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Some Inverse Problems in Ground Water Modelling

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TALK

Mr Chairman (Ms. Chairperson), Ladies and Gentlemen:

Let me thank the Conference organizers for the invitation.

1 Motivation for this work came in the past by the need to model the aquifer of this City and more recently by a joint *CNR – ENEL* research project, which I acknowledge for partial financial support.
The following items will be covered: the equation, which governs ground water flow in some circumstances, an inverse problem of coefficient identification and some recent results about uniqueness and stability and the properties of an identification algorithm.

2 From experimental evidence, ground water flows through a heterogeneous porous medium.
Flow through a porous medium can be classifed as a linear, irreversible process, like heat flow. Processes are usually described by a cause – effect relationship. Recall that heat flow is governed by Fourier' s law. Similarly, Ohm' s law governs the flow of electric charge through a resistive medium and Fick' s law governs diffusion.
Darcy' s law applies to ground water:

- $\mathbf{J}_w$ is the water flux density, a vector
- u is the hydraulic potential, a scalar,
- A is hydraulic conductivity, a matrix the entries of which are position dependent.

3 When Darcy' s law is recast in divergence form, the following equation is obtained. It relates the hydraulic potential u, a function of space and time, to the source term and to the medium properties.

Incidentally it should be noted that several situations of interest can be modelled by suitably selecting the function A. From now on, A will be considered only as a function of space, x. This is not the only possibility: time and space dependent coefficients have been dealt with, as well as state dependent ones, such that $A = A(u)$.

Once the model equation has been obtained, interest focusses on solving for the designated unknown i.e., the potential u. This is the direct problem. When supplementary information (data) is provided as initial and boundary conditions, a solution u can be shown to exist, to be unique and to continuously depend on all data.

4 What are models for (in ground water flow and elsewhere) ?
Description: interpret the natural system' s behaviour under the currently applied controls.
Prediction: determine the effects of different controls without carrying out an actual experiment.
In either case, the only unknown in the problem shall be the potential, u.
Whereas the domain, the initial and boundary conditions and the source term are easily determined, the medium properties are usually not directly measurable by any experiment. The first paradox is met: in seeking for u, A is needed first ! It looks as if the obstacle can be overcome by measuring u , with the purpose of determining A. This is the typical procedure of inverse problems, of coefficient identification in particular.

5 The idea beneath inverse problems is to interchange the roles of ∇u and A. The former becomes a datum and the latter the unknown. The relation is still formally expressed by the original equation. At this point the second paradox is met: improper use is made of the original equation, which was designed to yield u.

6 All kinds of difficulties show up: none of the properties of the direct problem are preserved.

7 Before illustrating some recent results about the inverse transmissivity problem, some related inverse problems of mathematical physics can be introduced.
One dimensional, time dependent flow is governed by this parabolic equation.
Time independent flow in two spatial dimensions is described by an elliptic equation, where transmissivity is either a matrix or a scalar.
In all cases transmissivity is the leading coefficient (i.e., coefficient of the highest derivative) in a differential equation.
Finally, there is a connection between the inverse transmissivity problem for the latter equation and the identification of potential energy in the following Schroedinger equation.

8 During the past 30 years, many methods have been devised to well pose the inverse problem and obtain a solution. Some results of interest can be obtained, when one assumes the problem has a solution and one wants to provide uniqueness conditions.
Once the solution is known to be unique, one wants to determine how it is affected by changes in the data: this is the scope of stability estimates.
The most challenging task is insuring the existence of a solution to the inverse problem: it usually requires a greater effort.
With reference to the identification of transmissivity, one shall properly translate the situations met in hydrogeology, then provide stability estimates, where the norms and the constants can be easily computed. Finally, and this applies to existence results, one shall aim at computational algorithms of practical interest.

9 Some recent results will be now summarized.
In the interval $D := (x_0, x_1)$, consider the one – dimensional, time dependent flow equation: at every time instant it is an ordinary differential equation of 1st order w.r. to a. Assume at least one positive, bounded solution $a(u; f)$ exists. The uniqueness of a is obtained by supplying a Cauchy datum. Cauchy problems can be either regular (u' vanishes nowhere in D) or singular (there exist isolated critical points for u). Uniqueness conditions can be classified according to the applicable type of Cauchy problem.

10 There is a relevant consequence for the achievable type of stability estimate: regular Cauchy problems lead to uniform estimates, whereas singular ones yield norm estimates of integral type.

11 This is an almost straightforward example, where uniqueness is due to an isolated critical point. The Lp norm of the difference between two transmissivities is estimated by the $W1,\infty$ norm of the difference between the two potentials.

12 Finally, constructive algorithms are been looked at from the dynamical system point of view.
The inverse problem is set in two spatial dimensions. The unknowns are the entries a and b of this matrix.
Iterative algorithms usually minimize some objective function. Here the equation error is considered. Iterative algorithms can be shown to have a "continuous time" counterpart i.e., an evolution differential equation.
By letting A depend on time as well as on position, the following example is obtained. The dynamical system which minimizes the equation error by the steepest descent rule is given by the following equations.
More interesting is the following result: assume that partial derivatives of u vanish nowhere, define the elliptic BVP w.r. to the auxiliary potential p, then the following evolution equation can be shown to be a gradient flow and to yield the following decay rate for the equation error.

This is all. Thank You very much for Your attention.