*AMS – IMS – SIAM*

*1992 Joint Summer Research Conferences*

**Control and Identification of PDEs**

Mount Holyoke College
South Hadley, MS
July 1992

# Some Stability Estimates for the Identification of Conductivity in a Parabolic PDE (*)

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(*) (the one-dimensional heat equation)

# INTRODUCTION

## The *distributed parameter system*

*conductivity* *potential* *source term*

$$\frac{\partial}{\partial x}\left[a(x)\,\frac{\partial}{\partial x}u(x,t)\right] = \frac{\partial}{\partial t}u(x,t) + f(x,t) \text{ in } D \times T$$

IC, BC



## An *inverse problem*

$$\{\, u, f \,\} \overset{?}{\longmapsto} a$$

## Why inverse problems ?

accessibility, ease of measurement, ...



## Difficulties

The ODE for the unknown conductivity:

$$a'(x)\frac{\partial}{\partial x}u(x,t) + a(x)\,\frac{\partial^2}{\partial x^2}u(x,t) = g(x,t) \text{ in } D \times T$$

Under– *vs*. overspecification => (non–)existence, (non–)uniqueness

## Key issues:

existence, uniqueness, stability

## How to attain *uniqueness*, given existence ?

Supply a Cauchy datum to the ODE

## What is *stability* (of the unique solution) ?

Given $a(u, f)$ , $b(v, f)$,

*i)* find suitable Banach spaces $\mathbb{X}$ (data), $\mathbb{Z}$ (parameters),

*ii)* try to relate $\| v - u \|_{\mathbb{X}}$ to $\| b - a \|_{\mathbb{Z}}$ by

$$\| b - a \|_{\mathbb{Z}} \leq const.\, \| v - u \|_{\mathbb{X}}$$

# DEFINITIONS AND PROBLEM STATEMENT

State equation:

$$Q := (x_0, x_1) \times (t_0, t_1) = D \times T$$

$$(a u_x)_x =_{\text{d.w.}} u_t + f \qquad (\text{d.w.} = \text{distribution} - \text{wise})$$

$$u(x_i, t) = u_i(t), \; i = 0, 1; \; \text{IC}$$

The inverse problem:

*given*

$$f \in C^0(\bar{T}; H^{-1}(D)) \,, \qquad \textit{(source term)}$$

$$u \in \mathbb{X} \qquad \textit{(potential)}$$

*find*

$$a \in \mathbb{A}_{ad} \qquad \textit{(conductivity)}$$

s.t.,

$$(a u_x)_x =_{\text{d.w.}} u_t + f \;\; \text{in } Q$$



where

$$\mathbb{X} := C^0(\bar{T}; W^{1,\infty}(D)) \cap C^1(\bar{T}; L^2(D))$$

$$\mathbb{A}_{ad} := \{ a \mid a \in L^\infty(D), \; \exists \lim_{x \to x_0^+} a \,, \; 0 < a_L = \operatorname*{ess\,inf}_D a \,; \; \operatorname*{ess\,sup}_D a = a_H \}$$

Hp. $\{ u, f \} \Rightarrow \exists \, a \in \mathbb{A}_{ad}$ *(reference solution)*

Defect equation:

$$(B v_x)_x =_{\text{d.w.}} V_t - (a V_x)_x \,, \; \forall \, t \in \bar{T}$$

where

second potential

$$V := v - u \;\; ; \;\; u, v \in \mathbb{X}$$

$$B := b(v, f) - a(u, f)$$

second conductivity ; $b \in L^\infty$, $\operatorname*{ess\,inf}_D b \geq a_L$

*ASIDE*:

## LESS REGULAR DATA PAIRS
## (*UNIQUENESS* CONDITIONS ONLY)

$f \in L^2(T; H^{-1}(D))$ , (*source term*)

$u \in \mathbb{U}_{ad}$ (*potential*)



where

$$\mathbb{U}_{ad} := \{u \mid u \in L^2(T; H^1(D));\ u_t \in L^2(T; H^{-1}(D));$$
$$u_x \in C^0(\bar{T}; C^0(\bar{D} \setminus S_{u(t)}))\}$$



the set of points where $u_x$ discontinuous is

$$S_u(t) := \{\ x(t) \mid x(t) \in D,\ x(.) \in C^0(\bar{T})\ \};$$
$$\text{meas}[S_u(t)] \stackrel{Hp}{=} 0,\ \forall t \in T$$



$\doteq u_x \in BV(\bar{D}), \forall t \in \bar{T}$

# UNIQUENESS CONDITIONS:
# 1 – GENERAL



## Classification

1) Type of information

LOCAL: Cauchy problem
for the state or the defect equation

NON–LOC.: *self–identifiability*

2) Type of Cauchy problem

REGULAR: $\frac{\partial u}{\partial x}(x, \tau) \neq 0, \forall x \in \bar{D}$

either $a(x_0)$ known or $\frac{da}{dx} = 0$

SINGULAR: $\frac{\partial u}{\partial x}(\xi(\tau), \tau) = 0$

3) Time span

data at $\tau \in T$

data in $T$

*ASIDE*:

## SPATIAL ANTIDERIVATIVES OF DISTRIBUTIONS

Let $R \in L^2(T; H^{-1}(D))$ a distribution

then $\{ R^{[-1]} \} \subset L^2(T; L^1_{loc}(D))$ s.t., the antiderivative

$$\langle R^{[-1]} \mid w \rangle = - \langle R \mid w_\varepsilon \rangle + \langle c_1 \mid w \rangle , \forall\, w \in \mathbf{W}$$

where

$$\mathbf{W} := \{ w \mid w \in L^2(T; H^1_0(D));\ w_t \in L^2(T; H^{-1}(D));\ w( .,\, t_1 ) = 0 \}$$

the space of test functions

$$w_\varepsilon(.,t) := \int_{x_0}^{x} [w(\sigma,\, t) - \omega_\varepsilon(\sigma - \sigma_0) \int_D w(\psi,\, t) d\psi\, ] d\sigma$$

$$w \in \mathbf{W} ,\ \omega_\varepsilon(x) = c_\varepsilon \exp[\tfrac{\varepsilon^2}{x^2 - \varepsilon^2}]$$ a *bell*–function



Rem.: antidifferentiation, $(.)^{[-1]}$, is *set valued*



Notation

$<h>$ := spatial average of $h \in C^0(\bar{T}; L^1(D))$ ,



$(\Rightarrow\ <h> \in C^0(\bar{T})\ )$

$R^{[-1]}_0$ := antiderivative, which vanishes @ $x_0$

(requires $\exists \lim_{x \to x_0^+} R^{[-1]}$ )