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$\mathbb{Z}$-linear Gale duality and poly weighted spaces (PWS)

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ABSTRACT

The present paper is devoted to discussing Gale duality from the $\mathbb{Z}$-linear algebraic point of view. This allows us to isolate the class of $\mathbb{Q}$-factorial complete toric varieties whose class group is torsion free, here called *poly weighted spaces* (PWS). It provides an interesting generalization of weighted projective spaces (WPS).

0. Introduction

The present paper is the first part of a longstanding tripartite study aimed to realize, for $\mathbb{Q}$-factorial projective toric varieties, a birational classification inspired by what

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V. Batyrev did in 1991 for smooth complete toric varieties [2]. This paper is then devoted to giving the necessary algebraic and geometric background. In particular we generalize many properties of weighted projective spaces (WPS, for short), as studied in [21], to the case of $\mathbb{Q}$-factorial and complete toric varieties of Picard number (in the following called *rank*) greater than 1.

The main subject of this paper will be the study of what we call *$\mathbb{Z}$-linear Gale Duality* and its applications to the geometry of toric varieties. Gale duality (or *Gale transform*) has the origin of its name to the article [14], by D. Gale, devoted to the study of polytopes, although the geometry connected with Gale duality is actually much older. D. Eisenbud and S. Popescu in [12, § 1] so write: "Perhaps the first result that belongs to the development of the Gale transform is the theorem of Pascal (from his "Essay Pour Les Coniques", from 1640, [...])". Gale duality has found its main application in the classification of polytopes [13, § 4, § 6] and probably for this reason it has been essentially developed on the real field $\mathbb{R}$. The pivotal paper in introducing Gale duality for toric varieties was [20], where T. Oda and H.S. Park employed Gale duality to study the toric geometric implication of the *Gel'fand–Kapranov–Zelevinsky decomposition* of the *secondary polytope* of a convex polytope [16,17]. D.A. Cox, J.B. Little and H.K. Schenck dedicated an entire chapter, of their comprehensive book on toric varieties, to the discussion of Gale duality and the secondary fan [9, Ch. 15].

Largely inspired by the latter, in this paper we consider Gale duality, as defined in [9, § 14.3], from the $\mathbb{Z}$-linear point of view (see § 3.1), which briefly is a duality, well defined up to left multiplication by a unimodular integer matrix, between a *fan matrix* of a toric variety, defined by taking the primitive generators of all the fan's rays, and a *weight matrix*: in the easiest case of a WPS, the latter is precisely the row vector of weights. In the more general case of a $\mathbb{Q}$-factorial complete toric variety, the weight matrix admits so many rows as the rank of the considered variety. Such a weight matrix should be compared with the *combined weight systems* (CWS) of M. Kreuzer and H. Skarke [18]: we believe that the Gale duality gives an algebraic background in which a CWS could be thought of.

For an extension of Gale duality to maps of finitely generated abelian groups see [4].

The core of this paper is § 2, which contains the motivations of our work and the main geometric results. It starts with a contextualization of the $\mathbb{Z}$-linear algebraic Gale duality in the toric geometric setup, allowing to associate to a complete $\mathbb{Q}$-factorial toric variety X a fan matrix V and a weight matrix Q. The first important geometric result is Theorem 2.4, which gives a linear algebraic characterization of the torsion of the class group $\mathrm{Cl}(X)$ in terms of the fan matrix V of X (see Proposition 2.6). In particular we see that the class group $\mathrm{Cl}(X)$ is a free abelian group exactly when we can recover the fan matrix V as a Gale dual of the weight matrix Q; this gives a first motivation to isolate the class of complete $\mathbb{Q}$-factorial toric varieties X such that $\mathrm{Tors}(\mathrm{Cl}(X)) = 0$, called *poly weighted spaces* (PWS, for short) (see Definition 2.7 and Proposition 2.6). In Remark 2.8 we list a number of motivations for such an appellation some of which can be briefly summarized as follows:

1. for rank 1, a PWS is a WPS,
2. the weight matrix of a PWS X allows one to completely determine the action of $\mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$ describing X as a Cox geometric quotient, just like the weight vector of a WPS does,
3. Gale duality generalizes to higher rank the well known characterization of a fan matrix of a PWS by means of the weights, as described e.g. in [21, Thm. 3].

Actually the main motivation to the study of PWS is given in § 2.3 and will be the subject of the forthcoming paper [22]: namely the Batyrev–Cox and Conrads result [3, Lemma 2.11], [7, Prop. 4.7] describing a $\mathbb{Q}$-factorial complete toric variety of rank 1 as a finite quotient of a WPS (in the literature called a *fake WPS*), can be generalized to higher rank by replacing the covering WPS with a suitable PWS: therefore every $\mathbb{Q}$-factorial complete toric variety is a *fake PWS* i.e. a finite quotient of a PWS.

An important result of § 2 is given by Theorem 2.9, which is the generalization, to higher rank, of [21, Prop. 8] and exhibits the bases of the subgroup of Cartier divisors inside the free group of Weil divisors and of the Picard subgroup inside the class group, respectively, for every PWS. Example 2.11 gives an account of all the described techniques.

Section 3 is devoted to present the algebraic background. When considered from the $\mathbb{Z}$-linear point of view, Gale duality turns out to relate properties of every submatrix of a fan matrix, defined by the choice of a subset of fan's generators, with properties of the *complementary* submatrix, in the sense of (13), of a weight matrix: here the main results are Theorem 3.2 and Corollary 3.3. As an example of an application of these results, one can control the singularities of a given projective $\mathbb{Q}$-factorial toric variety X, by looking at a suitable class of simplicial cones in $\mathrm{Cl}(X) \otimes \mathbb{R}$ containing the Kähler cone of X (this particular aspect will be deeply studied and applied in the forthcoming paper [23]). In section 3, $\mathbb{Z}$-linear Gale duality will be studied from the pure linear algebraic point of view, without any reference to the geometric properties of the underlying toric varieties, although speaking about *F-matrices* and *W-matrices* (see § 3.2) we are clearly referring to possible fan matrices and weight matrices, respectively. Further significant results of § 3 are

- Theorem 3.8 stating the equivalence, via Gale duality, between *F-complete matrices* and *W-positive matrices* (see Definition 3.4), geometrically meaning that a PWS is characterized by the choice of a positive weight matrix.
- Theorem 3.15 generalizing to any rank the well known reduction process for the weights of a WPS. It gives an easy interpretation by Gale duality: briefly a weight matrix is *reduced* if and only if a (and every) Gale dual matrix of it is a matrix whose columns admit only coprime entries.
- Theorem 3.18 whose geometric meaning is that a weight matrix of a complete $\mathbb{Q}$-factorial toric variety can always be assumed to be a positive matrix in row echelon form.

Section 4 contains the proofs of results presented in Section 2. It also provides some constructive techniques for dealing with fan and weight matrices. For example Proposition 4.3 gives a procedural recipe for Gale duality: a Gale dual matrix Q of a fan matrix V can be recovered by the bottom rows of the switching matrix giving the *Hermite Normal Form* (HNF, for short) of the transpose matrix V^T. It is the generalization to higher rank of [21, Prop. 5].

1. Preliminaries and notation

1.1. Toric varieties

A *n-dimensional toric variety* is an algebraic normal variety X containing the *torus* $T := (\mathbb{C}^*)^n$ as a Zariski open subset such that the natural multiplicative self-action of the torus can be extended to an action $T \times X \to X$.

Let us quickly recall the classical approach to toric varieties by means of *cones* and *fans*. For proofs and details the interested reader is referred to the extensive treatments [10,15,19] and the recent and quite comprehensive [9].

As usual M denotes the *group of characters* $\chi : T \to \mathbb{C}^*$ of T and N the *group of 1-parameter subgroups* $\lambda : \mathbb{C}^* \to T$. It follows that M and N are n-dimensional dual lattices via the pairing

$$\begin{array}{ccc} M \times N & \longrightarrow & \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{C}^* \\ (\chi, \lambda) & \longmapsto & \chi \circ \lambda \end{array}$$

which translates into the standard paring $\langle u, v\rangle = \sum u_i v_i$ under the identifications $M \cong \mathbb{Z}^n \cong N$ obtained by setting $\chi(\mathbf{t}) = \mathbf{t}^{\mathbf{u}} := \prod t_i^{u_i}$ and $\lambda(t) = t^{\mathbf{v}} := (t^{v_1}, \ldots, t^{v_n})$.

1.1.1. Cones and affine toric varieties

Define $N_{\mathbb{R}} := N \otimes \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes \mathbb{R} \cong \mathrm{Hom}(N, \mathbb{Z}) \otimes \mathbb{R} \cong \mathrm{Hom}(N_{\mathbb{R}}, \mathbb{R})$.

A *convex polyhedral cone* (or simply a *cone*) σ is the subset of $N_{\mathbb{R}}$ defined by

$$\sigma = \langle \mathbf{v}_1, \ldots, \mathbf{v}_s\rangle := \{r_1\mathbf{v}_1 + \cdots + r_s\mathbf{v}_s \in N_{\mathbb{R}} \mid r_i \in \mathbb{R}_{\geq 0}\}$$

The s vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s \in N_{\mathbb{R}}$ are said *to generate* σ. A cone $\sigma = \langle \mathbf{v}_1, \ldots, \mathbf{v}_s\rangle$ is called *rational* if $\mathbf{v}_1, \ldots, \mathbf{v}_s \in N$, *simplicial* if $\mathbf{v}_1, \ldots, \mathbf{v}_s$ are $\mathbb{R}$-linear independent and *non-singular* if $\mathbf{v}_1, \ldots, \mathbf{v}_s$ can be extended by $n - s$ further elements of N to give a basis of the lattice N. A 1-generated cone $\rho = \langle \mathbf{v}\rangle$ is also called a *ray*.

A cone σ is called *strictly convex* if it does not contain a linear subspace of positive dimension of $N_{\mathbb{R}}$. The linear span of a cone σ will be denoted by $\mathcal{L}(\sigma)$ and by definition $\dim(\sigma) := \dim(\mathcal{L}(\sigma))$. For a n-generated cone $\sigma = \langle \mathbf{v}_1, \ldots, \mathbf{v}_n\rangle$ we will set

$$\det(\sigma) := |\det(\mathbf{n}_1, \ldots, \mathbf{n}_n)|$$

where $\mathbf{n}_i$ is the generator of the monoid $\langle \mathbf{v}_i \rangle \cap N$. A n-generated cone σ will be called *unimodular* when $\det(\sigma) = 1$. Then a n-generated cone is simplicial if and only if its determinant is non-zero and it is non-singular if and only if it is unimodular.

The *dual cone* $\sigma^\vee$ *of* σ is the subset of $M_\mathbb{R}$ defined by

$$\sigma^\vee = \{\mathbf{u} \in M_\mathbb{R} \mid \forall\ \mathbf{v} \in \sigma \quad \langle \mathbf{u}, \mathbf{v} \rangle \geq 0\}$$

A *face* τ *of* σ (denoted by $\tau < \sigma$) is the subset defined by

$$\tau = \sigma \cap \mathbf{u}^\perp = \{\mathbf{v} \in \sigma \mid \langle \mathbf{u}, \mathbf{v} \rangle = 0\}$$

for some $\mathbf{u} \in \sigma^\vee$. Observe that also τ is a cone.

Gordon's Lemma (see [15] § 1.2, Proposition 1) ensures that the semigroup $S_\sigma := \sigma^\vee \cap M$ is *finitely generated*. Then also the associated $\mathbb{C}$-algebra $A_\sigma := \mathbb{C}[S_\sigma]$ is finitely generated. A choice of r generators gives a presentation of A_σ

$$A_\sigma \cong \mathbb{C}[X_1, \ldots, X_r]/I_\sigma$$

where I_σ is the ideal generated by the relations between generators. Then

$$U_\sigma := \mathcal{V}(I_\sigma) \subset \mathbb{C}^r$$

turns out to be an *affine toric variety*. In other terms an affine toric variety is given by $U_\sigma := \mathrm{Spec}(A_\sigma)$. Since a closed point $x \in U_\sigma$ is an evaluation of elements in $\mathbb{C}[S_\sigma]$ satisfying the relations generating I_σ, then it can be identified with a semigroup morphism $x : S_\sigma \to \mathbb{C}$ assigned by thinking of $\mathbb{C}$ as a multiplicative semigroup. In particular the *characteristic morphism*

$$\begin{array}{rccc} x_\sigma & : \sigma^\vee \cap M & \longrightarrow & \mathbb{C} \\ & \mathbf{u} & \longmapsto & \begin{cases} 1 \text{ if } \mathbf{u} \in \sigma^\perp \\ 0 \text{ otherwise} \end{cases} \end{array} \tag{1}$$

which is well defined since $\sigma^\perp < \sigma^\vee$, defines a *characteristic point* $x_\sigma \in U_\sigma$ whose torus orbit O_σ turns out to be a $(n - \dim(\sigma))$-dimensional torus embedded in U_σ (see e.g. [15] § 3).

1.1.2. Fans and toric varieties

A *fan* Σ is a finite set of cones $\sigma \subset N_\mathbb{R}$ such that

1. for any cone $\sigma \in \Sigma$ and for any face $\tau < \sigma$ then $\tau \in \Sigma$,
2. for any $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau < \sigma$ and $\sigma \cap \tau < \tau$.

For every i with $0 \leq i \leq n$ denote by $\Sigma(i) \subset \Sigma$ the subset of i-dimensional cones, called the *i-skeleton of* Σ. A fan Σ is called *simplicial* if every cone $\sigma \in \Sigma$ is simplicial and *non-singular* if every such cone is non-singular. The *support* of a fan Σ is the subset $|\Sigma| \subset N_{\mathbb{R}}$ obtained as the union of all of its cones i.e.

$$|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{R}} \ .$$

If $|\Sigma| = N_{\mathbb{R}}$ then Σ will be called *complete* or *compact*.

Since for any face $\tau < \sigma$ the semigroup S_σ turns out to be a sub-semigroup of S_τ, there is an induced immersion $U_\tau \hookrightarrow U_\sigma$ between the associated affine toric varieties which embeds U_τ as a principal open subset of U_σ. Given a fan Σ one can construct *an associated toric variety* $X(\Sigma)$ by patching all the affine toric varieties $\{U_\sigma \mid \sigma \in \Sigma\}$ along the principal open subsets associated with any common face. Moreover *for every toric variety X there exists a fan Σ such that $X \cong X(\Sigma)$* (see [19] Theorem 1.5). It turns out that ([19] Theorems 1.10 and 1.11; [15] § 2):

- *$X(\Sigma)$ is non-singular if and only if the fan Σ is non-singular*,
- *$X(\Sigma)$ is complete if and only if the fan Σ is complete.*

Let us now introduce some non-standard notation.

Definition 1.1. A 1*-generated rational fan* (1*-fan* for short) Σ is a fan whose 1-skeleton is given by a set $\Sigma(1) = \{\langle \mathbf{v}_1 \rangle, \ldots, \langle \mathbf{v}_s \rangle\} \subset N_{\mathbb{R}}$ of rational rays and whose cones $\sigma \subset N \otimes \mathbb{R}$ are generated by *any* proper subset of $\Sigma(1)$. When it makes sense, we will write

$$\Sigma = \mathrm{fan}(\mathbf{v}_1, \ldots, \mathbf{v}_s) \ . \tag{2}$$

Examples 1.2.

1. Given

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \in N = \mathbb{Z}^2$$

   then $\mathrm{fan}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is well defined and gives a fan of $\mathbb{P}^2$.

2. Given

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in N = \mathbb{Z}^2$$

   then $\mathrm{fan}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is not well defined.

Let us now introduce some non-standard notation.

Definition 1.3. A *rational simplicial fan* Σ is a fan whose 1-skeleton is given by a set $\Sigma(1) = \{\langle \mathbf{v}_1\rangle, \dots, \langle \mathbf{v}_s\rangle\} \subset N_{\mathbb{R}}$ of rational rays and whose cones $\sigma \subset N \otimes \mathbb{R}$ are generated by *a suitable choice of* proper subsets of $\Sigma(1)$, such that all the chosen subsets generate simplicial cones. In general, for a given set of generators these fans are not unique and $\mathcal{SF}(\mathbf{v}_1, \dots, \mathbf{v}_s)$ will denote the set of rational simplicial fans whose 1-skeleton is given by $\Sigma(1) = \{\langle \mathbf{v}_1\rangle, \dots, \langle \mathbf{v}_s\rangle\} \subset N_{\mathbb{R}}$ and whose support is $|\Sigma| = \langle \mathbf{v}_1, \dots, \mathbf{v}_s\rangle$. The matrix $V = (\mathbf{v}_1, \dots, \mathbf{v}_s)$ will be called *a fan matrix* for every fan in $\mathcal{SF}(V) := \mathcal{SF}(\mathbf{v}_1, \dots, \mathbf{v}_s)$: it is determined up to permutations of the s generators. A fan matrix V is called *reduced* if any column $\mathbf{v}_i$ of V is the generator of the monoid $\langle \mathbf{v}_i\rangle \cap N$.

Examples 1.4.

1. Consider $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subset N = \mathbb{Z}^2$ like in Example 1.2 (1). Then $\mathcal{SF}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \{\mathrm{fan}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\}$.
2. Consider $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subset N = \mathbb{Z}^2$ like in Example 1.2 (2). Then
$$\mathcal{SF}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \{\Sigma_1 = \{\langle \mathbf{v}_1, \mathbf{v}_3\rangle, \langle \mathbf{v}_2, \mathbf{v}_3\rangle, \langle \mathbf{v}_1\rangle, \langle \mathbf{v}_2\rangle, \langle \mathbf{v}_3\rangle, \langle 0\rangle\}\} \ .$$
3. Given
$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix}0\\1\\1\end{pmatrix} \in N = \mathbb{Z}^3 \ .$$
Then $\mathcal{SF}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \{\Sigma_1, \Sigma_2\}$ with
$$\begin{aligned}\Sigma_1 &= \{\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\rangle, \langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\rangle, \langle \mathbf{v}_1, \mathbf{v}_2\rangle, \langle \mathbf{v}_1, \mathbf{v}_3\rangle, \langle \mathbf{v}_2, \mathbf{v}_3\rangle, \langle \mathbf{v}_2, \mathbf{v}_4\rangle, \langle \mathbf{v}_3, \mathbf{v}_4\rangle,\\ &\qquad \langle \mathbf{v}_1\rangle, \langle \mathbf{v}_2\rangle, \langle \mathbf{v}_3\rangle, \langle \mathbf{v}_4\rangle, \langle 0\rangle\}\\ \Sigma_2 &= \{\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\rangle, \langle \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\rangle, \langle \mathbf{v}_1, \mathbf{v}_2\rangle, \langle \mathbf{v}_1, \mathbf{v}_3\rangle, \langle \mathbf{v}_1, \mathbf{v}_4\rangle, \langle \mathbf{v}_2, \mathbf{v}_4\rangle, \langle \mathbf{v}_3, \mathbf{v}_4\rangle,\\ &\qquad \langle \mathbf{v}_1\rangle, \langle \mathbf{v}_2\rangle, \langle \mathbf{v}_3\rangle, \langle \mathbf{v}_4\rangle, \langle 0\rangle\}\end{aligned}$$
giving two simplicial fans obtained by putting a diagonal facet inside the non-simplicial cone $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\rangle$.

Definition 1.5. If the set of generators $\{\mathbf{v}_1, \dots, \mathbf{v}_s\} \subset N$ is such that the cardinality $|\mathcal{SF}(\mathbf{v}_1, \dots, \mathbf{v}_s)| = 1$, then the unique element $\Sigma \in \mathcal{SF}(\mathbf{v}_1, \dots, \mathbf{v}_s)$ is called a 1-*detected fan* and the associated normal toric variety $X(\Sigma)$ is said a *divisorially detected variety* (or simply *dd-variety*).

Remarks 1.6. Some immediate observations are the following.

1. If $s \leq n = \operatorname{rk} N$ and $\mathbf{v}_1, \dots, \mathbf{v}_s$ are $\mathbb{R}$-linearly independent then
$$|\mathcal{SF}(\mathbf{v}_1, \dots, \mathbf{v}_s)| = 1$$
since its unique 1-detected fan is given by the 1-fan $\operatorname{fan}(\mathbf{v}_1, \dots, \mathbf{v}_s)$.
2. Clearly a 1-fan is always a 1-detected fan, but the converse is false: recall Example 1.4 (2).
3. If $n = 2$ and the fan matrix $V = (\mathbf{v}_1, \dots, \mathbf{v}_s)$ has maximum rank 2, then
$$|\mathcal{SF}(\mathbf{v}_1, \dots, \mathbf{v}_s)| = 1 \text{ for any } s \geq 2.$$
In fact for $s = 2$ we are in the previous case; for $s \geq 3$, construct the 1-detected fan $\Sigma \in \mathcal{SF}(\mathbf{v}_1, \dots, \mathbf{v}_s)$ by considering all the rank 2 sub-matrices $(\mathbf{v}_i, \mathbf{v}_j)$ of V such that every $\mathbf{v}_k$, with $k \neq i, j$, is not in the interior of the cone $\sigma_{ij} := \langle \mathbf{v}_i, \mathbf{v}_j \rangle$. Assume all the σ_{ij} to be the maximal cones of Σ and the remaining cones of Σ to be given by all their faces.
4. As a consequence of the previous fact, we get the non surprising assertion that *every 2-dimensional normal and $\mathbb{Q}$-factorial toric variety is a dd variety.*

1.2. Normal forms for matrices

Let us firstly recall the following:

Definition 1.7. A matrix is in *row echelon form* if

- all nonzero rows are above any rows of all zeroes (all zero rows, if any, lying on the bottom of the matrix);
- the first nonzero entry from the left of a nonzero row is always strictly on the right of the first nonzero entry of the previous row.

1.2.1. Hermite normal form

Hermite normal form is a particular case of row echelon form.

It is well known that Hermite algorithm provides an effective way to determine a basis of a subgroup of $\mathbb{Z}^n$. We briefly recall the definition and the main properties. For details, see for example [6].

Definition 1.8. An $m \times n$ matrix $M = (m_{ij})$ with integral coefficients is in *Hermite normal form* (abbreviated HNF) if there exists $r \leq m$ and a strictly increasing map $f : \{1, \dots, r\} \to \{1, \dots, n\}$ satisfying the following properties:

1. For $1 \leq i \leq r$, $m_{i,f(i)} \geq 1$, $m_{ij} = 0$ if $j < f(i)$ and $0 \leq m_{i,f(k)} < m_{k,f(k)}$ if $i < k$.
2. The last $m - r$ rows of M are equal to 0.

Theorem 1.9. *(See [6] Theorem 2.4.3.) Let A be an $m \times n$ matrix with coefficients in $\mathbb{Z}$. Then there exists a unique $m \times n$ matrix $B = (b_{ij})$ in* HNF *of the form $B = U \cdot A$ where $U \in \mathrm{GL}_m(\mathbb{Z})$.*

We will refer to matrix B as the HNF of matrix A. The construction of B and U is effective, see [6, Algorithm 2.4.4], based on Eulid's algorithm for greatest common divisor.

Proposition 1.10. *(See [6], § 2.4.3.)*

1. *Let $\mathcal{L}$ be a subgroup of $\mathbb{Z}^n$, $V = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ a set of generators, and let A be the $m \times n$ matrix having $\mathbf{v}_1, \dots, \mathbf{v}_m$ as rows. Let B be the* HNF *of A. Then the non-zero rows of B are a basis of $\mathcal{L}$.*
2. *Let A be a $m \times n$ matrix, and let $B = U \cdot A^T$ be the* HNF *of the transposed of A, and let r such that the first r rows of B are non-zero. Then a $\mathbb{Z}$-basis for the kernel of A is given by the last $m - r$ rows of U.*

Remark 1.11. The Hermite Normal Form can be defined also for $m \times n$ matrices with rational coefficients. Given such a matrix, one simply multiplies it by a common multiple D of denominators of the entries, compute the HNF and then divides by D.

1.2.2. Dual discrete subgroups in $\mathbb{Q}^n$

Definition 1.12. Given a discrete subgroup $\mathcal{L}$ of $\mathbb{Q}^n$, let $\mathcal{L}_{\mathbb{Q}}$ be the $\mathbb{Q}$-span of $\mathcal{L}$ in $\mathbb{Q}^n$. The *dual subgroup* of $\mathcal{L}$ is the discrete subgroup defined by

$$\mathcal{L}^* = \{\mathbf{x} \in \mathcal{L}_{\mathbb{Q}} \mid \mathbf{x} \cdot \mathbf{y} \in \mathbb{Z}, \forall \mathbf{y} \in \mathcal{L}\} \subseteq \mathbb{Q}^n$$

Definition 1.13. If A is a $m \times n$ matrix of rank m, we will call the matrix

$$A^* := (A \cdot A^T)^{-1} \cdot A$$

the *transverse* of A; it is also called the *contragredient matrix* by some authors. Notice that $A \cdot A^T$ is a non-singular square matrix if A is a $m \times n$ matrix of rank m; therefore A^* is well-defined.

In particular, when A is a non-singular square matrix, $A^* = (A^T)^{-1}$ (see [21, § 1.3]).

The following facts are well-known:

Proposition 1.14. *Let $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$ be discrete subgroup of $\mathbb{Q}^n$.*

1. *Let m be the rank of $\mathcal{L}$, and B be the $m \times n$ matrix having on the rows a basis of $\mathcal{L}$. Then a $\mathbb{Z}$-basis of $\mathcal{L}^*$ is given by the rows of the transverse matrix B^* of B.*

2. $\mathcal{L}^{**} = \mathcal{L}$.
3. $(\mathcal{L}_1 \cap \mathcal{L}_2)^*$ *is generated by* $\mathcal{L}_1^*$ *and* $\mathcal{L}_2^*$.
4. $\mathcal{L}_1 \cap \mathcal{L}_2$ *is the dual of the group generated by* $\mathcal{L}_1^*$ *and* $\mathcal{L}_2^*$.

Proof. (1) Notice first of all that the spaces spanned over $\mathbb{Q}$ by the rows of B^* and B are the same. We have $B^* \cdot B^T = \mathbf{I}_m$, so that the rows of B^* belong to $\mathcal{L}^*$. Conversely, let $\mathbf{x}$ be a (row) vector in $\mathcal{L}^*$; then $\mathbf{x} \cdot B^T = \mathbf{y} \in \mathbb{Z}^m$ and since $\mathbf{x} \in \mathcal{L}_{\mathbb{Q}}$ there exists $\mathbf{w} \in \mathbb{Q}^m$ such that $\mathbf{w} \cdot B = \mathbf{x}$. Then $\mathbf{y} = \mathbf{w} \cdot (B \cdot B^T)$, so that

$$\mathbf{x} = \mathbf{w} \cdot B = \mathbf{y} \cdot (B \cdot B^T)^{-1} \cdot B = \mathbf{y} \cdot B^* \in \mathcal{L}_r(B^*).$$

(2) Let B, B^* as above; then it is easily seen that $(B^* \cdot (B^*)^T)^{-1} \cdot B^* = B$, proving the claim.
(3) Is clear from the definition of duality.
(4) Follows from (2) and (3). □

1.2.3. Intersection of subgroups of $\mathbb{Z}^n$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subgroups of $\mathbb{Z}^n$. The previous Proposition 1.14 provides a constructive method to compute a basis of the intersection $\mathcal{L}_1 \cap \mathcal{L}_2$:

- compute matrices B_1 and B_2 having on the rows bases of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively;
- for $i = 1, 2$ compute B_i^*; define

$$M = \begin{pmatrix} B_1^* \\ B_2^* \end{pmatrix}$$

- compute $A' = \mathrm{HNF}(M)$ and extract from A' the submatrix A of all non-zero rows of A'.
- compute $B = A^*$; then the rows of B form a basis for $\mathcal{L}_1 \cap \mathcal{L}_2$.

Of course, the procedure can be adapted in order to calculate the intersection of more than two subgroups.

1.2.4. Smith Normal Form (SNF)

We recall the well-known Elementary Divisor Theorem, which will be often used in the sequel. For details see for example [6, § 2.4]

Theorem 1.15 *(Elementary Divisor Theorem). Let $\mathcal{L}$ be a subgroup of $\mathbb{Z}^m$. There exist a basis $\mathbf{f}_1, \ldots, \mathbf{f}_m$ of $\mathbb{Z}^m$, a positive integer $k \leq m$ (called the* rank *of $\mathcal{L}$) and integers $c_1, \ldots, c_k > 0$ such that c_i divides c_{i+1} for $i = 1, \ldots, k-1$, and $c_1\mathbf{f}_1, \ldots, c_k\mathbf{f}_k$ is a basis of $\mathcal{L}$.*

An effective version of Theorem 1.15 is given by the Smith Normal Form algorithm:

Definition 1.16. Let $A = (a_{ij})$ be a $d \times m$ matrix with integer entries. Then A is said in *Smith Normal Form* (abbreviated SNF) if there exist a positive integer $k \leq d$ and integers $c_1, \ldots, c_k > 0$ such that c_i divides c_{i+1} for $i = 1, \ldots, k-1$ such that $a_{ii} = c_i$ if $1 \leq i \leq k$ and $a_{ij} = 0$ if either $i \neq j$ or $i > k$.

Theorem 1.17. *(See [6], Algorithm 2.4.14.) Let $A = (a_{ij})$ be a $d \times m$ matrix with integer entries. Then it is possible to compute matrices $\alpha \in \mathrm{GL}_d(\mathbb{Z}), \beta \in \mathrm{GL}_m(\mathbb{Z})$ such that $\alpha \cdot A \cdot \beta$ is in* SNF.

Let $\mathcal{L}$ be a subgroup of $\mathbb{Z}^m$, and suppose to know generators $\mathbf{g}_1, \ldots, \mathbf{g}_d$ of $\mathcal{L}$. Let A be the matrix having $\mathbf{g}_1, \ldots, \mathbf{g}_d$ as rows, and let $\alpha \in \mathrm{GL}_d(\mathbb{Z}), \beta \in \mathrm{GL}_m(\mathbb{Z})$ such that $\alpha A \beta$ is in SNF, with diagonal coefficients $c_i, i = 1, \ldots, k$, as in Theorem 1.17. Then the rows of β^{-1} provide a basis $\mathbf{f}_1, \ldots, \mathbf{f}_m$ of $\mathbb{Z}_m$ such that $c_1\mathbf{f}_1, \ldots, c_k\mathbf{f}_k$ is a basis of $\mathcal{L}$, according to Theorem 1.15.

Let us fix some notation. If A is a $d \times m$ matrix with integer coefficients, $\mathcal{L}_r(A)$ ($\mathcal{L}_c(A)$ respectively) denotes the lattice spanned by the rows (resp. columns) of A.

A matrix, and in particular a vector, is said *positive* (resp. *strictly positive*) if each entry is ≥ 0 (resp. > 0).

We write $A_1 \sim A_2$ if A_1 and A_2 are in the same orbit by left multiplication by $\mathrm{GL}_d(\mathbb{Z})$.

2. Poly weighted spaces: motivations and main geometrical results

2.1. Divisors, fan matrices and weight matrices

Consider the normal toric variety $X = X(\Sigma)$. If $\mathcal{W}(X)$ denotes the group of Weil divisors of X then its subgroup of *torus-invariant Weil divisors* is given by

$$\mathcal{W}_T(X) = \langle D_\rho \mid \rho \in \Sigma(1) \rangle_{\mathbb{Z}} = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho$$

where $D_\rho = \overline{O}_\rho$ is the closure of the torus orbit of the ray ρ. Let $\mathcal{P}(X) \subset \mathcal{W}(X)$ be the subgroup of *principal divisors* and $\mathbf{v}_\rho$ be the minimal integral vector in ρ i.e. the generator of the monoid $\rho \cap N$. Then the morphism

$$\begin{array}{cccc} div: & M & \longrightarrow & \mathcal{P}(X) \cap \mathcal{W}_T(X) =: \mathcal{P}_T(X) \\ & \mathbf{u} & \longmapsto & div(\mathbf{u}) := \sum_{\rho \in \Sigma(1)} \langle \mathbf{u}, \mathbf{v}_\rho \rangle D_\rho \end{array} \tag{3}$$

is surjective. Let $\mathrm{Pic}(X)$ be the group of line bundles modulo isomorphism. It is well known that for an *irreducible* variety X the map $D \mapsto \mathcal{O}_X(D)$ induces an isomorphism $\mathcal{C}(X)/\mathcal{P}(X) \cong \mathrm{Pic}(X)$, where $\mathcal{C}(X) \subset \mathcal{W}(X)$ denotes the subgroup of Cartier divisors. The divisor class group is defined as the group of Weil divisors modulo rational (hence linear) equivalence, i.e. $\mathrm{Cl}(X) := \mathcal{W}(X)/\mathcal{P}(X)$. Then the inclusion $\mathcal{C}(X) \subset \mathcal{W}(X)$ passes through the quotient giving an immersion $\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X)$. One of main results on divisors of toric varieties is then the following

Theorem 2.1. *(See [15] § 3.4, [9] Proposition 4.2.5.) For a toric variety $X = X(\Sigma)$ the following sequence is exact*

$$M \xrightarrow{div} \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}\cdot D_\rho \xrightarrow{d} \mathrm{Cl}(X) \longrightarrow 0 \tag{4}$$

Moreover if $\Sigma(1)$ generates $N_{\mathbb{R}}$ then the morphism div is injective giving the following exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "div"] \arrow[d, "\|"] & \mathcal{C}_T(X) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "div"] \arrow[d] & \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}\cdot D_\rho \arrow[r, "d"] & \mathrm{Cl}(X) \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd} \tag{5}$$

where $\mathcal{C}_T(X) = \mathcal{C}(X) \cap \mathcal{W}_T(X)$. In particular $\mathrm{Pic}(X)$ and $\mathrm{Cl}(X)$ turn out to be completely described by means of torus invariant divisors and

$$\mathrm{rk}\,(\mathrm{Pic}\,(X)) \leq \mathrm{rk}\,(\mathrm{Cl}\,(X)) = |\Sigma(1)| - n\ .$$

Moreover if Σ contains a n-dimensional cone then the first sequence in (5) splits implying that $\mathrm{Pic}(X)$ is a free abelian group.

In the following $X = X(\Sigma)$ will be a $\mathbb{Q}$-factorial and complete n-dimensional toric variety, meaning that Σ is simplicial and its support $|\Sigma| = N_{\mathbb{R}}$, respectively. In particular completeness of X implies that it cannot admit torus factors, or equivalently that $\Sigma(1)$ generates the whole $N_{\mathbb{R}}$. Then diagram (5) in Theorem 2.1 holds and $|\Sigma(1)| = n + r$, where $r := \mathrm{rk}\,\mathrm{Pic}(X) \geq 1$ is the *Picard number of* X, simply called *the rank of* X. Recalling Definitions 1.1 and 1.3, let $V = (\mathbf{v}_1, \ldots, \mathbf{v}_{n+r})$ be the $n \times (n + r)$ *reduced fan matrix of* Σ i.e. $\mathbf{v}_i$ is the unique generator of the monoid $\langle \mathbf{v}_i \rangle \cap N$.

Observe that the transposed matrix V^T is a representative matrix of the $\mathbb{Z}$-linear morphism *div* defined in (3) and appearing in diagram (5).

Definition 2.2 *((Reduced) Weight matrix of a $\mathbb{Q}$-factorial complete toric variety).* Let X be a $\mathbb{Q}$-factorial complete n-dimensional toric variety of rank r and consider the exact sequence given by the second row of the diagram (5) in Theorem 2.1, which is

$$0 \longrightarrow M \xrightarrow{div} \mathcal{W}_T(X) = \mathbb{Z}^{n+r} \xrightarrow{d} \mathrm{Cl}(X) \longrightarrow 0 \ . \tag{6}$$

Dualizing this sequence, one gets the following exact sequence of free abelian groups

$$0 \longrightarrow \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{Z}) \xrightarrow{d^\vee} \mathrm{Hom}(\mathcal{W}_T(X), \mathbb{Z}) = \mathbb{Z}^{n+r} \xrightarrow{div^\vee} N \tag{7}$$

The transposed matrix Q of a $(n+r) \times r$ representative matrix Q^T of the $\mathbb{Z}$-linear morphism $d^\vee$ is called a *weight matrix of* X.

A weight matrix Q is called *reduced* if for every $i = 1, \ldots, n+r$, $\mathcal{L}_r(Q^i)$ has no cotorsion in $\mathbb{Z}^{n+r-1}$, where Q^i denotes the matrix obtained by removing from Q the i-th column and $\mathcal{L}_r(Q^i)$ the lattice spanned by the rows of Q^i, as defined at the end of §1.

Remark 2.3. Tensoring the short exact sequence (6) by the field $\mathbb{Q}$, the torsion subgroup $\mathrm{Tors}(\mathrm{Cl}(X))$ of the class group $\mathrm{Cl}(X)$ is killed and one can think of a weight matrix Q as a representative matrix of the $\mathbb{Q}$-linear morphism d. This is clearly no more true by the $\mathbb{Z}$-linear point of view, so giving the motivation for defining a weight matrix as the transposed of a representative matrix of the dual $\mathbb{Z}$-linear morphism $d^\vee$.

As observed [9, (9.5.8)], recalling the definition of Gale duality originally given by Oda and Park in [20], a (not necessarily reduced) weight matrix Q is a Gale dual matrix of the transposed V^T of a reduced fan matrix V. By abuse of notation we will say that V and Q are *Gale dual matrices*. In § 3 we will give a linear algebraic characterization of fan matrices, weight matrices and Gale duality, in terms of F-matrices, W-matrices and their reduction (see Definitions 3.10, 3.9, 3.13 and 3.14).

Let us finally notice that the reduction of a weight matrix as defined in the last part of Definition 2.2 is actually the generalization, on the rank r, of the standard definition of a reduced weight system of a weighted projective space, as observed in Remark 3.17. Nevertheless it is cumbersome and we think that the more natural definition of a reduced weight matrix is that given in Definition 3.14. Their equivalence is proved in the Reduction Theorem 3.15.

One of the main geometric results of this paper is the following:

Theorem 2.4. *Let $X = X(\Sigma)$ be a $\mathbb{Q}$-factorial complete toric variety. For any ray $\rho \in \Sigma(1)$ let $\mathbf{v}_\rho$ be the minimal integral vector of ρ and consider the $\mathbb{Z}$-module*

$$N_{\Sigma(1)} := \langle \mathbf{v}_\rho \mid \rho \in \Sigma(1) \rangle_{\mathbb{Z}} \tag{8}$$

as a subgroup of the lattice N. Let V be a fan matrix of Σ, and let T_n be the upper $n \times n$ part of the HNF *of V^T. Then*

$$N/N_{\Sigma(1)} \cong \mathrm{Tors}(\mathrm{Cl}(X)) \cong \mathbb{Z}^n/\mathcal{L}_r(T_n)\,, \tag{9}$$

where $\mathcal{L}_r(T_n)$ is the lattice spanned by the rows of T_n (as defined at the end of § 1).

A proof is given in §4.1.

Remark 2.5. The group (9) turns out to be isomorphic to the fundamental group in codimension 1 $\pi_1^1(X)$, as defined in [5].

2.2. Poly weighted spaces (PWS)

A first immediate consequence of previous Proposition 1.10 and Theorem 2.4 is that

Proposition 2.6. *Let $X = X(\Sigma)$ be a $\mathbb{Q}$-factorial complete n-dimensional toric variety of rank r. Let V be a fan matrix of Σ. Then the following are equivalent:*

1. *the group $N/N_{\Sigma(1)}$ is trivial,*
2. *the* HNF *of the transposed matrix V^T is given by $\begin{pmatrix} \mathbf{I}_n \\ \mathbf{0}_{r,n} \end{pmatrix}$,*
3. *the lattice $\mathcal{L}_c(V)$ spanned by the columns of V coincides with the whole $\mathbb{Z}^n$,*
4. *the fan matrix V has coprime $n \times n$ minors.*

A proof is given in 4.2.

Definition 2.7. A *poly weighted space* (PWS) is a $\mathbb{Q}$-factorial complete toric variety satisfying the equivalent conditions of Proposition 2.6.

Remarks 2.8. Let us list an amount of motivating reasons supporting the previous definition.

1. Let X be a n-dimensional PWS of rank 1. Then X is a weighted projective space (WPS) of weights given by the entries of the $1 \times (n+1)$ weight matrix $Q = (q_1, \dots, q_{n+1})$. In particular, the previous Remark 2.3 clarifies that $q_1, \dots, q_{n+1}$ give a reduced weight system of the WPS $X = \mathbb{P}(q_1, \dots, q_{n+1})$ if and only if Q is a reduced weight matrix.
2. Let $Q = (\mathbf{q}_1, \dots, \mathbf{q}_{n+r})$ be a $r \times (n+r)$ reduced weight matrix of a PWS $X = X(\Sigma)$. The Cox presentation of X as a geometric quotient [8] shows that

$$X \cong \left(\mathbb{C}^{n+r} \setminus Z_\Sigma\right) / \left(\mathbb{C}^*\right)^r$$

where the action of $(\mathbb{C}^*)^r$ is completely described by the weight matrix Q as follows

$$(\mathbf{t},\mathbf{z}) \in (\mathbb{C}^*)^r \times \mathbb{C}^{n+r} \longmapsto \left(\prod_{i=1}^r t_i^{q_{i,1}} z_1, \ldots, \prod_{i=1}^r t_i^{q_{i,n+r}} z_{n+r}\right) \in \mathbb{C}^{n+r} \ .$$

Notice that if $r = 1$ this gives the usual quotient $\left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^*$ by the weighted action $(t, \mathbf{z}) \mapsto (t^{q_1} z_1, \ldots, t^{q_{n+1}} z_{n+1})$ defining $\mathbb{P}(q_1, \ldots, q_{n+1})$.

3. Given a $d \times m$ matrix A, let us introduce the following notation: for every subset $I \subseteq \{1, \ldots, m\}$ let A_I (resp. A^I) be the submatrix of A given by the columns indexed by I (resp. indexed by the complementary subset $\{1, \ldots, m\} \backslash I$). Recalling [21, Thm. 3], a characterization of a fan matrix $V = (\mathbf{v}_1, \ldots, \mathbf{v}_{n+1})$ of $\mathbb{P}(q_1 \ldots, q_{n+1})$ is that:
   i. $\sum_{i=1}^{n+1} q_i \mathbf{v}_i = 0\,,$
   ii. $\forall\, i = 1, \ldots, n+1 \quad \left|\det\left(V^{\{i\}}\right)\right| = q_i\,.$
   The generalization of these conditions to the case $r > 1$ is given by
   I. $Q \cdot V^T = 0$, by the definition of Gale duality (see § 3.1),
   II. $\left|\det(V^I)\right| = |\det(Q_I)|$, for every $I \subset \{1, \ldots, n+r\}$ such that $|I| = r$ (by Corollary 3.3 below).
4. Recalling [3, Lemma 2.11] and [7, Prop. 4.7], a $\mathbb{Q}$-factorial complete toric variety X of rank 1 is a WPS if and only if its divisor class group $\mathrm{Cl}(X)$ is torsion free, hence $\mathrm{Cl}(X) \cong \mathbb{Z}$ and $\mathrm{Pic}(X)$ is a free subgroup of $\mathrm{Cl}(X)$.
   Analogously, by Proposition 2.6, a $\mathbb{Q}$-factorial complete toric variety X of rank r is a PWS if and only if its divisor class group $\mathrm{Cl}(X)$ is torsion free, hence $\mathrm{Cl}(X) \cong \mathbb{Z}^r$ and $\mathrm{Pic}(X)$ is a free subgroup, of maximum rank r, of $\mathrm{Cl}(X)$.

The last Remark 2.8.4 deserves further study. For a WPS $X = \mathbb{P}(q_1, \ldots, q_{n+1})$ with reduced weights, a generator L of $\mathrm{Cl}(X) \cong \mathbb{Z}$ can be found by taking any solution $(b_1, \ldots, b_{n+r})$ of the diophantine equation $\sum_{j=1}^{n+r} q_j x_j = 1$ and setting $L := \sum_{j=1}^{n+r} b_j D_j$, where D_j is the torus invariant divisor giving the closure of the torus orbit of the ray $\langle \mathbf{v}_j \rangle$. Then a generator of $\mathrm{Pic}(X)$ is given by the line bundle $\mathcal{O}_X(\delta L) \cong \mathcal{O}_X((\delta/q_j) D_j)$, for every $1 \leq j \leq n+1$, with $\delta := \mathrm{lcm}(q_1, \ldots, q_{n+1})$ [21, Prop. 8]. In particular the inclusion $\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X)$ is described by the multiplication by δ.

In the general case of a PWS of rank r these considerations admit the following generalization:

Theorem 2.9. *Let $X = X(\Sigma)$ be a n-dimensional PWS of rank r admitting a reduced weight matrix $Q = (\mathbf{q}_1, \ldots, \mathbf{q}_{n+r})$.*

1. *Consider the matrix*

$$U_Q = (u_{ij}) \in \mathrm{GL}_{n+r}(\mathbb{Z}) \ : \ U_Q \cdot Q^T \quad \textit{is a HNF matrix.}$$

Let us denote by ${}^rU_Q$ *the submatrix of* U_Q *given by the upper* r *rows of* U_Q*. Then the rows of* ${}^rU_Q$ *describe the following set of generators of* $\mathrm{Cl}(X)$

$$\forall\, 1 \leq i \leq r \quad L_i := \sum_{j=1}^{n+r} u_{ij} D_j \in \mathcal{W}_T(X) \quad \textit{and} \quad \mathrm{Cl}(X) = \bigoplus_{i=1}^{r} \mathbb{Z}[d(L_i)] \tag{10}$$

where $d : \mathcal{W}_T(X) \to \mathrm{Cl}(X)$ *is the morphism appearing in the exact sequence (6).*

2. *Let* V *be a fan matrix of* X*. Define*

$$\mathcal{I}_\Sigma = \{I \subset \{1, \dots, n+r\} : \langle V^I \rangle \in \Sigma(n)\}.$$

Through the identification of $\mathrm{Cl}(X)$ *with* $\mathbb{Z}^r$*, just fixed in (10),*

$$\mathrm{Pic}(X) = \bigcap_{I \in \mathcal{I}_\Sigma} \mathcal{L}_c(Q_I).$$

Therefore a basis of $\mathrm{Pic}(X) \subseteq \mathrm{Cl}(X) \simeq \mathbb{Z}^r$ *can be computed by applying the procedure described in § 1.2.3.*

3. *Let* $\mathbf{b}_1, \dots, \mathbf{b}_r$ *be a basis of* $\mathrm{Pic}(X)$ *in* $\mathbb{Z}^r \simeq \mathrm{Cl}(X)$*, and let* B *be the* $r \times r$ *matrix having* $\mathbf{b}_1, \dots, \mathbf{b}_r$ *on the rows. Then a basis of* $\mathcal{C}_T(X) \subseteq \mathbb{Z}^{n+r} \simeq \mathcal{W}_T(X)$ *is given by the rows of the matrix*

$$C = \begin{pmatrix} B & \mathbf{0}_{r,n} \\ \mathbf{0}_{n,r} & \mathbf{I}_n \end{pmatrix} \cdot U_Q = \begin{pmatrix} B \cdot {}^rU_Q \\ V \end{pmatrix}$$

where V *is the fan matrix of* X *given by the lower* n *rows of* U_Q *(see Proposition 4.3).*

4. *Setting* $\delta_\Sigma := \mathrm{lcm}\,(\det(Q_I) : I \in \mathcal{I}_\Sigma)$ *then*

$$\delta_\Sigma \mathcal{W}_T(X) \subseteq \mathcal{C}_T(X) \;\textit{and}$$
$$\delta_\Sigma \textit{ divides the index } [\mathrm{Cl}(X) : \mathrm{Pic}(X)] = [\mathcal{W}_T(X) : \mathcal{C}_T(X)]\,.$$

Remark 2.10. Let us underline that the HNF of a matrix and the associated switching matrix are obtained by a well known algorithm, based on Euclid's algorithm for greatest common divisor (see e.g. [6, Algorithm 2.4.4]). This algorithm is implemented in many computer algebra procedures. Due to the following Proposition 4.3 and part (1) of Theorem 2.9 the switching matrix $U_Q \in \mathrm{GL}_{n+r}(\mathbb{Z})$ turns out to encode both the generators of the divisor class group (given by the upper r rows) and the fan matrix (given by the lower n rows).

Parts (2) and (3) of Theorem 2.9 provide effective methods to compute generators of the subgroup $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X)$ and the subgroup $\mathcal{C}_T(X)$ of torus invariant Cartier divisor in $\mathcal{W}_T(X)$. Consequently, Theorem 2.9 allows us to determine the torsion group $\mathcal{T} := \mathcal{W}_T(X)/\mathcal{C}_T(X) \cong \mathrm{Cl}(X)/\mathrm{Pic}(X)$ of a PWS.

Moreover, Theorem 2.9 enables us to explicitly describe all the morphisms appearing in the commutative diagram (5), by giving all the representative matrices over suitable fixed bases. Namely, let us fix

- the basis of $\mathcal{W}_T(X)$ given by the torus invariant divisors $\{D_1, \ldots, D_{n+r}\}$,
- the basis of $\mathrm{Cl}(X)$ given by $\{d(L_1), \ldots, d(L_r)\}$, as in (10),

then the matrices Q, V, C, B defined in Theorem 2.9 completely describe diagram (5) as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\left(\mathbf{0}_{n,r}\,|\,\mathbf{I}_n\right)"] \arrow[d, "\|"] & \mathcal{C}_T(X) \cong \mathrm{Pic}(X) \oplus M \arrow[r, "\left(\mathbf{I}_r\,|\,\mathbf{0}_{r,n}\right)"] \arrow[d, "C^T"] & \mathrm{Pic}(X) \arrow[r] \arrow[d, "B^T"] & 0 \\
0 \arrow[r] & M \arrow[r, "div", "V^T"'] \arrow[d] & \mathcal{W}_T(X) = \bigoplus_{j=1}^{n+r} \mathbb{Z} \cdot D_j \arrow[r, "d", "Q"'] \arrow[d] & \mathrm{Cl}(X) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{T} \arrow[r, "="] \arrow[d] & \mathcal{T} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{11}$$

Example 2.11. Let us give an account of what observed until now by considering the following matrix in row echelon form: $Q = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$. One easily check that Q is reduced. The HNF of Q^T is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} \cdot Q^T = U_Q \cdot Q^T .$$

The matrix $V = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$ is a Gale dual of Q: it is a fan matrix satisfying the equivalent conditions of Proposition 2.6. As observed in Remark 1.6(3), $|\mathcal{SF}(V)| = 1$, which is that V determines a unique associated 2-dimensional PWS $X = X(\Sigma)$ of rank 2, whose fan Σ is described by all the faces of the following maximal cones:

$$\Sigma(2) = \left\{ \sigma_1 = \left\langle \begin{matrix} 1 & 1 \\ 0 & 2 \end{matrix} \right\rangle, \sigma_2 = \left\langle \begin{matrix} 1 & -1 \\ 2 & 0 \end{matrix} \right\rangle, \sigma_3 = \left\langle \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \right\rangle, \sigma_4 = \left\langle \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\rangle \right\}$$

Theorem 2.9 guarantees that the upper two rows of U_Q describe the generators of the divisor class group in terms of the torus invariant divisors $D_1, \ldots, D_4$, which is

$$\mathrm{Cl}(X) = \mathbb{Z}[d(L_1)] \oplus \mathbb{Z}[d(L_2)] \quad \text{with} \quad L_1 := D_1\ ,\ L_2 := -D_1 + D_2\,.$$

Notice that the column $\mathbf{q}_j$ of Q gives the components of the torus invariant D_j in terms of the generators L_i, namely:

$$D_1 = L_1\,,\ D_2 = L_1 + L_2\,,\ D_3 = L_2\,,\ D_4 = 2L_2\,. \tag{12}$$

Since $\delta_\Sigma = \mathrm{lcm}\,(\det(\sigma_i)\,|\,1 \le i \le 4) = 2$, the second part of Theorem 2.9 shows that $2L_1, 2L_2$ are certainly Cartier divisors. Actually one can say more, by solving easy 2×2 systems of linear equations and obtaining the *Cartier index* $c_\Sigma(D_j)$ of all the divisors D_j, which is *the least positive integer giving a Cartier multiple of D_j in $X(\Sigma)$*, namely:

$$c_\Sigma(D_1) = 2\,, c_\Sigma(D_2) = 2\,, c_\Sigma(D_3) = 2\,, c_\Sigma(D_2) = 1\,.$$

Now we want to explicitly exhibit the immersion $\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X)$ by using the method presented in § 1.2.3. With the notations of Theorem 2.9 we have

$$\mathcal{I}_\Sigma = \{I_1, I_2, I_3, I_4\},\ \text{with}$$
$$I_1 = [2,4], \quad I_2 = [1,4], \quad I_3 = [1,3], \quad I_4 = [2,3].$$

For $j = 1, \ldots 4$ we compute the transverse $Q_{I_j}^{T*} = Q_{I_j}^{-1}$:

$$Q_{I_1}^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad Q_{I_2}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad Q_{I_3}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_{I_4}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

We obtain

$$\mathrm{HNF}\left(\begin{pmatrix} Q_{I_1}^{-1} \\ Q_{I_2}^{-1} \\ Q_{I_3}^{-1} \\ Q_{I_4}^{-1} \end{pmatrix}\right) = \begin{pmatrix} A \\ \mathbf{0}_2 \end{pmatrix}, \text{ with } A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

so that a basis of $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X) \cong \mathbb{Z}^2$ is given by the rows of the matrix $A^* = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Therefore $\mathrm{Pic}(X) = 2\,\mathrm{Cl}(X)$. A basis of the subgroup of Cartier divisors $\mathcal{C}(X) \subseteq \mathcal{W}_T(X) \cong \mathbb{Z}^{n+r}$ is given by the rows of the matrix

$$C = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot U_Q = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}.$$

2.3. Further remarks and future purposes

In Remark 2.8(4) we already mentioned results [3, Lemma 2.11] and [7, Prop. 4.7] showing that a $\mathbb{Q}$-factorial complete toric variety of rank 1 is always a quotient of WPS. Actually this fact can be generalized to every rank r, by replacing the role of a WPS with that of a PWS i.e.

- *every $\mathbb{Q}$-factorial complete toric variety is always a quotient of a PWS.*

Clearly when $r = 1$ this fact just gives the already mentioned results of Batyrev–Cox and Conrads, since a PWS of rank 1 is a WPS. The proof of the above stated result will be one of the main purposes of our next paper [22] to which the interested reader is referred. Probably this is the main motivation for calling a $\mathbb{Q}$-factorial complete toric variety satisfying one of the equivalent conditions of Proposition 2.6, a *poly weighted space*.

A final observation is that

- *on the contrary of a WPS, a general PWS is no more the Proj of a graded algebra.*

In fact, a PWS can be even a *non-projective* variety, as the following Example 2.12 will show. Anyway, in the forthcoming paper [23] we will prove that, under suitable conditions on the weight matrix, a projective PWS will be birational equivalent and isomorphic in codimension 1 to a toric cover of the Proj of a suitable weighted graded algebra, by means of a finite chain of *wall crossings* in the secondary fan [9, § 15.3].

Example 2.12. Consider the following reduced weight matrix Q and its Gale dual V

$$Q = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

V satisfies conditions of Proposition 2.6. Then $X = X(\Sigma)$ is a PWS for every $\Sigma \in \mathcal{SF}(V)$. Then the reader can check that:

- $|\mathcal{SF}(V)| = 8$,
- only 6 among those 8 distinct fans give a projective PWS: in fact the *Moving Cone* $\mathrm{Mov}(X)$ in the secondary fan [9, (15.1.5)] admits only 6 chambers,
- in particular the fan Σ defined by taking the following maximal cones

$$\begin{aligned} \Sigma(3) := \{ &\langle \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5 \rangle, \langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5 \rangle, \langle \mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5 \rangle, \langle \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5 \rangle, \\ &\langle \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6 \rangle, \langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_6 \rangle, \langle \mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_6 \rangle, \langle \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6 \rangle \} \end{aligned}$$

and all their faces, defines a non-projective PWS, since it does not correspond to any chamber in $\mathrm{Mov}(X)$.

3. Linear algebra and Gale duality

3.1. $\mathbb{Z}$-linear Gale duality

Let V be a $n \times (n+r)$ integer matrix of rank n. If we think V as a linear application from $\mathbb{Z}^{n+r}$ to $\mathbb{Z}^n$ then $\ker(V)$ is a lattice in $\mathbb{Z}^{n+r}$, of rank r and without cotorsion. We shall denote $\mathcal{G}(V)$ the *Gale dual matrix of A*, which is an integral $r \times (n+r)$ matrix Q such that $\mathcal{L}_r(Q) = \ker(V)$; it is well-defined up to left multiplication by $\mathrm{GL}_r(\mathbb{Z})$. Notice that $\mathcal{L}_r(Q) = \mathcal{L}_r(V)^{\perp}$ w.r.t. the standard inner product in $\mathbb{R}^{n+r}$, and that $Q \cdot V^T = 0$; moreover Q can be characterized by the following universal property:

if $A \in \mathbf{M}_r(\mathbb{Z})$ is such that $A \cdot V^T = 0$ then $A = \alpha \cdot Q$ for some matrix $\alpha \in \mathbf{M}_r(\mathbb{Z})$.

Proposition 3.1. *With the notation just introduced:*

1. $\mathcal{L}_r(\mathcal{G}(V))$ *does not have cotorsion in* $\mathbb{Z}^{n+r}$*,*
2. $(\mathbb{Z}^{n+r}/\mathcal{L}_r(V))_{\mathrm{tors}} \cong (\mathbb{Z}^n/\mathcal{L}_c(V))_{\mathrm{tors}}$*; in particular* $\mathcal{L}_c(\mathcal{G}(V))$ *does not have cotorsion in* $\mathbb{Z}^r$*,*
3. *there exists a matrix* $\alpha \in \mathbf{M}_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$ *such that* $V = \alpha \cdot \mathcal{G}(\mathcal{G}(V))$*,*
4. $\mathcal{G}(\mathcal{G}(V)) \sim V$ *if and only if* $\mathcal{L}_r(V)$ *does not have cotorsion in* $\mathbb{Z}^{n+r}$*.*

Proof. (1) follows from the fact that $\mathcal{L}_r(\mathcal{G}(V)) = \ker(V)$ is a kernel; (2): by Theorem 1.17 there exist invertible matrices $\alpha \in \mathrm{GL}_n(\mathbb{Z})$, $\beta \in \mathrm{GL}_{n+r}(\mathbb{Z})$ such that $S = \alpha V \beta$ is in *SNF*; if $c_1, \ldots, c_k$ are the non-zero integers on the diagonal of S then $(\mathbb{Z}^{n+r}/\mathcal{L}_r(V))_{\mathrm{tors}} \cong \mathbb{Z}/c_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/c_k\mathbb{Z}$; on the other hand the *SNF* of V^T is S^T; so that the cotorsion of $\mathcal{L}_c(V)$ and that of $\mathcal{L}_c(V^T)$ are the same. (3) follows from the fact that $\mathcal{L}_r(V)$ is a subgroup of $\mathcal{L}_r(\mathcal{G}(\mathcal{G}(V)))$ of finite index. (4) follows from (1) and (3). $\square$

The following results probably give the core of $\mathbb{Z}$-linear Gale duality:

Theorem 3.2. *Let* $V = (\mathbf{v}_1, \ldots, \mathbf{v}_{n+r})$ *be a* $n \times (n+r)$ *matrix of rank* n*, and* $Q = \mathcal{G}(V) = (\mathbf{w}_1, \ldots, \mathbf{w}_{n+r})$*. Setting* $I := \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n+r\}$ *let* V_I *be the submatrix of* V *given by the columns indexed by* I *and* Q^I *be the submatrix of* Q *given by the columns indexed by* $\{1, \ldots, n+r\} \backslash I$*, i.e.*

$$V_I := (\mathbf{v}_i \mid i \in I)\,, \quad Q^I := (\mathbf{w}_j \mid j \in \{1, \ldots, n+r\} \backslash I)\ . \tag{13}$$

Then there is a natural isomorphism

$$\mathbb{Z}^{n+r-k}/\mathcal{L}_r(Q^I) \cong \mathcal{L}_c(V)/\mathcal{L}_c(V_I)\ .$$

Proof. We may assume $i_j = j$ for $j = 1, \ldots, k$. Notice that $\mathcal{L}_r(Q^I) = \pi(\mathcal{L}_r(Q))$ where π is the projection of $\mathbb{Z}^{n+r}$ on the $k+1, \ldots, n+r$ components. Consider the map $F : \mathbb{Z}^{n+r-k} \to \mathcal{L}_c(V)$ defined by $F(b_{k+1}, \ldots, b_{n+r}) = b_{k+1}\mathbf{v}_{k+1} + \ldots + b_{n+r}\mathbf{v}_{n+r}$. Since $\mathcal{L}_c(V)/\mathcal{L}_c(V_I)$ is generated by the images of $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{n+r}$, F induces a surjective map $\mathbb{Z}^{n+r-k} \to \mathcal{L}_c(V)/\mathcal{L}_c(V_I)$. Moreover we have $F(b_{k+1}, \ldots, b_{n+r}) \in \mathcal{L}_c(V_I)$ if and only if there is a vector of the form $\mathbf{b} = (b_1, \ldots, b_k, b_{k+1}, \ldots, b_n + r) \in \mathcal{L}_r(Q)$ and this happens if and only if $(b_{k+1}, \ldots, b_{n+r}) \in \mathcal{L}_r(Q^I)$; thus F induces the required isomorphism. □

Corollary 3.3. *Let V be a $n \times (n+r)$ matrix of maximal rank n and $Q = \mathcal{G}(V)$. If $|I| = n$ then V_I and Q^I are square matrices of order n and r, respectively. Then*

$$[\mathbb{Z}^n \ : \ \mathcal{L}_c(V)] \ \left|\det(Q^I)\right| = |\det(V_I)| \ .$$

In particular if $\mathcal{L}_c(V)$ does not have cotorsion in $\mathbb{Z}^n$ then $\left|\det(Q^I)\right| = |\det(V_I)|$.

Proof. We have

$$\begin{aligned}[\mathbb{Z}^n : \mathcal{L}_c(V_I)] &= [\mathbb{Z}^n : \mathcal{L}_c(V)] \cdot [\mathcal{L}_c(V) : \mathcal{L}_c(V_I)] \\ &= [\mathbb{Z}^n : \mathcal{L}_c(V)] \cdot [\mathbb{Z}^r : \mathcal{L}_r(Q^I)],\end{aligned}$$

by Theorem 3.2. Since V has maximal rank, $[\mathbb{Z}^n : \mathcal{L}_c(V)] < \infty$; then we see that $\det(Q^I) = 0$ if and only if $\det(V_I) = 0$; if both are non-zero then the theorem follows from the fact that $[\mathbb{Z}^r : \mathcal{L}_r(Q^I)] = \left|\det(Q^I)\right|$ and $[\mathbb{Z}^n : \mathcal{L}_c(V_I)] = |\det(V_I)|$. □

3.2. F-matrices and W-matrices

In this section we investigate some characterizing properties of fan matrices and their Gale duals.

Let d, m, n, r be positive integers.

Definition 3.4. Let A be a $d \times m$ matrix with integer entries.

We say that A is *W-positive* if $\mathcal{L}_r(A)$ has a basis consisting of positive vectors.

We say that A is *F-complete* if the cone generated by its columns is $\mathbb{R}^d$.

Notice that W-positiveness and F-completeness are both invariant by the left action of $\mathrm{GL}_d(\mathbb{Z})$.

Easy criteria for F-completeness and W-positiveness are given by the following propositions:

Proposition 3.5. *(See [11, Theorem 3.6, ii)].) Let A be a $d \times m$ matrix with integer entries. Then A is F-complete if and only if $\mathrm{rk}(A) = d$ and for each vector column $\mathbf{v}$ of A, $-\mathbf{v}$ is in the cone generated by the columns of A different from $\mathbf{v}$.*

Proposition 3.6. *Let A be a $d \times m$ matrix with integer entries such that each column of A is non-zero. Then A is W-positive if and only if there exists a strictly positive vector in $\mathcal{L}_r(A)$.*

Proof. If A is W-positive then $\mathcal{L}_r(A)$ is generated over $\mathbb{Z}$ by a finite set of positive vectors; by summing up all of them and using the fact that each column of A is non-zero we get a strictly positive vector. Conversely, let $\mathbf{a}$ be a strictly positive vector in $\mathcal{L}_r(A)$. By the elementary divisor theorem there exists a $\mathbb{Z}$-basis $(\mathbf{a_1}, \mathbf{a}_2, \dots, \mathbf{a}_r)$ of $\mathcal{L}_r(A)$ such that $\mathbf{a}$ is a positive multiple of $\mathbf{a}_1$. Take $k \in \mathbb{N}$ such that $\mathbf{a'}_i = \mathbf{a}_i + k\mathbf{a}_1$ is positive for $i = 2, \dots, r$. Then $(\mathbf{a}_1, \mathbf{a'}_2, \dots, \mathbf{a'}_r)$ is a positive basis of $\mathcal{L}_r(A)$. □

Corollary 3.7. *Let A be a $n \times (n+r)$ matrix with integer entries of rank n. Then*

i) *A is F-complete if and only if $\mathcal{G}(\mathcal{G}(A))$ is F-complete;*
ii) *A is W-positive if and only if $\mathcal{G}(\mathcal{G}(A))$ is W-positive.*

Proof. It follows directly from Propositions 3.5 and 3.6 by recalling that $\mathcal{L}_r(A)$ is a sublattice of $\mathcal{L}_r(\mathcal{G}(\mathcal{G}(A)))$ of finite index. □

The following proposition establishes a duality between the notions of F-completeness and W-positiveness.

Theorem 3.8. *Let V be a $n \times (n+r)$ matrix of maximal rank n such that each column of V is non-zero. Let $Q = \mathcal{G}(V)$, so that Q is an $r \times (n+r)$ integral matrix. Then V is F-complete (W-positive) if and only if Q is W-positive (F-complete).*

Proof. Let $\mathbf{v}_1, \dots, \mathbf{v}_{n+r} \in \mathbb{Z}^n$ be the columns of V; then

$$\mathcal{L}_r(Q) = \{(b_1, \dots, b_{n+r}) \in \mathbb{Z}^{n+r} \mid b_1\mathbf{v}_1 + \dots + b_{n+r}\mathbf{v}_{n+r} = 0\}.$$

Suppose that V is F-complete: then, by Proposition 3.5, for $i = 1, \dots, n+r$ there is a positive vector $\mathbf{u}_i \in \mathcal{L}_r(Q)$ such that its i-th entry is not zero. By summing up over all i we get a strictly positive vector $\mathbf{b} \in \mathcal{L}_r(Q)$, so that Q is W-positive by Proposition 3.6.

Suppose now that V is W-positive. By multiplying by a suitable matrix in $\mathrm{GL}_r(\mathbb{Z})$ we can suppose that all the coefficients of V are in $\mathbb{N}$. Let $\mathbf{w}_1, \dots, \mathbf{w}_{n+r} \in \mathbb{Z}^r$ be the columns of Q. Since each column of V is non-zero, for $i = 1, \dots, n+r$ there is a row in V having a non-zero coefficient at place i; moreover $\mathcal{L}_r(V) \subseteq \mathcal{L}_r(\mathcal{G}(Q))$. This means that $-\mathbf{w}_i$ is in the cone generated by the vectors $\mathbf{w}_j, j \neq i$ and therefore Q is F-complete.

The converse assertions then follow from Corollary 3.7. □

Definition 3.9. A *W-matrix* is an $r \times (n+r)$ matrix Q with integer entries, satisfying the following conditions:

a) $\mathrm{rk}(Q) = r$.
b) $\mathcal{L}_r(Q)$ does not have cotorsion in $\mathbb{Z}^{n+r}$.
c) Q is W-positive.
d) Every column of Q is non-zero.
e) $\mathcal{L}_r(Q)$ does not contain vectors of the form $(0, \dots, 0, 1, 0, \dots, 0)$.
f) $\mathcal{L}_r(Q)$ does not contain vectors of the form $(0, a, 0, \dots, 0, b, 0, \dots, 0)$, with $ab < 0$.

Conditions equivalent to the holdness of both conditions (a) and (b), which are useful in applications, are

- $\mathrm{HNF}(Q^T) = \begin{pmatrix} \mathbf{I}_r \\ \mathbf{0}_{n,r} \end{pmatrix}$, where $\mathbf{I}_r$ is the identity matrix in $\mathbf{M}_r(\mathbb{R})$ and $\mathbf{0}_{n,r}$ is the $n \times r$ zero-matrix.
- $Q \sim \mathcal{G}(\mathcal{G}(Q))$.

Definition 3.10. An *F-matrix* is a $n \times (n + r)$ matrix V with integer entries, satisfying the conditions:

a) $\mathrm{rk}(V) = n$;
b) V is F-complete;
c) all the columns of V are non-zero;
d) if $\mathbf{v}$ is a column of V, then V does not contain another column of the form $\lambda\mathbf{v}$ where $\lambda > 0$ is real number.

A *CF-matrix* is a F-matrix satisfying the further requirement

e) $\mathcal{L}_c(V) = \mathbb{Z}^n$.

Proposition 3.11. *V is an F-matrix if and only if $\mathcal{G}(\mathcal{G}(V))$ is a CF-matrix.*

Proof. All verifications are immediate, applying Proposition 3.1(3) and Theorem 3.8. □

Proposition 3.12. *Let A be an $n \times (n + r)$ integer matrix of rank n. Then*

1. *If A is a W-matrix then $\mathcal{G}(A)$ is a CF-matrix.*
2. *A is an F-matrix if and only if $\mathcal{G}(A)$ is a W-matrix.*

Proof. 1) suppose that A is a W-matrix; then $\mathcal{G}(A)$ is a $r \times (n + r)$ matrix of rank r: moreover it is F-complete by Theorem 3.8. If $\mathcal{G}(A)$ had a zero column, then $\mathcal{L}_r(A)$ should contain a vector of the form $(0, \dots 0, 1, 0, \dots, 0)$ and this would contradict condition (e) of the definition of W-matrix. If $\mathcal{G}(A)$ had two proportional columns, then A should violate condition (f). Thus $\mathcal{G}(A)$ is an F-matrix. Moreover $\mathcal{L}_r(\mathcal{G}(A))$ does not have cotorsion in $\mathbb{Z}^{n+r}$ (because so happens for every matrix in the image of $\mathcal{G}$), therefore it is a CF-matrix.

2) suppose that A is a F-matrix; then $\mathcal{G}(A)$ is a $r \times (n+r)$ matrix of rank r: it is W-positive by Theorem 3.8; it cannot contain a zero column, because A is F-complete; since A is in the image of $\mathcal{G}$, $\mathcal{L}_r(\mathcal{G}(A))$ does not have cotorsion in $\mathbb{Z}^{n+r}$. Moreover $\mathcal{L}_r(\mathcal{G}(A))$ must satisfy conditions e) and f) of Definition 3.9 because A satisfies conditions c) and d) of Definition 3.10. Then $\mathcal{G}(A)$ is a W-matrix. For the converse apply part 1) of this proposition and Proposition 3.11. □

3.3. W-positive and positive matrices

Let Q be a $r \times (n+r)$ W-matrix. We exhibit an effective procedure allowing to calculate a positive matrix $Q' \sim Q$. Put $V = \mathcal{G}(Q)$; it is F-complete by Theorem 3.8; therefore for $i = 1, \dots, n+r$ there is a relation $\sum_{j=1}^{n+r} c_{ij}\mathbf{v}_j = \mathbf{0}$ such that the c_{ij} are non negative integers and $c_{ii} > 0$. Such relations are computable by finding the components of $-\mathbf{v}_i$ w.r.t. each maximal system of linearly independent columns of V different from $\mathbf{v}_i$, and looking for non negative solutions. By summing up over all i's and dividing by the gcd of the resulting coefficients if necessary, we get a relation $\sum_{i=1}^{n+r} c_i\mathbf{v}_i$ such that $c_i > 0$ for every i and $\gcd(c_1, \dots, c_{n+r}) = 1$. Since $\mathcal{L}_r(Q)$ has no cotorsion, the vector $\mathbf{c} = (c_1, \dots, c_{n+r}) \in \mathcal{L}_r(Q)$. Let $\mathbf{r}_1, \dots, \mathbf{r}_r$ be the rows of Q. Then we can find coprime $\lambda_1, \dots, \lambda_r \in \mathbb{Z}$ such that $\sum_i \lambda_i\mathbf{r}_i = \mathbf{c}$. The HNF algorithm gives a matrix $\alpha \in \mathrm{GL}_r(\mathbb{Z})$ such that $\alpha \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then

$$\mathbf{c} = (\lambda_1, \dots, \lambda_r) \cdot Q = (1, 0, \dots, 0) \cdot (\alpha^{-1})^T \cdot Q,$$

so that $Q'' = (\alpha^{-1})^T \cdot Q$ is a matrix having $\mathbf{c}$ at the first row and such that $Q'' \sim Q$. By adding a suitable multiple of $\mathbf{c}$ to the other rows of Q'' we get a positive matrix $Q' \sim Q$.

3.4. Reduced F- and W-matrices

Definition 3.13. Let V be a F-matrix, and let d_i be the gcd of the elements on the i-th column, for $i = 1, \dots, n+r$.

- We say that V is *F-reduced* if $d_i = 1$ for $i = 1, .., n+r$.
- The F-matrix $V^{F\text{-}red} = V \cdot \mathrm{diag}(d_1, \dots, d_{n+r})^{-1}$ is called the *F-reduction* of V.

It is clear that the cones spanned by columns of V and V^{red} coincide.

Definition 3.14.

- A W-matrix Q is said to be W*-reduced* if $\mathcal{G}(Q)$ is F-reduced.
- The W-matrix $Q^{W\text{-}red} = \mathcal{G}(\mathcal{G}(Q)^{F\text{-}red})$ is called the W*-reduction* of Q.

We shall omit the superscripts F, W in reductions, when they will be clear from the context.

The next results provides an intrinsic criterion to verify if a W-matrix is reduced, and, if it is not the case, to obtain its reduction.

Theorem 3.15 *(Reduction theorem). Let Q be a $r \times (n+r)$ W-matrix and $V = \mathcal{G}(Q)$. For $i = 1, \dots, n+r$ denote by Q^i the matrix obtained by removing from Q the i-th column, by d_i the* gcd *of the elements on the i-th column of V, and by $V_{(i)}$ the matrix obtained from V by dividing by d_i the i-th column. Then*

1. *Q is reduced if and only if for every $i = 1, \dots, n+r$, $\mathcal{L}_r(Q^i)$ has no cotorsion in $\mathbb{Z}^{n+r-1}$.*
2. *$\mathbb{Z}^{n+r-1}/\mathcal{L}_r(Q^i)$ is a cyclic group of order d_i.*
3. *Let $\tilde{Q}^i$ be the* SNF *of Q^i, and $\alpha_i \in \mathrm{GL}_r(\mathbb{Z})$, $\beta_i \in \mathrm{GL}_{n+r-1}(\mathbb{Z})$ be such that $\alpha_i Q^i \beta_i = \tilde{Q}^i$. Let $Q_{(i)}$ be the matrix obtained from $\alpha_i Q$ by multiplying the i-th column by d_i and dividing the r-th row by d_i. Then $Q_{(i)} = \mathcal{G}(V_{(i)})$.*
4. *Call the matrix $Q_{(i)}$ obtained in (3) the i-reduction of Q. Then $Q^{W\text{-}red}$ can be obtained by iterating i-reductions, for $i = 1, \dots, n+r$.*

Proof. Of course (1) follows from (2) and (4) follows from (3). In order to prove (2), apply Theorem 3.2 with $I = \{i\}$; then we see that there is an isomorphism

$$\mathbb{Z}^{n+r-1}/\mathcal{L}_r(Q^i) \cong \mathcal{L}_c(V)/\mathcal{L}_c(\mathbf{v}_i) \cong \mathbb{Z}^n/\mathcal{L}_c(\mathbf{v}_i) \cong \mathbb{Z}/d_i\mathbb{Z} \oplus \mathbb{Z}^{n-1}.$$

Now we prove (3); notice that, by (2), the last row of $\alpha_i Q^i$ is divisible by d_i, so that $Q_{(i)}$ has integer entries. Set $\gamma_i = \mathrm{diag}(1, \dots, 1, d_i) \in \mathbf{M}_r(\mathbb{Z})$ and $\delta_i = \mathrm{diag}(1, \dots, 1, d_i, 1, \dots, 1) \in \mathbf{M}_{n+r}(\mathbb{Z})$, with d_i on the (i, i)-place. Then $Q_{(i)} = \gamma_i^{-1}\alpha_i Q \delta_i$ and $V_{(i)} = V\delta_i^{-1}$, so that $Q_{(i)} \cdot {V_{(i)}}^T = 0$. Moreover

$$\mathcal{L}_c(Q_{(i)}) \supseteq \mathcal{L}_c(\gamma_i^{-1}\alpha_i Q^i) = \mathbb{Z}^n$$

since the SNF of $\gamma_i^{-1}\alpha_i Q^i$ is the identity matrix. Therefore $\mathcal{L}_r(Q_{(i)})$ has no cotorsion in $\mathbb{Z}^{n+r}$, so that $Q_{(i)} = \mathcal{G}(V_{(i)})$ by Proposition 3.1 (2). □

An alternative reduction procedure has been described in [1] to which the interested reader is referred.

Example 3.16. Let Q be he 2×4 W-matrix

$$Q := \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 5 \end{pmatrix}$$

Then the *HNF* of $V = \mathcal{G}(Q)$ is

$$V := \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 5 & -3 \end{pmatrix}$$

With the notations of Theorem 3.15 we have $d_1 = 2$ and

$$Q^1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 5 \end{pmatrix}, \quad \tilde{Q}^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 & -1 & 0 \\ -3 & -6 & -5 \\ 2 & 4 & 3 \end{pmatrix}$$

so that the 1-reduction is given by

$$Q_{(1)} = \begin{pmatrix} 2 & 2 & 3 & 5 \\ -1 & -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{G}(Q_{(1)}) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 5 & -3 \end{pmatrix} = V_{(1)}.$$

By performing successively the 3- and 4-reductions we obtain

$$Q_{(1,3)} = \begin{pmatrix} 2 & 2 & 15 & 5 \\ -1 & -1 & -6 & -2 \end{pmatrix}, \quad \mathcal{G}(Q_{(1,3)}) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix} = V_{(1,3)};$$

$$Q^{W\text{-}red} = Q_{(1,3,4)} = \begin{pmatrix} 2 & 2 & 15 & 15 \\ -1 & -1 & -7 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix};$$

$$\mathcal{G}(Q_{(1,3,4)}) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = V_{(1,3,4)} = V^{F\text{-}red}.$$

Remark 3.17. In the case $r = 1$, Theorem 3.15 gives the well known criterion for reducing weights in weighted projective spaces, see [21, Definition 3] and references therein. In fact the matrices α_i are trivial in this case and the reduction process described in part (4) of Theorem 3.15 amounts to multiply on the right the weight matrix $Q = (q_0, \ldots, q_n)$ by the diagonal matrix $\frac{1}{\prod d_j} \operatorname{diag}(d_0, \ldots, d_n)$.

By [21, Lemma 1(c)] $d_i = \gcd(q_0, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n)$.

By [21, Prop. 3(2)] $\frac{\prod d_j}{d_i} = \operatorname{lcm}(d_0, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n) =: a_i$.

3.4.1. W-positiveness and row echelon form

Theorem 3.18. *Let $A = (a_{i,j})$ be a $d \times m$ W-positive matrix. Then there exist $\alpha \in \mathrm{GL}_d(\mathbb{Z})$, a permutation matrix $\beta \in \mathrm{GL}_m(\mathbb{Z})$ and a positive matrix $\underline{A}$ in row echelon form such that $\alpha A \beta = \underline{A}$.*

Proof. Up to the left multiplication by a matrix in $\mathrm{GL}_d(\mathbb{Z})$ we can think of A to be a positive matrix.

As a first step we show that:

(i) if A is a $d \times m$ positive matrix, with $d \geq 2$, then there exist $\alpha' \in \mathrm{GL}_d(\mathbb{Z})$ and a permutation matrix $\beta' \in \mathrm{GL}_m(\mathbb{Z})$ such that $\alpha' A \beta'$ is positive with a zero entry in the $(d, 1)$-place.

If the d-th row of A has some zero entry then we are done up to a permutation on columns. We can then assume that A has a strictly positive d-th row. Let $\beta' \in \mathrm{GL}_m(\mathbb{Z})$ be a permutation matrix such that $A\beta' = (a_{i,j})$ satisfies the condition

$$\frac{a_{d-1,j}}{a_{d,j}} \geq \frac{a_{d-1,j+1}}{a_{d,j+1}} \ . \tag{14}$$

Consider the last two rows of A, giving the submatrix

$$\widetilde{A} := \begin{pmatrix} a_{d-1,1} & a_{d-1,2} & \dots & a_{d-1,m} \\ a_{d,1} & a_{d,2} & \dots & a_{d,m} \end{pmatrix}$$

Set $D = \gcd(a_{d-1,1}, a_{d,1})$. The Bézout identity gives a matrix

$$\widetilde{\alpha} := \begin{pmatrix} x & y \\ -\frac{a_{d,1}}{D} & \frac{a_{d-1,1}}{D} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

such that

$$\widetilde{\alpha} \cdot \widetilde{A} = \begin{pmatrix} D & a'_{d-1,2} & \dots & a'_{d-1,m} \\ 0 & a'_{d,2} & \dots & a'_{d,m} \end{pmatrix}$$

where, for $2 \leq j \leq m$,

$$\begin{aligned} a'_{d-1,j} &= x a_{d-1,j} + y a_{d,j} \\ a'_{d,j} &= -\frac{a_{d,1} a_{d-1,j}}{D} + \frac{a_{d-1,1} a_{d,j}}{D} \end{aligned}$$

Notice that (14) gives $a'_{d,j} = \frac{a_{d,j} a_{d,1}}{D} \left(\frac{a_{d-1,1}}{a_{d,1}} - \frac{a_{d-1,j}}{a_{d,j}} \right) \geq 0$ and $a'_{d,j} = 0$ if and only if $\frac{a_{d-1,j}}{a_{d,j}} = \frac{a_{d-1,1}}{a_{d,1}}$, since also $A\beta'$ has a strictly positive d-th row; in this case

$$a'_{d-1,j} = a_{d,j} \left(x \frac{a_{d-1,j}}{a_{d,j}} + y \right) = a_{d,j} \left(x \frac{a_{d-1,1}}{a_{d,1}} + y \right) = \frac{a_{d,j}}{a_{d,1}} D > 0 \ .$$

Therefore $a'_{d-1,j} > 0$ when $a'_{d,j} = 0$. This means that, up to add to the $(d-1)$-st row a suitable multiple of the d-th row, which is up to the left multiplication of a suitable matrix

$\widetilde{\alpha}' \in \mathrm{GL}_d(\mathbb{Z})$, we can conclude the existence of a matrix $\alpha' = \widetilde{\alpha}' \cdot \begin{pmatrix} \mathbf{I}_{d-2} & \mathbf{0} \\ \mathbf{0} & \widetilde{\alpha} \end{pmatrix} \in \mathrm{GL}_d(\mathbb{Z})$ such that $\alpha' A \beta'$ is a positive matrix having 0 in the $(d,1)$-place, proving (i).

Now we prove

(ii) there exists $\alpha \in \mathrm{GL}_d(\mathbb{Z})$ and a permutation matrix $\beta \in \mathrm{GL}_m(\mathbb{Z})$ such that $\alpha A \beta$ is positive and the first column is zero except possibly for its first entry.

We prove (ii) by induction on d. The case $d = 1$ is obvious.

If $d > 1$ we can apply (i) in order to obtain the existence of a matrix $\alpha' \in \mathrm{GL}_d(\mathbb{Z})$, and a permutation matrix $\beta' \in \mathrm{GL}_m(\mathbb{Z})$ such that $A' = \alpha' \cdot A \cdot \beta'$ is a positive matrix having 0 at place $(d,1)$. Due to relations (14) and the structure of the matrix α', the entries $a'_{d,j}$ of the last row of A' satisfy the following condition

$$a'_{d,j} \quad \text{are} \quad \begin{matrix} = 0 & \text{for } 1 \leq j \leq j_0 \\ > 0 & \text{for } j > j_0 \end{matrix} \qquad \text{for some } j_0 \geq 1.$$

Consider now the $(d-1) \times j_0$ submatrix $\widehat{A}$ of A' consisting of the first $d-1$ rows of A' truncated at j_0. By induction there exist $\widehat{\alpha} \in \mathrm{GL}_{d-1}(\mathbb{Z})$ and a permutation matrix $\widehat{\beta} \in \mathrm{GL}_{j_0}(\mathbb{Z})$ such that $\widehat{\alpha} \cdot \widehat{A} \cdot \widehat{\beta}$ is positive and the first column is zero except possibly for its first entry. By putting $\alpha'' = \begin{pmatrix} \widehat{\alpha} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \beta'' = \begin{pmatrix} \widehat{\beta} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m-j_0} \end{pmatrix}$ we obtain that the first column of $A'' := \alpha'' \cdot A' \cdot \beta''$ is zero except possibly for its first entry. Moreover the d-th row of A'' and the columns from 1 to j_0 of A'' are positive. The columns from j_0+1 to m of A'' admit a strictly positive $(d-1)$-st entry. Then by summing a suitable multiple of the $(d-1)$-st row of A'' to the previous rows we get a positive matrix, meaning that there exists $\alpha''' \in \mathrm{GL}_d(\mathbb{Z})$ such that $A''' := \alpha''' \cdot A''$ is positive. Then, by setting $\alpha = \alpha'''\alpha''\alpha'$ and $\beta = \beta'\beta''$, the matrix $A''' = \alpha \cdot A \cdot \beta$ satisfies condition (ii).

Now we prove the theorem by induction on m. If $m = 1$ then (ii) suffices to conclude the proof. By induction let us now assume that the theorem holds for a positive matrix with $m-1$ columns. Let now A be a $d \times m$ positive matrix. By (ii) we may assume that the first column of A is zero except possibly for its first entry $a_{1,1}$. We have now two possibilities.

- $a_{1,1} = 0$: let A' be the submatrix of A obtained by deleting the first column of A. By induction there exist $\alpha \in \mathrm{GL}_d(\mathbb{Z})$ and a permutation matrix $\beta' \in \mathrm{GL}_{m-1}(\mathbb{Z})$ such that $\alpha A' \beta'$ is a positive matrix in row echelon form. Then we are done by setting $\beta = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \beta' \end{pmatrix}$.
- $a_{1,1} > 0$: let A'' be the submatrix of A obtained by deleting the first row and the first column of A. By induction there exist $\alpha'' \in \mathrm{GL}_{d-1}(\mathbb{Z})$ and a permutation matrix $\beta'' \in \mathrm{GL}_{m-1}(\mathbb{Z})$ such that $\alpha'' A'' \beta''$ is a positive matrix in row echelon form. Then we end up the proof by setting

$$\alpha = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \alpha'' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \beta'' \end{pmatrix}. \qquad \square$$

Remark 3.19. The above proof exhibits a procedure giving a row echelon form of a given W-positive matrix A, provided that A is already a positive matrix. When this is not the case, if A is a W-matrix, then one can apply the procedure described in § 3.3 in order to find an equivalent positive matrix.

4. Proofs of geometrical results

This section is devoted to proving results stated in § 2; let X, Σ, V, Q be as defined therein.

Remarks 4.1.

1. The fan matrix V associated to Σ is a *reduced F-matrix* in the sense of Definitions 3.10 and 3.13.
2. The weight matrix Q is a *W-matrix* in the sense of Definition 3.9.
3. In general the sequence (7) is not exact on the right. Anyway, if the divisor class group $\mathrm{Cl}(X)$ is a free abelian group, then (6) is a *splitting* exact sequence implying the right exactness in (7). In this case $\ker(div^\vee)$ is a co-torsion free subgroup meaning that the HNF of Q^T is given by $\begin{pmatrix} \mathbf{I}_r \\ \mathbf{0}_{n,r} \end{pmatrix}$.

Proposition 4.2. *Let us fix $\mathbb{Z}$-bases of M and $\mathcal{W}_T(X)$ in the sequence (6). Then:*

1. *a weight matrix Q is a Gale dual of the fan matrix V whose transpose V^T is the representative matrix of div, i.e. $Q = \mathcal{G}(V)$,*
2. *a weight matrix Q is a W-matrix in the sense of Definition 3.14,*
3. *Q is a W-reduced matrix if V is a F-reduced matrix,*
4. *a Gale dual $\mathcal{G}(Q)$ is a CF-matrix.*

Proof. (1), (2) and (3) follow immediately by the definition of Gale duality, given in 3.1, and the Definition 3.14. (4) is a direct consequence of the Proposition 3.11. $\square$

The following proposition provides a generalization of [21, Proposition 5].

Proposition 4.3. *Given a fan matrix V of X then $Q = \mathcal{G}(V)$ is a reduced weight matrix of X which can be obtained by the last r rows of a matrix $U_V \in \mathrm{GL}_{n+r}(\mathbb{Z})$ such that $U_V \cdot V^T$ is a* HNF *matrix. Conversely, if Q is a reduced weight matrix of X and the divisor class group* $\mathrm{Cl}(X)$ *is a free abelian group then $V = \mathcal{G}(Q)$ is a CF-matrix which* *is a fan matrix of X, given by the last n rows of a matrix $U_Q \in \mathrm{GL}_{n+r}(\mathbb{Z})$ such that $U_Q \cdot Q^T$ is a* HNF *matrix.*

Proof. The first part of the statement follows by applying point (2) in Proposition 1.10 to the morphism $div^\vee$ in the exact sequence (7), whose representative matrix is V. Since V is F-reduced then $Q = \mathcal{G}(V)$ is W-reduced by definition.

For the converse, if $\mathrm{Cl}(X)$ does not admit any torsion subgroup, then Remark 4.1.3 applies. Again Proposition 1.10(2) applied to the morphism d in (13), whose representative matrix is Q, ends up the proof. In particular $V = \mathcal{G}(Q)$ is a CF-matrix (see the previous Proposition 4.2 (4)) and it turns out to be a fan matrix of X: in fact, by Proposition 3.1 (4), $V = \mathcal{G}(Q) = \mathcal{G}(\mathcal{G}(V')) = V'$ where V' is the fan matrix of X whose transposed matrix is the representative matrix of the morphism div. Moreover $Q = \mathcal{G}(V')$ is W-reduced if V' is F-reduced, by definition. Then also V is F-reduced. □

4.1. Proof of Theorem 2.4

The exact sequence (6) gives that

$$\mathrm{Cl}(X) \cong \mathcal{W}_T(X)/\,\mathrm{Im}(div) \cong \mathbb{Z}^{n+r}/\mathcal{L}_r(V)$$

so that

$$\mathrm{Tors}(\mathrm{Cl}(X)) \cong \mathrm{Tors}(\mathbb{Z}^{n+r}/\mathcal{L}_r(V)) \cong \mathrm{Tors}(\mathbb{Z}^n/\mathcal{L}_c(V)) \cong \mathbb{Z}^n/\mathcal{L}_r(T_n).$$

The fact that Σ is simplicial ensures that $N_{\Sigma(1)}$ is still a full sublattice of N, and the rows of T_n give a basis of $N_{\Sigma(1)}$. Hence $N/N_{\Sigma(1)} \cong \mathbb{Z}^n/\mathcal{L}_r(T_n)$.

4.2. Proof of Proposition 2.6

The equivalences (1) ⇔ (2) ⇔ (3) follow from Theorem 2.4 and its proof. The equivalence (3) ⇔ (4) is a consequence of Corollary 4.5 below.

Lemma 4.4. *Let A be a $n \times (n+r)$ matrix of rank n with integral coefficients. Let d_A be the* gcd *of all minors of order n of A. Let $\beta \in \mathrm{GL}_{n+r}(\mathbb{Z})$ and put $A' = A\beta$. Then $d_{A'} = d_A$.*

Proof. Let $a_1, \dots, a_k$ be the sequence of minors of A of order n, and $a'_1, \dots, a'_k$ be the corresponding sequence of minors of A'. We can assume that A' is obtained by applying a step of Gauss $\mathbb{Z}$-reduction on the columns of A. It is clear that interchanging two columns or changing the sign of a column simply modifies some signs in the sequence of minors, so that $d_{A'} = d_A$ in these cases. Assume that A' is obtained from A by adding to a column (say $\mathbf{c_1}$) a multiple of another column (say $\mathbf{c_2}$), so that $\mathbf{c'_1} = \mathbf{c_1} + b\mathbf{c_2}$ for some $b \in \mathbb{Z}$. Let B be a square $n \times n$ submatrix of A, and B' be the corresponding submatrix of A'. Suppose that $a_1 = \det(B)$. Then: if either $\mathbf{c_1}$ is not a column of B or $\mathbf{c_2}$ is a column of B then $\det(B) = \det(B')$. If $\mathbf{c_1}$ is a column of B and $\mathbf{c_2}$ is not a column of B then $\det(B') = \det(B) + b\det(B'')$, where B'' is obtained from B by replacing $\mathbf{c_1}$ by $\mathbf{c_2}$. Then $\det(B'') = \pm a_i = \pm a'_i$ for some $i \neq 1$ and $a'_1 = \pm a_1 \pm b a'_i$ so that $\gcd(a_1, \dots, a_k) = \gcd(a'_1, \dots, a'_k)$. □

Corollary 4.5. *Let A, d_A be as in Lemma 4.4. Then*

$$d_A = \left|\mathrm{Tors}\left(\mathbb{Z}^{n+r}/\mathcal{L}_r(A)\right)\right| = \left|\mathbb{Z}^n/\mathcal{L}_c(A)\right| .$$

In particular $\mathcal{L}_c(A)$ has no cotorsion in $\mathbb{Z}^n$ if and only if the minors of A of order n are coprime.

Proof. Let $H = \mathrm{HNF}(A^T) = U \cdot A^T = \begin{pmatrix} T_n \\ \mathbf{0}_{r,n} \end{pmatrix}$. Then $H^T = A \cdot U^T$ and, by Lemma 4.4, $d_A = d_{H^T}$. But $d_{H^T} = \det(T_n) = |\mathbb{Z}^n/\mathcal{L}_c(A)|$. □

Proof of Theorem 2.9. (1) X is a PWS, meaning that $\mathrm{Cl}(X) \cong \mathbb{Z}^r$, by Proposition 2.6 and Theorem 2.4. Recalling Remark 4.1.3, we can then write

$$\begin{pmatrix} \mathbf{I}_r \\ \mathbf{0}_{n,r} \end{pmatrix} = U_Q \cdot Q^T \ \Rightarrow\ Q \cdot U_Q^T = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r,n} \end{pmatrix}$$

Then (10) follows by recalling that Q is a representative matrix of the morphism $d : \mathcal{W}_T(X) \to \mathrm{Cl}(X)$.

(2) Recall that, for any $k \in \mathbb{N}$, the Weil divisor $L = \sum_{j=1}^{n+r} a_j D_j$ is a Cartier divisor if it is locally principal, which is

$$\forall\, I \subset \{1, \dots, n+r\} : \left\langle V^I \right\rangle \in \Sigma(n) \quad \exists\, \mathbf{m}_I \in M : \forall\, j \notin I \ \mathbf{m}_I \cdot \mathbf{v}_j = a_j \,. \tag{15}$$

For $I \subset \{1, \dots, n+r\}$ define

$$\mathcal{P}^I = \{L = \sum_{j=1}^{n+r} a_j D_j \in \mathcal{W}_T(X) \mid \exists\, \mathbf{m} \in M : \forall\, j \notin I \ \mathbf{m} \cdot \mathbf{v}_j = a_j\}.$$

Then $\mathcal{P}^I$ contains $\mathrm{Im}(div)$ and a $\mathbb{Z}$-basis of $\mathcal{P}^I$ is given by

$$\{D_j, j \in I\} \cup \{\sum_{k=1}^{n+r} v_{ik} D_k, i = 1, \dots, n\},$$

where $\{v_{ik}\}$ is the i-th entry of $\mathbf{v}_k$. Let $\mathcal{B}_I = d(\mathcal{P}^I)$; then a basis of $\mathcal{B}_I$ in $\mathrm{Cl}(X)$ is

$$\{d(D_j), j \in I\}.$$

Through the fixed identification of $\mathrm{Cl}(X)$ with $\mathbb{Z}^r$, each $d(D_j)$ corresponds to the j-th column of the matrix Q. Therefore

$$\mathrm{Pic}(X) = \bigcap_{I \in \mathcal{I}_\Sigma} \mathcal{B}_I = \bigcap_{I \in \mathcal{I}_\Sigma} \mathcal{L}_c(Q_I).$$

(3) Observe that

$$Q \cdot C^T = Q \cdot U_Q^T \cdot \begin{pmatrix} B^T & \mathbf{0}_{r,n} \\ \mathbf{0}_{n,r} & \mathbf{I}_n \end{pmatrix} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r,n} \end{pmatrix} \cdot \begin{pmatrix} B^T & \mathbf{0}_{r,n} \\ \mathbf{0}_{n,r} & \mathbf{I}_n \end{pmatrix} = \begin{pmatrix} B^T & \mathbf{0}_{r,n} \end{pmatrix},$$

so that $\mathcal{L}_r(C) \subseteq \mathcal{C}_T(X)$; moreover

$$[\mathcal{W}_T(X) : \mathcal{L}_r(C)] = \det(B) = [\mathrm{Cl}(X) : \mathrm{Pic}(X)] = [\mathcal{W}_T(X) : \mathcal{C}_T(X)];$$

therefore $\mathcal{L}_r(C) = \mathcal{C}_T(X)$.

(4) is a direct consequence of (2). □

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