

BOONE–HIGMAN EMBEDDINGS FOR CONTRACTING SELF-SIMILAR GROUPS

JAMES BELK AND FRANCESCO MATUCCI

ABSTRACT. We give a short proof that every contracting self-similar group embeds into a finitely presented simple group. In particular, any contracting self-similar group embeds into the corresponding Röver–Nekrashevych group, and this in turn embeds into one of the twisted Brin–Thompson groups introduced by the first author and Matthew Zaremsky. The proof here is a simplification of a more general argument given by the authors, Collin Bleak, and Matthew Zaremsky for contracting rational similarity groups.

1. INTRODUCTION

In 1973, William Boone and Graham Higman conjectured that a finitely generated group has solvable word problem if and only if it embeds into a finitely presented simple group [6, 7]. Elizabeth Scott proved in 1984 that $\mathrm{GL}_n(\mathbb{Z})$ embeds into a finitely presented simple group for all $n \geq 2$ [14], and recent work has demonstrated Boone–Higman embeddings for all countable abelian groups [4] and for all hyperbolic groups [2]. See [1] for a survey of related results.

In 1999, Claas Röver described an embedding of Grigorchuk’s group into a finitely presented simple group [12]. Specifically, the action of Grigorchuk’s group \mathcal{G} on the infinite rooted binary tree induces an action of \mathcal{G} on the boundary Cantor set $\mathfrak{C}_2 = \{0, 1\}^\omega$, and Röver proved that the group $V\mathcal{G}$ of homeomorphisms of \mathfrak{C}_2 generated by \mathcal{G} and Thompson’s group V is finitely presented and simple. Note that Grigorchuk’s group \mathcal{G} is not itself finitely presented [8], which makes this result particularly surprising.

Grigorchuk’s group is the archetypical example of a self-similar group, as defined by Volodymyr Nekrashevych in [10]. Unfortunately, Röver’s proof does not extend easily to other self-similar groups. If G is a self-similar group acting on an infinite rooted d -ary tree \mathcal{T}_d , then G can be viewed as a group of homeomorphisms of the Cantor set $\mathfrak{C}_d = \{0, \dots, d-1\}^\omega$, and one can construct a Röver–Nekrashevych group V_dG analogous to Röver’s group [13, 9]. Unfortunately, V_dG is neither finitely presented nor simple in general, though in some cases this construction gives new finitely presented simple groups. For example, Rachel Skipper, Stefan Witzel, and Matthew Zaremsky used Röver–Nekrashevych groups to construct the first family of simple groups with arbitrary finiteness lengths [15].

The second author is a member of the Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni (GNSAGA) of the Istituto Nazionale di Alta Matematica (INdAM) and of the PRIN 2022 “Group theory and its applications” research group and gratefully acknowledges the support of the PRIN project 2022-NAZ-0286. Moreover, the second author gratefully acknowledges the support of the of the Università degli Studi di Milano–Bicocca (FA project 2021-ATE-0033 “Strutture Algebriche”).

In 2017, Nekrashevych proved that $V_d G$ is indeed finitely presented whenever the self-similar group G is contracting [9]. Contracting groups are arguably the most important class of self-similar groups, and play a central role in the theory of iterated monodromy groups and limit spaces developed by Nekrashevych [10]. In [2], the authors together with Collin Bleak and Matthew Zaremsky generalized Nekrashevych's result to arbitrary full, contracting rational similarity groups, which act on subshifts of finite type by asynchronous automata. It was also proven in [2] that any contracting rational similarity group embeds into a finitely presented simple group, yielding Boone–Higman embeddings for contracting self-similar groups as well as all hyperbolic groups.

In this note we give a short proof that contracting self-similar groups admit Boone–Higman embeddings. We use only Nekrashevych's finite presentation result, and we do not require the more general theory of rational similarity groups. Our main theorem is the following.

Theorem 1. *Every contracting self-similar group embeds into a finitely presented simple group.*

The strategy is to prove that the associated Röver–Nekrashevych group embeds into one of the twisted Brin–Thompson groups introduced by the first author and Matthew Zaremsky in 2022 [5]. Given any faithful action of a group H on a set S , there is an associated twisted Brin–Thompson group SV_H which is simple and has H as a subgroup.

These groups SV_H are defined as follows. Recall that Thompson's group $V = V_2$ acts naturally on the Cantor set $\mathfrak{C}_2 = \{0, 1\}^\omega$. Then the direct sum $\bigoplus_{s \in S} V$ acts coordinate-wise on the infinite product space $(\mathfrak{C}_2)^S$. The group H also acts on $(\mathfrak{C}_2)^S$ by permutation of coordinates, and together this defines an action of the restricted wreath product $V \wr H = (\bigoplus_{s \in S} V) \rtimes H$ on $(\mathfrak{C}_2)^S$. Then SV_H is the group of all homeomorphisms of $(\mathfrak{C}_2)^S$ that locally agree with elements of $V \wr H$. That is, a homeomorphism f lies in SV_H if and only if every point in $(\mathfrak{C}_2)^S$ has a neighborhood on which f agrees with some element of $V \wr H$.

In a recent article [16], Zaremsky proved that SV_H is finitely presented as long as the following conditions are satisfied:

- (1) H is finitely presented,
- (2) H acts oligomorphically on S , i.e. the induced action of H on S^n has finitely many orbits for all $n \geq 1$.
- (3) The stabilizer of any finite subset of S is finitely generated.

Here we prove that if G is a contracting self-similar group, then the action of the Röver–Nekrashevych group $H = V_d G$ on any orbit S of rational points in \mathfrak{C}_d satisfies conditions (2) and (3) above. Combining this with Nekrashevych's finite presentation result, it follows that the twisted Brin–Thompson group SV_H is a finitely presented simple group that contains H and hence G .

Incidentally, Scott proved in [13] that if a self-similar group $G \leq \text{Aut}(\mathcal{T}_d)$ is itself finitely presented, then so is the associated Röver–Nekrashevych group $V_d G$. Zaremsky has recently proven that all such groups embed into finitely presented simple groups [17], and his proof also works for contracting self-similar groups that are finitely generated.

Acknowledgments. The authors would like to thank Collin Bleak, James Hyde and Matthew Zaremsky for many helpful discussions. We would also like to thank the anonymous referee for helpful comments and suggestions.

2. THE PROOF

For $d \geq 2$, let $X_d = \{0, \dots, d-1\}$, and let \mathcal{T}_d be the infinite, rooted d -ary tree whose nodes are elements of X_d^* . If $g \in \text{Aut}(\mathcal{T}_d)$ and $\alpha \in X_d^*$, there exists a unique automorphism $g|_\alpha \in \text{Aut}(\mathcal{T}_d)$ such that

$$g(\alpha \cdot \beta) = g(\alpha) \cdot g|_\alpha(\beta)$$

for every $\beta \in X_d^*$, where \cdot denotes concatenation. This automorphism $g|_\alpha$ is known as the **local action** (or “restriction”) of g at α . A group $G \leq \text{Aut}(\mathcal{T}_d)$ is **self-similar** if $g|_\alpha \in G$ for all $g \in G$ and $\alpha \in X_d^*$. A self-similar group G is **contracting** if there exists a finite set $\mathcal{N} \subseteq G$, known as the **nucleus** of G , such that for every $g \in G$ the set

$$\{\alpha \in X_d^* \mid g|_\alpha \notin \mathcal{N}\}$$

is finite. Equivalently, for every $g \in G$ there exists an $M \geq 0$ so that $g|_\alpha \in \mathcal{N}$ for all words $\alpha \in X_d^*$ of length M or greater.

Now let \mathfrak{C}_d denote the Cantor space $\mathfrak{C}_d = X_d^\omega$, and note that $\text{Aut}(\mathcal{T}_d)$ acts faithfully on \mathfrak{C}_d by homeomorphisms. For each $\alpha \in X_d^*$, the corresponding **cone** in \mathfrak{C}_d is the set $\alpha\mathfrak{C}_d$ of all infinite d -ary sequences that have α as a prefix. These cones form a basis for the topology on \mathfrak{C}_d . Indeed, every open set is a disjoint union of cones, and a set is clopen if and only if it is a finite disjoint union of cones.

If $h \in \text{Homeo}(\mathfrak{C}_d)$ maps a cone $\alpha\mathfrak{C}_d$ to a cone $\beta\mathfrak{C}_d$, the corresponding **local action** (or “restriction”) of h is the homeomorphism $h|_\alpha$ of \mathfrak{C}_d defined by

$$h(\alpha \cdot \psi) = \beta \cdot h|_\alpha(\psi)$$

for all $\psi \in X_d^\omega$. This agrees with the earlier definition of a local action in the case where $h \in \text{Aut}(\mathcal{T}_d)$, and has the property that $h|_{\alpha\beta} = (h|_\alpha)|_\beta$ whenever $h|_\alpha$ is defined. If $G \leq \text{Aut}(\mathcal{T}_d)$ is a self-similar group, a homeomorphism $h \in \text{Homeo}(\mathfrak{C}_d)$ lies in the **Röver–Nekrashevych group** V_dG if for every point $p \in \mathfrak{C}_d$ there exists a cone $\alpha\mathfrak{C}_d$ containing p such that $h(\alpha\mathfrak{C}_d)$ is a cone and $h|_\alpha \in G$. We shall refer to such a cone $\alpha\mathfrak{C}_d$ as a **regular cone** for h .

We begin by verifying condition (2) given in the introduction.

Lemma 2. *Let $G \leq \text{Aut}(\mathcal{T}_d)$, let $g_1, \dots, g_n \in G$, and let $\alpha_1\mathfrak{C}_d, \dots, \alpha_n\mathfrak{C}_d$ and $\beta_1\mathfrak{C}_d, \dots, \beta_n\mathfrak{C}_d$ be two collections of cones in \mathfrak{C}_d , where each collection is pairwise disjoint and does not cover \mathfrak{C}_d . Then there exists an $h \in V_dG$ such that $h(\alpha_i\mathfrak{C}_d) = \beta_i\mathfrak{C}_d$ for each i , with $h|_{\alpha_i} = g_i$.*

Proof. Note that any partition of \mathfrak{C}_d into k cones must satisfy

$$k \equiv 1 \pmod{d-1}.$$

Since $\bigcup_{i=1}^n \alpha_i\mathfrak{C}_d$ is a clopen set, its complement is clopen, so we can find a partition $\gamma_1\mathfrak{C}_d, \dots, \gamma_r\mathfrak{C}_d$ of the complement into cones. Similarly, we can find a partition $\delta_1\mathfrak{C}_d, \dots, \delta_s\mathfrak{C}_d$ of the complement of $\bigcup_{i=1}^n \beta_i\mathfrak{C}_d$ into cones. Then $n+r \equiv n+s \equiv 1 \pmod{d-1}$, so $r \equiv s \pmod{d-1}$. Refining these partitions if necessary, we may therefore assume that $r = s$. Then the element $h \in V_dG$ which maps each $\alpha_i\mathfrak{C}_d$ to $\beta_i\mathfrak{C}_d$ with $h|_{\alpha_i} = g_i$ and maps each $\gamma_i\mathfrak{C}_d$ to $\delta_i\mathfrak{C}_d$ with $h|_{\gamma_i} = \text{id}$ has the desired properties. \square

Proposition 3. *If $G \leq \text{Aut}(\mathcal{T}_d)$ is a self-similar group, then V_dG acts oligomorphically on each of its orbits in \mathfrak{C}_d .*

Proof. Let S be such an orbit. It suffices to prove that for all $n \geq 1$ the group V_dG acts transitively on ordered n -tuples of distinct elements of S .

Let (p_1, \dots, p_n) and (q_1, \dots, q_n) be tuples of distinct points in S . Since p_i and q_i lie in the same orbit, there exists for each i an $h_i \in V_dG$ that maps p_i to q_i . Then each h_i has a regular cone $\alpha_i \mathfrak{C}_d$ that contains p_i . Extending the α_i 's if necessary, we may assume that the cones $\alpha_i \mathfrak{C}_d$ are pairwise disjoint and do not cover \mathfrak{C}_d , and that their images $h_i(\alpha_i \mathfrak{C}_d)$ have these same two properties. Then by Lemma 2, there exists an element $h \in V_dG$ such that $h(\alpha_i \mathfrak{C}_d) = h_i(\alpha_i \mathfrak{C}_d)$ for each i with $h|_{\alpha_i} = h_i|_{\alpha_i}$. Then $h(p_i) = h_i(p_i) = q_i$ for each i , as desired. \square

All that remains is condition (3) in the introduction. We will only be able to prove this condition under the hypothesis that G is contracting and S is the orbit of a rational point under V_dG , where a point $p \in \mathfrak{C}_d$ is **rational** if it has the form $\alpha\bar{\beta}$ for some $\alpha \in X_d^*$ and $\beta \in X_d^+$.

The proof of condition (3) requires a preliminary definition. If $H \leq \text{Homeo}(\mathfrak{C}_d)$ and $p \in \mathfrak{C}_d$, the **group of germs** of H at p is the quotient

$$(H)_p = \text{Stab}_H(p) / \text{Fix}_H^0(p)$$

where $\text{Fix}_H^0(p)$ is the subgroup of H consisting of elements that are the identity in a neighborhood of p . If $h \in \text{Stab}_H(p)$, we let $(h)_p$ denote its image in $(H)_p$.

Lemma 4. *If $G \leq \text{Aut}(\mathcal{T}_d)$ is a contracting self-similar group and $p \in \mathfrak{C}_d$ is a rational point, then $(V_dG)_p$ is virtually cyclic.*

Proof. Write $p = \alpha\bar{\beta}$ for some $\alpha, \beta \in X_d^+$, where β is not a power of a shorter word. By Lemma 2, there exists an $f \in V_dG$ such that

$$f(\alpha\psi) = \alpha\beta\psi$$

for all $\psi \in X_d^\omega$. We claim that the cyclic group $\langle (f)_p \rangle$ generated by $(f)_p$ has finite index in $(V_dG)_p$.

Let \mathcal{N} be the nucleus for G , and consider the mapping $g \mapsto g|_\beta$ on \mathcal{N} . Since \mathcal{N} is finite, every element of \mathcal{N} is periodic or pre-periodic under this mapping. Let $\mathcal{N}_\beta \subseteq \mathcal{N}$ denote the set of periodic points, and let M be the least common multiple of their periods.

Now, if h is any element of V_dG that fixes p , then for all sufficiently large i the cone $\alpha\beta^i \mathfrak{C}_d$ is regular for h and the local action $h|_{\alpha\beta^i}$ lies in the nucleus \mathcal{N} . Indeed, $h|_{\alpha\beta^i}$ must eventually lie in \mathcal{N}_β , and therefore the sequence of local actions $h|_{\alpha\beta^{Mi}}$ is eventually constant. This defines a function $\text{Stab}_{V_dG}(p) \rightarrow \mathcal{N}_\beta$, which descends to a well-defined function $(V_dG)_p \rightarrow \mathcal{N}_\beta$. We claim that each fiber lies in a single right coset of $\langle (f)_p \rangle$.

Let h and h' be any two elements of V_dG that fix p , and suppose that $h|_{\alpha\beta^{Mi}} = h'|_{\alpha\beta^{Mi}}$ for some i . Let $\zeta \mathfrak{C}_d = h(\alpha\beta^{Mi} \mathfrak{C}_d)$ and $\zeta' \mathfrak{C}_d = h'(\alpha\beta^{Mi} \mathfrak{C}_d)$. Since h and h' fix p , both these cones must contain p , so ζ and ζ' are finite prefixes of $\alpha\bar{\beta}$. Increasing i if necessary, we may assume that both ζ and ζ' have α as a prefix. But since

$$\alpha\bar{\beta} = h(\alpha\bar{\beta}) = \zeta \cdot h|_{\alpha\beta^{Mi}}(\bar{\beta})$$

and

$$\alpha\bar{\beta} = h'(\alpha\bar{\beta}) = \zeta' \cdot h'|_{\alpha\beta^{Mi}}(\bar{\beta}) = \zeta' \cdot h|_{\alpha\beta^{Mi}}(\bar{\beta})$$

and β is not a power of any shorter word, the only possibility is that $\zeta = \alpha\beta^j\gamma$ and $\zeta' = \alpha\beta^{j'}\gamma$ for some $j, j' \geq 0$ and some proper prefix γ of β . Without loss of generality, we may assume that $j \leq j'$. Then h' agrees with $f^{j'-j}h$ on $\alpha\beta^{M_i}\mathfrak{C}_d$, so $(h')_p = (f)_p^{j'-j}(h)_p$, and therefore (h) and (h') lie in the same right coset of $\langle (f)_p \rangle$. Thus every fiber under the mapping $(V_dG) \rightarrow \mathcal{N}_\beta$ defined above lies in a single right coset of $\langle (f)_p \rangle$, and since \mathcal{N}_β is finite we conclude that $\langle (f)_p \rangle$ has finite index in $(V_dG)_p$. \square

Remark 5. If V_d is the Higman–Thompson group, it is well-known that $(V_d)_p$ is infinite cyclic for any rational point p , and the lemma above shows that $(V_d)_p$ has finite index in $(V_dG)_p$. Using the methods of the proof above, it is also easy to show that $(G)_p$ is finite for any self-similar group G and any point p .

However, it is not true in general that $(V_dG)_p$ is generated by $(V_d)_p$ and $(G)_p$. For example, let $G \leq \text{Aut}(\mathcal{T}_2)$ be the two-element self-similar group $\{1, a\}$, where a is the reflection of \mathcal{T}_2 that switches all 0's and 1's in an infinite binary sequence, and let p be the point $\overline{01}$. Then $(G)_p$ is trivial but $(V_2)_p$ has index two in $(V_2G)_p$. In particular, if h is any element of V_2G such that $h(0\mathfrak{C}_2) = 01\mathfrak{C}_2$ and $h|_0 = a$, then $(h)_p$ lies in $(V_2G)_p$ but not $(V_2)_p$.

The following proposition completes the proof of Theorem 1. It follows the arguments given by the authors and James Hyde in [3], as well as some unpublished simplifications partially due to James Hyde.

Proposition 6. *If $G \leq \text{Aut}(\mathcal{T}_d)$ is a contracting self-similar group and S is a finite set of rational points in \mathfrak{C}_d , then the stabilizer of S in V_dG is finitely generated.*

Proof. Let $H = V_dG$ and let $\text{Fix}_H(S) = \bigcap_{p \in S} \text{Stab}_H(p)$ be the pointwise stabilizer of S . This has finite index in $\text{Stab}_H(S)$, so it suffices to prove that $\text{Fix}_H(S)$ is finitely generated.

Let $\pi: \text{Fix}_H(S) \rightarrow \prod_{p \in S} (H)_p$ be the homomorphism whose components are the quotient homomorphisms $\text{Stab}_H(p) \rightarrow (H)_p$ for $p \in S$. Since each $(H)_p$ is virtually cyclic by Lemma 4, the image of π is finitely generated, so it suffices to prove that $\ker(\pi) = \bigcap_{p \in S} \text{Fix}_H^0(p)$ is contained in some finitely generated subgroup of $\text{Fix}_H(S)$ (since then $\text{Fix}_H(S)$ is generated by this subgroup and lifts of the generators of $\text{im}(\pi)$.) Note that this kernel consists precisely of those elements of H that are the identity in a neighborhood of S .

Write $S = \{p_1, \dots, p_n\}$, where each $p_i = \alpha_i\overline{\beta_i}$ for some $\alpha_i, \beta_i \in X_d^+$. Extending each α_i if necessary, we may assume that the cones $\alpha_1\mathfrak{C}_d, \dots, \alpha_n\mathfrak{C}_d$ are pairwise disjoint and their union $E = \alpha_1\mathfrak{C}_d \cup \dots \cup \alpha_n\mathfrak{C}_d$ is not all of \mathfrak{C}_d . By Lemma 2, there exists an element $f \in H$ such that $f(\alpha_i\mathfrak{C}_d) = \alpha_i\beta_i\mathfrak{C}_d$ and $f|_{\alpha_i} = \text{id}$ for each i . Note that $f \in \text{Fix}_H(S)$. Furthermore, $E \supset f(E) \supset f^2(E) \supset \dots$, and every open neighborhood of S contains $f^k(E)$ for sufficiently large k . It follows that $\ker(\pi) = \bigcup_{i=0}^{\infty} f^i \text{Fix}_H(E) f^{-i}$, so it suffices to prove that $\text{Fix}_H(E)$ is contained in some finitely generated subgroup of $\text{Fix}_H(S)$.

Since E is clopen, its complement $\mathfrak{C}_d \setminus E$ can be written as a disjoint union of cones $\gamma_1\mathfrak{C}_d, \dots, \gamma_m\mathfrak{C}_d$. Let $k \in \mathbb{N}$ so that $m+k \equiv 1 \pmod{d-1}$, and choose pairwise disjoint cones $\gamma_{m+1}\mathfrak{C}_d, \dots, \gamma_{m+k}\mathfrak{C}_d$ which are contained in E but disjoint from S . Let $E' = \mathfrak{C}_d \setminus (\gamma_1\mathfrak{C}_d \cup \dots \cup \gamma_{m+k}\mathfrak{C}_d) = E \setminus (\gamma_{m+1}\mathfrak{C}_d \cup \dots \cup \gamma_{m+k}\mathfrak{C}_d)$. Then $\text{Fix}_H(E) \subseteq \text{Fix}_H(E') \subseteq \text{Fix}_H(S)$, so it suffices to prove that $\text{Fix}_H(E')$ is finitely generated. Indeed, we will prove that $\text{Fix}_H(E') \cong V_dG$, which is finitely generated by Nekrashevych's theorem [11].

Since $m + k \equiv 1 \pmod{d - 1}$, there exists a partition of \mathfrak{C}_d into $m + k$ cones $\delta_1\mathfrak{C}_d, \dots, \delta_{m+k}\mathfrak{C}_d$. Let $z: \mathfrak{C}_d \setminus E' \rightarrow \mathfrak{C}_d$ be the homeomorphism that maps each $\gamma_i\mathfrak{C}_d$ to $\delta_i\mathfrak{C}_d$ by the function $\gamma_i\psi \mapsto \delta_i\psi$.

For each $h \in V_dG$, let $\varphi(h)$ be the element of V_dG which is the identity on E' and agrees with $z^{-1}hz$ on $\mathfrak{C}_d \setminus E'$. Then $\varphi: V_dG \rightarrow \text{Fix}_H(E')$ is the desired isomorphism. We conclude that $\text{Fix}_H(E')$ is finitely generated, and therefore $\text{Fix}_H(S)$ is finitely generated. \square

Remark 7. It is important to use rational points here. For an irrational point $p \in \mathfrak{C}_d$, the group $(V_d)_p$ is trivial, and the groups $(G)_p$ and $(V_dG)_p$ are finite, but the analogue of Proposition 6 fails. Indeed, if $H = V_dG$ and $S \subset \mathfrak{C}_d$ is a finite, nonempty set of irrational points, then $\text{Stab}_H(S)$ is never finitely generated. To see this, observe that the subgroup $\text{Fix}_H^0(S)$ of elements that are the identity in a neighborhood of S cannot be finitely generated, and this has finite index in $\text{Stab}_H(S)$ if S consists of irrational points.

REFERENCES

- [1] J. Belk, C. Bleak, F. Matucci, and M. C. B. Zaremsky, Progress around the Boone–Higman conjecture. Preprint (2023). arXiv.
- [2] J. Belk, C. Bleak, F. Matucci, and M. C. B. Zaremsky, Hyperbolic groups satisfy the Boone–Higman conjecture. Preprint (2023). arXiv.
- [3] J. Belk, J. Hyde, and F. Matucci, Stabilizers in Higman–Thompson groups. Preprint (2021). arXiv.
- [4] J. Belk, J. Hyde, and F. Matucci, Finite germ extensions. Preprint (2024). arXiv.
- [5] J. Belk and M. C. B. Zaremsky, Twisted Brin–Thompson groups, *Geometry & Topology* **26.3** (2022): 1189–1223 Crossref
- [6] W. W. Boone, Between logic and group theory. *Proceedings of the Second International Conference on the Theory of Groups: Australian National University, August 13–24, 1973*. Lecture Notes in Mathematics **372**, Springer, 1974. Crossref.
- [7] W. W. Boone and G. Higman, An algebraic characterization of groups with soluble word problem. *Journal of the Australian Mathematical Society* **18.1** (1974): 41–53. Crossref.
- [8] R. I. Grigorchuk, On Burnside’s problem on periodic groups, *Funktsional. Anal. i Prilozhen*, **14.1** (1980), 53–54, Crossref.
- [9] V. V. Nekrashevych, Cuntz–Pimsner algebras of group actions. *Journal of Operator Theory* **52.2** (2004): 223–249. JSTOR.
- [10] V. Nekrashevych, *Self-Similar Groups*. Mathematical Surveys and Monographs **117**, American Mathematical Society, 2005. Crossref.
- [11] V. Nekrashevych, Finitely presented groups associated with expanding maps. In *Geometric and Cohomological Group Theory*, pp. 115–171, London Mathematical Society Lecture Note Series **444**, Cambridge University Press, 2018. Crossref.
- [12] C. E. Röver, Constructing finitely presented simple groups that contain Grigorchuk groups. *Journal of Algebra* **220.1** (1999): 284–313. Crossref.
- [13] E. A. Scott, A construction which can be used to produce finitely presented infinite simple groups. *Journal of Algebra* **90.2** (1984): 294–322. Crossref.
- [14] E. A. Scott, The embedding of certain linear and abelian groups in finitely presented simple groups. *Journal of Algebra* **90.2** (1984): 323–332. Crossref.
- [15] R. Skipper, S. Witzel, and M. C. B. Zaremsky, Simple groups separated by finiteness properties. *Inventiones Mathematicae* **215.2** (2019): 713–740. Crossref.
- [16] M. C. B. Zaremsky, A taste of twisted Brin–Thompson groups. In *Séminaires et Congrès 34: Geometric Methods in Group Theory: Papers Dedicated to Ruth Charney*, pp. 303–316, 2025. arXiv.

- [17] M. C. B. Zaremsky, Embedding finitely presented self-similar groups into finitely presented simple groups. *Bulletin of the London Mathematical Society* **57.4** (2025): 1150–1159. Cross-ref.

SCHOOL OF MATHEMATICS & STATISTICS, UNIVERSITY OF GLASGOW, GLASGOW, G12 8QQ, SCOTLAND.

Email address: `jim.belk@glasgow.ac.uk`

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO–BICOCCA, MILAN 20125, ITALY.

Email address: `francesco.matucci@unimib.it`