

Stochastic Control Problems with Unbounded Control Operators: solutions through generalized derivatives

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Abstract

This paper deals with a family of stochastic control problems in Hilbert spaces which arises in many engineering/economic/financial applications (in particular the ones featuring boundary control and control of delay equations with delay in the control) and for which it is difficult to apply the dynamic programming approach due to the unboundedness of the control operator and to the lack of regularity of the underlying transition semigroup. We introduce a specific concept of partial derivative, designed for this situation, and we develop a method to prove that the associated HJB equation has a solution with enough regularity to find optimal controls in feedback form.

Key words: Stochastic boundary control problems; Stochastic control of delay equation with delay in the control; Unbounded control operator; Second order Hamilton-Jacobi-Bellman equations in infinite dimension; Smoothing properties of transition semigroups.

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1 Introduction

1.1 Stochastic control in infinite dimension and its applications

Stochastic optimal control problems arise in a large variety of applications (see e.g. [74]), and have been (and currently are) the object of an extensive theoretical and applied literature. In recent years, due to progress in the methodologies and in the computational power there has been an increasing interest in studying also what is usually called “the infinite dimensional case”, i.e. the case when the state and control variables take their values in infinite dimensional spaces. The infinite dimensional case allows for more realistic modeling and substantially means that the state equation is a Stochastic Partial Differential Equation (SPDE from now on) or a Stochastic Differential Delay Equation (SDDE from now on).

Such types of state equations arise naturally in a wide range of applied models including physics, engineering, operations research, economics and finance.

On the one hand state equations of SPDE type are used when one wants to model control processes where the underlying dynamical system inherently depends on other basic variables beyond time. For example we recall: the control of SPDEs arising in fluid dynamics (see e.g. [23], [80]), in reaction-diffusion problems (see e.g. [73]), in modeling air pollution (see e.g. [27], [78]), in robotics (see e.g. [32], [33]), in elasticity theory and practice (see e.g. [21] [28],[29]), in spatio-temporal economics growth models (see e.g. [10, 11] in the deterministic case and [48] in the stochastic case), in advertising models (see e.g. [8], [57]).

On the other hand state equations of SDDE type are used when one wants to model control processes where the underlying dynamical system is not Markovian in the sense that the value of the state at time t depends on the past of the state/control variables. Such models arise in optimal advertising problems (see e.g. [19, 49, 50, 57, 72]); in optimal portfolio problems (see e.g. [15, 39, 76]); in optimal production planning (see e.g. [17], [83]); in the feedback stabilisation of engineering systems (see e.g. [66]).

1.2 The purpose of our paper: the “unbounded control case” and its interest

Despite its interest for applications, the theory of stochastic control in infinite dimension is still a young and incomplete area. For this reason we think it is interesting to continue to develop the theory for such problems, in particular trying to cover new problems arising in applications like the ones quoted above. For an up-to date account of the theory we send the reader see to the recent book [34]; other books which partly look at the subject are [24, Chapter 13], [16, Chapters 9-10] and [75, Chapters 5-6]).

This paper is devoted exactly to the above task in cases which are quite common in applications and for which there are, at the moment, few and incomplete results available: the cases, roughly speaking,

where the control enters in the state equation in an “unbounded” way. Such unboundedness translates into the fact that, while the state process takes values in a Hilbert space H (e.g. $L^2(0, 1)$), the term which brings the control action into the state equation, say Cu , takes values in a bigger Banach space \overline{H} (e.g. any Sobolev space of negative order) which strictly contains H . Typical applications where such unboundedness arises are **stochastic boundary control problems** and **stochastic control of delay equations with delay in the control**.

Concerning **stochastic boundary control problems**, i.e. problems where, for structural reasons, the control is applied only at the boundary of a given region, are quite common in a wide range of applied problems. In the engineering-related literature we recall some recent papers devoted to specific applications like, e.g., [31] in the field of robotics; [67, 64, 63, 73] for reaction-diffusion systems (also in the deterministic case); [65] for stochastic port-Hamiltonian systems; [28] [29] for marine risers and Timoshenko beams.

Stochastic boundary control problems also arise in advertising models when one wants to take account of the age structure of the products, see e.g. [8], [38], [56], [57].

On the other hand, **stochastic optimal control problems with delay in the control** arise since in many practical situations the effect of the control action persists in the future. We recall, in this respect, the so-called carryover effect in advertising, see e.g. [19], [49], [50], [72] and, for related deterministic problems, [43].

Applications to economics (delay in production due to the time to build) are studied in [18] where a problem with pointwise delay in the state and in the control is studied by means of the stochastic maximum principle (see also [6] in a related deterministic case).

Concerning applications to finance we recall [15] where a mean-field model of systemic risk with delay in the control is studied.

Finally stochastic control problems with delay in the control are also related to the problem of information delay (i.e. the time which may be necessary to implementing the control, which is studied e.g. in [77].

1.3 The novelty of our methodology

We use here the dynamic programming approach with the aim of finding solutions of the associated HJB equations which are regular enough to write optimal control strategies in feedback form and to prove verification theorems. In the literature one can, roughly speaking, distinguish three main methods to do this. The first is to look at the theory of viscosity solutions (which is partly developed in these cases, see e.g. [41, 53, 81] and [34, Section 3.12 and 4.8.3]); the viscosity solution is not differentiable in general, however in some infinite dimensional cases some type of differentiability can be proved (see e.g. [40], [76]) however such methods seems not applicable here due to the unboundedness of the control operator. The second approach (developed e.g. in [20], [1]) which we may call “variational” is based on the use of coercive bilinear forms associated to Ornstein-Uhlenbeck operators on a suitable Gelfand triple. Again this seems not applicable here since it seems not compatible with the unboundedness of our control operator and the lack of null controllability estimates of our cases.

The third approach, which is it the one we use here is the theory of mild solutions (see e.g. the papers [12], [13], [24], [46] [69], [45] and the book [34, Chapter 4]).

Roughly speaking such approach, which works only in the semilinear case, rewrites the HJB equation in a suitable integral form and tries to solve it using a fixed point argument. It is based on smoothing properties of an underlying transition semigroup and allows to find solution which are regular enough to define optimal feedback control strategies.

In the present case there are two technical barriers preventing the use of such approach:

- the need of giving a precise sense to the HJB equation, coming from the unboundedness explained above, which is the core of our setting;
- the lack of smoothing properties of the underlying transition semigroup.

To deal with such issues we introduce a specific concept of partial derivative, designed for this situation. We observe that in the literature various concepts of infinite dimensional derivative has been used, depending on the context. We mention, among others, the papers [55], [68] (which uses the so-called Fomin derivative), [44, 69, 70] (which use the so-called G -gradient for a bounded operator G), [34, Chapter 4]-[41] (which use the so-called G -gradient for a possibly unbounded operator G). Our definition extends the last one to take into account more general cases of unboundedness, more precise explanations

are given in Section 4.

Once such derivative is introduced we perform a nontrivial extension of an idea (which we call “partial smoothing”) that we used in our previous papers [51]-[52] in a delay case with bounded control operators. Details are given in Sections 5-6-7.

We must say that this paper is a first step to attack such difficult problems. Here we show how to find a regular solution to the HJB equation for a special type of cost functionals. A second step, which is the object of our current research, is to cover more general cost functionals, in particular the ones where the current cost can be state dependent. Moreover, to make our results more useful for applications we aim at proving verification type results: this would allow to construct the optimal controls in feedback form, on the line of what is done, e.g., in [2] or in our previous paper [52].

We also think that it would be interesting to study the case with partial observation (on the line of what is done, in cases which do not include ours, in [4], [54], [7]) or the case of Mc Kean - Vlasov dynamics (see e.g. [3], or [22]) and to study the applicability, to our setting, of numerical methods for infinite dimensional HJB equations like the ones developed in [5].

1.4 Plan of the paper

The plan of the paper is the following.

- Section 2 introduces some basic notations.
- Section 3 introduces our driving examples showing how to rewrite them in a suitable infinite dimensional setting.
- Section 4 provides the definition of our C -derivatives together with some comments to compare it with previous definitions.
- Section 5 presents our partial smoothing result for Ornstein-Uhlenbeck semigroups (Proposition 5.9).
- Section 6. generalizes the partial smoothing result to convolutions (Lemma 6.4).
- In Section 7 we first present a general control problem which includes both our driving examples (Subsection 7.1); then, in Subsection 7.2, we state and prove our main result of existence and uniqueness of mild solutions for the HJB equation (Theorem 7.5).
- Appendix A is devoted to show that our motivating examples satisfy the assumptions made.

2 Basic notation and spaces

For the reader’s convenience we collect here the basic notation used throughout the paper.

Let H be a Hilbert space. The norm of an element x in H will be denoted by $|x|_H$ or simply $|x|$, if no confusion is possible, and by $\langle \cdot, \cdot \rangle_H$, or simply by $\langle \cdot, \cdot \rangle$ we denote the inner product in H . We denote by H^* the dual space of H . If K is another Hilbert space, $\mathcal{L}(H, K)$ denotes the space of bounded linear operators from H to K endowed with the usual operator norm. All Hilbert spaces are assumed to be real and separable.

Let E be a Banach space. As for the Hilbert space case, the norm of an element x in E will be denoted by $|x|_E$ or simply $|x|$, if no confusion is possible. We denote by E^* the dual space of E , and by $\langle \cdot, \cdot \rangle_{E^*, E}$ the duality between E and E^* .

If F is another Banach space, $\mathcal{L}(E, F)$ denotes the space of bounded linear operators from E to F endowed with the usual operator norm. All Banach spaces are assumed to be real and separable.

In what follows we will often meet inverses of operators which are not one-to-one. Let $Q \in \mathcal{L}(H, K)$. Then $H_0 = \ker Q$ is a closed subspace of H . Let $H_1 := [\ker Q]^\perp$ be the orthogonal complement of H_0 in H : H_1 is closed, too. Denote by Q_1 the restriction of Q to H_1 : Q_1 is one-to-one and $\text{Im } Q_1 = \text{Im } Q$. For $k \in \text{Im } Q$, we define Q^{-1} by setting

$$Q^{-1}(k) := Q_1^{-1}(k).$$

The operator $Q^{-1} : \text{Im } Q \rightarrow H$ is called the pseudoinverse of Q . Q^{-1} is linear and closed but in general not continuous. Note that if $k \in \text{Im } Q$, then $Q_1^{-1}(k) \in [\ker Q]^\perp$. is the unique element of $\{h : Q(h) = k\}$ with minimal norm (see e.g. [84], p.209),

Next we introduce some spaces of functions. Let H and Z be real separable Hilbert spaces. By $B_b(H, Z)$ (respectively $C_b(H, Z)$, $UC_b(H, Z)$) we denote the space of all functions $f : H \rightarrow Z$ which are Borel measurable and bounded (respectively continuous and bounded, uniformly continuous and bounded).

Given an interval $I \subseteq \mathbb{R}$ we denote by $C(I \times H, Z)$ (respectively $C_b(I \times H, Z)$) the space of all functions $f : I \times H \rightarrow Z$ which are continuous (respectively continuous and bounded). $C^{0,1}(I \times H, Z)$ is the space of functions $f \in C(I \times H, Z)$ such that, for all $t \in I$, $f(t, \cdot)$ is continuously Fréchet differentiable with Fréchet derivative $\nabla f(t, x) \in \mathcal{L}(H, Z)$. By $UC_b^{1,2}(I \times H, Z)$ we denote the linear space of the mappings $f : I \times H \rightarrow Z$ which are uniformly continuous and bounded together with their first time derivative f_t and their first and second space derivatives $\nabla f, \nabla^2 f$.

If the destination space $Z = \mathbb{R}$ we do not write it in all the above spaces.

The same definitions can be given if H and Z are Banach spaces.

3 Two examples with unbounded control operator

We present here two stochastic controlled equations that motivates the introduction of generalized partial derivatives in Section 4. What they have in common is that, once they are reformulated as infinite dimensional stochastic controlled evolution equations, the control operator is unbounded.

3.1 Heat equations with boundary control

3.1.1 The state equation

Let (Ω, \mathcal{F}, P) be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t \geq 0}$. Fixed $0 \leq t \leq T < +\infty$, we consider, in an open connected set with smooth boundary $\mathcal{O} \subseteq \mathbb{R}^d$ ($d = 1, 2, 3$) the controlled stochastic heat equation with Dirichlet boundary conditions and with boundary control:

$$\begin{cases} \frac{\partial y}{\partial t}(s, \xi) = \Delta y(s, \xi) + \dot{W}_Q(s, \xi), & s \in [t, T], \xi \in \mathcal{O}, \\ y(t, \xi) = x(\xi), & \xi \in \mathcal{O}, \\ y(s, \xi) = u(s, \xi), & s \in [0, T], \xi \in \partial\mathcal{O}. \end{cases} \quad (3.1)$$

where Δ is the Laplace operator and we assume the following.

Hypothesis 3.1

- (i) The initial datum $x(\cdot)$ belongs to the state space $H := L^2(\mathcal{O})$. The set U of control values is a closed and bounded subset of the Hilbert space $K := L^2(\partial\mathcal{O})$.
- (ii) \dot{W}_Q is a so-called colored space-time noise (with space covariance $Q \in \mathcal{L}(H)$), the filtration $(\mathcal{F}_t)_{t \geq 0}$ coincides with the augmented filtration generated by W_Q ;
- (iii) the control strategy u belongs to \mathcal{U} where

$$\mathcal{U} := \{u(\cdot) : \Omega \times [0, T] \rightarrow U\} : \text{predictable}$$

Given any $(t, x) \in [0, T] \times \mathcal{O}$ and $u \in \mathcal{U}$, we denote, formally, by $y^{t,x,u}(s, \xi)$ the solution of (3.1) at $(s, \xi) \in [0, T] \times \mathcal{O}$.¹ We define the operator A_0 in H setting (here $H^2(\mathcal{O})$ and $H_0^1(\mathcal{O})$ are the usual Sobolev spaces)

$$\mathcal{D}(A_0) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}) \quad A_0 y = \Delta y \text{ for } y \in \mathcal{D}(A_0).$$

The operator A_0 is self-adjoint and diagonal with strictly negative eigenvalues $\{-\lambda_n\}_{n \in \mathbb{N}}$ (recall that $\lambda_n \sim n^{2/d}$ as $n \rightarrow +\infty$). We can endow H with a complete orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of eigenvectors of A_0 .² We recall that the linear trace operator $D : L^2(\partial\mathcal{O}) \rightarrow H$ is defined setting $Da = f$ where f is the unique solution of the Dirichlet problem

$$\begin{cases} \Delta f(\xi) = 0, & \xi \in \mathcal{O}, \\ f(\xi) = a(\xi), & \xi \in \partial\mathcal{O}. \end{cases}$$

¹Such solution could be defined with various methods. Here, similarly to [34, Appendix C] we define such solution as the unique mild solution (see (3.4) of the infinite dimensional system (3.2) below. Following the path outlined in [34, Appendix C] we give sense to (3.1) rewriting it as an evolution equation in the space $H := L^2(\mathcal{O})$. We assume that the initial condition $x(\cdot)$ belongs to H . The new state will be a process with values in H given, formally, by $X(s; t, x, u) = y^{t,x,u}(s, \cdot)$.

²We know that e_0 is constant and, when $d = 1$ and $\mathcal{O} = (0, \pi)$, $(e_n(\xi))_{n \geq 1} := (\sqrt{2} \sin(n\xi))_{n \geq 1}$.

Equation (3.1) can now be reformulated (see [34, Appendix C] for a proof) as

$$\begin{cases} dX(s) = A_0 X(s) ds + (-A_0) Du(s) dt + Q^{1/2} dW(s), \\ X(t) = x. \end{cases} \quad (3.2)$$

where $W(\cdot)$ is a cylindrical noise in H . We define

$$B_0 := (-A_0)D. \quad (3.3)$$

The operator B_0 , defined in $K = L^2(\partial\mathcal{O})$, does not take values in $H = L^2(\mathcal{O})$. Indeed for all $\varepsilon > 0$, the Dirichlet map takes its values in $\mathcal{D}((-A_0)^{1/4-\varepsilon})$: $D \in \mathcal{L}(K, \mathcal{D}((-A_0)^{1/4-\varepsilon}))$ (see again [34, Appendix C]). So,

$$B_0 = (-A_0)^{3/4+\varepsilon} (-A_0)^{1/4-\varepsilon} D : K \rightarrow \mathcal{D}((-A_0)^{-3/4-\varepsilon}) :$$

Here and from now on we take $0 < \varepsilon < 1/4$, indeed the point is to take ε as small as possible in order to have in B_0 a better unbounded part. Hence we have³ $B_0 \in \mathcal{L}(K, \mathcal{D}((-A_0)^{-3/4-\varepsilon}))$. With an abuse of language with respect to the standard use, we may say that B is unbounded on H , in the sense that its image is not contained in H but in a space larger than H (here $\mathcal{D}((-A_0)^{-3/4-\varepsilon}) = H^{-3/2-2\varepsilon}(\mathcal{O})$) which we will call \bar{H} .

The unique mild solution (which exists thanks e.g. to [34, Theorem 1.141]) of (3.2) is denoted by $X(\cdot; t, x, u)$ and is

$$X(s; t, x, u) = e^{(s-t)A_0} x + \int_t^s e^{(s-r)A_0} B_0 u(r) dr + \int_t^s e^{(s-r)A_0} Q^{1/2} dW(r), \quad s \in [t, T]. \quad (3.4)$$

Consequently, for any $(t, x) \in [0, T] \times \mathcal{O}$ and $u \in \mathcal{U}$, we give sense to $y^{t,x,u}$ by setting $y^{t,x,u}(s, \xi) := X(s; t, x, u)(\xi)$.

3.1.2 The optimal control problem

For any given $t \in [0, T]$ and $x \in H$, the objective is to minimize, over all control strategies in \mathcal{U} , the following finite horizon cost:

$$J(t, x; u) = \mathbb{E} \left[\int_t^T [\ell_0(s) + \ell_1(u(s))] ds + \phi(X(T; t, x, u)) \right], \quad (3.5)$$

under the following assumption

Hypothesis 3.2

- (i) $\ell_0 : [0, T] \rightarrow \mathbb{R}$, is measurable and bounded.
- (ii) $\ell_1 : U \rightarrow \mathbb{R}$ is measurable and bounded from below.
- (iii) $\phi : H \rightarrow \mathbb{R}$ is such that, for a suitable finite set $\{\mathbf{f}_1, \dots, \mathbf{f}_N\} \subseteq \mathcal{D}((-A_0)^\eta)$ (with $\eta > 1/4$) and a suitable $\bar{\phi} \in B_b(\mathbb{R}^n)$ we have

$$\phi_0(x) = \bar{\phi}(\langle x, \mathbf{f}_1 \rangle_H, \dots, \langle x, \mathbf{f}_N \rangle_H).$$

Such cost can be seen as the rewriting in H of a more "concrete" cost like

$$J_0(t, x; u) = \mathbb{E} \left[\int_t^T \int_{\mathcal{O}} [\bar{\ell}_0(s, \xi) + \bar{\ell}_1(u(s, \xi))] d\xi ds + \phi_0(y^{t,x,u}(T, \cdot)) \right], \quad (3.6)$$

where $\bar{\ell}_0, \bar{\ell}_1, \phi_0$ are chosen so that the corresponding ℓ_0, ℓ_1, ϕ satisfy Hypothesis 3.2 above.

Note that here the current cost does not depend on the state, this is due to the fact that putting the dependence on the state in the current cost would increase considerably the technical arguments in the solution of the HJB equation. Moreover, in order to bypass the lack of suitable smoothing properties of the underlying transition semigroup, we have to work on cost functionals which depend on the state only

³Here, for $\gamma > 0$ we denote by $\mathcal{D}((-A_0)^{-\gamma})$ the completion of H with respect to the norm $|\cdot|_{-\gamma} = |A_0^{-\gamma} \cdot|_H$.

through a suitable operator P , which here turns out to be defined by projections on $\mathcal{D}((-A_0)^\eta)$, with $\eta > 1/4$, see Section 7.1 and 7.2 for further details.

The value function of the problem is

$$V(t, x) := \inf_{u \in \mathcal{U}} J(t, x; u). \quad (3.7)$$

We define the Hamiltonians leaving aside the term not depending on the control u . For $p \in H$, $u \in U$, the current value Hamiltonian \hat{H}_{CV} is given by (this is formal since neither $B_0 u$ nor $B_0^* p$ belong to H , in general):

$$\hat{H}_{CV}(p; u) := \langle B_0 u, p \rangle_H + \ell_1(u) = \langle u, B_0^* p \rangle_H + \ell_1(u) \quad (3.8)$$

and the (minimum value) Hamiltonian by

$$\hat{H}_{min}(p) := \inf_{u \in U} \hat{H}_{CV}(p; u). \quad (3.9)$$

The associated HJB equation can then be formally written as

$$\begin{cases} -\frac{\partial v(t, x)}{\partial t} = \mathcal{A}[v(t, \cdot)](x) + \ell_0(t) + \hat{H}_{min}(\nabla v(t, x)), & t \in [0, T], x \in \mathcal{H}, \\ v(T, x) = \phi(x), \end{cases} \quad (3.10)$$

where B is defined in (3.3), and \mathcal{A} is the infinitesimal generator of the transition semigroup $(R_t)_{0 \leq t \leq T}$ associated to the process X when the control is zero: namely \mathcal{A} is formally defined by

$$\mathcal{A}[f](x) = \frac{1}{2} \text{Tr} Q \nabla^2 f(x) + \langle x, A^* \nabla f(x) \rangle. \quad (3.11)$$

From (3.8) we easily see that, still formally, $\hat{H}_{min}(p)$ depends not on p but on $B_0^* p$. On the same line also the minimum point in (3.8), when it exists, only depends on $B_0^* p$. This means that the candidate optimal feedback map, if it exists, is a function of $B_0^* \nabla v$. For this reason our main goal is to find a solution v of (3.10) for which $B_0^* \nabla v$ makes sense. For this reason in the sequel we will use the notation (for $p \in H$ and $q \in K$ such that the expressions below make sense):

$$H_{CV}(q; u) := \langle u, q \rangle_K + \ell_1(u) \quad \text{and} \quad H_{min}(q) := \inf_{u \in U} H_{CV}(q; u), \quad (3.12)$$

so that

$$\hat{H}_{CV}(p; u) = H_{CV}(B^* p; u) \quad \text{and} \quad \hat{H}_{min}(p) = H_{min}(B^* p)$$

3.2 SDEs with delay in the control

3.2.1 The state equation

In a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider the following controlled stochastic differential equation in \mathbb{R}^n with delay in the control:

$$\begin{cases} dy(s) = a_0 y(s) ds + b_0 u(s) ds + \int_{-d}^0 u(s + \xi) b_1(d\xi) ds + \sigma dW(s), & s \in [t, T] \\ y(t) = y_0, \\ u(t + \xi) = u_0(\xi), \quad \xi \in [-d, 0]. \end{cases} \quad (3.13)$$

Here we consider the case of delay in the control, the case with delay also in the state is more complicated and cannot be treated as an application of the techniques introduced in the present paper. We assume the following.

Hypothesis 3.3

- (i) W is a standard Brownian motion in \mathbb{R}^k , and $(\mathcal{F}_t)_{t \geq 0}$ is the augmented filtration generated by W ;
- (ii) the control strategy u belongs to \mathcal{U} where

$$\mathcal{U} := \{u(\cdot) : (\Omega \times [0, T] \rightarrow U) : \text{predictable}\}$$

with U a closed and bounded subset of \mathbb{R}^m ;

(iii) $a_0 \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$, $b_0 \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$, $\sigma \in \mathcal{L}(\mathbb{R}^k; \mathbb{R}^n)$, $d > 0$;

(iv) b_1 is an $m \times n$ matrix of signed Radon measures on $[-d, 0]$ (i.e. it is an element of the dual space of $C([-d, 0], \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n))$).

Given any initial datum $(y_0, u_0) \in \mathbb{R}^n \times L^2([-d, 0], \mathbb{R}^m)$ and any admissible control $u \in \mathcal{U}$ equation (3.13) admits a unique strong (in the probabilistic sense) solution which is continuous and predictable (see e.g. [58] Chapter 4, Sections 2 and 3).

Notice that Hypothesis 3.3-(iv) on b_1 covers, but it is not limited to, the very common case of pointwise delay⁴ but it is technically complicated to deal with: indeed it gives rise, as we are going to see in next subsection, to an unbounded control operator B . We underline the fact that in the case of pointwise delay the matrix b_1 is a matrix of discrete measures, like weighted Dirac measures. Recall that, in [51] and [52] the case of b_1 absolutely continuous with respect to the Lebesgue measure has been treated assuming

$$b_1(d\xi) = \bar{b}_1(\xi)d(\xi), \quad \bar{b}_1 \in L^2([-d, 0], \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)); \quad (3.14)$$

it is clear that such an assumption on b_1 leaves aside the pointwise delay case which we treat here.

3.2.2 Infinite dimensional reformulation

Now, using the approach of [82] (see [49] for the stochastic case), we reformulate equation (3.13) as an abstract stochastic differential equation in the Hilbert space $H = \mathbb{R}^n \times L^2([-d, 0], \mathbb{R}^n)$. To this end we introduce the operator $A_1 : \mathcal{D}(A_1) \subset H \rightarrow H$ as follows: for $x = (x_0, x_1) \in H$,

$$A_1 x = (a_0 x_0 + x_1(0), -x_1'), \quad \mathcal{D}(A_1) = \{x \in H : x_1 \in W^{1,2}([-d, 0], \mathbb{R}^n), x_1(-d) = 0\}. \quad (3.15)$$

We denote by A_1^* the adjoint operator of A_1 :

$$A_1^* x = (a_0^* x_0, x_1'), \quad \mathcal{D}(A_1^*) = \{x \in H : x_1 \in W^{1,2}([-d, 0], \mathbb{R}^n), x_1(0) = x_0\}. \quad (3.16)$$

We denote by e^{tA_1} the C_0 -semigroup generated by A_1 . For $x \in H$ we have

$$e^{tA_1} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} e^{ta_0} x_0 + \int_{-d}^0 1_{[-t, 0]} e^{(t+s)a_0} x_1(s) ds \\ x_1(\cdot - t) 1_{[-d+t, 0]}(\cdot) \end{pmatrix} \quad (3.17)$$

Similarly, denoting by $e^{tA_1^*} = (e^{tA_1})^*$ the C_0 -semigroup generated by A_1^* , we have for $z = (z_0, z_1) \in H$

$$e^{tA_1^*} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} e^{ta_0^*} z_0 \\ e^{(\cdot+t)a_0^*} z_0 1_{[-t, 0]}(\cdot) + z_1(\cdot + t) 1_{[-d, -t]}(\cdot) \end{pmatrix} \quad (3.18)$$

The infinite dimensional noise operator is defined as

$$G : \mathbb{R}^k \rightarrow H, \quad Gy = (\sigma y, 0), \quad y \in \mathbb{R}^k. \quad (3.19)$$

The control operator B_1 is defined as (here the control space is $K := \mathbb{R}^m$ and we denote by $C'([-d, 0], \mathbb{R}^n)$ the dual space of $C([-d, 0], \mathbb{R}^n)$)

$$B_1 : \mathbb{R}^m \rightarrow \mathbb{R}^n \times C'([-d, 0], \mathbb{R}^n), \quad (B_1 u)_0 = b_0 u, \quad \langle f, (B_1 u)_1 \rangle_{C, C'} = \int_{-d}^0 \langle f(\xi), b_1(d\xi)u \rangle, \quad u \in \mathbb{R}^m, \quad f \in C'([-d, 0], \mathbb{R}^n). \quad (3.20)$$

The adjoint B_1^* is

$$B_1^* : \mathbb{R}^n \times C''([-d, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^m, \quad B_1^*(x_0, x_1) = b_0^* x_0 + \int_{-d}^0 b_1^*(d\xi) x_1(\xi), \quad (x_0, x_1) \in \mathbb{R}^n \times C([-d, 0], \mathbb{R}^n), \quad (3.21)$$

⁴Our delayed SDE includes, for example, the state equation used in [15].

where we have denoted by $C''([-d, 0], \mathbb{R}^n)$ the dual space of $C'([-d, 0], \mathbb{R}^n)$, which contains $C([-d, 0], \mathbb{R}^n)$: here we consider B_1^* acting on $C([-d, 0], \mathbb{R}^n)$, for a characterization of $C''([-d, 0], \mathbb{R}^n)$, and for the inclusion of $C([-d, 0], \mathbb{R}^n)$ in $C''([-d, 0], \mathbb{R}^n)$ see e.g. [59], [60], [61] and [79]. If b_1 satisfies (3.14), B is a bounded operator from \mathbb{R}^n to H , and we can easily write $e^{tA_1}B$: see [51].

If b_1 is as in Hypothesis 3.3-(iv), then B is unbounded. Still it is possible to write $e^{tA_1}B$ by extending the semigroup, by extrapolation, to $\mathbb{R}^n \times C'([-d, 0]; \mathbb{R}^n)$. We have, for $u \in \mathbb{R}^m$

$$(e^{tA_1}B_1)_0 : \mathbb{R}^m \rightarrow \mathbb{R}^n, \quad (e^{tA_1}B_1)_0 u = e^{ta_0}b_0u + \int_{-d}^0 1_{[-t, 0]}e^{(t+r)a_0}b_1(dr)u, \quad (3.22)$$

$$(e^{tA_1}B_1)_1 : \mathbb{R}^m \rightarrow C'([-d, 0]; \mathbb{R}^n), \quad \langle f, (e^{tA_1}B_1)_1 u \rangle_{C, C'} = \int_{-d}^0 f(r+t)1_{[-d, -t]}b_1(dr)u. \quad (3.23)$$

Let us now define the predictable process $Y = (Y_0, Y_1) : \Omega \times [0, T] \rightarrow H$ as

$$Y_0(s) = y(s), \quad Y_1(s)(\xi) = \int_{-d}^{\xi} b_1(d\zeta)u(\zeta + s - \xi),$$

where y is the solution of (3.13) and $u \in \mathcal{U}$ is the control process. By [49, Proposition 2], the process Y is the unique mild solution of the abstract evolution equation in H

$$\begin{cases} dY(s) = A_1Y(s)ds + B_1u(s)ds + GdW(s), & t \in [0, T] \\ Y(0) = x = (x_0, x_1), \end{cases} \quad (3.24)$$

where $x_1(\xi) = \int_{-d}^{\xi} u_0(\zeta - \xi)b_1(d\zeta)u_0(\zeta - \xi)$, for $\xi \in [-d, 0]$, and u_0 has been introduced in (3.13) as the initial condition of the control process. Since we have assumed $u_0 \in L^2([-d, 0], \mathbb{R}^m)$. Note that we have $x_1 \in L^2([-d, 0]; \mathbb{R}^n)^5$. The mild (or integral) form of (3.24) is

$$Y(s) = e^{(s-t)A_1}x + \int_t^s e^{(s-r)A_1}B_1u(r)dr + \int_t^s e^{(s-r)A_1}GdW(r), \quad s \in [t, T]. \quad (3.25)$$

Here, similarly to what happen in the previous example (see Subsection 3.1.1), we may say that the image of B is not contained in H but in a space larger than H (here $\mathbb{R}^n \times C'([-d, 0], \mathbb{R}^n)$) which we will call \bar{H} .

3.2.3 The optimal control problem

Similarly to the previous section the objective is to minimize, over all control strategies in \mathcal{U} , a finite horizon cost:

$$\bar{J}(t, y_0, u_0; u(\cdot)) = \mathbb{E} \left[\int_t^T [\ell_0(s) + \ell_1(u(s))] ds + \bar{\phi}(y(T; t, x)) \right] \quad (3.26)$$

undert the following assumption

Hypothesis 3.4

- (i) $\ell_0 : [0, T] \rightarrow \mathbb{R}$, is measurable.
- (ii) $\ell_1 : U \rightarrow \mathbb{R}$ is measurable and bounded from below
- (iii) $\bar{\phi} : \mathbb{R}^n \rightarrow \mathbb{R}$ is measurable and bounded.

Such cost functional, using the infinite dimensional reformulation given above, can be rewritten as

$$J(t, x; u(\cdot)) = \mathbb{E} \left[\int_t^T [\ell_0(s) + \ell_1(u(s))] ds + \phi(Y(T; t, x)) \right] \quad (3.27)$$

where $\phi : H \rightarrow \mathbb{R}$ is defined as $\phi(x_0, x_1) = \bar{\phi}(x_0)$ for all $x = (x_0, x_1) \in H$. Note again that here the current cost does not depend on the state, again this is due to the fact that putting the dependence on the state in the current cost would increase considerably the technical arguments in the solution of the

⁵This can be seen, e.g., by a simple application of Jensen inequality and Fubini theorem.

HJB equation.

The value function of the problem is

$$V(t, x) := \inf_{u \in \mathcal{U}} J(t, x; u). \quad (3.28)$$

The Hamiltonians can be defined exactly in the same way as in Subsubsection 3.1.2 and (using the modified Hamiltonians introduced in (3.12)) the associated HJB equation is formally written as

$$\begin{cases} -\frac{\partial v(t, x)}{\partial t} = \mathcal{A}[v(t, \cdot)](x) + \ell_0(t) + H_{\min}(B_1^* \nabla v(t, x)), & t \in [0, T], x \in H, \\ v(T, x) = \phi(x), \end{cases} \quad (3.29)$$

where B_1 is defined in (3.20), and \mathcal{A} is the infinitesimal generator of the transition semigroup $(R_t)_{0 \leq t \leq T}$ associated to the process Y when the control is zero: namely \mathcal{A} is formally defined by

$$\mathcal{A}[f](x) = \frac{1}{2} \text{Tr} GG^* \nabla^2 f(x) + \langle x, A_1^* \nabla f(x) \rangle. \quad (3.30)$$

On the same line of Subsubsection 3.1.2 the candidate optimal feedback map, if it exists, is a function of $B_1^* \nabla v$.

4 C -derivatives

In this Section we introduce the definition of generalized partial derivatives (that we call C -directional derivatives, where C is a suitable linear operator) which is suitable for our needs. C -directional derivatives of functions have been introduced in [69, Section 2], [44] in the case when C is a bounded operator (see also [51, 52]), and in [41], [34, Section 4.2.1] in the case when C is possibly unbounded.

Our definition is different from the ones recalled above and is designed to cover a wider class of “unbounded” examples, in particular it makes it possible to treat the case when the intersection of image of C and the state space is just the origin, which is, e.g., the case of our examples of Section 3 which were not treatable within the previous setting.

We also recall that concepts which are connected to the one of C -directional derivative are the one of Fomin derivative (see, e.g., [9, Chapter 3] and, recently, [68]) and the one of derivative in the directions of a proper subspace (see e.g. [55]).

Here is our new definition. The operator C is still “possibly unbounded” in the sense that it does not take its values in the state space H but in a larger Banach space \overline{H} such that $H \subset \overline{H}$ with continuous embedding.

Definition 4.1 *Let H, Z, K and \overline{H} be Banach spaces such that $H \subset \overline{H}$ with continuous embedding. Let $C : K \rightarrow \overline{H}$ be a linear and bounded operator.*

- (i) *Let $k \in K$ and let $f : \overline{H} \rightarrow Z$. We say that f admits C -directional derivative at a point $x \in \overline{H}$ in the direction $k \in K$ (and we denote it by $\nabla^C f(x; k)$) if the limit, in the norm topology of Z ,*

$$\nabla^C f(x; k) := \lim_{s \rightarrow 0} \frac{f(x + sCk) - f(x)}{s}, \quad (4.1)$$

exists.

- (ii) *Let $f : \overline{H} \rightarrow Z$. We say that f is C -Gâteaux differentiable at a point $x \in \overline{H}$ if f admits the C -directional derivative in every direction $k \in K$ and there exists a **bounded** linear operator, the C -Gâteaux derivative $\nabla^C f(x) \in \mathcal{L}(K, Z)$, such that $\nabla^C f(x; k) = \nabla^C f(x)k$ for all $k \in K$. We say that f is C -Gâteaux differentiable on H (respectively \overline{H}) if it is C -Gâteaux differentiable at every point $x \in H$ (respectively $x \in \overline{H}$).*

- (iii) *Let $f : \overline{H} \rightarrow Z$. We say that f is C -Fréchet differentiable at a point $x \in \overline{H}$ if it is C -Gâteaux differentiable and if the limit in (4.1) is uniform for k in the unit ball of K . In this case we call $\nabla^C f(x)$ the C -Fréchet derivative (or simply the C -derivative) of f at x . We say that f is C -Fréchet differentiable on H (respectively \overline{H}) if it is C -Fréchet differentiable at every point $x \in H$ (respectively $x \in \overline{H}$).*

Remark 4.2 The main idea behind the use of C -derivatives (starting from the papers [69] and [44]) lies in the fact that, in applying the dynamic programming approach to optimal control problems which are linear in the control (with control operator $C : U \rightarrow H$ where U is the control space and H is the state space), the natural regularity requirement needed on the value function V to write the optimal feedbacks is that $\nabla^C V$ is well defined.⁶ This means that only directional derivatives in the directions of the image of C matter for the purpose of writing optimal feedback controls. In some cases, like the distributed control of heat equation (see e.g. [34, Section 2.6.1]), the image of C is contained in the state space (call it H), so $\nabla^C V$ is always well defined when ∇V exists. In some other cases, like the boundary control or the pointwise delayed control (see e.g. [34, Sections 2.6.2 and 2.6.8]) the image of the control operator C is not contained in H and it may even happen that the intersection of this image with H is only the origin, which is the case of the driving examples of this paper.

One strategy, used e.g. in [41, 42] and in [34, Section 4.8] to deal with such cases is to decompose the control operator C in the product $C_1 C_2$ where $C_2 : K \rightarrow H$ is bounded while the “unbounded part” C_1 is a closed unbounded operator $C_1 : D(C_1) \subseteq H \rightarrow H$ which usually is a power of the operator A driving the state equation. In this case the derivative needed to express the feedback control is $\nabla^{C_1} V$ which is defined exactly as in [41, Definition 2.2] or [34, Definition 4.4].⁷

In such setting, due to the boundedness required in [41, Definition 2.2-(ii)], asking that $\nabla^{C_1} V$ exists substantially means that we consider the directional derivatives of V in the directions of $\text{Im } \tilde{C}_1$ where \tilde{C}_1 is the extension of C_1 from the whole H to a suitable extrapolation space. The image of \tilde{C}_1 contains (but can be much larger than) the one of the control operator C .⁸

The approach used here is sharper in the sense that we look exactly at the derivatives in the directions of the image of C , even if they go out of the state space H . In this way we also avoid working with the decomposition of the operator C , which is not sharp for our purposes, in particular in the case of pointwise delayed control of Subsection 3.2 since in this case fractional powers of A are not well defined.⁹

Remark 4.3 Definition 4.1 is exactly the definition of C -derivative contained in [69, Section 2], [44] when $\bar{H} = H$. This means that, in this case, the classical Gâteaux or Fréchet differentiability implies the C -Gâteaux or C -Fréchet differentiability.

In [34, Definition 4.4] the operator C is a closed, possibly unbounded, linear operator $C : D(C) \subseteq K \rightarrow H$. This case can be partly embedded in the one we consider in Definition 4.1. We explain now why, restricting to the case when H is reflexive, which is true in our examples.

Let $C^* : D(C^*) \subseteq H' \rightarrow K'$ be the adjoint of C defined in the usual way through the duality $\langle C^* h, k \rangle_{K', K} = \langle h, Ck \rangle_{H, H}$, $\forall k \in D(C)$, $\forall h \in D(C^*)$. By [62, Theorem 5.29], since H is reflexive, we know that C^* is densely defined. Let

$$E := D(C^*) = \left\{ e \in H' : \exists a > 0 : \forall k \in D(C) \mid \langle Ck, e \rangle_{H, H'} \mid \leq a \|e\|_{H'} \right\} \subseteq H', \quad (4.2)$$

endowed with the usual graph norm, i.e.

$$\|w\|_E := \|w\|_{H'} + \|C^* w\|_{K'}, \quad \forall w \in E.$$

Let then $E' := D(C^*)'$. Clearly by (4.2) duality $H'' \subseteq E'$. Then, by the canonical embedding of the bidual we have $H \subseteq H'' \subseteq E'$. We extend, by extrapolation (see e.g. [30, §II.5] for the general theory and [37, §3.3] or [35, 36] for specific cases) C to a continuous operator $\tilde{C} : K \rightarrow E'$ setting, for $k \in K$ and $y \in E$,

$$\left\langle \tilde{C}k, y \right\rangle_{E', E} = \langle k, C^* y \rangle_{K, K'}$$

⁶Here we are simplifying a bit since, as one can read in [34, Section 4.8.1.4] (in particular equation (4.294)), the operator C in the gradient may be chosen a bit differently, and the linearity in the control can be weakened without affecting the main issues.

⁷Note that [34, Definition 4.4] is more general than our Definition in the sense that it allows the operator C to depend on the state variable $x \in H$. This could be performed here with ideas similar to what is done in [34, Section 4.2]. We do not do this since it would increase the technicalities without changing the main ideas which we want to make clear for the reader.

⁸For example, in the case of Neumann or Dirichlet boundary control in dimension 1, the image of C is two-dimensional while the one of \tilde{C}_1 is infinite dimensional

⁹A similar issue would arise if we consider boundary control problems where the driving operator A is of first order, like in the case of age-structured problems, see e.g., in the deterministic case, [37].

Continuity of \tilde{C} immediately follows observing that

$$|\langle \tilde{C}k, y \rangle_{E', E}| \leq |\langle k, C^*y \rangle_{K, K'}| \leq |k|_K |C^*y|_{K'}$$

and taking the supremum over all $y \in E$ in the unit ball.¹⁰

In this context we now compare [34, Definition 4.4] for C and Definition 4.1 for the corresponding extension \tilde{C} . Indeed we observe that [34, Definition 4.4-(i)] says, at point (i) (definition of directional derivatives):

“The C -directional derivative of f at a point $x \in H$ in the direction $k \in D(C) \subseteq K$ is defined as:

$$\nabla^C f(x; k) := \lim_{s \rightarrow 0} \frac{f(x + sCk) - f(x)}{s}, \quad s \in \mathbb{R}, \quad (4.3)$$

provided that the limit exists.”

But $k \in D(C)$, in the setting introduced above means that $\tilde{C}k \in H$. Hence, concerning point (i), Definition 4.1 extends [34, Definition 4.4].

Finally we observe that, when the image of the operator C crosses H only at the origin, then, [34, Definition 4.4] cannot be used while Definition 4.1 is still fit. ■

Remark 4.4 Observe that, similarly to what observed in [34, Remark 4.5] for Definition 4.4 (see also [41, Definition 2.2]), even if f is Fréchet differentiable at $x \in H$, the C -derivative may not exist in such point. This is obvious if we take, e.g., $f(x) = |x|^2$, $C : K \rightarrow \bar{H}$ with $\text{Im } C \not\subseteq H$. If $k \in K$ is such that $Ck \notin H$ clearly $\nabla^C f(x; k)$ does not exist. ■

We are now in position to define suitable spaces of C -differentiable functions.

Definition 4.5 Let I be an interval in \mathbb{R} , let H, \bar{H}, K and Z be suitable real Banach spaces. Moreover let $H \subset \bar{H}$ with continuous inclusion, and let $C \in \mathcal{L}(K, \bar{H})$.

- We call $C_b^{1,C}(\bar{H}, Z)$ the space of all continuous and bounded functions $f : \bar{H} \rightarrow Z$ which admit continuous and bounded C -Fréchet derivative. Moreover we call $C_b^{0,1,C}(I \times \bar{H}, Z)$ the space of continuous and bounded functions $f : I \times \bar{H} \rightarrow Z$ such that for every $t \in I$, $f(t, \cdot) \in C_b^{1,C}(\bar{H}, Z)$ and $\nabla^C f \in C_b(I \times \bar{H}, \mathcal{L}(K, Z))$. When $Z = \mathbb{R}$ we write $C_b^{1,C}(\bar{H})$ instead of $C_b^{1,C}(\bar{H}, Z)$, and it turns out that if $f \in C_b^{1,C}(\bar{H})$, then $\nabla^C f \in C_b(I \times \bar{H}, K')$.
- For any $\alpha \in (0, 1)$ and $T > 0$ (this time I is equal to $[0, T]$) we denote by $C_\alpha^{0,1,C}([0, T] \times \bar{H}, Z)$ the space of functions $f \in C_b([0, T] \times H, Z) \cap C_b^{0,1,C}([0, T] \times \bar{H}, Z)$ ¹¹ such that the map $(t, x) \mapsto t^\alpha \nabla^C f(t, x)$ belongs to $C_b([0, T] \times \bar{H}, \mathcal{L}(K, Z))$. When $Z = \mathbb{R}$ we omit it. The space $C_\alpha^{0,1,C}([0, T] \times \bar{H}, Z)$ is a Banach space when endowed with the norm

$$\|f\|_{C_\alpha^{0,1,C}([0, T] \times \bar{H}, Z)} = \sup_{(t, x) \in (0, T] \times \bar{H}} |f(t, x)| + \sup_{(t, x) \in (0, T] \times \bar{H}} t^\alpha \|\nabla^C f(t, x)\|_{\mathcal{L}(K, Z)}.$$

When clear from the context we will write simply $\|f\|_{C_\alpha^{0,1,C}}$.

¹⁰Notice that the second adjoint operator $C^{**} : D(C^{**}) \subset K'' \rightarrow H''$ is defined through the equality:

$$\langle k, C^*y \rangle_{K, K'} = \langle C^{**}k, y \rangle_{H'', H'}, \quad \forall k \in D(C^{**}), \forall y \in D(C^*);$$

with $D(C^{**})$ defined analogously to (4.2). So \tilde{C} and C^{**} are operator acting and taking values on different spaces:

$$\langle \tilde{C}k, y \rangle_{E', E} = \langle C^{**}k, y \rangle_{H'', H'}.$$

¹¹Note that here $f(t, \cdot)$ is well defined only in H when $t = 0$, while for $t > 0$ it is defined over \bar{H} . The reason is that the Ornstein-Uhlenbeck semigroup in our examples and in our setting (and consequently the solution of the HJB equation) satisfy the same property.

5 Partial smoothing for Ornstein-Uhlenbeck semigroups

In this section we study the “partial smoothing” properties of the Ornstein-Uhlenbeck semigroup (which we call R_t , for $t \geq 0$) applied to a generic function f weakening the definition of “smoothing” given, e.g., in [26] (see also [25, Chapter 9]). Note that a type of partial smoothing has been already developed, e.g., in [34, Ch.4] and in [51, 52]. As said above, the main difference here is that the directions along which we take the derivative can go out of the state space H and this allows to treat in sharper way the control problems exposed in Section 3.

The following basic assumption holds throughout this section.

Hypothesis 5.1

- (i) Let H, K, Ξ be three real separable Hilbert spaces¹².
- (ii) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions and let W be an $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ -cylindrical Wiener process in Ξ where $(\mathcal{F}_t)_{t \geq 0}$ is the augmented filtration generated by W .
- (iii) Let $A : D(A) \subseteq H \rightarrow H$ be the generator of a strongly continuous semigroup e^{tA} , $t \geq 0$ in H ,
- (iv) Let $G \in \mathcal{L}(\Xi, H)$ be such that the selfadjoint operator

$$Q_t = \int_0^t e^{sA} G G^* e^{sA^*} ds \quad (5.1)$$

is trace class. We call $Q = G G^* \in \mathcal{L}(H)$.

Let $Z(\cdot; x)$ be the Ornstein-Uhlenbeck process which solves the following SDE in H .

$$\begin{cases} dZ(t) = AZ(t)dt + GdW(t), \\ X(0) = x. \end{cases} \quad (5.2)$$

The process $Z(\cdot; x)$ is to be considered in its mild formulation:

$$Z(t; x) = e^{tA}x + \int_0^t e^{(t-s)A} G dW(s), \quad t \geq 0. \quad (5.3)$$

Z is a Gaussian process, namely for every $t > 0$, the law of $Z(t)$ is $\mathcal{N}(e^{tA}x, Q_t)$, the Gaussian measure with mean $e^{tA}x$ and covariance operator Q_t defined in (5.1). The convolution $\int_0^t e^{(t-s)A} G dW_s$ has law $\mathcal{N}(0, Q_t)$ and will be sometimes denoted by $W_A(t)$. The associated Ornstein-Uhlenbeck transition semigroup R_t is defined by setting, for every $\psi \in B_b(H)$ and $x \in H$,

$$R_t[\psi](x) = \mathbb{E}\psi(Z(t; x)) = \int_H \psi(z + e^{tA}x) \mathcal{N}(0, Q_t)(dz). \quad (5.4)$$

To study regularizing properties in the directions of an “unbounded” operator C (as introduced in Section 4), and for functions that have a special dependence on the state, through an operator P that we are going to introduce, we assume the following.

Hypothesis 5.2

- (i) Let \overline{H} be a real Banach space such that $H \subseteq \overline{H}$ with continuous and dense inclusion and that the semigroup e^{tA} admits an extension $e^{tA} : \overline{H} \rightarrow \overline{H}$ which is still a C_0 semigroup.
- (ii) Let $C \in \mathcal{L}(K, \overline{H})$.
- (iii) Let $P : H \rightarrow H$ be linear and continuous. Assume that, for every $t > 0$ the operator $Pe^{tA} : H \rightarrow H$ can be extended to a continuous linear operator $\overline{H} \rightarrow \overline{H}$, which will be denoted by $\overline{Pe^{tA}}$. With this notation the operator $\overline{Pe^{tA}}C : K \rightarrow H$ is well defined and continuous.

¹²These will be usually the state space, the control space and the noise space, respectively.

We now provide two remarks on the above hypothesis: the first on the adjoint of Pe^{tA} , the second one on the validity of such hypothesis in our examples.

Remark 5.3 *In the framework of the above Hypothesis 5.2 it is natural to identify H with its topological dual H' and consider the Gelfand triple*

$$\overline{H'} \subseteq H \subseteq \overline{H}.$$

The adjoint of the operator $Pe^{tA} : H \rightarrow H$ which is, clearly, $e^{tA} P^* : H \rightarrow H$, indeed takes its values in $\overline{H'}$ and is, consequently, the adjoint*

$$\left(\overline{Pe^{tA}}\right)^* : H \rightarrow \overline{H'}$$

of the extended operator $\overline{Pe^{tA}}$.

Indeed, consider $\{x_n\} \subset H$ such that, in the topology of \overline{H} , we have $x_n \rightarrow \bar{x} \in \overline{H}$. We know, by Hypothesis 5.2, that $Pe^{tA}x_n \rightarrow \overline{Pe^{tA}\bar{x}}$, hence, for every $y \in H$,

$$\left\langle x_n, e^{tA*} P^* y \right\rangle_H = \left\langle Pe^{tA} x_n, y \right\rangle_H \rightarrow \left\langle \overline{Pe^{tA}\bar{x}}, y \right\rangle_H.$$

Hence, the continuous linear form π_y on H (represented, with the Riesz identification on H , by $e^{tA} P^* y$) given by*

$$\pi_y : H \rightarrow \mathbb{R}, \quad \pi_y(h) = \left\langle h, e^{tA*} P^* y \right\rangle_H, \quad h \in H,$$

can be extended to a continuous linear form

$$\bar{\pi}_y : \overline{H} \rightarrow \mathbb{R}, \quad \bar{\pi}_y(\bar{h}) = \left\langle \overline{Pe^{tA}\bar{h}}, y \right\rangle_H \quad \bar{h} \in \overline{H},$$

with $|\bar{\pi}_y(\bar{h})| \leq |\overline{Pe^{tA}}|_{\mathcal{L}(\overline{H}, H)} |\bar{h}|_{\overline{H}} |y|_H$. This is equivalent to say that, under the Riesz identification of H with H' , $e^{tA} P^* y \in \overline{H'} \subseteq H$.*

Remark 5.4 *In the case of Subsection 3.1 the above Hypotheses 5.1 and 5.2 are satisfied if we choose, as seen in Subsection 3.1,*

$$H = L^2(\mathcal{O}), \quad \overline{H} = \mathcal{D}\left((-A_0)^{-3/4-\varepsilon}\right) = H^{-3/2-2\varepsilon}(\mathcal{O}) \quad (\text{for suitable small } \varepsilon > 0),$$

$A = A_0$, $C = B = (-A_0)D$ as from (3.3), P any continuous operator $H \rightarrow H$ (we will later take P to be a finite dimensional projection whose image is contained in $\mathcal{D}((-A_0)^{-\eta})$ for some $\eta \geq 0$). Since we can extend immediately e^{tA_0} to

$$\overline{e^{tA_0}} : \overline{H} \rightarrow H$$

then, in this case, $\overline{Pe^{tA}} = \overline{Pe^{tA}}$.

In the case of Subsection 3.2 the above Hypotheses 5.1 and 5.2 are satisfied if we choose

$$H = \mathbb{R}^n \times L^2(-d, 0; \mathbb{R}^n), \quad \overline{H} = \mathbb{R}^n \times C'([-d, 0]; \mathbb{R}^n),$$

(but also $\overline{H} = \mathbb{R}^n \times W^{-1,2}([-d, 0]; \mathbb{R}^n)$ can be chosen), $A = A_1$, $C = B$ as from (3.20), and $P(x_0, x_1) = (x_0, 0)$. Here the embedding of $L^2([-d, 0]; \mathbb{R}^n) \subset C'([-d, 0]; \mathbb{R}^n)$ is to be considered in the following sense: to any $f \in L^2([-d, 0]; \mathbb{R}^n)$ we associate the measure $\mu_f \in C'([-d, 0]; \mathbb{R}^n)$ such that $\mu_f(d\xi) = f(\xi)d\xi$. Note that

$$\text{Im } P = \mathbb{R}^n \times \{0\}.$$

Moreover, by (3.17), we have, for $x = (x_0, x_1) \in H$,

$$Pe^{tA}x = \left(e^{ta_0}x_0 + \int_{-d}^0 1_{[-t, 0]} e^{(t+s)a_0} x_1(s) ds, 0 \right)$$

Hence, also in this case, we can extend immediately Pe^{tA} to

$$\overline{Pe^{tA}} : \overline{H} \rightarrow H$$

by setting, for $x = (x_0, x_1) \in \mathbb{R}^n \times C'([-d, 0]; \mathbb{R}^n)$

$$\overline{Pe^{tA}}x = \left(e^{ta_0}x_0 + \int_{-d}^0 1_{[-t, 0]}e^{(t+s)a_0}x_1(ds), 0 \right). \quad (5.5)$$

Hence, also here Hypothesis 5.2-(iii) is satisfied.

Notice that in the second example P can be immediately extended to $\overline{P} : \overline{H} \rightarrow H$ so $\overline{Pe^{tA}} = \overline{P} \overline{e^{tA}}$ while in the first example P may not admit such an extension (it does when P is a finite dimensional projection).

Finally notice that in both examples we have $\overline{\text{Im}Pe^{tA}} \subseteq \text{Im}P$. ■

We pass to define the spaces where our “initial” data will belong.

Definition 5.5 We call $B_b^P(H)$ (respectively $C_b^P(H)$, $UC_b^P(H)$) the set of functions $\phi : H \rightarrow \mathbb{R}$ for which there exists $\bar{\phi} : \text{Im}(P) \rightarrow \mathbb{R}$ bounded and Borel measurable and (respectively continuous, uniformly continuous)¹³ such that

$$\phi(x) = \bar{\phi}(Px) \quad \forall x \in H. \quad (5.6)$$

Remark 5.6 We observe that, in the above Definition 5.5, when $\bar{\phi} : \text{Im}(P) \rightarrow \mathbb{R}$ is Borel measurable (respectively continuous, uniformly continuous), then also ϕ is Borel measurable (respectively continuous, uniformly continuous). Hence we can easily see that $B_b^P(H)$ (respectively $C_b^P(H)$, $UC_b^P(H)$) is a linear subspace of $B_b(H)$ (respectively $C_b(H)$, $UC_b(H)$).

We also observe that the choice of P in our driving examples (Subsections 3.1-3.2) will consider cases the case when $\text{Im}P$ is closed and finite dimensional. It is then useful to recall that, when the image of P is closed, we can identify the space $B_b^P(H)$ with $B_b(\text{Im}P)$ (and the same for the others). In particular, in the case of Subsection 3.2, when $\text{Im}P = \mathbb{R}^n \times \{0\}$, we immediately see that $B_b^P(H) \sim B_b(\mathbb{R}^n)$, $C_b^P(H) \sim C_b(\mathbb{R}^n)$, $UC_b^P(H) \sim UC_b(\mathbb{R}^n)$. This will be used in the sequel. ■

To prove our partial smoothing result we need the following controllability-like assumption.

Hypothesis 5.7

(i) We have

$$\text{Im} \overline{Pe^{tA}}C \subseteq \text{Im}(PQ_tP^*)^{1/2}, \quad \forall t > 0; \quad (5.7)$$

Consequently, by the Closed Graph Theorem, the operator

$$\Lambda^{P,C}(t) : K \rightarrow H, \quad \Lambda^{P,C}(t)k := (PQ_tP^*)^{-1/2} \overline{Pe^{tA}}Ck \quad \forall k \in K,$$

is well defined and bounded for all $t > 0$.

(ii) For every $T > 0$ there exists $\kappa_T > 0$ and $\gamma \in (0, 1)$ such that

$$\|\Lambda^{P,C}(t)\|_{\mathcal{L}(K, H)} \leq \kappa_T t^{-\gamma}, \quad \forall t \in (0, T].$$

Hypothesis 5.7-(i) is the analogous of the null controllability assumption which guarantees the strong Feller property of the associated Ornstein-Uhlenbeck transition semigroup, see e.g. [25] and [84], while 5.7-(ii) is an assumption that guarantees that for $t \rightarrow 0$, the operator norm of $\Lambda^{P,C}(t)$ blows up in an integrable way. Both the assumptions can be verified in some models, namely in the following we show that the motivating examples introduced in Section 3 satisfy Hypothesis 5.7.

Remark 5.8 In the case of Subsection 3.1 Hypothesis 5.7 is satisfied, e.g., if we choose:

- H, \overline{H}, A, C as in Remark 5.4,
- $Q = (-A)^{-2\beta}$ for some $\beta \geq 0$
- P a projection on a finite dimensional subspace contained in $\mathcal{D}(-A)^{-\alpha}$ for some $\alpha > \beta + \frac{1}{4}$.

¹³Here we endow $\text{Im}P \subseteq H$ with the topology inherited by H .

See Appendix A.1.

In the case of Subsection 3.2 the above Hypothesis 5.7 are satisfied if:

- we choose H, \overline{H}, A, C, P as in Remark 5.4;
- we assume that $\text{Im} \left(e^{ta_0} b_0 + \int_{-d}^0 1_{[-t,0]} e^{(t+r)a_0} b_1(dr) \right) \subseteq \text{Im} \sigma, \quad \forall t > 0$. We notice that this condition is verified when σ is invertible, and it is a weaker assumption.

See Appendix A.2. ■

Now we give the result.

Proposition 5.9 *Let Hypotheses 5.1, 5.2 and 5.7-(i) hold true.*

Then the semigroup $R_t, t > 0$ maps functions $\phi \in B_b^P(H)$ into functions which are C -Fréchet differentiable in \overline{H} , and the C -derivative is given, for all $x \in \overline{H}$, by

$$\nabla^C(R_t[\phi])(x)k = \int_H \bar{\phi}(z_1 + Pe^{tA}x) \left\langle \Lambda^{P,C}(t)k, (PQ_tP^*)^{-1/2}z_1 \right\rangle_H \mathcal{N}(0, PQ_tP^*)(dz_1) \quad (5.8)$$

$$= \mathbb{E} \left[\bar{\phi}(PX(t; x)) \left\langle \Lambda^{P,C}(t)k, (PQ_tP^*)^{-1/2}PW_A(t) \right\rangle \right] \quad (5.9)$$

Moreover, for any $\phi \in B_b^P(H)$ and any $k \in K$,

$$|\langle \nabla^C R_t[\phi](x), k \rangle| \leq \|\Lambda^{P,C}(t)\|_{\mathcal{L}(K,H)} \|\phi\|_\infty |k|. \quad (5.10)$$

Furthermore, if $\phi \in C_b^P(H)$, then $\nabla^C R_t[\phi] \in C((0, T] \times \overline{H}; K)$. Finally, if also Hypothesis 5.7-(ii) holds, then the map $(t, x) \rightarrow R_t[\phi](x)$ belongs to $C_\gamma^{0,1,C}([0, T] \times \overline{H})$.

Proof.

If $\phi \in B_b^P(H)$, then, by (5.4), for every $t > 0$ and $x \in H$,

$$R_t[\phi](x) = \int_H \bar{\phi}(Pz + Pe^{tA}x) \mathcal{N}(0, Q_t)(dz), = \int_H \bar{\phi}(z_1 + Pe^{tA}x) \mathcal{N}(0, PQ_tP^*)(dz_1), \quad (5.11)$$

where we adopt the change of variable $z_1 = Pz$ and we used that the image of the measure $\mathcal{N}(0, Q_t)$ through $P : H \rightarrow H$ is, clearly, $\mathcal{N}(0, PQ_tP^*)$. Now notice that, defining, for $t > 0$,

$$\mathcal{L}_t : L^2(0, t; K) \rightarrow H, \quad \mathcal{L}_t u = \int_0^t e^{(t-s)A} G u(s) ds,$$

we get, by simple computations, that

$$|(P\mathcal{L}_t)^* x|^2 = \langle Q_t P^* x, P^* x \rangle \quad \text{which implies} \quad \text{Im } P\mathcal{L}_t = \text{Im}(PQ_tP^*)^{1/2}.$$

Hence, in particular the image of $(PQ_tP^*)^{1/2}$ is contained in $\text{Im} P$. Moreover, if Hypothesis 5.7-(i) holds, the above also implies that $\text{Im } \overline{Pe^{tA}C} \subseteq \text{Im} P$ for all $t > 0$. Using this fact, for $\phi \in B_b^P(H)$, $t > 0$, $x \in H$, $k \in K$, $\alpha \in \mathbb{R}$,

$$\begin{aligned} R_t[\phi](x + \alpha Ck) &= \int_H \bar{\phi}(Pz + \overline{Pe^{tA}}(x + \alpha Ck)) \mathcal{N}(0, Q_t)(dz), \\ &= \int_H \bar{\phi}(z_1 + \overline{Pe^{tA}}(x + \alpha Ck)) \mathcal{N}(0, PQ_tP^*)(dz_1), \end{aligned} \quad (5.12)$$

where, in the second equality, we still use the change of variable $z_1 = Pz$. Now we apply the change of variable $z_2 = z_1 + \overline{Pe^{tA}}\alpha Ck$ to (5.12) getting that, for every $t > 0$ and $\phi \in B_b^P(H)$,

$$R_t[\phi](x + \alpha Ck) = \int_H \bar{\phi}(z_2 + Pe^{tA}x) \mathcal{N}(\alpha \overline{Pe^{tA}C}k, PQ_tP^*)(dz_2). \quad (5.13)$$

Now, for $\phi \in B_b^P(H)$, $x \in H$, $k \in K$, $\alpha \in \mathbb{R} - \{0\}$, we get, by (5.11)-(5.13),

$$\begin{aligned} \frac{1}{\alpha} [R_t[\phi](x + \alpha Ck) - R_t[\phi](x)] &= \\ &= \frac{1}{\alpha} \left[\int_H \bar{\phi}(z_1 + Pe^{tA}x) \mathcal{N}(\alpha \overline{Pe^{tA}Ck}, PQ_tP^*)(dz_1) - \int_H \bar{\phi}(z_1 + Pe^{tA}x) \mathcal{N}(0, PQ_tP^*)(dz_1) \right]. \end{aligned} \quad (5.14)$$

By the Cameron-Martin theorem, see e.g. [24], Theorem 1.3.6, the Gaussian measures $\mathcal{N}(\alpha \overline{Pe^{tA}Ck}, PQ_tP^*)$ and $\mathcal{N}(0, PQ_tP^*)$ are equivalent if and only if $\overline{Pe^{tA}Ck} \in \text{Im}(PQ_tP^*)^{1/2}$. In such case, setting, for $y \in \text{Im}(PQ_tP^*)^{1/2}$, the density is

$$\begin{aligned} d(t, y, z) &:= \frac{d\mathcal{N}(y, PQ_tP^*)}{d\mathcal{N}(0, PQ_tP^*)}(z) \\ &= \exp \left\{ \left\langle (PQ_tP^*)^{-1/2}y, (PQ_tP^*)^{-1/2}z \right\rangle_H - \frac{1}{2} \left| (PQ_tP^*)^{-1/2}z \right|_H^2 \right\}. \end{aligned} \quad (5.15)$$

Such density is well defined for $z \in (\ker PQ_tP^*)^\perp$ (see e.g. [34, Proposition 1.59]). Hence, by (5.14),

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} [R_t[\phi](x + \alpha Ck) - R_t[\phi](x)] &= \\ &= \lim_{\alpha \rightarrow 0} \int_H \bar{\phi}(z_1 + Pe^{tA}x) \frac{d(t, \alpha \overline{Pe^{tA}Ck}, z_1) - 1}{\alpha} \mathcal{N}(0, PQ_tP^*)(dz_1) \end{aligned} \quad (5.16)$$

Now we observe that, by the definition of $\Lambda^{P,C}(t)$,

$$\frac{d(t, \alpha \overline{Pe^{tA}Ck}, z_1) - 1}{\alpha} = \frac{1}{\alpha} \left[\exp \left\{ \alpha \left\langle \Lambda^{P,C}(t)k, (PQ_tP^*)^{-1/2}z_1 \right\rangle_H - \frac{\alpha^2}{2} \left| \Lambda^{P,C}(t)k \right|_H^2 \right\} - 1 \right].$$

When $\alpha \rightarrow 0$ the above limit is, $\left\langle \Lambda^{P,C}(t)k, (PQ_tP^*)^{-1/2}z_1 \right\rangle_H$, which makes sense for all $z_1 \in (\ker PQ_tP^*)^\perp$ and is an $L^2(H; \mathcal{N}(0, PQ_tP^*))$ function of z_1 (see again, e.g., [34, Proposition 1.59]). Moreover, with respect to the measure $\mathcal{N}(0, PQ_tP^*)(dz_1)$ the map

$$z_1 \mapsto \mathcal{Q}_t(z_1) := \left\langle \Lambda^{P,C}(t)k, (PQ_tP^*)^{-1/2}z_1 \right\rangle_H$$

is real valued Gaussian random variable with mean 0 and variance $\left| \Lambda^{P,C}(t)k \right|_H^2$ (see, e.g. [26, Remark 2.2]). So in particular, for all $L > 0$ $\mathbb{E}[e^{L|\mathcal{Q}_t|}] < +\infty$. Now it is easy to see that

$$\frac{d(t, \alpha \overline{Pe^{tA}Ck}, z_1) - 1}{\alpha} \leq e^{|\mathcal{Q}_t| + \left| \Lambda^{P,C}(t)k \right|_H^2}$$

Hence we can apply the dominated convergence theorem to (5.16) getting

$$\begin{aligned} &\exists \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} [R_t[\phi](x + \alpha Ck) - R_t[\phi](x)] = \\ &\lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \int_H \bar{\phi}(z_1 + Pe^{tA}x) \frac{d(t, \alpha \overline{Pe^{tA}Ck}, z_1) - 1}{\alpha} \mathcal{N}(0, PQ_tP^*)(dz_1) \\ &= \int_H \bar{\phi}(z_1 + Pe^{tA}x) \left\langle \Lambda^{P,C}(t)k, (PQ_tP^*)^{-1/2}z_1 \right\rangle_H \mathcal{N}(0, PQ_tP^*)(dz_1) \end{aligned}$$

Consequently, along Definition 4.1-(i), there exists the C -directional derivative $\nabla^C R_t[\phi](x; k)$ which is equal to the above right hand side. Using that $\Lambda^{P,C}(t)$ is continuous we see that the above limit is uniform for k in the unit ball of K , so there exists the C -Fréchet derivative $\nabla^C R_t[\phi](x)$. From the above and from [34, Proposition 1.59] we get

$$\begin{aligned} |\nabla^C R_t[\phi](x; k)| &\leq \|\bar{\phi}\|_\infty \left(\int_H \left\langle \Lambda^{P,C}(t)k, (PQ_tP^*)^{-1/2}z_1 \right\rangle_H^2 \mathcal{N}(0, PQ_tP^*)(dz_1) \right)^{1/2} \\ &= \|\bar{\phi}\|_\infty \|\Lambda^{P,C}(t)k\|_H \leq \|\phi\|_\infty \|\Lambda^{P,C}(t)\|_{\mathcal{L}(K;H)} \|k\|_K. \end{aligned}$$

This gives the required estimate. The statement on continuity follows using the same argument as in [34, Theorem 4.41-(ii)]. The last statement follows by the last part of Definition 4.5. \square

Remark 5.10 *The proof generalizes the one of Theorem 4.1 in [51] and the one of Theorem 4.41 in [34]. The main difference between Theorem 4.1 in [51] and the present proposition is that here we are able to handle an unbounded operator C by enlarging the space H . Notice that in the proof C appears only through the operator $\overline{Pe^{tA}C}$. Notice also that, as proved above, the image of such operator is contained in $\text{Im } P$, which is not obvious due to the presence of the closure. On the other hand, the difference with respect Theorem 4.41 in [34] is that there P is missing and the partial derivatives are taken in the unbounded but less general case of Definition [34, Definition 4.4].* \blacksquare

Remark 5.11 *Generalizing to our setting the ideas of Proposition 4.5 in [51] it is possible to prove that, if ϕ is more regular (i.e. $\phi \in C_b^1(H) \cap C_b^P(H)$), also $\nabla^C R_t[\phi]$ has more regularity, i.e. $\nabla \nabla^C R_t[\phi]$, $\nabla^C \nabla R_t[\phi]$ exist, coincide, and satisfy suitable formulae and estimates. We omit them here since we do not need them for the purpose of this paper. They will be useful to find optimal feedback controls, which will be the subject of a subsequent paper.* \blacksquare

6 Partial smoothing for convolutions

To solve HJB equations like (3.10) and (3.29) we need to extend the partial smoothing result of the previous section to convolutions.

We need first to introduce suitable spaces where such convolutions live and which will be useful later to perform the fixed point argument to find the solution of our HJB equations.

Definition 6.1 *Let $T > 0$, $\eta \in (0, 1)$. A function $g \in C_b([0, T] \times H) \cap C_b((0, T] \times \overline{H})$ belongs to $\Sigma_{T, \eta}^1$ if*

- *there exists a function $f \in C_b([0, T] \times H)$ such that¹⁴*

$$g(t, x) = f\left(t, \overline{Pe^{tA}x}\right), \quad \forall (t, x) \in (0, T] \times \overline{H};$$

- *for any $t \in (0, T]$ the function $g(t, \cdot)$ is C -Fréchet differentiable on \overline{H} and there exists a function $\overline{f} \in C_b((0, T] \times H; K)$ such that*

$$t^\eta \nabla^C g(t, x) = \overline{f}\left(t, \overline{Pe^{tA}x}\right), \quad \forall (t, x) \in (0, T] \times \overline{H}.$$

Remark 6.2 *Arguing as in [51, Section 5], it is possible to define a subspace of $\Sigma_{T, \eta}^1$ of functions g such that there exists the second order derivative $\nabla \nabla^C$ which depends in a special way on $x \in \overline{H}$. This could be useful to prove second order regularity of the solution of our HJB equations. As in Remark 5.11 we omit this step here: it will be useful to find optimal feedback controls, which will be the subject of a subsequent paper.* \blacksquare

Remark 6.3 *We observe that, using substantially the same argument as [51, Lemma 5.2], one can prove that*

$$\Sigma_{T, \eta}^1 \text{ is a closed subspace of } C_\eta^{0,1,C}([0, T] \times \overline{H}) \quad (6.1)$$

Moreover we also observe that, by Proposition 5.9, it is immediate to see that, under our Hypotheses 5.1-5.2-5.7, for any $\phi \in C_b^P(H)$, we have $R_t[\phi] \in \Sigma_{T, \gamma}^1$.

We now come back to the abstract common setting and we state a first lemma on the regularity of the convolution type terms.

¹⁴By continuity this also implies $g(0, x) = f(0, Px)$ for all $x \in H$.

Lemma 6.4 *Let Hypotheses 5.1, 5.2 and 5.7 hold. Let $T > 0$, $C \in \mathcal{L}(K, \overline{H})$ and let $\psi : K^* \rightarrow \mathbb{R}$ be a Lipschitz continuous function. For every $g \in \Sigma_{T,\gamma}^1$ (where γ is given in Hypothesis 5.7-(ii)), the function $\hat{g} : [0, T] \times \overline{H} \rightarrow \mathbb{R}$ belongs to $\Sigma_{T,\gamma}^1$ where*

$$\hat{g}(t, x) = \int_0^t R_{t-s} [\psi(\nabla^C g(s, \cdot))] (x) ds, \quad (t, x) \in [0, T] \times \overline{H}. \quad (6.2)$$

Hence, in particular, $\hat{g}(t, \cdot)$ is C -Fréchet differentiable on \overline{H} for every $t \in (0, T]$ and, for a suitable constant $\hat{\kappa}$ (depending only on T and ψ),

$$|\nabla^C(\hat{g}(t, \cdot))(x)|_{K^*} \leq \hat{\kappa} \left(t^{1-\gamma} + t^{1-2\gamma} \|g\|_{C_\gamma^{0,1,C}} \right), \quad \forall (t, x) \in (0, T] \times \overline{H}. \quad (6.3)$$

Moreover, for every $g_1, g_2 \in \Sigma_{T,\gamma}^1$ (where γ is given in Hypothesis 5.7-(ii)), the function $\hat{g}_1 - \hat{g}_2 : [0, T] \times \overline{H} \rightarrow \mathbb{R}$ belongs to $\Sigma_{T,\gamma}^1$ and, for a suitable constant κ (depending only on T and ψ),

$$|\hat{g}_1(t, x) - \hat{g}_2(t, x)| + t^\gamma |\nabla^C(\hat{g}_1(t, \cdot))(x) - \nabla^C(\hat{g}_2(t, \cdot))(x)|_{K^*} \leq \kappa (t + t^{1-\gamma}) \|g_1 - g_2\|_{C_\gamma^{0,1,C}}, \quad \forall (t, x) \in (0, T] \times \overline{H}. \quad (6.4)$$

Proof. We start by proving that \hat{g} from (6.2) is well defined, continuous, and C -Fréchet differentiable. First of all, for any $g \in \Sigma_{T,\gamma}^1$, we denote by f_g and \bar{f}_g the functions associated to it in Definition 6.1. Hence, given any $g \in \Sigma_{T,\gamma}^1$, we have, for $0 < s \leq t$, $x, z \in H$:

$$\psi \left(\nabla^C g(s, z + e^{(t-s)A} x) \right) = \psi \left(s^{-\gamma} \bar{f}_g \left(s, \overline{P e^{sA} z} + \overline{P e^{tA} x} \right) \right) \quad (6.5)$$

Hence we can give meaning to the left hand side also for $x \in \overline{H}$. So we can write

$$\begin{aligned} \int_0^t R_{t-s} [\psi(\nabla^C(g(s, \cdot)))] (x) ds &= \int_0^t \int_H \psi \left(\nabla^C g(s, z + e^{(t-s)A} x) \right) \mathcal{N}(0, Q_{t-s})(dz) \\ &= \int_0^t \int_H \psi \left(s^{-\gamma} \bar{f}_g \left(s, P e^{sA} z + \overline{P e^{tA} x} \right) \right) \mathcal{N}(0, Q_{t-s})(dz), \quad \forall (t, x) \in [0, T] \times \overline{H}. \end{aligned} \quad (6.6)$$

The above implies that \hat{g} is well defined on $[0, T] \times \overline{H}$. Continuity follows using the same argument as in [34, Proposition 4.50-(ii)]. Consequently the function $f_{\hat{g}}$ associated to \hat{g} along Definition 6.1 is

$$f_{\hat{g}}(t, y) = \int_0^t \int_H \psi \left(s^{-\gamma} \bar{f}_g \left(s, P e^{sA} z + y \right) \right) \mathcal{N}(0, Q_{t-s})(dz)$$

and, by the Lipschitz assumptions on ψ ,

$$\|\hat{f}_{\hat{g}}\|_\infty \leq \int_0^T [\psi]_{Lip} (|\psi(0)| + s^{-\gamma} \|\bar{f}_g\|_\infty) ds \leq [\psi]_{Lip} [|\psi(0)|T + \|\bar{f}_g\|_\infty (1-\gamma)^{-1} T^{1-\gamma}]$$

To compute the C -derivative we first compute, using what is given above,

$$\begin{aligned} &\int_0^t R_{t-s} [\psi(\nabla^C(g(s, \cdot)))] (x + \alpha Ck) ds \\ &= \int_0^t \int_H \psi \left(s^{-\gamma} \bar{f}_g \left(s, P e^{sA} z + \overline{P e^{tA} (x + \alpha Ck)} \right) \right) \mathcal{N}(0, Q_{t-s})(dz) ds \\ &= \int_0^t \int_H \psi \left(s^{-\gamma} \bar{f}_g \left(s, P e^{sA} z + \overline{P e^{tA} x} \right) \right) \mathcal{N} \left(\overline{P e^{tA} \alpha Ck}, Q_{t-s} \right) (dz) ds \\ &= \int_0^t \int_H \psi \left(s^{-\gamma} \bar{f}_g \left(s, P e^{sA} z + \overline{P e^{tA} x} \right) \right) d(t-s, \overline{P e^{tA} Ck}, z) \mathcal{N}(0, Q_{t-s})(dz) ds, \end{aligned} \quad (6.7)$$

where, in the last two equalities, we have used Cameron-Martin Theorem as in the proof of the above Proposition 5.9, and the density d is given by (5.15). Hence, using (6.6)-(6.7),

$$\begin{aligned} & \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \left[\int_0^t R_{t-s} [\psi (\nabla^C(g(s, \cdot)))] (x + \alpha Ck) ds - \int_0^t R_{t-s} [\psi (\nabla^C(g(s, \cdot)))] (x) ds \right] = \\ & = \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \int_0^t \int_H \psi \left(s^{-\gamma} \bar{f}_g \left(s, Pe^{sA} z + \overline{Pe^{tA} x} \right) \right) \frac{d(t-s, \alpha \overline{Pe^{tA} Ck}, z) - 1}{\alpha} \mathcal{N}(0, Q_{t-s})(dz) ds \end{aligned}$$

At this point we argue exactly as in the proof of the above Proposition 5.9 getting, uniformly for k in the unit sphere of K :

$$\begin{aligned} & \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \left[\int_0^t R_{t-s} [\psi (\nabla^C(g(s, \cdot)))] (x + \alpha Ck) ds - \int_0^t R_{t-s} [\psi (\nabla^C(g(s, \cdot)))] (x) ds \right] = \\ & = \int_0^t \int_H \psi \left(s^{-\gamma} \bar{f}_g \left(s, Pe^{sA} z + \overline{Pe^{tA} x} \right) \right) \left\langle Q_{t-s}^{-1/2} \overline{Pe^{tA} Ck}, Q_{t-s}^{-1/2} z \right\rangle_H \mathcal{N}(0, Q_{t-s})(dz) ds. \end{aligned}$$

This implies the required C -Fréchet differentiability and

$$\begin{aligned} & \langle \nabla^C \hat{g}(t, x), k \rangle_K = \tag{6.8} \\ & = \int_0^t \int_H \psi \left(s^{-\gamma} \bar{f}_g \left(s, Pe^{sA} z + \overline{Pe^{tA} x} \right) \right) \left\langle (Q_{t-s}^{-1/2} \overline{Pe^{tA} Ck}, Q_{t-s}^{-1/2} Pz) \right\rangle_H \mathcal{N}(0, Q_{t-s})(dz) ds. \end{aligned}$$

Moreover, the right hand side of (6.8) provides, when we substitute $\overline{Pe^{tA} x}$ with y , the function $\bar{f}_{\hat{g}}$ associated to \hat{g} along the second part of Definition 6.1.

At this point, in order to prove estimate (6.3), we use the above representation and the Holder inequality:

$$\begin{aligned} & |\langle \nabla^C \hat{g}(t, x), k \rangle_K| \leq \\ & \leq [\psi]_{Lip} \int_0^t \int_H \left(|\psi(0)| + \left| s^{-\gamma} \bar{f}_g \left(s, Pe^{sA} z + \overline{Pe^{tA} x} \right) \right| \right) \\ & \quad \left| \left\langle (Q_{t-s})^{-1/2} \overline{Pe^{tA} Ck}, Q_{t-s}^{-1/2} Pz \right\rangle_H \right| \mathcal{N}(0, Q_{t-s})(dz) ds \\ & \leq [\psi]_{Lip} \int_0^t \left(|\psi(0)| + s^{-\gamma} \|g\|_{C_\gamma^{0,1,C}} \right) \left\| Q_{t-s}^{-1/2} \overline{Pe^{tA} Ck} \right\|_{\mathcal{L}(K;H)} ds \\ & \leq \kappa_T [\psi]_{Lip} \int_0^t \left(|\psi(0)| + s^{-\gamma} \|g\|_{C_\gamma^{0,1,C}} \right) (t-s)^{-\gamma} |k|_K ds \end{aligned}$$

Since

$$\begin{aligned} & \int_0^t |\psi(0)| (t-s)^{-\gamma} |k|_K ds = |\psi(0)| |k|_K \frac{1}{1-\gamma} t^{1-\gamma} \\ & \int_0^t s^{-\gamma} \|g\|_{C_\gamma^{0,1,C}} (t-s)^{-\gamma} |k|_K ds = \|g\|_{C_\gamma^{0,1,C}} |k|_K \int_0^t s^{-\gamma} (t-s)^{-\gamma} ds = \|g\|_{C_\gamma^{0,1,C}} |k|_K \beta(1-\gamma, 1-\gamma) t^{1-2\gamma}, \end{aligned}$$

where by $\beta(\cdot, \cdot)$ we mean the Euler Beta function, the claim follows.

The proof of (6.4) follows in a similar way, taking into account estimate (6.3) and the fact that ψ is Lipschitz continuous:

$$\begin{aligned} & |\hat{g}_1(t, x) - \hat{g}_2(t, x)| + t^\gamma |\nabla^C(\hat{g}_1(t, \cdot))(x) - \nabla^C(\hat{g}_2(t, \cdot))(x)|_{K^*} \\ & \leq \left| \int_0^t R_{t-s} [\psi (\nabla^C(g_1(s, \cdot))) - \psi (\nabla^C(g_2(s, \cdot)))] (x) ds \right| \\ & \quad + t^\gamma |\nabla^C \int_0^t R_{t-s} [\psi (\nabla^C(g_1(s, \cdot))) - \psi (\nabla^C(g_2(s, \cdot)))] (x) ds| \\ & \leq t^\gamma \left| \int_0^t \int_H \left(\psi \left(s^{-\gamma} \bar{f}_{g_1} \left(s, Pe^{sA} z + \overline{Pe^{tA} x} \right) \right) - \psi \left(s^{-\gamma} \bar{f}_{g_2} \left(s, Pe^{sA} z + \overline{Pe^{tA} x} \right) \right) \right) \right. \\ & \quad \left. \left\langle (Q_{t-s}^{-1/2} \overline{Pe^{tA} Ck}, Q_{t-s}^{-1/2} Pz) \right\rangle_H \mathcal{N}(0, Q_{t-s})(dz) ds \right| + \kappa t \|g_1 - g_2\|_{C_\gamma^{0,1,C}} \\ & \leq \kappa (t + t^{1-\gamma}) \|g_1 - g_2\|_{C_\gamma^{0,1,C}}, \quad \forall (t, x) \in (0, T] \times \overline{H}. \end{aligned}$$

□

7 Applying partial smoothing to stochastic control problems

In this Section we first present the stochastic optimal control problem we aim to treat, then we show how the theory developed in the previous sections allows us to solve the associated HJB equation.

7.1 A stochastic control problem in the abstract setting

We consider the setting of Hypotheses 5.1-5.2-5.7 which we assume to hold. We present first the objective functional and then the state equation. The goal of the controller is to minimize the following finite horizon cost (here $X(\cdot; t, x)$ is the state process starting at time t with the datum x),

$$J(t, x; u) = \mathbb{E} \left(\int_t^T [\ell_0(s) + \ell_1(u(s))] ds + \phi(X(T; t, x)) \right) \quad (7.1)$$

over all controls $u(\cdot)$ in

$$\mathcal{U} := \{u : [0, T] \times \Omega \rightarrow U \subseteq K, \text{ progressively measurable}\} \quad (7.2)$$

under the following assumption.

Hypothesis 7.1 *We assume that:*

- (i) *the final cost ϕ belongs to $C_b^P(H)$ (see Definition 5.5);*
- (ii) *The current cost ℓ_0 is measurable and bounded;*
- (iii) *the set $U \subset K$ is closed and bounded and the current cost $\ell_1 : U \rightarrow \mathbb{R}$ is measurable and bounded from below.*

Remark 7.2 *Note that here the current cost does not depend on the state. Putting the dependence on the state in the current cost would increase considerably the technical arguments, namely the fixed point argument in the proof of Theorem 7.5, Section 7.2, wouldn't work, and it is left for a further paper.*

We also underline that the above technical problem cannot be overcome by transforming our Bolza type problem in a Mayer type problem (i.e. a problem with only the terminal cost) on the line of what is done e.g. in [14, Remark 7.4.1, p.714]. Indeed, using such transformation the state dependent running cost would disappear but the state equation would become nonlinear. This would prevent the use of our results on partial smoothing which, up to now, apply only to linear state equations.

Before introducing the state equation we observe that, due to Hypothesis 7.1-(i) above, what matters for the controller is the process $PX(\cdot)$. Now consider the following controlled SDE in the real separable Hilbert space H (here $0 \leq t \leq s \leq T$)

$$\begin{cases} dX(s) = [AX(s) + Cu(s)]ds + Q^{1/2}dW(s), & s \in (t, T], \\ X(s) = x \in H, & \forall s \in [0, t] \end{cases} \quad (7.3)$$

where A, G and W are as in Hypothesis 5.1 and $C \in \mathcal{L}(K, \overline{H})$ is as in Hypothesis 5.2. Equation (7.3) is formal and has to be considered in its mild formulation (using the so-called variation of constants, see e.g. [25, Chapter 7]) which still present some issues. Indeed the mild solution of (7.3) is, still formally,

$$X(s) = e^{(s-t)A}x + \int_t^s e^{(s-r)A}Cu(r)dr + \int_t^s e^{(s-r)A}Q^{1/2}dW(r), \quad s \in [t, T]. \quad (7.4)$$

Here the first and the third term belong to H , thanks to Hypothesis 5.1, while the second in general does not. By Hypothesis 5.2-(i) we see that the second term can be written as

$$\int_t^s \overline{e^{(s-r)A}Cu(r)}dr \in \overline{H}.$$

Hence, even when $x \in H$ the mild solution belongs to \overline{H} but not to H . Moreover, still using Hypothesis 5.2-(i), we see that the mild solution makes sense for all $x \in \overline{H}$ and belongs to \overline{H} .

On the other hand, thanks to Hypothesis 5.2-(iii), even when $x \in \overline{H}$ the process $PX(s)$ belongs to H and can be written as

$$PX(s) = \overline{Pe^{(s-t)A}x} + \int_t^s \overline{Pe^{(s-r)A}Cu(r)}dr + \int_t^s Pe^{(s-r)A}Q^{1/2}dW(r), \quad s \in (t, T]. \quad (7.5)$$

We define the value function related to this control problem, as usual, as

$$V(t, x) := \inf_{u \in \mathcal{U}} J(t, x; u). \quad (7.6)$$

7.2 Solution of the HJB equation

We define the Hamiltonian as follows: the current value Hamiltonian H_{CV} is

$$H_{CV}(p; u) := \langle p, u \rangle_K + \ell_1(u)$$

and the minimum value Hamiltonian is

$$H_{min}(p) = \inf_{u \in U} H_{CV}(p; u), \quad (7.7)$$

The HJB equation associated to the stochastic optimal control problem presented in the above Section 7.1 is then, formally,

$$\begin{cases} -\frac{\partial v(t, x)}{\partial t} = \mathcal{L}[v(t, \cdot)](x) + \ell_0(t) + H_{min}(\nabla^C v(t, x)), & t \in [0, T], x \in H, \\ v(T, x) = \phi(x), \end{cases} \quad (7.8)$$

Here the differential operator \mathcal{L} is the infinitesimal generator of the transition semigroup $(R_t)_{0 \leq t \leq T}$ defined in (5.4) related to the process Z solution of equation (5.2), namely \mathcal{L} is formally defined by

$$\mathcal{L}[f](x) = \frac{1}{2}Tr Q \nabla^2 f(x) + \langle x, A^* \nabla f(x) \rangle. \quad (7.9)$$

Definition 7.3 We say that a function $v : [0, T] \times H \rightarrow \mathbb{R}$ is a mild solution of the HJB equation (7.8) if the following are satisfied for some $\gamma \in (0, 1)$:

1. $v(T - \cdot, \cdot) \in C_\gamma^{0,1,C}([0, T] \times \overline{H})$;
2. the integral equation

$$v(t, x) = R_{T-t}[\phi](x) + \int_t^T R_{s-t} [H_{min}(\nabla^C v(s, \cdot)) + \ell_0(s)](x) ds, \quad (7.10)$$

is satisfied on $[0, T] \times H$.

We notice that the request in the previous definition 7.3, point 1, implies that $\nabla_x^C v(t, x)$ can blow up like $(T - t)^{-\gamma}$.

We now prove existence and uniqueness of a mild solution of the HJB equation (3.10) and (3.29) by a fixed point argument.

Remark 7.4 Since functions in $C_\gamma^{0,1,C}([0, T] \times \overline{H})$ are bounded (see Definition 4.5), the above Definition 7.3 requires, among other properties, that a mild solution is continuous and bounded up to T . This constrains the assumptions on the data, e.g. it implies that the final datum ϕ must be continuous and bounded. We may change this requirement in the above definition asking only measurability or only polynomial growth in x so allowing for more general datum ϕ in Hypothesis 7.1-(i). Our main results will remain true with straightforward modifications (see [34, Chapter 4] for a treatment of such cases in the case of bounded control operators).

Similarly we may weaken the request of Hypothesis 7.1-(iii) on the boundedness of the set U . This may result in the fact that the Hamiltonian H_{min} is not Lipschitz continuous. This case, even if more difficult, could still be treated using the ideas of [47, 71].

Theorem 7.5 *Let Hypotheses 5.1, 5.2, 5.7 and 7.1 hold true. Then the HJB equation (7.8) admits a mild solution v according to Definition 7.3. Moreover v is unique among the functions w such that $w(T - \cdot, \cdot) \in \Sigma_{T,\gamma}$ and it satisfies, for a suitable constant $\kappa_{1,T} > 0$, the estimate*

$$\|v(T - \cdot, \cdot)\|_{C_\gamma^{0,1,C}} \leq \kappa_{1,T} (\|\phi\|_\infty + \|\ell_0\|_\infty). \quad (7.11)$$

Proof. We first prove existence and uniqueness of a solution in $\Sigma_{T,\gamma}^1$, by using a fixed point argument in it. To this aim, first we rewrite (7.10) in a forward way. Namely if v satisfies (7.10) then, setting $w(t, x) := v(T - t, x)$ for any $(t, x) \in [0, T] \times H$, we get that w satisfies

$$w(t, x) = R_t[\phi](x) + \int_0^t R_{t-s}[H_{min}(\nabla^C w(s, \cdot)) + \ell_0(s)](x) ds, \quad t \in [0, T], x \in H, \quad (7.12)$$

which is the mild form of the forward HJB equation

$$\begin{cases} \frac{\partial w(t, x)}{\partial t} = \mathcal{L}[w(t, \cdot)](x) + \ell_0(t) + H_{min}(\nabla^C w(t, x)), & t \in [0, T], x \in H, \\ w(0, x) = \phi(x). \end{cases} \quad (7.13)$$

Referring to equation (7.12), which is the mild version of (7.10), define the map Υ on $\Sigma_{T,\gamma}^1$ by setting, for $g \in \Sigma_{T,\gamma}^1$,

$$\Upsilon[g](0, x) := \phi(x),$$

and, for $(t, x) \in (0, T] \times \overline{H}$,

$$\Upsilon[g](t, x) := R_t[\phi](x) + \int_0^t \ell_0(s) ds + \int_0^t R_{t-s}[H_{min}(\nabla^C g(s, \cdot))](x) ds. \quad (7.14)$$

Using Proposition 5.9, in particular (5.11), (5.8) and the last statement on continuity, we see that the sum of the first two terms of (7.14) belongs to $\Sigma_{T,\gamma}^1$ with

$$\begin{aligned} f(t, x) &= \int_H \bar{\phi}(z_1 + x) \mathcal{N}(0, PQ_t P^*)(dz_1) + \int_0^t \ell_0(s) ds, \\ \bar{f}(t, x) &= \int_H \bar{\phi}(z_1 + x) \left\langle \Lambda^{P,C}(t), (PQ_t P^*)^{-1/2} z_1 \right\rangle \mathcal{N}(0, PQ_t P^*)(dz_1), \end{aligned}$$

Moreover, we use Lemma 6.4 (simply substituting the generic function ψ with H_{min}) to deduce that the third term of (7.14) belongs to $\Sigma_{T,\gamma}^1$. Hence Υ is well defined in $\Sigma_{T,\gamma}^1$ with values in $\Sigma_{T,\gamma}^1$ itself.

As stated in Remark 6.3, $\Sigma_{T,\gamma}^1$ is a closed subspace of $C_\gamma^{0,1,C}([0, T] \times \overline{H})$, and so if Υ is a contraction, by the Contraction Mapping Principle there exists a unique (in $\Sigma_{T,\gamma}^1$) mild solution of (7.8). We then prove the contraction property of Υ .

Let $g_1, g_2 \in \Sigma_{T,\gamma}^1$. We evaluate

$$\|\Upsilon(g_1) - \Upsilon(g_2)\|_{\Sigma_{T,\gamma}^1} = \|\Upsilon(g_1) - \Upsilon(g_2)\|_{C_\gamma^{0,1,C}}.$$

For every $(t, x) \in (0, T] \times \overline{H}$, we have

$$\Upsilon(g_1)(t, x) - \Upsilon(g_2)(t, x) = \int_0^t R_{t-s} [H_{min}(\nabla^C g_1(s, \cdot)) - H_{min}(\nabla^C g_2(s, \cdot))](x) ds$$

Hence we can use the second part of Lemma 6.4, namely estimate (6.4), to get

$$|\Upsilon(g_1)(t, x) - \Upsilon(g_2)(t, x)| + t^\gamma |\nabla^C \Upsilon(g_1)(t, x) - \nabla^C \Upsilon(g_2)(t, x)|_{K^*} \leq \kappa(t + t^{1-\gamma}) \|g_1 - g_2\|_{C_\gamma^{0,1,C}}. \quad (7.15)$$

Hence, if T is sufficiently small, we get that the map Υ is a contraction in $\Sigma_{T,\gamma}^1$ and, if we denote by w its unique fixed point, then $v := w(T - \cdot, \cdot)$ turns out to be a mild solution of the HJB equation (7.8), according to Definition 7.3.

Since the constant C is independent of t , the case of generic $T > 0$ follows by dividing the interval $[0, T]$ into a finite number of subintervals of length δ sufficiently small, or equivalently, as done in [69], by taking an equivalent norm with an adequate exponential weight, such as

$$\|f\|_{\eta, C_\gamma^{0,1,C}} = \sup_{(t,x) \in (0,T] \times \overline{H}} |e^{\eta t} f(t, x)| + \sup_{(t,x) \in (0,T] \times \overline{H}} e^{\eta t} t^\gamma \|\nabla^C f(t, x)\|_{K^*}.$$

□

A Appendix: verifying Hypothesis 5.7 in our examples

A.1 The case of boundary control

We consider the stochastic heat equation with boundary control (3.1), reformulated as an abstract evolution equation (3.2), where we choose, according to Subsection 3.1 and Remark 5.4,

$$H = L^2(\mathcal{O}), \quad \bar{H} = \mathcal{D}\left((-A_0)^{-3/4-\varepsilon}\right) = H^{-3/2-2\varepsilon}(\mathcal{O}) \quad (\text{for suitable small } \varepsilon > 0),$$

$A = A_0$, $C = B = (-A_0)D$ as from (3.3). Moreover, as from Remark 5.8 we take $Q = (-A)^{-2\beta}$ for some $\beta \geq 0$ and P a projection on a finite dimensional subspace contained in $(-A)^{-\alpha}$ for some $\alpha > \beta + \frac{1}{4}$.

The covariance operator Q_t is given by

$$Q_t = \int_0^t (-A_0)^{-2\beta} e^{2sA_0} ds = (-A_0)^{-2\beta-1} (I - e^{2tA_0}). \quad (\text{A.1})$$

Notice that it can be deduced by the strong Feller property of the heat transition semigroup that $\text{Im } e^{tA} \subset \text{Im } Q_t^{1/2}$, see e.g. [25, Section 9.4 and Appendix B] for a comprehensive bibliography. Now we estimate $\|Q_t^{-1/2} e^{tA_0} (-A_0 D)\|$.

In the sequel we denote by $\lambda_k \geq 0$, $k \geq 1$, $\lambda_k \nearrow +\infty$, the opposite of the eigenvalues of the Laplace operator in \mathcal{O} :

$$A_0 e_k = -\lambda_k e_k, \quad k \geq 1.$$

Lemma A.1 *Let Q_t be defined in (A.1). For every $\varepsilon \in (0, \frac{1}{4})$, we get, for some $C_0 > 0$,*

$$\|Q_t^{-1/2} e^{tA_0} (-A_0 D)\| \leq C_0 t^{-\frac{5}{4}-\beta-\varepsilon}, \quad (\text{A.2})$$

Proof. We notice that $Q_t^{-1/2} e^{tA_0} (-A_0 D) = Q_t^{-1/2} e^{tA_0} (-A_0)^{\frac{3}{4}+\varepsilon} D_\varepsilon$, where $D_\varepsilon = (-A_0)^{\frac{1}{4}-\varepsilon} D$ is bounded $\forall \varepsilon \in (0, \frac{1}{4})$. Moreover for every $a \in \partial \mathcal{O}$

$$\begin{aligned} |Q_t^{-1/2} e^{tA_0} (-A_0)^{\frac{3}{4}+\varepsilon} D_\varepsilon a|^2 &= \sum_{k=1}^{+\infty} \frac{\lambda_k^{1+2\beta+\frac{3}{2}+2\varepsilon} e^{-2t\lambda_k}}{1 - e^{-2\lambda_k}} |(D_\varepsilon a)_k|^2 \\ &= \frac{1}{t^{\frac{5}{2}+2\beta+2\varepsilon}} \sum_{k=1}^{+\infty} \frac{(t\lambda_k)^{(\frac{5}{2}+2\beta+2\varepsilon)} e^{-2t\lambda_k}}{1 - e^{-2t\lambda_k}} |(D_\varepsilon a)_k|^2 \\ &= \frac{1}{t^{\frac{5}{2}+2\beta+2\varepsilon}} \sum_{k=1}^{+\infty} \frac{(t\lambda_k)^{(\frac{5}{2}+2\beta+2\varepsilon)}}{e^{2t\lambda_k} - 1} |(D_\varepsilon a)_k|^2 \\ &\leq \frac{1}{t^{\frac{5}{2}+2\beta+2\varepsilon}} \sup_{x \geq 0} \frac{x^{\frac{5}{2}+2\beta+2\varepsilon}}{e^x - 1} \sum_{k=1}^{+\infty} |(D_\varepsilon a)_k|^2 \\ &\leq \frac{1}{t^{\frac{5}{2}+2\varepsilon}} \sup_{x \geq 0} \frac{x^{\frac{5}{2}+2\varepsilon}}{e^x - 1} |(D_\varepsilon a)|^2 \leq C_0 \frac{1}{t^{\frac{5}{2}+2\beta+2\varepsilon}} |a|^2. \end{aligned}$$

So we can conclude that as $t \rightarrow 0$, $\forall a \in \partial \mathcal{O}$, $|Q_t^{-1/2} e^{tA_0} (-A_0 D) a|^2$ blows up at most like $t^{-5/2-2\beta-2\varepsilon}$ and so the claim follows. \square

Now we introduce the operator P . Let $\alpha > 0$, let $v_1, \dots, v_n \in D((-A_0)^\alpha)$ be linearly independent, and let P be the projection on the span of $\langle v_1, \dots, v_n \rangle$, namely

$$P : H \rightarrow H, \quad Px = \sum_{i=1}^n \langle x, v_i \rangle v_i, \quad \forall x \in H. \quad (\text{A.3})$$

We set moreover, noticing that $P = P^*$,

$$\bar{Q}_t := PQ_t P = P(-A)^{-1-\beta} (I - e^{2tA}) P \quad (\text{A.4})$$

Notice that $P_\alpha := (-A_0)^\alpha P$ is a continuous operator on H . Hence

$$\begin{aligned} \overline{Pe^{tA_0}B_0} &= Pe^{tA_0}(-A_0)^{\frac{3}{4}+\varepsilon}((-A_0)^{\frac{1}{4}-\varepsilon}D), & (\overline{Pe^{tA_0}B_0})^* &= ((-A_0)^{\frac{1}{4}-\varepsilon}D)^*(-A_0)^{\frac{3}{4}+\varepsilon-\alpha}e^{tA_0}P_\alpha \\ \langle \overline{Q_t}P^*x, P^*x \rangle &= \langle (I - e^{2tA_0})(-A_0)^{-1-2\alpha-\beta}P_\alpha x, P_\alpha x \rangle \end{aligned}$$

The aim now is to verify that $\text{Im } \overline{Pe^{tA_0}}(-A_0D) \subset \text{Im } \overline{Q_t}^{1/2}$ and to estimate $\|\overline{Q_t}^{-1/2}Pe^{tA_0}(-A_0D)\|$.

Lemma A.2 *Let $\overline{Q_t}$ be defined in (A.4). Let $\alpha > \beta + \frac{1}{4}$. Then, for $\varepsilon \in (0, \frac{1}{4})$,*

$$\text{Im } \overline{Pe^{tA}}(-A_0D) \subset \text{Im } \overline{Q_t}^{1/2}, \quad \|\overline{Q_t}^{-1/2}\overline{Pe^{tA}}(-A_0D)\| \leq \frac{C}{t^{1-\varepsilon}}. \quad (\text{A.5})$$

Proof. In the proof we consider the case of the projection on the space generated by only one element $v \in D((-A)^\alpha)$, namely in the proof $P : H \rightarrow H$, $Px = \langle x, v \rangle v, \forall x \in H$, the extension to a map as in (A.3) being direct.

We notice that

$$\langle \overline{Q_t}x, x \rangle = |\overline{Q_t}^{1/2}x|^2 = |\langle x, v \rangle|^2 \sum_{k \geq 1} \frac{1 - e^{-2t\lambda_k}}{\lambda_k^{1+2\beta}} v_k^2$$

and so

$$\begin{aligned} |\overline{Q_t}^{-1/2}v|^2 &= \frac{1}{|v|^2} \frac{1}{\sum_{k \geq 1} \frac{1 - e^{-2t\lambda_k}}{\lambda_k^{1+2\beta}} v_k^2} \leq \frac{1}{t^{1+2\beta}|v|^2} \frac{1}{\sum_{k \geq 1} \frac{e^{2t\lambda_k} - 1}{(t\lambda_k)^{1+2\beta} e^{2t\lambda_k}} v_k^2} \\ &\leq \frac{1}{t^{1+2\beta}|v|^2} \frac{1}{\sum_{k \geq 1} \frac{e^{2t\lambda_k} - 1}{(t\lambda_k)^{1+2\beta}} v_k^2} \leq \frac{1}{t^{1+2\beta}|v|^2} \frac{1}{\sum_{k \geq 1} v_k^2} \leq C_0 \frac{1}{t^{1+2\beta}|v|^4} \end{aligned} \quad (\text{A.6})$$

Moreover we can write, $\forall a \in \partial \mathcal{O}$ and setting $D_\varepsilon := (-A_0)^{\frac{1}{4}-\varepsilon}$

$$\overline{Pe^{tA_0}}(-A_0D)a = Pe^{tA_0}(-A_0)^{\frac{3}{4}+\varepsilon}D_\varepsilon a = \left(\sum_{k \geq 1} \lambda_k^{\frac{3}{4}+\varepsilon} e^{-t\lambda_k} (D_\varepsilon a)_k v_k \right) v,$$

so it is immediate to see that

$$\text{Im } \overline{Pe^{tA_0}B} \subset \text{Im } \overline{Q_t}^{1/2}.$$

Taking into account that D_ε is a bounded operator and that $v \in D((-A)^\alpha)$ we get

$$\begin{aligned} \left(\sum_{k \geq 1} \lambda_k^{\frac{3}{4}+\varepsilon} e^{-t\lambda_k} (D_\varepsilon a)_k v_k \right)^2 &= \left(\sum_{k \geq 1} \lambda_k^{\frac{3}{4}+\varepsilon-\alpha} e^{-t\lambda_k} (D_\varepsilon a)_k (\lambda_k)^\alpha v_k \right)^2 \\ &= \left| \sum_{k \geq 1} \lambda_k^{\frac{3}{4}+\varepsilon-\alpha} e^{-t\lambda_k} (D_\varepsilon a)_k (k^2)^\alpha v_k \right|^2 \\ &\leq \left(\sum_{k \geq 1} \lambda_k^{\frac{3}{2}+2\varepsilon-2\alpha} e^{-2t\lambda_k} |(D_\varepsilon a)_k|^2 \right) \left(\sum_{k \geq 1} \lambda_k^{2\alpha} v_k^2 \right) \\ &\leq C_0 \frac{1}{t^{\frac{3}{2}+2\varepsilon-2\alpha}} \left(\sum_{k \geq 1} (t\lambda_k)^{\frac{3}{2}+2\varepsilon-2\alpha} e^{-2t\lambda_k} |(D_\varepsilon a)_k|^2 \right) |A^\alpha v|^2. \end{aligned}$$

So by these calculations, and by estimate (A.6) we get, with the constant C_0 independent on t and that may change value from line to line,

$$\begin{aligned} |\overline{Q_t}^{-1/2}\overline{Pe^{tA}}(-A_0D)a|^2 &\leq C_0 \frac{1}{t^{1+2\beta}|v|^4} \frac{1}{t^{\frac{3}{2}+2\varepsilon+2\beta-2\alpha}} \sum_{k \geq 1} (t\lambda_k)^{\frac{1}{2}+2\varepsilon+2\beta-2\alpha} e^{-2t\lambda_k} |(D_\varepsilon a)_k|^2 |A^\alpha v|^2 \\ &\leq C_0 \frac{1}{|v|^4} \frac{1}{t^{\frac{3}{2}+2\beta+2\varepsilon-2\alpha}} \sup_{x \geq 0} x^{\frac{3}{2}+2\varepsilon-2\alpha} e^{-2x} |D_\varepsilon a|^2. \end{aligned}$$

Choosing $\alpha > \frac{1}{4} + \beta = 1 - 2\varepsilon$ we conclude the proof. \square

A.2 The case of delay in the control

We consider equation (3.24) in Section 3.2 and we notice that for every $(x_0, x_1) \in H$ the covariance operator Q_t of the stochastic convolution can be written as

$$Q_t(x_0, x_1) = (Q_t^0 x_0, 0), \quad (\text{A.7})$$

where Q_t^0 is the selfadjoint operator in \mathbb{R}^n defined as

$$Q_t^0 := \int_0^t e^{sa_0} \sigma \sigma^* e^{sa_0^*} ds, \quad (\text{A.8})$$

see [51], Lemma 4.6. So

$$\text{Im } Q_t = \text{Im } Q_t^0 \times \{0\} \subseteq \mathbb{R}^n \times \{0\}.$$

Lemma A.3 *The operator Q_t^0 defined in (A.8) is invertible for all $t > 0$ if and only if*

$$\text{Im}(\sigma, a_0 \sigma, \dots, a_0^{n-1} \sigma) = \mathbb{R}^n, \quad (\text{A.9})$$

Let P be the projection on the first component: $\forall (x_0, x_1) \in H$, $P(x_0, x_1) = (x_0, 0)$. Then (5.7) holds if and only if

$$\text{Im} \left(e^{ta_0} b_0 + \int_{-d}^0 1_{[-t, 0]} e^{(t+r)a_0} b_1(dr) \right) \subseteq \text{Im}(\sigma, a_0 \sigma, \dots, a_0^{n-1} \sigma). \quad (\text{A.10})$$

If moreover

$$\text{Im} \left(e^{ta_0} b_0 + \int_{-d}^0 1_{[-t, 0]} e^{(t+r)a_0} b_1(dr) \right) \subseteq \text{Im } \sigma, \quad \forall t > 0. \quad (\text{A.11})$$

then

$$\|(PQ_t P^*)^{-1/2} \overline{P e^{tA_1} B_1}\| \leq C t^{-1/2}. \quad (\text{A.12})$$

Proof. In this case (5.7) is written as

$$\text{Im } \overline{P e^{tA_1} B_1} \subseteq \text{Im}(PQ_t P^*)^{1/2}, \quad \forall t > 0.$$

Recalling (3.22) we see that $\overline{P e^{tA_1} B_1}$ is a bounded operator and can be written as

$$\overline{P e^{tA} B_1} = \left(e^{ta_0} x_0 + \int_{-d}^0 1_{[-t, 0]} e^{(t+s)a_0} b_1(ds), 0 \right):$$

Hence the inclusion follows exactly as in [51] in the case when B_1 is a bounded operator: indeed, in [51], formula (4.35), Proposition 4.11 the only difference is that in the present paper $b_1(\cdot)$ is not necessarily absolutely continuous with respect to the Lebesgue measure) but this does not affect the image of $\overline{P e^{tA} B_1}$. Hence, by [51], Proposition 4.11, it immediately follows (A.12). \square

References

- [1] N.U. Ahmed, *Optimal Control of Infinite Dimensional Stochastic Systems Via Generalized Solutions of HJB Equations*, Differential Inclusions, Control and Optimization, 21 (2001) 97-126.
- [2] N.U. Ahmed, *Generalized solutions of HJB equations applied to stochastic control on Hilbert space*, Nonlinear Analysis 54 (2003) 495 - 523.
- [3] N.U. Ahmed, *Nonlinear Diffusion Governed by McKean Vlasov Equation on HilbertSpace and Optimal Control*, SIAM J. Control Optim., 2007 Vol. 46, No. 1, pp. 356-378.
- [4] N.U. Ahmed, *Stochastic initial boundary value problems subject to distributed and boundary noise and their optimal control*, (JMAA) J. Math. Anal. Appl. 421 (2015) 157-179.
- [5] A. Alla, M. Falcone, D. Kalise, *A HJB-POD feedback synthesis approach for the wave equation*, Bull. Braz. Math. Soc. (N.S.) 47 (2016), no. 1, 51-64.

- [6] M. Bambi, C. Di Girolami, S. Federico, F. Gozzi, *Generically distributed investments on flexible projects and endogenous growth*. *Econom. Theory* 63 (2017), no. 2, 521-558.
- [7] E. Bandini, A. Cosso, M. Fuhrman, H. Pham, *Backward SDEs for optimal control of partially observed path-dependent stochastic systems: a control randomization approach*. *Ann. Appl. Probab.* 28 (2018), no. 3, 1634-1678.
- [8] E. Barucci, F. Gozzi *Optimal advertising with a continuum of goods*. *Annals of Operations Research* 88, 15-29, 1999.
- [9] V.I. Bogachev, *Differentiable Measures and the Malliavin Calculus*, American Mathematical Society, 2010.
- [10] Boucekkine, R., Camacho, C., Fabbri, G., *Spatial dynamics and convergence: The spatial AK model*, *Journal of Economic Theory*, Elsevier, vol. 148(6),2013, pages 2719-2736.
- [11] Brito, P. *The dynamics of growth and distribution in a spatially heterogeneous world*. Working Papers Department of Economics 2004/14. ISEG, University of Lisbon.
- [12] P. Cannarsa and G. Da Prato, *Second order Hamilton-Jacobi equations in infinite dimensions*, *SIAM J. Control Optim.* 29, 2, (1991), pp. 474-492.
- [13] P. Cannarsa and G. Da Prato, *Direct solution of a second order Hamilton-Jacobi equations in Hilbert spaces*, *Stochastic Partial Differential Equations and Applications*, (1992), pp. 72-85.
- [14] P. Cannarsa, C. Sinestrari *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control* Progress in nonlinear differential equations and their applications; v. 58. Birkhäuser Boston, Inc., Boston, MA, 2004.
- [15] R. Carmona, J.-P. Fouque, M. Mousavi, and L.-H. Sun. *Systemic Risk and Stochastic Games with Delay*, *Journal of Optimization and Applications (JOTA)*, Vol. 179(2), 2018, (p.336-399).
- [16] S. Cerrai, *Second Order PDEs in Finite and Infinite Dimension: a Probabilistic Approach*, *Lecture Notes in Mathematics*, vol. 1762 (Springer, Berlin, 2001) 107.
- [17] F. T.S. Chan, Z. Wang, J. Zhang. A two-level hedging point policy for controlling a manufacturing system with time-delay, demand uncertainty and extra capacity. *European Journal of Operational Research* 176 (2007) 1528-1558.
- [18] L. Chen, Z. Wu, *Maximum principle for the stochastic optimal control problem with delay and application*, *Automatica*, vol. 146, 6, pp. 1074-1080, (2010).
- [19] L. Chen, Z. Wu, *Stochastic Optimal Control Problem in Advertising Model with Delay*, *Journal of Systems Science and Complexity* volume 33, pages 968-987 (2020).
- [20] P.-L. Chow, J.-L. Menaldi, *Infinite-dimensional Hamilton-Jacobi-Bellman equations in Gauss-Sobolev spaces*. *Nonlinear Anal.* 29(4), 415-426 (1997).
- [21] P. L. Chow, J. L. Menaldi, *Stochastic PDE for nonlinear vibration of elastic panels*. *Differential Integral Equations*, 12, (2014), 419-434.
- [22] A Cosso, F Gozzi, I Kharroubi, H Pham, M Rosestolato. *Optimal control of path-dependent McKean-Vlasov SDEs in infinite dimension*. arXiv preprint arXiv:2012.14772
- [23] G. D. Prato and A. Debussche, *Control of the stochastic burgers model of turbulence*, *SIAM Journal on Control and Optimization*, vol. 37, no. 4, pp. 1123-1149, 1999.
- [24] G. Da Prato and J. Zabczyk, *Second order partial differential equations in Hilbert spaces*. London Mathematical Society Note Series, 293, Cambridge University Press, Cambridge, 2002.
- [25] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions. Second Edition*. *Encyclopedia of Mathematics and its Applications* 152, Cambridge University Press, 2014.
- [26] G. Da Prato and J. Zabczyk (1991) *Smoothing properties of transition semigroups in Hilbert spaces*, *Stochastics and Stochastic Reports*, 35:2, 63-77,

- [27] K.D. Do. *Hamilton-Jacobi Equation for Optimal Control of Nonlinear Stochastic Distributed Parameter Systems Applied to Air Pollution Process*. Applied Mathematical Sciences, Vol. 8, 2014, no. 57, 2801 - 2816.
- [28] K.D. Do. *Stochastic boundary control design for extensible marine risers in three dimensional space*. Automatica 77 (2017) 184-197.
- [29] K.D. Do. *Stochastic boundary control design for Timoshenko beams with large motions*. Journal of Sound and Vibration, 2017, 402-C, 164-184.
- [30] Engel, K.; Nagel, R., *One-parameter semigroups for linear evolution equations*. Graduate Texts in Mathematics, 194, Springer-Verlag, New York, 2000.
- [31] E.N. Evans, A.P. Kendall and E. A. Theodorou, *Stochastic Spatio-Temporal Optimization for Control and Co-Design of Systems in Robotics and Applied Physics*, Arxiv working paper, <https://arxiv.org/abs/2102.09144>, 2021.
- [32] E. N. Evans, M. A. Pereira, G. I. Boutselis, and E. A. Theodorou, *Variational optimization based reinforcement learning for infinite dimensional stochastic systems*, in Conference on Robot Learning, 2019.
- [33] E. N. Evans, A. P. Kendall, G. I. Boutselis, and E. A. Theodorou, *Spatio-temporal stochastic optimization: theory and applications to optimal control and co-design*, in Proceedings of Robotics: Science and Systems, 2020.
- [34] G. Fabbri, F. Gozzi, A. Swiech *Stochastic Optimal Control in Infinite Dimensions: Dynamic Programming and HJB Equations*. Springer 2017.
- [35] S. Faggian, *Regular solutions of first-order Hamilton-Jacobi equations for boundary control problems and applications to economics*. Appl. Math. Optim. 51 (2005), no. 2, 123–162.
- [36] S. Faggian, *Application of dynamic programming to economic problems with vintage capital*. Dynamics of Continuous, Discrete and Impulsive Systems
- [37] S. Faggian, F. Gozzi, P. Kort *Optimal investment with vintage capital: equilibrium distributions*. J.Math. Econ., forthcoming.
- [38] S. Faggian, L.Grosset, *Optimal advertising strategies with age-structured goodwill*. Math. Methods Oper. Res. 78 (2013), no. 2, 259-284.
- [39] S. Federico, *A stochastic control problem with delay arising in a pension fund model*. Finance Stoch. 15 (2011), no. 3, 421–459
- [40] S. Federico, B. Goldys, F. Gozzi *HJB equations for the optimal control of differential equations with delays and state constraints, I: regularity of viscosity solutions*. SIAM J. Control Optim. 48 (2010), no. 8, 4910–4937.
- [41] S. Federico, F. Gozzi, *Mild solutions of semilinear elliptic equations in Hilbert spaces*. J. Differential Equations 262 (2017), no. 5, 3343-3389.
- [42] S. Federico, F. Gozzi, *Verification theorems for stochastic optimal control problems in Hilbert spaces by means of a generalized Dynkin formula*. Ann. Appl. Probab. 28 (2018), no. 6, 3558-3599
- [43] G. Feichtinger, F. Hartl, S. P. Sethi, *Dynamic Optimal Control Models in Advertising: Recent Developments*. Management Science , Vol. 40, No. 2 (Feb., 1994), pp. 195-226.
- [44] M. Fuhrman and G. Tessitore, *Generalized directional gradients, backward stochastic differential equations and mild solutions of semilinear parabolic equations*. Appl. Math. Optim. 51 (2005), no. 3, 279–332.
- [45] B. Goldys and F.Gozzi, *Second order parabolic Hamilton-Jacobi-Bellman equations in Hilbert spaces and stochastic control: L^2_μ approach*. Stochastic processes and Applications, 116 (2006) 1932–1963.

- [46] F. Gozzi, *Regularity of solutions of second order Hamilton-Jacobi equations in Hilbert spaces and applications to a control problem*, (1995) Comm Partial Differential Equations 20, pp. 775-826.
- [47] F. Gozzi, *Global regular solutions of second order Hamilton-Jacobi equations in Hilbert spaces with locally Lipschitz nonlinearities*, (1996) J. Math. Anal. Appl. 198, pp. 399-443.
- [48] F. Gozzi, M. Leocata *A Stochastic Model of Economic Growth in Time-Space*, SIAM J. Control Optim. 60 (2022), no. 2, 620-651.
- [49] F. Gozzi and C. Marinelli, *Stochastic optimal control of delay equations arising in advertising models*. Stochastic partial differential equations and applications - VII, 133-148, Lect. Notes Pure Appl. Math., 245, Chapman & Hall/CRC, Boca Raton, FL, 2006.
- [50] F. Gozzi, C. Marinelli, S. Savin *On controlled linear diffusions with delay in a model of optimal advertising under uncertainty with memory effects*. J. Optim. Theory Appl. 142 (2009), no. 2, 29-321.
- [51] F. Gozzi, F. Masiero *Stochastic Optimal Control with Delay in the Control, I: solving the HJB equation through partial smoothing*. SIAM J. Control Optim. 55 (2017), no. 5, pp. 2981-3012.
- [52] F. Gozzi, F. Masiero *Stochastic Optimal Control with Delay in the Control, II: Verification Theorem and Optimal Feedback Controls*. SIAM J. Control Optim. 55 (2017), no. 5, pp. 3013-3038.
- [53] F. Gozzi, E. Rouy and A. Swiech. *Second Order Hamilton-Jacobi Equations in Hilbert Spaces and Stochastic Boundary Control* SIAM J. Control Optim., 38(2), 400-430. (31 pages)
- [54] F. Gozzi and A. Swiech. *Hamilton-Jacobi-Bellman equations for the optimal control of the Duncan-Mortensen-Zakai equation* Journal of Functional Analysis 172 (2), 466-510.
- [55] L. Gross, *Potential theory on Hilbert space*, J. Funct. Anal. 1 (1967), 123-181.
- [56] L. Grosset, B. Viscolani, *Advertising for the introduction of an age-sensitive product*. Optimal Control Appl. Methods 26 (2005), no. 3, 157-167.
- [57] J. Huang, M. Leng, L. Liang. *Recent developments in dynamic advertising research*. European Journal of Operational Research. Volume 220, Issue 3, 1 August 2012, Pages 591-609.
- [58] N. Ikeda, S. Watanabe, *Stochastic differential equations and diffusion processes. Second edition*. North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989.
- [59] S. Kaplan, *On the second dual of the space of continuous functions*. Trans. Amer. Math. Soc. 86 (1957), 70-90.
- [60] S. Kaplan, *The second dual of the space of continuous functions. II*. Trans. Amer. Math. Soc. 93 (1959), 329-350.
- [61] S. Kaplan, *Closure properties of $C(X)$ in its second dual*. Proc. Amer. Math. Soc. 17 (1966), 401-406.
- [62] T. Kato, *Perturbation Theory for Linear Operators*, Springer Verlag, New York, 1976.
- [63] H. Lhachemi, C. Prieur, E. Trélat, *PI regulation of a reaction-diffusion equation with delayed boundary control*. IEEE Trans. Automat. Control 66 (2021), no. 4, 1573-1587
- [64] H. Lhachemi, R. Shorten, *Boundary feedback stabilization of a reaction-diffusion equation with Robin boundary conditions and state-delay*. Automatica J. IFAC 116 (2020) 108931, 9 pp.
- [65] F. Lamoline, J.J. Winkin *Well-posedness of boundary controlled and observed stochastic port-Hamiltonian systems*. IEEE Trans. Automat. Control 65 (2020), no. 10, 4258-4264.
- [66] X. Li, X. Mao, *Stabilisation of highly nonlinear hybrid stochastic differential delay equations by delay feedback control*. Automatica 112 (2020), 108657, 11 pp.
- [67] X. Liu, C. Xie, *Boundary control of reaction-diffusion equations on higher-dimensional symmetric domains*. Automatica 114 (2020), 108832, 8 pp.

- [68] A. Lunardi, M. Roeckner *Schauder theorems for a class of (pseudo-)differential operators on finite and infinite dimensional state spaces-* Arxiv Preprint n. 1907.06237.
- [69] F. Masiero, *Semilinear Kolmogorov equations and applications to stochastic optimal control.* Appl. Math. Optim. 51 (2005), no. 2, 201-250.
- [70] F. Masiero, *Infinite horizon stochastic optimal control problems with degenerate noise and elliptic equations in Hilbert spaces.* Appl. Math. Optim. 55 (2007), no. 3, 285-326.
- [71] F. Masiero, A. Richou, *HJB equations in infinite dimensions with locally Lipschitz Hamiltonian and unbounded terminal condition.* J. Differential Equations 257 (2014), no. 6, 1989-2034.
- [72] D. Machowska. *Delayed effects of cooperative advertising in goodwill dynamics.* Operations Research Letters 47 (2019) 178-184.
- [73] S. J. Moura and H. K. Fathy, *Optimal boundary control of reaction-diffusion partial differential equations via weak variations,* Journal of Dynamic Systems, Measurement, and Control, vol. 135, no. 3, p. 034501, 2013.
- [74] R. Neck, *Stochastic control theory and operational research.* European Journal of Operational Research 17:3, (1984), 283-301.
- [75] M. Nisio, *Stochastic Control Theory: Dynamic Programming Principle,* Probability Theory and Stochastic Modelling, vol. 72 (Springer, Berlin, 2015)
- [76] M. Rosetolato, A. Swiech *Partial regularity of viscosity solutions for a class of Kolmogorov equations arising from mathematical finance,* Journal of Differential Equations 262 (3), 1897-1930.
- [77] Y. Saporito, J. Zhang, *Stochastic control with delayed information and related nonlinear master equation.* SIAM J. Control Optim. 57 (2019), no. 1, pp. 693-717.
- [78] J. H. Seinfeld and S. N. Pandis, *Atmospheric chemistry and physics: From air pollution to climate change,* John Wiley & Sons, New York, 1998.
- [79] TC. Shannon, *The second dual of $C(X)$.* Pacific J. Math. 72 (1977), no. 1, 237-253.
- [80] S.S. Sritharan, *An introduction to deterministic and stochastic control of viscous flow,* in Optimal Control of Viscous Flow, ed. by S.S. Sritharan (Society for Industrial and Applied Mathematics, Philadelphia, 1998), pp. 1-42.
- [81] A Swiech. *Viscosity Solutions to HJB Equations for Boundary-Noise and Boundary-Control Problems.* SIAM Journal on Control and Optimization, 58 (2020) 1, pp. 303-326.
- [82] R. B. Vinter and R. H. Kwong, *The infinite time quadratic control problem for linear systems with state and control delays: an evolution equation approach.* SIAM J. Control Optim., 19 (1):139-153, 1981.
- [83] H. S. Yan. *Hierarchical Stochastic Production Planning with Delay Interaction.* Journal of Optimization Theory and Application: Vol. 104, No. 3, 2000, pp. 659-689.
- [84] J. Zabczyk, *Mathematical Control Theory: an Introduction,* Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992.