

Distinguishing electroweak EFTs with $W_L W_L \rightarrow n \times h$ Raquel Gómez-Ambrosio *Dipartimento di Fisica “G. Occhialini,” Università degli Studi di Milano-Bicocca and INFN,
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(Received 19 April 2022; accepted 15 August 2022; published 20 September 2022)

The electroweak symmetry-breaking sector is one of the most promising and uncharted parts of the Standard Model (SM); but it seems likely that new electroweak physics may be out of reach of the present accelerator effort and the hope is to observe small deviations from the SM. Given that, effective field theory (EFT) becomes the logic method to use, and Standard Model effective field theory (SMEFT) has become the standard. However, the most general theory with the known particle content is Higgs effective field theory (HEFT), and whether SMEFT suffices should be investigated in future experimental efforts. Building on investigations by other groups that established geometric criteria to distinguish SMEFT from HEFT (useful for theorists examining specific beyond-SM completions), we seek more phenomenological understanding and present an analogous discussion aimed at a broader audience. We discuss various aspects of (multi)Higgs boson production from longitudinal electroweak gauge bosons $W_L W_L \rightarrow n \times h$ in the TeV region as the necessary information to characterize the flare function, $\mathcal{F}(h)$, that determines whether SMEFT or HEFT is needed. We also present tree-level amplitudes including contact and exchange channels, as well as a short discussion on accessing \mathcal{F} from the statistical limit of many bosons. We also discuss the status of the coefficients of the series expansion of $\mathcal{F}(h)$, its validity, whether its complex- h extension can be used to predict or not a tell-tale zero, and how they relate to the dimension-six and -eight SMEFT operators in the electroweak sector. We derive a set of new correlations among beyond the standard model corrections to the HEFT coefficients that help decide, from experimental data, whether we have a viable SMEFT. This analysis can be useful for machines beyond the LHC that could address the challenging final state with several Higgs bosons.

DOI: [10.1103/PhysRevD.106.053004](https://doi.org/10.1103/PhysRevD.106.053004)**I. INTRODUCTION**

Two effective field theories (EFTs) have come to the fore in trying to extend the successful Standard Model in the ignorance of which new physics may be present at a higher energy (if there is any). There are many aspects of the Standard Model that can be pursued at accelerators (the many-parameter flavor structure in both lepton and quark sectors, the Higgs couplings to the fermions, the CP violating phases, QCD processes, ...) but at the energy frontier the most important aspects of physics that are being clarified right now are the nature of the mechanism of electroweak (EW) symmetry breaking: whether it happens as the well-known discussion in the Standard Model

paradigm, or whether new particles or interactions influence the global $SU(2) \times SU(2) \rightarrow SU(2)$ breaking pattern that is at the crux of electroweak interactions. Several extended works have dealt with many of these aspects, for example [1–6].

Throughout most of this paper we will discuss these theories in a regime where the energies of the scattered particles are much higher than their masses $m_h \ll E \ll \Lambda$. A bit surprisingly perhaps for Standard Model practitioners, in such a regime and in the presence of new physics that would yield derivative couplings with $\partial \sim E \gg m_h$, the much discussed Higgs potential $V(h)$ is actually a correction that does not play the pivotal role it enjoys in the SM.

For a while, it was often stated that the Standard Model effective field theory (SMEFT) [7–12] and the Higgs effective field theory (HEFT) [13–18] must encode similar physics, just in different coordinates, with HEFT being perhaps a bit more general because it does not incorporate the Higgs boson h into an $SU(2)$ multiplet.

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However, the work of the San Diego and Oregon groups [19–22] has sharpened the differences between both formulations. Every Lagrangian in the shape in SMEFT form, such as that in Eq. (1) below, can be recast in HEFT form. An example in the relevant energy region is given in Eq. (3).

The converse statement need not always be true. From the work at San Diego it has become apparent that this can be achieved only if a certain function $\mathcal{F}(h)$ presented shortly in Eq. (4) has a zero for some real value of the classical field h , $\exists h_* \in \mathbb{R} | \mathcal{F}(h_*) = 0$. (The precise and complete conditions as presently understood for this SMEFT \leftrightarrow HEFT equivalence are presented in Sec. II.) This function controls high-energy processes with multiple Higgs bosons in the final state: achieving a good control over it requires measurements with an increasingly large number of them, which would look in a detector like a flare of Higgs bosons, whence the name of \mathcal{F} .

However, this is quite an obscure geometric requirement removed from accelerator physics. In recent work [23], an attempt has been made to bring up a phenomenological connection between an eventual breaking of unitarity at some scale and the Lagrangian to be chosen. Aware that such unitarity failure is a feature of perturbation theory in Taylor-series form, and that it does not occur if instead one expands the inverse partial-wave scattering amplitude [24], so that no information distinguishing the effective Lagrangians can really be gained that easily, we here continue exploring what relevant phenomenology there is.

A summary of our present understanding is given by the following scheme:

$$\left(\begin{array}{c} \text{Valid} \\ \text{SMEFT} \end{array} \right) \Rightarrow \left(\begin{array}{c} \text{Double zero} \\ \text{of } \mathcal{F}(h) \\ \text{at some } h_* \end{array} \right) \Rightarrow \left(\begin{array}{c} \text{Specific correlations} \\ \text{between } a_i \\ \text{coefficients} \\ \text{from expanding } \mathcal{F}(h) \\ \text{around } h=0 \end{array} \right)$$

Both aspects, the double zero of $\mathcal{F}(h)$ (see Sec. III C, for example) and the possibility of using correlations between a_i coefficients to distinguish SMEFT from HEFT from experimental data (see Sec. IV A 2) have been separately discussed in the past years. We here provide an integrated discussion with full detail, putting less weight on geometric aspects and more in field-theory and particle physics ones, more familiar to the typical reader, and make several new contributions. Finally, it is also worth commenting that similar relations exist between the coefficients of the nonderivative V_{HEFT} potential, since a valid SMEFT description of V must obey a series of conditions analogous to those for the flare function $\mathcal{F}(h)$ [22]. Assuming SMEFT’s validity would then also impose important correlations between the coefficients of the potential—trilinear, quartic, etc.—(see e.g., [25] for a HEFT phenomenological analysis). This is mostly beyond the scope of this article and

will certainly be studied in future work, but we present a slim discussion in Appendix C.

The rest of this article is organized as follows: we continue with introductory material in Secs. IA and IB, where some superficial differences between the electro-weak SMEFT and HEFT theories are studied at the level of the Lagrangian. Section II is devoted to identifying the effective operators in SMEFT (of dimension-six and -eight) with leading TeV-scale contributions and to expressing them in HEFT coordinates. In so doing we expose the correlations on the HEFT parameters induced by assuming SMEFT’s validity. Then, in Sec. III we review the field-redefinition invariant criteria to distinguish SMEFT from HEFT. We also present a novel and simple analytic way of deriving the conditions on the flare function, $\mathcal{F}(h)$, function that allows for the deployment of SMEFT around the symmetric point. Continuing, Sec. IV deals with the generic properties of the flare function $\mathcal{F}(h)$ such as its positivity, the general correlations (and their experimental bounds) induced on its coefficients by assuming the existence of a symmetric point [where $\mathcal{F}(h^*) = 0$] and analyticity around it, allowing a continuation to our physical vacuum. Also, some example functions are provided as illustration. Section V cursorily discusses how to experimentally extract the value of the coefficients of the flare function, presenting the amplitudes of the scattering of two Goldstone bosons to n Higgs bosons (with $n = 1, 2, 3, 4$, although an automated amplitude generator for $n \in \mathbb{N}$ can be provided on demand). We also observe that to first nontrivial order, the existence of SMEFT simplifies cross-section ratios of these processes with the counterterm Wilson coefficient canceling out, which is not the case in a generic HEFT. Section VI explores the status of the symmetric-point knowledge based on current bounds on HEFT parameters by ATLAS and CMS, and also under what conditions Schwarz’s lemma can guarantee the existence of such a point. Finally, Sec. VII takes on a more speculative road by studying the far-future possibility of producing a large number of Higgs bosons at nonzero temperature, hence giving access to an arbitrary expansion order of the flare function. Our final conclusions are provided in Sec. VIII and some technical details are relegated to Appendixes A–D.

A. Key aspects of the SMEFT Lagrangian

The first of those theories is the Standard Model EFT (SMEFT) electroweak Lagrangian. Its symmetry breaking sector is expressed in terms of the $SU(2)$ doublet $H = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2, \phi_4 + i\phi_3)$ [that can also be collected as an $O(4)$ quartet $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$] and takes the general form

$$\mathcal{L}_{\text{SMEFT}} = A(|H|^2)|\partial H|^2 + \frac{1}{2}B(|H|^2)(\partial(|H|^2))^2 - V(|H|^2) + \mathcal{O}(\partial^4). \quad (1)$$

where both functions $A(|H|^2)$ and $B(|H|^2)$ are real, and analytic around $|H|^2 = 0$. The SM is retrieved by choosing $A(|H|^2) = 1$ and $B(|H|^2) = 0$. Thus, the organization of the theory is carried around an electroweak symmetric vacuum, instead of the energy minimum at $\langle\phi_4\rangle = v$.

SMEFT arranges the order of usage of operators in terms of their canonical-dimension counting, so that the leading corrections to the SM are composed of dimension-six operators (each multiplied by a Wilson coefficient and divided by the new physics scale squared, Λ^2).

In the TeV region, the derivative terms multiplying B become much larger than V , and we can neglect this potential. It will be shown that the piece of most importance for this article is that function B , which contains the electroweak symmetry breaking physics in the TeV region in the presence of new physics, particularly the dimension-six operator $\mathcal{O}_{H\Box} = |H|^2\Box|H|^2$.

B. HEFT Lagrangian (for the electroweak symmetry breaking sector in the TeV region)

What has come to be called the Higgs EFT (HEFT) Lagrangian (the second to bear that name) is an evolution of the electroweak chiral Lagrangian. Its degrees of freedom are built from a Cartesian to spherical-like change of coordinates,

$$\phi = (1 + h/v)\mathbf{n}, \quad (2)$$

where $\mathbf{n} = (\omega_1, \omega_2, \omega_3, \sqrt{v^2 - \omega_1^2 - \omega_2^2 - \omega_3^2})$, so that $\mathbf{n} \cdot \mathbf{n} = v^2$. It couples the ω_i Goldstone bosons to an additional low-energy Higgs field singlet, h , which is not assumed to be part of the $SU(2)$ Goldstone triplet.

At leading order in the chiral counting, the scalar sector of the HEFT Lagrangian [in EW Goldstone spherical coordinates ω^i in Eq. (2)] is given by

$$\mathcal{L}_{\text{LO HEFT}} = \frac{1}{2} \mathcal{F}(h) \partial_\mu \omega^i \partial^\mu \omega^j \left(\delta_{ij} + \frac{\omega^i \omega^j}{v^2 - \omega^2} \right) + \frac{1}{2} \partial_\mu h \partial^\mu h, \quad (3)$$

where the function \mathcal{F} scales the scattering amplitudes involving two, four, and generically an even number of Goldstone bosons. Thus, the flare function $\mathcal{F}(h)$ relates the EW Goldstone processes to amplitudes with an arbitrary number of Higgs bosons: of the same order in the chiral counting appropriate for HEFT,

$$\mathcal{F}(h) = 1 + \sum_{n=1}^{\infty} a_n \left(\frac{h}{v} \right)^n. \quad (4)$$

Usually, since only the first terms of the \mathcal{F} function are known, the Lagrangian is expressed [26,27] in terms of $\mathcal{F}(h) \simeq [1 + 2a \frac{h}{v} + b(\frac{h}{v})^2]$, with $a_1 = 2a$ and $a_2 = b$. In the TeV regime, the leading corrections to this Lagrangian are not of order m_W or m_h , both in the 100 GeV range, but rather derivative couplings. This means that $V(h)$ is irrelevant and electroweak symmetry breaking in the TeV region is more naturally discussed in terms of the coefficients of the Higgs-flare function $\mathcal{F}(h)$.

At Next to Leading Order (NLO), the Lagrangian relevant to study unitarity and resonances in the TeV regime acquires two further derivatives (so that amplitudes receive terms of order s^2) and becomes

$$\begin{aligned} \mathcal{L}_{\text{NLO HEFT}} = & \frac{1}{2} \left[1 + 2a \frac{h}{v} + b \left(\frac{h}{v} \right)^2 \right] \partial_\mu \omega^i \partial^\mu \omega^j \left(\delta_{ij} + \frac{\omega^i \omega^j}{v^2 - \omega^2} \right) + \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{4\alpha_4}{v^4} \partial_\mu \omega^i \partial_\nu \omega^j \partial^\mu \omega^k \partial^\nu \omega^l \\ & + \frac{4\alpha_5}{v^4} \partial_\mu \omega^i \partial^\mu \omega^j \partial_\nu \omega^k \partial^\nu \omega^l + \frac{g}{v^4} (\partial_\mu h \partial^\mu h)^2 + \frac{2d}{v^4} \partial_\mu h \partial^\mu h \partial_\nu \omega^i \partial^\nu \omega^j + \frac{2e}{v^4} \partial_\mu h \partial^\nu h \partial^\mu \omega^i \partial_\nu \omega^j, \end{aligned} \quad (5)$$

which has been extensively studied in earlier work. Here we will concentrate on the LO Lagrangian (tree-level amplitudes $\propto s$) in Eq. (3) with the Taylor series of \mathcal{F} around the physical vacuum $h = 0$ (with zero number of physical Higgs particles, that is, with $\phi_i = \langle\phi_4\rangle\delta_{i4} = v\delta_{i4}$ in terms of the SMEFT coordinates) given by Eq. (4). The NLO coefficients in Eq. (5) should eventually encode similar physics to the B function in Eq. (1), but we will leave exploring this connection for future work, and here concentrate on the comparison between $A(|H|^2)$, $B(|H|^2)$ and $\mathcal{F}(h)$.

II. TeV-SCALE RELEVANT EW SMEFT IN HEFT FORM

We now show the explicit transformation to polar coordinates, and then to HEFT, of the SMEFT electroweak Lagrangian of Eq. (1) in terms of the $SU(2)$ doublet H , neglecting the gauge couplings and $V(H)$ as is appropriate for TeV-scale physics $E \gg m_{h,W,Z}$, following the discussion in [22], with $\mathcal{L}_{\text{SMEFT}} = A(|H|^2)|\partial H|^2 + \frac{1}{2}B(|H|^2)(\partial(|H|^2))^2$. This is achieved by decomposing the doublet H of the SMEFT framework in the spherical polar coordinates of Eq. (2),

$$H = \left(1 + \frac{h}{v}\right) U(\omega) \langle H \rangle, \quad (6)$$

where h denotes the radial Higgs-boson field in the SMEFT framework, $\langle H \rangle$ is the chosen Higgs doublet vacuum, commonly taken to be $(0 \quad v/\sqrt{2})^T$, $v = \sqrt{2\langle H^\dagger H \rangle}$ provides the modulus of the Higgs vacuum expectation value (VEV), the $SU(2)$ matrix $U(\omega)$ contains the EW Goldstone bosons. The VEV modification due to higher-order corrections can always be later incorporated to the analysis by considering a shift in the Higgs field¹ $h \rightarrow h + \Delta$. Substituting $H^\dagger H = (v + h)^2/2$ in A and B , this yields

$$\begin{aligned} \mathcal{L}_{\text{polar-SMEFT}} &= \frac{1}{2}(v + h)^2 A(h) (\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n}) \\ &\quad + \frac{1}{2}(A(h) + (v + h)^2 B(h)) (\partial h)^2 \\ &= \frac{1}{4}(v + h)^2 A(h) \langle \partial_\mu U^\dagger \partial^\mu U \rangle \\ &\quad + \frac{1}{2}(A(h) + (v + h)^2 B(h)) (\partial h)^2, \end{aligned} \quad (7)$$

with $A((h + v)^2/2) \equiv \tilde{A}(h) \rightarrow A(h)$ now a function of h to avoid cumbersome notation. The SM, with $A = 1$ and $B = 0$, is the first and simplest of the family of SMEFT Lagrangians in Eq. (7), and in this form it reads

$$\mathcal{L}_{\text{SM}} = |\partial H|^2 = \frac{1}{4}(v + h)^2 \langle \partial_\mu U^\dagger \partial^\mu U \rangle + \frac{1}{2}(\partial h)^2. \quad (8)$$

In the general case, even though the coordinates of Eq. (7) are now those of HEFT, the Lagrangian is not yet in its canonical form because, by convention, the HEFT Higgs field's h kinetic term needs to be fixed to its free-wave standard expression,

$$\mathcal{L}_{\text{HEFT}} = \frac{v^2}{4} \mathcal{F}(h_1) \langle \partial_\mu U^\dagger \partial^\mu U \rangle + \frac{1}{2}(\partial h_1)^2, \quad (9)$$

which requires a further change of the h variable. Finding an h_1 field that absorbs the multiplicative factor in Eq. (7) and that becomes the Higgs field in the HEFT framework, implies solving the differential condition [28]

$$dh_1 = \sqrt{A(h) + (v + h)^2 B(h)} dh, \quad (10)$$

which will collect all factors of h_1 to multiply only the Goldstone term to the right of Eq. (7), from which the \mathcal{F} of Eq. (3) can be read off,

$$v^2 \mathcal{F}(h_1) = (v + h(h_1))^2 A(h(h_1)), \quad (11)$$

¹This removes terms linear in h and recenters the Higgs field expansion around the potential minimum.

in terms of $h_1 = h_1[A, B](h)$. In Sec. II A we will show that nontrivial A terms are unnecessary, so we can set $A = 1$ and employ B alone, which will determine the relation $h_1 = h_1(h)$. Once h has been expressed in terms of h_1 , the Lagrangian will have reached its HEFT form and the subindex in h_1 may be dropped.² With this method, the coefficients of Eq. (4) expanding the generic HEFT Lagrangian radial function can be retrieved from the initial SMEFT.

To complete this discussion we will quickly digress, in the next subsection, to show that $A = 1$ can be consistently taken, afterwards proceeding to carry out the transformation $h \rightarrow h_1$ for the relevant SMEFT pieces for the electroweak sector in the TeV energy regime where $m_h \ll E \sim \partial \ll \Lambda$.

A. $A(H)$ is not really necessary for $\sqrt{s} \gg m_h$

We here quickly show that it is possible, and can be more convenient, to eliminate the n th-power operators (for $n \geq 1$) obtained in an expansion of A , by means of a partial integration. For this, note that, up to a total divergence,

$$\begin{aligned} (H^\dagger H)^n |\partial H|^2 &= -\frac{n}{2} (H^\dagger H)^{n-1} (\partial |H|^2)^2 \\ &\quad - \frac{1}{2} (H^\dagger H)^n ((\partial^2 H^\dagger) H + H^\dagger (\partial^2 H)), \end{aligned} \quad (12)$$

obtained by using the relation $\partial^2 |H|^2 = 2|\partial H|^2 + (\partial^2 H^\dagger) H + H^\dagger (\partial^2 H)$. This teaches us that we can always convert (by partial integration) any n -power operator of A type into an $(n - 1)$ -power operator of B type. The price to pay includes an irrelevant total derivative and a couple of terms proportional to $\partial^2 H$ and $\partial^2 H^\dagger$. However, the classical equations of motion of H trade the derivative operators for $\partial^2 H$ (and its conjugate) by operators without derivatives, up to correction of higher dimension in $1/\Lambda^2$. In this way, the A type of operators can be removed from the theory and transformed into B operators at fixed dimension-six, -eight, etc. Employing this freedom, we will set $\Delta A_{\text{BSM}} = 0$ and just keep the leading operator, $A = A_{\text{SM}} = 1$. Hence, SMEFT can be formulated in polar coordinates (h, ω^a) as

$$\begin{aligned} \mathcal{L}_{\text{polar-SMEFT}} &= \frac{v^2}{4} \left(1 + \frac{h}{v}\right)^2 \langle \partial_\mu U^\dagger \partial^\mu U \rangle \\ &\quad + \frac{1}{2}(1 + (v + h)^2 B(h)) (\partial h)^2, \end{aligned} \quad (13)$$

instead of Eq. (7). The change of variables in the Higgs field, $h = h(h_1)$ of Eq. (10), then becomes

²Note we are dropping here the possible shift $h_1 \rightarrow h_1 + \Delta$, required if there are modifications to the SM Higgs potential. We are interested in high-energy effects and ignore nonderivative operators in $\mathcal{L}_{\text{HEFT}}$. This shift can easily be incorporated if needed.

$$dh_1 = (1 + (v + h)^2 B(h))^{1/2} dh. \quad (14)$$

This change determines \mathcal{F} in the form

$$\mathcal{F}(h_1) = \left(1 + \frac{h(h_1)}{v}\right)^2, \quad (15)$$

with h implicitly given [28] by the relation

$$h_1 = \int_0^h (1 + (v + h)^2 B(h))^{1/2} dh. \quad (16)$$

B. Explicit computation with SMEFT's power expansion of $B(|H|^2)$

1. Order 6 in the SMEFT counting

The SMEFT Lagrangian is an alternative parametrization of SM deviations, which assumes the SM symmetries and fields, and particularly assumes the traditional doublet structure for the Higgs field. The Higgs sector of this Lagrangian was introduced in Eq. (1), and it can be written more generally as

$$\mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}} + \sum_{n=5}^{\infty} \sum_i \frac{c_i^{(n)}}{\Lambda^{n-4}} \mathcal{O}_i^{(n)}. \quad (17)$$

At dimension-six, there are three operators of the SMEFT Warsaw basis [29] that directly distort the Standard Model's electroweak symmetry breaking Lagrangian, which written in terms of the Higgs field doublet H appropriate for SMEFT are ($\partial^2 \equiv \square$)

$$\begin{aligned} \mathcal{O}_H &= (H^\dagger H)^3, & \mathcal{O}_{HD} &= (H^\dagger D_\mu H)^* (H^\dagger D^\mu H), \\ \mathcal{O}_{H\square} &= (H^\dagger H) \square (H^\dagger H). \end{aligned} \quad (18)$$

They can of course be reexpressed in terms of the singlet field for the Higgs boson via $(H^\dagger H) = (h + v)^2/2$ (in polar coordinates this is manifestly gauge independent). Those three operators are actually all that is needed for Higgs-Goldstone boson scattering up to dimension-six in the SMEFT counting. Moreover, \mathcal{O}_{HD} breaks custodial symmetry so that it can be counted as higher order due to the small size of the corrections to Peskin-Takeuchi observables in the SM at LEP.

We would like to remark that not only at dimension-six but also at dimension-eight there is an additional operator with two derivatives acting only on a product of Higgs doublets. However, these terms violate custodial symmetry and they actually contribute to an independent type of HEFT operator, Longhitano's a_0 Lagrangian term [30,31]. Consistently, this a_0 operator is related to the experimentally suppressed oblique T parameter. Thus, we will no longer consider this type of custodial breaking operators in

this article, although a similar study can be worked out if this kind of corrections needed to be included.

In turn, \mathcal{O}_H is not a derivative operator, so that it does not contribute to the flare function that we are pursuing (though it does affect the Higgs self-coupling, namely the Higgs SM potential, and the vacuum expectation value, important near threshold, its impact in the TeV region is much smaller than that of the derivative operator).

In summary, only the $\mathcal{O}_{H\square}$ operator contributes to $\mathcal{F}(h)$ at order $\mathcal{O}(\Lambda^{-2})$. Moreover, it has been shown in [32], by geometric arguments, that only one operator is needed at this order, which is consistent with our discussion. The rest of the electroweak operators of the Warsaw basis that the reader may be wondering about,

$$\begin{aligned} \mathcal{O}_W &= \epsilon_{ijk} W_\mu^{\nu i} W_\nu^{\rho j} W_\rho^{\mu k}, & \mathcal{O}_{HW} &= (H^\dagger H) W_{\mu\nu}^i W^{\mu\nu i}, \\ \mathcal{O}_{HB} &= (H^\dagger H) B_{\mu\nu} B^{\mu\nu}, & \mathcal{O}_{HWB} &= (H^\dagger \tau^i H) W_{\mu\nu}^i B^{\mu\nu}, \end{aligned} \quad (19)$$

are necessary only if one intends to couple the transverse electroweak gauge bosons [25,33]), but they are of no concern for our purposes of studying the TeV-region electroweak-symmetry breaking Lagrangian that requires only, by the equivalence theorem [34,35] in the TeV region, the Goldstone bosons \simeq longitudinal W_L, Z_L . Further, a generic basis could also contain an operator of the form $\partial_\mu (H^\dagger H) \partial^\mu (H^\dagger H)$, but this is eliminated in the standard Warsaw treatment because it is equivalent to $\mathcal{O}_{H\square}$ in Eq. (18) up to a total divergence, in analogy to Eq. (12),

$$(X^2) \square (X^2) = -\partial_\mu X^2 \partial^\mu X^2 + \underbrace{\partial_\mu (X^2 \partial^\mu X^2)}_{\text{surface term}}, \quad (20)$$

or in terms of the h singlet,

$$\begin{aligned} \mathcal{O}_{H\square} &= (H^\dagger H) \square (H^\dagger H) = -\partial_\mu (H^\dagger H) \partial^\mu (H^\dagger H) + \partial_\mu (\dots) \\ &= -(h + v)^2 \partial_\mu h \partial^\mu h + \partial_\mu (\dots). \end{aligned} \quad (21)$$

Therefore, the only contributing dimension-six operator of the Warsaw basis that preserves custodial symmetry is

$$\mathcal{O}_{H\square} = (H^\dagger H) \square (H^\dagger H) = -\partial_\mu (H^\dagger H) \partial^\mu (H^\dagger H), \quad (22)$$

that in the Lagrangian appears multiplied by the Wilson coefficient $c_{H\square}$ and is suppressed by two powers of the high-energy scale Λ respect to the dimension-four Lagrangian. Comparing with Eq. (1) we read the (constant) values $A(|H|^2) = 1$ and $B(|H|^2) = -2 \frac{c_{H\square}}{\Lambda^2}$, which contain no fields.

2. The role of $c_{H\square}$ in SMEFT and bounds on its size from experimental data

Because SMEFT has been used for a few years now to analyze LHC data, there already exist bounds on the

coefficient $c_{H\Box}$ from run 2 of the machine (even if the associated operator $\mathcal{O}_{H\Box}$ is quite elusive in LHC fits); we now recall those bounds.

The best overall constraints on the dimension-six basis arise from Higgs-sector observables (production and decay) [36,37], but it is only when combined with other electro-weak channels that this $c_{H\Box}$ coefficient can be well constrained. The reason for this is the way that $\mathcal{O}_{H\Box}$ enters in the B piece of the Lagrangian in Eq. (1). Its effect is to change the Higgs wave-function normalization

$$\mathcal{L}_{\text{SMEFT}} = \frac{1}{2} \left(1 - \frac{2c_{H\Box}v^2}{\Lambda^2} \right) \partial_\mu h \partial^\mu h + \dots \quad (23)$$

instead of the Higgs couplings to other particles, which are not directly affected. Hence, the contribution of this operator to any on-shell production or decay process of a single Higgs boson appears as a kinematics-independent shift, as evident from Eq. (23). In particular, for the reference value of $\Lambda = 1$ TeV used in most analysis, this overall shift for the several processes considered, whether decay width or production cross section, becomes

$$\frac{\sigma_{H,\text{SMEFT}}}{\sigma_{H,\text{SM}}} \propto \frac{\Gamma_{H,\text{SMEFT}}}{\Gamma_{H,\text{SM}}} \propto 1 + 2 \frac{c_{H\Box}v^2}{\Lambda^2} = 1 + 0.12c_{H\Box}, \quad (24)$$

which was already numerically observed by the ATLAS collaboration and reported, for example in Table 1 of [38]. It is obvious that the numbers there, between 0.115 and 0.125, just reflect the exact 0.12 factor of Eq. (24). This is true in any process involving only one on-shell Higgs boson [as will also be the case in our Eq. (91) below]; but events with two or multiple h particles such as Eq. (92) and following have different dependences on $c_{H\Box}$ and will allow a cleaner separation within SMEFT. Also, the cross sections above in $\frac{\sigma_{H,\text{SMEFT}}}{\sigma_{H,\text{SM}}}$ are implicitly understood as their narrow Higgs-width approximation, where one Higgs is produced on shell and then cascades into the final products. For off-shell Higgs studies the dependency would be different.

In consequence, this kinematically not very exciting $\mathcal{O}_{H\Box}$ operator is often overlooked and few works actually constrain it. Still, the works of Ellis *et al.* [36] and Ethier *et al.* [37] offer quite interesting bounds on $c_{H\Box}$, that at 95% confidence level, and rounding off to the precision of the leading digit of the uncertainty, read as follows:

$$c_{H\Box} \simeq -0.3 \pm 0.7 \text{ (individual)} \quad (25)$$

$$c_{H\Box} \simeq -1 \pm 2 \text{ (marginalized)}. \quad (26)$$

Both are compatible with the Standard Model value $c_{H\Box} = 0$.

3. Operator of order 8 in the SMEFT counting

Going one order further in the $1/\Lambda^2$ power counting makes the SMEFT parametrization more interesting [39–42]. In particular, the full dimension-eight basis in SMEFT was recently published in [43,44]. To find the dimension-eight operator that contributes $\mathcal{O}(\Lambda^{-4})$ corrections to the flare function $\mathcal{F}(h)$ we can return to the $B(H)$ term in Eq. (1) remembering that $A(H)$ is irrelevant as per Sec. II A, $\frac{1}{2}B(|H|^2)(\partial(|H|^2))^2$, and set $B(|H|^2) \propto |H|^2$ instead of 1. Therefore, at dimension-eight we only find the following operator:

$$\begin{aligned} \mathcal{O}_{H\Box}^{(8)} &= |H|^4 \Box |H|^2 = -2|H|^2 (\partial(|H|^2))^2 \\ &= -(h+v)^4 \partial_\mu h \partial^\mu h. \end{aligned} \quad (27)$$

We have chosen this operator's normalization for convenience and resemblance to $\mathcal{O}_{H\Box}^{(6)}$ in Eq. (22) when expressed in terms of the Higgs doublet modulus $(v+h)/\sqrt{2}$. Through partial integration, it can be easily rewritten in other forms considered in the bibliography (up to a total derivative):

$$\begin{aligned} \mathcal{O}_{H\Box}^{(8)} &= -2|H|^2 (\partial(|H|^2))^2 = |H|^4 \Box |H|^2 \\ &= 2|H|^4 |\partial H|^2 + |H|^4 ((\Box H^\dagger)H + H^\dagger(\Box H)), \end{aligned} \quad (28)$$

with the last contribution $|H|^4 ((\Box H^\dagger)H + H^\dagger(\Box H))$ being proportional to the Higgs equations of motion (EOM), so they can be removed from the effective action and transformed into operators including EW Goldstone bosons and also fermions, and $\mathcal{Q}_{H^6}^{(1)} = |H|^4 |DH|^2$ the operator chosen for the dimension-eight basis in Ref. [43]. The other possible two-derivative dimension-eight operator $\mathcal{Q}_{H^6}^{(2)}$ breaks custodial symmetry and will not be discussed in this article.

In the next section, we will first work out the precise change of Higgs variable $h \rightarrow h_1$ from SMEFT to HEFT up to dimension-six. Afterwards, in Sec. II C 2, we will proceed up to the next order and provide the Next-to-next to Leading Order (NNLO) modification induced by this dimension-eight operator.

C. Change of coordinates $h^{\text{SMEFT}} \rightarrow h_1^{\text{HEFT}}$

1. Derivation and result at dimension-six

The first step to put the dimension-six relevant SMEFT Lagrangian into HEFT form is then to change the variable as in Eq. (7) yielding

$$\begin{aligned} \mathcal{L}_{\text{SMEFT}} &= \frac{1}{2} \left(1 - 2(v+h)^2 \frac{c_{H\Box}}{\Lambda^2} \right) (\partial_\mu h)^2 \\ &\quad + \frac{1}{2} (v+h)^2 (\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n}). \end{aligned} \quad (29)$$

We next have to take the Higgs kinetic energy to canonical form. This requires integrating (10) (t being the integration variable taking the place of h):

$$h_1 = \int_0^h \sqrt{1 - (v+t)^2 \frac{2c_{H\Box}}{\Lambda^2}} dt = \int_v^{v+h} \sqrt{1 - t^2 \frac{2c_{H\Box}}{\Lambda^2}} dt = \sqrt{\frac{\Lambda^2}{2c_{H\Box}}} \int_{\theta_0}^{\theta_1} |\cos \theta| \cos \theta d\theta, \quad (30)$$

where $\theta_0 = \arcsin \sqrt{\frac{2c_{H\Box}}{\Lambda^2}} v$ and $\theta_1 = \arcsin \sqrt{\frac{2c_{H\Box}}{\Lambda^2}} (v+h)$. Now, since $0 \leq \theta_0 \leq \theta_1 \ll \pi/2$ (because the EFT coefficient is very small by current bounds [45]) we can assume that the cosine in Eq. (30) is positive and hence find

$$\begin{aligned} h_1 &= \frac{1}{2} \sqrt{\frac{\Lambda^2}{2c_{H\Box}}} \left(\theta + \frac{\sin 2\theta}{2} \right) \Big|_{\theta_0}^{\theta_1} = \frac{1}{2} \sqrt{\frac{\Lambda^2}{2c_{H\Box}}} \left(\theta + \sin \theta \sqrt{1 - \sin^2 \theta} \right) \Big|_{\theta_0}^{\theta_1} \\ &= \frac{1}{2} \left((v+h) \sqrt{1 - \frac{2c_{H\Box}}{\Lambda^2} (h+v)^2} - v \sqrt{1 - \frac{2c_{H\Box}}{\Lambda^2} v^2} \right) + \frac{1}{2} \sqrt{\frac{\Lambda^2}{2c_{H\Box}}} \left(\arcsin \sqrt{\frac{2c_{H\Box}}{\Lambda^2}} (v+h) - \arcsin \sqrt{\frac{2c_{H\Box}}{\Lambda^2}} v \right). \end{aligned} \quad (31)$$

Such an expression is not particularly useful, especially taking into account that we need to invert it to obtain $h(h_1)$, so we explore it by expanding Eq. (31) to leading order in $c_{H\Box}/\Lambda^2$, finding

$$h_1 = h - \frac{1}{3} \left(\frac{c_{H\Box}}{\Lambda^2} \right) (h^3 + 3h^2v + 3hv^2) + \mathcal{O} \left(\frac{c_{H\Box}^2}{\Lambda^4} \right), \quad (32)$$

which can be iteratively inverted, yielding

$$h = h_1 + \frac{1}{3} \left(\frac{c_{H\Box}}{\Lambda^2} \right) (h_1^3 + 3h_1^2v + 3h_1v^2) + \mathcal{O} \left(\frac{c_{H\Box}^2}{\Lambda^4} \right). \quad (33)$$

Finally, we can use Eqs. (11) and (33) to obtain $\mathcal{F}(h_1)$,

$$\begin{aligned} \mathcal{F}(h_1) &= \left(1 + \frac{h(h_1)}{v} \right)^2 \\ &= \left(1 + \frac{h_1}{v} \right)^2 + \frac{2v^2 c_{H\Box}}{\Lambda^2} \left(1 + \frac{h_1}{v} \right) \left(\frac{h_1^3}{3v^3} + \frac{h_1^2}{v^2} + \frac{h_1}{v} \right) + \mathcal{O} \left(\frac{c_{H\Box}^2}{\Lambda^4} \right) \\ &= 1 + \left(\frac{h_1}{v} \right) \left(2 + 2 \frac{c_{H\Box} v^2}{\Lambda^2} \right) + \left(\frac{h_1}{v} \right)^2 \left(1 + 4 \frac{c_{H\Box} v^2}{\Lambda^2} \right) + \left(\frac{h_1}{v} \right)^3 \left(8 \frac{c_{H\Box} v^2}{3\Lambda^2} \right) + \left(\frac{h_1}{v} \right)^4 \left(2 \frac{c_{H\Box} v^2}{3\Lambda^2} \right) + \mathcal{O} \left(\frac{c_{H\Box}^2}{\Lambda^4} \right), \end{aligned} \quad (34)$$

which expresses the expansion coefficients of \mathcal{F} in terms of the SMEFT Wilson coefficient (in the philosophy of the appendix of [46]),

$$\begin{aligned} a_1 = 2a = 2 \left(1 + v^2 \frac{c_{H\Box}}{\Lambda^2} \right), \quad a_2 = b = 1 + 4v^2 \frac{c_{H\Box}}{\Lambda^2}, \\ a_3 = \frac{8v^2 c_{H\Box}}{3\Lambda^2}, \quad a_4 = \frac{2v^2 c_{H\Box}}{3\Lambda^2}. \end{aligned} \quad (35)$$

These relations expose the inclusion of SMEFT into HEFT: the a_i coefficients, independent parameters in the latter, are correlated in SMEFT up to a given order, as all of the first four a_i are given in terms of only one Wilson coefficient $c_{H\Box}$. This feature has been suggested as a handle to discern, from upcoming experimental data, whether SMEFT will be

applicable later on (in the presence of any separation from the SM values $a = b = 1$). Measurements of the $\omega\omega \rightarrow nh$ scattering process (see Sec. V) would allow the determination of the a_i and can probe the SMEFT-predicted correlations [47] (or the SM ones [25,48]). In the absence of such correlations, it is plausible that a HEFT formulation would be needed.

However, this difference can be put into question in the presence of unnatural Wilson coefficients. If the dimension-eight operators contribute at an order similar to that of the dimension-six operator, because the coefficients are not of order 1 or because Λ is not large enough to significantly suppress them, additional SMEFT parameters would appear in the expressions of Eq. (35), decorrelating the a_i coefficients and voiding the analysis. Therefore, though perhaps orientative, given that naturalness may have

already failed as a safe guiding principle in view of the light Higgs boson mass, more robust criteria that helps systematize the correlations to distinguish SMEFT from HEFT have been provided, and to them we turn in the next section.

At last, the position of the symmetric point h_* that satisfies $\mathcal{F}(h_*) = 0$ is always given by $h(h_1) = -v$ [with $|H| = (h+v)/\sqrt{2}$], as seen directly from Eq. (15). In turn, the position of the symmetric point in HEFT coordinates becomes displaced from its SM value $h_*^{\text{SM}} = -v$ by an $\mathcal{O}(\Lambda^{-2})$ SMEFT correction,

$$\frac{h_*}{v} = \frac{h_1(h)}{v} \Big|_{h=-v} = -1 + \frac{c_{H\Box} v^2}{3\Lambda^2}, \quad (36)$$

where we use the relation $h_1 = h_1(h)$ in Eq. (32). This procedure provides the only real root of $F(h_1)$ up to $\mathcal{O}(1/\Lambda^2)$, with $\mathcal{F}(h_1) = F(h_1)^2$ (in addition, there are two more complex-conjugate roots). However, this information gets blurred if we instead consider \mathcal{F} up to $\mathcal{O}(1/\Lambda^2)$, which has the real roots $h_1/v = -1$ and $h_1/v = -1 + 2c_{H\Box}v^2/\Lambda^2$ (in addition to a pair of complex-conjugate roots which escape to infinity for $\Lambda \rightarrow \infty$). Notice that, due to

$$\begin{aligned} \mathcal{F}(h_1) &= \left(1 + \frac{h(h_1)}{v}\right)^2 \\ &= 1 + \left(\frac{h_1}{v}\right) \left(2 + 2\frac{c_{H\Box}^{(6)}v^2}{\Lambda^2} + 3\frac{(c_{H\Box}^{(6)})^2v^4}{\Lambda^4} + 2\frac{c_{H\Box}^{(8)}v^4}{\Lambda^4}\right) + \left(\frac{h_1}{v}\right)^2 \left(1 + 4\frac{c_{H\Box}^{(6)}v^2}{\Lambda^2} + 12\frac{(c_{H\Box}^{(6)})^2v^4}{\Lambda^4} + 6\frac{c_{H\Box}^{(8)}v^4}{\Lambda^4}\right) \\ &\quad + \left(\frac{h_1}{v}\right)^3 \left(8\frac{c_{H\Box}^{(6)}v^2}{3\Lambda^2} + 56\frac{(c_{H\Box}^{(6)})^2v^4}{3\Lambda^4} + 8\frac{c_{H\Box}^{(8)}v^4}{\Lambda^4}\right) + \left(\frac{h_1}{v}\right)^4 \left(2\frac{c_{H\Box}^{(6)}v^2}{3\Lambda^2} + 44\frac{(c_{H\Box}^{(6)})^2v^4}{3\Lambda^4} + 6\frac{c_{H\Box}^{(8)}v^4}{\Lambda^4}\right) \\ &\quad + \left(\frac{h_1}{v}\right)^5 \left(88\frac{(c_{H\Box}^{(6)})^2v^4}{15\Lambda^4} + 12\frac{c_{H\Box}^{(8)}v^4}{5\Lambda^4}\right) + \left(\frac{h_1}{v}\right)^6 \left(44\frac{(c_{H\Box}^{(6)})^2v^4}{45\Lambda^4} + 2\frac{c_{H\Box}^{(8)}v^4}{5\Lambda^4}\right) + \mathcal{O}(\Lambda^{-6}). \end{aligned} \quad (37)$$

Note that the bracket in each line provides the corresponding a_j up to $\mathcal{O}(\Lambda^{-4})$. Also, to make the order manifest and avoid confusion, we have denoted $c_{H\Box}$ by $c_{H\Box}^{(6)}$ in this paragraph and below whenever there might be any confusion.

As for the symmetric point around which SMEFT is built, Eq. (36) takes a further correction of $\mathcal{O}(\Lambda^{-4})$ that may take it away from the Standard Model value. Again, Eq. (15) shows that the $SU(2) \times SU(2)$ fixed-point condition $\mathcal{F}(h_*) = 0$ has always the solution $h = -v$, which in the HEFT coordinates up to $\mathcal{O}(\Lambda^{-4})$ SMEFT corrections is given by

$$\frac{h_*}{v} = \frac{h_1(h)}{v} \Big|_{h=-v} = -1 + \frac{c_{H\Box}^{(6)}v^2}{3\Lambda^2} + \frac{((c_{H\Box}^{(6)})^2 + 2c_{H\Box}^{(8)})v^4}{10\Lambda^4}, \quad (38)$$

the $1/\Lambda^2$ truncation in Eq. (34), none of the four zeros of Eq. (34), a fourth-order polynomial, seems to be double. This means that inside a small interval of values of h_1 of width suppressed by $1/\Lambda^2$, Eq. (34) can violate positivity (see Sec. IV A 1 below). Furthermore, we find that $\mathcal{F}(h_1) = \mathcal{O}(1/\Lambda^4)$ for $h_1/v = -1 + xv^2/\Lambda^2 \forall x \in \mathbb{R}$ (see Appendix D for further details). Hence, it looks like the perturbative analysis of \mathcal{F} alone does not allow us to extract any information beyond $h_* = -v + \mathcal{O}(1/\Lambda^2)$.

We note, however, that Eq. (36) is indeed the correct solution of $h(h_1) = -v$ up to order c_H/Λ^2 , even though the $\mathcal{O}(1/\Lambda^2)$ flare function in Eq. (34) fails to recover that precise expression at that perturbative order. This ambiguity will be solved in the $\mathcal{O}(1/\Lambda^4)$ analysis in the next subsection.

2. Result at dimension-eight

We now quote the result of adding the operator of dimension-eight in Eq. (II B 3); the calculation follows along the same lines so we only quote the combined result for the flare function \mathcal{F} , which reads

where we used the relation between SMEFT and HEFT coordinates at this order:

$$\begin{aligned} h_1 &= h + \frac{c_{H\Box}^{(6)}}{3\Lambda^2}(v^3 - (v+h)^3) \\ &\quad + \frac{((c_{H\Box}^{(6)})^2 + 2c_{H\Box}^{(8)})}{10\Lambda^4}(v^5 - (v+h)^5) + \mathcal{O}\left(\frac{1}{\Lambda^6}\right), \end{aligned} \quad (39)$$

which can be iteratively inverted, yielding

$$\begin{aligned} h &= h_1 + \frac{c_{H\Box}^{(6)}}{3\Lambda^2}((v+h)^3 - v^3) \\ &\quad + \frac{(c_{H\Box}^{(6)})^2}{30\Lambda^4}(13(v+h_1)^5 - 10v^3(v+h_1)^2 - 3v^5) \\ &\quad + \frac{c_{H\Box}^{(8)}}{5\Lambda^4}((v+h)^5 - v^5) + \mathcal{O}\left(\frac{1}{\Lambda^6}\right). \end{aligned} \quad (40)$$

The study of the $\mathcal{O}(1/\Lambda^4)$ flare function in Eq. (37) shows that $\mathcal{F}(h_1) = \mathcal{O}(\Lambda^{-4})$ for $h_1/v = -1 + xv^2/\Lambda^2$ for all x but for $h_1/v = -1 + c_{H\Box}^{(6)}v^2/3\Lambda^2$ where we find $\mathcal{F}(h_1) = \mathcal{O}(\Lambda^{-6})$. Hence, requiring $\mathcal{F}(h_1) = 0$ up to $\mathcal{O}(1/\Lambda^4)$ leads to the real solution $h_1/v = -1 + c_{H\Box}^{(6)}v^2/3\Lambda^2 + \mathcal{O}(1/\Lambda^4)$, solving the $\mathcal{O}(1/\Lambda^2)$ ambiguity of the previous subsection. However, the perturbative expression in Eq. (37) is not able to recover the $\mathcal{O}(1/\Lambda^4)$ of the symmetric point (38). In order to fix the $\mathcal{O}(1/\Lambda^4)$ term of h_* there are two ways to proceed: analyze the flare-function constraint $\mathcal{F}(h_1) = 0$ up to $1/\Lambda^6$; or solve the equation $F = 1 + h(h_1)/v = 0$ up to $\mathcal{O}(1/\Lambda^4)$.

III. GEOMETRIC AND ANALYTIC DISTINCTION BETWEEN HEFT AND SMEFT

This section succinctly exposes the precise theoretical conditions allowing to discern between SMEFT and HEFT, compiling the main results of several articles [16,19–22] in their geometrical aspects and adding an extended analytical discussion about the function \mathcal{F} of our own. Much of the past confusion between the two EFT formulations arises from the fact that there are two coordinate systems to describe the same system of fields; for this, the San Diego group advocated for employing a geometric perspective to be able to make coordinate-invariant statements.

A. Flat SM geometry

The $O(4)$ components in the scalar field of Eq. (2) used for the SM Higgs sector, $\boldsymbol{\phi} = (\phi^1, \phi^2, \phi^3, \phi^4)$ are taken to represent coordinates in a (momentarily flat, later in the next subsection curved) geometric manifold \mathcal{M} . $\boldsymbol{\phi}$ contains the Higgs field and the three “eaten” Goldstone bosons and has a Lagrangian (turning off gauge fields)

$$\mathcal{L}_{\text{SM}} = \frac{1}{2} \partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} - \frac{\lambda}{4} (\boldsymbol{\phi} \cdot \boldsymbol{\phi} - v^2)^2. \quad (41)$$

In these Cartesian coordinates the global $O(4)$ transformations should act *linearly*:

$$\boldsymbol{\phi} \rightarrow O\boldsymbol{\phi}, \quad O^T O = \mathbf{1}. \quad (42)$$

The field breaks the global electroweak symmetry $O(4)$ down to $O(3)$ by acquiring a vacuum expectation value,

$$\langle \boldsymbol{\phi} \cdot \boldsymbol{\phi} \rangle = v^2,$$

where $v \simeq 246$ GeV. Usually, the vacuum expectation value is chosen to be $\langle \phi_4 \rangle = v$ while $\langle \phi_1 \rangle = \langle \phi_2 \rangle = \langle \phi_3 \rangle = 0$ and the Higgs field h in these Cartesian coordinates is defined through the relation $\phi_4 = v + h$.

The alternative coordinate system in which HEFT is based expresses the SM Higgs sector Lagrangian in polar form:

$$\boldsymbol{\phi} = \left(1 + \frac{h}{v}\right) \mathbf{n}(\boldsymbol{\omega}), \quad \mathbf{n} \cdot \mathbf{n} = v^2. \quad (43)$$

Clearly, the constraint $\mathbf{n} \cdot \mathbf{n} = v^2$ makes the $O(4)$ symmetry to be realized in a *nonlinear* way. This comes about because the four components of $\mathbf{n} = (\omega^1, \omega^2, \omega^3, \pm\sqrt{v^2 - \boldsymbol{\omega}^2})$ rotate linearly with O in Eq. (42), imposing a nonlinear transformation law on the polar-coordinate Goldstone bosons, w^a ($a = 1, 2, 3$). In these polar coordinates, the Higgs sector SM Lagrangian is³

$$\begin{aligned} \mathcal{L}_{\text{SM}} = & \frac{1}{2} \left(1 + \frac{h}{v}\right)^2 (\partial_\mu \omega^i \partial_\mu \omega^j) \left(\delta_{ij} + \frac{\omega_i \omega_j}{v^2 - \boldsymbol{\omega}^2} \right) \\ & + \frac{1}{2} (\partial_\mu h)^2 - \frac{\lambda}{4} (h^2 + 2vh)^2. \end{aligned} \quad (44)$$

In the SM, we thus see that

$$\mathcal{F}(h = h_1) = \left(1 + \frac{h}{v}\right)^2. \quad (45)$$

This exercise enlightens the fact that the SM Higgs sector $O(4)$ symmetry can be realized both in a linear or a nonlinear manner. That is why, when studying EFTs that extend the SM, one should concentrate on objects which are invariant under field redefinitions [49]. Again, the important distinction between the SM, SMEFT and HEFT is not that the realization of the $O(4)$ symmetry is linear or not, since a field redefinition can turn one into the other. That is why the key aspect one must look for are the geometric invariants on the scalar manifold \mathcal{M} , which are by definition invariant under field redefinitions.

B. Beyond SM: Curved $\boldsymbol{\phi}$ geometry

The kinetic term of the Lagrangian (giving the classical equations of motion as the geodesics of that manifold) in Eq. (3) has the form

$$\frac{1}{2} g_{ij}(\boldsymbol{\phi}) \partial_\mu \phi^i \partial^\mu \phi^j$$

and can be interpreted in terms of a metric tensor g_{ij} that provides lengths in the geometrical space of the $\boldsymbol{\phi}$ fields, $g_{ij} d\phi_i d\phi_j$ in any of the different coordinate choices. This is the point of contact between physics experiments at colliders, i.e., production and scattering amplitudes derived from this Lagrangian, and the field geometry, where the former are represented in terms of curvature invariants. We now briefly recall the results presented in [19,22].

³Note the difference with Eq. (2.12) in [19] where the $\omega_i \omega_j$ piece is absent. It is unnecessary unless amplitudes with more than two Goldstone bosons are analyzed, which we leave for future investigation.

In the SM, the metric is just a Kronecker delta $g_{ij}(\boldsymbol{\phi})_{\text{SM}} = \delta_{ij}$, but the SMEFT of Eq. (1) extends it to a more general form:

$$g_{ij}|_{\text{SMEFT}} = A \left(\frac{\boldsymbol{\phi} \cdot \boldsymbol{\phi}}{\Lambda^2} \right) \delta_{ij} + B \left(\frac{\boldsymbol{\phi} \cdot \boldsymbol{\phi}}{\Lambda^2} \right) \frac{\phi_i \phi_j}{\Lambda^2}, \quad (46)$$

where Λ is the new physics scale. It is always possible to express the SMEFT metric (and in particular its SM limit) in HEFT form by changing to polar coordinates followed by an additional field redefinition to make the kinetic term of h canonical as in Sec. II. This makes the HEFT metric take the generic form

$$g_{ij}(h, \boldsymbol{\omega})_{\text{HEFT}} = \begin{bmatrix} \mathcal{F}(h)g_{ab}(\boldsymbol{\omega}) & 0 \\ 0 & 1 \end{bmatrix} \quad (47)$$

where $g_{ab}(\boldsymbol{\omega})$ is the $O(3)$ invariant metric on the scalar submanifold $O(4)/O(3) = S^3$ described by the Goldstone bosons in angular coordinates. The SM is the special case with a flat scalar manifold \mathcal{M} (since its metric is just δ_{ij} for all values of the field: there exist global Riemann coordinates). For both SMEFT and HEFT, \mathcal{M} has curvature. But what makes SMEFT different from HEFT?

Equation (47) allows to interpret the function $\mathcal{F}(h)$ in the manifold $\mathcal{M} \ni \boldsymbol{\phi}$ as a scale factor akin to the $a(t)$ one in the Friedmann-Robertson-Walker metric. For each value of h (in units of v) there is an S^3 submanifold (parametrized by the ω_i), away from the origin of \mathcal{M} by an amount $\mathcal{F}(h)$, which acts as a radial distance.

One can in that way always write SMEFT in HEFT form, but as we will see, the converse is not always true. This means that

$$\text{SM} \subset \text{SMEFT} \subset \text{HEFT}.$$

This is so because, in order to write a HEFT as a SMEFT, there must exist an invariant point in \mathcal{M} under the $O(4)$ symmetry. This translates into the condition that the function $\mathcal{F}(h)$ must vanish for some h_* , $\mathcal{F}(h_*) = 0$. Hence this h_* , an invariant point under the $O(4)$ symmetry, plays the role of an origin for the Cartesian-like SMEFT coordinates on \mathcal{M} .

This invariant point is a necessity to deploy a linear representation of the $O(4)$ group around it, just as in Eq. (42), and hence to write down HEFT as a SMEFT. It happens that the existence of such zero of $\mathcal{F}(h)$ is not a sufficient condition for SMEFT to be possible. It may happen that nonanalyticities arise at the fixed point h_* or between that h_* and $h = 0$ (the physical vacuum), spoiling the possibility of constructing a viable SMEFT around h_* which is applicable in particle physics. This is why we must require that the metric and thus $\mathcal{F}(h)$ be analytic in a sufficient domain (see [22] for further detail). In a lower energy regime $m_h \sim E$ where the potential V makes a

relevant contribution, the same considerations also apply to V .

Instead of obtaining these results by further following the powerful yet intricate geometric methods just mentioned, we will continue with coordinate-dependent field theory in the spirit of staying close to the phenomenological formulation that can be brought to bear at accelerator experiments.

C. Zero and analyticity of \mathcal{F} upon passing from HEFT to SMEFT

We are going to show in this subsection how to proceed from HEFT to SMEFT and under what condition this is possible in an analytical manner in terms of the field coordinates. For this we need to combine the Higgs h_1 and the EW Goldstone bosons ω^a appropriate for HEFT into the complex doublet Φ used in SMEFT. Problems can arise about whether the resulting Lagrangian will obey minimal physical requirements from the theoretical (e.g., analyticity) and the phenomenological point of view (e.g., perturbation theory convergence).

The first step to reconstruct the SMEFT form from the HEFT Lagrangian in Eq. (9) is to define a new Higgs variable h from the HEFT one, h_1 in this subsection, by the condition

$$\begin{aligned} \mathcal{F}(h_1(h)) &= F^2(h_1(h)) = \left(1 + \frac{h}{v}\right)^2, \\ F(h_1(h)) &= 1 + \frac{h}{v}, \end{aligned} \quad (48)$$

implying the inverse relations

$$h_1 = \mathcal{F}^{-1}((1 + h/v)^2), \quad h_1 = F^{-1}(1 + h/v). \quad (49)$$

This change of variable unravels the standard HEFT normalization of the Higgs kinetic term in Eq. (5), turning the Lagrangian in the ‘‘polar-SMEFT’’ form,

$$\begin{aligned} \mathcal{L}_{\text{polar-SMEFT}} &= \frac{v^2}{4} \left(1 + \frac{h}{v}\right)^2 \langle \partial_\mu U^\dagger \partial^\mu U \rangle \\ &+ \frac{1}{2} \left(\frac{1}{v} (F^{-1})'(1 + h/v) \right)^2 (\partial h)^2. \end{aligned} \quad (50)$$

The second, h -kinetic term can also be expressed in terms of the square of F , that is, \mathcal{F} . To achieve it, we can simply replace $(F^{-1})'(1 + h/v)$ by $2(1 + h/v)(\mathcal{F}^{-1})'((1 + h/v)^2)$. We note that F^{-1} and \mathcal{F}^{-1} are the inverse functions of F and \mathcal{F} , respectively, and $F(h_1) = 1 + h(h_1)/v$.

Up to this point there is no concerning issue; this polar-coordinate form halfway between HEFT and SMEFT, which we also find when calculating in the opposite direction in Eq. (13), is still a valid Lagrangian (if we postpone for a later moment the discussion on the

convergence of the perturbative series) totally equivalent to HEFT.

The possible obstacle to this conversion can however arise when trying to reconstruct the Higgs-doublet field H from the EW Goldstone bosons ω^a in U and the Higgs scalar field h , making use [22] of

$$\begin{aligned} |H|^2 &= \frac{(v+h)^2}{2}, \\ |\partial H|^2 &= \frac{(v+h)^2}{4} \langle \partial_\mu U^\dagger \partial^\mu U \rangle + \frac{1}{2} (\partial h)^2, \\ (\partial |H|^2)^2 &= (v+h)^2 (\partial h)^2 = 2 |H|^2 (\partial h)^2. \end{aligned} \quad (51)$$

The inversion of these linear equations to express Eq. (50) in terms of H brings about a possible $|H|^{-2}$ singularity in the SMEFT Lagrangian,

$$\mathcal{L}_{\text{SMEFT}} = \underbrace{|\partial H|^2}_{=\mathcal{L}_{\text{SM}}} + \frac{1}{2} \left[\underbrace{\left(\frac{1}{v} (F^{-1})' \left(\sqrt{2|H|^2/v^2} \right) \right)^2 - 1}_{=\Delta \mathcal{L}_{\text{BSM}}} \right] \frac{(\partial |H|^2)^2}{2|H|^2}. \quad (52)$$

Such divergence is incompatible with a power expansion as needed to deploy the SMEFT counting. The only way that Eq. (52) can provide an analytical Lagrangian around $|H| = 0$ to allow a valid SMEFT expansion in powers of H is by restricting the $\mathcal{F} = F^2$ function of Eq. (3) to fulfill the condition

$$\left[\frac{1}{v} (F^{-1})' \left(\sqrt{2|H|^2/v^2} \right) - 1 \right] \Big|_{|H| \rightarrow 0} \stackrel{!}{=} 0 + \mathcal{O}(|H|^2) \quad (53)$$

so that this zero cancels the $|H|^{-2}$ denominator in Eq. (52). Furthermore, even if the zero is canceled, the analyticity of the SMEFT Lagrangian at any order implies that the $\mathcal{O}(|H|^2)$ remnant must also have an analytic expansion in integer powers of $|H|^2$ (from the square root). Otherwise an expansion-breaking nonanalyticity is present and SMEFT becomes a theory that is not systematically improvable by furthering the expansion. This remarkable fact can be traced to Eq. (49), where the change of variables $h \rightarrow h_1$ happens at the level of individual singlet particles, whereas the doublet H employed in SMEFT (and in the SM) needs to be squared to $|H|^2$ to produce an electroweak singlet, forcing the square root upon us.

The relation (53) is a differential equation for (F^{-1}) in the variable $z := \sqrt{2|H|^2/v^2}$, whose integration leads to

$$F^{-1}(z) \stackrel{|H| \rightarrow 0}{=} F^{-1}(0) + vz + \mathcal{O}(z^3). \quad (54)$$

The analyticity of the SMEFT Lagrangian at all orders implies that the $\mathcal{O}(z^3)$ remainder has an analytic expansion that only contains odd powers of z . We solve for the z

variable around $z = 0$ in terms of $F^{-1}(z)$, and invert to recover the original function $F = (F^{-1})^{-1}$ around the point $h_1^* \equiv F^{-1}(0)$, remembering from Eq. (49) that F^{-1} is a HEFT-Higgs h_1 value:

$$z = F(h_1) \stackrel{|H| \rightarrow 0}{=} \frac{1}{v} (h_1 - h_1^*) + \mathcal{O}((h_1 - h_1^*)^3), \quad (55)$$

where the $\mathcal{O}(z^2)$ remnant can be put in $\mathcal{O}((h_1 - h_1^*)^2)$ form up to higher orders. In terms of \mathcal{F} the relation would be given by $(1 + h/v)^2 = 2|H|^2/v^2 = z^2 = F(h_1)^2 = \mathcal{F}(h_1)$. Moreover, the analyticity of the SMEFT Lagrangian at all orders implies that the solution of Eq. (54) for z —shown in (55)—has an analytical expansion around $h_1 = h_1^*$ that only contains odd powers of $(h_1 - h_1^*)$.

The existence of that zero $h_1^* \equiv F^{-1}(0)$ of F —and thus of its square \mathcal{F} —and the analyticity required for a power series expansion (both of \mathcal{F} and the Higgs potential V) broadly constitute the necessary and sufficient requirements for a given HEFT Lagrangian density characterized by \mathcal{F} to be expressible as a SMEFT. Let us summarize and make these findings, which agree with the ones presented in [22], somewhat more precise:

- (1) $F(h_1)$ must have at least a simple zero at some h_1^* , i.e., $F(h_1^*) = 0$. This implies that the function in the HEFT Lagrangian density of Eq. (3) [$\mathcal{F}(h_1^*) = F(h_1^*)^2 = 0$] must have a double zero.
- (2) At that point h_1^* , F must have the slope $F'(h_1^*) = \frac{1}{v}$. This translates into two conditions over \mathcal{F} , namely

$$\mathcal{F}'(h_1^*) = 0, \quad \mathcal{F}''(h_1^*) = \frac{2}{v^2}.$$

- (3) At that point h_1^* , F must have zero curvature $F''(h_1^*) = 0$, since the first correction must be $\mathcal{O}((h_1 - h_1^*)^3)$. From the point of view of \mathcal{F} this translates as the constraint $\mathcal{F}'''(h_1^*) = 0$.
- (4) Finally, it is possible to exploit the analyticity of the SMEFT Lagrangian at higher orders, if the expansion is to be continued and be systematically improvable. In general, analyticity as an all-order requirement forces all even derivatives to vanish at the symmetric point: $F^{(\ell)}(h_1^*) = 0$ for even ℓ . From the point of view of \mathcal{F} this implies the vanishing of all odd derivatives, $\mathcal{F}^{(2\ell+1)}(h_1^*) = 0$.

The first two conditions mean that the HEFT \mathcal{F} flare function must be an upward-bending parabola if an equivalent SMEFT is to exist. In the next subsection, Fig. 1 shall expose that current knowledge is compatible with it, and allows to estimate how intensely one of the HEFT coefficients needs to separate from the SM or SMEFT for the latter not to be applicable.

Should Eq. (55) fail, the SMEFT Lagrangian would not have an analytical expansion in powers of the doublet field H . Moreover, it is important to remark that in order to avoid

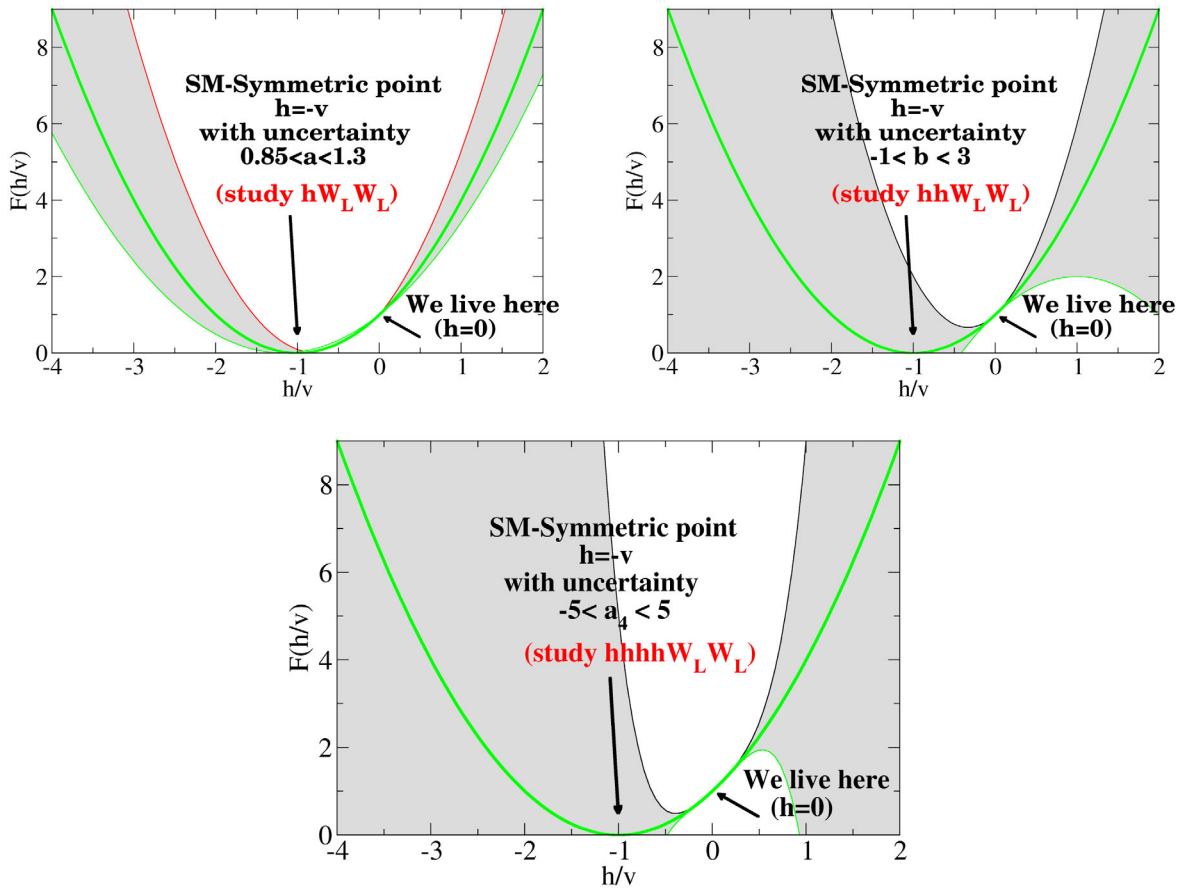


FIG. 1. Sensitivity of $\mathcal{F}(h)$ to typical parameter ranges. The solid parabola (green) is the SM prediction $1 + 2\frac{h}{v} + (\frac{h}{v})^2$. The gray bands, in the order given, show the uncertainty due to our present knowledge of the a_i coefficients that we vary one at a time around the SM value. Respectively, the coefficients $a_1 = 2a$, $a_2 = b$ and a_4 , couple $\omega\omega$ to h (the best constrained one, a), hh , and in view that a third order polynomial always has a zero, $hhhh$ with an even number of powers as the next most interesting one (and we ignore a_3 by itself).

a singularity, at least at dimension-six, the remnant in (53) must be at least $\mathcal{O}(|H|^2)$, or equivalently, the remnant in (55) must be at least $\mathcal{O}((h_1 - h_1^*)^3)$.

Various examples will be provided in Sec. IV B. We will deal with the possibility of experimentally finding such zeros h_1^* in Sec. VI.

IV. GENERIC PROPERTIES OF THE FLARE FUNCTION $\mathcal{F}(h)$

A. Current knowledge of $\mathcal{F}(h)$

In particle physics language, the appearance of the $\mathcal{F}(h)$ function in Eq. (3) controls the (derivative) coupling of a pair of $\omega \simeq W_L$ longitudinal gauge bosons to any number of Higgs bosons.

While this coefficient is the dominant Higgs production in the TeV region, multiboson processes at the LHC in the hundred GeV energy regime already constrain, although not tightly, the coefficients of the $\mathcal{F}(h_1)$ expansion. Since, for the rest of the main body of the article,

we will be concentrating on HEFT and there can be no confusion with the SMEFT h field, we will drop the subindex $h_1 \rightarrow h$ and make it explicit whenever a change of coordinates between SMEFT and HEFT is used. We give a graphical representation of the present status of \mathcal{F} in Fig. 1.

At the time of writing there is no significant deviation from the Standard Model, which is a particular case of SMEFT, meaning that there is no reason to doubt the applicability of SMEFT: the uncertainty bands by no means exclude a zero of \mathcal{F} , possibly where the Standard Model requires it, at $h = -v$. The SM line, as a particular SMEFT case, is a parabola with vertex at $\mathcal{F} = 0$, as discussed in Sec. III C. The reason why we have cut off values $\mathcal{F} < 0$ will become clear in the next comment.

1. Positivity from boundedness of the Hamiltonian

From the LO HEFT Lagrangian in Eq. (3) we can construct the Hamiltonian of the theory, finding

$$H_{\text{HEFT}} = \int d^3\mathbf{x} \frac{1}{2} \left[\partial_0 h \partial_0 h + \nabla h \cdot \nabla h + \mathcal{F}(h) (\partial_0 \omega^i \partial_0 \omega^i + \nabla \omega^i \cdot \nabla \omega^i) \left(\delta_{ij} + \frac{\omega_i \omega_j}{v^2 - \omega^2} \right) \right]. \quad (56)$$

From the condition that the Hamiltonian H_{HEFT} must be bounded from below for vacuum stability one obtains that $\mathcal{F}(h) \geq 0$. This justifies the common usage of the form $F^2(h)$ instead of simply \mathcal{F} . While a matter of taste, it is not clear what in particle physics is the quantity F being squared (a radial distance in the ω^a field space), so we prefer \mathcal{F} for most of the discussion.

One can also often see that, after expanding the function $\mathcal{F}(h)$ around our physical low-energy vacuum $h = 0$ as in Eq. (4), the positivity condition on $\mathcal{F}(h)$ is forgotten or not explicitly mentioned, although in those approaches employing $F(h)$ with $\mathcal{F} = F^2(h)$ it is automatically incorporated.

Therefore we are going to distinguish three cases. First let us mention that if \mathcal{F} requires an infinite expansion, the information about positiveness is intricately hidden in the coefficients a_i .

The second case that we next address corresponds to the treatment of experimental data within order by order EFT; \mathcal{F} is truncated to a few terms and the customary assumption that $\mathcal{F} = F^2$ is accepted. However, because the most general polynomial of degree n cannot be written as a square [50], we briefly discuss, as a third case, the possibility that \mathcal{F} is well approximated by a polynomial, but this needs to be decomposed as $\mathcal{F} = F_1^2 + F_2^2$ that holds in all generality (because it corresponds to $|F|^2$, the modulus square of a complex function). For the sake of

simplicity, we will express h in units of v in the discussion of this section.

Figure 2 displays, in its left plot, the (a_1, a_2) plane corresponding to a quadratic \mathcal{F} . The positivity-forbidden region is the lower, shaded area (blue). The white area at the top respects positivity and supports a HEFT formulation. The dividing line is that where the Higgs flare function presents a double zero, so it is of the form $\mathcal{F} = (1 + \frac{a}{2}h)^2$.

On the right panel of the figure we show a three-dimensional plot where, additionally to (a_1, a_2) , the value of h_* is represented. There are regions where there is no zero of \mathcal{F} (but HEFT is applicable), a line (along the lower surface's fold) where the zero of \mathcal{F} is double so SMEFT is an acceptable description, and regions where there are two simple real roots of $\mathcal{F} = 0$ that entail a sign change, and thus a violation of positivity. An example of this is shown by a vertical line piercing both sheets.

$\mathcal{F} = F^2$ is assumed.—In this situation, there are restrictions among the coefficients a_i that guarantee positivity of \mathcal{F} . We obtain them up to fourth order, by squaring the expansion of F . If the expansion ended at first order, that is, normalizing h by v ,

$$F(h) = 1 + ah \Rightarrow \mathcal{F}(h) = F^2(h) = 1 + 2ah + h^2 a^2, \quad (57)$$

we obtain the relation

$$a_2 = \frac{a_1^2}{4}, \quad (58)$$

or $b = a^2$ which is exactly the correct one for a and b in the Standard Model.

If that relation is experimentally found to be broken, then at least one more order is necessary in the expansion of F .

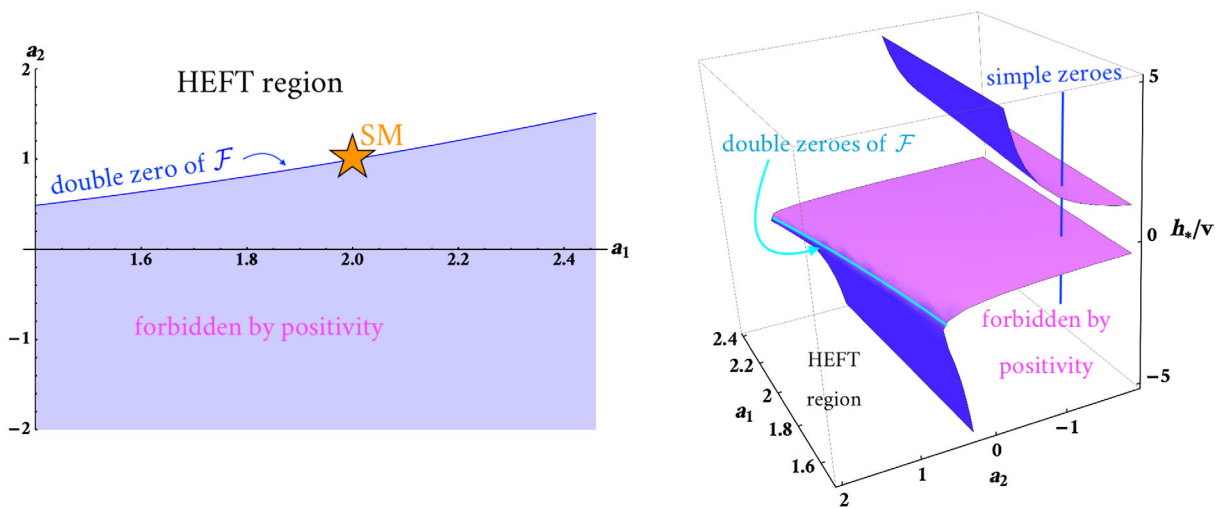


FIG. 2. (a_1, a_2) plane (left plot) and (a_1, a_2, h_*) three-dimensional representation (right plot), showing the regions where HEFT is valid; those where there is a double zero (that follow, e.g., from the dilaton model discussed in Sec. IV B 2 and SMEFT is deployable); and the region where there are two simple zeros and a quadratic polynomial \mathcal{F} corresponds to no valid theory.

This then implies that the degree of the flare function \mathcal{F} must be at least 2 orders higher,

$$\begin{aligned} F(h) &= 1 + \alpha_1 h + \alpha_2 h^2 \\ \Rightarrow \mathcal{F}(h) &= 1 + 2\alpha_1 h + h^2(\alpha_1^2 + 2\alpha_2) + 2\alpha_1\alpha_2 h^3 + \alpha_2^2 h^4, \end{aligned} \quad (59)$$

implying two correlations:

$$2a_3 = a_1 \left(a_2 - \frac{a_1^2}{4} \right), \quad 4a_4 = \left(a_2 - \frac{a_1^2}{4} \right)^2. \quad (60)$$

These new correlations (60) would substitute (58), being a smoking gun of the presence of further beyond the standard model (BSM) three-Higgs and four-Higgs vertices in the effective Lagrangian, which could then be measured in further collider experiments.

Iterating the analysis procedure, any eventual experimental deviations from the relations in Eq. (60) immediately imply the presence of higher order coefficients in the function F , and therefore also in the flare function \mathcal{F} that provides the $WW \rightarrow nh$ effective vertices.

To have enough freedom to accommodate the SMEFT values of the a_i coefficients in Eq. (35) to $\mathcal{O}(\Lambda^{-2})$, the expansion of F needs to be run up to fourth order in h ,

$$F(h) = 1 + \alpha_1 h + \alpha_2 h^2 + \alpha_3 h^3 + \alpha_4 h^4 \dots, \quad (61)$$

where the Greek names of the coefficients mimic those of the expansion of \mathcal{F} . The positivity conditions on the coefficients a_i of \mathcal{F} can be obtained from

$$\begin{aligned} \mathcal{F}(h) &= F^2(h) \\ &= 1 + 2\alpha_1 h + h^2(\alpha_1^2 + 2\alpha_2) + h^3(2\alpha_1\alpha_2 + 2\alpha_3) \\ &\quad + h^4(2\alpha_1\alpha_3 + \alpha_2^2 + 2\alpha_4) \dots \end{aligned} \quad (62)$$

and essentially leave the first four a_i undetermined, while those with $i = 5 \dots 8$ become dependent of those earlier four. Once more, an experiment that does not respect the corresponding correlations points to a higher term in the expansion of F and so on. There is a tower of positivity correlations that should be experimentally tested as multi-Higgs data in the correct kinematic region becomes available.

Most general non-negative polynomial satisfying $\mathcal{F} = F_1^2 + F_2^2$.—In this case, the flare function $\mathcal{F}(h)$ is the most general non-negative polynomial $\mathcal{F}(h) \geq 0 \forall h \in \mathbb{R}$, and therefore, of even degree $2d$, $\mathcal{F}(h) = \sum_{n=0}^{2d} a_n h^n$.

We invoke the theorem [50] that states that such a non-negative polynomial can be decomposed as

$\mathcal{F}(h) = F_1^2(h) + F_2^2(h)$ in terms of two polynomials⁴ $F_1(h)$ and $F_2(h)$ of degree d . The equivalent of Eq. (59) then becomes

$$\begin{aligned} \mathcal{F}(h) &= 1 + 2(\alpha_1 \cos \theta + \beta_1 \sin \theta)h \\ &\quad + (\alpha_1^2 + \beta_1^2 + 2\alpha_2 \cos \theta + 2\beta_2 \sin \theta)h^2 \\ &\quad + 2(\alpha_1\alpha_2 + \beta_1\beta_2)h^3 + (\alpha_2^2 + \beta_2^2)h^4, \end{aligned} \quad (63)$$

having expanded F_1 up to second order with coefficients α_i , F_2 with coefficients β_i , and having noted that because of the normalization of the kinetic term, $\mathcal{F}(0) = F_1(0)^2 + F_2(0)^2 = 1$, a practical parametrization is $F_1(0) = \cos \theta$, $F_2(0) = \sin \theta$ for some angle θ . The number of free parameters is now high enough (five, $\theta, \alpha_{1,2}, \beta_{1,2}$, for four orders that depend on them) so that, in an order by order expansion of the polynomial, the correlations are weaker than in the simplified case $\mathcal{F} = F^2$ with F real.

Nevertheless, one should note that if, given an experimental situation, the highest given order h^{2d} has a negative coefficient a_{2d} , then higher orders are necessary. This can be used in experiment to detect (further) new physics. Currently, the sign of $b = a_2$ is not known; discerning whether it is positive or negative is therefore an interesting experimental analysis exercise that, if it turned out to be negative, would immediately and of necessity point out to new coefficients a_3 and a_4 (and of course, indicate new physics, since in the Standard Model $b = 1$).

2. Restrictions on the coefficients a_i from the invertibility guaranteeing a SMEFT

The restrictions over \mathcal{F} at the symmetric point h_* that guarantees the existence of a SMEFT at the end of Sec. III C translate into conditions over the a_i at the physical vacuum $h = 0$, which are constrained by experiment. Here we will write down the known ones. Let us express the series expansion around $h = 0$ setting $a_0 := 1$, and take h normalized to v , so that $v = 1$,

$$\mathcal{F}(h) = \sum_{i=0}^n a_i h^i. \quad (64)$$

⁴To demonstrate it, we will note that the most general form of a polynomial of order $2d$, that is $\mathcal{F}(h) = a_n \prod_{i=1}^d (h - h_i^*)(h - \bar{h}_i^*)$, can be restricted because positivity and real analyticity demand that both the real and complex roots must be double,

$$\mathcal{F}(h) := |F(h)|^2 \quad \text{with} \quad F(h) = \sqrt{a_n} \prod_{i=1}^n (h - h_i^*);$$

the theorem then follows from taking the real and imaginary parts, $F_1(h) = \text{Re}[F(h)]$ and $F_2(h) = \text{Im}[F(h)]$. (That the real roots are double thwarts any sign change near them and guarantees positivity.)

Since the conditions over \mathcal{F} are taken at h_* , we reexpand around that point, and in terms of $a_j^* = \mathcal{F}^{(j)}(h_*)/j!$, find

$$\mathcal{F}(h) = \sum_{j=0}^n a_j^* (h - h_*)^j. \quad (65)$$

By matching the two expansions around the two different points it is easy to read off the coefficients a_j^* (on which the conditions over \mathcal{F} are expressed) in terms of the a_i (more directly accessible to experiment). The relation reads

$$a_j^* = \sum_{k=0}^{\infty} a_{k+j} h_*^k \cdot b_{jk}. \quad (66)$$

The coefficients of this expansion can be recursively obtained:

$$b_{0k} = 1 \quad \forall k, \quad b_{jk} = \sum_{l=0}^k b_{j-1l}. \quad (67)$$

The closed formula that solves this recursion,

$$a_{\ell}^* = A_{\ell j} a_j, \quad (68)$$

requires the following simple auxiliary matrix:

$$A_{\ell j} = \begin{cases} 0 & \text{if } j < \ell, \\ \binom{j}{\ell} h_*^{j-\ell} & \text{if } j \geq \ell. \end{cases} \quad (69)$$

We can now deploy the four straightforward conditions $\mathcal{F}(h_*) = \mathcal{F}'(h_*) = \mathcal{F}'''(h_*) = 0$, $\mathcal{F}''(h_*) = 2/v^2 \rightarrow 2$ as four linear constraints on the coefficients around the physical vacuum, namely

$$\begin{aligned} \sum_{k=0}^{\infty} h_*^k a_k \cdot b_{0k} = a_0^* = 0, & \quad \sum_{k=0}^{\infty} h_*^k a_{k+1} \cdot b_{1k} = a_1^* = 0, \\ \sum_{k=0}^{\infty} h_*^k a_{k+2} \cdot b_{2k} = a_2^* = 1, & \quad \sum_{k=0}^{\infty} h_*^k a_{k+3} \cdot b_{3k} = a_3^* = 0. \end{aligned} \quad (70)$$

Square-matrix four-coefficient truncation.—A first possible truncation of the series is to keep the terms with the first four (a_1, a_2, a_3, a_4) coefficients [the zeroth coefficient is identically $a_0 = \mathcal{F}(0) = 1$ by construction of the HEFT formalism so we include it in the inhomogeneous term together with the $(a_1^*, a_2^*, a_3^*, a_4^*)$ values from the conditions on \mathcal{F}]. These are the coefficients that collect dimension-six SMEFT corrections to the SM as shown in Eq. (34) or (B11), and the linear system becomes

$$\begin{pmatrix} h_* & h_*^2 & h_*^3 & h_*^4 \\ 1 & 2h_* & 3h_*^2 & 4h_*^3 \\ 0 & 1 & 3h_* & 6h_*^2 \\ 0 & 0 & 1 & 4h_* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad (71)$$

The matrix has determinant h_*^4 , so that barring the zero at $h_* = 0$ (the physical vacuum, where the coefficients a_i and a_i^* coincide), the system has a unique solution for each h_* . Such solution is that of Eq. (34), with the symmetric point of SMEFT $h^* = -v + v \frac{c_{H\Box}^{(6)} v^2}{3\Lambda^2}$ that of Eq. (36), as can be easily checked by substitution. At this order in $1/\Lambda^2$, if h_* would take a value different from that one, there would be a one-parameter family of coefficients that would yield a valid SMEFT.

Systematic order by order truncation.—Instead of that truncation, one could be more systematic and count powers of h on the left and right of Eq. (70), so that the respective left- and the right-hand sides of Eq. (71) are of the same order, say N . In that case the system $A\vec{a}_* = \vec{a}$ has N unknowns but $(N+1)$ equations and compatibility becomes an issue. The criterion of Rouché-Frobenius guaranteeing an algebraic solution then links possible values of the h_* with the unknown a_{2n}^* that can appear on the right-hand side of the equivalent system.

Taking $\mathcal{F}(h)$ as a polynomial of order h^4 , this compatibility condition is

$$1 = \mathcal{F}(0) = \frac{h_*^2}{v^2} + a_4^* \frac{h_*^4}{v^4}. \quad (72)$$

Without extra work, the vanishing of a_n^* for odd $n = 1, 3, 5, \dots$ yields the same relation even if $\mathcal{F}(h)$ is a polynomial of order h^5 . For a flare function $\mathcal{F}(h)$, still polynomial but now of order h^6 (or even h^7), the constraint takes one more term:

$$1 = \mathcal{F}(0) = \frac{h_*^2}{v^2} + a_4^* \frac{h_*^4}{v^4} + a_6^* \frac{h_*^6}{v^6}. \quad (73)$$

Let us recall that a nonsingular SMEFT Lagrangian requires $a_0^* = 0$, $a_2^* = 1$ and $a_n^* = 0$ for odd $n = 1, 3, 5, \dots$, as shown earlier in Sec. III C. It is not difficult to check that SMEFT fulfills these relations (72) and (73) at $\mathcal{O}(\Lambda^{-2})$ and $\mathcal{O}(\Lambda^{-4})$, respectively, as that effective theory predicts [see Eqs. (37) and (38)]:

$$\begin{aligned} \frac{h_*}{v} &= \frac{\mathcal{F}^{-1}(0)}{v} \\ &= -1 + \frac{c_{H\Box}^{(6)} v^2}{3\Lambda^2} + ((c_{H\Box}^{(6)})^2 + 2c_{H\Box}^{(8)}) \frac{v^4}{10\Lambda^4} + \mathcal{O}(\Lambda^{-6}), \\ a_4^* &= \frac{1}{4!} \mathcal{F}^{(4)}(h_*) = \frac{2c_{H\Box}^{(6)} v^2}{3\Lambda^2} + \mathcal{O}(\Lambda^{-6}), \\ a_6^* &= \frac{1}{6!} \mathcal{F}^{(6)}(h_*) \\ &= \left(\frac{44(c_{H\Box}^{(6)})^2}{45} + \frac{2c_{H\Box}^{(8)}}{5} \right) \frac{v^4}{\Lambda^4} + \mathcal{O}(\Lambda^{-6}), \end{aligned} \quad (74)$$

with all these properties determined by the precise form of the flare function \mathcal{F} .

If we extended the analysis to include a_5 and a_6 , which is easily done and omitted for brevity, we would have two more Lagrangian parameters but only one further constraint over \mathcal{F} , namely the vanishing of its fifth derivative. This means that SMEFT would have a second parametric degree of freedom that could take any value. In fact, this is precisely the case in Eq. (37), which depends on the additional parameter $c_{H\Box}^{(8)}$ from the dimension-eight relevant Lagrangian.

Resulting testable correlations.—Table I collects the correlations between the a_i coefficients of HEFT that we have worked out at order Λ^{-2} and Λ^{-4} (further correlations are possible from the higher odd derivatives of \mathcal{F} vanishing, and all become a bit weaker numerically if yet higher orders in $1/\Lambda$ are studied, by the need of introducing further a_i coefficients).

The correlation in the first row, second column of Table I originates in a quadratic one $2(\Delta a_2 - 2\Delta a_1) - \frac{3}{4}(a_3 - \frac{4}{3}\Delta a_1) = (-3\Delta a_1 + \frac{5}{2}\Delta a_2 - \frac{9}{8}a_3)^2$ with two solutions for a_3 , a small and a large one. In keeping near the SM value $a_3 = 0$ we take this second one and reexpand to linearize in a_3 so that it can be related to a_1 and a_2 in a straightforward manner; the difference is more suppressed than $\mathcal{O}(\Lambda^{-4})$ in the SMEFT expansion.

The remarkable property of these equations is that they are independent of the SMEFT parameters $c_{H\Box}^{(n)}$, that is, they are tests of the SMEFT theory framework itself, up to a given order in $1/\Lambda$, which cannot be rewritten away in terms of its parameters.

These equations can be experimentally tested looking for the consistency of SMEFT. Given tight experimental bounds on a_1 , these relations (and those from $\mathcal{F} \geq 0$) can already predict how the next HEFT coefficients will look like if SMEFT is valid. This we will delay until Sec. VI A 2.

The $1/\Lambda^2$ relations in the first column of Table I, all hanging from Δa_1 , are rather constraining given that one-Higgs production is well known. Those in the second column, as they depend also on Δa_2 , which is much less well measured, are not very useful; but they can be further tightened by imposing perturbativity of the SMEFT expansion.

Perturbativity constraints.—Perturbativity can be deployed by recalling that, at $\mathcal{O}(\Lambda^{-4})$,

$$\begin{aligned} a_1 &= \left(2 + 2 \frac{c_{H\Box}^{(6)} v^2}{\Lambda^2} + 3 \frac{(c_{H\Box}^{(6)})^2 v^4}{\Lambda^4} + 2 \frac{c_{H\Box}^{(8)} v^4}{\Lambda^4} \right) \\ a_2 &= \left(1 + 4 \frac{c_{H\Box}^{(6)} v^2}{\Lambda^2} + 12 \frac{(c_{H\Box}^{(6)})^2 v^4}{\Lambda^4} + 6 \frac{c_{H\Box}^{(8)} v^4}{\Lambda^4} \right). \end{aligned} \quad (75)$$

For clarity, let us shorten notation for the rest of the paragraph, writing

$$\begin{aligned} \Delta a_1 &= 2x + 3x^2 + 2y \\ \frac{\Delta a_2}{2} &= 2x + 6x^2 + 3y = \Delta a_1 + 3x^2 + y. \end{aligned} \quad (76)$$

In general, there are two free parameters, x and y . What perturbativity suggests is that each of the terms of the $\mathcal{O}(\Lambda^{-4})$ should not be larger than the $\mathcal{O}(\Lambda^{-2})$ term (this is akin to the Cauchy criterion for convergence of a sequence, but of course there is no guarantee that it will be satisfied at a fixed order; again, it is only a perturbativity argument, similar to the one in [41]). Taking this at face value, it must be that $3x^2 \leq 2|x|$ (by the way, this means that $|x| \leq 2/3$, which however is of little value as experimental constraints are much tighter) and that $|y| < |x|$.

Returning to the first of Eq. (76) and separately analyzing the positive and negative x cases, we find

TABLE I. Correlations between the a_i HEFT coefficients necessary for SMEFT to exist, at order Λ^{-2} and Λ^{-4} . They are given in terms of $\Delta a_1 := a_1 - 2 = 2a - 2$ and $\Delta a_2 := a_2 - 1 = b - 1$. This way, all the objects in the table vanish in the Standard Model, with all the equalities becoming $0 = 0$. Notice that the rhs of each identity in the second column shows the $\mathcal{O}(\Lambda^{-4})$ corrections to the relations of the first column. The third one assumes the perturbativity of the SMEFT expansion.

Correlations accurate at order Λ^{-2}	Correlations accurate at order Λ^{-4}	Λ^{-4} assuming SMEFT perturbativity
$\Delta a_2 = 2\Delta a_1$		$ \Delta a_2 \leq 5 \Delta a_1 $
$a_3 = \frac{4}{3}\Delta a_1$	$(a_3 - \frac{4}{3}\Delta a_1) = \frac{8}{3}(\Delta a_2 - 2\Delta a_1) - \frac{1}{3}(\Delta a_1)^2$	Those for a_3, a_4, a_5, a_6
$a_4 = \frac{1}{3}\Delta a_1$	$(a_4 - \frac{1}{3}\Delta a_1) = \frac{5}{3}\Delta a_1 - 2\Delta a_2 + \frac{7}{4}a_3$ $= \frac{8}{3}(\Delta a_2 - 2\Delta a_1) - \frac{7}{12}(\Delta a_1)^2$	
$a_5 = 0$	$a_5 = \frac{8}{5}\Delta a_1 - \frac{22}{15}\Delta a_2 + a_3$ $= \frac{6}{5}(\Delta a_2 - 2\Delta a_1) - \frac{1}{3}(\Delta a_1)^2$	All the same
$a_6 = 0$	$a_6 = \frac{1}{6}a_5$	

$$|x| < \max \left(|\Delta a_1^-|, \frac{1}{2} |\Delta a_1^+| \right) \quad (77)$$

and noting that half the upper 95% uncertainty $\Delta a_1^+/2$ is larger than the lower one Δa_1^- as discussed around Table II below, leads us to

$$|\Delta a_2| \leq 2|\Delta a_1| + 2 \cdot \frac{3}{2} |\Delta a_1^+| \Rightarrow |\Delta a_2| \leq 5|\Delta a_1^+|, \quad (78)$$

a relation which we elevate to the third column of Table I, in the understanding that the uncertainty there is the maximum (+) of the two asymmetric uncertainties, and where the absolute value bars have been at last dropped.

In the order in which experimental data can be used, we note the following:

- (i) A nonzero measurement of Δa_2 signals new physics. SMEFT or HEFT are needed.
- (ii) If additionally the stronger correlation $\Delta a_2 = 2\Delta a_1$ is violated, severe corrections to $1/\Lambda^2$ SMEFT are suggested.
- (iii) If the weaker correlation in Eq. (78), $\Delta a_2 \leq 5\Delta a_1$ is violated, those correlations make SMEFT unnatural and put its perturbative use into question but they do not necessarily rule it out as discussed in the next paragraph.
- (iv) If the weakest correlation in the second column of Table I is broken, the first two orders of SMEFT do not make much sense and the theory is falsified for all practical purposes.

To close this subsection, we note that the presence of the zero (and minimum) of \mathcal{F} at h^* is a distinguishing property in the TeV region, for near-threshold physics the Higgs potential $V(h)$ is also important. The SMEFT potential needs to be analytic too so that a power expansion makes sense. The relevant theory regarding V is briefly discussed in Appendix C.

3. Unitarity imposes no constraint on the coefficients, causality may

It has recently been proposed that unitarity violations in the effective theory could be used to describe the space of theories that can be characterized as HEFT but that, due to nonanalyticities, cannot be brought up to SMEFT form [23]. While this may deserve further study, we are not very sure about that program.

The reason is that the HEFT Lagrangian yields a properly Hermitian Hamiltonian, and therefore a unitary scattering matrix. Truncating an expansion of a partial wave amplitude in perturbation theory is indeed a procedure that violates unitarity, but this has nothing to do with the theory itself, but with the truncation. For example, in the well-known case of two-body elastic scattering one can, instead of the partial wave amplitude, expand first the inverse partial wave amplitude to one loop in the EFT,

$$\frac{1}{t^J} = \frac{1}{t_0^J + t_1^J}, \quad (79)$$

and then invert back to obtain

$$t^J \simeq \frac{(t_0^J)^2}{t_0^J - t_1^J}. \quad (80)$$

This expansion of the inverse amplitude, which can be carried out order by order, can also be derived from a dispersion relation, so it satisfies all analyticity properties expected from an elastic scattering amplitude. Additionally, elastic unitarity over the physical cut of the amplitude is exact, no matter how strong the interaction, as long as the low-energy theory has the structure of HEFT (or chiral perturbation theory or other similar theories with derivative couplings). This has been documented at length in the literature [24,26,54–57] so we will not delve any longer on the issue here. The point is that the failure (or not) of unitarity is not really about the theory, whether SMEFT, HEFT or another, but about the way to treat it to obtain observables. This is an ancient observation dating at least to the effective range expansion [58] that needs to be discussed more often in the context of high-energy physics.

On the other hand, causality does impose limits on the parameters of an effective Lagrangian, though they have not been very thoroughly studied and perhaps we will attempt this in future work. These come about because a scattered wave packet in the forward direction cannot precede the incoming wave packet (though this is possible at wide angles [59]). Perturbatively, Wigner's bound for the derivative of the phase shift of any partial wave δ_J respect to the center-of-mass three-momentum k , in terms of the scatterer's radius R is a well-known low-energy result [60],

$$\frac{d\delta_J}{dk} \geq -R. \quad (81)$$

However, what should be used for R in a relativistic scattering theory is less well understood. Such a set of bounds on the scattering matrix (one for each of its partial wave projections) yields one-sided bounds on the a_i coefficients. Employing unitarized methods one can immediately set constraints by demanding that no poles of the amplitude lay on the first Riemann sheet [26,55] of s , which also violate causality. But these poles typically fall in regions where the uncertainties of the unitarized amplitude [61] are large. In all, we think that this deserves a separate investigation.

B. Example functions to illustrate HEFT vs SMEFT differences

Let us illustrate the whole discussion with a few simple example functions (as opposed to the more ambitious construction of entire UV completions shown in [22,62]).

1. Example flare functions \mathcal{F} where SMEFT is applicable

A couple of examples of HEFT flare functions that lead to regular SMEFT Lagrangians are:

- (i) The SM has $\mathcal{F}(h) = (1 + h/v)^2$ that of course is analytic, possesses a zero at $h_* = -v$ and trivially fulfills all correlations in Table I since $\Delta a_1 = 0 = \Delta a_2$, $a_i = 0 \ \forall i > 2$.
- (ii) The minimally composite Higgs model with symmetry breaking pattern $SO(5)/SO(4)$ (Ref. [63]), with $\mathcal{F}(h) = \frac{f^2}{v^2} \sin^2(\frac{h}{f} + \arcsin \frac{v}{f})$, which expanded to fourth order in h/v and second in v/f yields⁵

$$\mathcal{F}(h) = 1 + \left(2 - \frac{v^2}{f^2}\right) \frac{h}{v} + \left(1 - \frac{2v^2}{f^2}\right) \frac{h^2}{v^2} - \frac{4v^2 h^3}{3f^2 v^3} - \frac{v^2 h^4}{3f^2 v^4}. \quad (82)$$

It is easy to observe that this is a particular case of the SMEFT flare function at $\mathcal{O}(\Lambda^{-2})$ in Eq. (34) after the identification $c_{H\Box} = -\Lambda^2/(2f^2)$.

2. Example flare functions \mathcal{F} where SMEFT is not applicable

Examples of HEFT Lagrangians that transform to non-regular SMEFT Lagrangians are given by the models with $\mathcal{F} = e^{2h/v}$ or $\mathcal{F} = 1 + \frac{1}{2} \sin(4h/v)$. Such models fail to have a zero of \mathcal{F} , in such a way that the behavior of its root function $F[h] = (h - h^*)/v$ is fulfilled by no real value of h^* (there is no zero).

However, as seen in Sec. III, there is more than this condition in order to have an appropriate SMEFT Lagrangian in terms of H : we illustrate this with the dilatonic model [64–66], which has a HEFT function $\mathcal{F} = (1 + ah/v)^2$ that does present a zero at $h^* = -v/a$. Nonetheless, the corresponding SMEFT Lagrangian (52) happens to be singular for $a \neq 1$, with a pole at $H^\dagger H = 0$:

$$\begin{aligned} \mathcal{L}_{\text{SMEFT}} &= \mathcal{L}_{\text{SM}} + \frac{1}{2} \left[\left(\frac{1}{v} (F^{-1})'(1 + h/v) \right)^2 - 1 \right] (\partial h)^2 \\ &= \mathcal{L}_{\text{SM}} + \frac{1}{2} \frac{(1-a)}{a} (\partial h)^2 \\ &= \mathcal{L}_{\text{SM}} + \frac{1}{2} \frac{(1-a)}{a} \frac{(\partial |H|^2)^2}{2|H|^2}. \end{aligned} \quad (83)$$

It might be tempting to consider that the divergence of the second term in the second line in Eq. (83) could be cured and

⁵Up to $\mathcal{O}(v^4/f^4)$, the flare function is given by the polynomial $\mathcal{F}(h) = 1 + (2 - \frac{v^2}{f^2} - \frac{v^4}{f^4}) \frac{h}{v} + (1 - \frac{2v^2}{f^2}) \frac{h^2}{v^2} + (-\frac{4v^2}{3f^2} + \frac{2v^4}{3f^4}) \frac{h^3}{v^3} + (-\frac{v^2}{3f^2} + \frac{2v^4}{3f^4}) \frac{h^4}{v^4} + (\frac{4v^4}{15f^4}) \frac{h^5}{v^5} + (\frac{2v^4}{45f^4}) \frac{h^6}{v^6}$. This result is fully consistent with the $\mathcal{O}(\Lambda^{-4})$ SMEFT flare function in Eq. (37) for the relations $c_{H\Box}^{(6)} = -\Lambda^2/(2f^2)$, $c_{H\Box}^{(8)} = -\Lambda^4/(2f^4)$.

removed by an appropriate rescaling of h , but this would disarray the operators in \mathcal{L}_{SM} , which would not come together anymore to conform \mathcal{L}_{SM} . In this case, it happens that there is a zero in $F(h) = 1 + \frac{ah}{v}$ at $h^* = -\frac{v}{a}$ but the slope of F is not $\frac{1}{v}$ but rather $F'(h^*) = \frac{a}{v} \neq \frac{1}{v}$ for $a \neq 1$.

From a completely different approach, based on the phenomenology of the effective couplings, we could observe that the dilaton is not compatible with the SMEFT expansion, since SMEFT—in Eq. (35)—predicts $\Delta b = 4\Delta a$ up to $1/\Lambda^4$ NNLO corrections [47,67], while the dilatonic model predicts that we should be observing $\Delta b = 2\Delta a$ [64–66], with $\Delta a \equiv a - 1$, $\Delta b \equiv b - 1$. The only way SMEFT could be able to reproduce the “dilatonic data” is through a 100% correction from operators of dimension-eight and greater, indicating a breakdown of the $1/\Lambda$ expansion.

3. Example of potentials V where SMEFT is applicable

Next, we propose two Higgs-Higgs self-interaction potentials that lead to regular SMEFT Lagrangians, for example:

- (i) The SM potential (with $\lambda, -\mu^2$ both positive) is given by

$$V_{\text{SM}}(H) = \mu^2 H^\dagger H + \lambda (H^\dagger H)^2, \quad (84)$$

which, in HEFT coordinates, becomes⁶

$$V_{\text{SM}}(h) = \frac{m_h^2}{2} \left(h^2 + \frac{h^3}{v} + \frac{h^4}{4v^2} \right), \quad (85)$$

with $v^2 = -\mu^2/\lambda$ and $m_h^2 = 2|\mu|^2$.

- (ii) The typical potential with the correlations obtained there in Appendix C will need to have an expansion which, up to $\mathcal{O}(\Lambda^{-2})$ in SMEFT, needs to have the form

$$\begin{aligned} V(h) &= \frac{m_h^2}{2} \left[h^2 + \frac{h^3}{v} (1 + \epsilon) + \frac{h^4}{v^2} \left(\frac{1}{4} + \frac{3}{2}\epsilon \right) \right. \\ &\quad \left. + \frac{3\epsilon h^5}{4v^3} + \frac{\epsilon h^6}{8v^4} \right]. \end{aligned} \quad (86)$$

It is possible to see that including the custodial-invariant SMEFT operator without derivatives, \mathcal{O}_H , in Eq. (18) one gets the potential

$$V_{\text{SMEFT}}(H) = \mu^2 H^\dagger H + \lambda (H^\dagger H)^2 - \frac{c_H}{\Lambda^2} (H^\dagger H)^3, \quad (87)$$

⁶In this case, the correlations of Table III in Appendix C are trivially satisfied, because the variables there defined $\Delta v_3 = \Delta v_4 = \dots = 0$ all vanish.

which reproduces the structure of the coefficients in Eq. (86). By expanding H around its minimum, the SMEFT potential in HEFT coordinates, finally produces the structure in Eq. (86) with $m_h^2 = -2\mu^2(1 + 3\epsilon/4)$ and $2\langle |H|^2 \rangle = v^2 = v_0^2(1 - 3\epsilon/4)$, where we made use of the lowest order VEV $v_0^2 = -\mu^2/\lambda$ and the $\mathcal{O}(\Lambda^{-2})$ correction $\epsilon = -2c_H v^4/m_h^2 \Lambda^2 = \mu^2 c_H/(\lambda^2 \Lambda^2)$. Notice that, for sake of clarity in the illustration, here we have taken $c_{H\Box} = 0$, so there is no Higgs field renormalization. (Notice also that treating only terms in the potential, i.e., nonderivative couplings implies, up to a constant shift, $h = h_1$.)

4. Example of potentials V where SMEFT is not applicable

An example of a potential which cannot be written as a SMEFT is

$$V(H) = V_{\text{SM}}(H) + \frac{\epsilon}{H^\dagger H} \quad (88)$$

with ϵ a constant small enough so as to avoid unsettling the potential away from $H = 0$ by a finite fraction of v . Now there is no symmetric $O(4)$ point where the function is analytic, there is a divergence at the origin. Consistently with the symmetric-point criterion, SMEFT cannot be used: this model does not reproduce Eq. (86).

V. $w\omega \rightarrow n \times h$ FOR ALL n IN HEFT AS THE TELLTALE PROCESS: EXTRACTION OF $\mathcal{F}(h)$ EXPANSION COEFFICIENTS

In this section we will indicate how to extract the coefficients of the flare function \mathcal{F} in a process where n Higgses are produced in the final state (see Fig. 3).

First we start by noticing that the measurement of the $\omega^+ \omega^- \rightarrow h$ total cross section gives us information on the value of the first nontrivial coefficient of $\mathcal{F}(h)$, $a_1 = 2a$. The value of a is well constrained and hence we move on to identify the processes where the subsequent coefficients of the flare function can be measured.

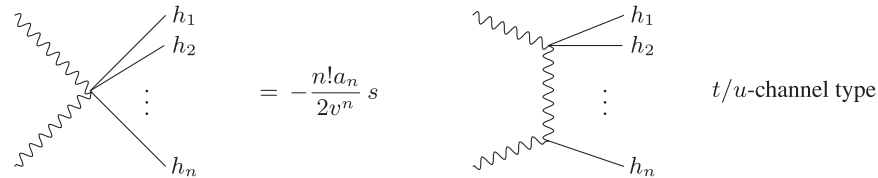


FIG. 3. Left: the a_n coefficients of the flare function \mathcal{F} control the contact piece of $\omega\omega \rightarrow n \times h$ processes. A large number n of Higgs bosons in the final state would appear as a flare of them in the detector readout, whence the nickname of the function. Right: t/u -channel type diagrams also contribute to the $\omega\omega \rightarrow n \times h$ process; they produce terms proportional to a product of a_i coefficients with $i < n$ as explained in the main text.

Generalizing to $n > 1$ Higgs bosons in the final state, the contributions to the amplitude will come from the contact diagram and the t -channel and u -channel diagrams. The contact diagram will give a contribution of $n!s a_n/(2v^n)$ whereas the t/u -channel will produce a string proportional to all the coefficients of $\mathcal{F}(h)$, a_m , for $1 \leq m \leq n-1$. So that, for generic n , the amplitude will take the form

$$T_{\omega\omega \rightarrow n \times h} = \frac{s}{v^n} \sum_{i=1}^{p(n)} \left(\psi_i(q_1, q_2, \{p_k\}) \prod_{j=1}^{|\text{IP}[n]_i|} a_{\text{IP}[n]_i^j} \right), \quad (89)$$

where $\psi_i(q_1, q_2, \{p_k\})$ are functions depending on all four-momenta involved in the process (the two Goldstone bosons having momenta q_1 and q_2 and the k th Higgs boson with momentum p_k) which will be made explicit below. These functions contribute to the angular integration used to obtain the total cross section of the process. The symbol $\text{IP}[n]$ represents the integer partitions of n and it is a collection of $p(n)$ vectors with length $|\text{IP}[n]_i|$ each, and components $\text{IP}[n]_i^j$. For example, for $n = 4$ [see Eq. (95) given shortly], $\text{IP}[4] = \{\{4\}, \{3, 1\}, \{2, 2\}, \{2, 1, 1\}, \{1, 1, 1, 1\}\}$ and hence $p(4) = 5$, and the lengths $|\text{IP}[4]_i| = \{1, 2, 2, 3, 4\}$ and $\text{IP}[4]_2^1 = 3$. In that case the amplitude takes the form

$$T_{\omega\omega \rightarrow 4 \times h} = \frac{s}{v^4} (4! a_4 + a_3 a_1 \psi_2(q_1, q_2, \{p_k\}) + a_2^2 \psi_3(q_1, q_2, \{p_k\}) + a_2 a_1^2 \psi_4(q_1, q_2, \{p_k\}) + a_1^4 \psi_5(q_1, q_2, \{p_k\})). \quad (90)$$

The strategy is to fit to data each a_n with increasing n starting from the one-Higgs boson production, then fit two-Higgs boson production, etc. We have developed a small program for the computation of the amplitudes $T_{\omega\omega \rightarrow n \times h}$ that can be provided by the authors on request. We present in the next Sec. VA the amplitudes for the production of one, two, three and four Higgs bosons.

A. Amplitudes of $\omega\omega \rightarrow n \times h$ with $n = 1, 2, 3, 4$

Formally, the amplitude $\omega\omega \rightarrow h$ with the LO HEFT Lagrangian in Eq. (3) is given by

$$T_{\omega\omega \rightarrow h} = -\frac{a_1 s}{2v}. \quad (91)$$

There is no on-shell cross section associated to this amplitude (because of the impossibility to satisfy four-momentum conservation with three on-shell massless particles). The amplitude cannot be used off shell because the Lagrangian of the EFT has been constructed on shell. Therefore we move on and quote the amplitude with two Higgs bosons in the final state, that is simply [26]

$$T_{\omega\omega \rightarrow hh} = \frac{s}{v^2}(a^2 - b) = \frac{s}{v^2}\left(\frac{a_1^2}{4} - a_2\right), \quad (92)$$

but it will be useful to introduce some notation to systematize what follows and give it in a more involved way:

$$T_{\omega\omega \rightarrow hh} = \frac{s}{v^2} \left(a_1^2 \frac{((z_1 - 2)f_1 + (z_2 - 2)f_2 + 2)}{4} - a_2 \right), \quad (93)$$

where we define, in the rest frame, the three-momentum fractions $f_i \equiv \|\vec{p}_i\|/\sqrt{s}$ ($s = 4\|\vec{q}_1\|^2$) for each Higgs boson; the angular functions $z_i \equiv 2\sin^2(\theta_i/2)$ with θ_i being the angle between the i th Higgs boson and the first ω Goldstone boson momenta, \vec{q}_1 (that is, $z_1 = 1 - \cos\theta$, $z_2 = 1 + \cos\theta$ as usual in a two-body problem with t and u channels). We also define $z_{ij} \equiv 2\sin^2(\theta_{ij}/2)$, θ_{ij} being the angle between the i th and j th Higgs bosons.

With this notation, the tree-level amplitude with a larger number of Higgs bosons can be obtained (by automated means); the one with three Higgs bosons in the final state is relatively manageable even when given in full,

$$\begin{aligned} T_{\omega\omega \rightarrow hhh} = & -\frac{s}{8v^3} \left(\mathbf{a}_1^3 \left[4f_1 f_3^2 \left(\frac{z_{23}(f_1 z_{23} - 1)}{f_3(z_3 - 2f_1 z_{23}) + f_2 z_2} + \frac{z_{13}(f_1 z_{13} - 1)}{f_1(z_1 - 2f_3 z_{13}) + f_3 z_3} \right) \right. \right. \\ & + 2f_3 \left(f_1 \left(\frac{z_{23} - 2f_2 z_{23}}{-2f_1 f_3 z_{23} + f_2 z_2 + f_3 z_3} + \frac{z_{13} - 2f_1 z_{13}}{-2f_1 f_3 z_{13} + f_1 z_1 + f_3 z_3} + z_{13} + z_{23} \right) + 3(z_3 - 2) \right) \\ & + \frac{2f_1 f_2 z_{12}(2f_1(f_2 z_{12} - 1) - 2f_2 + 1)}{f_1(z_1 - 2f_2 z_{12}) + f_2 z_2} + 2f_1(f_2 z_{12} + 3z_1 - 6) + 6f_2 z_2 - 12f_2 + 9 \left. \right] \\ & + 4\mathbf{a}_1 \mathbf{a}_2 \left[\frac{f_1^2(2z_1(-2f_2 z_{12} + f_3(z_{13} + z_{23}) - 3) - 4f_2 z_{12}(f_3(z_{13} + z_{23}) - 2) + 3z_1^2)}{2f_1 f_2 z_{12} - f_1 z_1 - f_2 z_2} \right. \\ & + \frac{2f_1 f_2(-2f_2 z_{12}(z_2 + 1) + z_2(f_3(z_{13} + z_{23}) + 3z_1 - 3) + z_{12}) + 3f_2^2 z_2^2}{2f_1 f_2 z_{12} - f_1 z_1 - f_2 z_2} + 6(f_2 + f_3 - 1) \\ & \left. - \frac{2f_1 f_3 z_{23}(2f_3(f_1 z_{23} - 1) - 2f_2 + 1)}{f_3(z_3 - 2f_1 z_{23}) + f_2 z_2} - \frac{2f_1 f_3 z_{13}(2f_1(f_3 z_{13} - 1) - 2f_3 + 1)}{f_1(z_1 - 2f_3 z_{13}) + f_3 z_3} - 3f_3 z_3 \right] + 24\mathbf{a}_3 \right). \quad (94) \end{aligned}$$

The amplitude with four Higgs bosons in the final state is complicated enough that it is worth quoting only one of the terms, corresponding to the ordering (p_1, p_2, p_3, p_4) of the four h momenta in the final state, with the other 23 permutations of these momenta not given. This one term reads

$$\begin{aligned} T_{\omega\omega \rightarrow hhhh} = & \frac{s}{16v^4} \left(\mathbf{a}_1^4 \left[\frac{(2f_1(z_1 - f_2 z_{12}) + f_2 z_2)(2f_1(-2f_2 z_{12} - f_3 z_{13} + z_1) + 2f_2(z_2 - f_3 z_{23}) + f_3 z_3)}{(f_1(z_1 - 2f_2 z_{12}) + f_2 z_2)(f_1(z_1 - 2(f_2 z_{12} + f_3 z_{13})) + f_2(z_2 - 2f_3 z_{23}) + f_3 z_3)} \right. \right. \\ & \times (f_1(z_1 - 2(2f_2 z_{12} + 2f_3 z_{13} + f_4 z_{14})) + f_2(-4f_3 z_{23} - 2f_4 z_{24} + z_2) - 2f_3 f_4 z_{34} + f_3 z_3 + 1) \left. \right] \\ & + 2\mathbf{a}_1^2 \mathbf{a}_2 \left[\frac{(f_1 z_1 + f_2 z_2)(2f_1(-2f_2 z_{12} - f_3 z_{13} + z_1) + 2f_2(z_2 - f_3 z_{23}) + f_3 z_3)}{(f_1(z_1 - 2f_2 z_{12}) + f_2 z_2)(f_1(z_1 - 2(f_2 z_{12} + f_3 z_{13})) + f_2(z_2 - 2f_3 z_{23}) + f_3 z_3)} \right. \\ & \times (f_1(z_1 - 2(2f_2 z_{12} + 2f_3 z_{13} + f_4 z_{14})) + f_2(-4f_3 z_{23} - 2f_4 z_{24} + z_2) - 2f_3 f_4 z_{34} + f_3 z_3 + 1) \\ & + \frac{(f_1(z_1 - 2(2f_2 z_{12} + 2f_3 z_{13} + f_4 z_{14})) + f_2(-4f_3 z_{23} - 2f_4 z_{24} + z_2) - 2f_3 f_4 z_{34} + f_3 z_3 + 1)}{f_1(z_1 - 2(f_2 z_{12} + f_3 z_{13})) + f_2(z_2 - 2f_3 z_{23}) + f_3 z_3} \\ & \left. \times (2f_1(-f_2 z_{12} - f_3 z_{13} + z_1) + f_2 z_2 + f_3 z_3) - \frac{(2f_1(z_1 - f_2 z_{12}) + f_2 z_2)}{f_1(z_1 - 2f_2 z_{12}) + f_2 z_2} (f_1(z_1 - 2(2f_2 z_{12} + f_3 z_{13} + f_4 z_{14})) \right. \end{aligned}$$

$$\begin{aligned}
& + f_2(-2f_3z_{23} - 2f_4z_{24} + z_2) + 1) \Big] + 4\mathbf{a}_1\mathbf{a}_3 \left[(f_1(z_1 - 2(f_2z_{12} + f_3z_{13} + f_4z_{14})) + 1) \right. \\
& + \frac{(f_1z_1 + f_2z_2 + f_3z_3)}{f_1(z_1 - 2(f_2z_{12} + f_3z_{13})) + f_2(z_2 - 2f_3z_{23}) + f_3z_3} (f_1(z_1 - 2(2f_2z_{12} + 2f_3z_{13} + f_4z_{14})) \\
& \left. + f_2(-4f_3z_{23} - 2f_4z_{24} + z_2) - 2f_3f_4z_{34} + f_3z_3 + 1) \right] \\
& + 4\mathbf{a}_2^2 \frac{s(f_1z_1 + f_2z_2)(f_1(z_1 - 2(2f_2z_{12} + f_3z_{13} + f_4z_{14})) + f_2(-2f_3z_{23} - 2f_4z_{24} + z_2) + 1)}{f_1(z_1 - 2f_2z_{12}) + f_2z_2} - 8\mathbf{a}_4 \Big) \\
& + \text{perm.} \tag{95}
\end{aligned}$$

and the 23 permutations of the four final-state Higgs momenta \mathbf{p}_i are to be taken in the computer code by invoking the amplitude with exchanged arguments. Permuting the i th Higgs with the j th Higgs will interchange $z_i \leftrightarrow z_j$, $f_i \leftrightarrow f_j$ and $z_{ik} \leftrightarrow z_{jk}$. (The indices of the a_i coefficients are of course not to be permuted, as they correspond to the terms in the Lagrangian, not the external boson legs.)

A check of these amplitudes is to take the limit to the Standard Model by setting the a_i coefficients to the values $a_1 = 2$, $a_2 = 1$, $a_3 = 0$, and $a_4 = 0$. Because the SM is renormalizable and unitary, these derivative terms must vanish, as indeed our computation reproduces, having Eqs. (92) and (94) above as well as Eq. (95) satisfy

$$T_{\omega\omega \rightarrow hh}^{\text{SM}} = 0; \quad T_{\omega\omega \rightarrow hhh}^{\text{SM}} = 0; \quad T_{\omega\omega \rightarrow hhhh}^{\text{SM}} = 0, \tag{96}$$

where conservation of momentum has been used. For a more detailed comparison, we refer to Ref. [48], which provides the physical $WW \rightarrow hh$ cross section in the SM, with $\sigma \sim 10^2$ pb [this cross section gets reduced to $\sigma \sim \mathcal{O}(\text{fb})$ for the actual LHC process $pp \rightarrow hh + 2$ jets containing this vector boson scattering].

B. Cross sections

Equations (91)–(95) and successive for an increasing number of Higgs bosons are what is needed for a phenomenological extraction of the a_i coefficients in the TeV region. From single Higgs production, through Eq. (91), a_1 is already constrained (see Sec. VI A 2), so current work focuses on two-Higgs processes which allows to address $a_2 = b$ in Eq. (92). The a_1 appears squared (and is known to 10% precision) and a_2 appears linearly, interference in this latter amplitude is possible and the sign of the deviations of a_2 from the SM value is at hand.

With a_1 and a_2 already constrained, it would become feasible to in turn constrain a_3 (null in the Standard Model) with Eq. (94) and so forth for higher coefficients with higher-point processes with more bosons in the final state. Since each successive amplitude is linear in the highest appearing coefficient, their signs can be determined if a separation from the SM value is found.

An important correlation that allows to ascertain whether SMEFT is at play comes from the observation that at order $1/\Lambda^2$ all the deviations from the SM in a_1 through a_4 stem from the same operator [see Eq. (35)]. Note also that the amplitudes in Sec. VA are the net deviations from the Standard Model in HEFT, since their SM prediction is zero. Then, all those amplitudes are necessarily linear in the same Wilson coefficient:

$$T_{\omega\omega \rightarrow nh} \propto \left(\frac{s}{v^{n-2}\Lambda^2} \right) c_{H\Box} \text{ in SMEFT up to } \mathcal{O}(\Lambda^{-2}). \tag{97}$$

This means that taking ratios of cross sections, the only parameter that encodes BSM physics in SMEFT in the relevant TeV energy region drops out, and what remains is a pure prediction, independent of the BSM physics scale, but dependent only on the structure of SMEFT:

$$\frac{\sigma(\omega\omega \rightarrow nh)}{\sigma(\omega\omega \rightarrow mh)} = \text{independent of } c_{H\Box}. \tag{98}$$

The ratios become weakly dependent on the value of the parameter when order- $1/\Lambda^4$, dimension-eight terms are included, as seen in Eq. (37). Additionally, there could be a contribution from the dimension-six custodial-symmetry breaking operator in both the SM and the SMEFT operator in Eq. (18). But if the SMEFT counting is sensible and custodial symmetry breaking suppressed, as seems to be the case, this effect should be small and a reasonable prediction is possible. It will be explored numerically in a follow-up document. This can be a way of distinguishing whether SMEFT is applicable or not, from “low” energy data, without access to the underlying UV completion of any new physics. To use the equation in practice, one can compare the predictions from the explicit computation of Eq. (97) with experimental data, and see whether indeed there is no room for a discrepancy at order $1/\Lambda^2$.

Scattering amplitudes based on Goldstone bosons can be related to interpreted as scattering amplitudes of longitudinal gauge bosons at the expense of $\mathcal{O}(m_W/\sqrt{s})$ corrections. At WW center-of-mass energy of 800 GeV

(challenging but not outlandish), this becomes a 1% expansion parameter, of the size of α_{EM} . It is therefore a very reasonable starting point for energies well enough over the threshold. The actual comparison with experimental data still needs to be eventually done with physical W states. However, the Goldstone couplings discussed in the article (a_1, a_2, a_3, \dots) provide the processes with physical W s as well. For example, as it happens with $\omega^+\omega^- \rightarrow hh$, $W^+W^- \rightarrow hh$ is also fully determined by $a_1 = 2a$ and $a_2 = b$ at LO in HEFT [48,68], even beyond the naive equivalence theorem [34,35]. Furthermore, a complete comparison with data needs to take these intermediate gauge bosons into account. This is no problem, nevertheless, as HEFT's leading order Lagrangian [17] already incorporates all possible W interactions, determined at that order by the same a_1, a_2, \dots . Still, the possibility of separating longitudinal and transverse W polarizations in data analysis would improve the efficiency of theory-experiment comparisons. Thus, precise enough experimental data should be able to recover the flare-function coefficients that set the amplitudes.

VI. FINDING OUT WHETHER THE $\mathcal{F}(h)$ FUNCTION HAS A ZERO

Among the precise conditions that allow to express a HEFT as a SMEFT, thoroughly studied in [22], the first necessary requirement among those spelled out in Sec. III C is the existence of an $O(4)$ symmetric point $h = h_*$. This requires a zero, which recalling the Taylor expansion in Eq. (4) yields the relation

$$\mathcal{F}(h_*) = 1 + \sum_{n=1}^{\infty} a_n \frac{h_*^n}{v^n} = 0.$$

In this section we will try to address what can be done, empirically and assuming that any UV physics is not known or understood (bottom-up approach) to improve the knowledge of whether such zero h_* could be present.

A. Finding the $O(4)$ fixed point candidate by looking at the polynomial approximation of $\mathcal{F}(h)$

1. From one or two Higgs production

Knowledge of the a_i coefficients is rapidly evolving, as they directly correspond to the κ_i scaling cross sections with respect to the Standard Model ones. A data-driven constraint for a_1 based on LHC run I data can be found in [69]; at 2σ , those authors conclude that $a \in [0.7, 1.23]$. A bound on b was originally obtained by examining the absence of a resonance in $W_L W_L$ scattering below 700 GeV [70] (according to [61], the dispersive methods used for obtaining these bounds have a 10%–20% uncertainty on the position of the resonance). Direct ATLAS and CMS work has improved those earlier limits, and the latest bounds

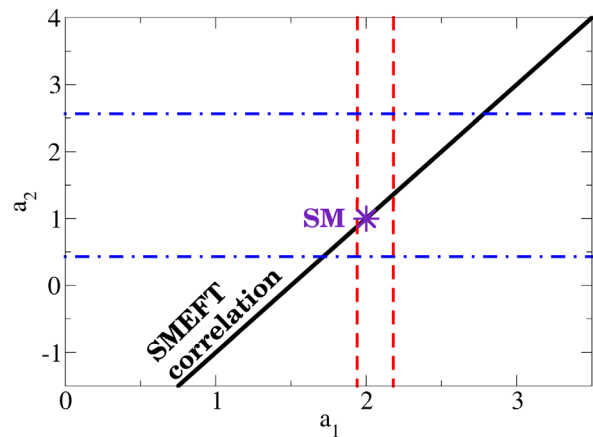


FIG. 4. SMEFT at order $1/\Lambda^2$ predicts the correlation $a_2 = 2a_1 - 3$ from the first column in Table I, which is plotted against the current 95% confidence intervals for these two HEFT parameters [51,53].

on the first two a_i coefficients are discussed next in Sec. VI A 2; those coefficients $a_1 = 2a$ and $a_2 = b$ remain the only ones with current experimental constraints.

In Fig. 4, a straight line shows the SMEFT correlation obtained in the first column of Table I. The rest of the plane corresponds at most to HEFT theory. The SM is the point in the center of the figure. Finally the 95% confidence bands for the a_1 and a_2 parameters are presented as dashed lines with the numbers taken from the caption of Table II.

TABLE II. We input the 95% confidence-level experimental bounds $a_1/2 = a \in [0.97, 1.09]$ [51] and, for the middle column, $a_2 = b = \kappa_{2V} \in [-0.43, 2.56]$ [52] (see the second erratum), by the ATLAS collaboration (top row) or the CMS collaboration (bottom row) interval of $a_2 = b = \kappa_{2V} \in [-0.1, 2.2]$ [53]. With them we have calculated and show here the expected corresponding 95% confidence limits (C.L.) intervals for several $W_L W_L \sim \omega\omega \rightarrow nh$ coupling, a_n , employing the relations of Table I. Violations of the intervals in the first column would sow doubt on the SMEFT adequacy at $\mathcal{O}(\Lambda^{-2})$; surpassing any in the third column, on its perturbativity; and those of the middle column would void SMEFT of much significance as an EFT. They can be further tightened with improved experimental data for κ_{2V} .

Consistent SMEFT range at order Λ^{-2}	Consistent SMEFT range at order Λ^{-4}	Perturbativity of Λ^{-4} SMEFT
$\Delta a_2 \in [-0.12, 0.36]$	ATLAS	ATLAS
$a_3 \in [-0.08, 0.24]$	$a_3 \in [-4.1, 4.0]$	$a_3 \in [-3.1, 1.7]$
$a_4 \in [-0.02, 0.06]$	$a_4 \in [-4.2, 3.9]$	$a_4 \in [-3.3, 1.5]$
$a_5 = 0$	$a_5 \in [-1.9, 1.8]$	$a_5 \in [-1.5, 0.6]$
$a_6 = 0$	$a_6 = a_5$	$a_6 = a_5$
	CMS	CMS
	$a_3 \in [-3.2, 3.0]$	$a_3 \in [-3.1, 1.7]$
	$a_4 \in [-3.3, 3.0]$	$a_4 \in [-3.3, 1.5]$
	$a_5 \in [-1.5, 1.3]$	$a_5 \in [-1.5, 0.6]$
	$a_6 = a_5$	$a_6 = a_5$

2. Multiple Higgs production: Testing the SMEFT-induced correlations over the HEFT function \mathcal{F} coefficients

We employ the correlations found earlier in Table I, in conjunction with current direct experimental bounds on deviations of a_1 and a_2 from their Standard Model values, to propagate the information to other coefficients of \mathcal{F} that are presently unconstrained *provided SMEFT holds*.

These are then quoted in Table II, an interesting new contribution of this article to the phenomenology of HEFT. If, for example, a_3 is measured to be different from zero, this would immediately establish new physics (which is known); but additionally, if it exceeds the bounds given in the Table, it would mean that SMEFT correlations are being violated and the EFT has to be extended into HEFT.

The constrains in the first column assume the validity of SMEFT up to order $1/\Lambda^2$, $\mathcal{O}(\Lambda^{-2})$; because of the tight experimental bounds on the $WW \rightarrow h$ coupling $a_1 = 2a$, the remaining a_n couplings are strongly limited. If SMEFT is considered up to $1/\Lambda^4$, $\mathcal{O}(\Lambda^{-4})$ (as we do in the second column), the $WW \rightarrow hh$ coupling $a_2 = b$ becomes independent, as seen in Table I; its experimental bounds are then also an input. Being poorly measured so far, it introduces a large uncertainty in the higher correlations. Thus, the bounds in the second column of Table II are much looser. Those large uncertainties can be much reduced by improving the experimental knowledge of a_2 : a decrease of its uncertainty by an order of magnitude scales almost linearly and makes these errors roughly a factor 10 smaller.

Notice that the values in the third column in Table II are similar for ATLAS and CMS. The reason is that when the experimental uncertainty of Δa_2 is very large, at the practical level, its only limitation comes from the constraint $|\Delta a_2| \leq 5|\Delta a_1|$; this is $\min(5a_1^-, -5a_1^+) \leq \Delta a_2 \leq \max(-5a_1^-, 5a_1^+)$. Since effectively the bounds just depend on the allowed values for a_1 we are obtaining the same outcomes for ATLAS and CMS in the third column.

B. When Schwarz's lemma guarantees a function's zero

In this subsection we examine and adapt a known result from complex-variable analysis that guarantees the existence of a zero of a complex function: in the case of $\mathcal{F}(h)$ this would be an $O(4)$ fixed point candidate around which SMEFT could be built.

The information that we would eventually need to have at hand to exploit the theorem would be a number of coefficients of the Taylor series, depending on any future accelerators energy reach (Sec. V). To avoid too large a mathematical digression, Schwarz's lemma and two of its corollaries are detailed in Appendix A. What can guarantee a zero of \mathcal{F} is the second corollary. The needed hypotheses are as follows:

- (i) First, the function $\mathcal{F}(h)$ (extended to be a complex function of a complex h argument, in units of v

throughout this whole section) needs to be analytic inside a disk of radius $|h| = R$ around the vacuum $h = 0$. This disk has to be large enough to reach the possible symmetric point (i.e., $h = h_*$ or, in SMEFT, $|H| = 0$) from the observed vacuum (i.e., $\langle |H| \rangle = 1/\sqrt{2}$, or in HEFT $\langle h \rangle = 0$), where one constructs the flare function $\mathcal{F}(h)$.

- (ii) Second, the image of that disk (the set of possible values of \mathcal{F}) has to be contained inside another disk of radius M (the maximum value of $|\mathcal{F}|$) centered at $\mathcal{F}(0) = 1$. Finally, the derivative of the function is assumed to have been measured, so that $\mathcal{F}'(0) = a_1$ is known.

The second corollary then guarantees that a disk of radius $R_{\min} := R^2 a_1^2 / \gamma M$ centered at $\mathcal{F} = 1$ is completely contained in the image of \mathcal{F} . Here γ is

$$\gamma = \frac{(\sqrt{2} + 2)(\sqrt{2} + 1)}{\sqrt{2}} \simeq 5.83. \quad (99)$$

Therefore, a zero of $\mathcal{F}(h)$ is ensured if that radius R_{\min} is greater than 1 (so that $\mathcal{F} = 0$ can be reached from $\mathcal{F} = 1$),

$$R^2 a_1^2 / \gamma M > 1 \Rightarrow \exists h^* |\mathcal{F}(h^*) = 0. \quad (100)$$

Depending on how large the a_i coefficients end up being, this lemma could provide a tool to extract a scale at which one is sure that there exists an $O(4)$ fixed point candidate.

To use that second corollary in Appendix A, notice that by construction we have that $\mathcal{F}(0) = 1$ and hence we can employ the auxiliary $g(h) \equiv \mathcal{F}(h) - 1$ satisfying $g(0) = 0$ and $g'(0) = \mathcal{F}'(0) = a_1$, which is the g to which the corollary applies. This means that, if the function $\mathcal{F}(h)$ is analytic in the open disk of radius R , denoted as $D(0, R)$, then we will have that the condition for the existence of at least one (complex) value of the Higgs field $h^* \in D(0, R)$ such that $\mathcal{F}(h^*) = 0$ is [see Eq. (A8) below]

$$R^2 > \frac{\gamma M}{a_1^2}, \quad (101)$$

where M is the maximum value that $|\mathcal{F}(h)|$ takes for $h \in D(0, R)$. For clarification see Fig. 5. Regrettably, the application of the lemma will give a definite positive answer to the existence of a zero if $M(R)$ is at most R^2 , which means that we can only profit from the lemma for polynomials of order up to 2 (due to analyticity). This still leaves room for some cases that we explore below, saliently including the variations around the SM that are conceivable in the near future, with \mathcal{F} up to order 4.

Experimentally, the full $\mathcal{F}(h)$ cannot be measured. It is only its Taylor expansion that can be accessed in practice (unless the SM UV completion is directly observed, of course). Hence, we must follow the logic:

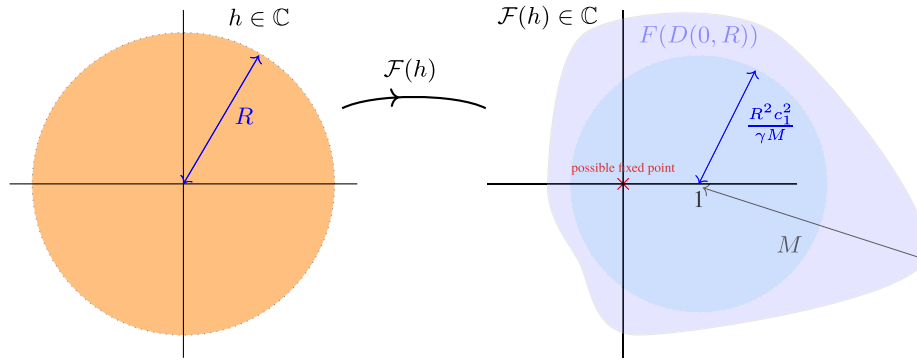


FIG. 5. Left-hand side: the disk of radius R , $D(0, R)$ (orange), is the region where the Taylor approximation of the $\mathcal{F}(h)$ function is supposedly trusted, which can only be experimentally assessed. Right-hand side: the grayish outer region is the image of $D(0, R)$, namely $\mathcal{F}(D(0, R))$, and M is the maximum distance of $\mathcal{F}(D(0, R))$ to 1 (thus, the maximum value of $|\mathcal{F} - 1|$ for $|h| \leq R$). Under the conditions of applicability for Schwarz's lemma, we can assure that the disk on the right (bluish), $D(1, \frac{R^2 c_1^2}{\gamma M})$, is contained in $\mathcal{F}(D(0, R))$, i.e., $D(1, \frac{R^2 c_1^2}{\gamma M}) \subset \mathcal{F}(D(0, R))$. When it happens that $\frac{R^2 c_1^2}{\gamma M} > 1$, the radius of the disk in the image \mathcal{F} plane around $\mathcal{F}(h = 0) = 1$, we are assured that $\mathcal{F}(h)$ has a zero for some $h^* \in D(0, R)$.

- (1) First we must assume that $\mathcal{F}(h)$ is analytic in a neighborhood of the $h = 0$ physical vacuum (and hence its Taylor expansion, and thus HEFT, makes sense). This region can be taken as the open disk $D(0, R)$.
- (2) Suppose we measure k coefficients of the Taylor expansion of the function $\mathcal{F}(h)$ such that, for h in units of v ,

$$\mathcal{F}(h) = 1 + \sum_{i=1}^k a_i h^i + R_k(h),$$

where of course we trust the expansion up to an energy scale such that we know that, for $h \in D(0, R)$, $\mathcal{F}(h)$ is analytic.⁷ Here $R_k(h)$ is the remainder of the Taylor expansion. We must neglect this Taylor remainder since, although it is bounded by $\{\max_{|h|=R} |\mathcal{F}(h)|\} \beta^{k+1} / (1 - \beta)$ for $\beta \in [|h|/R, 1)$, it cannot be experimentally accessed.

- (3) Assign

$$M = \max_{|h|=R} \left| \sum_{i=1}^k a_i h^i \right|.$$

The zeroth order coefficient is omitted because we must use the maximum of $g(h) \equiv \mathcal{F}(h) - 1$, as described in the Appendix A. Notice that the maximum modulus of $\sum_{i=1}^k a_i h^i$ is reached at the boundary of its domain thanks to the *maximum modulus principle*.

⁷The difficulty here is that, unlike in analyticity in Mandelstam s that ultimately follows from causality via Titchmarsh's theorem [59], it is hard to find a guiding principle in h space that justifies assuming analyticity. At least we are exposing the necessary hypothesis, which is often taken for granted when writing down a SMEFT.

- (4) Using the second corollary we will have that we can assure the presence of an $O(4)$ fixed point if we reach a field intensity such that

$$|h|^2 = R^2 > \frac{\gamma M}{a_1^2}. \quad (102)$$

1. Standard Model case

This discussion has been quite abstract, so let us try to apply Eq. (100) in practice. The first obvious example is the Standard Model.

We can apply Schwarz's lemma to either the $F(h)$ function, in the SM $F(h) = 1 + h$, or its square \mathcal{F} , the flare function. In the first case we see that $F(h)$ is analytic for all $h \in \mathbb{C}$, and hence we can take R as big as we want. It is immediate to see that $M = R$ so that we can assure the presence of a zero of $F(h)$ whenever

$$R^2 > \gamma R, \quad (103)$$

which can be met for $R = |h| > \gamma$ (in units of v).

If instead we apply Schwarz's lemma directly to the flare function $\mathcal{F}(h) = F^2(h)$, we find no useful information, as can be understood from the result in the next example.

2. Generic second-order polynomial

Taking $\mathcal{F}(h) = 1 + a_1 h + a_2 h^2$, the condition to assure the presence of a fixed-point candidate becomes

$$R^2 > \frac{\gamma}{a_1^2} (|a_1| R + |a_2| R^2) \geq \frac{\gamma}{a_1^2} \max_{|h|=R} (|a_1 h + a_2 h^2|). \quad (104)$$

So that for R sufficiently large, the condition will be met if $1 > \frac{\gamma |a_2|}{a_1^2}$, i.e., $a_1^2/a_2 > \gamma$ assures the presence of a fixed-point candidate. For the known $a_1 = 2$ central value we obtain that

$$\frac{4}{\gamma} > |a_2| \Rightarrow \text{if } a_2 \in (-0.68, 0.68) \Rightarrow \text{zero of } \mathcal{F} \text{ assured.} \quad (105)$$

This result is in agreement with the condition of positivity on the discriminant of the polynomial which gives $\frac{a_2^2}{4} > a_2$ (SMEFT region in Fig. 4) and hence guarantees a zero of $\mathcal{F}(h^*) = 0$ for $h^* \in \mathbb{R}$.

Comparing to the interval for a_2 given by experiment and quoted in Table II, we see that for negative a_2 , the experimental bound is already inside the Schwarz's lemma limit; if the upper experimental limit also drops into the 0.68 boundary (which is not unthinkable, only a factor 3 better than the current LHC extraction), then Schwarz's lemma will tell us that a zero of \mathcal{F} is at hand unless new discoveries of higher a_i coefficients require further scrutiny. Because a third-order polynomial always has a real zero, this takes us to a fourth order one, discussed in the next paragraph.

3. Perfect-square, fourth-order polynomial

Taking now a quadratic $F(h) = 1 + ah + \beta h^2$ entails a quartic flare function

$$\mathcal{F}(h) = 1 + 2ah + (\alpha^2 + 2\beta)h^2 + 2\alpha\beta h^3 + \beta^2 h^4. \quad (106)$$

In this case, the condition of Eq. (102) that guarantees the presence of a symmetric point candidate h_* becomes

$$\frac{\alpha^2}{\gamma} > \beta.$$

Squaring the above relation, for the central value $a_1 = 2$, we get the bound on the fourth order coefficient:

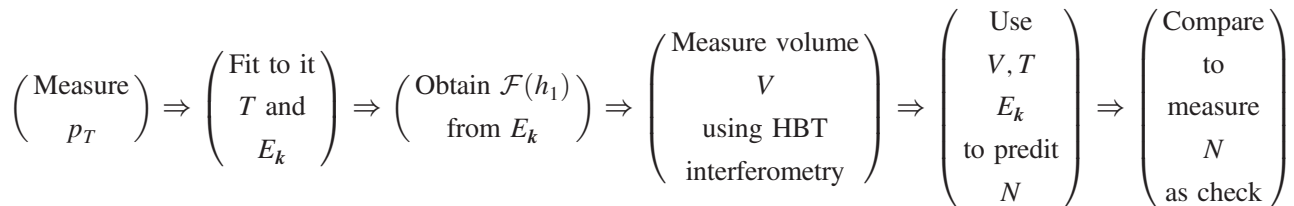
$$\frac{a_1^4}{4\gamma^2} > a_4 \Rightarrow \text{if } a_4 \in (-0.118, 0.118) \Rightarrow \text{zero of } \mathcal{F} \text{ assured.}$$

Notice of course that if a_4 is measured to be negative, higher order terms will be needed in the expansion of \mathcal{F} (see Sec. IVA 1) to guarantee its positivity.

VII. FAR FUTURE: MULTIPLE HIGGS PRODUCTION IN EXTREME- T COLLISIONS TO ACCESS THE SM SYMMETRIC POINT

The pion was first discovered in 1947 [71,72] when precious few events from cosmic rays were obtained in photographic emulsions taken at high altitudes; nowadays, they are routinely produced by the thousands per event in central heavy-ion collisions at the LHC [73]. Whereas currently multiple Higgs-boson events (or for that matter, multiple longitudinal gauge-boson ones) are not possible, one day they might come within reach. At that point, the entire $\mathcal{F}(h)$ function (or at least, to a very large order in the Taylor expansion) may become part of potential observables. We wish to illustrate the possibility of accessing it with such future work in this subsection.

The idea of a large number of Higgs bosons (which inspires the title of this article) has been put forward before [74–76] although in a different context, in a proposal to solve the hierarchy problem. Here, we notice that the appearance of $\mathcal{F}(h)$ in Eq. (3) makes it that, in a thermal medium with temperatures of order the hundreds of GeV (over 2 orders of magnitude beyond what is possible today, but not an arbitrarily large scale, and within the validity of the EFT), the process $X \rightarrow n \times h + m \times W_L/Z_L$ with a thermal distribution becomes possible. In the next lines we propose a very schematic analysis chain that proceeds according to the following flow diagram:



A. Transverse-momentum distribution

As pioneered by Hagedorn [77,78], an observable revealing the statistical distribution is the p_T distribution of the bosons produced.

In the case of a free-boson gas with Lagrangian $\frac{1}{2}((\partial h)^2 + (\partial \omega_i)^2)$ this is given [79] by

$$\frac{d^2 N}{2\pi p_T dp_T d\eta} \Big|_0 = m_T \frac{gV}{(2\pi)^3} e^{\frac{\mu-E}{T}}. \quad (107)$$

While μ is the chemical potential associated to a conserved particle number (which can be left out if events with a different number of bosons are considered), V is the volume

of the source and $g = 1, 3$ or 4 depending on what is measured (h , V_L or both), we want to call attention to the transverse masslike quantity $m_T = \sqrt{m^2 + p_T^2} \simeq p_T$.

Integrating Eq. (107) over the longitudinal momentum (or rapidity) then yields a typical p_T distribution:

$$f_{\text{Bose}}(p_T) dp_T = \text{constant} \times p_T dp_T \int_0^\infty dp_x \frac{1}{e^{\frac{1}{T} \sqrt{p_x^2 + m_T^2}} - 1}. \quad (108)$$

In the simplest free gas described by Eq. (107), $f_{\text{Bose}}(p_T)$ falls off as a simple exponential. This Boltzmann-like dependence is obtained from mean occupation numbers

$$\bar{n}_\alpha(k) = X_\alpha(k) \frac{\partial \log Z}{\partial X_\alpha(k)} \quad (109)$$

with $X_\alpha(k) = \exp(-E_\alpha(k)/T)$ and the partition function expressed [77] as

$$Z = \sum_n \prod_{ak} X_\alpha(k)^{\bar{n}_\alpha(k)}. \quad (110)$$

Here $k = 0, 1, 2, \dots, \infty$ as corresponds to a boson occupation number. The momentum distribution can then be obtained from the density of states $\frac{V}{2\pi^2} d^3 p$.

The p_T dependence of Eq. (108) is modified in the interacting theory: this is what gives access to the function $\mathcal{F}(h)$.

In case there is an interacting Hamiltonian, containing the $\mathcal{F}(h)$ function, it is possible, through the statistical distribution of bosons to access it almost completely or at least to a very high degree in the a_i expansion. Such statistical distribution [80] [see Eq. (26.6) in page 251 there, though in a nonrelativistic treatment) will amount to

$$\frac{dN}{d^3 \mathbf{k}} = \frac{-gV}{(2\pi)^3} T \sum_{n \in \mathbb{Z}} \frac{1}{i\omega_n - \sqrt{E_k^2 + \Sigma(\mathbf{k}, i\omega_n)}}, \quad (111)$$

where $E_k = \sqrt{\mathbf{k}^2 + m^2}$, the summation is carried over Matsubara frequencies $\omega_n = 2\pi n T$, and $\Sigma(\mathbf{k}, k_0)$ is the self-energy of the (Higgs or Goldstone) bosons defined through

$$[G(\mathbf{k}, k_0)]^{-1} = [[G^0(\mathbf{k}, k_0)]^{-1} - \Sigma(\mathbf{k}, k_0)]^{-1}. \quad (112)$$

The propagator $G(\mathbf{k}, i\omega_n)$ (see [81,82] for a detailed discussion on analytic continuation of propagators) can be computed from the Euclidean partition function, which has a path-integral representation,

$$Z = \int \mathcal{D}h \mathcal{D}\omega \exp \left\{ - \int_0^\beta d\tau \int d^3 x \left[\frac{1}{2} \mathcal{F}(h) \partial_\mu \omega^i \partial_\mu \omega^j \left(\delta_{ij} + \frac{\omega^i \omega^j}{v^2 - \omega^2} \right) + \frac{1}{2} \partial_\mu h \partial_\mu h \right] \right\}, \quad (113)$$

where summation in the Euclidean μ indices is assumed. We then directly see how the $\mathcal{F}(h)$ function affects the statistical distribution of bosons through their self-energy. The coordinate-space representation of this propagator for Higgs bosons will simply amount to

$$G(\mathbf{x}, \tau) = \int \mathcal{D}h \mathcal{D}\omega h(\mathbf{x}, \tau) h(\mathbf{0}, 0) \exp \left\{ - \int_0^\beta d\tau \int d^3 x \left[\frac{1}{2} \mathcal{F}(h) \partial_\mu \omega^i \partial_\mu \omega^j \left(\delta_{ij} + \frac{\omega^i \omega^j}{v^2 - \omega^2} \right) + \frac{1}{2} \partial_\mu h \partial_\mu h \right] \right\}. \quad (114)$$

Once this path integration has been estimated on the lattice or by other means, the self-energy from Eq. (112) can be extracted, and substituting it into Eq. (111), a p_T spectrum directly comparable with experiment can be obtained as a functional of \mathcal{F} .

B. Number of Higgs bosons

Additionally, we can try to get an idea of what is the number of Higgs bosons that should be produced in an experiment in order to access the SM $O(4)$ symmetric point $h_* = -v$. The SM Higgs potential is

$$V_{\text{SM}}(\phi) = \frac{1}{2} \mu^2 \phi \cdot \phi + \frac{1}{4} \lambda (\phi \cdot \phi)^2, \quad (115)$$

where $-\mu^2, \lambda > 0$ and $\phi \cdot \phi = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2$. Choosing the unitary gauge, the SM vacuum sits at $\phi_1 = \phi_2 = \phi_3 = 0$ and $\phi_4 = v$ and $\phi = \phi_4 = \sqrt{-\mu^2/\lambda}$. After redefining $\phi_4 = v + h$ the physical Higgs mass amounts to $m_h = \sqrt{2\lambda v^2}$, which using $m_h = 125.3$ GeV and $v = 246$ GeV gives $\lambda \simeq 0.13$.

The invariant point under $O(4)$ in field space is the origin $\phi = 0$. The difference of potential energy density between the SM vacuum and the SM $O(4)$ invariant point is

$$\begin{aligned} \Delta V &= V(0) - V(v) = \frac{1}{4} \lambda v^4 \simeq 1.19 \times 10^8 \text{ GeV}^4 \\ &= 1.49 \times 10^{10} \text{ GeV/fm}^3. \end{aligned}$$

Now we wish to translate this energy density into a temperature, for doing so we look for T such that

$$\varepsilon(T) \equiv \frac{1}{(2\pi)^3} \int d^3\mathbf{k} \frac{E_{\mathbf{k}}}{e^{E_{\mathbf{k}}/k_B T} - 1} = \Delta V, \quad (116)$$

where $E_{\mathbf{k}} = \sqrt{c^2 \hbar^2 \mathbf{k}^2 + m^2 c^4}$ is the relativistic energy of a boson with three-momentum \mathbf{k} . This gives a temperature of $k_B T = 140$ GeV (which matches the EW second order phase transition critical temperature [83]). Using this temperature we are ready to compute the number density of Higgs bosons at a temperature where the SM symmetric point is reached:

$$\begin{aligned} n(T) &= \frac{N}{V} = \frac{1}{(2\pi)^3} \int d^3\mathbf{k} \frac{1}{e^{E_{\mathbf{k}}/k_B T} - 1} \\ &= 258 \text{ Higgs bosons}/(0.1 \text{ fm})^3. \end{aligned} \quad (117)$$

This is certainly a daunting concentration of energy and particle number that is not expected in the foreseeable future. But when/if it is achieved, the absolute number can serve as cross-check of the p_T spectrum line shape to extract a temperature (hopefully the same) if the volume of the hot source, addressed next, is known.

C. Obtaining the volume of a multi-Higgs source

It remains to guess what would be the hot source volume that a future multi-Higgs factory (i.e., a collider capable of

producing statistically significant numbers of Higgs bosons) could achieve. This type of machine would allow us to explore the existence of such a symmetric point by directly heating the electroweak sector to populate it, and to explore the properties of the EW phase transition.

With the data in hand, the volume could be obtained by using Hanbury-Brown-Twiss (HBT) or its particle physics analog Goldhaber-Goldhaber-Lee-Pais (GGLP) interferometry: the Higgs bosons exiting the collision would retain memory (by interference) of the radius of the source that emitted them. The technique is routinely used in astrophysics to establish the size of astrophysical objects from the emitted photons, and in nuclear collisions by analyzing pions. That future electroweak collider could likewise obtain the radius of a hot electroweak ball from the Goldstone and Higgs bosons emitted.

Let us state schematically how this interferometry works (a comprehensive review can be found in [84]). Suppose that a source has emission points continuously distributed in a space-time volume V_4 with an emission probability amplitude, $\Pi(r; \mathbf{k})$, of emitting a particle with momentum \mathbf{k} [on the mass shell with plane-wave wave function $\psi_{\mathbf{k}}(\mathbf{r}) \propto e^{i\mathbf{k}\cdot\mathbf{r}}$] at the space-time point r . Hence, the total probability of observing the emission of one particle with momentum \mathbf{k} from the source is $P(\mathbf{k}) = \int_{V_4} d^4 r |\Pi(r; \mathbf{k})|^2$.

Likewise, the total probability of measuring two particles with momenta \mathbf{k}_1 and \mathbf{k}_2 , assuming the two emissions are uncorrelated, i.e., $\Pi(r_1, r_2; \mathbf{k}_1, \mathbf{k}_2) = \Pi(r_1; \mathbf{k}_1)\Pi(r_2; \mathbf{k}_2)$, amounts to

$$\begin{aligned} P(\mathbf{k}_1, \mathbf{k}_2) &= \int_{V_4} d^4 r_1 d^4 r_2 \left| \frac{\psi_{\mathbf{k}_1}(\mathbf{r}_1)\psi_{\mathbf{k}_2}(\mathbf{r}_2) + \psi_{\mathbf{k}_1}(\mathbf{r}_2)\psi_{\mathbf{k}_2}(\mathbf{r}_1)}{\sqrt{2}} \right|^2 |\Pi(r_1; \mathbf{k}_1)|^2 |\Pi(r_2; \mathbf{k}_2)|^2 \\ &= P(\mathbf{k}_1)P(\mathbf{k}_2) + \int_{V_4} d^4 r_1 d^4 r_2 \cos[(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{k}_1 - \mathbf{k}_2)] |\Pi(r_1; \mathbf{k}_1)|^2 |\Pi(r_2; \mathbf{k}_2)|^2. \end{aligned} \quad (118)$$

The GGLP experiment could be adapted to measuring, at two detection points \mathbf{d}_1 and \mathbf{d}_2 , two Higgs bosons with precise momentum \mathbf{k}_1 and \mathbf{k}_2 respectively (see Fig. 6). The correlation function among the two momenta is

$$C(\mathbf{k}_1, \mathbf{k}_2) := \frac{P(\mathbf{k}_1, \mathbf{k}_2)}{P(\mathbf{k}_1)P(\mathbf{k}_2)} = 1 + \frac{\int_{V_4} d^4 r_1 d^4 r_2 \cos[(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{k}_1 - \mathbf{k}_2)] |\Pi(r_1; \mathbf{k}_1)|^2 |\Pi(r_2; \mathbf{k}_2)|^2}{P(\mathbf{k}_1)P(\mathbf{k}_2)}. \quad (119)$$

Under the assumptions explained thoroughly in [85] [neglect of higher order symmetrization, smoothness and equal time approximations, useful for large (RHIC-like) sources] the correlation function in Eq. (119) simplifies to

$$\begin{aligned} C(\mathbf{k}_1, \mathbf{k}_2) - 1 &= \int d^3 \mathbf{r}' \mathcal{S}_{\mathbf{K}}(\mathbf{r}') \cos[(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{k}_1 - \mathbf{k}_2)] \\ \mathcal{S}_{\mathbf{K}}(\mathbf{r}') &:= \frac{\int_{V_4} d^4 r_1 d^4 r_2 |\Pi(r_1; \mathbf{k}_1)|^2 |\Pi(r_2; \mathbf{k}_2)|^2 \delta(\mathbf{r}' - \mathbf{r}_1 + \mathbf{r}_2)}{P(\mathbf{k}_1)P(\mathbf{k}_2)}, \end{aligned} \quad (120)$$

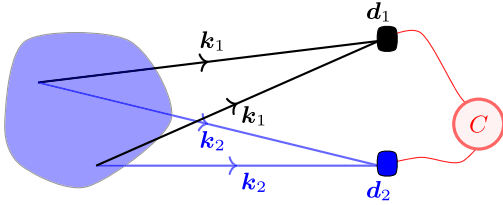


FIG. 6. HBT/GGLP interferometry: Detecting two particles with momenta \mathbf{k}_1 and \mathbf{k}_2 at the respective detection points \mathbf{d}_1 and \mathbf{d}_2 and studying their correlation gives information about the dimensions of the homogeneity region of source in bluish.

where $\mathbf{K} = \mathbf{k}_2 + \mathbf{k}_1$ is the total momentum of the pair of outgoing particles. The function $\mathcal{S}_{\mathbf{K}}(\mathbf{r}')$ encodes “the distribution of relative positions of particles with identical velocities and total momentum \mathbf{K} ” [85] and it gives information about the size of the region of homogeneity of a source (i.e., the region where the equilibrium assumptions can be taken). The curvature of $C(\mathbf{k}_1, \mathbf{k}_2)$ at $\mathbf{q} := \mathbf{k}_1 - \mathbf{k}_2 = 0$ is related to the mean-square separation of the three-dimensional quadrupolar moments [85]

$$-\left. \frac{C(\mathbf{k}_1, \mathbf{k}_2)}{dq_i dq_j} \right|_{\mathbf{q}=0} = \int d^3r \mathcal{S}_{\mathbf{K}}(\mathbf{r}) r_i r_j. \quad (121)$$

In this way we can obtain the volume of the region where the source can be considered homogeneous and the equilibrium conditions apply.

VIII. CONCLUSIONS

In this article we have bridged between the SMEFT and HEFT formalisms following the work of other groups [19–23]. We have focused on the Higgs-flare function \mathcal{F} that controls the derivative couplings of two Goldstone bosons ω_i to any number of Higgs bosons. We have exhaustively studied this flare function \mathcal{F} and particularly addressed the existence of its key zero at a symmetric point in the (ω_i, h) field space. In what follows we summarize the novel results presented here.

Saliently, we have provided a simple derivation of the previously known conditions on the flare function $\mathcal{F}(h)$ and its derivatives around the EW symmetric point. By comparing the Taylor expansions between the symmetric point and our vacuum, the correlations induced by SMEFT’s validity on HEFT coefficients are exposed at fixed order in the EFT expansion. We believe that this will extend the understandability of the criteria for SMEFT to exist, so far mostly discussed in geometric terms.

We extend previous results concerning the expression of a few coefficients of this function in terms of the $c_{H\Box}$ Wilson coefficient of SMEFT in Sec. II; in our work we have addressed a larger number of such coefficients, we have employed the Warsaw basis, in contrast with earlier analyses, and we have proceeded to the next order $(1/\Lambda^4)$

in the SMEFT expansion. We have identified the relevant TeV-scale SMEFT operators at dimensions-six and -eight; we have then employed the correlations that we here report, together with ATLAS and CMS constraints in order to identify the HEFT coefficient space where SMEFT at either dimension-six or dimension-eight can be deployed.

With the latest ATLAS and CMS bounds on the a_1 (also known as $2\kappa_V$) and a_2 (known as κ_{2V}) coefficients we have explicitly given 95% confidence intervals for a few a_i , $i > 2$ ones, that if exceeded would automatically rule out SMEFT, at least to the orders here considered, and point out the need of extending the SMEFT framework.

Further, we have completely eliminated the Wilson coefficients and obtained correlations that are solely expressed in terms of the HEFT parameters and can be used to falsify SMEFT itself from within the wrapping theory (Secs. III and IV). These correlations provide simple tests that analysts following upcoming and future experimental data can employ to test the framework of SMEFT. This is the thrust of a companion paper submitted together with this manuscript [86] that provides some additional discussion.

It may be useful for those analyses to have explicit expressions of the $\omega\omega \rightarrow nh$ amplitudes in HEFT and therefore we explicitly provide in Sec. VA the leading order in perturbation theory of those with lowest Higgs number $n = 1, 2, 3, 4$. An automated program applicable to generic n can be provided on demand. We have also noticed that, to leading order in the SMEFT counting and in the TeV scale, ratios of cross sections are pure numbers whose value is intrinsic to the structure of SMEFT and independent of the order-six Wilson coefficient. In future work, we additionally plan to address such cross sections with the physical experimental cuts so that the parameter space can directly be compared with data.

Finally, a few additional original contributions of this work are listed here: Section IV provides a thorough study of the flare function $\mathcal{F}(h)$ and its properties such as positivity from the boundedness from below of the HEFT Hamiltonian; Sec. VI studies what can be said about a possible zero of this function combining Schwarz’s lemmas of complex analysis and the current knowledge of the first two coefficients; and Sec. VII discusses how far-future colliders could access the full function by producing Higgs bosons at finite temperature. While $\omega\omega \rightarrow nh$ processes allow one to access \mathcal{F} order by order, a future collider that could substantially increase the temperature of the collision environment would open the entire function via the p_T spectrum of the emitted Higgs bosons.

In Appendix C and also in the companion paper [86] we have given similar correlations that we have extracted among the coefficients of the $V(h)$ nonderivative Higgs potential. This is attractive because it does not require the equivalence theorem [the process $\omega\omega \rightarrow nh$ needs to be extracted from $W_L W_L \rightarrow nh$ data and corrections are

needed at low energy, while $V(h)$ does not involve the Goldstone bosons] and is already accessible at LHC energies. Interestingly, it is affected by the properties of \mathcal{F} since it is this function which controls the change of variable between HEFT and SMEFT, $h_1 \rightarrow h$. The same reasoning applies to the Yukawa fermion-Higgs couplings which is another interesting alley of investigation for future works where activity is ongoing [87].

ACKNOWLEDGMENTS

This work was supported by Spanish MICINN No. PID2019–108655 GB-I00 Grant, and Universidad Complutense de Madrid under research group 910309 and the IPARCOS institute; ERC Starting Grants No. REINVENT-714788; No. UCM CT42/18-CT43/18; the Fondazione Cariplo and Regione Lombardia, Grant No. 2017-2070. A. S. B. thanks Professor Juan Ferrera-Cuesta for pointing out the relevance of using Schwarz's lemma. We also thank Carlos Quezada-Calonge for discussions on $WW \rightarrow hh$ scattering.

APPENDIX A: SCHWARZ'S LEMMA IN COMPLEX ANALYSIS AND ITS COROLLARIES

This brief Appendix provides a short overview of Schwarz's lemma, which we quickly use to demonstrate the corollary of interest for Sec. VI B. Again, the idea is whether the image set under \mathcal{F} includes or not a disk around $\mathcal{F} = 1$ large enough to encompass the origin $\mathcal{F} = 0$. The second corollary below gives a sufficient condition for this to be true. The point h^* which is the preimage of $\mathcal{F} = 0$ is the symmetric point around which the SMEFT expansion can be constructed. If the conditions of the second corollary are met, we know that \mathcal{F} will be analytic in a region broad enough to guarantee the power expansion.

To start, take a disk $D_z(0, 1)$ around the origin in the preimage complex space (in our application, the extension $h \rightarrow z$ of the singlet Higgs field to be a complex variable). Second, $D_f(0, 1)$ is a disk in the image complex space (also extending $\mathcal{F} \rightarrow f \in \mathbb{C}$), both disks having radius 1 and being centered around 0 as the notation indicates. We can then state the lemma [88] as follows.

1. Schwarz's lemma

Let $f: D_z(0, 1) \rightarrow D_f(0, 1)$ be holomorphic with $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Furthermore, if $|f(z_0)| = |z_0|$ for some $z_0 \in D_z(0, 1)$, then $|f(z)| = |z| \forall z \in D(0, 1)$.

a. Proof

Given those $f(0)$ and $f'(0)$, write $f(z) = zg(z)$: g is also holomorphic. Take $r < 1$, if $|z| = r$ we have that $|g(z)| = \frac{|f(z)|}{r}$ and hence $|g(z)| = \frac{|f(z)|}{r} \leq \frac{1}{r}$ [since the image of f is $D_f(0, 1)$]. The inequality $|g(z)| \leq \frac{1}{r}$ is satisfied for all $z \in \bar{D}(0, r)$ thanks to the *maximum*

modulus principle (if f is a holomorphic function, then the modulus $|f|$ cannot exhibit a strict local maximum that is in the interior of the domain of f). This means that if $g(z_0) = 1/r$, that is, it reaches its maximum for some z_0 satisfying $|z_0| < r$, then the function g must be a constant (and the maximum is reached at the boundary anyway). Now, taking the limit $r \rightarrow 1$ from the left we obtain $|g(z)| \leq 1$ and consequently $|f(z)| \leq |z|$ for all $z \in D_z(0, 1)$.

Noticing that $f'(z) = g(z) + zg'(z)$ it is immediate to prove that $|f'(0)| \leq 1$.

2. Corollaries

a. First Corollary

Let $f: D_z(0, 1) \rightarrow D_f(0, M)$ analytic such that $f(0) = 0$ and $|f'(0)| = 1$. Then we will have that $M \geq 1$ and $D_f(0, \frac{\sqrt{2}}{(\sqrt{2}+1)(\sqrt{2}+2)M}) \subset f(D_z(0, 1))$ (the open disk of radius $\frac{\sqrt{2}}{(\sqrt{2}+1)(\sqrt{2}+2)M}$ is contained in the image through f of the open unit disk).

b. Proof

Thanks to Schwarz's lemma we know that $M \geq 1$ since otherwise $|f'(0)| < 1$. We can then write f as

$$f(z) := z + \sum_{n=2}^{\infty} a_n z^n. \quad (\text{A1})$$

The triangular inequality of the complex norm yields

$$|z| = |f(z) - \sum_{n=2}^{\infty} a_n z^n| \leq |f(z)| + \left| \sum_{n=2}^{\infty} a_n z^n \right|. \quad (\text{A2})$$

Choosing to evaluate with $|z_M| := \frac{1}{\alpha M}$ with $\alpha > 1$, we find

$$\begin{aligned} |f(z_M)| &\geq |z_M| - \left| \sum_{n=2}^{\infty} a_n z_M^n \right| \geq |z_M| - \sum_{n=2}^{\infty} |a_n| |z_M|^n \\ &= \frac{1}{\alpha M} - \sum_{n=2}^{\infty} \frac{|a_n|}{(\alpha M)^n}. \end{aligned} \quad (\text{A3})$$

Thanks to Cauchy's estimates (canceling factorials) we know that for all $r < 1$, $|a_n| \leq \frac{M}{r^n}$ and hence $|a_n| \leq M$; we may take the worst bound with $r \rightarrow 1$. Then,

$$\begin{aligned} |f(z_M)| &\geq \frac{1}{\alpha M} - M \sum_{n=2}^{\infty} \frac{1}{(\alpha M)^n} = \frac{1}{\alpha M} - \frac{M}{(\alpha M)^2} \frac{1}{1 - \frac{1}{\alpha M}} \\ &= \frac{1}{\alpha M} \left(1 - \frac{M}{\alpha M - 1} \right) \end{aligned} \quad (\text{A4})$$

(having reconstructed the geometric series). Now, since $M \geq 1$ and we have taken $\alpha > 1$ we have that

$$\frac{1}{\alpha M} \left(1 - \frac{1}{\alpha - 1/M} \right) \geq \frac{1}{\alpha M} \left(1 - \frac{1}{\alpha - 1} \right) = \frac{1}{M} \left(\frac{\alpha - 2}{\alpha^2 - \alpha} \right), \quad (\text{A5})$$

giving the highest lower bound for $\alpha = 2 + \sqrt{2}$. Hence, taking $|z_M| = \frac{1}{(2+\sqrt{2})M}$, we have

$$|f(z_M)| \geq \frac{\sqrt{2}}{(\sqrt{2}+1)(\sqrt{2}+2)M}. \quad (\text{A6})$$

We now proceed to prove that the image of the disk in the Higgs field $f(D_z(0, 1))$ contains the disk of the \mathcal{F} function, namely $D_f(0, \frac{\sqrt{2}}{(\sqrt{2}+1)(\sqrt{2}+2)M})$.

Use for this an auxiliary $w_f \in D_f(0, \frac{\sqrt{2}}{(\sqrt{2}+1)(\sqrt{2}+2)M})$, that is, $|w_f| \leq |f(z_0)|$; the function $g(z) = w_f - f(z)$ verifies then

$$\begin{aligned} |f(z_M) + g(z_M)| &= |w| < \frac{\sqrt{2}}{(\sqrt{2}+1)(\sqrt{2}+2)M} \\ &\leq |f(z_M)| \text{ for } |z_M| = \frac{1}{(2+\sqrt{2})M}. \end{aligned} \quad (\text{A7})$$

Now we make use of Rouché's theorem to state that f and g have the same number of zeros in $D(0, \frac{1}{(2+\sqrt{2})M})$, i.e., at least one by definition [because by hypothesis $f(0) = 0$]. As a consequence there exists $z_0 \in D(0, 1)$ such that $f(z_0) = w_f$, in other words $f(D_z(0, 1))$ contains $D_f(0, \frac{\sqrt{2}}{(\sqrt{2}+1)(\sqrt{2}+2)M})$.

c. Second Corollary

Let $g: D(0, R) \rightarrow D_g(0, M)$ analytic, such that $g(0) = 0$ and $|g'(0)| = \mu > 0$. Then $g(D(0, R))$ contains another disk where g is analytic,

$$g(D(0, R)) \supset D_g\left(0, \frac{\sqrt{2}R^2\mu^2}{(\sqrt{2}+1)(\sqrt{2}+2)M}\right). \quad (\text{A8})$$

The R appearing there is what can be tested to guarantee that the image includes a disk that in turn includes 0 [and therefore, $\exists h_* \mathcal{F}(h_*) = 0$], and the function is analytic between the vacuum and that symmetric point.

d. Proof

It follows in a relatively straightforward manner by applying the first corollary to the auxiliary function:

$$f(z) := \frac{1}{Rg'(0)}g(Rz) \text{ for } |z| < 1. \quad (\text{A9})$$

APPENDIX B: FROM SMEFT TO HEFT

In this Appendix we show the two alternative (complementary) procedures employed to obtain the effective Lagrangian simplifications discussed along the article: first by means of Higgs field redefinitions; in a second subsection, we show how to get this same simplifications of the

effective action by means of partial integration and the use of the field equation of motion for the classical fields.

These two approaches for the simplifications of the effective theory are but two sides of the same coin: field redefinitions in the integration of the generating functional modifies the classical action in the exponential e^{iS} , introducing new operators that are proportional to the classical EOM (this is, through an appropriate field redefinition one can remove operators in S that are proportional to the EOM, while keeping the same generating functional); on the other hand, at tree level the effective action Γ coincides with the classical action S evaluated in the (quantum) classical field, where, by construction, any operator in Γ proportional to the EOM is deemed to vanish. Further details can be found, e.g., in Appendix B in Ref. [28], Appendix A in Refs. [89], and [90].

1. Simplification of the action by field redefinitions

We will take the SMEFT Lagrangian as our starting point, which written in the modulus-phase form has the structure

$$\begin{aligned} \mathcal{L}_{\text{SMEFT}} &= \frac{v^2}{4} \left(1 + \frac{h}{v}\right)^2 \langle D_\mu U^\dagger D^\mu U \rangle \\ &\quad + \frac{1}{2} (1 + (v+h)^2 B(h)) (\partial_\mu h)^2 \\ &\quad - V_{\text{SM}}(h) - \Delta V_{\text{dim-6}}(h), \end{aligned} \quad (\text{B1})$$

which, up to $\mathcal{O}(\Lambda^{-2})$, is given by $B(h) = -\frac{2c_H \square}{\Lambda^2}$.

We want to make the Higgs kinetic term canonical, as it is customary in the HEFT Lagrangian. Thus, one has that the relation between the SMEFT Higgs modulus field h and the HEFT Higgs singlet h_1 is provided by

$$\begin{aligned} (1 + (v+h)^2 B(h))^{1/2} \partial_\mu h &= \partial_\mu h_1 \\ \Rightarrow (1 + (v+h)^2 B(h))^{1/2} \frac{dh}{dh_1} &= 1, \end{aligned} \quad (\text{B2})$$

which leads to

$$\begin{aligned} dh_1 &= (1 + (v+h)^2 B(h))^{1/2} dh \\ \Rightarrow h_1 &= \int_0^h (1 + (v+h)^2 B(h))^{1/2} dh. \end{aligned}$$

With this integration range prescription, one recovers the LO result $h_1 = h$.⁸ If we go up to NLO in $1/\Lambda^2$ we can compute this redefinition in a perturbative way: first, use the perturbative expansion of the integrand in Eq. (B3),

⁸Here we are not performing the necessary Higgs field shift required to take into account that the potential minimum of $V_{\text{HEFT}}(h)$ is not the same as the minimum for the tree-level SM potential ($\langle h \rangle = 0$). This can be always performed in a second field redefinition afterwards if needed.

e.g., $(1 + (v + h)^2 B(h))^{1/2} = 1 - c_{H\Box}(v + h)^2/\Lambda^2$ up to NLO in SMEFT; second, trivially integrate each term up to the considered perturbative order. These two steps lead to the $h_1 = h_1(h)$ relation up to the desired perturbative order where, e.g., Eq. (33) in the text provides the expression up to $\mathcal{O}(\Lambda^{-2})$ in the SMEFT expansion.

Finally, through an iterative perturbative procedure, it is possible to invert this relation and extract the relation $h = h(h_1)$ as an expansion in powers of $1/\Lambda^2$, as it was shown in Eq. (33) for SMEFT at NLO.

Under this field redefinition the SMEFT Lagrangian then becomes

$$\begin{aligned} \mathcal{L}_{\text{SMEFT}} &= \frac{v^2}{4} \left(1 + \frac{h(h_1)}{v}\right)^2 \langle D_\mu U^\dagger D^\mu U \rangle \\ &\quad + \frac{1}{2} (\partial_\mu h_1)^2 - V_{\text{SM}}(h(h_1)) - \Delta V_{\text{dim-6}}(h(h_1)) \\ &= \frac{v^2}{4} \mathcal{F}(h_1) \langle D_\mu U^\dagger D^\mu U \rangle + \frac{1}{2} (\partial_\mu h_1)^2 - V_{\text{HEFT}}(h_1). \end{aligned} \quad (\text{B3})$$

with the flare function $\mathcal{F}(h_1)$ provided in Eq. (4).

2. Alternative route: Partial integration and equations of motion on the field h

The previous transformations to cast a SMEFT Lagrangian into a HEFT, particularly Eq. (12) for the partial integration in terms of H fields and Eq. (30) changing the variable from h (SMEFT's) to h_1 (HEFT's), lay a path that can be traded for a different one. We can easily start with the quantum effective action Γ in terms of h , then apply partial integration (Green's theorem) and finally use the classical Higgs EOM of h . The resulting effective action is then valid only up to a given order, and the field appearing therein should be interpreted as h_1 . This will allow us to transform and remove any operator with only Higgs fields and two derivatives on the same field, for example to express $\mathcal{O}_2^m := h^m \partial^2 h$ in terms of $\mathcal{O}_1^n = h^n (\partial_\mu h)^2$. First, we will show that the former ones can always be converted in the second ones up to a total derivative. For this, we will once more use partial integration (Leibniz's rule) to rewrite a Lagrangian term of this form as

$$\mathcal{O}_1^n = -n\mathcal{O}_1^{n-1} - \mathcal{O}_2^{n+1} + \partial^\mu (h^{n+1} \partial_\mu h), \quad (\text{B4})$$

which, as $n > 0$ entails $n \neq -1$, and up to the now omitted total divergence, can be recast in the form

$$\mathcal{O}_1^n = h^n (\partial_\mu h)^2 = -\frac{1}{n+1} \mathcal{O}_2^{n+1} = -\frac{1}{n+1} h^{n+1} \partial^2 h. \quad (\text{B5})$$

Thus, we find that all the possible \mathcal{O}_1^n operators of our Lagrangian ($n \geq 0$) can be always rewritten in the \mathcal{O}_2^m form (up to a total derivative) and vice versa. Specifically, up to a total derivative,

$$(h + v)^m \partial_\mu h \partial^\mu h = -\frac{[(v + h)^{m+1} - v^{m+1}]}{m + 1} \partial^2 h. \quad (\text{B6})$$

This is equivalent to Eq. (22) but written in terms of the singlet and now for operators with arbitrarily large n powers of h ; we can use it to rewrite more general SMEFT operators in the HEFT form.⁹

Moving on, we transform and remove this second type of operators including $\partial^2 h$ that are generated in SMEFT; we will make use of the EOM of the Higgs field, which causes a difference that is pushed to higher orders in the expansion that are not included anyway, so that at fixed order they are equivalent. The EOM (within the electroweak sector alone) reads [19,21,90,91]

$$\partial^2 h = \frac{(v + h)}{2} \langle D_\mu U^\dagger D^\mu U \rangle - V'(h), \quad (\text{B7})$$

with Higgs potential (ignorable at high s) given by $V(h) = V_{\text{SM}}(h) = m_h^2 \left(-\frac{v^2}{8} + \frac{h^2}{2} + \frac{h^3}{2v} + \frac{h^4}{8v^2}\right)$ at lowest order in the $1/\Lambda^2$ SMEFT expansion, i.e., the tree-level SM potential, and $V'_0(h) = m_h^2 \left(h + \frac{3h^2}{2v} + \frac{h^3}{2v^2}\right)$. For further detail, see Eqs. (3) and (10) in [90], where one can indeed see that we also get additional fermion operators whose discussion is beyond the scope of this article. Thus, we can make use of the EOM of the classical Higgs field to simplify the operators in the quantum effective action Γ (which amounts to appropriate field redefinitions in the generating functional).

The construction derived from a $\mathcal{O}_{H\Box}$ -type operator can then be transformed as, up to a total derivative,

$$\begin{aligned} (v + h)^n (\partial_\mu h)^2 &= -\frac{1}{n+1} [(v + h)^{n+1} - v^{n+1}] \partial^2 h \\ &= -\frac{(v + h_1) [(v + h_1)^{n+1} - v^{n+1}]}{2(n+1)} \langle D_\mu U^\dagger D^\mu U \rangle \\ &\quad + \frac{1}{n+1} [(v + h_1)^{n+1} - v^{n+1}] V'(h_1), \end{aligned} \quad (\text{B8})$$

with h_1 now understood as the HEFT field. In the simplest nontrivial case, $n = 2$, the term $\Delta\mathcal{L} = c_{H\Box} \mathcal{O}_{H\Box}$ yields, up to a total derivative,

⁹Another useful relation for this type of partial integration simplifications is given by the identity: $\Delta\mathcal{L} = A \partial_\mu B \partial^\mu C = \frac{1}{2} ((\partial^2 A)BC - A(\partial^2 B)C - AB(\partial^2 C)) + \partial_\mu \chi^\mu$, with the irrelevant total derivative term given by $\chi^\mu = ((\partial^\mu A)BC - A(\partial^\mu B)C - AB(\partial^\mu C))/2$. This relation is essentially the position-space representation of the relation $p_B p_C = \frac{1}{2}(p_A^2 - p_B^2 - p_C^2)$, which is the square of the momentum conservation equation with all $p_{A,B,C}$ incoming, $p_B + p_C = -p_A$. By means of it one also can rewrite the \mathcal{O}_1^n Lagrangian terms into \mathcal{O}_2^m operators up to a total derivative (although it is a little less efficient than the simplifications in the main text).

$$\begin{aligned}
 \Delta\mathcal{L} &= \frac{c_{H\Box}}{\Lambda^2} \mathcal{O}_{H\Box} = -\frac{c_{H\Box}}{\Lambda^2} (v+h)^2 (\partial_\mu h)^2 \\
 &= \frac{c_{H\Box}(v+h)}{6\Lambda^2} [(v+h)^3 - v^3] \langle D_\mu U^\dagger D^\mu U \rangle \\
 &\quad - \frac{c_{H\Box}}{3\Lambda^2} [(v+h)^3 - v^3] V'(h). \tag{B9}
 \end{aligned}$$

With this, the SMEFT Lagrangian up to dimension-six can be rewritten in the form

$$\begin{aligned}
 \mathcal{L}_{\text{SMEFT}} &= \frac{v^2}{4} \left(1 + \frac{h_1}{v}\right)^2 \langle D_\mu U^\dagger D^\mu U \rangle \\
 &\quad + \frac{1}{2} \left(1 - \frac{2c_{H\Box}(h_1+v)^2}{\Lambda^2}\right) (\partial_\mu h_1)^2 - V(h_1) \\
 &= \frac{v^2}{4} \mathcal{F}(h_1) \langle D_\mu U^\dagger D^\mu U \rangle + \frac{1}{2} (\partial_\mu h_1)^2 \\
 &\quad - V(h) - \frac{c_{H\Box}[(v+h_1)^3 - v^3]}{3\Lambda^2} V'(h_1), \tag{B10}
 \end{aligned}$$

with \mathcal{F} the flare function shown in the main text, given at this order by

$$\begin{aligned}
 \mathcal{F}(h_1) &= 1 + \frac{2h_1}{v} \left(1 + \frac{c_{H\Box}v^2}{\Lambda^2}\right) + \frac{h_1^2}{v^2} \left(1 + \frac{4c_{H\Box}v^2}{\Lambda^2}\right) \\
 &\quad + \frac{h_1^3}{v^3} \left(\frac{8c_{H\Box}v^2}{3\Lambda^2}\right) + \frac{h_1^4}{v^4} \left(\frac{2c_{H\Box}v^2}{3\Lambda^2}\right). \tag{B11}
 \end{aligned}$$

Satisfactorily, this result is in agreement with earlier investigations [47,67], although now we are extending the relation up to $\mathcal{O}(h_1^4)$. Terms of $\mathcal{O}(h_1^5)$ in $\mathcal{F}(h_1)$ and higher start at $\mathcal{O}(\Lambda^{-4})$ or above, and they are suppressed in the SMEFT counting.

APPENDIX C: RESTRICTIONS ON THE COEFFICIENTS OF THE HEFT POTENTIAL $V(h_1)$ REQUIRED FOR VALID SMEFT

Though a full phenomenological analysis will be left for other work (see [92] for a precis), we here want to call attention to restrictions over the coefficients of the generic HEFT Higgs potential $V(h)$ that are required for SMEFT to be applicable, and are thus of interest to falsify it too, beyond those for \mathcal{F} that we have concentrated on for most of the manuscript. These relations may be interesting at lower $E < 0.5$ TeV which appears to make them more attractive; but overall, the physics is more contrived due to the important corrections that the equivalence theorem takes, so that distinguishing ω and W_L becomes necessary. A full standard model analysis of the relations is needed, which we will not address now. In contrast, the equivalence theorem allows a much cleaner analysis of \mathcal{F} in the TeV region (at the prize of exacting experimental conditions).

Again, as in Sec. IV A 2, a proper expansion of the SMEFT Lagrangian that is analytic, and may thus be

approximated by a polynomial of the quadratic singlet $H^\dagger H$, requires to avoid any singularities in passing to it from HEFT. After the transformation to SMEFT coordinates as in Eq. (51), we must recover a SMEFT potential usable in Eq. (1),

$$\begin{aligned}
 V_{\text{SMEFT}} &= \text{const} \times (H^\dagger H) + \text{const}' \times (H^\dagger H)^2 \\
 &\quad + \text{const}'' \times (H^\dagger H)^3 + \dots \tag{C1}
 \end{aligned}$$

In a nutshell, when the change of variable from HEFT to SMEFT is attempted, making the Higgs kinetic term canonical involves \mathcal{F} intruding in the relation between the potentials as per Eq. (49),

$$V_{\text{SMEFT}}(H^\dagger H) = V_{\text{HEFT}}(h_1(z)) = V_{\text{HEFT}}(\mathcal{F}^{-1}(z)), \tag{C2}$$

with $z := \sqrt{2H^\dagger H/v^2}$ as in Eq. (54).

Given a generic V_{HEFT} and any coordinate transformation F , odd powers of z appear in V_{SMEFT} upon expanding in this variable around $z \simeq 0$. But, as such odd-power terms are nonanalytic (odd powers of the squared root $\sqrt{H^\dagger H}$) and cannot appear in SMEFT, which is a Taylor expansion, we need to demand that they vanish.

Thus, for SMEFT to be valid, the expansion of $V_{\text{HEFT}}(h_1)$ can only contain even terms in its $(h_1 - h_*)^n$ expansion. This mimicks our discussion of \mathcal{F} , and thus the nonderivative potential density $V_{\text{HEFT}}(h_1)$ must be an even function when expanded around the symmetric point h_* ; ignoring the potential's zero-point value, that expansion starts at order two, namely

$$\begin{aligned}
 V_{\text{HEFT}}(h_1) &= \frac{m_h^2 v^2}{2} \left[\frac{v_2^*}{v^2} (h_1 - h_*)^2 + \frac{v_4^*}{v^4} (h_1 - h_*)^4 \right. \\
 &\quad \left. + \frac{v_6^*}{v^6} (h_1 - h_*)^6 + \dots \right]. \tag{C3}
 \end{aligned}$$

This is completely analogous to the properties of \mathcal{F} found in Sec. III C, which upon expanding around the symmetric point where $\mathcal{F}(h_*) = 0$ needed to have an expansion with only even powers of $h_1 - h_*$:

$$\begin{aligned}
 \mathcal{F}(h_1) &= \frac{1}{v^2} (h_1 - h_*)^2 + \frac{a_4^*}{v^4} (h_1 - h_*)^4 \\
 &\quad + \frac{a_6^*}{v^6} (h_1 - h_*)^6 + \dots \tag{C4}
 \end{aligned}$$

The global factor $\frac{1}{2} m_h^2 v^2$ that has been extracted in Eq. (C3) helps us make a direct comparison between correlations constraining the potential constants and the very same correlations that we find for the a_i coefficients of the flare function \mathcal{F} .

That is, the trilinear, quadrilinear, pentilinear ... coefficients of the potential expanded around the physical vacuum,

$$V_{\text{HEFT}} = \frac{m_h^2 v^2}{2} \left[\left(\frac{h_1}{v} \right)^2 + v_3 \left(\frac{h_1}{v} \right)^3 + v_4 \left(\frac{h_1}{v} \right)^4 + \dots \right] \quad (\text{C5})$$

are determined by the v_4^* , v_6^* , ..., coefficients around the symmetric vacuum, in analogy to Eq. (70) where the a_i were determined by the a_i^* .

One difference among the two discussions is that, while \mathcal{F} in the Standard Model Eq. (45) (and hence, the zeroth order SMEFT) is only of order h^2 , the SM potential V reaches order h^4 . Likewise, SMEFT at one more order takes \mathcal{F} up to $\mathcal{O}(h^4)$ while the potential V reaches $\mathcal{O}(h^6)$ and so on. Translated in terms of the expansion coefficients of \mathcal{F} , the SM limit implies $a_2^* \neq 0$ while $a_4^*, a_6^* \dots = 0$; but those of $V_{\text{HEFT}}(h_1)$ are one higher order, $v_2^*, v_4^* \neq 0$ with $v_6^*, v_8^* \dots = 0$,

Truncating both series around the symmetric and the physical vacua to order h_1^6 we can solve for the first three coefficients around the physical vacuum, yielding (where $h_*/v \rightarrow h_*$ absorbs the normalization for visibility)

$$\begin{pmatrix} v_2^* \\ v_4^* \\ v_6^* \end{pmatrix} = -\frac{1}{8h_*^4} \begin{pmatrix} \frac{15}{2}h_*^3 & 7h_*^4 & 3h_*^5 \\ \frac{5}{2}h_* & 5h_*^2 & 3h_*^3 \\ \frac{1}{2}h_*^{-1} & 1 & h_* \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad (\text{C6})$$

with $v_1 = 0$ and $v_2 = 1$ at the potential minimum. This can then be employed to solve the relation, at the same order, for v_4, v_5, v_6 that then yields

$$\begin{pmatrix} v_4 \\ v_5 \\ v_6 \end{pmatrix} = \frac{1}{4h_*^4} \begin{pmatrix} -\frac{5}{2}h_* & -5h_*^2 & -6h_*^3 \\ \frac{3}{2} & 3h_* & 3h_*^2 \\ -\frac{1}{4}h_*^{-1} & -\frac{1}{2} & -\frac{h_*}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \quad (\text{C7})$$

These relations allow expressing v_4, v_5 , and v_6 in terms of $v_1 = 0, v_2 = 1$, the trilinear coupling v_3 and the symmetric point position h_* .

TABLE III. Correlations among the HEFT Higgs potential $V(h)$ around the physical vacuum (thus, directly tree-level observables) derived from assuming that the flare function $\mathcal{F}(h)$ has a zero at $h^* = -v$; the field is normalized with the physical v corresponding to the observation and not the bare Lagrangian parameter, additionally, $c_{H\Box} = 0$. If these correlations are violated, SMEFT needs to be extended to a more general HEFT. In the last column we introduce the SMEFT-allowed bounds on v_4, v_5 and v_6 from the CMS bound $v_3 \in [-2.5, 5.7]$ [53].

$v_4 = \frac{1}{4}(-5 + 6v_3)$	$\Delta v_4 = \frac{3}{2}\Delta v_3$	$v_4 \in [-5.0, 7.3]$
$v_5 = \frac{3}{4}(v_3 - 1)$	$\Delta v_5 = \frac{3}{4}\Delta v_3$	$v_5 \in [2.6, 3.5]$
$v_6 = \frac{1}{8}(v_3 - 1)$	$\Delta v_6 = \frac{1}{8}\Delta v_3$	$v_6 \in [-0.4, 0.6]$

In SMEFT the symmetric point is always set at $\sqrt{2}|H| = (h + v) = 0$. In the absence of derivative operators of the $c_{H\Box}^{(j)}$ type, the SMEFT radial coordinate $\sqrt{2}|H| = (h + v)$ coincides with the HEFT field combination $(h_1 + v)$. Thus, $h = h_1 = 0$ at the potential minimum $|H| = v/\sqrt{2}$ and $h = h_1 = h_* = -v$ at the symmetric point $H = 0$. Note that v is given by the minimum of the V potential at the given SMEFT order [see Eq. (87) and below]. Setting $h_*/v = -1$ we have the three correlations in Table III.

If we take the particular value $v_3 \rightarrow 1$, we recover $v_4 = \frac{1}{4}$ and $v_5 = 0 = v_6$ that imply no new physics.

This result can be easily generalized to the case with SMEFT operators of the form $c_{H\Box}^{(6)}, c_{H\Box}^{(8)}$, etc. One must simply substitute the expression for h_* at $\mathcal{O}(\Lambda^{-2})$ [at $\mathcal{O}(\Lambda^{-4})$] in Eq. (36) [in Eq. (38)], instead of the value $h_* = -v$ employed in Table III.

APPENDIX D: ZEROS OF THE FLARE FUNCTION IN SMEFT

The flare function to order $1/\Lambda^2$ is

$$\begin{aligned} \mathcal{F}(h_1) &= 1 + 2 \frac{h_1}{v} \underbrace{\left(1 + \frac{c_{H\Box} v^2}{\Lambda^2} \right)}_a + \left(\frac{h_1}{v} \right)^2 \underbrace{\left(1 + \frac{4c_{H\Box} v^2}{\Lambda^2} \right)}_b + \left(\frac{h_1}{v} \right)^3 \underbrace{\left(\frac{8c_{H\Box} v^2}{3\Lambda^2} \right)}_c + \left(\frac{h_1}{v} \right)^4 \underbrace{\left(\frac{2c_{H\Box} v^2}{3\Lambda^2} \right)}_d \\ &= \left(1 + \frac{h_1}{v} \right)^2 + \left(\frac{h_1}{v} \right) \left(\frac{2c_{H\Box} v^2}{\Lambda^2} \right) + \left(\frac{h_1}{v} \right)^2 \left(\frac{4c_{H\Box} v^2}{\Lambda^2} \right) + \left(\frac{h_1}{v} \right)^3 \left(\frac{8c_{H\Box} v^2}{3\Lambda^2} \right) + \left(\frac{h_1}{v} \right)^4 \left(\frac{2c_{H\Box} v^2}{3\Lambda^2} \right) \\ &= \left(1 + \frac{h_1}{v} \right)^2 + \left(\frac{c_{H\Box} v^2}{\Lambda^2} \right) \left[2 \left(\frac{h_1}{v} \right) + 4 \left(\frac{h_1}{v} \right)^2 + \frac{8}{3} \left(\frac{h_1}{v} \right)^3 + \frac{2}{3} \left(\frac{h_1}{v} \right)^4 \right]. \end{aligned} \quad (\text{D1})$$

We can see that, at this order in Λ , $h_1/v = -1$ is a solution of $\mathcal{F}(h_1) = 0$ [notice that also $h_1/v = -1 + 2c_{H\Box}/(3\Lambda^2)$ is also an exact root]. Moreover, we can see that, up to higher order corrections, $h_1 = -v + x/\Lambda^2$ is also a zero:

$$\mathcal{F}(-v + x/\Lambda^2) = \left(\frac{x}{\Lambda^2}\right)^2 + \left(\frac{c_{H\Box}v^2}{\Lambda^2}\right) \left[\underbrace{-2 + 4 - 8/3 + 2/3}_{=0} + \mathcal{O}\left(\frac{x}{\Lambda^2}\right) \right] = \mathcal{O}(\Lambda^{-4}). \quad (\text{D2})$$

The coefficients in front of $c_{H\Box}$ conspire to make the fixed point of $\mathcal{F}(h_1)$ be the one of the SM (up to higher order corrections). That is, this cancellation would not hold if $a, b, c,$ and d would not be related the way they are in Eq. (D1), as $(-2, 4, -8/3, 2/3)$.

Using the SMEFT basis, it is only at quadratic level in $c_{H\Box}/\Lambda^2$ that a shift is allowed. Observing the contribution from the dimension-six operator $\mathcal{O}_{H\Box}$ to the flare function up to $\mathcal{O}(c_{H\Box}^2/\Lambda^4)$ [see Eq. (37) ignoring $c_{H\Box}^{(8)}$], we can test again the solution $h_1 \rightarrow -v$:

$$\begin{aligned} \mathcal{F}(-v) &= (-1) \left[2 \left(\frac{c_{H\Box}v^2}{\Lambda^2} + \frac{3c_{H\Box}^2v^4}{2\Lambda^4} \right) - \left(\frac{4c_{H\Box}v^2}{\Lambda^2} + \frac{12c_{H\Box}^2v^4}{\Lambda^4} \right) + \left(\frac{8c_{H\Box}v^2}{3\Lambda^2} + \frac{56c_{H\Box}^2v^4}{3\Lambda^4} \right) \right. \\ &\quad \left. - \left(\frac{2c_{H\Box}v^2}{3\Lambda^2} + \frac{44c_{H\Box}^2v^4}{3\Lambda^4} \right) + \left(\frac{88c_{H\Box}^2v^4}{15\Lambda^4} \right) - \left(\frac{44c_{H\Box}^2v^4}{45\Lambda^4} \right) \right] \\ &= \left(-\frac{c_{H\Box}v^2}{\Lambda^2} \right) \left[\underbrace{2 - 4 + \frac{8}{3} - \frac{2}{3}}_{=0} \right] - \left(\frac{c_{H\Box}v^2}{\Lambda^2} \right)^2 \left[\underbrace{3 - 12 + \frac{56}{3} - \frac{44}{3} + \frac{88}{15} - \frac{44}{45}}_{=-1/9} \right]. \end{aligned} \quad (\text{D3})$$

In this case we can see that $h_1 = -v$ is no longer a root, as $\mathcal{F}(-v) = \left(\frac{c_{H\Box}v^2}{3\Lambda^2}\right)^2 \neq 0$ and an $\mathcal{O}(1/\Lambda^4)$ remainder survives. A similar thing happens for $h_1/v = -1 + 2c_{H\Box}/(3\Lambda^2)$. The sixth-order polynomial Eq. (37) (ignoring $c_{H\Box}^{(8)}$) has no calculable solutions. However, it is simple to prove that $h_1/v = -1 + c_{H\Box}v^2/(3\Lambda^2)$ [given in Eq. (36)] is an approximate root of $\mathcal{F}(h_1)$ that makes it vanish up to $\mathcal{O}(\Lambda^{-6})$ corrections.

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