# COHERENT SYSTEMS ON CURVES OF COMPACT TYPE

SONIA BRIVIO AND FILIPPO F. FAVALE

Abstract. Let $C$ be a polarized nodal curve of compact type. In this paper we study coherent systems $(E,V)$ on $C$ given by a depth one sheaf $E$ having rank $r$ on each irreducible component of $C$ and a subspace $V \subset H^0(E)$ of dimension $k$. Moduli spaces of stable coherent systems have been introduced by [KN95] and depend on a real parameter $\alpha$. We show that when $k \geq r$, these moduli spaces coincide for $\alpha$ big enough. Then we deal with the case $k = r+1$: when the degrees of the restrictions of $E$ are big enough we are able to describe an irreducible component of this moduli space by using the dual span construction.

## Introduction

Coherent systems on smooth curves can be seen as the generalisation of classical linear systems. They were studied first, under different names, by Bradlow ([BD91]), Bertram ([Ber94]) and Le Potier ([LP93]). They are closely related to higher rank Brill-Noether theory: for relevant results on this argument one can see for example [Bra09], [New11] and [BGPMN03]. Moreover, coherent systems have been a useful tool in studying theta divisors and the geometry of moduli spaces of vector bundles on curves (see for instance, some results of the authors: [Bri15], [Bri17], [BF19] and [BV12]). Nevertheless, they are already interesting by themselves since a notion of stability can be defined, depending on a real parameter $\alpha$. Varying $\alpha$ one gets a family of moduli spaces, providing examples of higher dimensional algebraic varieties with a rich and interesting geometry. For comparison of different notions of stability arising in moduli theory see for instance [BB12].

Questions concerning emptiness or non emptiness, smoothness, irreducibility, and singularities have been deeply studied by many authors. Among them, we can point out some relevant results in [BPGN97] and [BGPMN03].

Coherent systems can be defined even on a singular curve. A notion of semistability has been introduced, depending on a polarization $\underline{w}$ on the curve and a real parameter $\alpha$, and coarse moduli spaces can be constructed as well as in the smooth case (see [KN95]). Nevertheless, there has been little work in the singular case, (for example see [Bal06] and [Bal06b]). The situation becomes much better in the nodal case. In fact, many results of [BGPMN03] and [BPGN97] have been extended to irreducible nodal curves by Bhosle in [Bho09].

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In this paper we start the study of coherent systems on a reducible nodal curve $C$ of compact type. By a coherent system on $C$ we mean a pair $(E,V)$, given by a depth one sheaf $E$ on $C$ and by a subspace $V \subseteq H^0(E)$. Fix a polarization $\underline{w}$ on the curve $C$, then we can define $\underline{w}$-rank and $\underline{w}$-degree of $E$ (denoted as $\mathrm{rk}_{\underline{w}}(E)$ and $\deg_{\underline{w}}(E)$). For any $\alpha \in \mathbb{R}$, the notion of $(\underline{w}, \alpha)$-stability has been defined, see [KN95].

A pair $(E,V)$ is called generated (resp. generically generated) if the map of evaluation of global sections of $V$ is surjective (resp. generically surjective), see Section 3 for details. We focus our attention on generated pairs $(E,V)$ on the curve $C$ of multitype $(\underline{r}, d, k)$: i.e. $E$ has rank $r$ on each irreducible component of $C$, $\deg_{\underline{w}}(E) = d$ and $\dim V = k \geq r$. We denote by $\mathcal{G}_{(\underline{w},\alpha)}(\underline{r}, d, k)$ the moduli space parametrizing families of $(\underline{w}, \alpha)$-stable coherent systems as above. As in the case of smooth curves and of irreducible nodal curves, we have the following results, which are proved in Proposition 3.5, Theorem 3.9 and Corollary 3.10 and are summarized as follows.

**Theorem A** *Let $C$ be a reducible nodal curve of compact type and let $\underline{w}$ be a polarization on it. Let $r \geq 1$, $d \geq 0$ and $k \geq r$ integers. There exists $\alpha_l \in \mathbb{R}$, depending on $\underline{w}$, $r$, $d$, $k$ and $p_a(C)$, such that the moduli spaces $\mathcal{G}_{(\underline{w},\alpha)}(\underline{r}, d, k)$ coincide for $\alpha > \alpha_l$. Moreover, for any $\alpha > \alpha_l$, any $(\underline{w}, \alpha)$-stable $(E,V)$ is generically generated.*

Let $(E,V)$ be a coherent system on the curve $C$, we can define in a natural way, coherent systems $(E_i, V_i)$ which are the restrictions of $(E,V)$ to each irreducible component $C_i$. As in the case of $\underline{w}$-stability for depth one sheaves on $C$ (see [TiB11] and [BF19b]), $\alpha$-stability on the restrictions does not imply, in general, $(\underline{w}, \alpha)$-stability. Nevertheless, when $k$ and $r$ are coprime, we give a sufficient condition in order to ensure that $\alpha$-stability of restrictions implies $(\underline{w}, \alpha)$-stability: this is proved in Theorem 3.12.

In the second part of this paper we will concentrate on coherent systems with $k = r+1$. In the case of smooth curves, they have been studied by using the dual span construction, which was introduced by Butler in [But]. For Petri curves, it is now completely known when such moduli spaces are non-empty (see [BBPN08] and [BBPN15]). In this case, for any $r \geq 2$, $d > 0$, the moduli space $\mathcal{G}_\alpha(r, d, r+1)$ is birational to the moduli space $\mathcal{G}_\alpha(1, d, r+1)$ for large $\alpha$ ([BGPMN03]).

We generalize the dual span costruction to generated coherent systems $(L, W)$ on a nodal curve of compact type $C$ where $L$ is a line bundle. More precisely, we assume that $C$ has $\gamma$ irreducible components $C_i$ of genus $g_i \geq 2$. For any $(d_1, \dots, d_\gamma) \in \mathbb{N}^\gamma$, we can consider the subvariety

$$X_{d_1,\dots,d_\gamma} \subset \mathcal{G}_{(\underline{w},\alpha)}(\underline{1}, d, r+1),$$

parametrizing all coherent systems $(L, W)$ where $L$ is a line bundle whose restriction on $C_i$ has degree $d_i$. The first result for this part is Theorem 5.3 which is summarized as follows.

**Theorem B** *Under the hypothesis of Theorem A, if $d_i \geq max(2g_i + 1, g_i + r)$ and $d = \sum_{i=1}^{\gamma} d_i$, then, for $\alpha$ big enough, the closure $\overline{X_{d_1,\dots,d_\gamma}} \subset \mathcal{G}_{(\underline{w},\alpha)}(\underline{1}, d, r+1)$ is an irreducible component of dimension equal to the Brill-Noether number $\beta_C(1, d, r+1)$. Any $(L, W) \in X_{d_1,\dots,d_\gamma}$ is a smooth point of the moduli space.*

Then by applying the dual span construction to coherent systems in $X_{d_1,\dots,d_\gamma}$ we obtain the main results of the second part of the paper. These are Theorem 5.4 and Theorem 5.5 and are summarized as follows.

**Theorem C** *Under the hypothesis of Theorem B, for $\alpha$ big enough, there exists an irreducible component $Y_{d_1,\dots,d_\gamma} \subset \mathcal{G}_{(\underline{w},\alpha)}(\underline{r}, d, r+1)$ which is birational to $X_{d_1,\dots,d_\gamma}$, with dimension equal to the Brill-Nother number $\beta_C(r, d, r+1)$.*

*We have the following commutative diagram:*

$$\begin{array}{ccc} \overline{X_{d_1,\cdots,d_\gamma}} & \overset{\mathcal{D}}{\dashrightarrow} & Y_{d_1,\dots,d_\gamma} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \Pi_{i=1}^{\gamma}\mathcal{G}_{C_i,\alpha}(1, d_i, r+1) & \underset{\Pi D_i}{\dashrightarrow} & \Pi_{i=1}^{\gamma}\mathcal{G}_{C_i,\alpha}(r, d_i, r+1) \end{array}$$

*where $\mathcal{G}_{C_i,\alpha}(s, d_i, r+1)$ is the moduli space of $\alpha$-stable coherent systems of type $(s, d_i, r+1)$ on the curve $C_i$, $\mathcal{D}$ and $D_i$ are the maps sending a coherent system to its dual span and the vertical maps are restrictions to the components of $C$. Finally, the maps $\pi_1$ and $\pi_2$ are dominant.*

The paper is organized as follows. In Section 1 we recall basic properties of nodal curves and depth one sheaves. In Section 2 we introduce the notion of coherent system, $(\underline{w}, \alpha)$-stability and we recall some results concerning their moduli spaces. In Section 3 we focus on generated coherent systems of multitype $(\underline{r}, d, k)$ and we prove Theorem A. In Section 4, by using the dual span construction, we produce $(\underline{w}, \alpha)$-stable coherent systems of multitype $(\underline{r}, d, r+1)$. Finally, in Section 5 we prove the results stated in Theorem B and Theorem C.

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## 1. Reducible nodal curves and depth one sheaves

In this paper we will consider connected reduced and reducible curves $C$ over the complex numbers which are *nodal*, i.e. complete algebraic curves whose singularities are at most *ordinary double points*. We recall that a connected nodal curve is said to be of *compact type* if every irreducible component of $C$ is smooth and its dual graph is a tree. For the theory of nodal curves see [ACG11]. We will always assume that $C$ is a nodal curve of compact type and that each irreducible component $C_i$ of $C$ is smooth of genus $g_i \geq 2$. If we denote by $\gamma$ the numbers of irreducible components of $C$ and by $\delta$ the number of nodes of $C$, we have $\gamma = \delta + 1$. The normalization map of $C$ is

$$\nu \colon C^\nu \to C,$$

where $C^\nu = \bigsqcup_{i=1}^{\gamma} C_i$ and $\nu$ induces an isomorphism $\mathrm{Pic}(C) \simeq \bigoplus_{i=1}^{\gamma} \mathrm{Pic}(C_i)$ between the Picard groups. In particular, we will denote by $\mathrm{Pic}^{(d_1,\dots,d_\gamma)}(C)$ the subgroup of line bundles $L$ on $C$

whose restriction to $C_i$ is in $\mathrm{Pic}^{d_i}(C_i)$. The arithmetic genus of $C$ is

$$p_a(C) = 1 - \chi(\mathcal{O}_C) = \sum_{i=1}^{\gamma} g_i.$$

We recall that, since $C$ is nodal, then it can be embedded in a smooth projective surface $S$, see [AK79]. Let $B$ be any subcurve of $C$. The complementary curve of $B$, denoted by $C - B$, is defined as the closure of $C \setminus B$ and it is actually the difference of $C - B$ as divisors on $S$. We will denote by $\Delta_B$ the intersection of $B$ with its complementary curve, it is given by double points common to a component of $B$ and one of $C - B$. In particular, when $C_i$ is a component of $C$, $\Delta_{C_i}$ is given by double points on $C_i$. To simplify notations we set: $\Delta_{C_i} = \Delta_i$, $C - C_i = C_i^c$ and $\delta_i = \#\Delta_i$. For any subcurve $B$ of $C$ we have the following exact sequence:

$$0 \to \mathcal{O}_{C-B}(-\Delta_B) \to \mathcal{O}_C \to \mathcal{O}_B \to 0, \tag{1.1}$$

from which we deduce

$$p_a(C) = p_a(B) + p_a(C - B) + deg(\Delta_B) - 1.$$

In particular, when $B = C_i$, we have:

$$0 \to \mathcal{O}_{C_i^c}(-\Delta_i) \to \mathcal{O}_C \to \mathcal{O}_{C_i} \to 0, \tag{1.2}$$

which gives $p_a(C_i^c) = \sum_{j \neq i} g_j + 1 - \delta_i$.

We stress a useful fact that we will use a lot in the following.

**Remark 1.1.** Since $C$ is of compact type, we can find a component $C_i$ such $C_i^c$ is a connected curve of compact type too. Actually, it is enough to show that there exists a component $C_i$ with $\delta_i = 1$, i.e. such that only one node of $C$ lies on $C_i$. Assume on the contrary that, for all $i = 1 \dots \gamma$, we have $\delta_i \geq 2$. As every node lies on two components, we have

$$\delta = \frac{1}{2}\sum_{i=1}^{\gamma} \delta_i \geq \frac{1}{2}\sum_{i=1}^{\gamma} 2 = \gamma.$$

But this is impossible as $\gamma = \delta + 1$.

The dualizing sheaf $\omega_C$ is an invertible sheaf. Moreover, for any subcurve $B$ of $C$ the dualizing sheaf $\omega_B$ is invertible too and we have:

$$\omega_{C|B} = \omega_B \otimes \mathcal{O}_B(\Delta_B), \tag{1.3}$$

in particular for any component $C_i$ we have: $\omega_C|_{C_i} = \omega_{C_i} \otimes \mathcal{O}_{C_i}(\Delta_i)$.

**Definition 1.2.** *A polarization on the curve $C$ is a vector $\underline{w} = (w_1, \dots, w_\gamma) \in \mathbb{Q}^\gamma$, with*

$$0 < w_i < 1, \quad \sum_{i=1}^{\gamma} w_i = 1. \tag{1.4}$$

We will fix an ample primitive invertible sheaf $\mathcal{O}_C(1)$ on the curve $C$, with $a_i = \deg(\mathcal{O}_C(1)|_{C_i})$. It determines a polarization $\underline{w}$ by defining $w_i = \frac{a_i}{\sum_{k=1}^{\gamma} a_k}$. Note that since $\mathcal{O}_C(1)$ is ample, $a_i \geq 1$ and $\gcd(a_1, \ldots, a_\gamma) = 1$ since $\mathcal{O}_C(1)$ is primitive.

We recall the notion of depth one sheaves on nodal curves, for details see [Ses82](Chapter VII). A coherent sheaf $E$ on $C$ is said to be of *depth one*[1] if $\dim F = \dim \mathit{supp}(F) = 1$ for every subsheaf $F$ of $E$. A coherent sheaf $E$ on $C$ is of depth one if the stalk of $E$ at the node $p = C_i \cap C_j$ is isomorphic to $\mathcal{O}_p^a \oplus \mathcal{O}_{q_i}^{b_i} \oplus \mathcal{O}_{q_j}^{b_j}$, where $\nu^{-1}(p) = \{q_i, q_j\}$ and $\mathcal{O}_{q_t} = \mathcal{O}_{C_t, q_t}$. In particular, any vector bundle on $C$ is a sheaf of depth one. Let $E$ be a sheaf of depth one on $C$, its restriction $E|_{C_i \setminus \Delta_i}$ is either zero or it is locally free; moreover, any subsheaf of $E$ is of depth one too.

Let $E$ be a sheaf of depth one on $C$, we set

$$E_i = E \otimes \mathcal{O}_{C_i}/Torsion, \tag{1.5}$$

which is called the *restriction of $E$ modulo torsion* on the component $C_i$. If $E_i$ is not zero, we set $r_i = \mathrm{rk}(E_i)$; otherwise we set $r_i = 0$. We associate to $E$:

$$\underline{\mathrm{rk}}(E) = (r_1, \ldots, r_\gamma), \tag{1.6}$$

which is called the *multirank* of $E$;

$$\mathrm{rk}_{\underline{w}}(E) = \sum_{i=1}^{\gamma} w_i r_i \tag{1.7}$$

which is called the *$\underline{w}$-rank* of $E$;

$$\deg_{\underline{w}} E = \chi(E) - \mathrm{rk}_{\underline{w}}(E)\chi(\mathcal{O}_C), \tag{1.8}$$

which is called the *$\underline{w}$-degree* of $E$.

Note that $\underline{w}$-rank and $\underline{w}$-degree are not necessary integers. When $E$ is a vector bundle on $C$, i.e. it is locally isomorphic to $\mathcal{O}_C^r$, then the $\underline{w}$-rank of $E$ is actually $r$ and the $\underline{w}$-degree of $E$ is an integer too.

**Lemma 1.3.** *Let $E$ be a depth one sheaf on $C$ and let $E_i$ be the restriction modulo torsion of $E$ to $C_i$. Then we have:*

(1) *let $\underline{\mathrm{rk}}(E) = (r_1, \ldots, r_\gamma)$ and $r_M = \max(r_1, \cdots, r_\gamma)$:*

$$\sum_{i=1}^{\gamma} \chi(E_i) - r_M(\gamma - 1) \leq \chi(E) \leq \sum_{i=1}^{\gamma} \chi(E_i);$$

(2) *if $\underline{\mathrm{rk}}(E) = (r, \ldots, r)$:*

$$\sum_{i=1}^{\gamma} \deg(E_i) \leq \deg_{\underline{w}}(E) \leq \sum_{i=1}^{\gamma} \deg(E_i) + r(\gamma - 1);$$

[1] Different terms are used to refer to such sheaves. As $C$ is a scheme of pure dimension 1, this is equivalent to ask that $E$ is *pure of dimension* 1 or that $E$ is *torsion free*.

*(3) if $E$ is locally free of rank $r$, then we have:*

$$\chi(E)=\sum_{i=1}^{\gamma}\chi(E_i)-r(\gamma-1)\quad \deg_{\underline{w}}(E)=\sum_{i=1}^{\gamma}\deg(E_i);$$

*(4) if $E$ is locally free and $h^0(E_i)=0$ for any $i=1,\cdots,\gamma$, then we also have $h^0(E)=0$.*

*Proof.* (1) We have an exact sequence (see [Ses82])

$$0\to E\to\bigoplus_{i=1}^{\gamma}E_i\to T\to 0$$

where $T$ is a torsion sheaf whose support in contained in the set of nodes of $C$. We deduce that $\chi(E)=\sum_{i=1}^{\gamma}\chi(E_i)-\chi(T)$. Note that $\chi(T)=l(T)\geq 0$, hence $\chi(E)\leq\sum_{i=1}^{\gamma}\chi(E_i)$. Let $p\in C_i\cap C_j$ be a node, such that $\nu^{-1}(p)=\{q_i,q_j\}$. If $E_p\simeq\mathcal{O}_p^s\oplus\mathcal{O}_{q_i}^{b_i}\oplus\mathcal{O}_{q_j}^{b_j}$ with $0\leq s\leq\min(r_i,r_j)$, $s+b_i=r_i$ and $s+b_j=r_j$, then $T_p\simeq\mathbb{C}^s$, see [Ses82]. This implies that $l(T)\leq r_M(\gamma-1)$ and we can conclude that $\chi(E)\geq\sum_{i=1}^{\gamma}\chi(E_i)-r_M(\gamma-1)$.

(2) From the above sequence we obtain $\deg_{\underline{w}}(E)=\deg_{\underline{w}}(\bigoplus_{i=1}^{\gamma}E_i)-l(T)$. We have:

$$\deg_{\underline{w}}\left(\bigoplus_{i=1}^{\gamma}E_i\right)=\sum_{i=1}^{\gamma}(\deg(E_i)+r(1-g_i))-r(1-p_a(C))=\sum_{i=1}^{\gamma}\deg(E_i)+r(\gamma-1).$$

As $0\leq l(T)\leq r(\gamma-1)$, the assertion follows.

(3) If $E$ is locally free, for any node $p\in C_i\cap C_j$, we have $E_p\simeq\mathcal{O}_p^r$. This implies that $l(T)=r(\gamma-1)$ and the first claim follows. By definition we have:

$$\deg_{\underline{w}}(E)=\chi(E)-\mathrm{rk}_{\underline{w}}(E)\chi(\mathcal{O}_C)=\sum_{i=1}^{\gamma}\chi(E_i)-r(\gamma-1)-r\chi(O_C).$$

Since $\chi(E_i)=\deg(E_i)+r(1-g_i)$ we obtain: $\deg_{\underline{w}}(E)=\sum_{i=1}^{\gamma}\deg(E_i)$.

(4) We prove the assertion by induction on the number $\gamma$ of irreducible components of $C$. If $\gamma=2$, then $C$ has two irreducible components and a single node $p$. By tensoring (1.2) with $E$ we have the exact sequence

$$0\to E_1(-p)\to E\to E_2\to 0.$$

If we pass to global sections we obtain

$$0\to H^0(E_1(-p))\to H^0(E)\to H^0(E_2)\to\ldots$$

and $h^0(E)=0$ since $h^0(E_2)=h^0(E_1(-p))=0$.

Assume now that $C$ is a nodal curve with $\gamma\geq 3$ irreducible components. As we have seen, there exists an irreducible component $C_i$ having a single node $p_{ij}$. We can consider the exact sequence:

$$0\to O_{C_i^c}(-p_{ij})\to O_C\to O_{C_i}\to 0,$$

tensoring with $E$ we obtain:

$$0\to E|_{C_i^c}(-p_{ij})\to E\to E_i\to 0,$$

passing to global sections:

$$0 \to H^0(E|_{C_i^c}(-p_{ij})) \to H^0(E) \to H^0(E_i) \to \ldots$$

Notice that $p_{ij}$ is a smooth point for $C_i^c$, so $E|_{C_i^c}(-p_{ij})$ is locally free on the curve $C_i^c$. Moreover we have: $E|_{C_i^c}(-p_{ij})_{|C_j} = E_j(-p_{ij})$ and $E|_{C_i^c}(-p_{ij})_{|C_k} = E_k$ for $k \neq i, j$. The curve $C_i^c$ is a nodal connected curve of compact type with $\gamma - 1$ components, by induction hypothesis we have $h^0(E|_{C_i^c}(-p_{ij})) = 0$. Since $h^0(E_i) = 0$ too, this implies that $h^0(E) = 0$.

□

**Lemma 1.4.** *Let $L$ be a line bundle on $C$, let $L_i$ be the restriction of $L$ to the component $C_i$ and $d_i = deg(L_i)$. Then*

*(1) $L$ is ample if and only if $d_i > 0$ for all $i$;*
*(2) if $d_i \geq 2g_i$ for all $i$, then $h^1(L) = 0$ and $L$ is globally generated;*
*(3) if $d_i \geq 2g_i + 1$ for all $i$, then the restriction map $\rho_i : H^0(L) \to H^0(L_i)$ is surjective and $L$ is very ample.*

*Proof.* (1) See [ACG11](Ch X, Lemma 2.15).

(2) By [CF96](Lemma 2.1), in order to have $h^1(L) = 0$ and that $L$ is globally generated, it is enough to prove that for any subcurve $B$ of $C$ we have $\deg(L|_B) \geq 2p_a(B)$. Let $B$ be a subcurve of $C$ and assume that $B$ is connected. Then $B = \bigcup_{k=1}^{\gamma_B} C_{i_k}$ is a curve of compact type so $p_a(B) = \sum_{k=1}^{\gamma_B} g_{i_k}$ and $\deg(L|_B) = \sum_{k=1}^{\gamma_B} d_{i_k}$ by Lemma 1.3. Since we are assuming $d_i \geq 2g_i$ for all $i$, we have

$$\deg(L|_B) = \sum_{k=1}^{\gamma_B} d_{i_k} \geq 2 \sum_{k=1}^{\gamma_B} g_{i_k} = 2p_a(B).$$

Assume now that $B$ is not connected. Then $B$ is the disjoint union $B = \bigsqcup_{k=1}^{c} B_k$ of connected curves $B_1, \ldots, B_c$ which are of compact type. It is easy to see that

$$\deg(L|_B) = \sum_{k=1}^{c} \deg(L|_{B_k}) \qquad p_a(B) = \sum_{k=1}^{c} p_a(B_k) - (c-1).$$

Then, since we have $\deg(L|_{B_k}) \geq 2p_a(B_k)$, as we just proved, we have

$$\deg(L|_B) = \sum_{k=1}^{c} \deg(L|_{B_k}) \geq 2 \sum_{k=1}^{c} p_a(B_k) > 2p_a(B).$$

(3) If we tensor the exact sequence (1.2) with $L$ and consider the long exact sequence in cohomology we have

$$0 \to H^0(L|_{C_i^c}(-\Delta_i)) \to H^0(L) \xrightarrow{\rho_i} H^0(L_i) \to H^1(L|_{C_i^c}(-\Delta_i)) \to H^1(L).$$

By the previous point we have that $h^1(L) = 0$ so the surjectivity of $\rho_i$ is equivalent to $h^1(L|_{C_i^c}(-\Delta_i)) = 0$. Denote by $C'$ the curve $C_i^c$ (which is a finite disjoint union of connected curves of compact type) and by $L'$ its line bundle $L|_{C_i^c}(-\Delta_i)$. Note that for any $j \neq i$ we have

$$d'_j = \deg(L'|_{C_j}) \geq d_j - 1$$

since at most one of the points of $\Delta_i$ lies in $C_j$ (as each connected component of $C'$ is of compact type). Since, by assumption, we have $d'_j \geq 2g_j$, then $h^1(L|_{C_i^c}(-\Delta_i)) = 0$ by (2). This implies that $\rho_i$ is surjective.

Finally, the very ampleness of $L$ follows from [CFHR99]: in fact, by using the same arguments of (2), one can prove that for any subcurve $B$ of $C$ we have $\deg(L|_B) \geq 2p_a(B) + 1$. □

## 2. Coherent systems on nodal curves

Let $C$ be a reducible nodal curve as in Section 1. In this section we will recall the notion of coherent systems[2] on the curve $C$, for details see [KN95]. A *coherent system* on the curve $C$ is given by a pair $(E, V)$, where $E$ is a depth one sheaf on $C$ and $V$ is a subspace of $H^0(E)$. A *coherent subsystem* $(F, U)$ of $(E, V)$ is a coherent system which consists of a subsheaf $F \subseteq E$ and a subspace $U \subseteq V \cap H^0(F)$. We say that $(F, U)$ is a *proper* subsystem if $(F, U) \neq (0, 0)$ and $(F, U) \neq (E, V)$. A coherent system $(E, V)$ is said to be of *type* $(r, d, k)$ if $\mathrm{rk}_{\underline{w}}(E) = r$, $\deg_{\underline{w}} E = d$ and $\dim V = k$; if the multirank of $E$ is $\underline{\mathrm{rk}}(E) = (r_1, \cdots, r_\gamma)$ then it is said to be of *multitype* $((r_1, \cdots, r_\gamma), d, k)$.

**Definition 2.1.** *A family of coherent systems parametrized by a scheme $T$ is given by a triplet $(\mathcal{E}, \mathcal{V}, \xi)$ where*

- *$\mathcal{E}$ is a sheaf on $C \times T$ flat over $T$ such that for any $t \in T$ the sheaf $E_t = \mathcal{E}_{|C \times t}$ is of depth one;*
- *$\mathcal{V}$ is a locally free sheaf on $T$ whose fiber at $t$ is $V_t$;*
- *$\xi \colon \pi^*\mathcal{V} \to \mathcal{E}$ is a map of sheaves, where $\pi \colon C \times T \to T$ is the projection, and, for any $t \in T$, the map*
$$\xi_t \colon V_t \otimes \mathcal{O}_{C \times t} \to E_t,$$
*induces an injective map $H^0(\xi_t) \colon V_t \to H^0(E_t)$.*

*Two families $(\mathcal{E}, \mathcal{V}, \xi)$ and $(\mathcal{F}, \mathcal{U}, \eta)$ are isomorphic if and only if there exists an invertible sheaf $\mathcal{L}$ on $T$ such that $\mathcal{F} \simeq \mathcal{E} \otimes \pi^*\mathcal{L}$, $\mathcal{U} \simeq \mathcal{V} \otimes \mathcal{L}$ and $\eta = \xi \otimes \pi^*\mathcal{L}$.*

**Remark 2.2.** Let $(\mathcal{E}, \mathcal{V}, \xi)$ be a family of coherent systems parametrized by a connected scheme $T$. Note that the restriction $\mathcal{E}|_{(C_i \setminus \Delta_i) \times T}$ is flat over $T$ too, so we have that
$$\mathrm{rk}(E_t|_{C_i \setminus \Delta_i}) = r_i, \quad \forall t \in T.$$
This implies that all coherent systems of the family have the same multitype. Moreover, the set of $t \in T$ such that $E_t$ is locally free is an open subset of $T$. Finally, if $\mathcal{E}$ is locally free, then $\mathcal{E}|_{C_i \times T}$ is flat over $T$ so it gives a family of vector bundles on the curve $C_i$ of rank $r_i$ and degree $d_i$.

We recall the notion of $\underline{w}$-slope for depth one sheaves on $C$ and the definition of $\underline{w}$-stability.

[2]Note that the authors of [KN95] use the term *Brill-Noether pairs*.

**Definition 2.3.** *Let $E$ be a depth one sheaf on $C$. For any polarization $\underline{w}$ on $C$ we define the $\underline{w}$-slope of $E$ as:*

$$\mu_{\underline{w}}(E) = \frac{\chi(E)}{\mathrm{rk}_{\underline{w}}(E)} = \chi(\mathcal{O}_C) + \frac{\deg_{\underline{w}}(E)}{\mathrm{rk}_{\underline{w}}(E)}.$$

*$E$ is said to be $\underline{w}$-semistable (respectively $\underline{w}$-stable) if for any proper subsheaf $F$ of $E$ we have $\mu_{\underline{w}}(F) \leq \mu_{\underline{w}}(E)$ (respectively $<$).*

The notion of $(\underline{w}, \alpha)$-slope and $(\underline{w}, \alpha)$-stability for coherent systems is defined as follows.

**Definition 2.4.** *Let $(E, V)$ be a coherent system of type $(\mathrm{rk}_{\underline{w}}(E), \deg_{\underline{w}}(E), k)$ on the curve $C$. For any positive $\alpha \in \mathbb{R}$ and for any polarization $\underline{w}$ on the curve $C$, we define the $(\underline{w}, \alpha)$-slope of $(E, V)$:*

$$\mu_{\underline{w},\alpha}(E, V) = \frac{\deg_{\underline{w}}(E)}{\mathrm{rk}_{\underline{w}}(E)} + \alpha\frac{k}{\mathrm{rk}_{\underline{w}}(E)} = \frac{\chi(E)}{\mathrm{rk}_{\underline{w}}(E)} - \chi(\mathcal{O}_C) + \alpha\frac{k}{\mathrm{rk}_{\underline{w}}(E)}.$$

**Definition 2.5.** *A coherent system $(E, V)$ is said $(\underline{w}, \alpha)$-semistable (resp. stable) if for any proper coherent subsystem $(F, U)$ we have:*

$$\mu_{\underline{w},\alpha}(F, U) \leq \mu_{\underline{w},\alpha}(E, V) \quad (resp. <).$$

Fix $(r, d, k)$ with $r, d \in \mathbb{R}$, $r > 0$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{Q}$ positive. In [KN95] it is proved that there exists a projective scheme $\tilde{\mathcal{G}}_{\underline{w},\alpha}(r, d, k)$ which is a coarse moduli space for families of $(\underline{w}, \alpha)$-semistable coherent systems of type $(r, d, k)$ on the curve $C$. Moreover, let $\mathcal{G}_{\underline{w},\alpha}(r, d, k)$ denote the subscheme parametrizing $(\underline{w}, \alpha)$-stable coherent systems, it is an open subscheme of $\tilde{\mathcal{G}}_{\underline{w},\alpha}(r, d, k)$. As $C$ is a reducible curve, these spaces are reducible too, different components correspond to possible multiranks $(r_1, .., r_\gamma)$, see Remark 2.2. We are interested in those components containing coherent systems arising from locally free sheaves of rank $r$. With this aim, we set $\underline{r} = (r, \dots, r)$ and let $\mathcal{G}_{\underline{w},\alpha}(\underline{r}, d, k)$ denote the subscheme of $\mathcal{G}_{\underline{w},\alpha}(r, d, k)$ parametrizing families of $(\underline{w}, \alpha)$-stable coherent systems of multitype $(\underline{r}, d, k)$. We will denote by

$$\mathcal{G}'_{\underline{w},\alpha}(\underline{r}, d, k) \subset \mathcal{G}_{\underline{w},\alpha}(\underline{r}, d, k)$$

the open subset corresponding to $(\underline{w}, \alpha)$-stable coherent systems $(E, V)$ with $E$ locally free.

We have the following fundamental result:

**Theorem 2.6.** *Let $(E, V)$ be a coherent system which is $(\underline{w}, \alpha)$-stable and let $\Lambda \in \mathcal{G}_{\underline{w},\alpha}(r, d, k)$ be the corresponding point.*

1. *The Zariski tangent space of $\mathcal{G}_{\underline{w},\alpha}(r, d, k)$ at the point $\Lambda$ is isomorphic to $\mathrm{Ext}^1(\Lambda, \Lambda)$;*
2. *if $\mathrm{Ext}^2(\Lambda, \Lambda) = 0$, then $\mathcal{G}_{\underline{w},\alpha}(r, d, k)$ is smooth of dimension $\dim \mathrm{Ext}^1(\Lambda, \Lambda)$ at the point $\Lambda$;*
3. *for every irreducible component $S$ of $\mathcal{G}_{\underline{w},\alpha}(r, d, k)$ through $\Lambda$ we have:*

$$\dim \mathrm{Ext}^1(\Lambda, \Lambda) - \dim \mathrm{Ext}^2(\Lambda, \Lambda) \leq \dim S \leq \dim \mathrm{Ext}^1(\Lambda, \Lambda).$$

This theorem has been proved in the case of smooth curves in [He98]. Actually, the machinery introduced by the author in order to prove the result also works for arbitrary reduced nodal curves. This has been also noted in [Bho09] (where the author is interested in the irreducible case).

When $r, k \in \mathbb{N}$ and $d \in \mathbb{Z}$, as in the smooth case, we can define the *Brill-Noether number* :

$$\beta_C(r,d,k) = r^2(p_a(C)-1) + 1 - k(k-d+r(p_a(C)-1)). \tag{2.1}$$

If $\Lambda \in \mathcal{G}_{\underline{w},\alpha}(r,d,k)$ corresponds to a coherent system $(E,V)$ with $E$ locally free, then we can define the *Petri map* $\mu_{E,V}$ of $(E,V)$ as follows:

$$\mu_{E,V} \colon V \otimes H^0(\omega_C \otimes E^*) \to H^0(\omega_C \otimes E \otimes E^*) \tag{2.2}$$

which is given by multiplication of global sections. For coherent systems $(E,V)$ with $E$ locally free, we have the following result:

**Proposition 2.7.** *Let $\Lambda \in \mathcal{G}_{\underline{w},\alpha}(r,d,k)$ corresponding to a coherent system $(E,V)$ with $E$ locally free. Then, if the Petri map of $(E,V)$ is injective, $\Lambda$ is a smooth point of $\mathcal{G}_{\underline{w},\alpha}(r,d,k)$ and the dimension of $\mathcal{G}_{\underline{w},\alpha}(r,d,k)$ at $\Lambda$ is given by the Brill-Noether number.*

This result has been proved for smooth curves in [BGPMN03](Prop. 3.10) and has been generalized to a nodal irreducible curve in [Bho09](Prop. 3.7). Actually, as previously noted, the arguments involved in the proof of this proposition still works for a reducible nodal curve too.

## 3. Generated coherent systems on nodal curves

Let $C$ be a reducible nodal curve as in Section 1 with $\gamma$ components.

**Definition 3.1.** *A coherent system $(E,V)$ on $C$ of type $(r,d,k)$ is said to be generated if the evaluation map of global sections*

$$ev_V \colon V \otimes O_C \to E$$

*is surjective. It is said to be generically generated if either it is generated or* $\operatorname{coker} ev_V$ *is a sheaf whose support is 0-dimensional.*

Assume that $(E,V)$ is a coherent system on $C$. For any connected subcurve $B$ of $C$ we can define the restriction of $(E,V)$ to $B$ as follows. From the exact sequence

$$0 \to \mathcal{O}_{C-B}(-\Delta_B) \to \mathcal{O}_C \to \mathcal{O}_B \to 0$$

by tensoring with $E$, we have a surjective map $E \to E \otimes \mathcal{O}_B$ which is actually the restriction map. Then, if we set $E_B = E \otimes \mathcal{O}_B/torsion$, we have a surjective map $E \to E_B$ which induces the following map of global sections:

$$\rho_B \colon H^0(E) \to H^0(E_B).$$

We define $V_B$ as the image of $V$ by the map $\rho_B$. Then $(E_B, V_B)$ is a coherent system on the subcurve $B$. Notice that when $E$ is a vector bundle then $E_B = E \otimes \mathcal{O}_B$ and it is a vector bundle too on $B$.

**Definition 3.2.** *We will call $(E_B, V_B)$ the restriction of $(E,V)$ to the subcurve $B$. When $B = C_i$, to simplify notations, we will denote it by $(E_i, V_i)$.*

**Lemma 3.3.** *Let $(E,V)$ be a generated (respectively generically generated) coherent system on $C$. If $B$ is a connected subcurve of $C$, then $(E_B, V_B)$ is generated (respectively generically generated) too.*

*Proof.* Consider the evaluation map $ev_V : V \otimes \mathcal{O}_C \to E$. As both maps $V \to V_B$ and $\mathcal{O}_C \to \mathcal{O}_B$ are restriction maps we have a commutative diagram

$$\begin{array}{ccccc} V \otimes \mathcal{O}_C & \xrightarrow{ev_V} & E & \twoheadrightarrow & \operatorname{coker} ev_V \\ \downarrow & & \downarrow & & \downarrow \\ V_B \otimes \mathcal{O}_B & \xrightarrow{ev_{V_B}} & E_B & \twoheadrightarrow & \operatorname{coker} ev_{V_B} \end{array}$$

where vertical maps are surjective. If $(E,V)$ is generated, then $ev_V$ is surjective and $ev_{V_B}$ is surjective, so $(E_B, V_B)$ is generated. If $(E,V)$ is generically generated (but not generated), by the above diagram, $\dim \operatorname{Supp}(\operatorname{coker} e_V) = 0$ implies $\dim \operatorname{Supp}(\operatorname{coker} e_{V_B}) = 0$. □

From now on, we will restrict our attention to coherent systems on $C$ of multitype $(\underline{r}, d, k)$, where we set $\underline{r} = (r, \cdots, r)$.

**Remark 3.4.** Let $(E,V)$ be a coherent system of multitype $(\underline{r}, d, k)$. If it is generically generated, then by Lemma 3.3, $(E_i, V_i)$ is a generically generated coherent system on the curve $C_i$ with $d_i \geq 0$ and $\dim V_i \geq r$. Hence, by Lemma 1.3, we also have $d \geq 0$ and $k \geq r$, as $\dim V \geq \dim V_i$.

The following property is a generalization of [BGPMN03](Prop. 4.4) and of [Bho09](Cor. 3.15).

**Proposition 3.5.** *Let $C$ be a nodal curve as in Section 1 and let $\underline{w}$ be a polarization on it. Fix integers $r \geq 1$, $d \geq 0$ and $k \geq r$. There exists $\alpha_g \in \mathbb{R}$ such that, for any $\alpha \geq \alpha_g$, any coherent system $(E,V)$ of type $(\underline{r}, d, k)$, which is $(\underline{w}, \alpha)$-semistable, is generically generated and the kernel of the evaluation map $ev_V$ has no global sections.*

*Proof.* Let $(E,V)$ be a coherent system which is $(\underline{w}, \alpha)$-semistable. Assume that $(E,V)$ is not generically generated. We denote by $G$ the image of the evaluation map $ev_V$. Then $G$ is a depth one subsheaf of $E$, with $\operatorname{rk}(G_i) = r_i \leq r$ for any $i$, which fits into the exact sequence

$$0 \to G \to E \to \operatorname{coker} ev_V \to 0$$

and $\dim \operatorname{Supp}(\operatorname{coker} ev_V) = 1$. There exists a component $C_j$ such that $C_j \subseteq \operatorname{Supp}(\operatorname{coker} ev_V)$, so we have $\dim \operatorname{Supp}(\operatorname{coker} ev_{V_j}) = 1$ too and $r_j < r$. This implies $\operatorname{rk}_{\underline{w}}(G) < \operatorname{rk}_{\underline{w}}(E) = r$. Consider the coherent system $(G,V)$, it is a proper subsystem of $(E,V)$ and it is also generated by construction. From the $(\underline{w}, \alpha)$-semistability of $(E,V)$ we have

$$\mu_{(\underline{w},\alpha)}(G,V) \leq \mu_{(\underline{w},\alpha)}(E,V),$$

equivalently

$$\frac{\deg_{\underline{w}}(G)}{\mathrm{rk}_{\underline{w}}(G)} + \alpha\frac{k}{\mathrm{rk}_{\underline{w}}(G)} \le \frac{d}{r} + \alpha\frac{k}{r} \qquad \text{i.e.} \qquad \alpha k\left(\frac{1}{\mathrm{rk}_{\underline{w}}(G)} - \frac{1}{r}\right) \le \frac{d}{r} - \frac{\deg_{\underline{w}}(G)}{\mathrm{rk}_{\underline{w}}(G)}.$$

Note that the coefficient of $\alpha$ is positive, since $\mathrm{rk}_{\underline{w}}(G) < r$, so we can write

$$\alpha \le \frac{d\,\mathrm{rk}_{\underline{w}}(G) - r\deg_{\underline{w}}(G)}{k(r - \mathrm{rk}_{\underline{w}}(G))}.$$

We recall that for any $i$ we have, see Section 1, that $w_i = \frac{a_i}{\sum_{m=1}^{\gamma} a_m}$, with $a_m \ge 1$. Note that, since $r_j < r$, we have

$$r - \mathrm{rk}_{\underline{w}}(G) = \sum_{i=1}^{\gamma} w_i(r - r_i) \ge w_j(r - r_j) \ge w_j = \frac{a_j}{\sum_{m=1}^{\gamma} a_m},$$

so

$$\alpha \le \frac{\sum_{m=1}^{\gamma} a_m}{k a_j}(d\,\mathrm{rk}_{\underline{w}}(G) - r\deg_{\underline{w}}(G)).$$

By Lemma 1.3, we have

$$\deg_{\underline{w}}(G) = \chi(G) - \mathrm{rk}_{\underline{w}}(G)\chi(\mathcal{O}_C) \ge \sum_{i=1}^{\gamma}\chi(G_i) - r_M\delta - (1 - p_a)\sum_{i=1}^{\gamma} w_i r_i$$

and we obtain:

$$\alpha \le \frac{\sum_{m=1}^{\gamma} a_m}{k a_j}\left(d\sum_{i=1}^{\gamma} w_i r_i - r\sum_{i=1}^{\gamma}\deg(G_i) + r\sum_{i=1}^{\gamma} r_i(g_i - 1) + r r_M\delta - r(p_a - 1)\sum_{i=1}^{\gamma} w_i r_i\right).$$

As $(G_i, V_i)$ is generated $\deg(G_i) \ge 0$. Since $r_i \le r$ and $r_j < r$ we have: $r_M \le r$ and $\sum_{i=1}^{\gamma} w_i r_i < r$. Finally, $a_j \ge 1$ by construction, so the above inequality become:

$$\alpha < \frac{\sum_{m=1}^{\gamma} a_m}{k}(dr + r^2(p_a(C) - 1)) = \alpha_g.$$

Hence the first claim is proved.

Let $(E, V)$ be generically generated. Then we have the exact sequence

$$0 \to \mathrm{Ker}\, ev_V \to V \otimes O_C \to E \to \mathrm{coker}\, ev_V \to 0,$$

where $\mathrm{coker}\, ev_V$ has 0-dimensional support. Since $H^0(ev_V)\colon V \to H^0(E)$ is injective, then we have $H^0(\mathrm{Ker}\, ev_V) = 0$. □

Note that in the proof of the above Theorem we have defined

$$\alpha_g = \frac{\sum_{m=1}^{\gamma} a_m}{k}(dr + r^2(p_a(C) - 1)). \tag{3.1}$$

This number depends only on the arithmetic genus of $C$, the polarization $\underline{w}$ and the multitype $(\underline{r}, d, k)$.

The following property generalizes [BGPMN03](Prop. 4.5 (i)).

**Proposition 3.6.** *Let $C$ be a nodal curve as in Section 1 and let $\underline{w}$ be a polarization on it. Let $(E,V)$ be a generically generated coherent system of multitype $(\underline{r},d,k)$. If there exists a proper subsystem $(F,U)$ of $(E,V)$ such that*

$$\frac{\dim U}{\mathrm{rk}_{\underline{w}}(F)} > \frac{\dim V}{\mathrm{rk}_{\underline{w}}(E)}, \tag{3.2}$$

*then $(E,V)$ is not $(\underline{w},\alpha)$-semistable for $\alpha > k\alpha_g$.*

*Proof.* Let $(E,V)$ be a generically generated coherent system which is $(\underline{w},\alpha)$-semistable. Assume that there exists a proper subsystem $(F,U)$ of $(E,V)$ satisfying (3.2). Since $U \subseteq V$, then we have $\mathrm{rk}_{\underline{w}}(F) < \mathrm{rk}_{\underline{w}}(E) = r$. We can assume that $(F,U)$ is generated, otherwise we can consider the subsystem $(\mathrm{Im}\, ev_U, U)$ of $(E,V)$ which satisfies (3.2) too.

Let $h = \dim U$. Since $(E,V)$ is $(\underline{w},\alpha)$-semistable we have

$$\frac{\deg_{\underline{w}}(F)}{\mathrm{rk}_{\underline{w}}(F)} + \alpha\frac{h}{\mathrm{rk}_{\underline{w}}(F)} \leq \frac{d}{r} + \alpha\frac{k}{r}.$$

This is equivalent to:

$$\alpha\left(\frac{h}{\mathrm{rk}_{\underline{w}}(F)} - \frac{k}{r}\right) \leq \frac{d}{r} - \frac{\deg_{\underline{w}}(F)}{\mathrm{rk}_{\underline{w}}(F)}.$$

Then by 3.2 we get

$$\alpha \leq \frac{d\,\mathrm{rk}_{\underline{w}}(F) - r\deg_{\underline{w}}(F)}{hr - k\,\mathrm{rk}_{\underline{w}}(F)}.$$

Note that $k\,\mathrm{rk}_{\underline{w}}(F)$ is a rational number, we denote by $\lfloor k\,\mathrm{rk}_{\underline{w}}(F)\rfloor$ its integral part and by $\{k\,\mathrm{rk}_{\underline{w}}(F)\}$ its fractional part. By (3.2) we get $hr \geq \lfloor k\,\mathrm{rk}_{\underline{w}}(F)\rfloor + 1$, which implies that:

$$hr - k\,\mathrm{rk}_{\underline{w}}(F) \geq 1 - \{k\,\mathrm{rk}_{\underline{w}}(F)\}.$$

We recall that for any $i$ we have, see Section (1), that $w_i = \frac{a_i}{\sum_{m=1}^{\gamma} a_m}$, with $a_m \geq 1$. So we have $\{k\,\mathrm{rk}_{\underline{w}}(F)\} = \frac{b}{\sum_{m=1}^{\gamma} a_m}$ with $0 \leq b \leq \sum_{m=1}^{\gamma} a_m - 1$. This allows us to prove the bound

$$hr - k\,\mathrm{rk}_{\underline{w}}(F) \geq \frac{1}{\sum_{m=1}^{\gamma} a_m}.$$

Hence we obtain

$$\alpha \leq \left(\sum_{m=1}^{\gamma} a_m\right)(d\,\mathrm{rk}_{\underline{w}}(F) - r\deg_{\underline{w}}(F)).$$

Since $(F,U)$ is generated we can proceed as in the proof of Proposition 3.5 in order to obtain

$$\alpha \leq \left(\sum_{m=1}^{\gamma} a_m\right)(dr + r^2(p_a(C)-1)) = k\alpha_g.$$

□

As in the case of smooth curves and of irreducible nodal curves, this gives us a necessary condition for $(\underline{w},\alpha)$-semistability.

**Definition 3.7.** *Let $(E,V)$ be a coherent system of multitype $(\underline{r},d,k)$. We say that $(E,V)$ satisfies property $(\star)$ (property $(\star')$ respectively) if for any proper coherent subsystem $(F,U)$ of $(E,V)$ we have either $(\star_1)$ or $(\star_2)$ ($(\star_1)$ or $(\star'_2)$ respectively) where*

($\star_1$): $\frac{\dim U}{\mathrm{rk}_{\underline{w}}(F)} < \frac{\dim V}{\mathrm{rk}(E)}$

($\star_2$): $\frac{\dim U}{\mathrm{rk}_{\underline{w}}(F)} = \frac{\dim V}{\mathrm{rk}(E)}$ *and* $\frac{\deg_{\underline{w}}(F)}{\mathrm{rk}_{\underline{w}}(F)} < \frac{\deg_{\underline{w}}(E)}{\mathrm{rk}(E)}$

($\star_2'$): $\frac{\dim U}{\mathrm{rk}_{\underline{w}}(F)} = \frac{\dim V}{\mathrm{rk}(E)}$ *and* $\frac{\deg_{\underline{w}}(F)}{\mathrm{rk}_{\underline{w}}(F)} \leq \frac{\deg_{\underline{w}}(E)}{\mathrm{rk}(E)}$

**Remark 3.8.** Under the hypothesis of Proposition 3.6, we have the following properties:

(1) if $(E,V)$ is a coherent system which is $(\underline{w},\alpha)$-stable ($(\underline{w},\alpha)$-semistable respectively) for any $\alpha > k\alpha_g$, then $(E,V)$ satisfies property $(\star)$ ($(\star')$ respectively).

(2) if $(E,V)$ is a coherent system which satisfies property $(\star)$ (property $(\star')$ respectively) and $E$ is $\underline{w}$-stable ($\underline{w}$-semistable respectively), then $(E,V)$ is $(\underline{w},\alpha)$-stable ( $(\underline{w},\alpha)$-semistable respectively) for any $\alpha > 0$.

The following Theorem generalizes the results of [BGPMN03](Prop. 4.5) and [Bho09](Prop. 3.16).

**Theorem 3.9.** *Let $C$ be a reducible nodal curve as in Section 1 and let $\underline{w}$ be a polarization on it. Let $(E,V)$ be a coherent system of multitype $(\underline{r},d,k)$ as above. If $(E,V)$ is generically generated and satisfies property $(\star)$ ($(\star')$ respectively), then $\forall\alpha > k\alpha_g$, $(E,V)$ is $(\underline{w},\alpha)$-stable ($(\underline{w},\alpha)$-semistable respectively).*

*Proof.* Let $(E,V)$ be a generically generated coherent system of multitype $(\underline{r},d,k)$ satisfying property $(\star)$. Assume that there exists a proper coherent subsystem $(F,U)$ destabilizing $(E,V)$. Then $F \subseteq E$ is a subsheaf of depth one, $U \subseteq V$ with $\dim U = h \leq k$. Since $(E,V)$ satisfies property $(\star)$, we have

$$\frac{h}{\mathrm{rk}_{\underline{w}}(F)} \leq \frac{k}{r}.$$

If equality holds, then we are in case $(\star_2)$ so $\frac{\deg_{\underline{w}}(F)}{\mathrm{rk}_{\underline{w}}(F)} < \frac{d}{r}$. This implies

$$\frac{\deg_{\underline{w}}(F)}{\mathrm{rk}_{\underline{w}}(F)} + \alpha\frac{h}{\mathrm{rk}_{\underline{w}}(F)} < \frac{d}{r} + \alpha\frac{k}{r},$$

for any $\alpha > 0$ and thus we have a contradiction.

So we are in case $(\star_1)$. The inequality

$$\mu_{(\underline{w},\alpha)}(F,U) = \frac{\deg_{\underline{w}}(F)}{\mathrm{rk}_{\underline{w}}(F)} + \alpha\frac{h}{\mathrm{rk}_{\underline{w}}(F)} \geq \frac{d}{r} + \alpha\frac{k}{r} = \mu_{(\underline{w},\alpha)}(E,V), \tag{3.3}$$

is equivalent to

$$\alpha\left(\frac{k}{r} - \frac{h}{\mathrm{rk}_{\underline{w}}(F)}\right) \leq \frac{\deg_{\underline{w}}(F)}{\mathrm{rk}_{\underline{w}}(F)} - \frac{d}{r},$$

which gives

$$\alpha \leq \frac{r\deg_{\underline{w}}(F) - \mathrm{rk}_{\underline{w}}(F)d}{k\,\mathrm{rk}_{\underline{w}}(F) - hr}. \tag{3.4}$$

As in the proof of Proposition 3.6, we denote by $\lfloor k\,\mathrm{rk}_{\underline{w}}(F)\rfloor$ the integral part of $k\,\mathrm{rk}_{\underline{w}}(F)$ and by $\{k\,\mathrm{rk}_{\underline{w}}(F)\}$ its fractional part. Since $\frac{h}{\mathrm{rk}_{\underline{w}}(F)} < \frac{k}{r}$, we get

$$hr \leq \begin{cases} \lfloor k\,\mathrm{rk}_{\underline{w}}(F)\rfloor - 1 & \text{if } \{k\,\mathrm{rk}_{\underline{w}}(F)\} = 0 \\ \lfloor k\,\mathrm{rk}_{\underline{w}}(F)\rfloor & \text{if } \{k\,\mathrm{rk}_{\underline{w}}(F)\} \neq 0 \end{cases} \implies -hr \geq -\lfloor k\,\mathrm{rk}_{\underline{w}}(F)\rfloor.$$

This implies that $k\,\mathrm{rk}_{\underline{w}}(F) - hr \geq \{k\,\mathrm{rk}_{\underline{w}}(F)\} = \frac{b}{\sum_{m=1}^{\gamma} a_m}$, with $1 \leq b \leq \sum_{m=1}^{\gamma} a_m - 1$. Hence we obtain

$$k\,\mathrm{rk}_{\underline{w}}(F) - hr \geq \frac{1}{\sum_{m=1}^{\gamma} a_m},$$

and Inequality (3.4) becomes:

$$\alpha \leq \left(\sum_{m=1}^{\gamma} a_m\right)(r\deg_{\underline{w}}(F) - d\,\mathrm{rk}_{\underline{w}}(F)) \leq \left(\sum_{m=1}^{\gamma} a_m\right)(r\deg_{\underline{w}}(F)),$$

as $\mathrm{rk}_{\underline{w}}(F) > 0$ and $d \geq 0$, since $(E,V)$ is generically generated, see Remark 3.4. By definition, we have: $\deg_{\underline{w}}(F) = \chi(F) - \mathrm{rk}_{\underline{w}}(F)\chi(O_C)$, and by Lemma 1.3 (1) we have:

$$\chi(F) \leq \sum_{i=1}^{\gamma} \chi(F_i) = \sum_{i}^{\gamma} (\deg(F_i) + r_i(1-g_i)).$$

Since $F_i$ is a subsheaf of $E_i$, which is generically generated, the quotient $E_i/F_i$ is generically generated too and so has non negative degree. This implies that $\deg(F_i) \leq \deg(E_i)$. By Lemma 1.3 (2), $\sum_{i=1}^{\gamma} \deg(E_i) \leq \deg_{\underline{w}}(E) = d$, so that

$$\chi(F) \leq d - \sum_{i=1}^{\gamma} r_i(g_i - 1) \leq d,$$

since $r_i \geq 0$ and $g_i \geq 2$. Finally, we obtain:

$$\alpha \leq \left(\sum_{m=1}^{\gamma} a_m\right)(rd + r\,\mathrm{rk}_{\underline{w}}(F)(p_a(C) - 1)) \leq \left(\sum_{m=1}^{\gamma} a_m\right)(rd + r^2(p_a(C) - 1)) = k\alpha_g.$$

□

**Corollary 3.10.** *Let $C$ and $\underline{w}$ be as in Theorem 3.9. For any $r \geq 1$, $d \geq 0$, $k \geq r$ integers, the moduli spaces $\mathcal{G}_{(\underline{w},\alpha)}(\underline{r}, d, k)$ coincide for any $\alpha > k\alpha_g$.*

This completes the proof of Theorem $A$ as stated in the introduction.

**Corollary 3.11.** *Under the hypothesis of Theorem 3.9, if $(E,V)$ satisfies property $(\star)$, then each restriction $(E_i, V_i)$ satisfies the condition $\dim V_i \geq w_i \dim V$.*

*Proof.* Consider the subsheaf $G \subset E$ which is the kernel of the restriction map $\rho_i \colon E \to E_i$:

$$0 \to G \to E \to E_i \to 0.$$

It is a depth one sheaf and $\mathrm{rk}_{\underline{w}}(G) = \mathrm{rk}_{\underline{w}}(E) - \mathrm{rk}_{\underline{w}}(E_i) = r(1 - w_i)$. Let $U \subset V$ be the kernel of the restriction map $\rho_i : V \to V_i$. Then $\dim U = \dim V - \dim V_i \geq 0$ and $U \subseteq H^0(G)$. Hence $(G, U)$ is a proper coherent subsystem of $(E, V)$. Since it satisfies property $(\star)$ we have

$$\frac{\dim U}{\mathrm{rk}_{\underline{w}}(G)} \leq \frac{\dim V}{\mathrm{rk}_{\underline{w}}(E)},$$

which is equivalent to

$$r(\dim V - \dim V_i) \leq r(1 - w_i) \dim V,$$

which gives us $\dim V_i \geq w_i \dim V$. □

Finally we have the following sufficient condition for $(\underline{w}, \alpha)$-stability.

**Theorem 3.12.** *Let $C$ be a reducible nodal curve as in Section 1 and let $\underline{w}$ be a polarization on it. Let $(E, V)$ be a coherent system of multitype $(\underline{r}, d, k)$ and denote by $(E_i, V_i)$ its restriction to $C_i$. Assume that:*

*(1) $(E, V)$ is generically generated;*
*(2) for all $i$, $(E_i, V_i)$ is a coherent system of type $(r, d_i, k)$ on $C_i$;*
*(3) for all $i$, $(E_i, V_i)$ is $\alpha$-stable for any $\alpha > d_i(r - 1)$;*
*(4) either $r$ and $k$ are coprime or $E$ is $\underline{w}$-stable.*

*Then $(E, V)$ is $(\underline{w}, \alpha)$-stable $\forall \alpha > k\alpha_g$.*

*Proof.* By Theorem 3.9 it is enough to prove that $(E, V)$ satisfies property $(\star)$. Let $(F, U)$ be a proper coherent subsystem of $(E, V)$. First of all we have to prove that:

$$\frac{\dim U}{\mathrm{rk}_{\underline{w}}(F)} \leq \frac{\dim V}{\mathrm{rk}_{\underline{w}}(E)}. \tag{3.5}$$

We recall that $F \subseteq E$ is a subsheaf of depth one and $U \subseteq V$ with $\dim U = h \leq k$. If $h = 0$ then (3.5) is satisfied. So we can assume $h \geq 1$. For any $i$ we consider the restriction $(F_i, U_i)$, it is a coherent subsystem of $(E_i, V_i)$. In particular, since by assumption (2), the restriction map $\rho_i|_V : V \to V_i$ is an isomorphism, then $\rho_i|_U : U \to U_i$ is an isomorphism too. This implies $\dim U_i = \dim U = h$. Let $r_i = \mathrm{rk}(F_i)$, as $\dim U_i \geq 1$, then $r_i \geq 1$. Since $(E_i, V_i)$ is $\alpha$-stable for $\alpha > d_i(r - 1)$, then, by [BGPMN03](Prop. 4.5), it satisfies property $(\star)$, in particular:

$$\frac{h}{r_i} \leq \frac{k}{r}, \tag{3.6}$$

equivalentely $hr \leq kr_i$. From the above inequality we deduce the following:

$$hr = \sum_{i=1}^{\gamma} w_i hr \leq \sum_{i=1}^{\gamma} w_i k r_i = k\, \mathrm{rk}_{\underline{w}}(F),$$

which is equivalent to (3.5).

Finally, if $r$ and $k$ are coprime, then both the inequalities (3.5) and (3.6) are strict, so $(E, V)$ satisfies property $(\star_1)$ (and $(\star_2)$ cannot occur). If $r$ and $k$ are not coprime, by hypotesis, we have that $E$ is $\underline{w}$-stable so

$$\frac{\deg_{\underline{w}}(F)}{\mathrm{rk}_{\underline{w}}(F)} < \frac{\deg_{\underline{w}}(E)}{\mathrm{rk}_{\underline{w}}(E)}.$$

This, with the inequality (3.5) guarantees that $(E, V)$ satisfies $(\star)$ as claimed. $\square$

## 4. Construction of coherent systems of type $(r, d, r+1)$

Let $C$ be a nodal curve as in Section 1, with $\gamma$ irreducible components. Let $L \in \mathrm{Pic}^{(d_1,\cdots,d_\gamma)}(C)$ be a globally generated line bundle on $C$. From Lemma 1.3 we have that

$$\chi(L) = \sum_{i=1}^{\gamma} \chi(L_i) - \gamma + 1 \quad deg_{\underline{w}}(L) = d = \sum_{i=1}^{\gamma} d_i.$$

Let $r \geq 1$ and consider a subspace $W \subseteq H^0(L)$ of dimension $r+1$ such that the evaluation map

$$ev_W : W \otimes \mathcal{O}_C \to L$$

is surjective. Then $(L, W)$ is a generated coherent system of multitype $(\underline{1}, d, r+1)$ on the curve $C$. Let $(L_i, W_i)$ be the restriction of $(L, W)$ to the component $C_i$. By Lemma 3.3, $(L_i, W_i)$ is a generated coherent system on $C_i$ of type $(1, d_i, k_i)$. This implies $d_i \geq 0$, $\forall i = 1, \cdots, \gamma$. For any $i$, we define $R_i$ as follows

$$R_i = H^0(L|_{C_i^c}(-\Delta_i)) \cap W, \tag{4.1}$$

where $\mathcal{O}_{C_i{}^c}(-\Delta_i)$ is defined in the exact sequence (1.2).

**Proposition 4.1.** *Let $C$ be a nodal curve as in Section 1, with $\gamma$ irreducible components and $\underline{w}$ a polarization on it. Let $L \in \mathrm{Pic}^{(d_1,\cdots,d_\gamma)}(C)$ be a line bundle on the curve $C$. Let $(L, W)$ be a generated coherent system of multitype $(\underline{1}, d, r+1)$ and assume that $R_i = 0$ for any $i = 1, \cdots, \gamma$. Then we have:*

*(1) $(L_i, W_i)$ is a generated coherent system on $C_i$ of type $(1, d_i, r+1)$;*
*(2) $(L, W)$ is $(\underline{w}, \alpha)$-stable for any $\alpha > (r+1)\alpha_g$.*

*Proof.* (1) From the exact sequence (1.2), by tensoring with $L$ and passing to global sections we have:

$$0 \to H^0(L|_{C_i^c}(-\Delta_i)) \to H^0(L) \xrightarrow{\rho_i} H^0(L_i) \to \cdots.$$

When we restrict $\rho_i$ to $W$ we obtain

$$0 \to R_i \to W \to W_i \to 0. \tag{4.2}$$

Then $W \simeq W_i$ if and only if $R_i = \{0\}$.

(2) The assertion follows from Theorem 3.12, since $(L_i, W_i)$ is $\alpha$-stable for any $\alpha > 0$. $\square$

Let $L \in \mathrm{Pic}^{(d_1,\cdots,d_\gamma)}(C)$ be a line bundle with $d_i \geq 1$. Consider a generated coherent system $(L, W)$ of multitype $(\underline{1}, d, r+1)$. We can associate to it a coherent system $(E, V)$ of multitype $(\underline{r}, d, r+1)$ on $C$, with $E$ locally free. As $ev_W : W \otimes \mathcal{O}_C \to L$ is surjective it defines an exact sequence of vector bundles on $C$:

$$0 \to \mathrm{Ker}\, ev_W \to W \otimes \mathcal{O}_C \to L \to 0, \tag{4.3}$$

we will denote it *the exact sequence defined by* $(L, W)$. Its dual gives the exact sequence

$$0 \to L^{-1} \to W^* \otimes \mathcal{O}_C \to E \to 0, \tag{4.4}$$

where we set $E = (\operatorname{Ker} ev_W)^*$. Note that by construction $E$ is a vector bundle on $C$ of rank $r$ with determinant $\det(E) = L$. By taking the induced exact sequence on cohomology, as $d_i \geq 1$, by Lemma 1.3 we have $h^0(L^{-1}) = 0$, so we have an injective map

$$0 \to W^* \to H^0(E) \to \cdots$$

We define $V$ as the image of $W^*$ in $H^0(E)$ by the above inclusion. As $V \simeq W^*$, we have $\dim V = r + 1$ . By the exactness of sequence (4.4), we also have that $V$ generates $E$, so $(E, V)$ is a generated coherent system of multitype $(\underline{r}, d, r + 1)$ on $C$.

**Definition 4.2.** *The coherent system* $(E, V)$ *is called the dual span of* $(L, W)$ *and we will denote it as* $D((L, W))$.

Vector bundles arising as kernel of the evaluation map of a generated coherent system $(E, V)$, on a smooth curve, are called *kernel bundles* or *Lazarsfeld bundles*. They were introduced by Butler and their stability has been deeply studied by many authors. For recent results on nodal curves with a node see [BF20].

**Remark 4.3.** Since $(L, W)$ is generated it defines a morphism $\Phi_{|W|} : C \to \mathbb{P}W^* = \mathbb{P}^r$. Consider the Euler sequence on $\mathbb{P}W^*$:

$$0 \to \mathcal{O}_{\mathbb{P}^r}(-1) \to W^* \otimes \mathcal{O}_{\mathbb{P}^r} \to T_{\mathbb{P}^r}(-1) \to 0.$$

By taking the pullback of this sequence with respect to $\Phi_{|W|}$ we obtain the exact sequence (4.4). Hence we have $E = \Phi^*_{|W|} T_{\mathbb{P}^r}(-1)$ and global sections of $V$ are the pullback of the global sections of $T_{\mathbb{P}^r}(-1)$ which are in $W^*$.

**Remark 4.4.** If $(E, V) = D((L, W)) = D((L', W'))$, then $(L, W) = (L', W')$. In fact, by the exact sequence defining $E$ we have that $L = \det(E) = L'$. We get the equality $W = W'$ by the dualizing exact sequence (4.4) and considering the cohomology sequence.

**Proposition 4.5.** *Let* $C$ *be a nodal curve as in Section 1, with* $\gamma$ *irreducible components. Let* $L \in \operatorname{Pic}^{(d_1, \cdots, d_\gamma)}(C)$ *be a line bundle on the curve* $C$*, with* $d_i \geq 1$. *Let* $(L, W)$ *be a generated coherent system of multitype* $(\underline{1}, d, r + 1)$. *Consider its dual span* $D((L, W)) = (E, V)$ *and its restriction* $(E_i, V_i)$ *to the component* $C_i$. *Then we have:*

1. *if* $R_i = 0$, *then* $(E_i, V_i) = D((L_i, W_i))$ *and it is* $\alpha$*-stable for any* $\alpha > (r - 1)d_i$;
2. *if* $R_i \neq 0$, *then* $(E_i, V_i)$ *is* $\alpha$*-unstable for any* $\alpha > 0$.

*Proof.* If we tensor the exact sequence (4.3) defined by $(L, W)$ with $\mathcal{O}_{C_i}$, we have again an exact sequence. Its relation with the exact sequence defined by the restriction $(L_i, W_i)$ is described in the following diagram.

(4.5)
$$\begin{array}{ccccccccc} & & R_i\otimes\mathcal{O}_{C_i} & = & R_i\otimes\mathcal{O}_{C_i} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & (\operatorname{Ker} ev_W)\otimes\mathcal{O}_{C_i} & \hookrightarrow & W\otimes\mathcal{O}_{C_i} & \xrightarrow{ev_W|_{C_i}} & L_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \rho_i} & & \| & & \\ 0 & \longrightarrow & \operatorname{Ker} ev_{W_i} & \hookrightarrow & W_i\otimes\mathcal{O}_{C_i} & \xrightarrow{ev_{W_i}} & L_i & \longrightarrow & 0 \end{array}$$

where the second column is simply the exact sequence (4.2) tensored again with $\mathcal{O}_{C_i}$. It is easy to see that the above diagram is indeed commutative. Finally, $\dim W_i \geq 2$ as $L_i$ is globally generated of degree $d_i \geq 1$ and $\operatorname{Ker} ev_{W_i}$ is a vector bundle on $C_i$.

If we dualize diagram (4.5) we can clearly see the relations between the dual span $(G_i, W_i^*)$ of $(L_i, W_i)$ and the restriction $(E_i, V_i)$ of the dual span $(E, V)$ of $(L, W)$. More precisely we have, as claimed

$$(G_i, W_i^*) = (E_i, V_i) \Longleftrightarrow R_i = 0.$$

Moreover, if $R_i \neq 0$, then $(G_i, W_i^*)$ is a non trivial coherent subsystem of $(E_i, V_i)$.

Now we prove the statements about $\alpha$-stability of $(E_i, V_i)$.

First of all, assume that $R_i = 0$. Then we have $W_i \simeq W$ and, as we have seen, the restriction $(E_i, V_i)$ of $(E, V)$ is indeed the dual span of $(L_i, W_i)$. By [BGPMN03](Cor. 5.10), it follows that $(E_i, V_i)$ is $\alpha$-stable for all $\alpha > d_i(r-1)$.

Assume now that $R_i \neq 0$. Then $(G_i, W_i^*)$ is a non trivial coherent subsystem of $(E_i, V_i)$ of type $(s_i, d_i, s_i+1)$, we prove that it is a destabilizing subsystem of $(E_i, V_i)$. First of all note that since $s_i < r$ we have $\frac{(s_i+1)}{s_i} > \frac{(r+1)}{r}$. On the other hand, as $d_i \geq 1$, we have also $\frac{d_i}{s_i} > \frac{d_i}{r}$ hence

$$\mu_{(\underline{w},\alpha)}(G_i, W_i^*) = \frac{d_i}{s_i} + \alpha\frac{s_i+1}{s_i} > \frac{d_i}{r} + \alpha\frac{r+1}{r} = \mu_{(\underline{w},\alpha)}(E_i, V_i).$$

Hence, for all $\alpha > 0$ we have that $(E_i, V_i)$ is $\alpha$-unstable as claimed. □

**Theorem 4.6.** *Let $C$ be a nodal curve as in Section 1, with $\gamma$ irreducible components and let $\underline{w}$ be a polarization on it. Let $L \in \operatorname{Pic}^{(d_1,\cdots,d_\gamma)}(C)$ be a line bundle on the curve $C$, with $d_i \geq 1$. Let $(L, W)$ be a generated coherent system of multitype $(\underline{1}, d, r+1)$ satisfying $R_i = 0$ for $i = 1\cdots\gamma$. Then the dual span $(E, V)$ of $(L, W)$ is $(\underline{w}, \alpha)$-stable for any $\alpha > (r+1)\alpha_g$.*

*Proof.* Since $R_i = 0$ for any $i = 1, \ldots \gamma$, by Proposition 4.1 we have that $(L_i, W_i)$ is a generated coherent system of type $(1, d_i, r+1)$. By Proposition 4.5, its dual span is $(E_i, V_i)$, it is a coherent system of type $(r, d_i, r+1)$ and it is $\alpha$-stable for $\alpha > (r-1)d_i$. As $r$ and $r+1$ are coprime, we can apply Theorem 3.12 and conclude the proof. □

## 5. Moduli spaces of coherent systems of type $(r, d, r+1)$

Let $C$ be a nodal curve as in Section 1 with $\gamma$ irreducible components. Let $L \in \operatorname{Pic}^{(d_1,\cdots,d_\gamma)}(C)$ be a globally generated line bundle on $C$. For any $r \geq 1$ consider the Grassmannian variety

$\mathrm{Gr}(r+1, H^0(L))$, parametrizing $(r+1)$-dimensional subspaces of $H^0(L)$. For any subspace $W \in \mathrm{Gr}(r+1, H^0(L))$, $(L, W)$ is a coherent system of multitype $(\underline{1}, d, r+1)$ on the curve $C$, with $d = \sum_{i=1}^{\gamma} d_i$.

**Proposition 5.1.** *Let $C$ be a nodal curve as in Section 1 and let $\underline{w}$ be a polarization on it. Let $r \geq 1$ and let $L \in \mathrm{Pic}^{(d_1,\cdots,d_\gamma)}(C)$ be a line bundle on $C$ with $d_i \geq \max(2g_i+1, g_i+r)$ for all $i$. Then, for a general $W \in \mathrm{Gr}(r+1, H^0(L))$ and for $\alpha > (r+1)\alpha_g$ we have that $(L, W) \in \mathcal{G}'_{\underline{w},\alpha}(\underline{1}, d, r+1)$ and its dual span $(E, V) \in \mathcal{G}'_{\underline{w},\alpha}(\underline{r}, d, r+1)$.*

*Proof.* By Lemma 1.4, since $d_i \geq 2g_i+1$, $h^1(L) = 0$ and $L$ is globally generated. Since $r \geq 1$ we have $\dim(W) \geq 2$. Hence, for $W$ general in $\mathrm{Gr}(r+1, H^0(L))$, $(L, W)$ is generated. By Lemma 1.4 $\rho_i \colon H^0(L) \to H^0(L_i)$ is surjective and its kernel has dimension

$$\dim \mathrm{Ker}(\rho_i) = h^0(L|_{C_i^c}(-\Delta_i)) = h^0(L) - h^0(L_i).$$

Since by assumption, $r+1 \leq d_i - g_i + 1 = h^0(L_i)$, then for a general $W$ we have that $R_i = W \cap \mathrm{Ker}(\rho_i) = \{0\}$. Hence we can conclude using Proposition 4.1 and Theorem 4.6. □

As a corollary we have the following:

**Theorem 5.2.** *Let $C$ be a nodal curve as in Section 1 with $\gamma$ components and let $\underline{w}$ be a polarization on it. For any integer $r \geq 1$ and for any integer $d \geq \max(2p_a(C)+\gamma, p_a(C)+r\gamma)$ we have $\mathcal{G}'_{\underline{w},\alpha}(\underline{r}, d, r+1) \neq \emptyset$ for any $\alpha > (r+1)\alpha_g$.*

*Proof.* It is enough to choose a line bundle $L \in \mathrm{Pic}^{(d_1,\cdots,d_\gamma)}(C)$ with $d_i \geq \max(2g_i+1, g_i+r)$ and $\sum_{i=1}^{\gamma} d_i = d$, then the assertion follows from Proposition 5.1. □

Assume that $\mathcal{G}'_{\underline{w},\alpha}(\underline{1}, d, r+1) \neq \emptyset$. For any $(d_1, \cdots, d_\gamma) \in \mathbb{Z}^\gamma$ with $\sum_{i=1}^{\gamma} d_i = d$, we define the following subscheme of $\mathcal{G}'_{\underline{w},\alpha}(\underline{1}, d, r+1)$

$$X_{d_1,\ldots,d_\gamma} = \{(L, W) \in \mathcal{G}'_{\underline{w},\alpha}(\underline{1}, d, r+1) \mid L_i \in \mathrm{Pic}^{d_i}(C_i)\}, \tag{5.1}$$

and consider its closure $\overline{X_{d_1,\ldots,d_\gamma}}$ in $\mathcal{G}_{\underline{w},\alpha}(\underline{1}, d, r+1)$. Then we have the following Theorem (which is Theorem $B$ of the Introduction).

**Theorem 5.3.** *Let $C$ be a nodal curve as in Section 1 with $\gamma$ components and let $\underline{w}$ be a polarization on it. Let $r \geq 1$ and $d_i \geq \max(2g_i+1, g_i+r)$. Then, for any $\alpha > (r+1)\alpha_g$, $\overline{X_{d_1,\ldots,d_\gamma}}$ is an irreducible component of $\mathcal{G}_{\underline{w},\alpha}(\underline{1}, d, r+1)$ of dimension equal to the Brill-Noether number*

$$\beta_C(1, d, r+1) = p_a(C) + (r+1)(d - r - p_a(C)).$$

*Moreover, each $(L, W) \in X_{d_1,\cdots,d_\gamma}$ is a smooth point of the moduli space.*

*Proof.* Since we assumed $d_i \geq \max(2g_i+1, g_i+r)$, then by Proposition 5.1 we have that $X_{d_1,\ldots,d_\gamma} \neq \emptyset$. Let $(L, W) \in X_{d_1,\ldots,d_\gamma}$. Then as we have seen before, we have $h^1(L) = 0$, so the Petri map

$$\mu_{L,W} \colon W \otimes H^0(\omega_C \otimes L^*) \to H^0(\omega_C \otimes L \otimes L^*)$$

is injective. By Theorem 2.7, any $(L,W) \in X_{d_1,\cdots,d_\gamma}$ is a smooth point of the moduli space $\mathcal{G}_{\underline{w},\alpha}(1,d,r+1)$ and the dimension of $\mathcal{G}_{\underline{w},\alpha}(1,d,r+1)$ at $(L,W)$ is given by the Brill-Noether number

$$\beta_C(1,d,r+1) = p_a(C) + (r+1)(d-r-p_a(C)).$$

In order to prove the assertion, we can consider the natural morphism

$$\pi \colon X_{d_1,\ldots,d_\gamma} \to \mathrm{Pic}^{d_1}(C_1) \times \cdots \times \mathrm{Pic}^{d_\gamma}(C_\gamma),$$

sending $(L,W) \to (L_1,\ldots,L_\gamma)$. We recall that, since $C$ is a curve of compact type, we have an isomorphism

$$\mathrm{Pic}^{d_1}(C_1) \times \cdots \times \mathrm{Pic}^{d_\gamma}(C_\gamma) \simeq \mathrm{Pic}^{(d_1,\cdots,d_\gamma)}(C).$$

By Proposition 5.1, $\pi$ is surjective and each fiber $\pi^{-1}(L_1,\cdots,L_\gamma)$ is an open subset of the Grassmannian variety $Gr(r+1,H^0(L))$, where $L \in \mathrm{Pic}^{(d_1,\cdots,d_\gamma)}(C)$ is the unique line bundle on the curve $C$ corresponding to $(L_1,\cdots,L_\gamma)$. Hence all the fibers of $\pi$ are irreducible and equidimensional of dimension $(r+1)(h^0(L)-r-1) = (r+1)(d-r-p_a(C))$.

We will denote by $Z_i$ the irreducible components of $X_{d_1,\cdots,d_\gamma}$. Since all fibers are irreducible and equidimensional, we have

$$Z_j = \bigcup \pi^{-1}(L_1,\cdots,L_\gamma), \qquad (L_1,\cdots,L_\gamma) \in \pi(Z_j)$$

so $\dim Z_j = \dim \pi(Z_j) + (r+1)(d-r-p_a(C))$.

Since $\pi$ is surjective, there is an irreducible component $Z_0$ of $X_{d_1,\cdots,d_\gamma}$ such that the restriction $\pi|_{Z_0}$ is dominant. Hence we have

$$\dim Z_0 = p_a(C) + (r+1)(d-r-p_a(C)) = \beta_C(1,d,r+1).$$

This implies that the closure $\overline{Z_0}$ is an irreducible component of $\mathcal{G}_{\underline{w},\alpha}(\underline{1},d,r+1)$.

In order to prove that $X_{d_1,\cdots,d_\gamma}$ is a component of the moduli space as claimed, it is enough to show that $Z_0 = X_{d_1,\cdots,d_\gamma}$. First of all we will prove that for any other possible irreducible component $Z_j$ of $X_{d_1,\ldots,d_\gamma}$ we have $\dim Z_j < \dim Z_0$. Indeed, otherwise, $\overline{Z_j}$ would be an irreducible component of the moduli space $\mathcal{G}_{\underline{w},\alpha}(\underline{1},d,r+1)$, moreover since $\pi(Z_j)$ would be an open subset of $\mathrm{Pic}^{d_1}(C_1) \times \cdots \times \mathrm{Pic}^{d_\gamma}(C_\gamma)$ we would have $Z_j \cap Z_0 \neq \emptyset$. This is impossible since all points of $Z_j$ and $Z_0$ are smooth points of the moduli space and thus they cannot be common to two irreducible components.

Assume that $Z_j$ is an irreducible component of $X_{d_1,\cdots,d_\gamma}$ different from $Z_0$. Let $Y_j$ be the unique irreducible component of $\mathcal{G}_{\underline{w},\alpha}(\underline{1},d,r+1)$ containing $Z_j$. By construction the dimension of $Y_j$ is equal to the Brill-Noether number (as it contains $Z_j$ whose points are smooth for the moduli space) and it is strictly bigger then the dimension of $Z_j$. By the same argument as above we have $Y_j \cap Z_0 = \emptyset$.

Consider the intersection $U_j = Y_j \cap \mathcal{G}'_{\underline{w},\alpha}(\underline{1},d,r+1)$. It is a non-empty open subset of $Y_j$ since it contains $Z_j$. Then the generic point of $U_j$ cannot lie on $X_{d_1,\ldots,d_\gamma}$ as $U_j$ is disjoint from $Z_0$. This implies that $U_j$ contains coherent systems with different multidegrees which is impossible as $Y_j$ is irreducible (see Remark 2.2). □

By the dual span construction we obtain the following result:

**Theorem 5.4.** *Let $C$ be a nodal curve as in Section 1 with $\gamma$ components and let $\underline{w}$ be a polarization on it. Let $r \geq 1$, $d_i \geq \max(2g_i+1, g_i+r)$ and set $d = \sum_{i=1}^{\gamma} d_i$. Then, for any $\alpha > (r+1)\alpha_g$, there exists an irreducible component $Y_{d_1,\cdots,d_\gamma}$ of $\mathcal{G}_{\underline{w},\alpha}(\underline{r}, d, r+1)$ which is birational to $X_{d_1,\dots,d_\gamma}$, of dimension equal to the Brill-Noether number*

$$\beta_C(r,d,r+1) = p_a(C) + (r+1)(d-r-p_a(C)).$$

*Moreover, a general $(E,V) \in Y_{d_1,\cdots,d_\gamma}$ satisfies the following properties:*

*(1) it is a generated coherent system with $E$ locally free and $\deg(E_i) = d_i$, $i = 1,\cdots,\gamma$;*
*(2) it is a smooth point of the moduli space $\mathcal{G}_{\underline{w},\alpha}(\underline{r}, d, r+1)$;*
*(3) for any $i = 1,\cdots,\gamma$ the restriction $(E_i, V_i)$ is a generated coherent system on $C_i$ of type $(r, d_i, r+1)$ which is $\alpha$-stable for any $\alpha > (r-1)d_i$.*

*Proof.* We consider the irreducible component $\overline{X_{d_1,\cdots,d_\gamma}}$ of $\mathcal{G}_{\underline{w},\alpha}(\underline{1}, d, r+1)$ described in Theorem 5.3. Let $(L,W) \in X_{d_1,\cdots,d_\gamma}$, assume that it is a generated coherent system, then we can define its dual span $D((L,W)) = (E,V)$, see Definition 4.2. It is a generated coherent system of multitype $(\underline{r}, d, r+1)$ and $E$ is locally free. If moreover, $R_i = 0$ for any $i = 1,\dots,\gamma$, then by Theorem 4.6, it follows that $(E,V) \in \mathcal{G}_{\underline{w},\alpha}(\underline{r}, d, r+1)$, for any $\alpha > (r+1)\alpha_g$. This allows us to define a map

$$\mathcal{D} : \overline{X_{d_1,\cdots,d_\gamma}} \dashrightarrow \mathcal{G}_{\underline{w},\alpha}(\underline{r}, d, r+1),$$

sending $(L,W) \to D((L,W))$. Actually, $\mathcal{D}$ is defined on the subset

$$\mathcal{U} = \{(L,W) \in X_{d_1,\cdots,d_\gamma} | \ (L,W) \text{ generated and } R_i = 0, \ i = 1,\dots,\gamma\},$$

which is a non-empty open subset by Proposition 5.1. We will prove that the restriction $\mathcal{D}_{|\mathcal{U}}$ is a birational morphism onto its image $\mathcal{D}(\mathcal{U})$. Let $(\mathcal{L}, \mathcal{W}, \xi)$ be a family of coherent systems of $\mathcal{U}$ parametrized by a a connected scheme $T$. Then, $\mathcal{L}$ is a locally free sheaf on $C \times T$ such that $\mathcal{L}_{|C_i \times t} \in \mathrm{Pic}^{d_i}(C_i)$. Let $\pi \colon C \times T \to T$ be the projection, $\mathcal{W}$ is a locally free sheaf on $T$ and $\xi \colon \pi^*\mathcal{W} \to \mathcal{L}$ is a map of locally free sheaves such that for any $t \in T$ the map

$$\xi_t : W_t \otimes \mathcal{O}_{C \times t} \to L_t$$

induces the following injective map $H^0(\xi_t) \colon W_t \to H^0(L_t)$. Since $(L_t, W_t)$ is generated, $\xi_t$ is surjective for any $t \in T$, then it follows that $\xi$ is a surjective map of locally free sheaves on $C \times T$. We can consider its kernel $\mathrm{Ker}\,\xi$, it is a locally free sheaf on $C \times T$. We have then an exact sequence of locally free sheaves on $C \times T$

$$0 \to \mathrm{Ker}(\xi) \xrightarrow{\eta} \pi^*\mathcal{W} \xrightarrow{\xi} \mathcal{L} \to 0.$$

and its dual

$$0 \to \mathcal{L}^* \xrightarrow{\xi^*} \pi^*\mathcal{W}^* \xrightarrow{\eta^*} \mathcal{E} \to 0,$$

where we have denoted by $\mathcal{E}$ the sheaf $\mathrm{Ker}(\xi)^*$. This implies that, for all $t \in T$ the map

$$\eta_t^* : W_t^* \otimes \mathcal{O}_{C\times t} \to E_t$$

is surjective and the map $H^0(\eta_t^*)\colon W_t^* \to H^0(E_t)$ is injective. This implies that $(\mathcal{E}, \pi^*\mathcal{W}^*, \eta^*)$ is a family of generated coherent systems of multitype $(\underline{r}, d, r+1)$, which are $(\underline{w}, \alpha)$-stable for any $\alpha > (r+1)\alpha_g$. This ensures that the map $\mathcal{D}|_{\mathcal{U}}$ is a morphism and it is injective by construction, see Remark 4.4. This proves that $\mathcal{D}$ is a birational map onto its image. We denote by $Y_{d_1,\cdots,d_\gamma}$ the closure of $\mathcal{D}(\mathcal{U})$ in $\mathcal{G}_{\underline{w},\alpha}(\underline{r}, d, r+1)$. It is an irreducible subscheme of $\mathcal{G}_{\underline{w},\alpha}(\underline{r}, d, r+1)$ of dimension $\beta_C(1, d, r+1) = p_a(C) + (r+1)(d - r - p_a(C))$. Note that we have

$$\beta_C(1, d, r+1) = \beta_C(r, d, r+1),$$

so, in order to prove the assertion, we will show that for each coherent system $(E, V) \in \mathcal{D}(\mathcal{U})$ the Petri map

$$\mu_{E,V}\colon V \otimes H^0(\omega_C \otimes E^*) \to H^0(\omega_C \otimes E \otimes E^*)$$

is injective. Consider the exact sequence defining $(E, V)$, i.e.

$$0 \to L^{-1} \to V \otimes O_C \to E \to 0$$

and tensor it with $L$. This yields a surjective map $V \otimes H^1(L) \to H^1(E \otimes L)$. Under our assumptions we have $h^1(L) = 0$ by Proposition 5.1 so $H^1(E \otimes L) = 0$ too. In particular, $H^0(\omega_C \otimes E^* \otimes L^{-1}) = 0$ by Serre duality. If we consider again the above exact sequence and tensor it with $E^* \otimes \omega_C$ and take cohomology we obtain

$$0 \to H^0(\omega_C \otimes E^* \otimes L^{-1}) \to V \otimes H^0(\omega_C \otimes E^*) \xrightarrow{\mu_{E,V}} H^0(\omega_C \otimes E \otimes E^*) \to \cdots$$

which implies that $\mu_{E,V}$ is indeed injective as claimed. □

In what follows we will denote by $\mathcal{G}_{C_i,\alpha}(s, d_i, r+1)$ the moduli space of $\alpha$-stable coherent systems of type $(s, d_i, r+1)$ on the curve $C_i$. Assume that it is not empty and denote by $D_i$ the map

$$\mathcal{G}_{C_i,\alpha}(1, d_i, r+1) \overset{D_i}{\dashrightarrow} \mathcal{G}_{C_i,\alpha}(r, d_i, r+1).$$

sending a generated coherent system $(L_i, W_i)$ to its dual span $D_i((L_i, W_i))$. The map $D_i$ is birational, see [BGPMN03](Cor. 5.10).

Let $\overline{X_{d_1,\cdots,d_\gamma}}$ and $Y_{d_1,\dots,d_\gamma}$ be the irreducible components of $\mathcal{G}_{\underline{w},\alpha}(\underline{1}, d, r+1)$ and $\mathcal{G}_{\underline{w},\alpha}(\underline{r}, d, r+1)$ respectively described in Theorems 5.3 and 5.4. We can consider the diagram

$$\begin{array}{ccc} \overline{X_{d_1,\cdots,d_\gamma}} & \overset{\mathcal{D}}{\dashrightarrow} & Y_{d_1,\dots,d_\gamma} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \Pi_{i=1}^{\gamma}\mathcal{G}_{C_i,\alpha}(1, d_i, r+1) & \underset{\Pi D_i}{\dashrightarrow} & \Pi_{i=1}^{\gamma}\mathcal{G}_{C_i,\alpha}(r, d_i, r+1) \end{array} \tag{5.2}$$

where the vertical maps $\pi_1$ and $\pi_2$ are restrictions to the components of the curve $C$.

Then we have the following result:

**Theorem 5.5.** *Under the hypothesis of Theorem 5.4, the Diagram (5.2) is commutative and the maps $\pi_1$ and $\pi_2$ are both dominant. Moreover, the general fiber of $\pi_i$ has dimension $\delta \cdot r(r+1)$, where $\delta$ denotes the number of nodes on the curve $C$.*

*Proof.* Let $\mathcal{U} \subset X_{d_1,\cdots,d_\gamma}$ be the open subset where $\mathcal{D}$ is defined. Note that if $(L,W) \in \mathcal{U}$, then then by Theorems 5.3 and 5.4, we have: $D((L,W)) = (E,V) \in Y_{d_1,\cdots,d_\gamma}$, the restrictions $(E_i,V_i)$ are $\alpha$-stable coherent systems of type $(r,d_i,r+1)$ for any $\alpha > (r-1)d_i$, $(L_i,W_i)$ are coherent systems of type $(1,d_i,r+1)$ and we have $D_i((L_i,W_i)) = (E_i,V_i)$. Hence, the diagram commutes and in particular we have that both $\mathcal{G}_{C_i,\alpha}(1,d_i,r+1)$ and $\mathcal{G}_{C_i,\alpha}(r,d_i,r+1)$ are non-empty.

Since $D_i$ is birational, see [BGPMN03], we have:

$$\dim \mathcal{G}_{C_i,\alpha}(r,d_i,r+1) = \dim \mathcal{G}_{C_i,\alpha}(1,d_i,r+1) = \beta_{C_i}(1,d_i,r+1) = g_i + (r+1)(d_i - g_i - r).$$

This implies:

$$\dim \Pi_{i=1}^{\gamma} \mathcal{G}_{C_i,\alpha}(1,d_i,r+1) = \dim \Pi_{i=1}^{\gamma} \mathcal{G}_{C_i,\alpha}(r,d_i,r+1) = \\ = p_a(C) + (r+1)(d - p_a(C) - r) - (r+1)r\delta$$

where $\delta = \gamma - 1$, denotes the number of nodes of $C$.

We will prove that $\pi_1$ is dominant. Since it is a rational map between two irreducible varieties and $\dim X_{d_1,\cdots,d_\gamma} - \dim \Pi_{i=1}^{\gamma} \mathcal{G}_{C_i,\alpha}(1,d_i,r+1) = r(r+1)\delta$, then it is enough to show that a general fiber has dimension $r(r+1)\delta$. Let $(L,W) \in \mathcal{U}$. We will compute the dimension of the fiber $\mathcal{F}$ of $\pi_1$ over $\pi_1((L,W))$. Let $(L_i,W_i)$ be the restrictions of $(L,W)$ to the components of $C$. Since there is a unique line bundle on the curve $C$ having restrictions $L_i$, then we have:

$$\mathcal{F} = \{(L,W') \in \mathcal{U} \quad | \quad W'|_{C_i} = W_i\}.$$

Note that $\mathcal{F} \neq \emptyset$ since $(L,W) \in \mathcal{F}$. We recall that we have an exact sequence as follows, see Lemma 1.3:

$$0 \to L \to \bigoplus_{i=1}^{\gamma} L_i \to T \to 0$$

where $T = \bigoplus_{j=1}^{\delta} \mathbb{C}_{p_j}$. As $h^1(L) = 0$ we also have the exact sequence

$$0 \to H^0(L) \to \bigoplus_{i=1}^{\gamma} H^0(L_i) \xrightarrow{\alpha} \mathbb{C}^\delta \to 0,$$

where $\alpha$ at the node $p_j = C_{j1} \cap C_{j2}$ is the map sending $(s_1,\cdots,s_\gamma) \to s_{j1}(p_j) - s_{j2}(p_j)$. We can consider the restriction $\alpha'$ of $\alpha$ to $\bigoplus_{i=1}^{\gamma} W_i$. It is a surjective map, since $(L_i,W_i)$ is generated. So we have:

$$0 \to S \to \bigoplus_{i=1}^{\gamma} W_i \xrightarrow{\alpha'} \mathbb{C}^\delta \to 0$$

and $\dim S = \sum_{i=1}^{\gamma} \dim W_i - \delta = \delta r + r + 1$. So we have:

$$\mathcal{F} \simeq \{W' \in Gr(r+1,S) \mid (L,W') \in \mathcal{U}\}.$$

Since by Proposition 5.1 the subset $\{W' \in Gr(r+1, H^0(L)) \mid (L, W') \in \mathcal{U}\}$ is a non-empty open subset of the variety $Gr(r+1, H^0(L))$, then $\mathcal{F}$ is a non-empty open subset of $Gr(r+1, S)$. Hence $\dim \mathcal{F} = \dim Gr(r+1, S) = \delta r(r+1)$ as claimed. This concludes the proof. □

This completes the proof of Theorem $C$ of the Introduction.

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Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via Roberto Cozzi, 55, I-20125 Milano, Italy

*Email address*: `sonia.brivio@unimib.it`

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via Roberto Cozzi, 55, I-20125 Milano, Italy

*Email address*: `filippo.favale@unimib.it`