



The $*$ -Exponential as a Covering Map

Amedeo Altavilla¹ · Samuele Mongodi²

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Abstract

We employ tools from complex analysis to construct the $*$ -logarithm of a quaternionic slice regular function. Our approach enables us to achieve three main objectives: we compute the monodromy associated with the $*$ -exponential; we establish sufficient conditions for the $*$ -product of two $*$ -exponentials to also be a $*$ -exponential; we calculate the slice derivative of the $*$ -exponential of a regular function.

Keywords Slice-regular functions · Quaternionic exponential · Quaternionic logarithm · Baker–Campbell–Hausdorff · Covering maps · Monodromy

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1 Introduction

One of the fundamental ideas in the early stages of real analytic geometry and in the study of flat real analytic CR manifolds is complexification: real analyticity ensures that most of the features we are interested in will be reflected in the complexification, at least on a formal and algebraic level.

In this respect, the theory of slice regular functions benefits no less, if not more, from this idea. As is known, slice regular functions on a symmetric open domain $U \subseteq \mathbb{H}$ can be viewed as a special family of holomorphic functions (namely *stem*

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Amedeo Altavilla and Samuele Mongodi contributed equally to this work.

✉ Amedeo Altavilla
amedeo.altavilla@uniba.it
Samuele Mongodi
samuele.mongodi@unimib.it

¹ Dipartimento di Matematica, Università degli Studi di Bari “Aldo Moro”, Via Edoardo Orabona, 4, 70125 Bari, Italy

² Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via Roberto Cozzi, 55, 20126 Milan, Italy

functions) from an open domain in \mathbb{C} to $\mathbb{C} \otimes \mathbb{H}$ (i.e. the *complexification* of \mathbb{H} as an associative algebra with unity).

In [14], this idea was employed to account for the many similarities between the theory of slice regular functions and the theory of holomorphic functions of one complex variable; this argument was pushed further in [13] to describe the link between the complex geometry of the set of square roots of -1 of an associative algebra and the space of slice regular functions.

On the other hand, the purely algebraic properties resulting from the structure of $\mathbb{C} \otimes \mathbb{H}$ were used to a great extent in [1–4], where many properties of the $*$ -exponential were studied, the structure of the $*$ -product was better understood, and the problem of the existence of a $*$ -logarithm was initiated.

Building on the same intuition and exploiting more thoroughly the equality of the analytic expressions of algebraic operations of \mathbb{H} and $\mathbb{C} \otimes \mathbb{H}$, in [5] we showed how the problem of finding $*$ -roots of a slice regular function can be translated into a problem of lifting functions through a holomorphic covering map. The number and the structure of such $*$ -roots were then linked to the group of deck transformations of the covering map.

Again, we would like to emphasize that this is possible because the analytic expressions of the multiplication in \mathbb{H} and in $\mathbb{C} \otimes \mathbb{H}$ in terms of the coordinates with respect to some basis of \mathbb{H} and the corresponding complexified basis of $\mathbb{C} \otimes \mathbb{H}$ are the same, reflecting the fact that a real analytic function on, say, the real line has a unique extension to the complexification of the real line given by a power series with the same coefficients.

We present here another instantiation of this phenomena: we treat the case of $*$ -logarithms by considering the map $\exp : \mathbb{H} \rightarrow \mathbb{H}$ and lifting it to the complexification, to a map with the same analytic expression. The study of the local inverses of \exp again becomes a problem in complex analytic covering maps, from whose solution we also recover what we already proved in the case of $*$ -roots. Using a geometric approach we will see that under natural topological hypotheses the exponential map in $\mathbb{C} \otimes \mathbb{H}$ is a covering map (see Theorem 4.2) and we will be able to write down its monodromy. Then, thanks to the standard relation between holomorphic stem functions and slice regular functions, given a never-vanishing slice regular function $f : U \rightarrow \mathbb{H}$ such that its “vector part” is never-vanishing, we will be able to construct a 2-parameter family of $*$ -logarithms (1-parameter family if $U \cap \mathbb{R} \neq \emptyset$), see Corollaries 5.2 and 5.3 for the results and Remark 5.1 for the explicit description of the monodromy.

We note that the study of the $*$ -exponential is closely related to many problems in geometric function theory, such as the possible ranges of analytic functions. In this context, a quaternionic version of the Picard theorem has been discussed in [7].

Such a study extends what is already contained in [4, 9, 11] by showing the geometric nature of the many problems encountered in the search for a good notion of logarithm in the non-commutative setting.

As already mentioned, the proofs of many results contained in the present paper follow topological strategies, which then produce natural hypotheses and conditions, simplifying many proofs contained in the aforementioned papers. On the other hand, since it is not the specific aim of this work, we will only give a glimpse into how the

remaining residual cases should be treated, i.e. how some of the hypotheses could be relaxed.

In an effort to highlight the impact that this simple idea can produce, we analyze the problem of when a product of exponentials is an exponential itself; the question for quaternions is easily settled by using a simplified version of the Baker–Campbell–Hausdorff formula (or, if one interprets quaternions as rotations, by a standard application of Rodrigues’ formula). Once the problem is *analytically* solved for quaternions, we formally consider the same solution for the same problem in $\mathbb{C} \otimes \mathbb{H}$, where stem functions take their values. This gives us a solution to the same problem at the level of stem functions, hence for slice regular functions. The same idea is used to compute the slice derivative of the $*$ -exponential of a slice regular function. Even though this computation is quite natural, it has not been implemented yet, possibly due to the lack of a strategy like the one we use here. In these last two tasks, we will use a simple formula inspired by standard linear algebra, which allows us, given a generic slice regular function f , to write any other function g as a sum of a component in the “direction” of f and another “orthogonal” part. This will allow us to write much simpler formulas and to identify possible future generalizations.

2 Preliminaries

2.1 Algebraic Structures of \mathbb{H} and of $\mathbb{H} \otimes \mathbb{C}$

In this paper we will deal with many different imaginary units, not only those contained in the space of quaternions, but also with others coming from different algebras. Starting from complex numbers, the symbol ‘ i ’ will denote the standard imaginary unit in \mathbb{C} (and hence will be used when working in \mathbb{C}^N , $N \geq 1$). The symbol ‘ i ’ will denote the first imaginary unit in the definition of the space \mathbb{H} of quaternions:

$$\mathbb{H} := \{q = q_0 + q_1i + q_2j + q_3k \mid q_\ell \in \mathbb{R}, \ell = 0, 1, 2, 3, \\ i^2 = j^2 = k^2 = -1, ij = k = -ji\}.$$

We will make use of the standard conjugation in \mathbb{H} denoted by the superscript c :

$$q = q_0 + q_1i + q_2j + q_3k \mapsto q^c = q_0 - (q_1i + q_2j + q_3k).$$

Using this conjugation, given any quaternion q , it is possible to define its scalar and vector parts as follows

$$q_0 = \frac{q + q^c}{2}, \quad q_v = \frac{q - q^c}{2},$$

so that $q = q_0 + q_v$. Obviously, if q is represented in the form $q = q_0 + q_1i + q_2j + q_3k$, then q_0 is the scalar part of q and $q_v = q_1i + q_2j + q_3k$. Using this representation, we can express the product of two quaternions $q = q_0 + q_v$ and $p = p_0 + p_v$ in a more understandable way:

$$qP = q_0p_0 - \langle q_v, p_v \rangle + q_0p_v + p_0q_v + q_v \wedge p_v, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ and \wedge denote the standard Euclidean and cross products. In particular, the square norm of q can be computed as $|q|^2 = qq^c$ and we have that $q_v^2 = -|q_v|^2$.

Whenever $q_v \neq 0$, we are able to represent q in another convenient form:

$$q = \alpha + I\beta,$$

where $\alpha = q_0$, $I = \frac{q_v}{|q_v|}$ and $\beta = |q_v|$. In particular, if we denote the set of imaginary units as follows

$$\mathbb{S} := \{I \in \mathbb{H} \mid I^2 = -1\} = \{\alpha_1i + \alpha_2j + \alpha_3k \mid \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1\},$$

and we denote by $\mathbb{C}_I = span(1, I) = \{\alpha + I\beta \mid \alpha, \beta \in \mathbb{R}\}$ the complex plane generated by 1 and I , we have that

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I.$$

This last representation comes in handy when working with slice functions, and in order to do that we need to discuss the complexification of \mathbb{H} (in particular, we will follow the approach of [12]).

The symbol ‘ $\sqrt{-1}$ ’ will denote the complex imaginary unit defining the complexification of \mathbb{H} , i.e.

$$\mathbb{C} \otimes \mathbb{H} := \{q + \sqrt{-1}p \mid q, p \in \mathbb{H}\}.$$

The algebraic structure of $\mathbb{C} \otimes \mathbb{H}$ is defined in the usual way: if $q_1 + \sqrt{-1}p_1, q_2 + \sqrt{-1}p_2 \in \mathbb{C} \otimes \mathbb{H}$, then:

$$(q_1 + \sqrt{-1}p_1)(q_2 + \sqrt{-1}p_2) = q_1q_2 - p_1p_2 + \sqrt{-1}(q_1p_2 + p_1q_2).$$

By fixing a (orthogonal) basis of \mathbb{H} containing 1 and by writing any quaternion in its 4 real coordinates we get a biholomorphism between $\mathbb{C} \otimes \mathbb{H}$ and \mathbb{C}^4 : if $q = q_0 + q_1i + q_2j + q_3k$ and $p = p_0 + p_1i + p_2j + p_3k$, then we define $\phi : \mathbb{C} \otimes \mathbb{H} \rightarrow \mathbb{C}^4$ as

$$\phi(q + \sqrt{-1}p) = (q_0 + \iota p_0, q_1 + \iota p_1, q_2 + \iota p_2, q_3 + \iota p_3).$$

In particular, this biholomorphism induces an algebraic structure on \mathbb{C}^4 that is defined exactly as that of \mathbb{H} :

$$\begin{aligned} \mathbb{C} \otimes \mathbb{H} = \{z = z_0 + z_1i + z_2j + z_3k \mid z_\ell \in \mathbb{C}, \ell = 0, 1, 2, 3, \\ i^2 = j^2 = k^2 = -1, ij = k = -ji\}. \end{aligned}$$

In $\mathbb{C} \otimes \mathbb{H}$ it is possible to define two commuting conjugations:

$$q + \sqrt{-1}p \mapsto (q + \sqrt{-1}p)^c = q^c + \sqrt{-1}p^c,$$

$$q + \sqrt{-1}p \mapsto \overline{q + \sqrt{-1}p} = q - \sqrt{-1}p.$$

If we work in \mathbb{C}^4 , these two conjugations translate as follows

$$z = z_0 + z_1i + z_2j + z_3k \mapsto z^c = z_0 - (z_1i + z_2j + z_3k),$$

$$z = z_0 + z_1i + z_2j + z_3k \mapsto \bar{z} = \bar{z}_0 + \bar{z}_1i + \bar{z}_2j + \bar{z}_3k.$$

Exactly as before, we can define the ‘‘scalar’’ and ‘‘vector’’ part of $z \in \mathbb{C} \otimes \mathbb{H}$ as

$$z_0 = \frac{z + z^c}{2}, \quad \underline{z} = \frac{z - z^c}{2},$$

Within this language, the product of two elements $z, w \in \mathbb{C} \otimes \mathbb{H}$ can be written formally as in Formula 1:

$$zw = z_0w_0 - \langle \underline{z}, \underline{w} \rangle + z_0\underline{w} + w_0\underline{z} + \underline{z} \wedge \underline{w},$$

where $\langle \cdot, \cdot \rangle$ and \wedge are the formal generalization of the Euclidean and cross product. In particular if $z = z_0 + \underline{z} = z_0 + z_1i + z_2j + z_3k$, setting

$$|\underline{z}|^2 = z_1^2 + z_2^2 + z_3^2,$$

we have that $zz^c = \langle z, z \rangle = z_0^2 + |\underline{z}|^2 \in \mathbb{C}$ and zz^c is a real number only if the four components of z are real numbers, i.e. only if $z \in \mathbb{H}$. We remark that the function $z \in \mathbb{C} \otimes \mathbb{H} \mapsto |\underline{z}|^2 \in \mathbb{C}$ is not a *norm* (because it takes values in \mathbb{C}), but it is the formal generalization of the Euclidean norm. However, since the product in \mathbb{C} is commutative, for any $z, w \in \mathbb{C} \otimes \mathbb{H}$ we have

$$(zw)(zw)^c = zww^cz^c = (z^c)(ww^c). \tag{2}$$

If z is such that $zz^c \neq 0$, then $z \neq 0$, but unfortunately $\mathbb{C} \otimes \mathbb{H}$ contains zero divisors. So, in particular, there are $z \neq 0$ such that $zz^c = 0$.

2.2 Stem Functions, Slice Functions and Regularity

We are now ready to introduce and discuss slice functions. As already mentioned, we will rely on the approach of stem functions developed in [12] and in subsequent works by the same authors. We also refer to [5] to deepen our specific point of view. We start with the following definition.

Definition 2.1 Let $U \subset \mathbb{C}$ be such that $\overline{\overline{U}} = U$. A function $F : U \rightarrow \mathbb{C} \otimes \mathbb{H}$ is said to be a *stem function* if, for any $z \in U$, we have $F(\bar{z}) = \overline{F(z)}$.

If we write $F : \mathcal{U} \rightarrow \mathbb{C} \otimes \mathbb{H}$ as $F(z) = F_{ev}(z) + \sqrt{-1}F_{od}(z)$, then the condition $F(\bar{z}) = \overline{F(z)}$ is reflected in the following two equalities $F_{ev}(\bar{z}) = F_{ev}(z)$ and $F_{od}(\bar{z}) = -F_{od}(z)$. If, instead, we read F as a function taking values in \mathbb{C}^4 , $F(z) = (F_0(z), F_1(z), F_2(z), F_3(z))$, then the stem condition must be satisfied by all four components, i.e. $F_\ell(\bar{z}) = \overline{F_\ell(z)}$, for $\ell = 0, 1, 2, 3$.

Definition 2.2 Let $U \subset \mathbb{H}$ be such that if $q = \alpha + I\beta \in U$ then $\alpha + J\beta \in U$, for any $J \in \mathbb{S}$ and let $\mathcal{U} = \{\alpha + i\beta \mid \alpha + I\beta \in U\}$. A function $f : U \rightarrow \mathbb{H}$ is said to be a *slice function* if there exists a stem function $F = F_{ev} + \sqrt{-1}F_{od} : \mathcal{U} \rightarrow \mathbb{C} \otimes \mathbb{H}$ such that $f(\alpha + I\beta) = F_{ev}(\alpha + i\beta) + IF_{od}(\alpha + \sqrt{-1}\beta)$; in this case we will write $f = \mathcal{I}(F)$ and we will say that f is induced by F .

If U is a domain and F is a holomorphic function, then f is said to be a *slice regular function*.

The definition of stem functions guarantees the well-definition of slice functions: in fact, since $F(\bar{z}) = \overline{F(z)}$, then the value of f at $\alpha + (-I)(-\beta)$ is not different from that of f at $\alpha + I\beta$. Examples of slice regular functions are polynomials and converging power series in the quaternionic variable q with right quaternionic coefficients.

The main property of slice functions is the so-called *Representation Formula* contained in the following statement (see [10, Thm. 1.16]). It essentially says that a slice function can be recovered from its values on two different semislices \mathbb{C}_J^+ and \mathbb{C}_K^+ , where the exponent ‘+’ indicates the upper half plane.

Theorem 2.3 (Representation Formula) *Let $f : U \rightarrow \mathbb{H}$ be a slice function and let $J, K \in \mathbb{S}$ be such that $J \neq K$. Then, for every $\alpha + I\beta \in U$ the following formula holds*

$$f(\alpha + I\beta) = (I - K)((J - K)^{-1} f(\alpha + J\beta)) - (I - J)((J - K)^{-1} f(\alpha + K\beta)).$$

It is well known that the pointwise product of two slice functions does not preserve regularity, however, the pointwise product of two stem functions is a stem function. Therefore it is natural to introduce a new notion of product as follows.

Definition 2.4 Let $f = \mathcal{I}(F)$ and $g = \mathcal{I}(G)$ be two stem functions defined on the same domain U . The **-product* of f and g is defined as the slice function

$$f * g = \mathcal{I}(FG) : U \rightarrow \mathbb{H}.$$

Since the *-product is defined from the pointwise product in a non-commutative algebra (namely in $\mathbb{C} \otimes \mathbb{H}$), it is non-commutative itself. However, if we consider a slice function $f = \mathcal{I}(F_{ev} + \sqrt{-1}F_{od}) : U \rightarrow \mathbb{H}$, such that F_{ev} and F_{od} take only real values, then F is a genuine complex function of one complex variable and, for any other slice function g defined on U , we have that

$$f * g = fg = g * f.$$

A function $f = \mathcal{I}(F)$ with the above property is said to be *slice preserving*. In fact, as F_{ev} and F_{od} are real valued, then, for any $q = \alpha + I\beta \in U$, the element $f(q)$ belongs

to the same slice \mathbb{C}_I of q . Written as a complex curve in \mathbb{C}^4 the stem function F of a slice preserving function takes the following form

$$F(z) = (F_{ev} + \iota F_{od}, 0, 0, 0) = (F_0, 0, 0, 0).$$

At this stage we can apply all the formalism and properties described in the previous part of this Section and obtain that, if $f = \mathcal{I}(F)$ and $g = \mathcal{I}(G)$, $F = F_0 + F_1i + F_2j + F_3k$ and $G = G_0 + G_1i + G_2j + G_3k$, $f_\ell = \mathcal{I}(F_\ell)$ and $g_\ell = \mathcal{I}(G_\ell)$ for $\ell = 0, 1, 2, 3$, $f_v = \mathcal{I}((F + F^c)/2)$ and $g_v = \mathcal{I}((G + G^c)/2)$, then

$$f * g = f_0g_0 - \langle f_v, g_v \rangle_* + f_0g_v + g_0f_v + f_v \mathbb{A} g_v,$$

where $\langle f_v, g_v \rangle_* = f_1g_1 + f_2g_2 + f_3g_3$ and $f_v \mathbb{A} g_v = (f_2g_3 - f_3g_2)i + (f_3g_1 - f_1g_3)j + (f_1g_2 - f_2g_1)k$ and, of course, all f_ℓ and g_ℓ are slice preserving functions. These last two operators can be defined in an intrinsic way by means of the so-called regular conjugation: given a slice function $f = \mathcal{I}(F) : U \rightarrow \mathbb{H}$, we define its *regular conjugate* as the function $f^c : U \rightarrow \mathbb{H}$ defined as $f^c = \mathcal{I}(F^c)$. Then, if $g : U \rightarrow \mathbb{H}$ is another slice function, we have that

$$\langle f, g \rangle_* = \frac{f * g^c + g * f^c}{2}, \quad f \mathbb{A} g = f_v \mathbb{A} g_v = \frac{f * g - g * f}{2} = \frac{[f, g]}{2}.$$

This representation of the product highlights how many algebraic features of a slice functions directly come from those of quaternions (or of quaternionic curves). For instance, two non slice preserving functions f and g commute if and only if $f_v \mathbb{A} g_v \equiv 0$ if and only if there exist two slice preserving functions α and β not both identically zero, such that $\alpha f_v + \beta g_v \equiv 0$ (see e.g. [1, Prop. 2.10]). A particular instance of this phenomenon is when f and g are both \mathbb{C}_I -preserving for some $I \in \mathbb{S}$, i.e., for any $q = \alpha + I\beta$ in the domain of f and g , we have that $f(q), g(q) \in \mathbb{C}_I$. Slice preserving functions are \mathbb{C}_I -preserving for any $I \in \mathbb{S}$. Keeping this parallelism between the algebraic features of \mathbb{H} and those of $\mathbb{C} \otimes \mathbb{H}$, we notice that the role of the Euclidean norm of $\mathbb{R}^4 \simeq \mathbb{H}$ is taken here by the so-called *symmetrization* of f : given a slice function $f = f_0 + f_1i + f_2j + f_3k$, its symmetrization is the function $f^s := \mathcal{I}(FF^c) = f_0^2 + f_1^2 + f_2^2 + f_3^2$. The symmetrization of f has an important role in the study of zeroes of f (see [10, Ch. 3]).

We now recall an important example of slice preserving regular function.

Example 2.1 Let $\mathcal{J} : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$ be the function such that

$$\mathcal{J}(q) = \frac{q_v}{|q_v|}.$$

This is clearly a slice preserving function and it is constant on each semislice \mathbb{C}_I^+ . In fact, if $q = \alpha + I\beta \in \mathbb{H} \setminus \mathbb{R}$, and $\beta > 0$, then $\mathcal{J}(q) = I$. This particular function plays an important role in the theory of slice regular functions with domain that does not intersect the real axis. In fact, thanks to the fact that $\mathcal{J} * \mathcal{J} = \mathcal{J}^2 \equiv -1$, we are

able to define slice regular idempotent functions (and hence zero divisors) as

$$\ell_+, \ell_- : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}, \quad \ell_{\pm} = \frac{1 \mp \mathcal{J}i}{2}.$$

For a complete study of these functions see [3].

Starting from the previous example it is worth noticing that, given any slice regular function $f = f_0 + f_v$, such that $f_v \not\equiv 0$ and $\sqrt{f_v^s}$ is well defined (e.g. when f_v^s is never vanishing see [1, Cor. 3.2]), then the function

$$\frac{f_v}{\sqrt{f_v^s}},$$

is such that

$$\frac{f_v}{\sqrt{f_v^s}} * \frac{f_v}{\sqrt{f_v^s}} \equiv -1.$$

So, at least two (intrinsically) different functions, take the role of the imaginary unit in the setting of slice functions.

As mentioned earlier, a slice regular function is a slice function such that its stem function is holomorphic. In fact, if $f = \mathcal{I}(F) : U \rightarrow \mathbb{H}$ is a slice function of class \mathcal{C}^1 defined on a domain U , then the functions $\partial F / \partial \bar{z}$ and $\partial F / \partial z$ are stem functions as well. In particular, the function $\partial_c f := \mathcal{I}(\partial F / \partial z)$ is called the *slice derivative* of f . As is clear from the definition, the slice derivative of a slice regular function controls the behavior “along slices”. Thus, to have complete information of first order of a slice regular function f we need to consider another operator, namely the *spherical derivative*: given a slice function $f = \mathcal{I}(F) : U \rightarrow \mathbb{R}$, we define $\partial_s f : U \setminus \mathbb{R} \rightarrow \mathbb{R}$ as the slice function

$$\partial_s f(\alpha + I\beta) = \mathcal{I} \left(\frac{F_{od}(\alpha + \iota\beta)}{\beta} \right).$$

Even if it does not look like a derivative, the spherical derivative can also be obtained as the result of a differential operator applied to f (see [15]).

We close this preliminary section by recalling the definition of the $*$ -exponential of a slice regular function.

Definition 2.5 Let $f : U \rightarrow \mathbb{H}$ be any slice function. We denote by $f^{*2} = f * f$ and, for any $N > 2$ we define

$$f^{*N} = f * f^{*(N-1)}.$$

If f is slice regular, then we define the function $\exp_*(f) : U \rightarrow \mathbb{H}$ as

$$\exp_*(f) = \sum_{n \in \mathbb{N}} \frac{f^{*n}}{n!}.$$

Many properties and representations of the $*$ -exponential of a slice regular function are discussed in [1, 4, 9, 11].

3 The Quaternionic Exponential as a Covering Map

It is well known that the quaternionic exponential map $\exp : \mathbb{H} \rightarrow \mathbb{H}$ is a covering map. However, in order to be self-contained we propose here a proof of this fact using its slice regular nature. In fact, the function \exp is induced by the stem function $E : \mathbb{C} \rightarrow \mathbb{C} \otimes \mathbb{H}$ defined as $E(\alpha + \iota\beta) = e^{\alpha + \sqrt{-1}\beta} = e^\alpha (\cos \beta + \sqrt{-1} \sin \beta)$ or, with our usual abuse of notation, viewed as a curve in \mathbb{C}^4 , as $E(z) = (e^z, 0, 0, 0)$. Notice that, from the definition, the function \exp is slice preserving.

We now introduce the following family of sets where \exp will turn out to be non-singular. For any $k \in \mathbb{N}$ set

$$\mathcal{U}_k := \{q \in \mathbb{H} \mid k\pi < |q_v| < (k + 1)\pi\}.$$

We are now ready to state and prove the first result.

Theorem 3.1 *The real differential of \exp is non-singular at q if and only if $q \in \mathcal{U}_k$ for some $k \in \mathbb{N}$. Moreover, for each $k \in \mathbb{N}$, the restriction $\exp|_{\mathcal{U}_k}$ is a diffeomorphism onto its image, which is $\mathbb{H} \setminus \mathbb{R}$.*

Proof Following [5, Lem. 3.1 and Lem. 3.3] or [10, Prop. 8.19], if $f : \Omega \rightarrow \mathbb{H}$ is slice regular and $q = \alpha + I\beta \in \Omega$, then the real differential of f is singular at q if and only if $\partial_c f(q) = 0$ or $\partial_s f(q) = 0$ or $\partial_c f(q)(\partial_s f(q))^c \in (\mathbb{C}_I)^\perp$. We have that $\partial_c \exp = \exp$, so it is never vanishing. The spherical derivative of \exp at $q = \alpha + I\beta$ is equal to $e^\alpha \frac{\sin \beta}{\beta}$ and therefore it vanishes if and only if $\beta = h\pi$, where $h \in \mathbb{Z}^*$. As

$$((\partial_c \exp)(q)) \cdot ((\partial_s \exp)(q))^c = (\exp(q)) \cdot e^\alpha \frac{\sin \beta}{\beta} \in \mathbb{C}_I,$$

for all $q \in \mathbb{H}$, we have that the set of critical points, i.e. the set of points where the real differential does not have maximum rank, is given by

$$C_0(\exp) = \{q \in \mathbb{H} \mid |q_v| = h\pi, h \in \mathbb{Z}^*\} = \{\pi(z, I) \mid \text{Im}(z) = h\pi, h \in \mathbb{Z}^*, I \in \mathbb{S}\}.$$

It is easy to see that $\exp(C_0(\exp)) = \mathbb{R}$ and therefore, the set of singular points $S(\exp) = \exp^{-1}(\exp(C_0(\exp)))$ is given by

$$S(\exp) = \{\pi(z, I) \mid \text{Im}(z) = h\pi, h \in \mathbb{Z}, I \in \mathbb{S}\}. \tag{3}$$

Collecting everything, we get the following equality

$$\mathbb{H} \setminus S(\exp) = \bigcup_{h \in \mathbb{N}} \mathcal{U}_h.$$

Now, as \exp is slice-preserving and $\exp(\mathbb{H} \setminus S(\exp)) \subseteq \mathbb{H} \setminus \mathbb{R}$, given $q = \alpha + I\beta, q' = \alpha' + I'\beta' \in \mathbb{H} \setminus S(\exp)$, we have that $\exp(q) = \exp(q')$ if $I = I'$. In fact, if we denote by $\mathcal{T} : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H} \setminus \mathbb{R}$ the map defined by

$$\mathcal{T}(\alpha + I\beta) = \alpha + I(\beta + \pi)$$

for $\beta > 0$ and $I \in \mathbb{S}$, by the standard properties of the complex exponential, we obtain that $q = \mathcal{T}^{(h)}(q')$ for some $h \in \mathbb{N}$, where the superscript denotes iterates.

More generally, \mathcal{T} is a diffeomorphism from \mathcal{U}_h and \mathcal{U}_{h+1} for all h ; therefore, \exp is injective from \mathcal{U}_h to $\mathbb{H} \setminus \mathbb{R}$ and, again by the properties of the complex exponential, also surjective. Being a bijective local diffeomorphism, \exp is a diffeomorphism between \mathcal{U}_h and $\mathbb{H} \setminus \mathbb{R}$. □

Remark 3.1 The open domains \mathcal{U}_h are all disjoint, therefore the map

$$\exp|_{\mathbb{H} \setminus S(\exp)} : \mathbb{H} \setminus S(\exp) \rightarrow \mathbb{H} \setminus \mathbb{R}$$

is a covering map in the trivial way (i.e. there is no ramification).

4 The Exponential in $\mathbb{C} \otimes \mathbb{H}$

In this section we are going to study the exponential function of the algebra $\mathbb{C} \otimes \mathbb{H}$. As $\mathbb{C} \otimes \mathbb{H}$ is biholomorphic to \mathbb{C}^4 , this study is made in order to apply the ideas of complex analysis to our quaternionic context. Given $z = z_0 + z_1i + z_2j + z_3k \in \mathbb{C}^4 \cong \mathbb{C} \otimes \mathbb{H}$ we recall that, $\underline{z} = z_1i + z_2j + z_3k$ and $|\underline{z}|^2 = z_1^2 + z_2^2 + z_3^2$. Moreover, given a slice regular function $f = \mathcal{I}(F)$, we also recall the following set of relations already introduced in the preliminary section:

$$f_0 = \mathcal{I}(F_0), \quad f_v = \mathcal{I}(F), \quad f^c = \mathcal{I}(F_0 - F), \quad f^s = \mathcal{I}(F_0^2 + |F|^2), \quad f_v^s = \mathcal{I}(|F|^2).$$

In [5] we introduced the analog of the quaternionic n -th $*$ -power as $\sigma_n : \mathbb{C} \otimes \mathbb{H} \rightarrow \mathbb{C} \otimes \mathbb{H}$, where

$$\sigma_n(z) = \left(p_0^n(z_0, |\underline{z}|^2), z_1 p_1^{n-1}(z_0, |\underline{z}|^2), z_2 p_1^{n-1}(z_0, |\underline{z}|^2), z_3 p_1^{n-1}(z_0, |\underline{z}|^2) \right),$$

where p_0^n and p_1^{n-1} are the usual Chebischev polynomials, such that

$$(x + iy)^n = p_0^n(x, y^2) + i p_1^{n-1}(x, y^2). \tag{4}$$

We are now able to introduce the exponential function of the algebra $\mathbb{C} \otimes \mathbb{H}$ as the function $\varepsilon : \mathbb{C} \otimes \mathbb{H} \rightarrow \mathbb{C} \otimes \mathbb{H}$ defined by

$$\varepsilon = \sum_{n=0}^{\infty} \frac{\sigma_n}{n!}.$$

From the definition of σ_n , we have that

$$\varepsilon(z) = \sum_{n=0}^{\infty} \frac{p_0^n(z_0, |z|^2)}{n!} + \underline{z} \sum_{n=0}^{\infty} \frac{p_1^{n-1}(z_0, |z|^2)}{n!}.$$

Moreover, thanks to Formula 4, we have that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{p_0^n(x, y^2)}{n!} &= \operatorname{Re} \sum_{n=0}^{\infty} \frac{(x + Iy)^n}{n!} = e^x \cos(y), \\ y \sum_{n=0}^{\infty} \frac{p_1^{n-1}(x, y^2)}{n!} &= \operatorname{Im} \sum_{n=0}^{\infty} \frac{(x + Iy)^n}{n!} = e^x \sin(y). \end{aligned}$$

The aim of the following pages is to prove that, under suitable hypotheses, ε is a covering map. In order to obtain such a result, we ‘lift’ our construction to another space where we are able to use standard techniques from complex analysis. This was the fruitful strategy already used in [5] in order to better understand $*$ -roots of slice functions.

4.1 Lift to $\mathbb{C}^2 \times \mathcal{S}$

As in [5] we consider the set \mathcal{S} of imaginary units contained in $\mathbb{C} \otimes \mathbb{H}$

$$\mathcal{S} := \{z = \underline{z} \in \mathbb{C} \otimes \mathbb{H} \mid z^2 = |\underline{z}|^2 = z_1^2 + z_2^2 + z_3^2 = -1\},$$

and the map $\rho : \mathbb{C}^2 \times \mathcal{S} \rightarrow \mathbb{C} \otimes \mathbb{H}$ defined by

$$\rho((u_0, u_1), s) = u_0 + u_1 s.$$

Remark 4.1 With the language and symbols of tensor product the set \mathcal{S} contains those elements $\underline{z} = p + \sqrt{-1}q$, with $p, q \in \mathbb{H}$, such that $-1 = (p + \sqrt{-1}q)^2 = p^2 - q^2 + \sqrt{-1}(pq + qp)$. Therefore, p and q satisfy the following system (see also [5, Rem. 4.2]),

$$\begin{cases} p^2 - q^2 = -1, \\ pq + qp = 0. \end{cases}$$

The map ρ is a local diffeomorphism and, if we set $\mathcal{W}' := \{(u_0, u_1) \in \mathbb{C}^2 \mid u_1 \neq 0\}$ and

$$\Omega' = \rho(\mathcal{W}' \times \mathcal{S}) = \{(z_0, \underline{z}) \in \mathbb{C}^4 \mid |\underline{z}|^2 \neq 0\},$$

then, the restriction of ρ to $\mathcal{W}' \times \mathcal{S}$ is, in fact, a double cover of its image Ω' . In fact, for any $(z_0, \underline{z}) \in \Omega'$, we have that

$$\rho^{-1}(z_0, \underline{z}) = \left(\left(z_0, \pm\sqrt{|z|^2} \right), \pm\frac{z}{\sqrt{|z|^2}} \right).$$

In [5] we made a large use of this double cover because the total space is ‘large enough’ to allow many classical properties to hold. Now we consider the map $\epsilon : \mathbb{C}^2 \times \mathcal{S} \rightarrow \mathbb{C}^2 \times \mathcal{S}$ defined as

$$\epsilon((u_0, u_1), s) = ((e^{u_0} \cos(u_1), e^{u_0} \sin(u_1)), s).$$

This map is defined in order to have that $\epsilon \circ \rho = \rho \circ \epsilon$,

$$\begin{array}{ccc} \mathbb{C}^2 \times \mathcal{S} & \xrightarrow{\epsilon} & \mathbb{C}^2 \times \mathcal{S} \\ \rho \downarrow & & \downarrow \rho \\ \mathbb{C} \otimes \mathbb{H} & \xrightarrow{\epsilon} & \mathbb{C} \otimes \mathbb{H} \end{array}$$

i.e., ϵ can be viewed as the lift of ϵ in $\mathbb{C}^2 \times \mathcal{S}$. We want to prove that ϵ is a covering map onto its image. This result will allow us to state that ϵ is a covering map as well. Before stating the result, we introduce the following set

$$\mathcal{W} := \{(u_0, u_1) \in \mathbb{C}^2 \mid u_0^2 + u_1^2 = 0\}.$$

Theorem 4.1 *The complex differential of ϵ is everywhere non-singular. Moreover, the function ϵ is a covering map onto its image, which is $\epsilon(\mathbb{C}^2 \times \mathcal{S}) = (\mathbb{C}^2 \setminus \mathcal{W}) \times \mathcal{S}$.*

Proof Setting $(u', s) = \epsilon(u, s)$, the differential of ϵ at (u, s) is a map

$$D\epsilon_{(u,s)} : T_u\mathbb{C}^2 \times T_s\mathcal{S} \rightarrow T_{u'}\mathbb{C}^2 \times T_s\mathcal{S}$$

that can be represented by the following matrix

$$\begin{pmatrix} e^{u_0} \cos(u_1) & -e^{u_0} \sin(u_1) & \mathbf{0} \\ e^{u_0} \sin(u_1) & e^{u_0} \cos(u_1) & \mathbf{0} \\ \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{I}_S \end{pmatrix},$$

where $\mathbf{0} = (0, 0)$ and, as $\dim_{\mathbb{C}} \mathcal{S} = 2$, \mathbf{I}_S is the 2×2 identity matrix. We have $\det(D\epsilon_{(u,s)}) = e^{2u_0}$, so $D\epsilon$ is always invertible, i.e. ϵ is a local diffeomorphism between $\mathbb{C}^2 \times \mathcal{S}$ and itself.

We now pass on to look at the image of ϵ . Given $(w_0, w_1) \in \mathbb{C}^2$ we consider the following system

$$\begin{cases} e^{u_0} \cos(u_1) = w_0 \\ e^{u_0} \sin(u_1) = w_1. \end{cases} \tag{5}$$

We have that

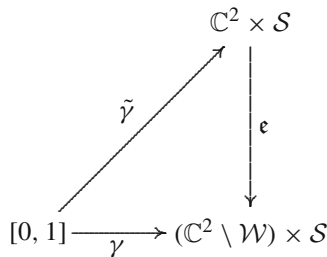
$$w_0 + iw_1 = e^{u_0} e^{iu_1}, \quad w_0 - iw_1 = e^{u_0} e^{-iu_1},$$

and hence

$$\begin{aligned} u_0 &= \frac{\log(w_0 + iw_1) + \log(w_0 - iw_1)}{2} + (h_1 + h_2)i\pi, \\ u_1 &= \frac{\log(w_0 + iw_1) - \log(w_0 - iw_1)}{2i} + (h_1 - h_2)\pi, \end{aligned}$$

with $h_1, h_2 \in \mathbb{Z}$. Therefore, the system in Formula (5) has a solution if and only if $w_0 \pm iw_1 \neq 0$, i.e. if and only if $w_0^2 + w_1^2 \neq 0$. Hence, $\epsilon(\mathbb{C}^2 \times \mathcal{S}) = (\mathbb{C}^2 \setminus \mathcal{W}) \times \mathcal{S}$.

We now pass on to prove that ϵ is a covering map onto its image. Having proved that it is a local diffeomorphism, we are left to prove that the *lifting property* is satisfied, i.e., given a continuous curve $\gamma : [0, 1] \rightarrow (\mathbb{C}^2 \setminus \mathcal{W}) \times \mathcal{S}$, we will show that it is possible to construct a continuous $\tilde{\gamma} : [0, 1] \rightarrow \mathbb{C}^2 \times \mathcal{S}$ such that $\epsilon \circ \tilde{\gamma} = \gamma$, i.e., such that the following diagram commutes.



Given $((\tilde{u}_0, \tilde{u}_1), \tilde{s}) \in \mathbb{C}^2 \times \mathcal{S}$ let us consider a curve $\gamma = ((\gamma_0, \gamma_1), \gamma_S) : [0, 1] \rightarrow (\mathbb{C}^2 \setminus \mathcal{W}) \times \mathcal{S}$, such that $\gamma(0) = \epsilon((\tilde{u}_0, \tilde{u}_1), \tilde{s})$, then $\gamma_0(t)^2 + \gamma_1(t)^2 \neq 0$ for all $t \in [0, 1]$. Thus, if we define

$$\alpha(t) = \gamma_0(t) + i\gamma_1(t), \quad \beta(t) = \gamma_0(t) - i\gamma_1(t),$$

we have that α and β are continuous paths in \mathbb{C}^* . But then, as $z \mapsto e^z$ is a covering map from \mathbb{C} to \mathbb{C}^* , we can construct $\tilde{\alpha}$ and $\tilde{\beta}$ such that

$$e^{\tilde{\alpha}} = \alpha, \quad e^{\tilde{\beta}} = \beta,$$

with $\tilde{\alpha}(0) = \tilde{u}_0 + i\tilde{u}_1$ and $\tilde{\beta}(0) = \tilde{u}_0 - i\tilde{u}_1$.

In conclusion, the path $\tilde{\gamma} : [0, 1] \rightarrow \mathbb{C}^2 \times \mathcal{S}$ defined by

$$\tilde{\gamma}(t) = \left(\left(\frac{\tilde{\alpha}(t) + \tilde{\beta}(t)}{2}, \frac{\tilde{\alpha}(t) - \tilde{\beta}(t)}{2} \right), \gamma_{\mathcal{S}}(t) \right),$$

is continuous, is such that $\tilde{\gamma}(0) = ((\tilde{u}_0, \tilde{u}_1), \tilde{s}) \in \epsilon^{-1}(\gamma(0))$ and $\epsilon \circ \tilde{\gamma} = \gamma$, i.e. the map $\epsilon : \mathbb{C}^2 \times \mathcal{S} \rightarrow (\mathbb{C}^2 \setminus \mathcal{W}) \times \mathcal{S}$ is a covering map. \square

Remark 4.2 As a byproduct of the proof of Theorem 4.1, we get that the fundamental group of $\mathbb{C}^2 \setminus \mathcal{W}$ is isomorphic to \mathbb{Z}^2 ; indeed, for $(h_1, h_2) \in \mathbb{Z}^2$, the monodromy is given by the following action

$$(h_1, h_2) \cdot (u_0, u_1) = (u_0 + (h_1 + h_2)t\pi, u_1 + (h_1 - h_2)\pi).$$

Moreover, notice that as $h_1 + h_2$ and $h_1 - h_2$ have the same parity, this result is coherent with [4, Thm. 1.2] (see the relation between the indices m and n at point (2) in the second bullet of the referred result).

4.2 Back to $\mathbb{C} \otimes \mathbb{H}$

We now move back to the study of ϵ . Let us recall from [5, Sect. 4] the definition of the following two sets:

$$\begin{aligned} V_{-1} &:= \{(z_0, \underline{z}) \in \mathbb{C} \otimes \mathbb{H} \mid z_0^2 + |\underline{z}|^2 = 0\} = \rho(\mathcal{W} \times \mathcal{S}), \\ V_{\infty} &:= \{(z_0, \underline{z}) \in \mathbb{C} \otimes \mathbb{H} \mid |\underline{z}|^2 = 0\} = (\mathbb{C} \otimes \mathbb{H}) \setminus \Omega'. \end{aligned}$$

From these two sets, we define

$$\Omega := \epsilon^{-1}(\mathbb{C} \otimes \mathbb{H} \setminus (V_{-1} \cup V_{\infty})) = \{(z_0, \underline{z}) \in \mathbb{C} \otimes \mathbb{H} \mid |\underline{z}|^2 \neq h^2\pi^2, \text{ for } h \in \mathbb{Z}\}.$$

Notice that Ω is the exact transpose in the context of $\mathbb{C} \otimes \mathbb{H}$ of the set $\mathbb{H} \setminus S(\text{exp})$, where $S(\text{exp})$ is defined in Formula (3). Eventually, recalling that $\epsilon \circ \rho = \rho \circ \epsilon$ and collecting the fact that $\epsilon : \mathbb{C}^2 \times \mathcal{S} \rightarrow (\mathbb{C}^2 \setminus \mathcal{W}) \times \mathcal{S}$ and $\rho : \mathcal{W}' \times \mathcal{S} \rightarrow \Omega'$ are covering maps, we have proved the following theorem.

Theorem 4.2 *The function $\epsilon : \Omega \rightarrow \mathbb{C} \otimes \mathbb{H} \setminus (V_{-1} \cup V_{\infty})$ is a covering map and its monodromy group is isomorphic to \mathbb{Z}^2 .*

The case when $h = 0$ is somehow special as described in the following remark.

Remark 4.3 The further restriction of ϵ to V_{∞} is a covering map onto its image $\epsilon(V_{\infty}) = V_{\infty} \setminus \{(0, \underline{z}) \mid |\underline{z}|^2 = 0\}$. In fact, if $(z_0, \underline{z}) \in V_{\infty}$, i.e. when $h = 0$, we have that $\epsilon(z_0, \underline{z}) = e^{z_0}(1, \underline{z})$ (compare with [1, Cor. 4.6]), which, again, belongs to V_{∞} ; moreover it is easy to see that $(z_0, \underline{z}) \mapsto e^{z_0}(1, \underline{z})$ is a covering map. However, while in this case each element in V_{∞} has a one-parameter family of preimages, thanks to Remark 4.2, in the general case of Theorem 4.2 any element in $\mathbb{C} \otimes \mathbb{H} \setminus (V_{-1} \cup V_{\infty})$

has a two-parameter family of preimages in Ω . So, in a sense, in the case described in this remark, we lose a bunch of preimages. In particular, we have the following isomorphisms of the fundamental groups:

$$\pi_1(\mathbb{C} \otimes \mathbb{H} \setminus V_\infty) \simeq \mathbb{Z}, \quad \pi_1(\mathbb{C} \otimes \mathbb{H} \setminus (V_{-1} \cup V_\infty)) \simeq \mathbb{Z}^2.$$

Thanks to the previous theorem, we can construct global ‘logarithms’ with respect to ε .

Corollary 4.3 *Let \mathcal{U} be a simply connected domain and let $F : \mathcal{U} \rightarrow \mathbb{C} \otimes \mathbb{H} \setminus (V_{-1} \cup V_\infty)$ be a continuous function. Then there exist a two-parameter family of continuous functions $F_{(h_1, h_2)} : \mathcal{U} \rightarrow \mathbb{C} \otimes \mathbb{H}$, for $(h_1, h_2) \in \mathbb{Z}^2$, such that $\varepsilon \circ F_{h_1, h_2} = F$.*

Of course the previous corollary applies, in particular, to stem functions.

5 Global $*$ -Logarithms

In this short section we collect a series of consequences of the previous section, allowing us to define global $*$ -logarithms of a slice regular function $f = \mathcal{I}(F)$ such that $f^s \neq 0 \neq f_v^s$ or, equivalently, such that $F \in \mathbb{C} \otimes \mathbb{H} \setminus (V_{-1} \cup V_\infty)$. As in [5, Sect. 5] we declare the following assumption.

Assumption 5.1 From now on, the set of definition $\mathcal{U} = \overline{\mathcal{U}}$ of our stem functions will be open and simply connected or the union of two simply connected domains (if $\mathcal{U} \cap \mathbb{R} = \emptyset$).

In [9, 11] the quaternionic domains U coming from the sets \mathcal{U} just described in the previous Assumption are called *basic domains*.

Thanks to Corollary 4.3, if f is a slice function, such that its stem function F does not intersect $V_{-1} \cup V_\infty$, we virtually have a two-parameters countable family of $*$ -logarithms. We only need to check that the resulting logarithms at the level of $\mathbb{C} \otimes \mathbb{H}$ are stem functions as well. Later we will see that if the domain of f contains real points, then we lose a parameter, obtaining a closer analogy with the complex case.

As done in [5] for the case of n -th $*$ -powers, given a stem function $F : \mathcal{U} \rightarrow \mathbb{C} \otimes \mathbb{H} \setminus (V_{-1} \cup V_\infty)$, we define the following set

$$\mathcal{G} := \{G : \mathcal{U} \rightarrow \mathbb{C} \otimes \mathbb{H} \mid \varepsilon \circ G = F\}. \tag{6}$$

Since $\varepsilon(\bar{z}) = \overline{\varepsilon(z)}$, if $G \in \mathcal{G}$ then the function $\hat{G} : \mathcal{U} \rightarrow \mathbb{C} \otimes \mathbb{H}$ defined by $\hat{G}(z) = \overline{G(\bar{z})}$ belongs to \mathcal{G} as well, in fact

$$(\varepsilon \circ \hat{G})(z) = \varepsilon(\hat{G}(z)) = \varepsilon(\overline{G(\bar{z})}) = \overline{\varepsilon(G(\bar{z}))} = \overline{F(\bar{z})} = F(z).$$

Note that G is a stem function if and only if $\hat{G} = G$. Therefore, we can prove the following result, the proof of which goes exactly as that of [5, Thm. 5.3].

Corollary 5.2 *Let U be a basic domain such that $U \cap \mathbb{R} = \emptyset$ and let $f : U \rightarrow \mathbb{H}$ be a slice regular function such that $f^s(q) \neq 0 \neq f_v^s(q)$, for all $q \in U$. Then, there exists a two-parameter family of slice functions $f_{(h_1, h_2)} : U \rightarrow \mathbb{H}$, for $(h_1, h_2) \in \mathbb{Z}^2$, such that*

$$\exp_*(f_{h_1, h_2}) = f.$$

Remark 5.1 Following Remark 4.2, if $g = g_0 + g_v$ is a $*$ -logarithm of f , then, for any pair of integers h_1 and h_2 , the function

$$\begin{aligned} &g_0 + (h_1 + h_2)\mathcal{J}\pi + \left(\sqrt{g_v^s} + (h_1 - h_2)\pi\right) \frac{g_v}{\sqrt{g_v^s}} \\ &= g + \left[(h_1 + h_2)\mathcal{J} + (h_1 - h_2) \frac{g_v}{\sqrt{g_v^s}} \right] \pi, \end{aligned}$$

is a $*$ -logarithm of f as well (see [4, Thm. 1.2]).

We now pass on to analyze the case in which the function f is defined on a domain which intersects the real axis. Under this hypothesis, it is clear that the function \mathcal{J} cannot appear in the set of solutions. In fact, as explained later, we will obtain that the two parameters h_1 and h_2 shall be related by the equality $h_1 = -h_2$.

The proof of the following corollary goes as that of [5, Thm. 5.4].

Corollary 5.3 *Let U be a basic domain such that $U \cap \mathbb{R} \neq \emptyset$ and let $f : U \rightarrow \mathbb{H}$ be a slice regular function such that $f^s(q) \neq 0 \neq f_v^s(q)$, for all $q \in U$. Then, there exist a one-parameter family of slice functions $f_h : U \rightarrow \mathbb{H}$, for $h \in \mathbb{Z}$, such that*

$$\exp_*(f_h) = f.$$

Exactly as in [5], thanks to our construction, the previous two corollaries can be stated without the hypothesis of regularity.

Remark 5.2 As anticipated before, if the domain of f contains real points then we loose a one-parameter family of solution. In fact, if $x^0 \in U \cap \mathbb{R}$, then $F(x^0) \in \mathbb{R}^4 \subset \mathbb{C}^4 \simeq \mathbb{C} \otimes \mathbb{H}$ and there exists $y^0 \in \mathbb{R}^4$ such that $\varepsilon(y^0) = F(x^0)$. We have $y^0 = \rho((u_0, u_1), s)$ with $(u_0, u_1) \in \mathbb{R}^2$. Therefore, we obtain

$$\begin{aligned} \varepsilon^{-1}(F(x^0)) &= \{\rho((h_1, h_2) \cdot ((u_0, u_1), s)) \mid (h_1, h_2) \in \mathbb{Z}^2\} \\ &= \{\rho((u_0 + (h_1 + h_2)\iota\pi, u_1 + (h_1 - h_2)\pi), s) \mid (h_1, h_2) \in \mathbb{Z}^2\}, \end{aligned}$$

whose only real points are those obtained when $h_1 + h_2 = 0$ (see the first component), i.e. when $h_2 = -h_1$. To each point y in $\varepsilon^{-1}(F(x_0))$ we associate $G \in \mathcal{G}$ such that $G(x^0) = y$; the G 's described in the previous result are those corresponding to real points in $\varepsilon^{-1}(F(x^0))$.

All the results contained in this paper so far are consistent with those contained in [4, 9]. As already pointed out in the introduction, the main difference here is the idea of using the complex geometry of \mathbb{C}^4 and of $\mathbb{C}^2 \times \mathcal{S}$ to reveal the nature of ε and of ϵ as covering maps. In a broad sense, the strategy of [4] is that of “solving” the $*$ -logarithm mostly in algebraic terms, while in [9] the same problem is addressed by considering a sort of “ $*$ -logarithm variety” and by analyzing the geometry of curves contained in it. As already noted in [5], in our opinion our approach seems to be more suitable to generalizations to other contexts, while giving a global view of the geometric structure lying beneath the specific issue.

We conclude this section by highlighting how it is possible to recover results about the $*$ -roots, starting from the $*$ -logarithm. The starting point is a quite standard argument from one complex variable but, as we will see, the computation of the monodromy needs some deeper investigation.

Remark 5.3 As already said, in [5] we widely studied the existence of a n -th $*$ -root of a slice regular function. Exactly as in the complex case, almost all the work done in [5] can be recovered from the study of the $*$ -logarithm. In fact, if U is a basic domain and $f : U \rightarrow \mathbb{H}$ is a slice function such that $f^s(q) \neq 0 \neq f_v^s(q)$, for all $q \in U$, then, for any $n \in \mathbb{R}$ we are able to define

$$(f)^{*1/n} := \exp_* \left(\frac{1}{n} \log_*(f) \right),$$

where the exponent $*$, means that we are considering the power with respect to the $*$ -product. We will show in the next section how to recover the monodromy of the n -th $*$ -root from that of the $*$ -logarithm.

6 Automorphisms of ϵ and of ε

In this section we will give a description of the deck transformations of ϵ and of ε , i.e. the automorphisms of $\mathbb{C}^2 \times \mathcal{S}$ and of Ω fixing the fibers of ϵ and of ε , respectively. To be precise, given a covering map $\pi : X \rightarrow Y$, we are interested in the set

$$\text{Aut}_\pi := \{f : X \rightarrow X \mid \pi \circ f = \pi\}.$$

In particular, we will study Aut_ϵ and Aut_ε . We recall from [5] that $\text{Aut}_\rho = \{\text{id}, \Gamma\}$, where $\Gamma : \mathbb{C}^2 \times \mathcal{S} \rightarrow \mathbb{C}^2 \times \mathcal{S}$ is given by

$$\Gamma((u_0, u_1), s) = ((u_0, -u_1), -s).$$

Thanks to the content of the previous section we are able to represent Aut_ϵ in a convenient way. In fact, let us define $T_\ell : \mathbb{C}^2 \times \mathcal{S} \rightarrow \mathbb{C}^2 \times \mathcal{S}$ as the function

$$T_\ell((u_0, u_1), s) = ((u_0 + i\pi, u_1 + \ell\pi), s), \quad \ell = -1, 1.$$

Then, following the proof of Theorem 4.1, we have that

$$\text{Aut}_\epsilon = \{T_{(a,b)} := aT_1 + bT_{-1} \mid a, b \in \mathbb{Z}, a \equiv_2 b\}.$$

In particular, given $(h_1, h_2) \in \mathbb{Z}^2$ from Remark 4.2, we get $a = h_1 + h_2$ and $b = h_1 - h_2$, while, given $T_{(a,b)}$, then $h_1 = (a + b)/2$ and $h_2 = (a - b)/2$.

We now pass to study Aut_ϵ . Recall that $\epsilon \circ \rho = \rho \circ \epsilon$ and notice that $\Gamma \circ T_1 = T_{-1} \circ \Gamma$. Therefore, if $(a, b) \in \mathbb{Z}^2$, we get

$$\Gamma \circ (aT_1 + bT_{-1}) = a\Gamma \circ T_1 + b\Gamma \circ T_{-1} = aT_{-1} \circ \Gamma + bT_1 \circ \Gamma = (aT_{-1} + bT_1) \circ \Gamma.$$

Hence, $T_{(a,b)}$ descends to a map $S \in \text{Aut}_\epsilon$, if and only if $(a, b) = (b, a)$, i.e. if and only if $T_{(a,b)} = T_{(a,a)} = a(T_1 + T_{-1})$, $a \in \mathbb{Z}$. But then it follows that, if we define $S_0 : \mathbb{C} \otimes \mathbb{H} \rightarrow \mathbb{C} \otimes \mathbb{H}$ as the map

$$S_0(z_0, \underline{z}) = (z_0 + 2i\pi, \underline{z}),$$

then

$$\text{Aut}_\epsilon = \langle S_0 \rangle_{\mathbb{Z}} \simeq \mathbb{Z}.$$

We have just proven the following result which is analogous to [5, Cor. 6.4 and Prop. 6.5]

Proposition 6.1 *The covering map ϵ is regular, while ε is not.*

6.1 Monodromy of $*$ -roots

We now want to recover the monodromy of $*$ -roots by means of that of the $*$ -logarithm. This computation was already performed in [5] with different techniques. Here we will use what we just learned from the study of the $*$ -logarithm.

Since, for any $z \in \mathbb{C} \otimes \mathbb{H}$, we have that $\varepsilon(2z) = \varepsilon(z)^2$, where sum and product are the algebra operations of $\mathbb{C} \otimes \mathbb{H}$, for any $n \in \mathbb{N}$, then we also have that

$$\varepsilon(nz) = \sigma_n(\varepsilon(z)).$$

Let us recall the set \mathcal{G} from Formula (6). Given $G \in \mathcal{G}$, let $H_G : \mathcal{U} \rightarrow \mathbb{C}^2 \times \mathcal{S}$ be such that $\rho \circ H_G = G$. Then we have

$$\varepsilon \circ \rho \circ T_\ell \circ H_G = \rho \circ \epsilon \circ T_\ell \circ H_G = \rho \circ \epsilon \circ H_G = \varepsilon \circ \rho \circ H_G = \varepsilon \circ G = F.$$

Under the hypotheses of Corollary 4.3, the set \mathcal{G} is a two-parameters family of logarithms with respect to ε . Therefore, we can represent each element of \mathcal{G} as $G_{(a,b)} = \rho \circ T_{(a,b)} \circ H_{\tilde{G}}$, with $(a, b) \in \mathbb{Z}^2$, where $G_{(0,0)} =: \tilde{G}$ is any particular solution of $\varepsilon \circ X = F$.

Now, $\varepsilon \circ (\frac{1}{n}G_{(a,b)}) = \varepsilon \circ (\frac{1}{n}G_{(c,d)})$ with $(a, b), (c, d) \in \mathbb{Z}^2$, if and only if

$$\varepsilon \circ \rho \circ \frac{1}{n}T_{(a,b)} \circ H_{\tilde{G}} = \varepsilon \circ \rho \circ \frac{1}{n}T_{(c,d)} \circ H_{\tilde{G}}.$$

Set $H_{\tilde{G}}(z) = ((u_0(z), u_1(z)), s(z))$, then

$$\frac{1}{n}T_{(a,b)} \circ H_{\tilde{G}} = \left(\left(\frac{1}{n}u_0(z) + \frac{a+b}{n}i\pi, \frac{1}{n}u_1(z) + \frac{a-b}{n}\pi \right), s(z) \right).$$

Assume, also, that $u_1 \neq 0$, then the equality

$$\rho \circ \varepsilon \circ \frac{1}{n}T_{(a,b)} \circ H_{\tilde{G}} = \rho \circ \varepsilon \circ \frac{1}{n}T_{(c,d)} \circ H_{\tilde{G}}$$

holds true if and only if

$$\varepsilon \circ \frac{1}{n}T_{(a,b)} \circ H_{\tilde{G}} = \varepsilon \circ \frac{1}{n}T_{(c,d)} \circ H_{\tilde{G}}$$

which is equivalent to saying that there exists $(e, f) \in \mathbb{Z}^2$ such that

$$T_{(e,f)} \circ \frac{1}{n}T_{(a,b)} \circ H_{\tilde{G}} = \frac{1}{n}T_{(c,d)} \circ H_{\tilde{G}},$$

i.e.

$$\begin{aligned} & \left(\left(\frac{1}{n}u_0(z) + \left((e+f) + \frac{a+b}{n} \right) i\pi, \frac{1}{n}u_1(z) + \left((e-f) + \frac{a-b}{n} \right) \pi \right), s(z) \right) \\ &= \left(\left(\frac{1}{n}u_0(z) + \frac{c+d}{n}i\pi, \frac{1}{n}u_1(z) + \frac{c-d}{n}\pi \right), s(z) \right), \end{aligned}$$

i.e.

$$\begin{aligned} \frac{(c+d) - (a+b)}{n} &= e + f, & \frac{(c-d) - (a-b)}{n} &= e - f \\ \frac{c-a}{n} + \frac{d-b}{n} &= e + f, & \frac{c-a}{n} - \frac{d-b}{n} &= e - f \end{aligned}$$

i.e.

$$\frac{c-a}{n} \in \mathbb{Z}, \quad \frac{d-b}{n} \in \mathbb{Z},$$

i.e. $n|(c-a)$ and $n|(d-b)$, i.e. $(a, b) \equiv (c, d) \pmod{n}$.

Consider the subgroup of automorphisms

$$I_n = \langle T_{(a,b)} \mid a \equiv b \equiv 0 \pmod{n} \rangle$$

and let $W_n = \mathbb{C}^2 \times \mathcal{S}/I_n$; the projection $\epsilon_n : \mathbb{C}^2 \times \mathcal{S} \rightarrow W_n$ is a covering map. As I_n is a subgroup of Aut_ϵ and as ϵ is a Galois covering, we can factor ϵ via ϵ_n : we consider the map $\mathfrak{s}_n : W_n \rightarrow (\mathbb{C}^2 \setminus \mathcal{W}) \times \mathcal{S}$ such that $\epsilon = \mathfrak{s}_n \circ \epsilon_n$; the map \mathfrak{s}_n is in fact a covering map of degree n^2 (the index of I_n in Aut_ϵ).

As Aut_ρ does not intersect Aut_ϵ , we can induce a map $\tilde{\rho} : W'_n \rightarrow \Omega_n$ on a suitable open set $W'_n \subseteq W_n$ such that

- $\tilde{\rho}$ is a double cover,
- there is a (unique) covering map $\sigma_n : \Omega_n \rightarrow (\mathbb{C} \otimes \mathbb{H}) \setminus (V_{-1} \cup V_\infty)$ with $\sigma_n \circ \tilde{\rho} = \rho \circ \mathfrak{s}_n$,
- there is a (unique) covering map $\epsilon_n : \Omega \rightarrow \Omega_n$ such that $\epsilon = \sigma_n \circ \epsilon_n$.

It is easy to see that $\epsilon_n((u_0, u_1), s) = ((u_0/n, u_1/n), s)$, $\epsilon_n(z) = \epsilon(z/n)$, $\tilde{\rho} = \rho$, $\sigma_n(z) = z^n$ (and the induced \mathfrak{s}_n) satisfy the previous requirements. Therefore, W'_n and Ω_n can be realized as subdomains of $\mathbb{C}^2 \times \mathcal{S}$ and $\mathbb{C} \otimes \mathbb{H}$ respectively.

Carrying out the computations, one can find the definitions given in [5].

Finally, we compute the monodromy of \mathfrak{s}_n . By simple arithmetic, the group $\mathbb{Z}_n \times \mathbb{Z}_n = \mathbb{Z}^2/I$ is generated by the classes $[(1, 1), (1, -1)]$ when n is odd and by the classes $[(1, 1), (1, -1), (1, 0)]$ when n is even.

Given $a, b \in \{0, \dots, n - 1\}$

$$\begin{aligned} \epsilon_n(T_{(a,b)}((u_0, u_1), s)) &= \epsilon \left(\left(\frac{1}{n}u_0 + \frac{a+b}{n}i\pi, \frac{1}{n}u_1 + \frac{a-b}{n}\pi \right), s \right) \\ &= \left(\left(e^{\frac{u_0}{n}} e^{\frac{a+b}{n}i\pi} \cos \left(\frac{u_1}{n} + \frac{a-b}{n}\pi \right), \right. \right. \\ &\quad \left. \left. e^{\frac{u_0}{n}} e^{\frac{a+b}{n}i\pi} \sin \left(\frac{u_1}{n} + \frac{a-b}{n}\pi \right) \right), s \right) \\ &= \xi \cdot \left(A_\eta \frac{(u_0, u_1)}{n}, s \right) \end{aligned}$$

where $\xi = e^{\frac{a+b}{n}i\pi}$ is a n -th root of unity and

$$A_\eta = \begin{pmatrix} \cos(\frac{a-b}{n}\pi) & -\sin(\frac{a-b}{n}\pi) \\ \sin(\frac{a-b}{n}\pi) & \cos(\frac{a-b}{n}\pi) \end{pmatrix}$$

is the 2×2 matrix representation of the complex number $\eta = e^{\frac{a-b}{n}i\pi}$.

So, for n odd, the generators of the deck transformations of \mathfrak{s}_n are ξ (corresponding to $[(1, 1)]$) and A_η (corresponding to $[(1, -1)]$) with ξ, η primitive n -th roots of 1; for n even we have these two and $\xi \cdot A_\eta$ (corresponding to $[(1, 0)]$) with ξ, η primitive $2n$ -th roots of 1.

7 The Product of Two $*$ -Exponentials

In this section we will give sufficient conditions for the product of two exponentials to be an exponential. This topic clearly deals with the so-called Baker–Campbell–Hausdorff (BCH) formula for the $*$ -exponential.

In its more general formulation the BCH formula states that, whenever it exists, the product $e^X e^Y$ equals e^Z , where

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots \tag{7}$$

Clearly, depending on the context, it is possible to give sufficient conditions for the sum on the right hand side of Formula (7) to be convergent (see for instance [6, Prop. 2.2] for Banach algebras or [8] for a general overview). In the context of quaternions, the situation is much clearer: if $p = p_0 + p_v$ and $q = q_0 + q_v$, with $p_v \neq 0 \neq q_v$, then

$$\begin{aligned} \exp(p) \exp(q) &= \left[e^{p_0} \left(\cos |p_v| + \sin |p_v| \frac{p_v}{|p_v|} \right) \right] \left[e^{q_0} \left(\cos |q_v| + \sin |q_v| \frac{q_v}{|q_v|} \right) \right] \\ &= e^{p_0+q_0} \left[\cos |p_v| \cos |q_v| - \sin |p_v| \sin |q_v| \left\langle \frac{p_v}{|p_v|}, \frac{q_v}{|q_v|} \right\rangle + \right. \\ &\quad \left. + \cos |p_v| \sin |q_v| \frac{q_v}{|q_v|} + \cos |q_v| \sin |p_v| \frac{p_v}{|p_v|} \right. \\ &\quad \left. + \sin |p_v| \sin |q_v| \frac{p_v}{|p_v|} \wedge \frac{q_v}{|q_v|} \right] \\ &= \exp(w_0 + w_v), \end{aligned}$$

where $w_0 = p_0 + q_0$ and w_v solves

$$\begin{cases} \cos |w_v| = \cos |p_v| \cos |q_v| - \sin |p_v| \sin |q_v| \left\langle \frac{p_v}{|p_v|}, \frac{q_v}{|q_v|} \right\rangle, \\ \sin |w_v| \frac{w_v}{|w_v|} = \cos |p_v| \sin |q_v| \frac{q_v}{|q_v|} + \cos |q_v| \sin |p_v| \frac{p_v}{|p_v|} + \sin |p_v| \sin |q_v| \frac{p_v}{|p_v|} \wedge \frac{q_v}{|q_v|}. \end{cases}$$

Notice that, in this case, the existence of the solution is ensured because $\exp(p) \exp(q) \neq 0$ and hence it is possible to define w .

Now, as \mathbb{H} and $\mathbb{C} \otimes \mathbb{H}$ have the same algebraic structure, the same equalities hold true for the complexification. However, recall that the euclidean norm must be translated into its purely algebraic form, i.e., if $u = u_0 + \underline{u}$, $u' = u'_0 + \underline{u}' \in \mathbb{C} \otimes \mathbb{H}$, then

$$\begin{aligned} \exp(u) \exp(u') &= e^{u_0+u'_0} \left[\cos \sqrt{|\underline{u}|^2} \cos \sqrt{|\underline{u}'|^2} - \sin \sqrt{|\underline{u}|^2} \sin \sqrt{|\underline{u}'|^2} \left\langle \frac{\underline{u}}{\sqrt{|\underline{u}|^2}}, \frac{\underline{u}'}{\sqrt{|\underline{u}'|^2}} \right\rangle + \right. \\ &\quad \left. + \cos \sqrt{|\underline{u}|^2} \sin \sqrt{|\underline{u}'|^2} \frac{\underline{u}'}{\sqrt{|\underline{u}'|^2}} + \cos \sqrt{|\underline{u}'|^2} \sin \sqrt{|\underline{u}|^2} \frac{\underline{u}}{\sqrt{|\underline{u}|^2}} + \right. \\ &\quad \left. + \sin \sqrt{|\underline{u}|^2} \sin \sqrt{|\underline{u}'|^2} \frac{\underline{u}}{\sqrt{|\underline{u}|^2}} \wedge \frac{\underline{u}'}{\sqrt{|\underline{u}'|^2}} \right]. \end{aligned}$$

In this case, the solution $p = p_0 + \underline{p} \in \mathbb{C} \otimes \mathbb{H}$ of the equation $\exp(u) \exp(u') = \exp(p)$ exists provided $\exp(u) \exp(u') \in \mathbb{C} \otimes \mathbb{H} \setminus (V_{-1} \cup V_\infty)$. From the previous computations and also from [1, Thm. 4.14], we already know that if u commutes with u' or, if $|\underline{u}|^2 = \pi^2 n^2$ and $|\underline{u}'|^2 = \pi^2 m^2$ with n and m satisfying a certain parity condition, then $\exp(u) \exp(u') = \exp(u + u')$. Therefore, we are interested in understanding when these conditions are satisfied excluding the cases listed in the already mentioned result [1, Thm. 4.14]. In order to proceed, we need a couple of preliminary lemmas.

Lemma 7.1 *Let $\underline{z} \in \mathbb{C} \otimes \text{Im}(\mathbb{H})$ be such that $|\underline{z}|^2 \neq 0$. Then, for any $\underline{w} \in \mathbb{C} \otimes \text{Im}(\mathbb{H})$, there exist $w_1 \in \mathbb{C}$ and $w_\perp \in \mathbb{C} \otimes \text{Im}(\mathbb{H})$, such that*

$$\underline{w} = w_1 \underline{v} + w_\perp, \quad \text{and} \quad \langle \underline{z}, w_\perp \rangle = 0.$$

Moreover, we have

$$|\langle \underline{z} \wedge w_\perp \rangle|^2 = |\underline{z}|^2 |w_\perp|^2.$$

Proof By standard linear algebra, it is sufficient to define

$$w_1 = \frac{\langle \underline{w}, \underline{z} \rangle}{\langle \underline{z}, \underline{z} \rangle} = \frac{\langle \underline{w}, \underline{z} \rangle}{|\underline{z}|^2}.$$

□

Now, if $z, w \in \mathbb{C} \otimes \mathbb{H}$, then $(zw)_0^2 + |\langle zw \rangle|^2 = (z_0^2 + |\underline{z}|^2)(w_0^2 + |\underline{w}|^2)$ (see Formula (2)), therefore, if $z_0^2 + |\underline{z}|^2 \neq 0 \neq w_0^2 + |\underline{w}|^2$, then $(zw)_0^2 + |\langle zw \rangle|^2 \neq 0$. Hence, the following result completes the characterization we are looking for.

Theorem 7.2 *Let $z, w \in \mathbb{C} \otimes \mathbb{H} \setminus (V_{-1} \cup V_\infty)$, then, $|\langle zw \rangle|^2 = 0$ if and only if*

$$w_0 = -z_0 w_1 \pm \sqrt{-1} \sqrt{\frac{z_0^2 + |\underline{z}|^2}{|\underline{z}|^2}} \sqrt{|w_\perp|^2},$$

where w_1 and w_\perp are the elements defined in Lemma 7.1.

Proof First of all, thanks to Lemma 7.1, we can write $\underline{w} = w_1 \underline{z} + w_\perp$, with $w_1 \in \mathbb{C}$ and $\langle w_\perp, \underline{z} \rangle = 0$. Therefore,

$$\begin{aligned} zw &= (z_0 + \underline{z})(w_0 + w_1 \underline{z} + w_\perp) \\ &= (z_0 w_0 - |\underline{z}|^2 w_1) + (z_0 w_1 \underline{z} + z_0 w_\perp + w_0 \underline{z} + \underline{z} \wedge w_\perp), \end{aligned}$$

and so

$$\begin{aligned} |\langle zw \rangle|^2 &= |\langle z_0 w_1 \underline{z} + z_0 w_\perp + w_0 \underline{z} + \underline{z} \wedge w_\perp \rangle|^2 \\ &= |\langle z_0 w_1 \underline{z} + z_0 w_\perp + w_0 \underline{z} \rangle|^2 + |\langle \underline{z} \wedge w_\perp \rangle|^2 \end{aligned}$$

$$= (z_0 w_1 + w_0)^2 |z|^2 + (z_0^2 + |z|^2) |w_\perp|^2.$$

Therefore, $|(zw)|^2 = 0$ if and only if $(z_0 w_1 + w_0)^2 |z|^2 + (z_0^2 + |z|^2) |w_\perp|^2 = 0$, which is equivalent to

$$(z_0 w_1 + w_0)^2 = -\frac{z_0^2 + |z|^2}{|z|^2} |w_\perp|^2,$$

and hence, we get the result. □

The previous result can be applied to slice regular functions recalling their decomposition in “scalar–vector” parts. We start by rewriting Lemma 7.1 in terms of slice regular functions.

Corollary 7.3 *Let $f = f_0 + f_v : U \rightarrow \mathbb{H}$ be a slice regular function such that, for any $q \in U$, $f_v^s(q) \neq 0$. Then, for any slice regular function $g = g_0 + g_v$, there exist two slice regular functions g_1, g_\perp, g_1 being slice preserving, such that*

$$g = g_1 f_v + g_\perp, \quad \text{and} \quad \langle f_v, g_\perp \rangle = 0.$$

Moreover, we have

$$(f_v \wedge g_\perp)^s = f_v^s g_\perp^s.$$

Proof Exactly as in the previous result, for any $q \in U$ such that $f_v^s(q) \neq 0$ it is sufficient to define $g_1 := \frac{\langle g_v, f_v \rangle}{f_v^s}$. Assume now that $q_0 = \alpha_0 + i\beta_0 \notin \mathbb{R}$ and $f_v^s(q_0) = 0$. Define $D_{q_0}(\epsilon)$ as the disk in $U \cap \mathbb{C}_i$ centered at q_0 with radius ϵ , such that $\overline{D_{q_0}(\epsilon)} \subset U \cap \mathbb{C}_i$ such that $f_v^s(q) \neq 0$ for any $q \in \overline{D_{q_0}(\epsilon)} \setminus \{q_0\}$. Then we can define $g_1(q_0)$ by means of the Cauchy formula

$$g_1(q_0) := \frac{1}{2\pi i} \int_{\partial D_{q_0}(\epsilon)} \frac{g_1(\alpha + i\beta)}{\alpha + i\beta - q_0} d(\alpha + i\beta).$$

By repeating the same construction at q_0^c and using the Representation Formula, we obtain the result. The previous argument can be performed at $q_0 \in \mathbb{R}$. □

Corollary 7.4 *Let f, g be two slice regular functions defined on U such that $f^s \neq 0 \neq g^s$ and $f_v^s \neq 0 \neq g_v^s$, then, $(f * g)_v^s(q) = 0$ if and only if*

$$(f_0(q)g_1(q) + g_0(q))^2 + \frac{f^s(q)}{f_v^s(q)} g_\perp^s(q) = 0 \tag{8}$$

where g_1 and g_\perp are the functions defined in Corollary 7.3. In particular, if $U \cap \mathbb{R} \neq \emptyset$, then there exists an open neighborhood U of $U \cap \mathbb{R}$ such that $(f * g)_v^s|_U \neq 0$.

Proof The first part of the statement is a direct consequence of Theorem 7.2. For the second part, assume that $x \in U \cap \mathbb{R}$. Then Formula (8) evaluated at x , gives no solutions since the left hand side is strictly positive. Therefore, there exists an open neighborhood U of $U \cap \mathbb{R}$ where the function $(f_0g_1 + g_0)^2 + \frac{f^s}{f_v^s}g_\perp^s$ is never vanishing, and hence $(f * g)_v^s \neq 0$ on U . \square

Thanks to the previous two corollaries we can reverse engineer several examples of slice regular functions f, g with $f^s \neq 0 \neq g^s$ and $f_v^s \neq 0 \neq g_v^s$ but $(f * g)_v^s(q) = 0$.

Example 7.1 Assume for simplicity that $U \cap \mathbb{R} = \emptyset$. Then, given f satisfying the hypotheses of the previous corollary, we define $g = g_0 + g_v = g_0 + g_1f_v + g_\perp$ as follows:

$$g_0 = -f_0g_1 \pm \mathcal{J} \sqrt{\frac{f^s}{f_v^s}} \sqrt{g_\perp^s},$$

with g_1 and g_\perp be such that $g_v^s = g_1^2f_v^s + g_\perp^s \neq 0$ and $g_0^2 + g_1^2f_v^s + g_\perp^s \neq 0$. Clearly if $g_\perp^s \equiv 0$, then it is sufficient to take $g = g_1(-f_0 + f_v) + g_\perp$. For instance, if f is \mathbb{C}_i -preserving, i.e. $f = f_0 + f_1i$, with $f_0^2 + f_1^2 \neq 0 \neq f_1$, then, if $g = -f^c + \ell_{+,i} * j$, we have that $f * g = -f^s + f * \ell_{+,i} * j$, $(f * g)_v = f * \ell_{+,i} * j$ and, therefore $(f * g)_v^s \equiv 0$.

Another possible case is when $f_0 \equiv 0$. In this case, given $f = f_v$, it is sufficient to consider $g = \pm \mathcal{J} \sqrt{g_\perp^s} + g_1f_v + g_\perp$, with $g^s = g_1^2f_v^s \neq 0$, i.e. $g_1 \neq 0$, and $g_1^2f_v^s + g_\perp^s \neq 0$.

Clearly, the previous example allows us to construct functions f and g , such that $(f * g)_v^s \equiv 0$, while the condition in Corollary 7.4 is given pointwise.

At this point we are able to give sufficient conditions for f and g in order to have that $\exp_*(f) * \exp_*(g)$ is an exponential function. As we said, this happens if $(\exp_*(f) * \exp_*(g))_v^s(q) \neq 0$ for all q . Clearly, we can separate the ‘‘scalar’’ part of $\exp_*(f)$ and of $\exp_*(g)$ and only consider $(\exp_*(f)_v * \exp_*(g)_v)_v^s$.

We get the following result.

Corollary 7.5 *Let $f, g : U \rightarrow \mathbb{H}$ be slice regular functions such that f_v does not commute with g_v and for all $q \in U$ $f_v^s(q), g_v^s(q) \notin \{\pi^2n^2 \mid n \in \mathbb{Z}\}$. Write $g_v = g_1 \frac{f_v}{\sqrt{f_v^s}} + g_\perp$. If for any $q \in U$*

$$\left(g_1 \cos \sqrt{f_v^s} \sin \sqrt{g_v^s} + \cos \sqrt{g_v^s} \right)^2 \left(\sin \sqrt{f_v^s} \right)^2 + \left(\sin \sqrt{g_v^s} \right)^2 g_\perp^s \neq 0,$$

then there exists a slice regular function $h : U \rightarrow \mathbb{H}$ such that

$$\exp_*(f) * \exp_*(g) = \exp_*(h).$$

Notice that in the last result the function $h = h_0 + h_v$ is determined by $h_0 = f_0 + g_0$ and h_v is a solution to the system

$$\begin{cases} \cos \sqrt{h_v^s} = \cos \sqrt{f_v^s} \cos \sqrt{g_v^s} - \sin \sqrt{f_v^s} \sin \sqrt{g_v^s} \left\langle \frac{f_v}{\sqrt{f_v^s}}, \frac{g_v}{\sqrt{g_v^s}} \right\rangle, \\ \sin \sqrt{h_v^s} \frac{h_v}{\sqrt{h_v^s}} = \cos \sqrt{f_v^s} \sin \sqrt{g_v^s} \frac{g_v}{\sqrt{g_v^s}} + \cos \sqrt{g_v^s} \sin \sqrt{f_v^s} \frac{f_v}{\sqrt{f_v^s}} \\ \quad + \sin \sqrt{f_v^s} \sin \sqrt{g_v^s} \frac{f_v}{\sqrt{f_v^s}} \wedge \frac{g_v}{\sqrt{g_v^s}}. \end{cases}$$

8 Slice Derivative of the $*$ -Exponential

In this section we will provide a formula for the slice derivative of $\exp_*(f)$, f being a slice regular function. As in the previous section, let us begin with a short description of the general algebraic case. If X is a matrix, the differential of e^X , is given by the following formula

$$e^{-X} de^X = dX - \frac{1}{2!} [X, dX] + \frac{1}{3!} [X, [X, dX]] - \frac{1}{4!} [X, [X, [X, dX]]] + \dots \quad (9)$$

Assume now that $q : [0, 1] \rightarrow \mathbb{H}$ is a differentiable curve and denote $\dot{q} = dq/dt$. Therefore, Formula 9 can be written in the quaternionic setting as

$$\begin{aligned} e^{-q(t)} \frac{de^{q(t)}}{dt} &= \dot{q}(t) - \frac{1}{2!} [q(t), \dot{q}(t)] + \frac{1}{3!} [q(t), [q(t), \dot{q}(t)]] \\ &\quad - \frac{1}{4!} [q(t), [q(t), [q(t), \dot{q}(t)]]] + \dots \\ &= \dot{q}(t) + \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m!} [q^{(m-1)}, \dot{q}](t), \end{aligned}$$

where $[q^{(n)}, \dot{q}](t)$ stands for

$$\underbrace{[q(t)[q(t)[\dots [q(t), \dot{q}(t)]]]}_{n \text{ times}}.$$

Therefore,

$$e^{-q(t)} \frac{de^{q(t)}}{dt} = \dot{q}(t) - \sum_{h=1}^{\infty} \frac{1}{(2h)!} [q^{(2h-1)}, \dot{q}](t) + \sum_{h=1}^{\infty} \frac{1}{(2h+1)!} [q^{(2h)}, \dot{q}](t).$$

Now, as for any $p, q \in \mathbb{H}$, we have that $[p, q] = 2p \wedge q = 2p_v \wedge q_v$. Then we have

$$\begin{aligned} [q, \dot{q}] &= 2q_v \wedge \dot{q}_v \\ [q^{(2)}, \dot{q}] &= 2^2 [\langle q_v, \dot{q}_v \rangle q_v - |q_v|^2 \dot{q}_v] \\ [q^{(3)}, \dot{q}] &= 2^3 (-1) |q_v|^2 q_v \wedge \dot{q}_v \end{aligned}$$

$$\begin{aligned}
 [q^{(4)}, \dot{q}] &= 2^4(-1)|q_v|^2[\langle q_v, \dot{q}_v \rangle q_v - |q_v|^2 \dot{q}_v] \\
 [q^{(5)}, \dot{q}] &= 2^5(-1)^2(|q_v|^2)^2 q_v \wedge \dot{q}_v \\
 [q^{(6)}, \dot{q}] &= 2^6(-1)^2(|q_v|^2)^2[\langle q_v, \dot{q}_v \rangle q_v - |q_v|^2 \dot{q}_v] \\
 &\vdots
 \end{aligned}$$

where, in order to simplify the notation, we removed the dependence from the parameter t . Hence,

$$\begin{aligned}
 e^{-q} \frac{de^q}{dt} &= \dot{q} - \left[\sum_{h=1}^{\infty} \frac{(-1)^{h-1} 2^{2h-1}}{(2h)!} |q_v|^{2(h-1)} \right] q_v \wedge \dot{q}_v \\
 &\quad + \left[\sum_{h=1}^{\infty} \frac{(-1)^{h-1} 2^{2h}}{(2h+1)!} |q_v|^{2(h-1)} \right] [\langle q_v, \dot{q}_v \rangle q_v - |q_v|^2 \dot{q}_v] \\
 &= \dot{q} - \frac{\sin^2(|q_v|)}{|q_v|^2} q_v \wedge \dot{q}_v + \frac{|q_v| - \cos(|q_v|) \sin(|q_v|)}{|q_v|^3} [\langle q_v, \dot{q}_v \rangle q_v - |q_v|^2 \dot{q}_v] \\
 &= \dot{q} - \frac{1 - \cos(2|q_v|)}{2|q_v|} \frac{q_v}{|q_v|} \wedge \dot{q}_v + \left[1 - \frac{\sin(2|q_v|)}{2|q_v|} \right] \left[\left\langle \frac{q_v}{|q_v|}, \dot{q}_v \right\rangle \frac{q_v}{|q_v|} - \dot{q}_v \right] \\
 &= \dot{q} + \left[1 - \frac{\sin(2|q_v|)}{2|q_v|} \right] \left[\left\langle \frac{q_v}{|q_v|}, \dot{q}_v \right\rangle \frac{q_v}{|q_v|} - \dot{q}_v \right] - \frac{1 - \cos(2|q_v|)}{2|q_v|} \frac{q_v}{|q_v|} \wedge \dot{q}_v.
 \end{aligned}$$

Clearly, if $q_v = 0$, we obtain the usual formula. Moreover, if $q_v \neq 0$ and \dot{q}_v commutes with q_v , i.e. there exists a real valued function α such that $\dot{q}_v = \alpha q_v$, then

$$e^{-q} \frac{de^q}{dt} = \dot{q},$$

as expected.

If we write $\dot{q} = q_1 \frac{q_v}{|q_v|} + q_{\perp}$, then we obtain

$$e^{-q} \frac{de^q}{dt} = \dot{q} - \frac{1 - \cos(2|q_v|)}{2|q_v|} \frac{q_v}{|q_v|} \wedge q_{\perp} - \left[1 - \frac{\sin(2|q_v|)}{2|q_v|} \right] q_{\perp} \tag{10}$$

$$= \dot{q}_0 + q_1 \frac{q_v}{|q_v|} + \frac{\sin(2|q_v|)}{2|q_v|} q_{\perp} - \frac{1 - \cos(2|q_v|)}{2|q_v|} \frac{q_v}{|q_v|} \wedge q_{\perp}. \tag{11}$$

Now, exactly as we have done before, this last relation extends to the complexification of \mathbb{H} , where in place of a curve $q : [0, 1] \rightarrow \mathbb{H}$ we consider a complex curve $F : D \subset \mathbb{C} \rightarrow \mathbb{C} \otimes \mathbb{H}$, the derivative with respect to t is changed into the derivative with respect to $z \in D$ and the usual exponential function is changed into ε . After these modifications we have the following formula.

Proposition 8.1 *Let $f : U \rightarrow \mathbb{H}$ be a slice regular function. Then we have the following formula*

$$\begin{aligned} \exp_*(f)^{-*} * \partial_c(\exp_*(f)) &= \partial_c f + \left[\sum_{h=1}^{\infty} \frac{(-1)^{h-1} 2^{2h}}{(2h+1)!} (f_v^s)^{(h-1)} \right] \\ &\quad [\langle f_v, (\partial_c f)_v \rangle_* f_v - f_v^s (\partial_c f)_v] + \\ &\quad - \left[\sum_{h=1}^{\infty} \frac{(-1)^{h-1} 2^{2h-1}}{(2h)!} (f_v^s)^{(h-1)} \right] f_v \mathbb{A} (\partial_c f)_v. \end{aligned}$$

From this proposition it is possible to derive some convenient corollaries.

Corollary 8.2 *Let $f : U \rightarrow \mathbb{H}$ be a slice regular function and let $q_0 \in U$ be any point. If $f_v^s(q_0) = 0$, then*

$$\begin{aligned} \left(\exp_*(f)^{-*} * \partial_c(\exp_*(f)) \right)(q_0) &= (\partial_c f)(q_0) - (f_v \mathbb{A} (\partial_c f)_v)(q_0) \\ &\quad + \frac{2}{3} \left(\langle f_v, (\partial_c f)_v \rangle_* \right)(q_0) f_v(q_0). \end{aligned}$$

In the case when f_v^s is never-vanishing, many equivalent formulas can be derived.

Corollary 8.3 *Let $f : U \rightarrow \mathbb{H}$ be a slice regular function such that f_v^s is never-vanishing, then we have*

$$\begin{aligned} \partial_c(\exp_*(f)) &= \exp_*(f) * \left\{ \partial_c f + \left[1 - \frac{\sin(2\sqrt{f_v^s})}{2\sqrt{f_v^s}} \right] \left[\left\langle \frac{f_v}{\sqrt{f_v^s}}, (\partial_c f)_v \right\rangle \frac{f_v}{\sqrt{f_v^s}} - (\partial_c f)_v \right] + \right. \\ &\quad \left. - \frac{1 - \cos(2\sqrt{f_v^s})}{2\sqrt{f_v^s}} \frac{f_v}{\sqrt{f_v^s}} \mathbb{A} (\partial_c f)_v \right\}. \end{aligned}$$

As mentioned before, the formula contained in the last corollary is just one of the possible generalizations we have seen in this section. In the same spirit, it is clearly possible to generalize Formula (10) or (11).

Remark 8.1 Many of the previous formulas can also be related to the function $v : \mathbb{H} \rightarrow \mathbb{H}$ introduced in [4, Def. 2.16] as

$$v(q) = \sum_{m \in \mathbb{N}} \frac{(-1)^m q^{2m+1}}{(2m+1)!},$$

and noticing that $v(q^2)q = \sin(q)$.

Remark 8.2 Exactly as in the case of a quaternionic curve, even in this case the formula for the slice derivative of the $*$ -exponential of a slice regular function simplifies to

the usual one when $(\partial_c f)_v$ and f_v commute, i.e., getting rid of the trivial cases, when there exists a slice preserving function γ such that

$$(\partial_c f)_v = \gamma f_v.$$

Examples of functions satisfying this relation are slice constant functions (i.e. functions with everywhere vanishing slice derivative), \mathbb{C}_I -preserving function (for any $I \in \mathbb{S}$) or functions of the form $f_v = \exp(\gamma(q)q)c$, where c is any purely imaginary quaternion.

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