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Analyzing the greenium term structure of twin government bonds

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ABSTRACT

Green bonds provide financial backing for low-carbon initiatives and facilitate the transition towards a greener economy. The greenium effect refers to the potential premium that bondholders are willing to forgo when investing in green securities compared to investments with similar characteristics such as maturity, coupon rate, and issuer credit profile. Despite recent interest in the literature on this topic, the determinants and dynamics of the greenium effect remain inadequately understood, particularly regarding its term structure and geographical dependencies. In this paper, we propose a mathematical framework for the definition of the greenium term structure by employing an affine GARCH model for the interest rate. We then compute semi-analytical bond price formulas that are used for the likelihood estimation of model parameters. Empirical analyses using German and Danish twin bonds show that greenium is maturity specific. In particular, for short maturities, fluctuations in greenium dynamics align with the implementation of European directives aimed at accelerating renewable energy adoption.

1. Introduction

It is well known that substantial economic effort is needed for the transition to net zero emissions and climate-resilient economies by 2050 to achieve the targets set by the Paris Climate Agreement. In this context, government financing through green bonds is crucial in meeting these objectives. These instruments enable the funding of various environmentally sustainable projects and the European Commission has highlighted the potential and functioning of the green bond market by presenting the Clean Energy for All Europeans (CEP) legislative package, finalised in May 2019 with the aim to promote a regulatory framework that will help the EU meet its 2030 climate and energy targets.[1]
The greenium effect refers to the potential premium that bondholders are willing to pay to invest in green securities compared to investments with similar characteristics such as maturity, coupon rate, and issuer credit profile. Consistent with Zerbib (2019), we assume that the greenium is a premium that investors are willing to forgo in order to finance green projects. As highlighted for example in Pástor et al. (2021) and Pástor et al. (2022), returns on green assets are lower than those on conventional assets and fluctuate with changes in environmental concerns. Estimates of the greenium in corporate and municipal bond markets vary significantly, see, for example, Baker et al. (2022), Caramichael and Rapp (2024), Flammer (2021), Larcker and Watts (2020), and Tang and Zhang (2020) who document that stock prices respond positively to the issuance of green bonds. Pástor et al. (2022) are the first to use German twin bonds, despite focusing on the US stock market, to illustrate the widening of the green spread. These authors and many others, see, e.g. Bachelet et al. (2019) and Hachenberg and Schiereck (2018), emphasize that environmental preferences can influence the premium of green bonds, which motivates greenium research on green government bonds.
Indeed, despite the interest of recent literature on this topic, e.g. Pietsch and Salakhova (2022) and Fandella and Cociancich (2024), the determinants and dynamics of the greenium effect remain inadequately understood, particularly concerning its term structure. Relevant recent contributions on this topic are De Vincentiis and Abis (2025) and Mercuri et al. (2026). The first paper claims that greenium is larger for the German government bond issuances with nearer maturities while the second claim that the slope of the greenium term structure can be used to grasp the perceived timeline to a low carbon economy. In the latter, it is further proposed that the greenium term structure may be influenced by legislative interventions and/or political events. It is suggested that

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[1] Directive (EU) 2018/2001 on the promotion of the use of energy from renewable sources (recast) [2018] OJ L328/82 (RED II).

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a steeper term structure can be indicative of an acceleration in the perceived time to transition.

As concerns theoretical frameworks for the pricing of green bonds that focus on exogenous variables we find in literature a limited number of contributions. Zhang et al. (2020) considers a model where the coupon rate is a double barrier option written on a quoted market quantity (the carbon price) while Agliardi and Agliardi (2021) assume that a jump diffusion model can be used to describe the firm value of the bond issuer where jumps depend on an abrupt change of climate policies. In the latter, the arrival rate of the Poisson shock reflects uncertainty due to climate change policy, which is determined by cumulative carbon emissions and thus the global temperature anomaly. However, recent events suggest that jumps in the firm value should include additional factors and that their effect may be not strictly monotonically decreasing. The pricing of green bonds for banks is important as these collected sums are invested for example in green mortgages where the applied interest rate is lower than in classical ones. The lower remuneration in the case of green mortgages reflects the lower risk of mortgage default as observed in Billio et al. (2022). Kanamura (2020) defines the green bond price based on its dependence on energy prices by including supply and demand uncertainties in energy and green bond markets. The aforementioned papers do not directly model interest rates and do not deal with stylized facts such as mean-reversion observed in interest rates. Recently, Kanamura (2024) introduced an affine model for green bond prices that accounts not only for the interest rate dynamics but also for the green premium dynamics which is assumed to be orthogonal to interest rate. In particular, the interest rate is described through a Cox Ingersoll Ross (CIR) model (Cox et al., 1985) while the green bond premium is assumed simply to be mean-reverting. Although affine models are quite used in financial markets, the proposed two factor model is not straightforward to estimate. The parameters are estimated using the Kalman filter applied to Bloomberg Fixed Income indices with data that cover a period with low interest rates, that is, 2014–2020.

To overcome the gap in the literature regarding the determinants of greenium and the estimation problems mentioned above, in this paper we propose a mathematical framework employing an affine Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model for the *short* rate.

The latter quantity, modeled here as a discrete-time autoregressive process with affine GARCH innovations, is neither intended nor expected to coincide with the underlying short risk-free rate that governs the entire term structure and remains unobservable in the market. Rather, both the proposed model and the estimation procedure aim to describe effective short rates—brown and green—for each bond maturity, that implicitly combine the underlying short risk-free rate with additional bond-specific factors such as maturity, liquidity, and seniority.

Affine GARCH models have been considered mainly for option pricing, see, for example, Bellini and Mercuri (2014), Christoffersen et al. (2006), Heston and Nandi (2000), Mercuri (2008, 2011) and Hitaj et al. (2018). Recent applications of these models refer respectively to the pricing of variance swaps (Badescu et al., 2019), VIX (Cao et al., 2020) and VIX futures (Wang & Wang, 2021). We consider a similar structure for detecting possible yield premium for twin bonds.[2]

Our class of models has several advantages in capturing the dynamics of the interest rates for green and brown bonds. First, it is able to reproduce asymmetric and leptokurtic conditional distributions observed in financial time series. It exhibits mean-reverting behavior that allows us to capture the long-run mean of interest rates. It is able to reproduce a time-varying variance process, similarly to continuous-time stochastic volatility models. However, in contrast to stochastic volatility models, there is a single source of randomness. In particular, the variance process is modeled directly using its past values and past innovations of the return process. In stochastic volatility models, the presence of two sources of randomness combined with an unobservable variance process makes parameter estimation cumbersome in the absence of a dense term structure. On the other hand, affine GARCH innovations inherently exhibit time-varying skewness and kurtosis, as highlighted in Christoffersen et al. (2006), Mercuri (2008). In contrast, non-affine GARCH setups require more complex specifications to capture such dynamics. For instance, Harvey and Siddique (1999) necessitates an additional predictable process specifically to control the evolution of skewness.

Compared to an autoregressive model driven by standard GARCH models, the affine GARCH framework offers a closed-form expression for the conditional moment-generating function (m.g.f.). This feature ensures the availability of a pricing formula for a zero-coupon bond, which serves as a fundamental component for the construction of a maximum likelihood estimation procedure. Our methodology employs likelihood estimation techniques and quoted bond prices from both the green and conventional markets with the objective of establishing a comprehensive methodology to investigate the greenium effect. In contrast, under a classical GARCH specification, the price of a zero-coupon bond must be evaluated via Monte Carlo simulation. This requirement makes both estimation and pricing more time-consuming and introduces simulation-based errors that propagate to the estimated parameters, bond prices, and the inferred greenium term structure. Another noteworthy attribute of our model is its convergence towards the Vasicek model (Vasicek, 1977), which is a well-known model in fixed income markets. In the Appendix, we generalise the proposed model by introducing a dependence structure between the short rates of green and brown bonds. The bivariate framework is composed of a vector autoregressive model of order one, i.e. a VAR(1) model, for the risk-free rate and for the short rate of each bond. In this framework, it is assumed that the level of the risk-free rate impacts the greenium, but not vice versa. However, given the highly parametrised nature of the model and the relatively short series of twin bonds, the univariate model is employed for the empirical analyses.

We test our univariate model by considering daily prices of twin German sovereign zero coupon and coupon bonds in order to define the greenium term structure. The interest in sovereign green bonds is recent, as green bond issuances from governments are relatively recent as discussed in Ando et al. (2023). In particular, our empirical analyses leverages German federal securities' 'twin' structure. We also checked the results with Danish bond pairs, which are issued with a structure similar to that of German bonds.

Furthermore, we propose a framework for identifying the term structure of greenium based on coupon bonds. In the case of twin bonds, the interest rate spread may be taken as an indication of the value of greenium, since the yield of conventional bonds should account for all factors influencing sovereign yields, except those related to environmental issues. The idea of using different bonds for the nodes in the term structure goes in parallel with the idea of multicurves for interest rates. Indeed, as for forward rates we consider inputs for the bootstrapping procedure that are coherent in terms of credit risk and market liquidity, here for each node we use bond prices that are influenced from the same expected deadlines in terms of policy interventions towards a zero carbon economy. Within this framework, for any new issuance of twin bonds, we can implicitly define the coherent value for the greenium term.

We emphasize that our approach is quite general and can be applied to corporate and/or financial issuances for any pair of matched green and brown bonds, minimizing weighted dissimilarity functions that account for both continuous features (such as duration, amount issued) and for discrete ones (for example, rating, currency) as done in Pietsch and Salakhova (2022). In this paper, we deliberately focus on government bonds for which an explicit twin structure already exists, i.e., pairs of green and conventional bonds that are identical in terms of issuer, maturity, coupon structure, and credit quality. Thus, we isolate the

[2] A hybrid Heston model with a common stochastic volatility to describe government bond yield dynamics is presented in Recchioni and Tedeschi (2017). Notice that our model is described in discrete time.

greenium component in a clean empirical setting. Our empirical analyses confirm the fact that the greenium is time varying and that it disappears for longer maturities, at least for the sovereign bonds included in this analysis. This result offers a partial explanation for the absence of concordance with regard to the presence of greenium in the literature. Indeed, the majority of the aforementioned papers address bonds with varying durations, which may result in the disappearance of greenium in some of these bonds. Another important contribution of our work is to elucidate the underlying causes of the significant price fluctuations observed in the green bond market. We first consider the impact of incremental issuances of green bonds. Empirical evidence suggests that these events do not have a significant impact on the price of green bonds. Instead, we empirically observe that green bond price jumps of significant magnitude are evident when EU-wide directive on the acceleration of renewable energy is issued, which change investor attitudes.

The paper is organized as follows. In Section 2, we introduce a model for the interest rate dynamics and semi-analytical formulas for bond prices. Section 3 is devoted to the estimation procedure. In the same section, we detail the routine for the construction of the greenium term structure in the univariate case. In particular, it develops the estimation procedures, respectively, in the case of zero coupon bonds and coupon bonds. In Section 4 we present our numerical analysis and empirical results considering twin German and Danish bonds. Section 5 concludes the paper.

2. Definition and modelling of greenium

We assume that at each observation date t and for each maturity T we have market quotations of zero coupon bonds of a pair of twin bonds issued from the same institution (sovereign or private), same currency, same level of subordination and no embedded optionalities. The *greenium* $g_r(t,T)$ at day t for the time horizon T, seen as the difference of the two yield to maturities, can be defined as:

$$g_r(t,T) := -\frac{1}{T-t}\left\{\log\left[D_b(t,T)\right] - \log\left[D_g(t,T)\right]\right\} \tag{2.1}$$

where $D_g(t,T)$ and $D_b(t,T)$ are the green and the brown zero coupon bond prices with maturity T.

Let $\left(\Omega, \mathcal{F}, \left\{\mathcal{F}_{t_i}\right\}_{i\geq 1}, \mathbb{P}\right)$ be a filtered probability space, we consider a discrete time grid $\{t_i\}_{i\geq 1}$, (eventually) equally spaced. Under the martingale measure $\mathbb{Q}$, for any natural number $n \geq 1$, the zero coupon bond prices are computed as:

$$D\left(t_i, t_{n+1}\right) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\sum_{j=i}^{n} r_{t_j}\Delta_{j+1}}\middle|\mathcal{F}_{t_i}\right], \tag{2.2}$$

with $\Delta_{i+1} = t_{i+1} - t_i$ and r_t is the short rate assumed to be constant in the time interval Δ_{i+1}. The short rates for a given tenor of pairs of twin bonds are supposed to be influenced from the same factors, except from the impact of policy interventions that lead to a net zero carbon economy. In the following section, we will consider an affine GARCH model for the dynamics of the short rate that provides semi-analytical formulas for zero coupon bond prices. With these prices we can use Eq. (2.1) to extrapolate the greenium term from a set composed of historical prices of a pair of bonds.[3]

2.1. An affine model for the short rate

Let r_{t_i} be modelled as follows:

$$r_{t_i} = b + a\, r_{t_{i-1}} + c\left(\epsilon_{t_i} - h_{t_i}\right) \tag{2.3}$$

[3] The formula presented in (2.2) can be extended by considering the implicit default probabilities from the credit default swap spreads of the issuer. In the context of highly rated sovereign twin bonds the extension can be neglected as we will consider the same issuer for each pair of AAA bonds. Moreover, the duration of the most liquid green sovereign bonds manture in less than ten years.

where the predictable process h_{t_i} is defined as:

$$h_{t_i} = c_0 + c_2\, h_{t_{i-1}} + d\, \epsilon_{t_{i-1}}, \tag{2.4}$$

with $b, a \in \mathbb{R}$ and $c_0, c, c_2, d \in (0, +\infty)$, while ϵ_{t_i} is the innovation term. We assume that the conditional moment generating function of ϵ_{t_i} has the following form:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{\mathrm{w}\epsilon_{t_i}}\middle|\mathcal{F}_{t_{i-1}}\right] = e^{h_{t_i} f(\mathrm{w},\theta)} \tag{2.5}$$

where θ is a vector containing parameters of ϵ_{t_i} and w belongs to an open interval that contains the origin. Specifically $\exists\, \delta > 0$ such that $\forall \mathrm{w} \in (-\delta, +\delta)$ we have $\mathbb{E}^{\mathbb{Q}}\left[e^{\mathrm{w}\epsilon_{t_i}}\middle|\mathcal{F}_{t_{i-1}}\right] < +\infty$.

Remark 1. The couple $\left(r_{t_i}, h_{t_{i+1}}\right)$ can be written as a bivariate process $\mathbf{X}_i := \left[r_{t_i}, h_{t_{i+1}}\right]^\top$ with dynamics:

$$\mathbf{X}_i = \gamma_0 + \gamma_1 \mathbf{X}_{i-1} + \gamma_2 \epsilon_{t_i}, \tag{2.6}$$

where

$$\gamma_0 = \begin{bmatrix} b \\ c_0 \end{bmatrix},\ \gamma_1 = \begin{bmatrix} a & -c \\ 0 & c_2 \end{bmatrix} \text{ and } \gamma_2 = \begin{bmatrix} c \\ d \end{bmatrix}.$$

Adding and subtracting $\gamma_2 h_{t_i}$ in (2.6), we get:

$$\begin{aligned}\mathbf{X}_i &= \gamma_0 + \gamma_1 \mathbf{X}_{i-1} + \gamma_2 h_{t_i} + \gamma_2(\epsilon_{t_i} - h_{t_i}) \\ &= \gamma_0 + \mathbf{A}\mathbf{X}_{i-1} + \gamma_2 Z_{t_i}\end{aligned} \tag{2.7}$$

where the diagonal matrix $\mathbf{A}$ is defined as:

$$\mathbf{A} := \begin{bmatrix} a & 0 \\ 0 & c_2 + d \end{bmatrix}$$

while

$$Z_{t_i} := \epsilon_{t_i} - h_{t_i},$$

is the noise term in (2.7) that has zero conditional mean[4] under the assumption

$$\left.\frac{\partial f(\mathrm{w},\theta)}{\partial \theta}\right|_{\mathrm{w}=0} = 1.$$

The process X_i is a vector autoregressive model of order one, VAR(1), driven by an heteroskedastic error Z_{t_i}. For $c_0 > 0$, the conditions $|a| < 1$ and $0 < c_2 + d < 1$ are sufficient for the stationary and ergodic behaviour of the process r_{t_i}, we refer to Hamilton (1994), Lütkepohl (2005) for the stationary and ergodic conditions in the VAR model. Obviously, for $|a| \geq 1$, the process r_{t_i} exhibits a non-autoregressive behaviour. Conversely, when $|a| < 1$ but $c_2 + d \geq 1$, the predictable process h_{t_i} does not admit a finite unconditional mean; however, since the conditional mean of the innovations Z_{t_i} is zero, the process r_{t_i} may still be strictly stationary and ergodic. A similar situation arises in the integrated GARCH (IGARCH) model (Bougerol & Picard, 1992; Nelson, 1990).

From a theoretical point of view the list of innovations, whose conditional moment generating function satisfies the form in Eq. (2.5) is very large. Among all possible candidates, e.g. Compound Poisson (Mikosch, 2009), Positively Tempered Stable (Hougaard, 1986; Tweedie, 1984), Gamma and Inverse Gaussian,[5] the last two are the most tractable since

[4] The condition $0 < c_2 + d < 1$ ensures the existence of the unconditional mean of the term ϵ_{t_i} i.e. $\mathbb{E}\left(\epsilon_{t_i}\right) = \mathbb{E}\left(h_{t_i}\right) = \frac{c_0}{1-(c_2+d)}$ for any $i = 0, 1, \ldots$.

[5] X is a single parameter IG r.v., i.e. $X \sim \mathrm{IG}(\delta)$ with $\delta > 0$, if its density is

$$f(x,\delta) = \frac{\delta}{\sqrt{2\pi x^3}} \exp\left[-\frac{(x-\delta)^2}{2x}\right]$$

and the moment generating function is:

$$\mathbb{E}\left(e^{tX}\right) = \exp\left[\delta\left(1 - \sqrt{1-2t}\right)\right].$$

they also admit a closed-form formula for the density and consequently, the likelihood function can be written explicitly.

Choosing

$$f(\mathrm{w},\theta) = -\ln(1-\mathrm{w}), \tag{2.8}$$

ϵ_{t_i} is conditionally Gamma distributed, i.e. $\epsilon_{t_i}\big|\mathcal{F}_{t_{i-1}} \sim \Gamma\left(h_{t_i},1\right)$ with h_t shape and unitary scale. On the other hand, the conditional Inverse Gaussian innovations are obtained by setting

$$f(\mathrm{w},\theta) = 1-\sqrt{1-2\mathrm{w}}. \tag{2.9}$$

We notice that the innovations $\left\{Z_{t_i}\right\}_{i\geq 1}$ are uncorrelated and their conditional distribution is time-varying. Indeed, we have that:

$$\mathbb{E}^{\mathbb{Q}}\left[Z_{t_{i-1}}Z_{t_i}\right] = \mathbb{E}^{\mathbb{Q}}\left[Z_{t_{i-1}}\mathbb{E}^{\mathbb{Q}}\left[Z_{t_i}\big|\mathcal{F}_{t_{i-1}}\right]\right] = 0.$$

Moreover, as in the standard GARCH case, the squared innovations $\left\{Z^2_{t_i}\right\}_{i\geq 1}$ are not uncorrelated. Indeed, we first notice that there is a dependence structure on the ϵ_{t_i}'s for $c_2 \neq 0$, $d_2 \neq 0$. Then, it is straightforward to show that the autocovariance:

$$\begin{aligned} cov\left[Z^2_{t_{i-1}},Z^2_{t_i}\right] &:= \mathbb{E}^{\mathbb{Q}}\left[Z^2_{t_{i-1}}Z^2_{t_i}\right] - \mathbb{E}^{\mathbb{Q}}\left[Z^2_{t_{i-1}}\right]\mathbb{E}^{\mathbb{Q}}\left[Z^2_{t_i}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[Z^2_{t_{i-1}}\mathbb{E}^{\mathbb{Q}}\left[Z^2_{t_i}\big|\mathcal{F}_{t_{i-1}}\right]\right] - \mathbb{E}^{\mathbb{Q}}\left[Z^2_{t_{i-1}}\right]\mathbb{E}^{\mathbb{Q}}\left[Z^2_{t_i}\right], \end{aligned}$$

is different from zero as $\mathbb{E}^{\mathbb{Q}}\left[Z^2_{t_i}\big|\mathcal{F}_{t_{i-1}}\right]$ is not independent of $Z^2_{t_{i-1}}$. The dynamics for the short rate in (2.3) is thus similar to a GARCH(1,1) model but as stated in the following theorem the structure of $\left\{Z_{t_i}\right\}_{i\geq 1}$ allows for a log-affine form of the conditional moment generating function hence, it belongs to the class of affine-GARCH models.

Theorem 1. *The conditional moment generating function of the random variable $\sum_{j=i}^{n} r_{t_j}\Delta_{j+1}$ has a log-affine form that is expressed as follows:* [6]

$$\begin{aligned} \phi_{t_i}(\mathrm{u}) &= \mathbb{E}^{\mathbb{Q}}\left[e^{\mathrm{u}\sum_{j=i}^{n} r_{t_j}\Delta_{j+1}}\Big|\mathcal{F}_{t_i}\right] \\ &= \exp\left[\alpha_0\left(t_i,t_{n+1}\right)+\alpha_1\left(t_i,t_{n+1}\right)r_{t_i}+\beta_1\left(t_i,t_{n+1}\right)h_{t_{i+1}}\right] \end{aligned} \tag{2.10}$$

where the time-dependent coefficients satisfy the following finite difference equations:

$$\begin{cases} \alpha_0\left(t_i,t_{n+1}\right) = \alpha_0\left(t_{i+1},t_{n+1}\right)+\alpha_1\left(t_{i+1},t_{n+1}\right)b+\beta_1\left(t_{i+1},t_{n+1}\right)c_0 \\ \alpha_1\left(t_i,t_{n+1}\right) = \mathrm{u}\Delta_{i+1}+\alpha_1\left(t_{i+1},t_{n+1}\right)a \\ \beta_1\left(t_i,t_{n+1}\right) = \beta_1\left(t_{i+1},t_{n+1}\right)c_2-\alpha_1\left(t_{i+1},t_{n+1}\right)c \\ \qquad +f\left(\alpha_1\left(t_{i+1},t_{n+1}\right)c+\beta_1\left(t_{i+1},t_{n+1}\right)d,\theta\right) \end{cases} \tag{2.11}$$

with final conditions

$$\begin{cases} \alpha_0\left(t_n,t_{n+1}\right) = 0 \\ \alpha_1\left(t_n,t_{n+1}\right) = \mathrm{u}\left(t_{n+1}-t_n\right). \\ \beta_1\left(t_n,t_{n+1}\right) = 0 \end{cases} \tag{2.12}$$

Proof. We assume that the relation in (2.10) is valid for t_{i+1} and we determine the conditional m.g.f at t_i using the iterative property of the conditional expectation. Thus, we have:

$$\begin{aligned} \phi_{t_i}(u) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{u\sum_{j=i}^{n} r_{t_j}\Delta_{j+1}}\Big|\mathcal{F}_{t_{i+1}}\right]\Big|\mathcal{F}_{t_i}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{\mathrm{u}r_{t_i}\left(t_{i+1}-t_i\right)}\mathbb{E}^{\mathbb{Q}}\left[e^{u\sum_{j=i+1}^{n} r_{t_j}\Delta_{j+1}}\Big|\mathcal{F}_{t_{i+1}}\right]\Big|\mathcal{F}_{t_i}\right]. \end{aligned}$$

[6] The result holds under the integrability condition, i.e. if u belongs to the domain of the conditional moment generating function of the random variable $\sum_{j=i}^{n} r_{t_j}\Delta_{j+1}$.

From (2.10), we get:

$$\begin{aligned} \phi_{t_i}(\mathrm{u}) = \mathbb{E}^{\mathbb{Q}}\Big[& e^{\mathrm{u}r_{t_i}\Delta_{i+1}}\exp\left[\alpha_0\left(t_{i+1},t_{n+1}\right)+\alpha_1\left(t_{i+1},t_{n+1}\right)r_{t_{i+1}}\right. \\ &\left.+\beta_1\left(t_{i+1},t_{n+1}\right)h_{t_{i+2}}\right]\Big|\mathcal{F}_{t_i}\Big]. \end{aligned}$$

Using (2.3) and (2.4), we obtain:

$$\begin{aligned} \phi_{t_i}(\mathrm{u}) = & \exp\left[\mathrm{u}\Delta_{i+1}r_{t_i}+\alpha_0\left(t_{i+1},t_{n+1}\right)+\alpha_1\left(t_{i+1},t_{n+1}\right)b+\alpha_1\left(t_{i+1},t_{n+1}\right)a\, r_{t_i}\right] \\ & \times\exp\left[\beta_1\left(t_{i+1},t_{n+1}\right)c_0+\beta_1\left(t_{i+1},t_{n+1}\right)c_2\, h_{t_{i+1}}-\alpha_1\left(t_{i+1},t_{n+1}\right)c\, h_{t_{i+1}}\right] \\ & \times\mathbb{E}^{\mathbb{Q}}\left[\exp\left[\alpha_1\left(t_{i+1},t_{n+1}\right)c\,\epsilon_{t_{i+1}}+\beta_1\left(t_{i+1},t_{n+1}\right)d\,\epsilon_{t_{i+1}}\right]\Big|\mathcal{F}_{t_i}\right]. \end{aligned}$$

Exploiting (2.5), we have:

$$\begin{aligned} \phi_{t_i}(u) = & \exp\left[\alpha_0\left(t_{i+1},t_{n+1}\right)+\alpha_1\left(t_{i+1},t_{n+1}\right)b+\beta_1\left(t_{i+1},t_{n+1}\right)c_0\right] \\ & \times\exp\left[\left(u\Delta_{i+1}+\alpha_1\left(t_{i+1},t_{n+1}\right)a\right)r_{t_i}\right] \\ & \times\exp\left[\left(\beta_1\left(t_{i+1},t_{n+1}\right)c_2-\alpha_1\left(t_{i+1},t_{n+1}\right)c\right)h_{t_{i+1}}\right] \\ & \times\exp\left[f\left(\alpha_1\left(t_{i+1},t_{n+1}\right)c\epsilon_{t_{i+1}}+\beta_1\left(t_{i+1},t_{n+1}\right)d;\theta\right)h_{t_{i+1}}\right]. \end{aligned}$$

Recalling that the final conditions are given by $\phi_{t_n} = e^{\mathrm{u}r_{t_n}\Delta_{n+1}}$, we obtain the final condition in system (2.11). □

Using Theorem 1, we obtain the price of a zero coupon bond for a given specification of the conditional distribution of the innovations ϵ_{t_i}. Indeed, the price of a zero coupon bond coincides with (2.5) for $\mathrm{u}=-1$.

Corollary 1. *If ϵ_{t_i} is conditionally Gamma distributed, the price of a zero coupon bond is log-affine and the time coefficients can be obtained using Theorem 1, that is:*

$$\begin{cases} \alpha_0\left(t_i,t_{n+1}\right) = \alpha_0\left(t_{i+1},t_{n+1}\right)+\alpha_1\left(t_{i+1},t_{n+1}\right)b+\beta_1\left(t_{i+1},t_{n+1}\right)c_0 \\ \alpha_1\left(t_i,t_{n+1}\right) = -\Delta_{i+1}+\alpha_1\left(t_{i+1},t_{n+1}\right)a \\ \beta_1\left(t_i,t_{n+1}\right) = \beta_1\left(t_{i+1},t_{n+1}\right)c_2-\alpha_1\left(t_{i+1},t_{n+1}\right)c \\ \quad +\ln\left(1-\alpha_1\left(t_{i+1},t_{n+1}\right)c-\beta_1\left(t_{i+1},t_{n+1}\right)d\right) \end{cases} \tag{2.13}$$

with final conditions:

$$\begin{cases} \alpha_0\left(t_n,t_{n+1}\right) = 0 \\ \alpha_1\left(t_n,t_{n+1}\right) = -\left(t_{n+1}-t_n\right). \\ \beta_1\left(t_n,t_{n+1}\right) = 0 \end{cases} \tag{2.14}$$

Corollary 2. *If the conditional distribution of ϵ_{t_i} is an Inverse Gaussian, i.e. $\epsilon_{t_i}\big|\mathcal{F}_{t_{i-1}} \sim IG\left(h_t\right)$, the price of a zero coupon bond is log-affine and the time coefficients can be obtained using Theorem 1, that is:*

$$\begin{cases} \alpha_0\left(t_i,t_{n+1}\right) = \alpha_0\left(t_{i+1},t_{n+1}\right)+\alpha_1\left(t_{i+1},t_{n+1}\right)b+\beta_1\left(t_{i+1},t_{n+1}\right)c_0 \\ \alpha_1\left(t_i,t_{n+1}\right) = -\Delta_{i+1}+\alpha_1\left(t_{i+1},t_{n+1}\right)a \\ \beta_1\left(t_i,t_{n+1}\right) = \beta_1\left(t_{i+1},t_{n+1}\right)c_2-\alpha_1\left(t_{i+1},t_{n+1}\right)c \\ \quad +\left(1-\sqrt{1-2\alpha_1\left(t_{i+1},t_{n+1}\right)c+2\beta_1\left(t_{i+1},t_{n+1}\right)d}\right) \end{cases} \tag{2.15}$$

with final conditions as in (2.14).

2.2. Convergence to Vasicek's bond pricing formula

Here we present the convergence of the pricing formula introduced in Section 2.1 to Vasicek's bond pricing formula under a suitable choice of model parameters. Although our proposed model with affine-GARCH innovations captures complex dynamics that go beyond classical frameworks, showing that it recovers the Vasicek formula as a special limit case serves as a theoretical validation of the model's structure. This result provides a formal link between our framework and the well-established literature on the interest rate term structure. First, we use a heuristic argument for parameter specification. Then, we show that system (2.13) converges to the Vasicek ODE's system.

We start with the heuristic approach, where we consider the autoregressive model of the short term interest rate introduced in Section 2.1. In particular we set $c_2 = d = 0$ thus $\forall\, t_i \geq 0$ we get $h_{t_i} = c_0$ that implies

$$r_{t_i} = \left(b + a\, r_{t_{i-1}}\right) + c\left(\epsilon_{t_i} - c_0\right),$$

where we assume the innovations $\left\{\epsilon_{t_i}\right\}_{i=0}^{n}$ to be Gamma distributed. [7]
It is worthwhile to note that for $c_2 = d = 0$ the effect of past events on the innovations is removed. It follows that these innovations are also independent and identically distributed.
On an equally spaced time grid with $\Delta := t_i - t_{i-1}$ and $i = 1, \ldots, n$, the increment of the interest rate is defined as follows:

$$r_{t_i} - r_{t_{i-1}} = b + (a-1) r_{t_{i-1}} + c\left(\epsilon_{t_i} - c_0\right).$$

Assuming b, a c and c_0 to be functions of Δ, it is possible to retrieve a dynamics that can be seen a discrete version of the Vasicek model. Indeed, for

$$b(\Delta) := \bar{b}\Delta \ \text{ with } \ \bar{b} > 0$$
$$a(\Delta) := \bar{a}\Delta + 1 \ \text{ with } \ \bar{a} < 0$$
$$c_0(\Delta) := \frac{1}{\Delta}$$
$$c(\Delta) := \sigma\Delta \ \text{ with } \ \sigma > 0,$$

the increment of the interest rate reads:

$$r_{t_i} - r_{t_{i-1}} = \left(\bar{b} + \bar{a} r_{t_{i-1}}\right)\Delta + \sigma\Delta\left(\epsilon_{t_i} - \frac{1}{\Delta}\right). \tag{2.16}$$

Notice that the innovation in (2.16) is $\epsilon_{t_i} \sim \Gamma\left(\frac{1}{\Delta}, 1\right)$ $\forall\, i = 1, \ldots, n$ while the mean and the variance of the centered innovations $\left(\epsilon_{t_i} - \frac{1}{\Delta}\right)$ are respectively

$$\mathbb{E}^{\mathbb{Q}}\left[\left(\epsilon_{t_i} - \frac{1}{\Delta}\right)\right] = 0 \text{ and } \operatorname{Var}\left[\left(\epsilon_{t_i} - \frac{1}{\Delta}\right)\right] = \frac{1}{\Delta}.$$

Since $\sigma\Delta\left(\epsilon_{t_i} - \frac{1}{\Delta}\right)$ in (2.16) converges in distribution to a Normal as $\Delta \to 0$, we deduce the following approximation for small values of Δ:

$$\sigma\Delta\left(\epsilon_{t_i} - \frac{1}{\Delta}\right) \overset{d}{\approx} \sigma\sqrt{\Delta}\, Z, \tag{2.17}$$

where $Z \sim N(0,1)$.

Theorem 2. *Posing $c_2 = d = 0$ in (2.4) and setting:*

$$b(\Delta) = \bar{b}\Delta \ \textit{ with } \ \bar{b} > 0$$
$$a(\Delta) = \bar{a}\Delta + 1 \ \textit{ with } \ \bar{a} < 0$$
$$c_0(\Delta) = \frac{1}{\Delta}$$
$$c(\Delta) = \sigma\Delta \ \textit{ with } \ \sigma > 0 \tag{2.18}$$

the zero coupon bond pricing formula in (2.10) converges to Vasicek's zero coupon bond pricing formula.

Proof. We notice that since $h_{t_i} = \frac{1}{\Delta}$, the formula in (2.10) reduces to:

$$D\left(t_i, t_{n+1}\right) = \exp\left(\alpha_0(t_i, T) + \beta_1(t_i, T)\frac{1}{\Delta} + \alpha_1(t_i, T) r_{t_i}\right).$$

Defining the following quantity:

$$\bar{\alpha}_0(t_i, T) := \alpha_0(t_i, T) + \beta_1(t_i, T)\frac{1}{\Delta},$$

we observe that the pricing formula becomes:

$$D\left(t_i, t_{n+1}\right) = \exp\left(\bar{\alpha}_0(t_i, T) + \alpha_1(t_i, T) r_{t_i}\right).$$

We then compute the variation of the time-dependent coefficient $\left\{\alpha_1(t_i, T)\right\}_{i=1}^{n}$ on the equally spaced time grid $\left\{t_i\right\}_{i=1}^{n}$ with $t_i := t_0 + i\Delta$ and $T := t_{n+1}$. We thus determine the following quantity:

$$\begin{aligned}\Delta\alpha_1(t_{i+1}, T) :=& \quad \alpha_1\left(t_{i+1}, T\right) - \alpha_1\left(t_i, T\right)\\ =& \quad \alpha_1\left(t_{i+1}, T\right) - \mathrm{u}\Delta - \alpha_1(t_{i+\Delta})(\bar{a}\Delta + 1)\end{aligned} \tag{2.19}$$

where the last equality is due to (2.13) and the Δ-dependent parameters in (2.18). We further simplify the result in (2.19) and, taking the limit $\Delta \to \mathrm{d}t$, we get the following ordinary linear differential equation (ODE):

$$\mathrm{d}\alpha_1(t, T) = -(\mathrm{u} + \alpha_1(t, T)\bar{a})\mathrm{d}t. \tag{2.20}$$

Using the final condition in (2.14), the ODE in (2.20) admits a solution that at time t_i is

$$\alpha_1(t_i, T) = \frac{1 - e^{\bar{a}(T - t_i)}}{\bar{a}}. \tag{2.21}$$

We recognize that, as $\bar{a} < 0$, the solution in (2.21) is the time-dependent coefficient that multiplies the short rate in the Vasicek's zero coupon bond pricing formula (Vasicek, 1977).

We now focus on the increment of the coefficient $\bar{\alpha}_0(t_i, T)$ that reads:

$$\begin{aligned}\Delta\bar{\alpha}_0(t_{i+1}, T) :=& \ \bar{\alpha}_0(t_{i+1}, T) - \bar{\alpha}_0(t_i, T)\\ =& \ \frac{1}{\Delta}\left[\alpha_1(t_{i+i}, T)\sigma\Delta + \ln\left(1 - \alpha_1\left(t_{i+i}, T\right)\right)\sigma\Delta\right) + \beta(t_{i+i}, T)\right]\\ &- \left[\alpha_1(t_{i+\Delta}, T)\bar{b}\Delta + \beta_1(t_{i+i}, T)\frac{1}{\Delta}\right].\end{aligned} \tag{2.22}$$

Using the second order MacLaurin expansion, we get:

$$\begin{aligned}\ln(1 - \alpha_1(t_{i+i}, T)\sigma\Delta) =& - \alpha_1(t_{i+i}, T)\sigma\Delta - \frac{1}{2}\alpha_1^2(t_{i+i}, T)\sigma^2\Delta^2\\ &+ o(\alpha_1^2(t_{i+i}, T)\sigma^2\Delta^2).\end{aligned} \tag{2.23}$$

Substituting (2.23) in (2.22), we get:

$$\begin{aligned}\Delta\bar{\alpha}_0(t_{i+1}, T) =& -\frac{\frac{1}{2}\alpha_1^2(t_{i+i}, T)\sigma^2\Delta^2 + o\left(\alpha_1^2(t_{i+i}, T)\sigma^2\Delta^2\right)}{\Delta} - \alpha_1(t_{i+i}, T)\bar{b}\Delta\\ =& -\left[\alpha_1\left(t_{i+i}, T\right)\bar{b} + \frac{1}{2}\alpha_1^2\left(t_{i+i}, T\right)\sigma^2\right]\Delta + o(\Delta).\end{aligned} \tag{2.24}$$

As $\Delta \to \mathrm{d}t$, (2.24) becomes:

$$\mathrm{d}\bar{\alpha}_0(t, T) = -\left[\alpha_1(t, T)\bar{b} + \frac{1}{2}\alpha_1^2(t, T)\sigma^2\right]\mathrm{d}t. \tag{2.25}$$

Given the final condition $\bar{\alpha}_0(T, T) = 0$, we obtain the first time dependent coefficient in Vasicek's formula for the zero coupon bond price. □

3. Estimation

In Section 3.1 we present the maximum likelihood estimation procedure for model parameters from quoted zero coupon bond prices starting from the observation that the bivariate process of the discount rate r_{t_i} and the predictable process $h_{t_{i-1}}$ can be represented through a VAR(1) model. Then, we extend in Section 3.2 the maximum estimation procedure for the case of coupon bonds.

3.1. Likelihood of a zero coupon bond

Defining the quantity:

$$\tilde{\alpha}\left(t_i, t_{n+1}\right) := \begin{bmatrix} \alpha_1\left(t_i, t_{n+1}\right) \\ \beta_1\left(t_i, t_{n+1}\right) \end{bmatrix} \tag{3.1}$$

we notice that denoting with $\mathrm{zcb}_{t_i} := D(t_i, t_{n+1})$, the log price of a zero coupon bond can be written in the following form:

$$\ln\left(\mathrm{zcb}_{t_i}\right) = \alpha_0\left(t_i, t_{n+1}\right) + \tilde{\alpha}\left(t_i, t_{n+1}\right)^{\top}\mathbf{X}_i, \tag{3.2}$$

where the process $\mathbf{X}_i$ is defined in (2.6) while the coefficients $\alpha_0\left(t_i, t_{n+1}\right)$, $\alpha_1\left(t_i, t_{n+1}\right)$ and $\beta_1\left(t_i, t_{n+1}\right)$ are computed recursively as in Theorem 1.

[7] The convergence is obtained also assuming that the innovations $\left\{\epsilon_{t_i}\right\}_{i=1}^{n}$ have either an Inverse Gaussian (Christoffersen et al., 2006) or a positively skewed Tempered Stable (Kawai & Masuda, 2011; Mercuri, 2008) distribution. Indeed, as for the Gamma, their centered version admits the Gaussian distribution as a special case.

From the conditional density of the innovation ϵ_{t_i}, it is possible to determine the density of $\ln\left(\text{zcb}_{t_i}\right)$ given the information at time t_{i-1}. In particular, denoting with $\varphi_{\epsilon_{t_i}}(\cdot)$ the conditional density of the innovation, the conditional density $\varphi_{\text{lzcb}_{t_i}}(\cdot)$ of the $\ln\left(\text{zcb}_{t_i}\right)$ has the following form:

$$\varphi_{\text{lzcb}_{t_i}}\left(\ln\left(\text{zcb}_{t_i}\right)\right) = \varphi_{\epsilon_{t_i}}\left(\epsilon_{t_i}\right)\left|\frac{1}{\frac{\partial \ln\left(\text{zcb}_{t_i}\right)}{\partial \epsilon_{t_i}}}\right| \tag{3.3}$$

where ϵ_{t_i} is obtained by inverting (3.2), that is:

$$\epsilon_{t_i} := \frac{\ln\left(\text{zcb}_{t_i}\right) - \tilde{a}_{t_i}}{\tilde{c}_{t_i}}, \tag{3.4}$$

while the quantities $\tilde{a}_{t_i}$ and $\tilde{c}_{t_i}$ are respectively defined as:

$$\tilde{a}_{t_i} := \alpha_0\left(t_i, t_{n+1}\right) + \tilde{\alpha}\left(t_i, t_{n+1}\right)^{\top}\gamma_0 + \tilde{\alpha}\left(t_i, t_{n+1}\right)^{\top}\gamma_1 \mathbf{X}_{i-1} \tag{3.5}$$

$$\tilde{c}_{t_i} := \tilde{\alpha}\left(t_i, t_{n+1}\right)^{\top}\gamma_2. \tag{3.6}$$

Below we recap the routine used to compute the density in (3.3) and the log-likelihood of observed log-prices.

1. Fix $u = -1$ and choose a vector $\theta := \left(a, b, c, c_0, c_2, d, r_{t_0}\right)$.
2. Compute with the parameters in the previous point the coefficients $\alpha_0\left(t_i, t_{n+1}\right)$, $\alpha_1\left(t_i, t_{n+1}\right)$, $\beta_1\left(t_i, t_{n+1}\right)$ using the recursions in (2.12) for each t_i.
3. Define $\tilde{\alpha}(t_i, t_{n+1})$ as in (3.1) using the quantities $\alpha_1\left(t_i, t_{n+1}\right)$ and $\beta_1\left(t_i, t_{n+1}\right)$ in the previous point.
4. Compute the quantities $\left\{\tilde{c}_{t_i}\right\}_{i=1}^{n}$ and $\mathbf{X}_0 = \left[r_{t_0}, h_{t_1}\right]$ where h_{t_1} is obtained from the price of the zero coupon bond quoted at t_0, that is:
   $$h_{t_1} = \frac{\ln\left(\text{zcb}_{t_0}\right) - \alpha_0\left(t_0, t_{n+1}\right) - \alpha_1\left(t_0, t_{n+1}\right) r_{t_0}}{\beta_1\left(t_0, t_{n+1}\right)}.$$
5. Iteratively compute the following quantities in the order described below for each iteration:
   - $\mathbf{X}_{i-1}$ as in (2.6)
   - $\tilde{a}_{t_i}$ as in (3.5)
   - Using the zero coupon bond price at time t_i and the current values for $\tilde{a}_{t_i}$ and $\tilde{c}_{t_i}$, extract ϵ_{t_i} as in (3.4).
   - Compute the conditional density $\varphi_{\text{lzcb}_{t_i}}\left(\ln\left(\text{zcb}_{t_i}\right)\right)$ in (3.3) noticing that
     $$\frac{\partial \ln\left(\text{zcb}_{t_i}\right)}{\partial \epsilon_{t_i}} = \tilde{c}_{t_i}.$$
   - Update $\mathbf{X}_i$ using (2.6).
6. Let $\left\{\text{zcb}_{t_i}\right\}_{i=0}^{\bar{n}}$ be the historical series of market zero coupon bond prices, we compute the log-likelihood $\mathcal{L}\left(\theta; \left\{\text{zcb}_{t_i}\right\}_{i=0}^{\bar{n}}\right)$ as a function of θ, i.e.:

$$\mathcal{L}\left(\theta; \left\{\text{zcb}_{t_i}\right\}_{i=0}^{\bar{n}}\right) = \sum_{i=1}^{\bar{n}} \ln\left(\varphi_{\text{lzcb}_{t_i}}\left(\ln\left(\text{zcb}_{t_i}\right)\right)\right). \tag{3.7}$$

The maximum likelihood estimator $\hat{\theta}_{\bar{n}}$ is then obtained as:

$$\hat{\theta}_{\bar{n}} := \operatorname{argmax} \mathcal{L}\left(\theta; \left\{\text{zcb}_{t_i}\right\}_{i=0}^{\bar{n}}\right). \tag{3.8}$$

The numerical procedure for solving the problem in (3.8) returns the sequence of filtered noises $\{\hat{\epsilon}_{t_i}\}_{i=1}^{\bar{n}}$, the filtered interest rates $\{\hat{r}_{t_i}\}_{i=1}^{\bar{n}}$ and the filtered $\{\hat{h}_{t_{i+1}}\}_{i=1}^{\bar{n}}$. These series are useful both for gaining insight about the dynamics of the short rate and for computing fictitious prices of zero coupon bonds for artificial maturities using the couple $\left(\hat{r}_{t_{\bar{n}}}, \hat{h}_{t_{\bar{n}+1}}\right)$ and the estimates of a, b, c, c_0, c_2 and d in the pricing formula obtained by fixing $u = -1$ in (2.10).

Applying this procedure to a couple of the twin bonds, we generate a term structure for fixed tenors for the greenium, as from these fictitious prices with some artificial bond maturity t_{art} we use the formula in (2.1).

3.2. Likelihood function for a coupon bond

The likelihood is computed from historical coupon bond prices with maturity t_m, i.e. $\left\{\text{cb}^{\text{mkt}}_{t_i,t_m}\right\}_{i=1}^{\bar{n}}$, $\bar{n} < m$ with similar steps to those described in Section 3.1 for the likelihood based on zero coupon bonds. The main difference concerns the filtering procedure for the innovations $\left\{\epsilon_{t_i}\right\}_{i=1}^{\bar{n}}$. Indeed the inversion in (3.4) is no longer possible and the term ϵ_{t_i} is obtained by solving numerically the following non-linear equation:

$$\text{cb}^{\text{theo}}_{t_i,t_m}\left(\epsilon_{t_i}\right) = \text{cb}^{\text{mkt}}_{t_i,t_m} \tag{3.9}$$

where $\text{cb}^{\text{theo}}_{t_i,t_m}\left(\epsilon_{t_i}\right)$ is the pricing formula in our model of the coupon bond quoted at time t_i and maturity t_m. To determine the theoretical price of a coupon bond, we use the fact that it is a sum of the prices of the zero coupon bonds with maturities that coincide with the coupon payment dates. Thus $\text{cb}^{\text{theo}}_{t_i,t_m}\left(\epsilon_{t_i}\right)$ is given by:

$$\text{cb}^{\text{theo}}_{t_i,t_m}(\epsilon_{t_i}) = \sum_{k=i+1}^{m} C_k\, \text{zcb}_{t_i,t_k}(\epsilon_{t_i}) + N\, \text{zcb}_{t_i,t_m}(\epsilon_{t_i}) \tag{3.10}$$

where the stream of payments is $\left\{C_1, \dots, C_m + N\right\}$ with maturities $\left\{t_1, \dots, t_m\right\}$; the generic term zcb_{t_i,t_k} denotes the theoretical price of the zero coupon bond with maturity t_k as a function of the innovation ϵ_{t_i} and is obtained as follows:

$$\text{zcb}_{t_i,t_k} = \exp\left[\alpha_0\left(t_i, t_k\right) + \tilde{\alpha}\left(t_i, t_k\right)^{\top}\mathbf{X}_i\right] \tag{3.11}$$

where $\mathbf{X}_i$ is defined recursively in (2.6). To solve Eq. (3.9) we apply the Newton-Raphson algorithm reported below.

Algorithm 1 Newton-Raphson method for the filtration of the innovation ϵ_{t_i}.

Require:
$\text{cb}^{theo}_{t_i,t_m}(\epsilon_{t_i}) - \text{cb}^{mkt}_{t_i,t_m}$: The function to find the root of
ϵ_0: Initial guess for the root
δ: Desired tolerance for the root
$N_{\max}$: Maximum number of iterations
Ensure:
$\hat{\epsilon}_{t_i}$: Approximation of the root of $\text{cb}^{theo}_{t_i,t_m}(\epsilon_{t_i}) - \text{cb}^{mkt}_{t_i,t_m}$

```
1: h ← 1
2: ε̂_{t_i}(1) ← ε_0
3: while h ≤ N_max and |cb^{theo}_{t_i,t_m}(ε̂_{t_i}(h)) − cb^{mkt}_{t_i,t_m}| > δ do
4:     ε̂_{t_i}(h+1) ← ε̂_{t_i}(h) − (cb^{theo}_{t_i,t_m}(ε̂_{t_i}(h)) − cb^{mkt}_{t_i,t_m}) / (∂cb_{t_i,t_m}/∂ε_{t_i} |_{ε_{t_i}=ε̂_{t_i}(h)})     ▷ Update the approximation
5:     h ← h + 1                                  ▷ Increment the iteration counter
6: end while
7: if |cb^{theo}_{t_i,t_m}(ε̂_{t_i}(h)) − cb^{mkt}_{t_i,t_m}| > δ then
8:     Error: Method did not converge within N_max iterations
9: end if
10: Output: ε̂_{t_i}(h)
```

To implement Algorithm 1, we compute the derivative of $\text{cb}^{\text{theo}}_{t_i,t_m}(\epsilon_{t_i})$ w.r.t the innovation ϵ_{t_i} that results to be;

$$\frac{\partial \text{cb}^{\text{theo}}_{t_i,t_m}(\epsilon_{t_i})}{\partial \epsilon_{t_i}} = \sum_{k=1}^{m} C_k \frac{\partial \text{zcb}_{t_i,t_k}(\epsilon_{t_i})}{\partial \epsilon_{t_i}} + N \frac{\partial \text{zcb}_{t_i,t_m}(\epsilon_{t_i})}{\partial \epsilon_{t_i}} \tag{3.12}$$

where

$$\begin{aligned}\frac{\partial \mathrm{zcb}_{t_i,t_k}(\epsilon_{t_i})}{\partial \epsilon_{t_i}} &= \mathrm{zcb}_{t_i,t_k}(\epsilon_{t_i})\frac{\partial \ln\left(\mathrm{zcb}_{t_i,t_k}(\epsilon_{t_i})\right)}{\partial \epsilon_{t_i}} \\ &= \mathrm{zcb}_{t_i,t_k}(\epsilon_{t_i})\tilde{\alpha}\left(t_i,t_k\right)^{\top}\gamma_2.\end{aligned}$$

To obtain the Maximum Likelihood estimator (MLE) of the parameter vector θ, here we consider the conditional density of the coupon bond price cb_{t_i,t_m} given the information available at time t_{i-1}:

$$\varphi_{\mathrm{cb}_{t_i,t_m}}\left(\mathrm{cb}_{t_i,t_m}\right) = \varphi_{\epsilon_{t_i}}\left(\epsilon_{t_i}\right)\left|\frac{1}{\frac{\partial \mathrm{cb}_{t_i,t_m}}{\partial \epsilon_{t_i}}}\right|.$$

Thus, the log-likelihood reads:

$$\mathcal{L}\left(\theta,\left\{\mathrm{cb}^{\mathrm{mkt}}_{t_i,t_m}\right\}_{i=1}^{\bar{n}}\right) = \sum_{i=1}^{\bar{n}} \ln\left(\varphi_{\mathrm{cb}_{t_i,t_m}}\left(\mathrm{cb}^{\mathrm{mkt}}_{t_i,t_m}\right)\right).$$

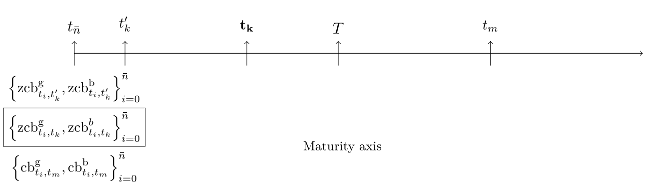
The MLE estimator is then obtained as:

$$\hat{\theta}_{\bar{n}} = \operatorname{argmax} \mathcal{L}\left(\theta,\left\{\mathrm{cb}^{\mathrm{mkt}}_{t_i,t_m}\right\}_{i=1}^{\bar{n}}\right).$$

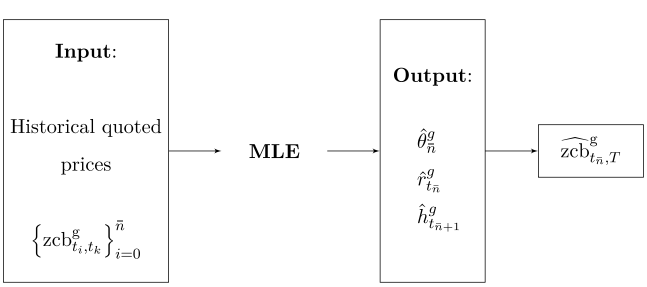
3.3. Implicit greenium term structure

As in the market usually we do not have bonds with a duration that coincides with the tenor of a given node in the term structure, we highlight here all the steps required for the computation of the greenium $g_r\left(t_{\bar{n}},T\right)$ at a specific maturity T; $t_{\bar{n}}$ denotes the date where we define the node in the term structure. For the date $t_{\bar{n}}$, we suppose that set of twin bond prices with different durations are collected. We consider the case where there are quotations for maturities t'_k, t_k and t_m. This procedure is composed of the following stages:

1. **Choose a pair of twin bonds with duration close to T.** See the example here, where the twin couple $\left\{\mathrm{zcb}^{g}_{t_i,t_k},\mathrm{zcb}^{b}_{t_i,t_k}\right\}_{i=0}^{\bar{n}}$ is chosen for the estimation of model parameters as this is the maturity t_k that precedes the node T .



2. **Estimation procedure using the maximum likelihood estimation for green bond.** The procedure returns $\hat{\theta}^{g}_{\bar{n}}$, $\hat{r}^{g}_{t_{\bar{n}}}$, $\hat{h}^{g}_{t_{\bar{n}+1}}$ and the estimated artificial zero coupon price $\widehat{\mathrm{zcb}}^{g}_{t_{\bar{n}},T}$ by means of (2.10) with $u=-1$ as reported in the following diagram.



3. **Repeat Step 2 for the brown bond.**
4. **Estimate greenium at time $t_{\bar{n}}$ with tenor T as:**

$$\hat{g}_r\left(t_{\bar{n}},T\right) := -\frac{1}{T-t_{\bar{n}}}\left\{\log\left[\widehat{\mathrm{zcb}}^{b}_{t_{\bar{n}},T}\right] - \log\left[\widehat{\mathrm{zcb}}^{g}_{t_{\bar{n}},T}\right]\right\}.$$

Table 1
Features of German twin bonds.

ISIN	Maturity	Type	First Coupon	Coupon Rate
DE0001141828	10/10/2025	Brown	–	–
DE0001030716	10/10/2025	Green	–	–
DE0001141869	15/10/2027	Brown	15/10/2023	1.3
DE0001030740	15/10/2027	Green	15/10/2023	1.3
DE0001102507	15/08/2030	Brown	–	–
DE0001030708	15/08/2030	Green	–	–
DE0001102564	15/08/2031	Brown	–	–
DE0001030732	15/08/2031	Green	–	–
DE000BU2Z007	15/02/2033	Brown	15/02/2024	2.3
DE000BU3Z005	15/02/2033	Green	15/02/2024	2.3
DE0001102481	15/08/2050	Brown	–	–
DE0001030724	15/08/2050	Green	–	–
DE0001102614	15/08/2053	Brown	15/08/2023	1.8
DE0001030757	15/08/2053	Green	15/08/2023	1.8

Table 2
The first four moments of the returns of the German green bonds, computed over the sample period from June 15, 2023, to May 21, 2024.

ISIN	Maturity	Mean	Variance	Skewness	Kurtosis
DE0001030716	10/10/2025	9.22×10^{-5}	8.26×10^{-7}	0.24	3.12
DE0001030740	15/10/2027	2.08×10^{-5}	3.89×10^{-6}	0.26	2.86
DE0001030708	15/08/2030	7.67×10^{-5}	1.26×10^{-5}	0.25	2.67
DE0001030732	15/08/2031	7.88×10^{-5}	1.65×10^{-5}	0.25	2.67
DE000BU3Z005	15/02/2033	-2.80×10^{-6}	2.00×10^{-5}	0.23	2.66
DE0001030724	15/08/2050	-3.30×10^{-6}	1.78×10^{-4}	0.25	2.71
DE0001030757	15/08/2053	-3.46×10^{-6}	1.26×10^{-4}	0.26	2.72

4. Empirical analysis

The idea of using different bonds for the nodes in the greenium term structure goes in parallel with the concept of multicurves for interest rates, as done for example in Atkins and Cummins (2023). Indeed, as for the bootstrap of forward rates, we consider market data homogeneous in terms of credit risk and market liquidity. For each node in the term structure, we use bonds with duration close to the maturity we are interested at. It is natural to expect that policy interventions towards a zero-carbon economy will have some implications on part of the greenium term structure.
Within this framework, our aim is for any new issuance of twin bonds, to implicitly define the coherent value for the green bond premium. We focus on the univariate case of the affine GARCH model introduced in the paper. Specifically, we consider green bonds issued by the German Finance Agency (GFA), which pairs each green bond with a conventional twin bond with the same maturity and coupon structure. We consider daily prices of zero coupon and coupon bonds in order to extract the greenium term structure in a given observation date. The analysis refers to data collected from 01/06/2022 to 21/05/2024. These bonds have no embdedded optionalities and are AAA rated.

With regard to the distribution of innovations, a selection is conducted between Gamma and Inverse Gaussian using the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC). As reported in Tables 4 and 5 for German bonds (and analogously for Danish bonds, Tables 10 and 11), the Gamma specification systematically outperforms IG in both AIC and BIC. In light of these results, we use the innovations with conditional Gamma distribution in the extraction of the greenium term structure.

Table 1 presents the key features of the seven German green bonds and their conventional twins issued between September 2020 and September 2022.

Tables 2 and 3 present the descriptive statistics (mean, variance, skewness, and kurtosis) of daily log returns of the green and brown bonds analyzed, respectively.

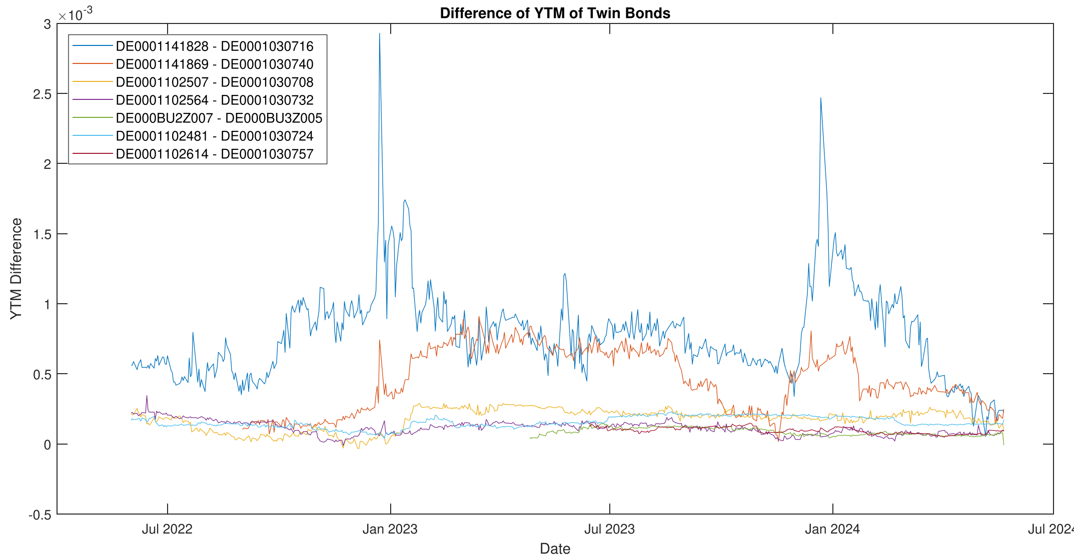


Fig. 1. Differences in yield to maturity for each pair of twin German bonds, maturing in 1yr (blue line), 3yrs (orange line), 6yrs (yellow line), 7yrs (purple line), 9yrs (green line), 27yrs (light blue line) and 30yrs (brown line) respectively. The first ISIN of each pair corresponds to the brown bond, while the second ISIN corresponds to the green bond. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

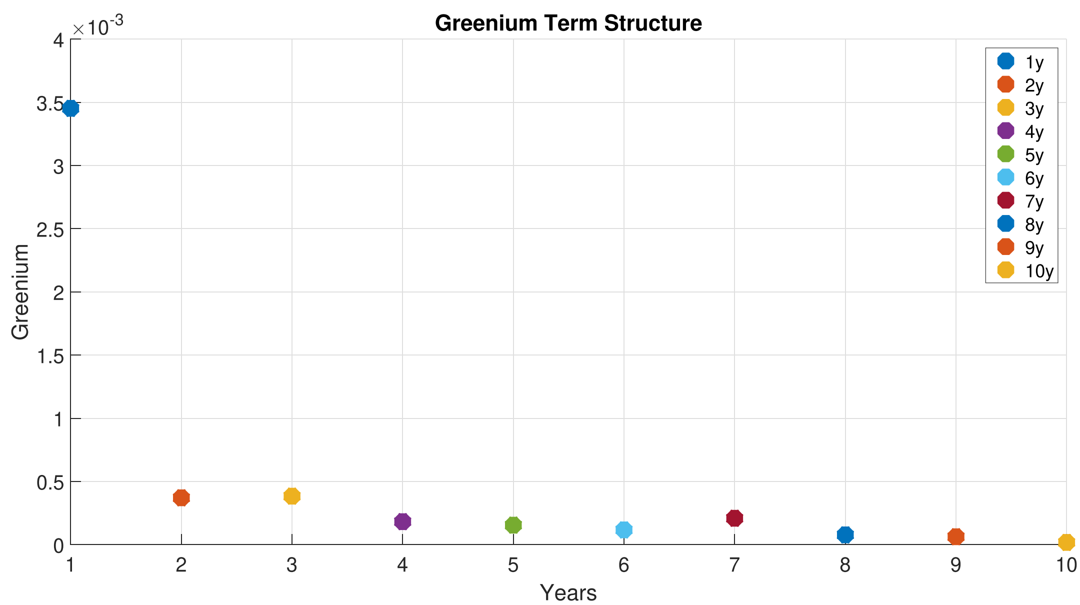


Fig. 2. German greenium term structure at January 3, 2024 for different maturities of artificial bonds. The term structure was reconstructed considering the estimated parameters in Table 4, considering the last 100 observations.

Table 3
The first four moments of the returns of the German brown bonds, computed over the sample period from June 15, 2023, to May 21, 2024.

ISIN	Maturity	Mean	Variance	Skewness	Kurtosis
DE0001141828	10/10/2025	9.86×10^{-5}	7.96×10^{-7}	0.26	3.15
DE0001141869	15/10/2027	3.11×10^{-5}	3.87×10^{-6}	0.24	2.84
DE0001102507	15/08/2030	8.13×10^{-5}	1.25×10^{-5}	0.25	2.69
DE0001102564	15/08/2031	8.06×10^{-5}	1.66×10^{-5}	0.25	2.66
DE000BU2Z007	15/02/2033	1.48×10^{-6}	2.00×10^{-5}	0.23	2.65
DE0001102481	15/08/2050	-3.45×10^{-5}	1.78×10^{-4}	0.25	2.71
DE0001102614	15/08/2053	-3.11×10^{-5}	1.26×10^{-4}	0.26	2.70

Fig. 1 shows the difference between the yield to maturity (YTM) of the various pairs of twin bonds and provides a first intuition of why we should define greenium as a stochastic process. We can observe that greenium is maturity specific and it has a stochastic dynamics with mean reversion. Table 4 shows the MLE parameters for different pairs of bonds, using the last 100 observations before January 3, 2024.

Results in Tables 4, 5, 10 and 11, show that the condition $|a| < 1$ stated in Remark 1 is always guaranteed. However, the condition for the existence of the unconditional first moment for the process h_t, namely $0 < c_2 + d < 1$, is occasionally violated. It is evident that this violation may have implications for the asymptotic properties of the estimators obtained via the procedure delineated in Section 3. Consequently, the consistency of these estimators cannot be assured.

In order to assess this property empirically, a parametric bootstrap is implemented, as outlined in Efron and Tibshirani (1994). In particular, i) bond prices were simulated for cases where $c_2 + d > 1$. (ii) the model parameters were re-estimated for each simulated series; (iii) the empirical distribution of each estimator was constructed. Table 6 reports summary statistics of 1000 bootstrap replications for the German

Table 4
Estimated parameters assuming affine Gamma-GARCH innovations. ISIN for each German zero coupon or coupon bond (first column); maturity (second column); bond type (third column); estimated model parameters (rest of columns). For the estimation, we consider the last 100 observations before January 3, 2024.

ISIN	Maturity	Type	b	a	c	c_0	c_2	d	r_0	AIC	BIC
DE0001141828	10/10/2025	Brown	0.0443	−0.4341	0.0266	0.0694	0.2277	0.7800	0.0453	−1797.2	−1778.9
DE0001030716	10/10/2025	Green	0.0257	−0.1590	0.0021	0.0010	0.6336	0.3810	0.0059	−1868.3	−1850.0
DE0001141869	15/10/2027	Brown	0.0131	0.5534	0.0705	4.0000	0.8704	0.1027	0.0169	−1739.1	−1720.8
DE0001030740	15/10/2027	Green	0.0074	0.7024	0.0330	3.0997	0.8619	0.1154	0.0123	−1750.8	−1.7326
DE0001102507	15/08/2030	Brown	0.0122	0.5798	0.0657	0.0039	0.9790	0.0211	0.0994	−1642.3	−1624.1
DE0001030708	15/08/2030	Green	0.0022	0.8551	0.0067	0.0526	0.9984	0.0002	0.0386	−1748.7	−1730.4
DE0001102564	15/08/2031	Brown	0.0490	0.5195	0.1900	3.9995	0.9777	0.0005	0.0349	−1506.1	−1487.9
DE0001030732	15/08/2031	Green	0.0204	0.7730	0.0862	1.8490	0.9081	0.0818	0.0869	−1540.8	−1522.6
DE000BU2Z007	15/02/2033	Brown	0.0184	0.7034	0.0948	3.8999	0.8693	0.1079	0.0200	−1598.4	−1580.1
DE000BU3Z005	15/02/2033	Green	0.0335	0.5418	0.1788	3.9966	0.8751	0.0983	0.0515	−1563.2	−1544.9
DE0001102481	15/08/2050	Brown	0.0198	0.8653	0.0710	1.0970	0.9931	0.0002	0.0703	−1554.4	−1536.1
DE0001030724	15/08/2050	Green	0.0124	0.8899	0.0440	0.9252	0.9948	0.0008	0.0582	−1463.1	−1444.8
DE0001102614	15/08/2053	Brown	0.0066	0.8889	0.0376	3.9997	0.8291	0.1432	0.0186	−1587.2	−1568.9
DE0001030757	15/08/2053	Green	0.0103	0.8096	0.0613	4.0000	0.8738	0.0967	0.0152	−1525.4	−1507.1

Table 5
Estimated parameters assuming Inverse Gaussian distribution for the innovations. ISIN for each German zero coupon or coupon bond (first column); maturity (second column); bond type (third column); estimated model parameters (rest of columns). For the estimation, we consider the last 100 observations before January 3, 2024.

ISIN	Maturity	Type	b	a	c	c_0	c_2	d	r_0	AIC	BIC
DE0001141828	10/10/2025	Brown	0.0824	−0.9040	4.5389	0.1807	0.5937	0.0767	0.0828	−918.0034	−899.7672
DE0001030716	10/10/2025	Green	0.0814	−0.9098	3.7010	0.0798	0.8764	0.0240	0.0620	−942.3697	−924.1336
DE0001141869	15/10/2027	Brown	0.0012	0.944	0.0100	0.0011	0.9994	0.0005	0.0945	−1110.2	−1090.9
DE0001030740	15/10/2027	Green	0.0013	0.9421	0.0100	0.0045	0.9991	0.0005	0.0664	−1107.2	−1088.9
DE0001102507	15/08/2030	Brown	0.0013	0.9100	0.0101	0.0243	0.9896	0.0185	0.0990	−987.6497	−969.4135
DE0001030708	15/08/2030	Green	0.0013	0.9100	0.0100	0.0021	0.9999	0.0083	0.0997	−988.3568	−970.1206
DE0001102564	15/08/2031	Brown	0.0013	0.9099	0.0101	0.0260	0.9967	0.0119	0.0971	−960.3860	−942.1498
DE0001030732	15/08/2031	Green	0.0013	0.9100	0.0100	0.0030	0.9968	0.0120	0.0990	−961.4055	−943.1693
DE000BU2Z007	15/02/2033	Brown	0.0009	0.9661	0.0100	0.0040	0.9992	0.0006	0.0501	−948.1741	−929.9379
DE000BU3Z005	15/02/2033	Green	0.0873	0.5582	0.3054	1.0543	0.9924	0.0014	0.0011	−1485.1	−1466.9
DE0001102481	15/08/2050	Brown	0.0016	0.9098	0.0120	0.133	0.9811	0.0284	0.0257	−716.2738	−698.0376
DE0001030724	15/08/2050	Green	0.0015	0.9098	0.0119	0.0349	0.9831	0.0274	0.0196	−716.4360	−698.1998
DE0001102614	15/08/2053	Brown	0.0004	0.9890	0.0010	3.9993	0.9965	0.0001	0.0319	−801.7103	−783.4741
DE0001030757	5/08/2053	Green	0.0003	0.9888	0.0010	2.3520	0.8232	0.1749	0.0744	−794.1468	−775.9106

Fig. 3. Time of issuance and outstanding volumes of green German Federal securities at the end of 2023. Quantities are in billions. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

brown bond with maturity 10 October 2025. In this case, the empirical distributions of c_2 and d exhibit means and medians close to the original estimates. Analogous outcomes persist for all instances where the condition $0 < c_2 + d < 1$ is not met. The results for these cases are available upon request.

In Fig. 2 we plot the greenium term structure from artificial zero coupon bonds using the estimated parameters obtained under the assumption of affine GARCH innovations conditionally Gamma distributed and the procedure discussed in Section 3.3; results on the conditional Inverse Gaussian innovations are available upon request. The first three nodes of the greenium term assume significantly positive values. Moreover, it can be observed that the greenium disappears for longer maturities, at least for the analysis performed on German sovereign bonds. This result may explain the lack of agreement on the presence of greenium in the literature as they include of bonds with different durations, i.e. in some of these bonds the greenium potentially is not significant.

The need of financing transition to a carbon free economy from

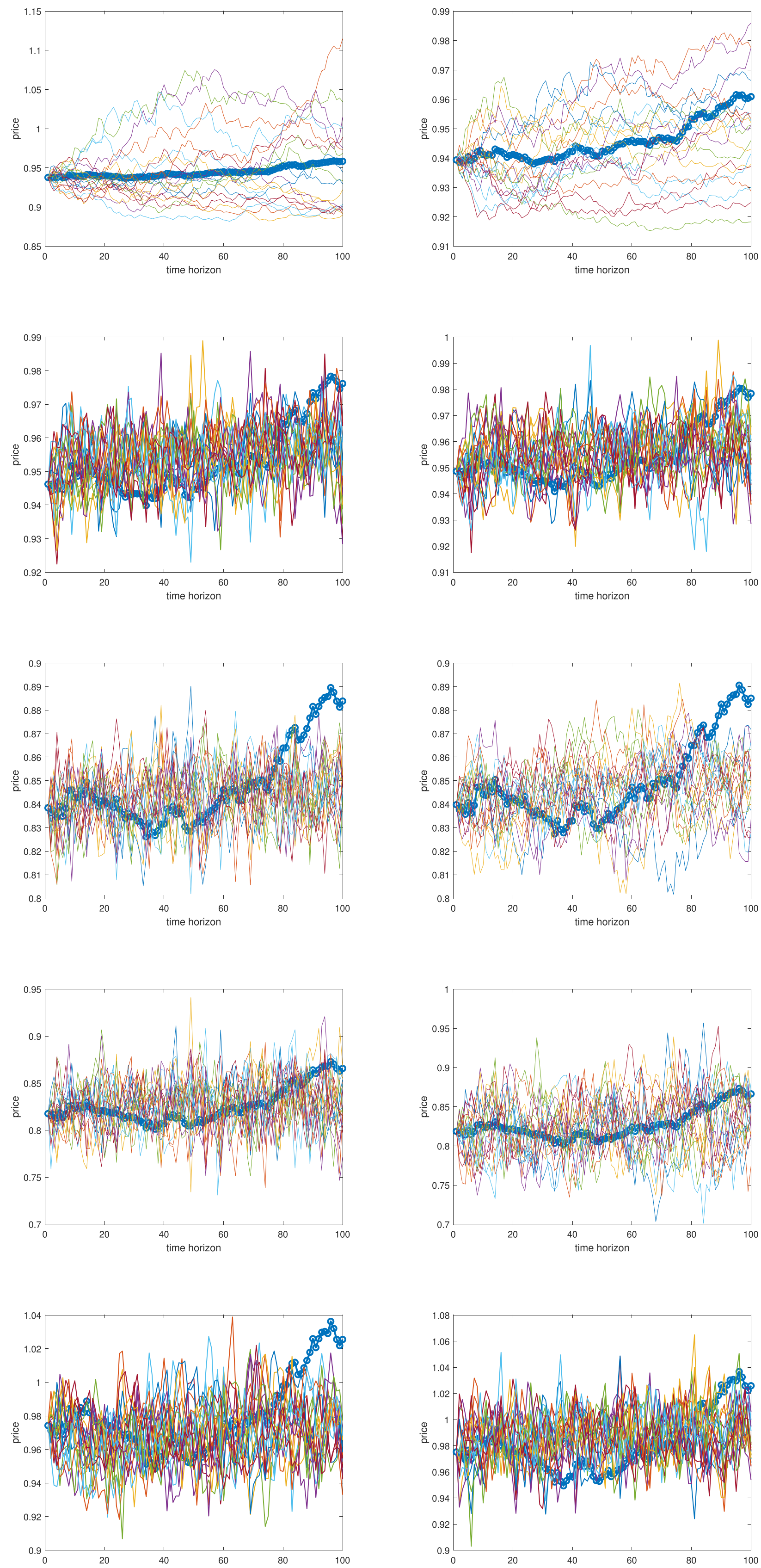


Fig. 4. Each plot shows the market price (dotted dashed line with dot) and 20 artificially simulated price trajectories of the synthetic instrument (solid lines, colored) for each German zero coupon bond and coupon bond. Specifically, each line shows the pair, brown bond (left graph) and green bond (right graph) respectively, in ascending order of maturity (up to the 10-year maturity bond), as in Table 1. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

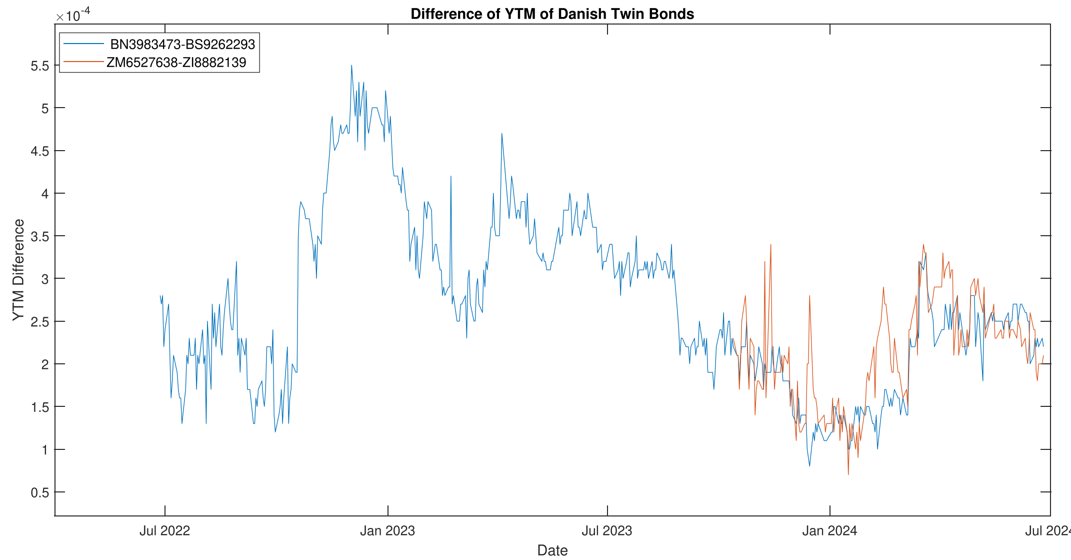


Fig. 5. Differences in yield to maturity for each pair of the Danish twin bonds maturing respectively in 2031 (blue line) and in 2033 (orange line). The first ISIN of each pair corresponds to the brown bond, while the second ISIN corresponds to the green bond. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

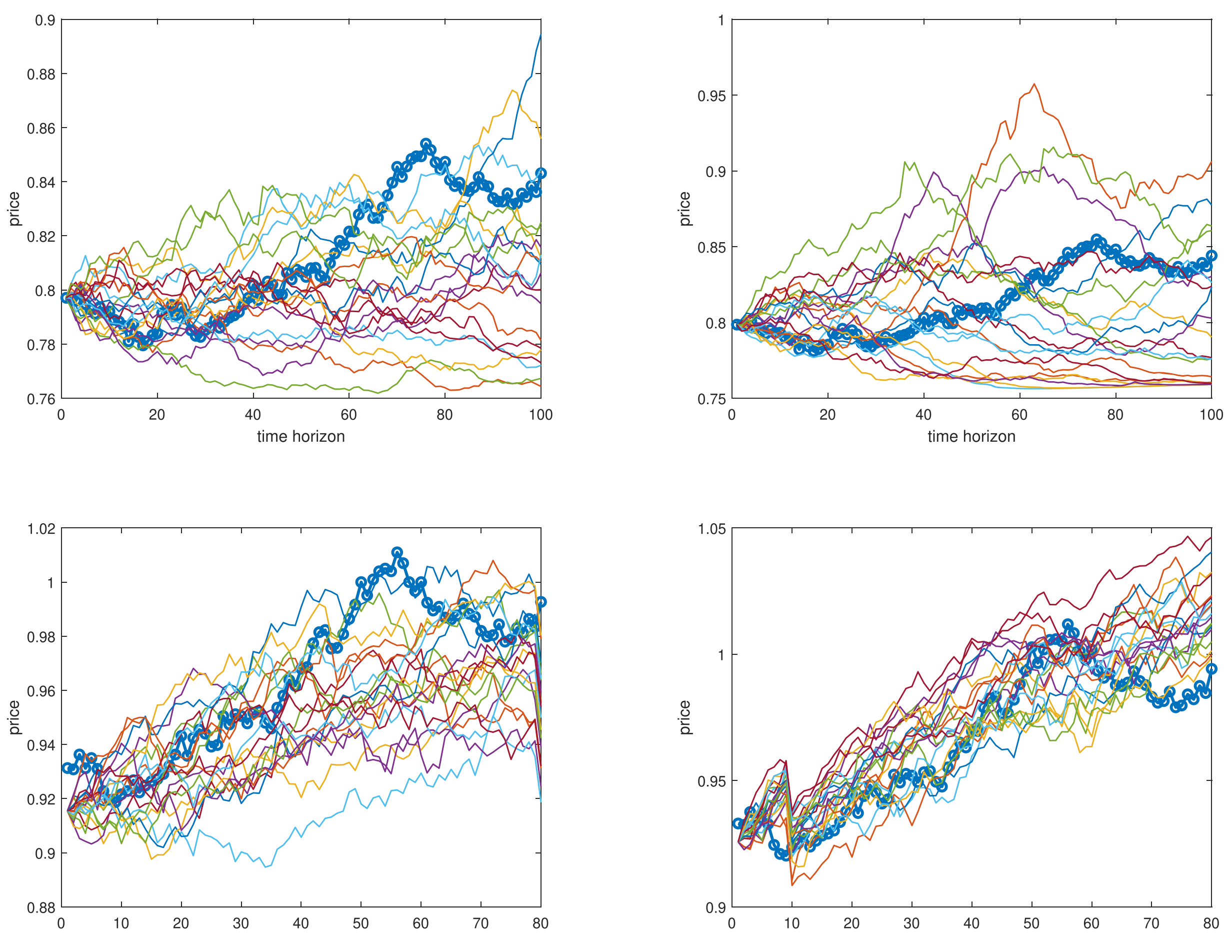


Fig. 6. Each figure shows the market price (dotted dashed line) and the 20 artificially simulated price trajectories of the synthetic instrument (solid lines, colored) for each Danish zero coupon bond and coupon bond. Specifically, each line shows the pair, brown bond (left plot) and Green bond (right plot) respectively, in ascending order of maturity, as in Table 10.

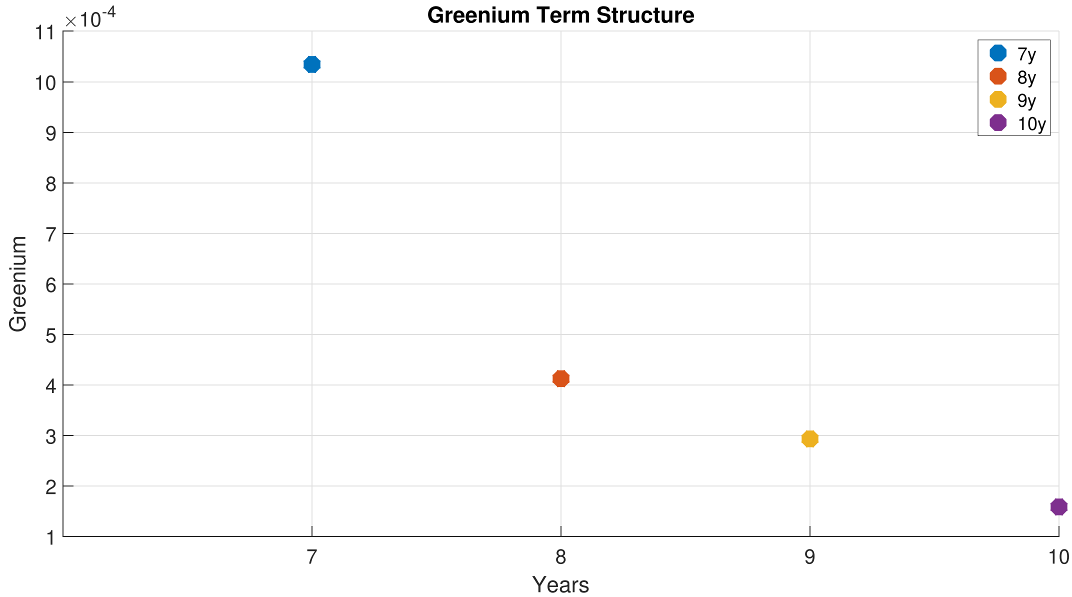


Fig. 7. Danish term structure at January 3, 2024 of greenium for different maturities of artificial bonds. The term structure was reconstructed considering the estimated parameters in Table 10 considering the last 80 observations.

Table 6
Parametric bootstrap analysis of the estimated parameters $\bar{\theta}$ for the German bond DE0001141828. The table reports the bootstrap mean $\mathbb{E}(\hat{\theta})$, the absolute bias $\left|\bar{\theta} - \mathbb{E}(\hat{\theta})\right|$, the median $q_{0.5}(\hat{\theta})$, and the standard deviation (std) of the bootstrap distribution.

	$\bar{\theta}$	$\mathbb{E}(\hat{\theta})$	$\left\|\bar{\theta} - \mathbb{E}(\hat{\theta})\right\|$	$q_{0.5}(\hat{\theta})$	$\left\|\bar{\theta} - q_{0.5}(\hat{\theta})\right\|$	std
b	0.0443	0.2536	0.2092	0.0593	0.0150	0.9086
a	−0.4341	−0.4360	0.0019	−0.4399	0.0058	0.0752
c	0.0266	0.0190	0.0076	0.0012	0.0254	0.4176
c_0	0.0694	0.0578	0.0116	0.0689	0.0005	0.5131
c_2	0.2277	0.2257	0.0020	0.2295	0.0018	0.2555
d	0.7800	0.8078	0.0277	0.7907	0.0107	0.2525
r_0	0.0453	0.0507	0.0054	0.0441	0.0012	0.1056

Table 7
Features of Danish twin bonds.

ISIN	Maturity	Type	First Coupon	Coupon Rate
BN3983473	15/11/2031	Brown	–	0
BS9262293	15/11/2031	Green	–	0
ZM6527638	15/11/2033	Brown	15/11/2023	2.25
ZI8882139	15/11/2033	Green	15/11/2023	2.25

Table 8
First four moments for Danish green bonds, computed over the sample period from October 2, 2023, to June 25, 2024.

ISIN	Maturity	Mean	Variance	Skewness	Kurtosis
BS9262293	15/11/2031	3.76×10^{-4}	1.48×10^{-5}	0.09	2.42
ZI8882139	15/11/2033	3.23×10^{-4}	2.06×10^{-5}	0.06	2.36

Table 9
First four moments for Danish brown bonds, computed over the sample period from October 2, 2023, to June 25, 2024.

ISIN	Maturity	Mean	Variance	Skewness	Kurtosis
BN3983473	15/11/2031	3.76×10^{-4}	1.48×10^{-5}	0.08	2.40
ZM6527638	15/11/2033	3.29×10^{-4}	1.98×10^{-5}	0.06	2.41

governments is expected to increase volumes of issued green bonds. Fig. 3 shows the volumes of new emissions for each green bond in Germany between 2020 and 2023. As the price of green bonds exhibit significant fluctuations compared to their brown counterparts, it is interesting to investigate the impact of these additional volumes of green bonds on the greenium dynamics. We empirically find that announcements of new volumes do not cause substantial jumps in the yield to maturity difference dynamics.[8] The jump dates correspond to the introduction of two directives: the first one is from Regulation (EU) 2022/2577 of the Council, dated December 22, 2022, which establishes a framework to accelerate the development of renewable energies and the second one is from Regulation (EU) 2024/223 of the Council, dated December 22, 2023, which amends Regulation (EU) 2022/2577 for the same purpose. Specifically, the latter regulation named “Accelerating the deployment of renewable energy”,[9] establishes a framework to accelerate the diffusion of renewable energy in the short term. It is an emergency measure based on Article 122 of the *Treaty on the Functioning of the European Union* and applies from December 30, 2022. Initially, it was valid for 18 months and has been extended by Regulation (EU) 2024/223 following a review by the European Commission. In particular, the price of bonds with 1 year and 2 years maturity seem to be significantly affected by these directives, as can be seen in Fig. 1. For longer maturities the effect of these directives is mitigated.

Fig. 4 illustrates the market prices and 20 artificially simulated price trajectories, from the MLE parameters, of the bonds (green and brown) with maturity up to 10 years over the specified estimation time period.

[8] To clarify this concept further, we refer to the document concerning “Announcement of a multi-ISIN auction: Reopening of two Green German Federal securities”, available at the link :https://www.bundesbank.de/resource/blob/928208/081b78034864242158724ba155059577/mL/2024-03-20-ankuendigung-download.pdf, where, for example, an announcement made on March 26, 2024, regarding ISIN *DE00011030716*, did not lead to a notable spike in the price near that date or in the adjacent dates, as can be seen from Fig. 1. We have highlighted that for bonds with 1-year and 3-year maturities, the most significant jumps occur around two dates: December 23, 2022 and December 22, 2023.

[9] Available at url: https://eur-lex.europa.eu/EN/legal-content/summary/accelerating-the-deployment-of-renewable-energy.html

Table 10
Estimated parameters assuming affine Gamma-GARCH innovations. ISIN for each Danish zero coupon or coupon bond (first column); maturity (second column); bond type (third column); estimated model parameters (rest of columns). For the estimation, we consider the last 80 observations before January 3, 2024.

ISIN	Maturity	Type	b	a	c	c_0	c_2	d	r_0	AIC	BIC
BN3983473	15/11/2031	Brown	0.0069	0.6696	0.0083	0.0001	0.3608	0.6414	0.0948	−1667.5	−1649.2
BS9262293	15/11/2031	Green	0.0038	0.8180	0.0125	0.0000	0.0111	0.9902	0.0990	−1684.7	−1666.5
ZM6527638	15/11/2033	Brown	0.0004	0.9822	0.0016	3.9954	0.9766	0.0001	0.0804	−1431.2	−1414.5
ZI8882139	15/11/2033	Green	0.0004	0.9743	0.0039	0.0262	0.6719	0.3277	0.0500	−1334.0	−1317.3

Table 11
Estimated parameters considering an Inverse Gaussian distribution for the innovations. ISIN for each Danish zero coupon or coupon bond (first column); maturity (second column); bond type (third column); estimated model parameters (rest of columns). For the estimation, we consider the last 80 observations before January 3, 2024.

ISIN	Maturity	Type	b	a	c	c_0	c_2	d	r_0	AIC	BIC
BN3983473	15/11/2031	Brown	0.0101	0.4190	0.0699	6.9999	0.7591	0.2544	0.0536	−717.8057	−699.5695
BS9262293	15/11/2031	Green	0.0025	0.8771	0.0250	6.1200	0.8967	0.1002	0.0499	−799.2814	−781.0452
ZM6527638	15/11/2033	Brown	0.0010	0.9626	0.0100	0.0050	0.9994	0.0004	0.0698	−761.5968	−744.9226
ZI8882139	15/11/2033	Green	0.0009	0.9618	0.0100	0.0005	0.6585	0.3415	0.0506	−738.8902	−722.2161

The simulated trajectories for the first bond display a large variability although the observed market price fall inside the confidence interval at each simulation date. The range of variation of the other simulated bond prices is similar to that of the observed sequences. The uncertainty in the simulated bond prices for the first maturity can be explained with the jump in the green bond price that we expected to reflect the introduction of Regulation (EU) 2024/223 of the Council, dated December 22, 2023.

This is confirmed by a similar analysis on Danish twin bonds. Indeed, Denmark has issued twin bonds, aligning with the twin bond concept introduced by Germany in 2020. This means that a green bond is issued with the same financial characteristics as one of the central government's conventional on-the-run issues. The idea behind the twin bond concept is to support the liquidity of the green bond. This is achieved by providing a green bond with characteristics identical to an existing general reference bond, to enable investors to switch from a green bond to the corresponding, more liquid conventional twin bond at any time. Table 7 presents the various characteristics of the pairs of Danish bonds we examined. Tables 8 and 9 present the descriptive statistics (mean, variance, skewness, and kurtosis) computed on the returns of Danish green and brown bonds, respectively. Fig. 5 displays the difference between the yield to maturity (YTM) of the Danish twin bonds. For Denmark, we have only two maturities with the coupon bond maturing in 2033 with first issuance dated to October 2023. The two series of yield to maturity differences for twin bonds seem to have the same trend while there is a clear difference in terms of volatility. Indeed, the series computed on the coupon bonds fluctuates more. Notice that, as the maturity of the bonds exceeds the deadline of EU short term policy interventions, we do not observe a jump in the yield to maturity difference on December 22, 2023 as in the case of German twin bonds. Considering 80 observations,[10] from the issuance of the green coupon bond until January 3, 2024, we estimate the parameters of the affine GARCH model for the short rate. The MLE parameters are included in Table 10, while the resulting greenium term structure is presented in Fig. 7. We observe that the greenium for Danish twin bonds decreases over time and tends to flatten as the time horizon increases, confirming the time-varying nature of the greenium although we do not have a large set of maturities. In Fig. 6, we have depicted in the upper panels the market price zero coupon bond versus 20 simulated zero coupon bond trajectories for conventional and green bond prices, respectively. In the bottom panels, we present the same analysis for coupon bonds. Here we observe the simulated trajectories reproduce well the observed prices suggesting ad adequate fitting of model parameters.

[10] Here we use a shorter dataset as the issuance of the twin coupon bonds in Denmark is quite recent

5. Conclusion

In this paper, we have reconstructed the green premium using an autoregressive model for the dynamics of interest rates. In particular, a simple mathematical framework was employed that provides semi-analytical bond prices. The maturity specific greenium term is then obtained as the difference between the yield to maturity of artificial bond prices computed with fitted parameters. The structure of the model permitted the definition of an optimisation routine for the fitting of model parameters, both for zero-coupon and coupon bonds. In accordance with our initial hypothesis, namely that the set of initial inputs for the term structure should be maturity-specific, we considered a variety of bond pairs for nodes in the term structure with a maturity that was close to that of the bond to be priced. The empirical analysis yielded confirmation that the greenium is nonnegative, time-varying, and that it tends to diminish for longer bond duration, at least for European sovereign debt with a high credit rating.
These findings offer an explanation for the lack of consensus in the literature on the presence of greenium. In fact, most of the existing papers on this topic consider panel regression exercises involving bonds with varying maturities, which can produce misleading results. The proposed modeling framework is capable of capturing the differences that emerge in the cases of pairs of bonds with similar characteristics, but that are influenced by specific legislative innovations or monetary policy interventions.
Indeed, although we observed an increase in the volume of green bonds due to new issuances from the German government, no significant price fluctuations were detected. The leading cause of these jumps is attributable to European directives aimed at accelerating renewable energy development, which evidently alter investors' return expectations for these instruments.

CRediT authorship contribution statement

Lorenzo Mercuri: Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization; **Edit Rroji:** Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization; **Ilaria Stefani:** Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Acknowledgement

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Supplementary material

Supplementary material associated with this article can be found in the online version at 10.1016/j.ejor.2026.03.008.

References

Agliardi, E., & Agliardi, R. (2021). Pricing climate-related risks in the bond market. *Journal of Financial Stability*, *54*, 100868.

Ando, S., Fu, C., Roch, F., & Wiriadinata, U. (2023). How large is the sovereign greenium? *Oxford Bulletin of Economics and Statistics 86*(6), 1472–1483.

Atkins, P. J., & Cummins, M. (2023). Improved scalability and risk factor proxying with a two-step principal component analysis for multi-curve modelling. *European Journal of Operational Research 304*(3), 1331–1348.

Bachelet, M. J., Becchetti, L., & Manfredonia, S. (2019). The green bonds premium puzzle: The role of issuer characteristics and third-party verification. *Sustainability*, *11*(4), 1098.

Badescu, A., Cui, Z., & Ortega, J. P. (2019). Closed-form variance swap prices under general affine GARCH models and their continuous-time limits. *Annals of Operations Research 282*, 27–57.

Baker, M., Bergstresser, D., Serafeim, G., & Wurgler, J. (2022). The pricing and ownership of US green bonds. *Annual Review of Financial Economics*, *14*(1), 415–437.

Bellini, F., & Mercuri, L. (2014). Option pricing in a conditional bilateral gamma model. *Central European Journal of Operations Research*, *22*, 373–390.

Billio, M., Costola, M., Pelizzon, L., & Riedel, M. (2022). Buildings energy efficiency and the probability of mortgage default: The dutch case. *The Journal of Real Estate Finance and Economics*, *65*(3), 419–450.

Bougerol, P., & Picard, N. (1992). Strict stationarity of generalized autoregressive processes. *Annals of Probability*, *20*(4), 1714–1730. https://doi.org/10.1214/aop/1176989526

Cao, H., Badescu, A., Cui, Z., & Jayaraman, S. (2020). Valuation of VIX and target volatility options with affine GARCH models. *Journal of Futures Markets*, *40*(12), 1880–1917.

Caramichael, J., & Rapp, A. C. (2024). The green corporate bond issuance premium. *Journal of Banking & Finance*, *162*, 107126.

Christoffersen, P., Heston, S., & Jacobs, K. (2006). Option valuation with conditional skewness. *Journal of Econometrics*, *131*(1-2), 253–284.

Cox, J. C., Ingersoll, J. E., & Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica 53*(2), 385–407.

De Vincentiis, P., & Abis, D. (2025). Non-identical twins: Evidence on greenium in the German treasury bond market. *Research in International Business and Finance*, (p. 103057).

Efron, B., & Tibshirani, R. J. (1994). An introduction to the bootstrap. (1st ed.). Chapman and Hall/CRC. https://doi.org/10.1201/9780429246593

Fandella, P., & Cociancich, V. (2024). Uncovering the greenium: Investigating the yield spread between green nad conventional bonds. *Investment Management & Financial Innovations*, *21*(2), 1–14.

Flammer, C. (2021). Corporate green bonds. *Journal of Financial Economics 142*(2), 499–516.

Hachenberg, B., & Schiereck, D. (2018). Are green bonds priced differently from conventional bonds? *Journal of Asset Management*, *19*, 371–383.

Hamilton, J. D. (1994). Time series analysis. Princeton, NJ: Princeton University Press.

Harvey, C. R., & Siddique, A. (1999). Autoregressive conditional skewness. *Journal of Financial and Quantitative Analysis*, *34*(4), 465–487.

Heston, S. L., & Nandi, S. (2000). A closed-form GARCH option valuation model. *The Review of Financial Studies 13*(3), 585–625.

Hitaj, A., Mercuri, L., & Rroji, E. (2018). VIX computation based on affine stochastic volatility models in discrete time. Springer.

Hougaard, P. (1986). Survival models for heterogeneous populations derived from stable distributions. *Biometrika*, *73*, 387–396. https://doi.org/10.1093/biomet/73.2.387

Kanamura, T. (2020). Are green bonds environmentally friendly and good performing assets? *Energy Economics*, *88*, 104767.

Kanamura, T. (2024). Stochastic behavior of green bond premiums. *International Review of Financial Analysis*, (p. 103836).

Kawai, R., & Masuda, H. (2011). On simulation of tempered stable random variates. *Journal of Computational and Applied Mathematics 235*(8), 2873–2887.

Larcker, D. F., & Watts, E. M. (2020). Where's the greenium? *Journal of Accounting and Economics*, *69*(2-3), 101312.

Lütkepohl, H. (2005). New introduction to multiple time series analysis. Berlin, Heidelberg: Springer.

Mercuri, L. (2008). Option pricing in a GARCH model with tempered stable innovations. *Finance Research Letters*, *5*(3), 172–182. https://www.sciencedirect.com/science/article/pii/S1544612308000329. https://doi.org/10.1016/j.frl.2008.05.003

Mercuri, L. (2011). Pricing Asian options in Affine GARCH models. *International Journal of Theoretical and Applied Finance*, *14*(02), 313–333. https://doi.org/10.1142/S0219024911006371

Mercuri, L., Perchiazzo, A., Rroji, E., & Stefani, I. (2026). Market-implied time to transition to a low-carbon economy from the greenium term structure. Available at SSRN. https://ssrn.com/abstract=6135666.

Mikosch, T. (2009). Non-life insurance mathematics: An introduction with the poisson process. Universitext (2nd ed.). Berlin: Springer. https://doi.org/10.1007/978-3-540-88233-6

Nelson, D. B. (1990). Stationarity and persistence in the GARCH(1,1) model. *Econometric Theory*, *6*(3), 318–334. http://www.jstor.org/stable/3532198.

Pástor, L., Stambaugh, R. F., & Taylor, L. A. (2021). Sustainable investing in equilibrium. *Journal of Financial Economics 142*(2), 550–571.

Pástor, L., Stambaugh, R. F., & Taylor, L. A. (2022). Dissecting green returns. *Journal of Financial Economics 146*(2), 403–424.

Pietsch, A., & Salakhova, D. (2022). Pricing of green bonds: Drivers and dynamics of the greenium. *ECB Working Paper*.

Recchioni, M. C., & Tedeschi, G. (2017). From bond yield to macroeconomic instability: A parsimonious affine model. *European Journal of Operational Research 262*(3), 1116–1135.

Tang, D. Y., & Zhang, Y. (2020). Do shareholders benefit from green bonds? *Journal of Corporate Finance*, *61*, 101427.

Tweedie, M. C. K. (1984). An index which distinguishes between some important exponential families. In J. Ghosh, & J. Roy (Eds.), *Statistics: Applications and new directions* (pp. 579–604). Indian Statistical Institute. Proc. Indian Statistical Institute Golden Jubilee International Conference.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics 5*(2), 177–188. https://doi.org/10.1016/0304-405X(77)90016-2

Wang, Q., & Wang, Z. (2021). VIX futures and its closed-form pricing through an affine GARCH model with realized variance. *Journal of Futures Markets*, *41*(1), 135–156.

Zerbib, O. D. (2019). The effect of pro-environmental preferences on bond prices: Evidence from green bonds. *Journal of Banking & Finance*, *98*, 39–60.

Zhang, S., Yang, Z., & Wang, S. (2020). Design of green bonds by double-barrier options. *Discrete & Continuous Dynamical Systems-Series S*, *13*(6), 1867–1882.