ln\frac{w}{2} + \gamma\right)I_0(w) + \sum_{k=1}^{+\infty} A_k \frac{w^{2k}}{(2k!)^{2k}}, \tag{C.21}$$

where $A_k = \sum_{l=1}^{k} \frac{1}{k}$. From last equation it follows (recalling that $\psi(1) = -\gamma$, with γ being the Euler constant)

$$K_0(w) = -\ln\frac{w}{2} - \gamma + O(w^2 \ln w) \quad \text{for } w \to 0. \tag{C.22}$$

- From (C.19) and (C.20) it follows that

$$I_n(w) \sim \frac{1}{n!}\left(\frac{w}{2}\right)^n \quad \text{and} \quad K_n(w) \sim \frac{(n-1)!}{2}\left(\frac{w}{2}\right)^{-n} \quad \text{for} \quad w \to 0 \quad n = 1, 2, \ldots. \tag{C.23}$$

- Their derivatives satisfy

$$I_n'(w) = I_{n-1}(w) - \frac{n}{w}I_n(w) \quad n = 1, 2, \ldots \tag{C.24}$$

$$K_n'(w) = -K_{n-1}(w) - \frac{n}{w}K_n(w) \quad n = 1, 2, \ldots \tag{C.25}$$

and $I_0'(w) = I_1(w)$ and $K_0'(w) = -K_1(w)$.

- For x positive real, $I_n(x)$ and $K_n(x)$ are positive real. In particular $I_0(x) \geq 1 \ \forall x \geq 0$ and $I_0(0) = 1$.
- The following cross-product identity

$$I_\nu(w)K_{\nu+1}(w) + I_{\nu+1}(w)K_\nu(w) = \frac{1}{w}. \tag{C.26}$$

- The connection formulas

$$K_0(-w) = K_0(w) - i\pi I_0(w) \tag{C.27}$$

$$J_0(w) = I_0(iw), \quad -\pi \leq \operatorname{Arg} w \leq \frac{\pi}{2} \tag{C.28}$$

$$Y_0(w) = -iI_0(iw) - \frac{2}{\pi}K_0(iw), \quad -\pi \leq \operatorname{Arg} w \leq \frac{\pi}{2}. \tag{C.29}$$

- The integral representation

$$K_\nu(w) = \int_0^{+\infty} e^{-w\cosh t}\cosh(\nu t)\,\mathrm{d}t, \quad \left|\operatorname{Arg} w\right| < \frac{\pi}{2}. \tag{C.30}$$

- The inequality (see [48])

$$I_n(w) < \frac{\cosh w}{n!}\left(\frac{w}{2}\right)^n \quad n = 1, 2 \dots . \tag{C.31}$$

- The inequality (see [24])

$$2^{n-1}(n-1)!e^{-w} < w^n K_n(w) < 2^{n-1}(n-1)! \quad n = 1, 2 \dots . \tag{C.32}$$

- The following asymptotics

$$I_\nu(w) \sim \frac{1}{\sqrt{2\pi w}}\left(\frac{ew}{2\nu}\right)^\nu \quad K_\nu \sim \sqrt{\frac{\pi}{2\nu}}\left(\frac{ew}{2\nu}\right)^{-\nu} \quad \text{for} \quad \nu \to +\infty \tag{C.33}$$

$$I_n(w) \sim \frac{e^w}{\sqrt{2\pi w}} \quad K_n(w) \sim \sqrt{\frac{\pi}{2w}}e^{-w} \quad |w| \to +\infty. \tag{C.34}$$

C.3.4. Spherical Bessel Functions

We define the spherical Bessel functions of non-negative order n as

$$j_n(w) = \sqrt{\frac{\pi}{2w}}J_{n+\frac{1}{2}}(w) \tag{C.35}$$

$$y_n(w) = \sqrt{\frac{\pi}{2w}}Y_{n+\frac{1}{2}}(w), \tag{C.36}$$

where j_n and y_n denote respectively the spherical Bessel function of the first and of the second kind of order ν. It holds that, for n non-negative integer, if $u, v, w \in \mathbb{R}^n$ with $w = u - v$ and let α be the angle between u and v, then

$$\frac{\cos|w|}{|w|} = -\sum_{n=0}^{+\infty}(2n+1)P_n(\cos\alpha)j_n^{(1)}\Big(|v|\Big)y_n\Big(|u|\Big), \quad |v| < |u| \tag{C.37}$$

$$\frac{\sin|w|}{|w|} = \sum_{n=0}^{+\infty}(2n+1)P_n(\cos\alpha)j_n^{(1)}\Big(|v|\Big)j_n\Big(|u|\Big), \tag{C.38}$$

where P_n denotes the n-th Legendre polynomial.

C.3.5. Spherical Hankel functions

The Spherical Hankel function of the first kind of order n is defined as

$$h_n^{(1)}(w) = j_n(w) + iy_n(w). \tag{C.39}$$

C.3.6. Modified Spherical Bessel Functions

We define the modified spherical Bessel functions as follows

$$i_n^{(1)}(w) = i^{-n} j_n(iw) \tag{C.40}$$

$$k_n(w) = -\frac{\pi}{2} i^n h_n^{(1)}(iw). \tag{C.41}$$

These functions have the following properties:

- for n non-negative integer the following power series representations hold

$$i_n^{(1)}(w) = w^n \sum_{k=0}^{+\infty} \frac{2^{-k} z^{2k}}{k!(2n+2k+1)!!} \tag{C.42}$$

$$\begin{aligned} k_n(w) = {} & \frac{\pi}{2w^{n+1}} \sum_{k=0}^{n} \frac{(2n-2k-1)!!(-1)^k 2^{-k} w^{2k}}{k!} \\ & + (-1)^{n+1} \frac{\pi}{2} w^n \sum_{k=0}^{+\infty} \frac{2^{-k} w^{2k}}{k!(2n+2k+1)!!} + (-1)^n \frac{\pi}{2w^{n+1}} \sum_{k=n+1}^{+\infty} \frac{2^{-k} w^{2k}}{k!(2k-2n-1)!!}; \end{aligned} \tag{C.43}$$

- they and their derivatives have this asymptotic behaviour towards 0

$$i_n^{(1)}(w) \sim \frac{w^n}{(2n+1)!!} \quad \text{and} \quad k_n(w) \sim \frac{\pi(2n-1)!!}{2w^{n+1}} \tag{C.44}$$

$$i_0^{(1)\prime}(w) = \frac{w}{3} + o(w) \quad i_n^{(1)\prime}(w) = \frac{n}{(2n+1)!!} w^{n-1} + o(w^{n-1}) \quad \text{for} \quad n = 1, 2, \ldots \tag{C.45}$$

$$k_n'(w) = -\frac{\pi}{2}(n+1)(2n-1)!!\, w^{-n-2} + o(w^{-n-2}) \quad \text{for} \quad n = 0, 1, \ldots; \tag{C.46}$$

- they are linked to the modified Bessel functions I and K, through

$$i_n^{(1)}(w) = \sqrt{\frac{\pi}{2w}} I_{n+\frac{1}{2}}(w) \quad \text{and} \quad k_n(w) = \sqrt{\frac{\pi}{2w}} K_{n+\frac{1}{2}}(w); \tag{C.47}$$

- their derivatives satisfy

$$i_n^{(1)\prime}(w) = i_{n-1}^{(1)}(w) - \frac{n+1}{w} i_n^{(1)}(w) \quad \text{for} \quad n = 1, 2, \ldots \tag{C.48}$$

$$k_n'(w) = -k_{n-1}(w) - \frac{n+1}{w} k_n(w) \quad \text{for} \quad n = 1, 2, \ldots. \tag{C.49}$$

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