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Rationality of Boundaries

Surname PEREGO Name DAVIDE

Registration number 861022

Supervisor: Prof. Francesco Matucci

Co-Supervisor: Prof. James Belk

Coordinator: Prof. Pierluigi Colli

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RATIONALITY OF BOUNDARIES

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CONTENTS

LIST OF SYMBOLS

Metric Spaces and Graphs

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{S}^1$, $[0,l]$, $[0,\infty[$	naturals, integers, real numbers, circle, interval, half-line
i, j, k, n, m	elements of $\mathbb{N}$ and indices
t, $\tilde{t}$, t_1, t_2, l	elements of $\mathbb{R}$ and $[0,l]$
X, Y	metric spaces
Γ, Γ_0	graph and subgraph
$\lvert\Gamma\rvert$	geometric realization of Γ
A, B	subsets
K	compact subset
x, y, z	elements of metric spaces and graphs
x_0	distinguished point of a metric space or a graph
$B_k(x)$	(closed) ball of radius k centered in x
B_k	(closed) ball of radius k centered in x_0
S_k	sphere of radius k centered in x_0
$[x,y]$	(combinatorial) geodesic between x and y
$[x,y](t)$	point on a geodesic between x and y
$c:[0,l]\to X$	path
$\ell(c)$	path length
d, $\underline{\mathrm{d}}$	distance, distance using sequences
$\widehat{\mathrm{d}}$	intrinsic metric
X/d, $\sim_{\mathrm{d}}$	metric quotient, quotient relation
$C(x)$	cone of x
$\mathcal{C}_k$	connected component of $\Gamma - B_{k-1}$
$\mathcal{E}_{\mathrm{T}}(\Gamma)$, $\mathcal{E}(\Gamma)$	topological ends of Γ, ends of Γ

Groups and Hyperbolic Groups

G	(finitely presented) group
$\mathcal{R}$	group of rational homeomorphisms
X/G	quotient by an action
S, S_1, S_2	(symmetric) sets of generators
$\Gamma(G,S)$, $\Gamma(G)$	Cayley Graph of G with respect to S
δ	hyperbolicity constant in the triangle condition
$\tilde{\delta}$	hyperbolicity constant for the hyperbolic inequality ($\mathfrak{H}$)

$\sim_\Gamma$	Gromov sequences equivalence relation
γ, η	geodesic rays
$\gamma(t)$	element of the geodesic ray
$\partial\Gamma$	Gromov boundary
x_∞, y_∞, $[\gamma]$, $[\eta]$	elements of $\partial\Gamma$
x_*, y_*	elements of $\Gamma \cup \partial\Gamma$
Horofunction Boundary and Atoms	
$\mathcal{F}$	functions from Γ to $\mathbb{Z}$
$\overline{\mathcal{F}}$	quotient of $\mathcal{F}$ over constant functions
d_x, $\overline{d}_x$	distance-like function, image of d_x into $\overline{\mathcal{F}}$
f_x	canonical representative of $\overline{d}_x$
ι	canonical embedding of Γ into $\overline{\mathcal{F}}$
$\partial_h\Gamma$	horofunction boundary
u, v	horofunctions
π_h	projection of $\partial_h\Gamma$ onto $\partial\Gamma$
$\mathcal{A}(\Gamma)$, $\mathcal{A}$	tree of atoms
$\mathcal{A}_k$	k-level of $\mathcal{A}$
$a, \overline{a}, b$	atoms
$(u_k)_{k=1}^\infty$	atom-coding of the horofunction u
$\overline{x} \in N(x, B_k)$, $N(a, B_k)$	nearest neighbor of x (resp. of a)
$p \in V(x, B_k)$, $V(a, B_k)$	visible points of x (resp. of a)
$p \in P(x, S_k)$, $P(a, S_k)$	k-proximal points of x (resp. of a)
$T(a)$	tip of a
$\widehat{x}, \widehat{y}$	elements of $T(a)$
$\mathbf{T}$	set of all tips
$\Gamma_\mathcal{A}$, $\Gamma_{T(\mathcal{A})}$	graph of atoms, graph of tips
$(\Gamma_\mathcal{A})_n$	n-sphere in the graph of atoms

Languages and trees

Σ, $\overline{\Sigma}$, Ξ, $\overline{\Xi}$	alphabets
ε, σ, σ_1, σ_2	empty string, elements of the alphabet
Σ^*, Σ^ω	set of finite strings, set of infinite strings
$\mathcal{L}$	language
$\mathcal{G}$	equivalence relation on Σ^ω
Θ, θ_0	set of states of an automaton, initial state
R	set of rigid types
r_1, r_2, r_i	rigid types
$\widehat{\Omega}$, Ω, τ_o, τ_i, ψ_x	elementary markings, markings
$\mathcal{M}$	automaton recognizing horofunction gluing
Δ	set of gluing atoms
$\mathcal{T}$	rooted tree
d	degree of a regular rooted tree
$\mathcal{T}_x$	subtree with root x
$\varphi_{x,y}$	rigid morphism

Distances and Gromov products

$\lvert t_1 - t_2\rvert$	Euclidean metric on $\mathbb{R}$
$\mathrm{d}_{\mathcal{H}}$	Hausdorff metric
d_h	infinity norm between horofunctons
d_Γ	metric on Γ, semi-metric on atoms
d_{F}	limit semi-metric on atoms
d_{B}	metric on $\Gamma - B_k$, semi-metric on atoms
$T^*\,\mathrm{d}$	semi-metric d on tips
$(\cdot \mid \cdot)$	Gromov product on Γ
$\curlyvee$	visual metric on Γ
β, $\mathcal{B}$	base for $\curlyvee$, maximum value of β
$(\cdot \mid \cdot)_k$	Gromov product on the tips of $\mathcal{A}_k$
$\curlyvee_k$	visual metric on $\mathcal{A}_k$
$(\cdot \mid \cdot)_{\mathcal{A}}$	Gromov product on the tree $\mathcal{A}$
$\curlyvee_{\mathcal{A}}$	visual metric on the tree $\mathcal{A}$
$\curlyvee_h$	visual metric on $\partial_h\Gamma$

Constants

λ	gluing constant
λ_a	hooking constant
λ_h	analytic constant
λ_∞	infinite cones constant
λ_e	defining constant of $\Gamma_{\mathcal{A}}$
C_λ	number of λ-types
L, L_1, L_2, L_3	constants for quasi-isometries
E, E_1, E_2	exponential divergence constants
C	used for definitions and proofs involving gluing
$M, W, D, \tilde{D}$	other constants used in proofs
ϵ	infinitesimal
e	Euler's number

Note: Maps and actions are written on the right. The only exceptions are listed above and concern functions related to distances or geometric objects as geodesics. Symbols for general maps are ϕ and ψ. For the case of specific maps appearing in proofs we adopt $\mathcal{S}_h$, $\mathcal{S}_{\mathbf{T}}$, $\mathcal{S}_{\mathcal{A}}$ to suggest sections of quotients and Φ for quasi-isometries.

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INTRODUCTION

The Gromov boundary of a hyperbolic group is an object which has been widely studied in the past decades. Examples of this past and ongoing interest can be found in the survey [KB02]. A particular effort has been made to detect "recursive" presentations of such boundary: in their works [CP93, CP01], Coornaert and Papadopoulos show how it can be seen as a *semi-Markovian space* when the group is torsion free, while Pawlik [Paw15] provides a way to describe it as a *Markov-compacta* and completes the work on semi-Markovian presentations in the general case, and Barrett [Bar18] gives an algorithm to determine if the boundary is a circle and investigates other topological properties. Also the well studied tool of *subdivision rules* plays a role in this context, see e.g. [Rus14, Rus17].

The concept of *rationality* that we follow can be found in the work [GNS00] of Grigorchuk, Nekrashevych and Sushchanskiĭ. The idea is to describe sets and hence relations, and functions, by using *finite state machines*. One of the main goals is to define homeomorphisms of the *Cantor set* $\{0,1\}^\omega$ via asynchronous machines (one bit, i.e. 0 or 1, as input and a finite string written using $\{0,1\}$ as output at each step of the computation), these are *rational functions*. On the other hand, synchronous machines, which for us have just inputs, at each step can read exactly one bit, are used to define *rational sets* and *rational relations*.

In [BBM21] Belk, Bleak and Matucci associate a *self-similar tree* called the *tree of atoms* $\mathcal{A}(\Gamma)$ to any hyperbolic graph Γ, and they proceed to prove that the action of a hyperbolic group on the boundary of such a tree is rational, that is any element of the group can be regarded as a finite state machine that has a boundary point as input and its image according to the action as output. They also show that the boundary $\partial\mathcal{A}(\Gamma)$ projects onto the Gromov boundary $\partial\Gamma$ of Γ

(exploiting Webster and Winchester's work [WW05]). Since any self-similar tree defines a language, i.e. a subset of Σ^ω where Σ is a finite set of symbols, and the language is also rational, then the projection induces a coding of any element of the Gromov boundary. Here we mean that to any boundary point we associate some (possibly more than one) elements of Σ^ω. It is natural to ask whether the equivalence relation given by the projection is a rational relation.
In this dissertation, we tried to answer these and other questions about the relation between the tree of atoms and the Gromov boundary.
In order to keep the treatment self-contained, we recall the main tools in metric geometry, geometric group theory and language theory we need in Chapter 1. Actually, Section 1.5 contains the first original results of this thesis, which regard the relation of atoms with cones and balls and in some cases are an improvement of what is pointed out in [BBM21]. Furthermore, we introduce *tips of atoms* (Definition 1.53), which turn out to be useful in our study.

In Chapter 2 we show how infinite sequences of atoms behave like geodesic rays in hyperbolic graphs. The most useful result for the rest of the discussion is the following.

Theorem 2.4 *Let Γ be a hyperbolic graph and let $\mathcal{A}(\Gamma)$ be its tree of atoms. Let $u = (u_k)_{k=1}^\infty$ and $v = (v_k)_{k=1}^\infty$ be two elements of $\partial\mathcal{A}(\Gamma)$. Then there exists a constant C and a family of distances $\{{\mathrm{d}_\Gamma}^k\}_{k=1}^\infty$ each defined on a level of the tree such that the sequences u and v are mapped in the same element of the Gromov boundary $\partial\Gamma$ if and only if ${\mathrm{d}_\Gamma}^k(u_k, v_k) \leq C$ for all $k \geq 1$.*

Roughly speaking, this is an analog to the fellow traveler property of geodesic rays. We also prove an atom-version of the exponential divergence for geodesics (see Proposition 2.19) and we define the Gromov product for atoms and for infinite sequences of atoms, providing an explicit relation between the latter and the Gromov product of elements of $\partial\Gamma$ (see Lemma 1.59 and the discussion before it). Moreover, we bound the fibers of the projection $\partial\mathcal{A}(\Gamma) \twoheadrightarrow \partial\Gamma$ (Theorem 2.22) and, consequently, we provide another way to bound the topological dimension of the Gromov boundary using Theorem 2.25.

One can construct, starting from the tree of atoms and the main results of Chapter 2, the *set of tips* $(\mathbf{T}, \mathrm{d}_{\mathcal{H}})$, where $\mathrm{d}_{\mathcal{H}}$ is the Hausdorff metric, and the *graph of atoms* $\Gamma_{\mathcal{A}}$ endowed with the standard metric on graphs (Definition 3.4). In particular, the graph is an *augmented tree* in the sense of [Kai03]. In Chapter 3 we provide a quasi-isometry between the Cayley graph of a hyperbolic group and the set of tips (see Proposition 3.3 for both the definition of the set and the quasi-isometry). Furthermore,

Theorem 3.7. *Let G be a hyperbolic group and let Γ be its Cayley graph. Then the graph of atoms $\Gamma_{\mathcal{A}}$ and Γ are quasi-isometric.*

We provide examples to better understand these results also from the point of view of approximation of the Gromov boundary via finite graphs.

Finally, in Chapter 4 we present the machine which describes the equivalence relation given by the projection $\partial\mathcal{A}(\Gamma) \twoheadrightarrow \partial\Gamma$. The language is based on the *rigid structure* of the tree of atoms, which is a particular self-similar structure that assigns to each edge in the tree an element of Σ. The construction of the machine uses, again, results from Chapter 2. The whole chapter can be summarized obtaining the following

Theorem 4.23. *The quotient map $\partial\mathcal{A}(\Gamma) \twoheadrightarrow \partial\Gamma$ defines a rational equivalence relation.*

We point out that being a semi-Markovian space implies the existence of such a map with such a property, generally the two notions do not coincide and our case seems to fail being semi-Markovian .

Section 4.2 and Section 4.3 are devoted to the complete or partial description of gluing machines for a group acting on a regular tiling of the hyperbolic plane, see Figure 22, and a group with an Apollonian gasket as Gromov boundary (see also Section 3.3).

CHAPTER 1

BACKGROUND

1.1 Metric Geometry

We will introduce some basic notions of metric geometry which will be useful later on. The first definitions will be stated in the more general context of sets, but soon we will restrict our attention to graphs.

Definition 1.1. Let X be a set. A function $\mathrm{d} : X \times X \to \mathbb{R} \cup \{\infty\}$ such that $\mathrm{d}(x, y) \geq 0$ or $\mathrm{d}(x, y) = \infty$ is called **distance** if the following conditions hold:

(a) $\mathrm{d}(x, x) = 0$ for all $x \in X$;

(b) $\mathrm{d}(x, y) = \mathrm{d}(y, x)$ for all $x, y \in X$.

In some cases, we add further conditions

($\overline{a}$) if $\mathrm{d}(x, y) = 0$ with $x, y \in X$, then $x = y$;

(c) $\mathrm{d}(x, y) \leq \mathrm{d}(x, z) + \mathrm{d}(z, y)$ for all $x, y, z \in X$.

In particular, a **semi-metric** is a distance which satisfies $(\overline{a})$ and a **pseudo-metric** is a distance which satisfies (c).

Definition 1.2. A **metric** is a semi-metric that is also a pseudometric.

Note that this definition of metric allows two points to be at infinite distance. If $\mathrm{d}(X \times X) \subseteq \mathbb{R}$, then we say that d is **finite**. A **metric space** (X, d) is a set X and d a metric on X. We set

$$B_k(x) = \{y \in X \mid \mathrm{d}(x, y) \leq k\}$$

to be the **ball** of radius k centered in x.

Example 1.3 (Hausdorff metric). Let (X, d) be a metric space and let A and B two non-empty subsets. We define the **Hausdorff metric** between A and B as follows

$$\mathrm{d}_{\mathcal{H}}(A, B) := \max\{\sup_{x\in A} \inf_{y\in B} \mathrm{d}(x, y), \sup_{y\in B} \inf_{x\in A} \mathrm{d}(x, y)\}.$$

Note that it is actually a metric and when we deal with finite sets, all $\inf$ and $\sup$ become $\min$ and $\max$.

In metric geometry there are some techniques that allow a distance to become a metric by changing the underlying set in a reasonable sense. We are interested in the following two operations (for more details see [BH13, I.1.24] and [BBI01, Proposition 1.1.5]).

First Move. Take a distance d on a space X and define $\underline{\mathrm{d}}$ in the following way: consider all the possible finite sequences of elements that start from x and end in y and take the minimum of the sums of the distances between two consecutive elements of the sequence. We get that $\underline{\mathrm{d}}$ is a pseudo-metric. We do not modify the space X in this case.

Second Move. Take a pseudo-metric d on a space X and consider the quotient $X/\,\mathrm{d}$ where elements are the equivalence classes of the following relation: $x \sim_{\mathrm{d}} y$ if and only if $\mathrm{d}(x, y) = 0$. This leads to a metric space, which is a quotient of X.

A **geodesic** between x and y in a metric space (X, d) is a map $[x, y] : [0, l] \to X$ with $[0, l] \subset \mathbb{R}$ such that $[x, y](0) = x$, $[x, y](l) = y$ and $\mathrm{d}([x, y](t_1), [x, y](t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. There can be multiple geodesics between two points, in general, though we use this interval notation when we only care about one between them. When we want to put emphasis on the metric space, we write $[x, y]_X$. With a slight abuse of notation, we use $[x, y]_X$ meaning $[x, y]([0, l]) \subseteq X$. A metric space is said to be **geodesic** if given two elements $x, y \in X$, then there exists a geodesic between them.

A **path** is a continuous map from $[0, l]$ for some $l \in \mathbb{R}$ to X. Note that any geodesic is a path. On the other hand, we can define a distance starting from paths. What we get is a so called *length space*. We start with the following

Definition 1.4. Let (X, d) be a metric space and let $c : [0, l] \to X$ with $l \in \mathbb{R}$ a path. The **length** of c is

$$\ell(c) := \sup_{0=t_0 \leq t_1 \leq \ldots \leq t_n = l} \sum_{i=0}^{n-1} \mathrm{d}(t_{i+1}, t_i)$$

for all possible partitions $0 = t_0 \leq t_1 \leq \ldots \leq t_n = l$ of $[0, l]$ and no bound on n.

A path c is said to be **rectifiable** if $\ell(c)$ is finite.

Definition 1.5. A **length space** is a metric space such that the distance between every pair of elements is equal to the infimum of the lengths of rectifiable paths joining them.

Remark 1.6. Since a geodesic is a rectifiable path, it follows that a geodesic space is a length space.

Given a metric space (X, d) it can be possible to turn it into a length space defining a new metric $\widehat{\mathrm{d}}$ such that $\widehat{\mathrm{d}}(x, y)$ coincides with the infimum of the lengths (with respect to d) of rectifiable paths joining them.

Assume now that (X, d) is a length space and Y is a subset of X. In general the metric space $(Y, \mathrm{d}_{|_Y})$ is not a length space. But, as presented above, we can consider $(Y, \widehat{\mathrm{d}}_{|_Y})$ and we call $\widehat{\mathrm{d}}_{|_Y}$ the **intrinsic metric** on Y with respect to d.

An example to keep in mind is $X = \mathbb{R}^2$ with the Euclidean metric and $Y = \mathbb{S}^1$. The intrinsic metric is the arc length metric on Y.

Remark 1.7. Let (X, d) be a length space and let Y be a subset endowed with the intrinsic metric $\widehat{\mathrm{d}}$. Then the following hold.

– If $y_1, y_2 \in Y$, then $\mathrm{d}(y_1, y_2) \leq \widehat{\mathrm{d}}(y_1, y_2)$.

– Let $y_1, y_2 \in Y$. Suppose that there exists a geodesic $[y_1, y_2]_X$ (with respect to d) which is fully contained in Y. Then $\mathrm{d}(y_1, y_2) = \widehat{\mathrm{d}}(y_1, y_2)$.

To conclude we note that a natural way to define maps between two metric spaces is to require that distances are preserved, namely if $\phi : (X, \mathrm{d}_X) \to (Y, \mathrm{d}_Y)$ is a map between metric spaces, we call it an **isometry** if it is bijective and

$$\mathrm{d}_Y(x_1\phi, x_2\phi) = \mathrm{d}_X(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Note that a geodesic can be seen as an isometry between $[0, l]$ and its image.

1.2 Graphs and Coarse Geometry

Despite the generality of the context described in the previous paragraph, we are looking for a combinatorial setting to work with. In particular, we give the following

Definition 1.8. A **graph** Γ consists of sets V and E where E is a subset of the collection of all unordered pairs in V. An element of V is called **vertex** and an element of the structure E, i.e. a pair $\{x, y\}$ with $x, y \in V$, is called **edge**.

By an abuse of notation we will often refer to Γ, and we will write $x \in \Gamma$, to mean the vertex set. The reason will be clear soon, once we introduce a metric on Γ. Note that, considering unordered pairs, we do not allow loops or multiple edges.

A **subgraph** Γ_0 is a subset of vertices together with a subcollection of the edges such that if an edge $\{x, y\}$ is in the subgraph collection of edges, then x and y belong to the subset of vertices. A subgraph Γ_0 is said to be **induced** if its collection of edges consists of all the $\{x, y\} \in E$ such that $x, y \in \Gamma_0$.

A **combinatorial path** in a graph Γ is a sequence $x_1, \dots, x_{n+1}$ of vertices such that $\{x_i, x_{i+1}\}$ is an edge for all $i = 1, \dots, n$. We say that such a combinatorial path has length n. A combinatorial path is called a **combinatorial geodesic** if its length is the minimum length among all possible paths between the first vertex and the last vertex.

Definition 1.9. Let Γ be a graph and $x, y \in \Gamma$. We define $\mathrm{d}_\Gamma(x, y)$ to be the length of any combinatorial geodesic that connects x and y or ∞ if x and y are in different path components.

The map $\mathrm{d}_\Gamma : \Gamma \times \Gamma \to \mathbb{Z} \cup \{\infty\}$ is actually a metric and we call it the **standard metric** of the graph.

Now we present three properties of graphs, which we will often assume for the rest of this dissertation. Let x be a vertex of Γ. The **degree** of x is the number of edges in which it is a vertex. We say that Γ is **locally finite** if every vertex has finite degree. Moreover, if the degree is the same (and finite) for all vertices, the graph is called **regular**. Finally, a graph is **connected** if every pair of vertices has a combinatorial path joining them. A subgraph of a connected graph need not be connected. Please note that a graph Γ is connected if and only if d_Γ is finite. If a graph Γ has a distinguished vertex x_0, a vertex y in Γ that shares an edge with another vertex x such that $x \in [x_0, y]$ is called a **successor** of x.

Definition 1.10. A locally finite connected graph such that for each pair of vertices there exists exactly one combinatorial path joining them is called **tree**. Whenever we choose a distinguished vertex x_0, we say that the tree is **rooted** and x_0 is the root.

Similarly to graph with a distinguished vertex, a vertex y in a rooted tree that shares an edge with another vertex x such that $x \in [x_0, y]$ is called a **child** of x. A rooted regular tree is a tree such that each vertex has the same number of children.

Trees are some of the main characters of the dissertation, together with their boundaries. But we will discuss them later.

Assumption. From now on, when we talk about graphs we only refer to locally finite connected ones.

In order to relate the discrete point of view to the general one and better encode algebraic properties, we need a notion of map that is less rigid than that of isometry.

Definition 1.11. Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $\phi : X \to Y$

is called a **quasi-isometric embedding** if there exist $L_1 \geq 1$ and $L_2 \geq 0$ such that

$$\frac{1}{L_1}\,\mathrm{d}_X(x_1, x_2) - L_2 \leq \mathrm{d}_Y(x_1\phi, x_2\phi) \leq L_1\,\mathrm{d}_X(x_1, x_2) + L_2$$

for all $x_1, x_2 \in X$.

A quasi-isometric embedding $\phi : X \to Y$ is a **quasi-isometry** if there exists a quasi-isometric embedding $\psi : Y \to X$ and a constant L such that

$$\mathrm{d}_X(x\phi\psi, x) \leq L \text{ and } \mathrm{d}_Y(y\psi\phi, y) \leq L$$

for all $x \in X$ and $y \in Y$. An equivalent and useful definition follows from

Proposition 1.12 ([Lö17], Proposition 5.1.10)**.** *A quasi-isometric embedding ϕ : $X \to Y$ is a quasi-isometry if and only if its image is **quasi-dense**, i.e. there exists a constant L_3 such that for all $y \in Y$ there exists $x \in X$ with $\mathrm{d}_Y(x\phi, y) \leq L_3$.*

Let Γ be a graph. We define its **geometric realization** $|\Gamma|$ to be the metric space constructed in the following way: we associate to each edge a copy of $[0, 1]$ and we identify the two vertices x and y of an edge $\{x, y\}$ respectively with 0 and 1 (note that, by definition of edge, the pair is unordered, so we arbitrarily choose one identification). The metric $\mathrm{d}_{|\Gamma|}$ defined on $|\Gamma|$ comes from the Euclidean metric on $[0, 1]$.

Example 1.13. We consider the map $|\cdot| : (\Gamma, \mathrm{d}_\Gamma) \to (|\Gamma|, \mathrm{d}_{|\Gamma|})$ such that $x \mapsto x$. The map is an isometric embedding with $L_1 = 1$ and $L_2 = 0$. In fact, it is an isometry when restricted on its image. We can construct a quasi-isometric embedding of $|\Gamma|$ onto Γ by shrinking one half on a geometric edge (the copy of $[0, 1]$) to the vertex identified with 0 and the second half to the vertex identified with 1. One can arbirarily choose where to send $\frac{1}{2}$. In this way $|\cdot|$ is a quasi-isometry.

It is straightforward that combinatorial paths in Γ are quasi-isometric to paths in $|\Gamma|$ and, in particular, combinatorial geodesics are quasi-isometric to geodesics in $|\Gamma|$ and the lengths are preserved. With a slight abuse of notation, we drop the word "combinatorial" and we consider a graph Γ to be a length space (actually a geodesic space) with respect to d_Γ. Let $n \in \mathbb{N}$. We define the n-**sphere** S_n (centered at a distinguished point x_0) to be the collection of all vertices x such that there exists a geodesic between x and x_0 of length n in Γ.

The focus on quasi-isometries is due to our forthcoming study of groups as metric spaces.
First, we make an assumption which is very common in geometric group theory and is related to the assumption we made on graphs.

Assumption. A group is always assumed to be finitely generated and a set of generators is always symmetric, namely if s belongs to a set a generators S then also $s^{-1} \in S$, and it does not contain 1_G.

We turn groups into graphs in the following way

Definition 1.14. Let G be group and let S be a set of generators for G. The **Cayley Graph** $\Gamma(G, S)$ of G with respect to S is a graph that has G as the collection of vertices and $\{\{g, sg\} \mid g \in G, s \in S\}$ as edges.

When S is clear from the context, we will write $\Gamma(G)$. If m is the cardinality of S, then $\Gamma(G, S)$ is regular with degree m. Moreover, it is connected. So it satisfies the assumption we made on graphs.

Since $\Gamma(G, S)$ is a graph, we can endow it with its standard metric and turn it into a metric space. In this particular case, we will refer to the standard metric as the **word metric**. The reason for this choice is that vertices are reduced words in S and, given two vertices g and h, their distance is the length of a shortest word w such that $wg = h$ in G.

Since we want to consider an action of G on $\Gamma(G, S)$, we need to formalize what we intend with a group action on a graph.

Definition 1.15. Let G be a group and let Γ be a graph. We say that G **acts** on Γ if G acts on the collection of vertices and $\{xg, yg\}$ is an edge for every edge $\{x, y\}$ in Γ and for every $g \in G$.

Please note that each element of G is an isometry of Γ with respect to to the standard metric.

The natural action of G on $\Gamma(G, S)$ is induced by the right multiplication in G. As we noted before, the action is right invariant with respect to the word metric i.e. the group acts by isometries.

As mentioned above, quasi-isometries are a good way to see algebraic objects as metric spaces.

Proposition 1.16 ([Lö17], Proposition 5.2.5)**.** *Let G be a group. If S_1 and S_2 are two sets of generators, then there exists a quasi-isometry between $\Gamma(G, S_1)$ and $\Gamma(G, S_2)$.*

We conclude this section discussing the so called fundamental theorem of geometric group theory. To do that, we need some properties of a group action on a metric space.

- A group action of G on a metric space (X, d_X) is said to be **cocompact** if the quotient space X/G is compact.
- A group G acts **properly discontinuously** on a metric space (X, d_X) if $\{g \in$

$G \mid K \cap Kg \neq \emptyset\}$ is finite for every compact subspace K of X.

– If a group acts properly discontinuously and cocompactly on a metric space, then we say that the action is **geometric**.

The last remark we want to do before stating the theorem is the following: a metric space is said to be **proper** if any closed ball is compact, so, in our description of graphs as metric spaces, a graph is always proper.

Theorem 1.17 (Milnor-Schwarz Lemma)**.** *Let G be a group acting geometrically on a proper geodesic metric space (X, d_X). Then the $(\Gamma(G, S), \mathrm{d}_\Gamma)$ and (X, d_X) are quasi-isometric.*

Proof. See Proposition I.8.19 in [BH13]. □

The actual statement also says that the group is finitely generated, but since we made that assumption before, we did not mention it.

This gives us the possibility to study certain aspects of groups via spaces on which they act geometrically. On the other hand, we can reduce our coarse study of spaces to that of graphs quasi-isometric to them.

1.3 Hyperbolic Groups and their Boundaries

For the first notions of this section we deal with spaces, but soon we will return to our discrete setting.

Let (X, d_X) be a metric space. We fix an element x_0 and we consider the quantity

$$(\Pi) \qquad (x \mid y) := \frac{1}{2}\left[\mathrm{d}_X(x, x_0) + \mathrm{d}_X(y, x_0) - \mathrm{d}_X(x, y)\right],$$

for any two elements x, y in X. We call it the **Gromov product** between x and y with respect to the base-point x_0.
Note that the notation $(x \mid y)$ does not mention the basepoint, this happens because at some point we will fix one and we will never change it.

Definition 1.18. Let (X, d_X) be a geodesic metric space with a distinguished point x_0. Then X is **hyperbolic** if one of the following (equivalent) conditions holds:

– there exists a constant $\tilde{\delta} \geq 0$ such that

$$(\mathfrak{H}) \qquad (x \mid z) \geq \min\{(x \mid y), (y \mid z)\} - \tilde{\delta}$$

for all $x, y, z \in X$;

– there exists a constant δ such that for every triple $([x, y], [y, z], [z, x])$ one of the geodesics is contained in the union of the δ-neighborhoods of the other two.

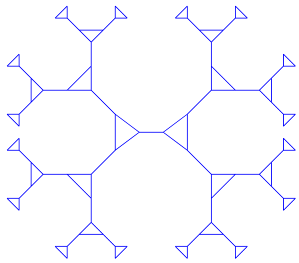
We call ($\mathfrak{H}$) the **hyperbolic inequality**.

A first step towards graphs is the following property.

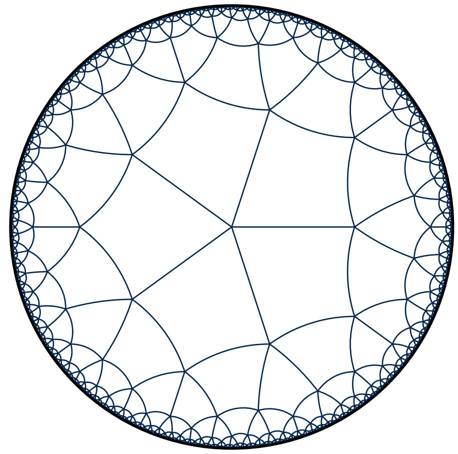
Proposition 1.19 ([BH13], Theorem H.1.9)**.** *If a hyperbolic space X is quasi-isometric to a space Y, then Y is hyperbolic.*

The main reason to pass to discrete structures is given by this definition.

Definition 1.20. A group is said to be hyperbolic if one (and hence all) of its Cayley graphs is hyperbolic.



(a) The portion of Cayley graph.



(b) The 1-skeleton of the tiling.

Figure 1: Two examples of hyperbolic graphs: a portion of a Cayley graph for the free product between the cylic groups of order 2 and 3 and the 1-skeleton of the uniform tiling of the hyperbolic plane given by five squares meeting in each vertex.

Elementary examples of hyperbolic groups are virtually cyclic groups (i.e. with a finite-index cyclic subgroup). A hyperbolic group which is not virtually cyclic is called **non-elementary**. Another example useful to understand the definitions is the free group. It is well known that the Cayley graph of a free group (with respect to free generators) looks like a rooted tree (in the identity). We can see hyperbolic groups as groups such that their Cayley graphs behave like a tree "up to a small error δ" (see Figure 2).

Through this informal point of view, we get a good interpretation of the concepts involved in Definition 1.18: in rooted trees the Gromov product between two vertices measures the length of their greatest common prefix or, in other words, how far the geodesic between the two vertices is from the base-point and it is straightforward that the hyperbolic property holds with $\tilde{\delta} = 0$, while the condition on triangles holds for $\delta = 0$. In hyperbolic graphs, we have that two geodesics, starting from the base-point and reaching two vertices x and y,

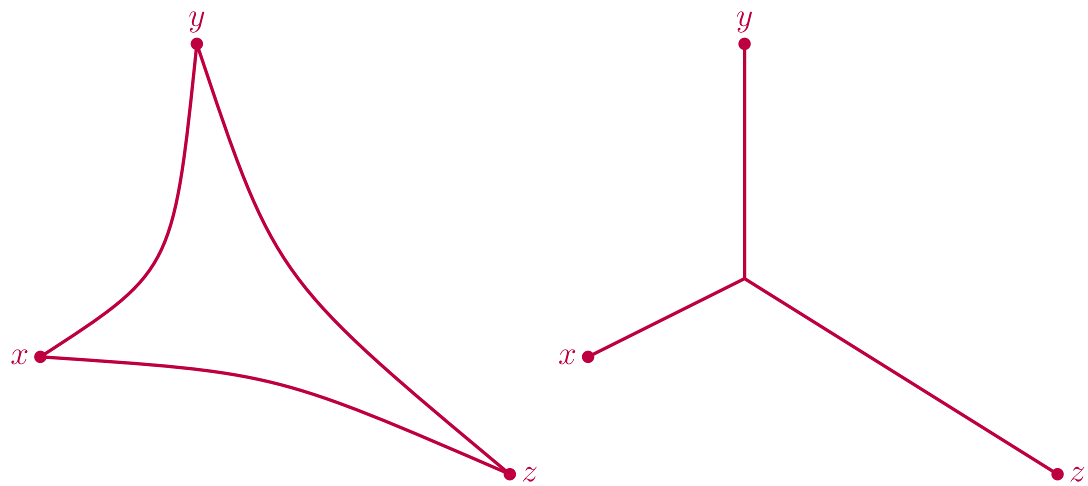


Figure 2: On the left an example of a hyperbolic triangle. On the right, a triangle in a tree.

travel together (within a distance of 2δ) in their initial part of length $(x \mid y)$ or, more formally, we can generalize what we have just said about trees (see Figure 3).

Lemma 1.21 ([CDP90], Ch. 3, Lemma 2.7)**.** *Let* Γ *be a* δ*-hyperbolic graph with a distinguished point* x_0*. If* x *and* y *are two vertices of* Γ *and* $[x, y]$ *is any geodesic connecting* x *to* y*, then*

$$\mathrm{d}_\Gamma(x_0, [x, y]) - 4\delta \leq (x \mid y) \leq \mathrm{d}_\Gamma(x_0, [x, y])$$

with $\mathrm{d}_\Gamma(x_0, [x, y])$ *defined as the minimum distance over all the points in* $[x, y]$*.*

This way of thinking about the Gromov product will be stressed throughout the whole dissertation, for example when we will meet boundaries and metrics on them, so it may be convenient keep it in mind whenever a new definition will occur. Moreover, it is one of the many interesting features of hyperbolic groups and it helps them having such nice properties.

In fact, we can say something more general about distances between two

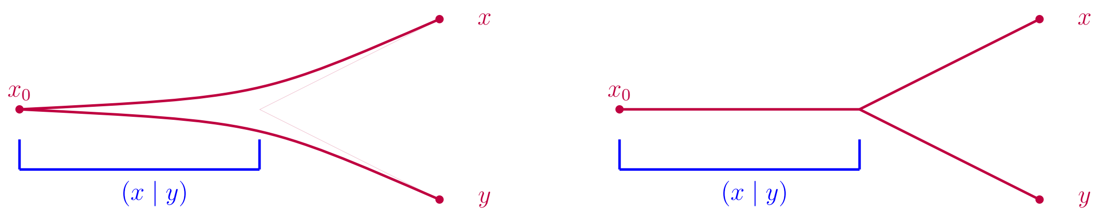


Figure 3: On the left the geometric meaning of the Gromov product and the behavior of two geodesics in a hyperbolic graph. On the right, the same picture in a tree.

geodesics starting from the same point and about their behavior after their bounded distance initial segments (see Figure 4).

Proposition 1.22. *Let Γ be a hyperbolic graph and let $[z,x]$ and $[z,y]$ be two geodesics such that $\mathrm{d}_\Gamma(z,x) = t_x$ and $\mathrm{d}_\Gamma(z,y) = t_y$. Put $t_{max} = \max\{t_x, t_y\}$ and extend the shorter one to $[0, t_{max}]$ by the constant map. Then*

$$\mathrm{d}_\Gamma([z,x](t), [z,y](t)) \leq 2\,\mathrm{d}_\Gamma(x,y) + 4\delta$$

for all $0 \leq t \leq t_{max}$.

For the proof see e.g. [BH13, Lemma H.1.15].

Proposition 1.23 (Exponential divergence)**.** *Let Γ be a hyperbolic graph. There exist three constants E, E_1 and E_2 such that for any two geodesics $[z,x]$ and $[z,y]$ and given $t, \tilde{t}$ such that $\tilde{t} + t \leq \min\{\mathrm{d}_\Gamma(z,x), \mathrm{d}_\Gamma(z,y)\}$, if $\mathrm{d}_\Gamma([z,x](\tilde{t}), [z,y](\tilde{t})) > E$ and c is a rectifiable path fully contained in $\Gamma - B_{\tilde{t}+t}(z)$ from $[z,x](\tilde{t}+t)$ to $[z,y](\tilde{t}+t)$, then $\ell(c) > E_1 e^{E_2 t}$.*
Moreover, E_1 and E_2 only depend on δ.

As a matter of fact, by the Milnor-Schwarz Lemma, we can give an (apparently) more general characterization of hyperbolic groups. Namely, a hyperbolic group is a group which acts geometrically on a hyperbolic space. A

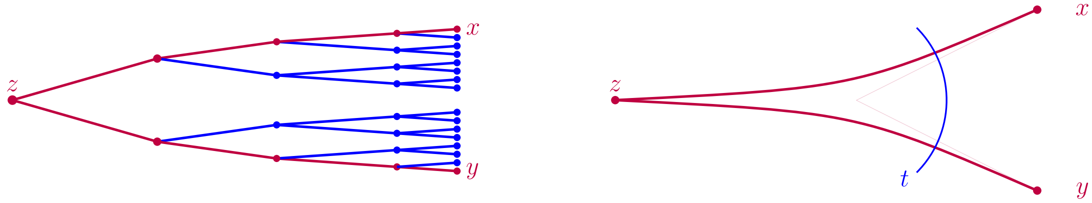


Figure 4: On the left, we can see that on a rooted tree the number of vertices between two points at the same level grows exponentially with respect to the level. On the right, we have similar behavior on a hyperbolic graph, as any path between the two points of the geodesics that are on the sphere of radius t is longer than an exponential with respect to t.

first instance of the previous connection with trees is that geometric actions are closely related to free actions and that a free group is a group that acts freely on a tree providing there is no inversion of edges.

So from now on we are interested in hyperbolic graphs related to hyperbolic groups and we choose a distinguished point x_0 which will always remain implicit.

We are going to introduce the first type of boundary. Despite the definitions will be stated for hyperbolic graphs, they can be given in the general setting (see e.g. [BH13]).

Let $\{x_k\}_{k=0}^{\infty}$ be a sequence of points in a graph Γ. We say that it **goes to infinity in the sense of Gromov** if $(x_k \mid x_j)$ diverges to infinity whenever k and j go to infinity. Sequences that go to infinity in a suitable way are good candidates to be representatives of some boundaries.

If $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ are two sequences as above with $y_0 = x_0$, we define the following relation

$$\{x_k\}_{k=0}^{\infty} \sim_\Gamma \{y_k\}_{k=0}^{\infty} \Leftrightarrow (x_k \mid y_k) \xrightarrow[k\to\infty]{} \infty.$$

In general it is not an equivalence relation, but if the graph is hyperbolic, it is. In fact, this is the first boundary we are interested in.

Definition 1.24. The **Gromov boundary** $\partial\Gamma$ of a hyperbolic graph Γ is the quotient of the collection of all sequences that go to infinity in the Gromov sense over the relation $\sim_\Gamma$.

Some remarks are needed before we continue. First of all, $\partial\Gamma$ is always naturally a metrizable space and we will be discussing metrics on $\partial\Gamma$ later on. The boundary does not depend on the choice of x_0, that is if you start the construction from another point, the resulting boundary is homeomorphic to the one defined with x_0.

More in general, a quasi-isometry of hyperbolic spaces induces a homeomorphism of Gromov boundaries. In particular, the Gromov boundary of a hyperbolic group is well-defined.

We now present a more geometric way of describing the Gromov boundary. To do that we need to slightly generalize the concept of (combinatorial) geodesic.

Definition 1.25. Let Γ be a hyperbolic graph with a distinguished point x_0. A **geodesic ray** is a function $\gamma : [0, \infty[\to |\Gamma|$ such that $\gamma(0) = x_0$ and the restriction $\gamma_{|[t_1,t_2]}$ is a geodesic for every interval $[t_1, t_2]$. As for geodesics, we will often drop the geometric realization and consider combinatorial geodesic rays via the standard quasi-isometry.

Note that the notion we just introduced can be stated in more generality, indeed it is usually not required that the path starts from a distinguished point.

Our further assumption is merely a choice of convenience for our purposes.

Having a collection of sequences, we need a suitable equivalence relation: we say that two geodesic rays γ and η are **asymptotic** if there exists a constant C such that $\mathrm{d}_\Gamma(\gamma(t), \eta(t)) \leq C$ for all t and with C not depending on t. In fact, we can see that this is a generalization of what we said for geodesics and that if two geodesic rays are asymptotic then we take $C = 2\delta$.

Lemma 1.26 ([BH13], Lemma H.3.13)**.** *Let Γ be a hyperbolic graph. Then there exists a bijection between $\partial\Gamma$ and the collection of equivalence classes of asymptotic geodesic rays.*

By virtue of this lemma, we will refer to both sets with an abuse of notation as $\partial\Gamma$.

Example 1.27*.* Using this second characterization, it is easy to argue that the Gromov boundary of a free group is a **Cantor set**, that is a compact, perfect, totally disconnected metric space. We can show that two spaces with this topological description are homeomorphic (see [Wil70, Theorem 30.3]), so the Cantor set is well-defined in this way. One can also see that the boundary of a rooted regular tree is a Cantor set and given a Cantor set one can always construct a rooted regular tree whose Gromov boundary is homeomorphic to the given Cantor set (see Figure 5).

We are aware that the concepts involved in the previous example lie in a far more general setting, but, for our purpose, it was easier to exploit the Gromov boundary.

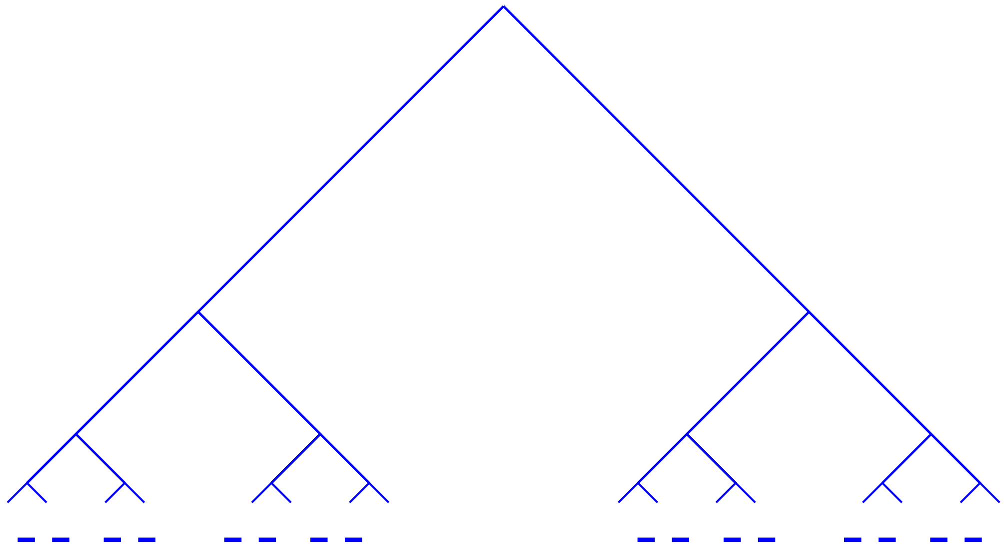


Figure 5: The fourth level of a (rooted) tree and its corresponding approximation of a Cantor set.

It is worth noticing that adding the boundary to the graph Γ, you get a compact space (the fact that $\Gamma \cup \partial\Gamma$ is a topological space will be stated later on). Actually every compact space can be seen as a boundary of a hyperbolic space. But since we are focused on groups, there are fewer possibilities. Indeed, the Gromov boundary of a non-elementary hyperbolic group is a compact metrizable space without isolated points. For all these facts see [KB02, Section 2] for further details. Something more specific can be said, once we introduce *visual metrics*. In order to do so, we need to generalize the notion of Gromov product to elements of the Gromov boundary.

Definition 1.28. Let Γ be a hyperbolic graph and let $x_\infty, y_\infty \in \partial\Gamma$. Then the Gromov product between x_∞ and y_∞ is

$$(x_\infty \mid y_\infty) := \inf \liminf_k (x_k \mid y_k)$$

where the infimum is taken over all the Gromov sequences $\{x_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$

that converge respectively to x_∞ and y_∞.

There are some considerations that are worth pointing out before we continue. For a complete treatment on the argument, we recommend to see [Vä05, Section 5].

Remark 1.29.

(a) In the same way, we can consider the product between an element $x_\infty \in \partial\Gamma$ and an element $y \in \Gamma$ by taking $y_n = y$, for every n. Note that $\liminf_n (x_\infty \mid y_n)$ for some Gromov sequence $\{y_n\}_{n=1}^\infty$ is infinite if and only if the sequence converges to x_∞.

(b) The hyperbolic inequality ($\mathfrak{H}$) can be extended to $\Gamma \cup \partial\Gamma$.

(c) To define the Gromov product on the boundary, one can consider also the $\inf \limsup$ and even $\sup \liminf$ or $\sup \limsup$. We choose the smallest one, but they are all related. Indeed, they all lie in a $2\tilde{\delta}$-interval where $\tilde{\delta}$ is the hyperbolic constant involved in ($\mathfrak{H}$) (see [Vä05, Definition 5.7] for details).

To stress the connection between trees and hyperbolic spaces, we give this result that will be useful later. The proof can be found in Lemma 3.7 of [GMS19].

Lemma 1.30. *Let Γ be a hyperbolic graph with a distinguished point x_0. Let γ and η be two geodesic rays of Γ. Then there exists a quasi-isometry, with $L_1 = 1$ and $L_2 = 5\delta$ and δ the hyperbolic constant, between $\gamma \cup \eta$ and the tripod consisting of the rays glued together along an initial segment of length $([\gamma] \mid [\eta])$.*

See again Figure 3 for a reference.

Now that we have a tool that measure how long two geodesics or geodesic rays fellow travel, we can define a distance such that the more they travel to-

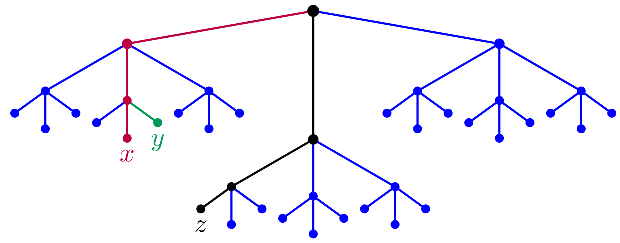


Figure 6: Fix $\beta = 3$. The distance between x and y is $3^{-2} = \frac{1}{9}$ while $\curlyvee(x, z) = 3^{-0} = 1$.

gether, the more the two points are close and viceversa.

Definition 1.31. Let $\beta > 1$ be a fixed constant. The **visual metric** on the completion $\Gamma \cup \partial\Gamma$ of a hyperbolic graph is defined as follow

$$\curlyvee(x_*, y_*) := \beta^{-(x_*|y_*)} \text{ with } x_*, y_* \in \Gamma \cup \partial\Gamma,$$

with the convention that $\beta^{-\infty} = 0$.

Again, we suggest to keep in mind the case of free groups (or the case of rooted trees), where two geodesic rays, that are two elements of the boundary since we are working with a tree, are close if they share a long common prefix. Note that in this particular context, the visual metric on the boundary is in fact a metric (see Figure 6).

For the general setting, we need some more properties that will also help us with the fact that in the notation we do not explicitly mention β.

Lemma 1.32. *Let Γ be a hyperbolic graph. Then*

(a) the function $\curlyvee$ defines a topology on $\Gamma \cup \partial\Gamma$;

(b) the function $\curlyvee$ is a semi-metric when restricted to $\partial\Gamma$;

(c) using the First Move explained just after Definition 1.2 and with an abuse of notation, we get a new function that we call again $\curlyvee$ *and that is a metric on* $\partial\Gamma$*;*

(d) taking $\curlyvee$ *as in the previous point, there exists a constant* $\mathcal{B}$*, depending only on* δ*, such that for all* $1 < \beta \leq \mathcal{B}$ *we have*

$$\frac{1}{2}\beta^{-(x_*|y_*)} \leq \curlyvee(x_*, y_*) \leq \beta^{-(x_*|y_*)}.$$

Proof. See Proposition 5.16 in [Vä05]. □

Please note that point (d) partially explains the ambiguity of point (c) and the abuse of notation in the Definition, indeed from now on we will assume β will satisfy the constraint just introduced.

With the metric, and hence the topology, just introduced on $\partial\Gamma$, we can ask ourselves about connectivity properties of the boundary.
One well-investigated quasi-isometric invariant is the *ends*. We recall them briefly giving two possible definitions and we state that they are the connected component of $\partial\Gamma$.

We start with the notion that can be generalized to any topological space:

Definition 1.33 (Topological Ends)**.** Let Γ be a graph with a distinguished point x_0. A sequence $\{\mathcal{C}_k\}_{k=1}^{\infty}$ such that $\mathcal{C}_{k+1} \subseteq \mathcal{C}_k$ and $\mathcal{C}_k$ is a connected component of $\Gamma - B_{k-1}$ is called an **end** of Γ. We denote the collection of all ends with $\mathcal{E}_{\mathrm{T}}(\Gamma)$.

While the second is specific for graphs:

Definition 1.34 (Graph Ends)**.** Let Γ be a graph with a distinguished point x_0. We put an equivalence relation on geodesic rays in this way: two geodesic rays

γ_1 and γ_2 are equivalent if $\gamma_1([k, \infty[)$ and $\gamma_2([k, \infty[)$ belong to the same connected component of $\Gamma - B_{k-1}$ for all k. Given a geodesic ray γ, we denote its equivalence class with $end(\gamma)$ and call this an **end** of Γ and denote the collection of all ends with $\mathcal{E}(\Gamma)$.

In our case, i.e. hyperbolic graphs, it is not difficult to see that the second definition induces a surjective map $\partial\Gamma \twoheadrightarrow \mathcal{E}(\Gamma)$ such that $[\gamma] \mapsto end(\gamma)$. This map gives a correspondence between graph ends and connected components of the Gromov boundary. More in general, the two definitions are equivalent, in the sense that there exists a bijection between $\mathcal{E}(\Gamma)$ and $\mathcal{E}_{\mathrm{T}}(\Gamma)$ whenever Γ is a graph (see [DK03]).

The famous Freudenthal-Hopf theorem (see [Lö17, Theorem 8.2.11]) tells us that finitely generated groups can have only 0,1,2 or ∞ ends. As we also know that groups with zero ends are finite and groups with two ends are virtually $\mathbb{Z}$ (see [Lö17, Theorem 8.2.14(1) and 8.2.14(2)]); only two classes remains to study. A model for groups with an infinite number of ends is the free groups on $n \geq 2$ generators ([Lö17, Example 8.2.10]), while any group acting geometrically on a tiling of the hyperbolic plane has one end (see [KB02, Theorem 5.4] and recall that ends are connected components of the Gromov boundary). There are several works in this direction. We mention, for example, the PhD Thesis of Barrett [Bar18] in which it is shown that, given the presentation of a hyperbolic group, one can determine with an algorithm if the Gromov boundary is a circle.

1.4 Trees associated to Hyperbolic Graphs

Continuing with the connection between trees and hyperbolic graphs, we introduce what we can call a **tree structure**. Namely, a rooted tree (usually built from our graph Γ) together with a quotient map from its boundary onto the Gromov boundary. In this section, we will present one of the many tree structures introduced by Coornaert and Papadopoulos in [CP93] and a less intuitive structure provided by [BBM21].

The first idea is to consider all the geodesic rays of a hyperbolic graph Γ and x_0 as the root. In order to get a tree, we define the vertices as all the finite geodesics starting from x_0. Two geodesics $x_0, x_1, \ldots, x_n$ and $x_0, y_1, \ldots, y_{n+1}$ are connected by an edge if and only if $x_i = y_i$ for all $i \leq n$. It easily follows by the construction that it is a tree and that the boundary is the collection of all geodesic rays (see Figure 7 for an example). Moreover, it is straightforward to define a map from the vertices of this tree onto Γ sending each geodesic to its endpoint. This induces a map from a boundary (the one of the tree) to another (the Gromov boundary of Γ) given by the usual quotient $\gamma \mapsto [\gamma] \in \partial\Gamma$ where γ is a geodesic ray in the tree, which define also a geodesic ray in Γ. In fact, this map is Lipschitz (hence continuous) and surjective. Even if it does not have finite fibers, in the case of groups one can consider a suitable subtree to get even this property. This construction exploits a lexicographic order on the set of generators and it is explained in detail in [CP93, Section 5.4].

We now consider the abelian group $\mathcal{F} = \mathbb{Z}^\Gamma$ of functions from the vertices of Γ (which we will identify with Γ) to $\mathbb{Z}$. This space is endowed with the product

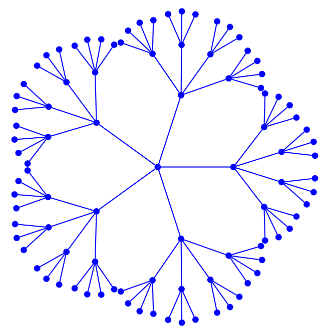


Figure 7: Ball of radius 3 of the tree built from all the geodesic rays in Figure 1.(b).

topology (which is also the compact-open topology, since the set of vertices of Γ is discrete). In particular, if we take the quotient $\overline{\mathcal{F}}$ of $\mathcal{F}$ over the subgroup of constant functions, it inherits the quotient topology.

We consider the function $d_x : \Gamma \to \mathbb{Z}$ defined as $d_x(-) := \mathrm{d}_\Gamma(x, -)$ with $x \in \Gamma$. The class $\overline{d}_x$ in $\overline{\mathcal{F}}$ has a global minimum at x. Hence there is a canonical embedding $\iota : \Gamma \to \overline{\mathcal{F}}$ since $\overline{d}_x$ is an isolated point in $\Gamma\iota$.

Definition 1.35. Let Γ be a hyperbolic graph. The **horofunction boundary** $\partial_h\Gamma$ of Γ is the set of limit points of $\Gamma\iota$.

An element of $\partial_h\Gamma$ is called **horofunction**. Using the definition, we say that a sequence of vertices $\{x_k\}_{k=1}^{\infty}$ converges to a horofunction u if and only if the sequence $\{\overline{d}_{x_k}\}_{k=1}^{\infty}$ converges to u (in the compact-open sense). Horofunctions were initially introduced by Gromov in [BGS85], he refers to them as the "metric boundary" and since then, they were widely studied. For a different, but related notion see Definition 2.23.

Among horofunctions there are also the so-called **Busemann points**. Namely, a horofunction is a Busemann point if it is the limit (in the sense explained above) of a geodesic ray. This notion is a first step towards the next result, but please note that in the general hyperbolic case there are horofunctions that are not

Busemann points (see e.g. [WW06]).

This second boundary can always be constructed, instead of the first one which requires hyperbolicity. Moreover, when the two are well-defined, we can retrieve $\partial\Gamma$ starting from horofunctions. Indeed, we have the following

Proposition 1.36. *Let Γ be a hyperbolic graph. Then there exists a continuous surjective map $\pi_h : \partial_h\Gamma \twoheadrightarrow \partial\Gamma$.*

Proof. See [WW05, Section 4]. $\square$

In particular, if a sequence converges to a point in $\partial_h\Gamma$ then it goes to infinity in the sense of Gromov.

There is another way to represent horofunctions, it is the so called **tree of atoms** ([BBM21, Definition 3.4]). The idea is to construct a suitable collection of partitions of Γ (seen as the set of vertices) and then to endow it with a tree structure.

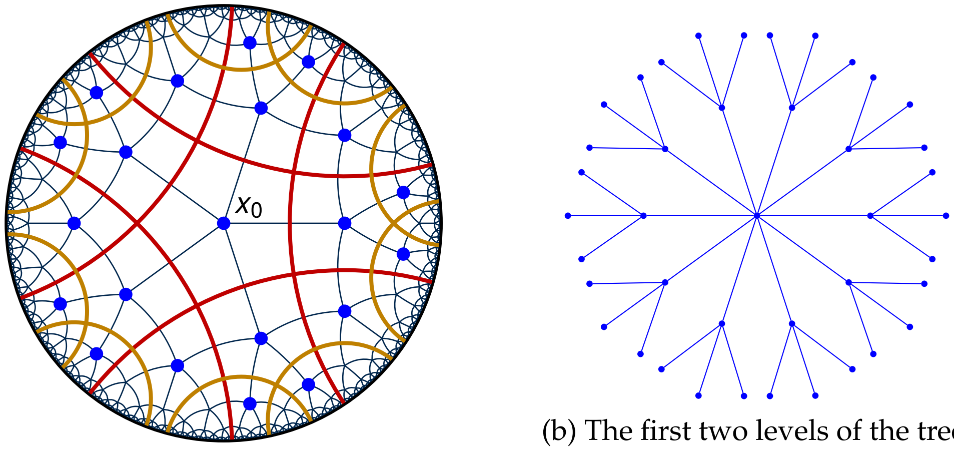
Let x be an element of a hyperbolic graph Γ. We consider the function

$$f_x(-) := \mathrm{d}_\Gamma(-, x) - \mathrm{d}_\Gamma(x_0, x) : \Gamma \to \mathbb{Z} \tag{$\mathfrak{F}$}$$

for all $x \in \Gamma$.

Now we fix k to be a non-negative integer. The k-partition comes from the following equivalence relation: two vertices x and y are equivalent if and only if f_x and f_y agree on the ball B_k of radius k centered in x_0 (this means that $\overline{d}_x = \overline{d}_y$). We call the equivalence classes that contain an infinite number of vertices k**-level atoms** and we denote the collection of such classes with $\mathcal{A}_k(\Gamma)$. When Γ will be clear, we will drop it in the notation.

It can be shown that each partition is finite and it is a refinement of the previous one. Indeed, given $x \in \Gamma$ and $k \in \mathbb{N}$, there are only finitely many possibilities



(a) The atoms seen as subsets in the graph together with the ball of radius 2 (in blue).

(b) The first two levels of the tree.

Figure 8: The first two levels of the atoms for the 1-skeleton in Figure 1.(b).

for the restriction of f_x to B_k. Moreover, since we are dealing with restriction, if f_x and f_y agree on B_{k+1}, then they agree on B_k. In particular, if we consider atoms, they have a structure of an infinite tree

$$\mathcal{A}(\Gamma) := \coprod_{k=1}^{\infty} \mathcal{A}_k(\Gamma).$$

For further details about this construction see [BBM21, Subsection 3.1].

Example 1.37. Consider the 1-skeleton of the hyperbolic tiling depicted in Figure 1.(b). The first level consists of ten atoms: in Figure 8 the subdivision given by the red lines gives all ten of them and a finite region in which x_0 is the only element. The second level can be described as follows: every 1-level atom has three children, given by the intersection between the atom and the brown lines.

A first property which says something about the asymptotic behavior of atoms is the following

Proposition 1.38 ([BBM21], Proposition 3.5)**.** *Every k-level atom is contained in*

$\Gamma - B_{k-1}$.

But the key aspect of this structure is the following

Theorem 1.39 ([BBM21], Theorem 3.6)**.** *Let Γ be a hyperbolic graph. Then the boundary of $\mathcal{A}(\Gamma)$ is homeomorphic to $\partial_h\Gamma$.*

This means that we can represent horofunctions via infinite nested sequences of atoms, namely if u is an element of $\partial_h\Gamma$, then there exists a unique nested sequence $(u_k)_{k=1}^{\infty}$ such that u_k is a k-level atom and the horofunction corresponds to the sequence by virtue of the homeomorphism. In symbols we will simply write $u = (u_k)_{k=1}^{\infty}$. We will frequently refer to this representation as the **atom-coding of the horofunction**.

Remark 1.40. Let $u = (u_k)_{k=1}^{\infty}$ be a horofunction. If we denote with f_u the representative of u such that $f_u(x_0) = 0$, then $f_u|_{B_k} = f_{u_k}$ with f_{u_k} defined as the restriction on B_k of f_x for all $x \in u_k$, where f_x is the function introduced in ($\mathfrak{F}$).

1.5 Balls, Cones and Atoms

This section is devoted to describing some further properties of atoms. In particular, we will present a first relation with the Gromov boundary and others with cones and balls which will be useful to establish a full connection with $\partial\Gamma$. We will also define Gromov products on atoms and horofunctions in a different (and improper, but very useful) way. Unless specified, the results contained in this section are to be considered new.

Before starting, we clarify a notation that will be used from now on. If $x \in \Gamma$ and $B \subseteq \Gamma$, then with $\mathrm{d}_\Gamma(x, B)$ we mean the mininum over all elements of B of their distances with respect to d_Γ from x.

Also, in this section we deal with two different definitions of *type* in two different situations. This occurs because usually in the literature the collection of types refers to a partition of the vertices.

We begin defining some collections of points.

Definition 1.41. Let Γ be a hyperbolic graph with a distinguished point x_0 and let x be an element in $\Gamma - B_k$ for some $k \in \mathbb{N}$.

(a) A **nearest neighbor** for x is a vertex $\overline{x}$ in B_k such that $\mathrm{d}_\Gamma(B_k, x) = \mathrm{d}_\Gamma(\overline{x}, x)$.

(b) A point $p \in B_k$ is **visible** for x if $[p, x] \cap B_k = \{p\}$ for every geodesic $[p, x]$ from p to x.

(c) A point $p \in S_n$ with $n \leq k$ is said to be n-proximal (or simply **proximal** when the n is clear) to x if there exists a geodesic $[x_0, p]$ such that for every geodesic $[x_0, x]$ we have

$$\mathrm{d}_\Gamma(p_i, x_i) \leq 4\delta + 2 \text{ for } 1 \leq i \leq n$$

where p_i is the i-th vertex of $[x_0, p]$ and x_i is the i-th vertex of $[x_0, x]$.

We denote with $N(x, B_k)$ the collection of nearest neighbors in B_k of x; and with $V(x, B_k)$ and $P(x, S_n)$ the collections of visible points and proximal points respectively.

Lemma 1.42. *Let Γ be a hyperbolic graph and let $x \in a$ for some k-level atom a. Then the following properties hold.*

(a) Every nearest neighbor of x is visible and every visible point is proximal, in short $N(x, B_k) \subseteq V(x, B_k) \subseteq P(x, S_k)$.

(b) A point p is k-proximal if and only if there exists a $(k-1)$-proximal point at

distance 1 *from* p *and* $\mathrm{d}_\Gamma(p,q) \leq 4\delta + 2$ *for all* $q \in V(x, B_k)$*. In particular, the diameter of* $P(x, S_k)$ *is bounded by* $8\delta + 4$*.*

(c) *If* p *is a nearest neighbor in* B_k *for* $x \in a$*, then it is a nearest neighbor for all elements in* a*. The same statement is true for* $V(x, B_k)$ *and* $P(x, S_k)$*.*

An immediate consequence is that $N(a, B_k)$, $V(a, B_k)$ and $P(a, S_k)$ are well-defined.

Proof. The proofs of all these statements can be found in [BBM21]: the first part of (a) is straightforward, for the second see Proposition 3.19, for (b) see Proposition 3.21 and Corollary 3.22, while for (c) we need to put together Proposition 3.15 and Corollary 3.24. □

We turn now to the other important notion we will care about, but before doing that we note that if $\overline{x}$ belongs to $N(x, B_k)$ for some $x \in \Gamma - B_k$, then there exists a geodesic $[x_0, x]$ via $\overline{x}$, i.e. $\overline{x} \in [x_0, x]$. In fact, this can be seen as an alternative definition.

Definition 1.43. Let Γ be a hyperbolic graph with a distinguished point x_0. If p is a vertex of Γ, then we define its **cone** to be

$$C(p) := \{x \in X \mid \mathrm{d}_\Gamma(x_0, x) = \mathrm{d}_\Gamma(x_0, p) + \mathrm{d}_\Gamma(p, x)\}.$$

Equivalently, a point x is in the cone of p if and only if there is a geodesic $[x_0, x]$ such that $p \in [x_0, x]$.

For the case $\Gamma = \Gamma(G, S)$ and $x_0 = id$ with G some hyperbolic group, we can define the so-called *cone types*.

Definition 1.44. Let G be a hyperbolic group and let S be one of its generating set. If $g \in G$ then its **cone type** is the collection

$$\{h \in G \mid \ell(hg) = \ell(h) + \ell(g)\}$$

where g is intended as a geodesic between id and g.

In particular, if g_1 and g_2 have the same cone type then the map $g_1^{-1}g_2$ is an isometry of cones.

Even if they are largely studied, we just report the main result about them (due to Cannon)

Proposition 1.45. *Let G be a hyperbolic group. Then the number of cone types of its Cayley graph is finite.*

Proof. See e.g. Proposition 7.5.4 in [Lö17]. For a different approach, see [Can84]. □

We now come back to our purpose, that is to link atoms with the Gromov boundary and so to the Gromov product. A property which will describe the behavior of the product over an atom is the following

Remark 1.46. If $p \in S_k$, i.e. $\mathrm{d}_\Gamma(x_0, p) = k$, then the Gromov product restricted to its cone is more than or equal to k. Indeed, taken $x, y \in C(p)$ we have

$$(x \mid y) = \frac{1}{2}\left[k + \mathrm{d}_\Gamma(p, x) + k + \mathrm{d}_\Gamma(p, y) - \mathrm{d}_\Gamma(x, y)\right]$$

and a triangle inequality yields the claim.

One can see that there is a weak connection between cones and atoms, namely

Proposition 1.47. *Let* Γ *be a hyperbolic graph with a distinguished point* x_0 *and let* a *be a* k*-level atom. Then*

$$a \subseteq \bigcap_{p \in N(a,B_k)} C(p) - \bigcup_{q \in S_k - N(a,B_k)} C(q).$$

Proof. The fact that a lies in the intersection of its nearest neighbors cones follows immediately from the definition.

Now suppose that $x \in a$ and $x \in C(q)$ with $q \in S_k - N(a, B_k)$. This means that there exists $p \in B_k$ such that $\mathrm{d}_\Gamma(q,x) > \mathrm{d}_\Gamma(p,x)$. Hence $\mathrm{d}_\Gamma(x_0,x) = \mathrm{d}_\Gamma(x_0,q) + \mathrm{d}_\Gamma(q,x) \geq \mathrm{d}_\Gamma(x_0,p) + \mathrm{d}_\Gamma(q,x) > \mathrm{d}_\Gamma(x_0,p) + \mathrm{d}_\Gamma(p,x) \geq \mathrm{d}_\Gamma(x_0,x)$, and this proves the claim. □

The goal should be to fully characterize atoms in geometric terms (i.e. via cones). Even though this question is still open, we can bind atoms with the so called N*-types* introduced by Cannon (see [Can84] and [CP93] for other applications). If Γ is the Cayley graph of an hyperbolic group, we say that two elements x and y **have the same N-type** if $\mathrm{d}_\Gamma(xz, id) - \mathrm{d}_\Gamma(x, id) = \mathrm{d}_\Gamma(yz, id) - \mathrm{d}_\Gamma(y, id)$ for all $z \in B_N(id)$. In fact, they are a useful tool to prove Proposition 1.45.

Remark 1.48. Let Γ be the Cayley graph of a hyperbolic group. Then $x, y \in \Gamma$ belong to the same N-level atom if and only if x^{-1} and y^{-1} are of the same N-type. This is straightforward once we notice that $\mathrm{d}_\Gamma(x, id) = \mathrm{d}_\Gamma(x^{-1}, id)$ and $\mathrm{d}_\Gamma(xz, id) = \mathrm{d}_\Gamma(z, x^{-1})$.

The problem is that, again, the connection between N-types and cones is yet to be fully understood.

A topic on which we can say something is the topology of atoms in $\partial\Gamma$. More precisely, exploiting the correspondence given by π_h, we can define the **shadows** of a k-level atom a as

$$\partial a := \{u \in \partial_h \Gamma \mid f_u = f_{a_k} \text{ on } B_k\}.$$

We know that they are closed subsets of $\partial_h \Gamma$ and since π_h is a closed map (it is continuous from a compact space to a metrizable space) we get that $\partial a \pi_h$ is closed in $\partial \Gamma$.
In this context, the diameter of $\partial a \pi_h$ with respect to the visual metric is bounded above by β^{-k} with $a \in \mathcal{A}_k$.

We return for a moment to our (otherwise implicit) group action of G on Γ. In particular, we use it to induce a notion of morphisms between subtrees of $\mathcal{A}(\Gamma)$.

Definition 1.49. Let G be a group that acts geometrically on a hyperbolic graph Γ with a distinguished point x_0. Let $a_n \in \mathcal{A}_n$ and $a_m \in \mathcal{A}_m$. We say that an element $g \in G$ **induces a morphism** between a_n and a_m if

- $a_n g = a_m$,
- $(a_n \cap B_{n+k})g = a_m \cap B_{m+k}$ for all $k \geq 0$,
- for each $k > 0$ and each atom $\tilde{a}_{n+k} \in \mathcal{A}_{n+k}$ contained in a_n, there exists an atom $\tilde{a}_{m+k} \in \mathcal{A}_{n+k}$ contained in a_m such that $\tilde{a}_{n+k} g = \tilde{a}_{m+k}$.

The second condition can be restated in these terms

$$(\mathfrak{L}) \qquad \mathrm{d}_\Gamma(x_0, xg) - m = \mathrm{d}_\Gamma(x_0, x) - n \ \ \forall x \in a_n.$$

The third condition fits into the context of subtrees of $\mathcal{A}(\Gamma)$. In fact, we can see the atoms that are contained in a fixed one a as a subtree $\mathcal{A}(\Gamma)_a$ rooted in

a and the condition admits the existence of an isomorphism between two such subtrees given by g. With this in mind, we give the following

Definition 1.50. Let G be a group that acts geometrically on a hyperbolic graph Γ with a distinguished point x_0. Two atoms of Γ are of the same **type** if there exists a morphism (given by an element of G) between them.

As for cones, the following important property holds (see [BBM21] for a complete proof and Chapter 4 for further details).

Theorem 1.51. *Let G be a group that acts geometrically on a hyperbolic graph Γ with a distinguished point x_0. Then the number of different types of atoms in $\mathcal{A}(\Gamma)$ is finite.*

Using these considerations, we get a tool, useful in the next chapter, about atoms and their balls:

Hooking Lemma 1.52. *There exists a constant λ_a that bounds the distances between atoms and their balls.*

The symbol λ_a will be used from now on to denote the **hooking constant**.

Proof. Condition ($\mathfrak{L}$) says that the distance between an atom and its ball depends only on the type of the atom. Since the graph is hyperbolic we know it has a finite number of types (by Theorem 1.51). These two facts combined together allow us to consider the maximum over all the types of the distances and to get the constant. □

Open Question. *Is there a way to express λ_a with respect to δ?*

The last collection of vertices we define are tips.

Definition 1.53. We call the **tip** of an atom $a \in \mathcal{A}_k$ the collection $T(a) := \{x \in a \mid \mathrm{d}_\Gamma(B_k, x) = \mathrm{d}_\Gamma(B_k, a)\}$, or, equivalently, the first non-empty intersection $a \cap S_{k+i}$ with $i \geq 0$.

Proposition 1.54. *Let a be an atom of Γ. Then*

$$\operatorname{diam} T(a) := \max_{x,y \in T(a)} \mathrm{d}_\Gamma(x, y)$$

is bounded by $2\lambda_a$.

Proof. It suffices to construct a triangle made by geodesics with two elements of $T(a)$ and a nearest neighbor of a as vertices, namely if $\widehat{x}, \widehat{y} \in T(a)$ and $\overline{x} \in N(a, B_k)$ then $\mathrm{d}_\Gamma(\widehat{x}, \widehat{y}) \leq \mathrm{d}_\Gamma(\widehat{x}, \overline{x}) + \mathrm{d}_\Gamma(\overline{x}, \widehat{y})$. The claim follows by the Hooking Lemma. □

To understand how small a tip could be, we give the following

Lemma 1.55. *Let $a_k \in \mathcal{A}_k$. Suppose $B_k \cap a_k \neq \emptyset$, then $B_k \cap a_k$ consists of one point. Which means that $T(a_k) = N(a_k, B_k)$ consists of one point.*

Proof. The fact that $B_k \cap a_k = T(a_k) = N(a_k, B_k)$ is straightforward.
It suffices to show that $T(a_k)$ is a point: given $x, y \in T(a_k)$, and hence to S_k, by definition of atom we get

$$\mathrm{d}_\Gamma(x, y) - k = \mathrm{d}_\Gamma(x, x) - k = -k$$

that is $\mathrm{d}_\Gamma(x, y) = 0$ and the claim holds. □

Example 1.56. We consider again the uniform tiling of the hyperbolic plane made of squares such that each vertex has degree 5. Recall that we have ten 1-level atoms. They divide equally into two different types. One type has the property

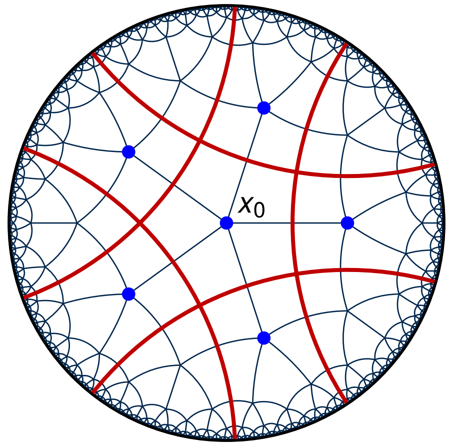


Figure 9: In blue the ball of radius 1. Each element of the sphere coincides with the tip of a 1-level atom.

described above. Indeed, each element of the sphere S_1 is the tip of one of the five atoms. See Figure 9.

Remark 1.57. It is worth pointing out that two different atoms a and b such that b is a child of a can have the same tip. This could happen in one of the following situations

– the atom b is the only child of a and it is isometric to it. For an example see narrow type atoms in Section 3.3.

– the atom a splits in different children, but one of them has the same tip. See type C and D atoms in Example 4.8.

A first application of tips is that the two projections π_h and $\partial\Gamma \twoheadrightarrow \mathcal{E}_{\mathrm{T}}(\Gamma)$ are compatible.

Remark 1.58. We induce the map $\widehat{\iota} : \partial_h\Gamma \twoheadrightarrow \mathcal{E}_{\mathrm{T}}(\Gamma)$ from the family of surjective maps ι_k from $\mathcal{A}_k$ to the collection of connected components of $\Gamma - B_{k-1}$ defined in the following way: $a_k \mapsto \mathcal{C}_k$ if $a_k \subseteq \mathcal{C}_k$. Note that any k-level atom a is fully contained in a connected component of $\Gamma - B_{k-1}$ since, if $x, y \in a$, in order to get a path between the two of them, one can consider a geodesic going from x to a nearest neighbor $\overline{x}$ and then a geodesic from $\overline{x}$ to y. Now, taking a horofunction $u = (u_k)_{k=1}^{\infty}$ its image is $\mathcal{C}_u = (\mathcal{C}_k^u)_{k=1}^{\infty}$ with $u_k \subseteq \mathcal{C}_k^u$ and $u_{k+1} \subsetneq u_k \subsetneq \mathcal{C}_k^u$, so $\mathcal{C}_{k+1}^u \subseteq \mathcal{C}_k^u$. Moreover, we claim that the diagram below commutes.

$$\begin{tikzcd}
\partial_h \Gamma \arrow[d, two heads, "\pi_h"] \arrow[dr, two heads, "\widehat{\iota}"] & \\
\partial \Gamma \arrow[r, two heads] & \mathcal{E}_{\mathrm{T}}(\Gamma)
\end{tikzcd}$$

To prove this, let $[\gamma_u]$ be a point in the Gromov boundary such that $u\pi_h = [\gamma_u]$. Let $\mathcal{C}_\gamma \in \mathcal{E}(\Gamma)$ such that $end(\gamma) = \mathcal{C}_\gamma$ and $\mathcal{C}_\gamma$ corresponds to some sequence $(\mathcal{C}_k)_{k=1}^{\infty} \in \mathcal{E}_{\mathrm{T}}(\Gamma)$. We want $\mathcal{C}_\gamma \simeq \mathcal{C}_u$.
Now $\mathcal{C}_n$ is such that $\gamma([0, \infty[) \subseteq \mathcal{C}_n$ and there exists a sequence $\{\widehat{x}_k\}_{k=1}^{\infty}$ in Γ such that $\widehat{x}_k \in T(u_k)$, hence going to infinity in the sense of Gromov, that converges to $u \in \partial_h \Gamma$ such that $(\widehat{x}_k \mid \gamma(k)) \to \infty$. So $\widehat{x}_k$ and $\gamma(k)$ need to be in the same connected component. Otherwise

$$\mathrm{d}_\Gamma(\widehat{x}_k, x_0) + \mathrm{d}_\Gamma(\gamma(k), x_0) - \mathrm{d}_\Gamma(\widehat{x}_k, \gamma(k)) \leq \lambda_a + 2\overline{k} \qquad \forall k > \overline{k}.$$

Indeed, $\mathrm{d}_\Gamma(\widehat{x}_k, x_0) \leq k + \lambda_a$ for the Hooking Lemma, $\mathrm{d}_\Gamma(\gamma(k), x_0) = k$ since γ is a geodesic ray and $\mathrm{d}_\Gamma(\widehat{x}_k, \gamma(k)) \geq 2k - 2\overline{k}$ where $\overline{k} \in \mathbb{N}$ is such that $\mathcal{C}_k$ is the first component not containing $\widehat{x}_{\overline{k}}$. It follows that $\mathcal{C}_n = \mathcal{C}_n^u$ and hence the claim.

The conclusion of this section is devoted to describing a different Gromov product for atoms. Definition (Π) of product that we gave can be applied to $\mathcal{A}$ as a tree, we will denote it with $(\cdot \mid \cdot)_{\mathcal{A}}$. It will be useful in the next chapters and it has an associated visual metric $\Upsilon_{\mathcal{A}}$. What we want to do here is to define a Gromov product between atoms of the same level and one between horofunctions (recall Theorem 1.39 and the atom-coding) that keeps track of what is happening in the underlying hyperbolic graph. Namely, we put

$$(a_k \mid b_k)_k := \max_{\widehat{x} \in T(a_k), \widehat{y} \in T(b_k)} (\widehat{x} \mid \widehat{y}) \text{ and } (u \mid v) := \liminf_k (u_k \mid v_k)_k$$

We will always drop the k in the notation, as it will be clear from the context.

We need to check that this notion somehow agrees with the one regarding points of the Gromov boundary.

Lemma 1.59. *Let u and v be two horofunctions of Γ. Then*

$$(u\pi_h \mid v\pi_h) \leq (u \mid v) \leq (u\pi_h \mid v\pi_h) + 2\tilde{\delta}.$$

Proof. By Definition 1.28, we have $(u\pi_h \mid v\pi_h) \leq (u \mid v)$. For the second inequality, we start by saying that

$$(u \mid v) \leq \max_{x_k \in T(u_k), y_k \in T(v_k)} \liminf_k (x_k | y_k),$$

here we mean that the maximum has to be taken over all the possible pairs of sequences of vertices such that $x_k \in T(u_k)$ and $y_k \in T(v_k)$.
Now

$$\max_{x_k \in T(u_k), y_k \in T(v_k)} \liminf_k (x_k | y_k) \leq \sup_{\{x_k\} \in u\pi_h, \{y_k\} \in v\pi_h} \liminf_k (x_k | y_k),$$

hence combining the two inequalities and using Remark 1.29(c), we get $(u \mid v) \leq (u\pi_h \mid v\pi_h) + 2\tilde{\delta}$. □

1.6 Languages and Automata

This section is devoted to recalling some basic definitions in Language Theory, to fix some notations, but most importantly to discuss the connection between (self-similar) trees and languages and to formalize what it means for a compact metrizable space to be encoded via a Cantor set.

We start by setting Σ as our finite **alphabet**, that is a finite collection of symbols $\sigma \in \Sigma$. Since we need more alphabets at the same time, we will use also $\tilde{\Sigma}$, Ξ and $\tilde{\Xi}$. From an alphabet Σ, we can construct two different objects: the collection of all finite strings Σ^* and the collection of all infinite strings Σ^ω. A language is a subcollection of Σ^* or of Σ^ω. We will usually deal with infinite strings, the reason is that Σ^ω is a Cantor set (see Example 1.27) and hence it provides the geometric feature we need. Even if it is clear intuitively, we will make it more formal later.

Since a languange can be any subcollection, we need tools to represent and to classify it in a reasonable and constructible way. These tools are called *machines*. Among all the machines one can find in the literature, one of the most famous is the *synchronous deterministic finite state automaton*. Before giving the formal definition, we recall that a *partial function* is a binary relation between two sets that associates to every element of the first set at most one element of the second.

Definition 1.60. A **synchronous deterministic finite state automaton** is a quadruple $(\Sigma, \Theta, \rightarrow, \theta_0)$ with Θ a finite set called the *set of states*, $\theta_0 \in \Theta$ called *initial state* and a partial function $\rightarrow$ between $\Theta \times \Sigma$ and Θ which is called the *transition function*.

The informal idea is that one moves through the states and processes an element of Σ every time a movement occurs and the element determines the direction of the movement. We use the word "process", since it can be seen as reading a (finite or infinite) string in input or as producing a string while moving on the machine.

The machine is referred to as finite state for obvious reasons, while it is deterministic since $\to$ is an actual (partial) function, which means that there cannot be two different transitions starting from a state and processing the same element of the alphabet, and it is synchronous because it processes one element of Σ at each step. We will drop all the adjectives and we will simply call it an automaton, this because it will be the only machine of this type.

We say that an automaton **recognizes** a language when a string belongs to the language if and only if it is processed by the automaton. Note that the fact the function is partial plays a key role here. In general, at each given state of an automaton, we may not have transition functions for all possible elements of Σ (i.e. there might be letters that are not sent to anything).

Definition 1.61. If a language of infinite strings is recognized by an automaton, it is called a **rational subset** of Σ^ω.

We also want machines that turn languages into other languages. Again, we will consider just one of them:

Definition 1.62. An **asynchronous deterministic finite state transducer** is a quintuple $(\Sigma, \Xi \cup \{\varepsilon\}, \Theta, \to, out, \theta_0)$ with Θ a finite set called the *set of states*, $\theta_0 \in \Theta$ called *initial state*, a *transition function* defined as $(\theta_1, \sigma) \to \theta_2$ with $\theta_1, \theta_2 \in \Theta$ and $\sigma \in \Sigma$ and the *output function* defined as $(\theta, \sigma)out = \xi_1 \dots \xi_n$ with $\theta \in \Theta$, $\sigma \in \Sigma$ and $\xi_i \in \Xi$ or $\xi_i = \varepsilon$ the **empty string**.

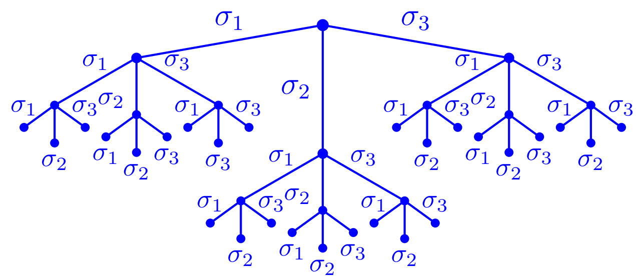


Figure 10: Strings of length 3 based on the alphabet $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ in their geometric representation.

In this case, the interpretation is that we have an input alphabet and an output one, we read in input a string and we determine a string in output by using the machine.

As before, we will drop all the adjectives and we will simply call it a transducer.

A first step towards geometry is the following: we can see the set of finite strings Σ^* as a rooted regular tree where the number of children of each vertex is the cardinality of Σ and the root is the empty string ε. Two strings share an edge if and only if the difference of their lengths is 1 and one is the prefix of the other (see Figure 10).

It follows immediately that the boundary of the tree is Σ^ω and hence the latter is a Cantor set. We now want to define maps between Cantor sets by using transducers:

Definition 1.63. A map ϕ between two Cantor sets Σ^ω and Ξ^ω is called **rational** if there exists a transducer such that $w\phi = (\theta_0, w)out$ for all $w \in \Sigma^\omega$.

One can show that these maps form a category. Moreover, they are continuous with respect to the product topology and if they are bijective, then they are homeomorphisms (see [GNS00, Subsection 2.3]).

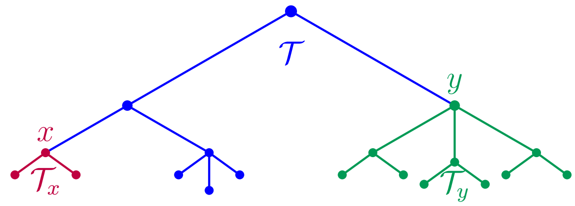


Figure 11: Two examples of rooted subtrees.

Definition 1.64. A bijective rational map is called a **rational homeomorphism**.

It is also true that the inverse of a rational homeomorphism is a rational homeomorphism. So, fixing an alphabet Σ, we can define the *group of rational homeomorphisms* $\mathcal{R}$ over Σ. Since two different alphabets (with at least two elements) give two isomorphic groups, we refer to one of them simply as $\mathcal{R}$, without mentioning the underlying alphabet.

For a generalization of this setting to non-finite state machines and to better understand the topic see [GNS00]. Here, we just mention the following result as it will be useful later.

Proposition 1.65 ([GNS00], Proposition 2.11)**.** *A set $\mathcal{L} \subseteq \Xi^\omega$ is rational if and only if it is the image of a rational map $\phi : \overline{\Xi}^\omega \to \Xi^\omega$ with $\overline{\Xi}$ a finite alphabet.*

A language is any subcollection of Σ^ω, but we are only interested in the rational ones. And we have a geometric interpretation of Σ^*. We want to consider rooted trees that are not necessarily regular, but still have some nice properties. Given a rooted tree $\mathcal{T}$ and a vertex x, we can consider the rooted subtree $\mathcal{T}_x$ such that the root is x (see Figure 11).

Definition 1.66. A **self-similar structure** on a rooted tree $\mathcal{T}$ is a partition of the vertices into finitely many classes together with a finite set of rooted tree isomorphisms between $\mathcal{T}_x$ and $\mathcal{T}_y$ for each pair of vertices x and y in the same class and where each isomorphism maps vertices of $\mathcal{T}_x$ to vertices of $\mathcal{T}_y$ in the same class.

Moreover, we want the isomorphisms to satisfy some natural conditions. They are closed under taking the inverse, composition and restriction. Namely, if $\varphi : \mathcal{T}_x \to \mathcal{T}_y$ and $\psi : \mathcal{T}_y \to \mathcal{T}_z$ belong to the self-similar structure, then φ^{-1}, $\varphi\psi$ and $\varphi_{|\mathcal{T}_{x'}} : \mathcal{T}_{x'} \to \mathcal{T}_{x'\varphi}$ belong to the structure too.

A tree $\mathcal{T}$ with a self-similar structure is called **self-similar**. Note that the equivalence classes are originally called types, though we do not use that terminology, to avoid confusion.

If we set Σ to be the set of classes of a self-similar tree $\mathcal{T}$, it can be seen, but we will not give further details here, that there is a projection of the Gromov boundary $\partial\mathcal{T}$ onto a subset $\mathcal{L}$ of Σ^ω. Furthermore, one can see that $\mathcal{L}$ is a rational subset and, in fact, that any rational subset can be characterized geometrically in this way (see e.g. [BBM21, Subsection 2.1]). In order to make the projection bijective, which means that $\partial\mathcal{T}$ is a subset of Σ^ω, we need to rely on a different coding that comes from *rigid structures*. These are again self-similar structures, but with some further hypothesis, and we will see that we can get a rigid one starting from a self-similar structure which is not rigid a priori. All the details are provided in Chapter 4.

An important example of self-similar tree associated to a hyperbolic graph is the tree of atoms. In fact, it was introduced to prove the following

Theorem 1.67. *Let G be a hyperbolic group. Then every element of G acts on $\mathcal{A}(G)$ as a rational homeomorphism.*

Again, see ([BBM21]) for a full treatment of the topic.

The last connection between the geometric and the symbolic points of view is due to this well-known property of Cantor sets:

Proposition 1.68 ([Wil70], Theorem 30.7)**.** *Let X be a compact metrizable space. Then there exists a continuous surjective map from a Cantor set onto X.*

Exploiting the discussions we made in this section, every compact metrizable space can be *encoded* by a collection of infinite strings Σ^ω (it could be a proper subcollection of the whole space as far as the map is still surjective). Namely, every element is represented by a (non-necessarily unique) infinite string. Such a string is called a **coding**.

Example 1.69. Set $\Sigma = \{0, 1, \ldots, 9\}$ and let $[0, 1]$ be the unit interval. Then each element of $[0, 1]$ has at most 2 codings due to the decimal expansion: exactly one if the number is irrational and exactly two if the number is rational. Namely, $\sigma_1\sigma_2 \ldots \sigma_n\overline{0}$ and $\sigma_1\sigma_2 \ldots (\sigma_n - 1)\overline{9}$ where $\sigma_n - 1$ is the difference *mod 10* and $\overline{\sigma} = \sigma\sigma \ldots \sigma \ldots$.

Remark 1.70. Despite the presence of a tree and choice of using the same word, to avoid ambiguity, it is worth mentioning that atom-coding is not coding in this sense.

CHAPTER 2

GEODESIC BEHAVIOR OF ATOM-CODINGS

In this chapter we will present some properties of atom-codings that are similar to the ones of geodesic rays. In the first section, we will deal with a coarse version of the so called fellow traveler property using only elementary definitions and a result that comes from an analytic point of view. The second section exploits the notion of tips, which can be seen as a coarse version of the spheres in the following sense: considering the set of tips for a fixed level k, one get a subset of a bounded annulus. Tips lead to the exponential divergence result. Despite the differences, this way of thinking shares many points with [CP93], in particular the proof of finiteness of fibers of π_h uses the same technique provided there. The last section contains the stronger results and is based on the following strategy: first we will associate quasi-geodesic rays to atom-codings, then finite geodesics and in the end also geodesic rays. This approach allows us to put together and generalize the previous sections and gives a first approximation of the Gromov boundary via atoms.

2.1 Gluing Relation via Atoms

Now that we developed the two points of view for the horofunction boundary of a hyperbolic graph Γ and some tools regarding the tree of atoms, we want to find a connection between them to understand the gluing relation given by the quotient map $\pi_h : \partial_h\Gamma \twoheadrightarrow \partial\Gamma$. More precisely, the goal is a way to determine when two horofunctions are glued by looking at the tree of atoms.

In order to do that, we introduce the following

Definition 2.1. Let Γ be a hyperbolic graph and let $\mathcal{A}(\Gamma)$ be its tree of atoms. For all k, we choose d^k a semi-metric defined on the k-level of $\mathcal{A}(\Gamma)$. We say that d^k **represents the gluing** if there exists a constant C such that given two horofunctions $u = (u_k)_{k=1}^{\infty}$ and $v = (v_k)_{k=1}^{\infty}$, then

$$u\pi_h = v\pi_h \quad \Leftrightarrow \quad \forall k \geq 0 \ \mathrm{d}^k(u_k, v_k) \leq C.$$

In this way, we see that, under such a family of semi-metrics, the horofunctions which glue have a similar, or maybe we should say generalized, behavior of two geodesics that glue on $\partial\Gamma$.

We are going to introduce three semi-metrics and to prove that they represent the gluing.

The first one we consider comes from a natural way to think about distances between atoms. Let a and b two k-level atoms, then

$$\mathrm{d}_\Gamma{}^k(a, b) := \min_{x \in a, y \in b} \mathrm{d}_\Gamma(x, y).$$

Since $\mathrm{d}_\Gamma{}^k$ is defined between atoms of the same level k, we will always refer to $\mathrm{d}_\Gamma{}^k$ simply as d_Γ when k will be already specified by a and b. Whenever we will deal with atoms of different levels and hence with $\mathrm{d}_\Gamma{}^k$ and $\mathrm{d}_\Gamma{}^{k+1}$, we will make it clear. Moreover, in statements and discussions we may consider the collection of semi-metrics $\{\mathrm{d}_\Gamma{}^k\}_{k=1}^{\infty}$ and we will simply say *the semi-metric* d_Γ (note that d_Γ defined on Γ is instead a metric).

Since we are interested in horofunctions, given $u = (u_k)_{k=1}^{\infty}$ and $v = (v_k)_{k=1}^{\infty}$, we will consider $\{\mathrm{d}_\Gamma(u_k, v_k)\}_{k=1}^{\infty}$. The first thing we note is that the sequence of distances is non-decreasing. Indeed, we take $x_{k+1} \in u_{k+1}$ and $y_{k+1} \in v_{k+1}$ such that $\mathrm{d}_\Gamma(u_{k+1}, v_{k+1}) = \mathrm{d}_\Gamma(x_{k+1}, y_{k+1})$. But $x_{k+1} \in u_k$ and $y_{k+1} \in v_k$, hence by definition

$$\mathrm{d}_\Gamma(u_k, v_k) \leq \mathrm{d}_\Gamma(x_{k+1}, y_{k+1}).$$

Despite the fact that we are looking to $\partial_h\Gamma$ from a geometric viewpoint, we can adopt a more analytic distance

$$\mathrm{d}_h(u, v) := \|f_u - f_v\|_\infty.$$

In particular, we have the following result which is clearly related to our goal and will be useful for our proof.

Proposition 2.2. *Let Γ be a hyperbolic graph. Let u and v be two horofunctions of Γ and let $f_u \in u$ and $f_v \in v$ be two representatives such that $f_u(x_0) = f_v(x_0) = 0$. Then u and v are glued on $\partial\Gamma$ if and only if there exists λ_h (independent from u and v) such that*

$$d_h(u, v) := \|f_u - f_v\|_\infty < \lambda_h.$$

We will always denote this constant with λ_h.

Proof. See [WW05, Proposition 4.4] □

Note that if $u = (u_k)_{k=1}^\infty$ is a horofunction and f_u is the representative such that $f_u(x_0) = 0$. When we consider its restriction to B_k, we get

$$f_{u_k} := \mathrm{d}_\Gamma(-, u_k) - \mathrm{d}_\Gamma(x_0, u_k)$$

and $\mathrm{d}_h(u_k, v_k) = \|f_{u_k} - f_{v_k}\|_\infty$.

With the following results we link the geometric distance with the analytic one.

Lemma 2.3. *For all $k \in \mathbb{N}$ and for all pairs of k-level atoms u_k and v_k we have*

$$\frac{1}{2}\,\mathrm{d}_h(u_k, v_k) \leq \mathrm{d}_\Gamma(u_k, v_k) \leq \mathrm{d}_h(u_k, v_k) + 2\lambda_a.$$

Proof. The first part follows by two triangle inequalities. Indeed, for all $p \in \Gamma$ we have

$$|\,\mathrm{d}_\Gamma(u_k, p) - \mathrm{d}_\Gamma(v_k, p)| \leq \mathrm{d}_\Gamma(u_k, v_k),$$

and since we also have

$$\begin{aligned}
&|\,\mathrm{d}_\Gamma(u_k, p) - \mathrm{d}_\Gamma(u_k, x_0) - \mathrm{d}_\Gamma(v_k, p) + \mathrm{d}_\Gamma(v_k, x_0)| \leq \\
&\qquad\qquad |\,\mathrm{d}_\Gamma(u_k, p) - \mathrm{d}_\Gamma(v_k, p)| + |\,\mathrm{d}_\Gamma(v_k, x_0) - \mathrm{d}_\Gamma(u_k, x_0)|,
\end{aligned}$$

with $p \in B_k$, by applying the first inequality with two times, we get $\mathrm{d}_h(u_k, v_k) \leq 2\,\mathrm{d}_\Gamma(u_k, v_k)$.

For the second part, we start by taking $x \in u_k$ and $\overline{x} \in B_k$ (resp. $y \in v_k$ and $\overline{y} \in B_k$) such that

$$\mathrm{d}_\Gamma(B_k, u_k) = \mathrm{d}_\Gamma(\overline{x}, x) \;\; (\text{resp. } \mathrm{d}_\Gamma(B_k, v_k) = \mathrm{d}_\Gamma(\overline{y}, y)).$$

By definition of f_{v_k} and $\overline{y}$ we know that $f_{v_k}(\overline{y}) = -k$. This implies that $f_{u_k}(\overline{y}) + k \leq d_h(u_k, v_k)$. Equivalently, we have $\mathrm{d}_\Gamma(\overline{y}, x) - \mathrm{d}_\Gamma(x_0, x) + k \leq d_h(u_k, v_k)$ and by using the definition of $\overline{x}$, we get

$$\begin{aligned}
&\mathrm{d}_\Gamma(\overline{y}, x) - [\mathrm{d}_\Gamma(x_0, \overline{x}) + \mathrm{d}_\Gamma(\overline{x}, x)] + k = \\
&\qquad\qquad \mathrm{d}_\Gamma(\overline{y}, x) - [k + \mathrm{d}_\Gamma(\overline{x}, x)] + k \leq d_h(u_k, v_k).
\end{aligned}$$

Which means that $\mathrm{d}_\Gamma(\overline{y}, x) \leq \mathrm{d}_\Gamma(x, \overline{x}) + d_h(u_k, v_k)$.

Now

$$\mathrm{d}_\Gamma(u_k, v_k) \leq \mathrm{d}_\Gamma(x, y) \leq \mathrm{d}_\Gamma(x, \overline{y}) + \mathrm{d}_\Gamma(\overline{y}, y),$$

by the discussion in the previous paragraph we get

$$\mathrm{d}_\Gamma(u_k, v_k) \leq \mathrm{d}_\Gamma(x, \overline{x}) + d_h(u_k, v_k) + \mathrm{d}_\Gamma(\overline{y}, y)$$

and by the Hooking Lemma we have the thesis. □

By combining the previous facts, we get the main result about d_Γ.

Theorem 2.4. *Let Γ be a hyperbolic graph and let $\pi_h : \partial_h\Gamma \twoheadrightarrow \partial\Gamma$ be the projection of the horofunction boundary onto the Gromov boundary. Let $u = (u_k)_{k=1}^{\infty}$ and $v = (v_k)_{k=1}^{\infty}$ be two horofunctions expressed via their infinite sequences of infinite atoms. The following are equivalent.*

(A) The horofunctions are glued on $\partial\Gamma$ i.e. $u\pi_h = z_\infty = v\pi_h$ for some $z_\infty \in \partial\Gamma$.

(B) There exists a constant C and two sequences of vertices $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ such that $x_k \in u_k$ and $y_k \in v_k$, and $\mathrm{d}_\Gamma(x_k, y_k) \leq C$ for all $k \geq 1$.

From now on λ will be the **gluing constant**, that is the smallest constant such that the claim holds. Note that, since $\mathrm{d}_\Gamma(x_k, y_k)$ is a natural number, λ is well-defined and is a natural number.

Proof. Suppose that (B) holds. The fact that $x_k \in u_k$ for all $k \geq 1$ implies that $\{x_k\}_{k=1}^{\infty}$ converges to the horofunction u. It follows that $\{x_k\}_{k=1}^{\infty}$ goes to infinity in the sense of Gromov. Analogously we have that $\{y_k\}_{k=1}^{\infty}$ converges to v and goes to infinity in the sense of Gromov. A fortiori, the distances $\mathrm{d}_\Gamma(x_0, x_k)$ and $\mathrm{d}_\Gamma(x_0, y_k)$ go to infinity as $k \to \infty$. By hypothesis

$$(x_k \mid y_k) \geq \frac{1}{2}\left[\mathrm{d}_\Gamma(x_k, x_0) + \mathrm{d}_\Gamma(y_k, x_0) - C\right]$$

and (A) follows.

All that is left is to combine the previous results to get (B) starting from (A). By Lemma 2.3 we know that $\mathrm{d}_\Gamma(u_k, v_k) \leq d_h(u_k, v_k) + 2\lambda_a$ and by Proposition 2.2 we have that $d_h(u, v) < \lambda_h$, hence $d_h(u_k, v_k) < \lambda_h$. To conclude, we have that $\mathrm{d}_\Gamma(u_k, v_k) < \lambda_h + 2\lambda_a$ for all $k \geq 1$, so there exist $x_k \in u_k$ and $y_k \in v_k$ such that $\mathrm{d}_\Gamma(x_k, y_k) < C$ with $C = \lambda_h + 2\lambda_a$. $\square$

A straightforward consequence is what we were looking for.

Corollary 2.5. *The semi-metric* d_Γ *represents the gluing.*

The second distance we introduce is aimed to illustrate the fact that atoms which glue are near in an asymptotic way i.e. the distance occurs to be less than the gluing constant λ in an infinite number of points which lie in the atoms.

Definition 2.6. We define

$$\mathrm{d}_{\mathsf{F}}{}^{k}(a,b) := \sup_{A\subset a, B\subset b} \min_{\substack{x\in a-A \\ y\in b-B}} \mathrm{d}_\Gamma(x,y)$$

with a and b atoms of the k-level and the A and B finite sets.

Again, we will write d_{F} when k is clear or to mean the collection of $\{\mathrm{d}_{\mathsf{F}}{}^{k}\}_{k=1}^{\infty}$. Note that if $\mathrm{d}_{\mathsf{F}}(a,b)$ is finite, then the supremum is actually a maximum.

The aim, the definition and the discussion that follow show how we can think d_{F} as a limit of d_Γ.

A particular benefit of this distance is that, for a fixed level, it can be calculated by looking at the level below. In fact, $\mathrm{d}_{\mathsf{F}}(a,b)$ is the minimum over all the children of a and b of the distances between such children.

We start a comparison with d_Γ by stating some properties.

(1) The distance d_{F} is a semi-metric.

(2) If $a_{k+1} \subseteq a_k$ and $b_{k+1} \subseteq b_k$, then $\mathrm{d}_{\mathsf{F}}(a_k, b_k) \leq \mathrm{d}_{\mathsf{F}}(a_{k+1}, b_{k+1})$. This follows immediately from the definition (as for d_Γ) or using the discussion we made above.

(3) The distance d_Γ is less than or equal to the distance d_{F}. Indeed, if we put $A = B = \emptyset$ in the definition of d_{F} we recover d_Γ.

In general the converse of Property (3) is not true: it can happen that two atoms are d_Γ-adjacent but not d_{F}-adjacent. But something more specific can be stated.

Lemma 2.7. *Let $u = (u_k)_{k=1}^{\infty}$ and $v = (v_k)_{k=1}^{\infty}$ two horofunctions. If $\mathrm{d}_\Gamma(u_k, v_k) \leq C$ for all $k \geq 1$, then $\mathrm{d}_{\mathsf{F}}(u_k, v_k) \leq C$ for all $k \geq 1$.*

Proof. We fix k. We know that for all $n \geq k$ there exists a $\widehat{k} \geq k$ such that $u_{\widehat{k}} \subseteq u_k - B_n$ and $v_{\widehat{k}} \subseteq v_k - B_n$. So that

$$\min_{\substack{x \in u_k - B_n \\ y \in v_k - B_n}} \mathrm{d}_\Gamma(x, y) \leq \min_{\substack{x \in u_{\widehat{k}} \\ y \in v_{\widehat{k}}}} \mathrm{d}_\Gamma(x, y) = \mathrm{d}_\Gamma(u_{\widehat{k}}, v_{\widehat{k}}) \leq C.$$

It follows that $\sup_{n \geq k} \min_{\substack{x \in u_k - B_n \\ y \in v_k - B_n}} \mathrm{d}_\Gamma(x, y) \leq C$.

Taking a finite subset A of u_k and a finite subset B of v_k, there exists an n such that $A \cup B \subseteq B_n$. Hence $u_k - A \supseteq u_k - B_n$ and $v_k - B \supseteq v_k - B_n$, that implies

$$\min_{\substack{x \in u_k - A \\ y \in v_k - B}} \mathrm{d}_\Gamma(x, y) \leq \min_{\substack{x \in u_k - B_n \\ y \in v_k - B_n}} \mathrm{d}_\Gamma(x, y).$$

Finally, we get $\mathrm{d}_{\mathsf{F}}(u_k, v_k) \leq \sup_{n \geq k} \min_{\substack{x \in u_k - B_n \\ y \in v_k - B_n}} \mathrm{d}_\Gamma(x, y)$ and the claim follows. □

As an immediate consequence of Theorem 2.4 and Lemma 2.7 we have

Corollary 2.8. *The semi-metric d_{F} represents the gluing.*

Remark 2.9.

- In fact, we have proven that if there exists a level k such that $\mathrm{d}_{\mathsf{F}}(u_k, v_k) \leq C$, then there exist two horofunctions u and v with u_k and v_k respectively in their sequences such that $u\pi_h = v\pi_h$.
- If $u\pi_h \neq v\pi_h$, then $\mathrm{d}_\Gamma(u_k, v_k) \to \infty$. So there exists an index $\overline{k}$ such that $\mathrm{d}_{\mathsf{F}}(u_{\overline{k}}, v_{\overline{k}})$ is finite and $\mathrm{d}_{\mathsf{F}}(u_{\overline{k}+1}, v_{\overline{k}+1}) = \infty$.

We are interested in a distance which can represent the gluing, but also that increases exponentially when there is no gluing. It can easily be seen that d_Γ is too slow. Indeed, if we suppose that a and b are two k-level atoms, then ${\mathrm{d}_\Gamma}^k(a,b) \leq \mathrm{d}_\Gamma(x,x_0) + \mathrm{d}_\Gamma(x_0,y) \leq 2k + 2\lambda_a$ for some $x \in a$ and $y \in b$. On the other hand, d_F is too fast, as stated in the second point of the previous Remark. With this purpose in mind, we give the following

Definition 2.10. Let k be an integer greater or equal to 1. We consider $\Gamma - B_{k-1}$ as the induced subgraph with respect to the subset of vertices $\Gamma - B_{k-1}$. We can put a metric on $\Gamma - B_{k-1}$ which is the standard metric on the graph and we call it ${\mathrm{d}_\mathsf{B}}^k$.

The notation is subject to the same convention used before with d_Γ and d_F. Please note that ${\mathrm{d}_\mathsf{B}}^k$ is finite if and only if $\Gamma - B_{k-1}$ is connected and that, according to Proposition 1.38, it holds $\mathcal{A}_k \subseteq \Gamma - B_{k-1}$.
In the same way we defined the semi-metric d_Γ, we put

$$\mathrm{d}_\mathsf{B}(a,b) = \min_{x\in a, y\in b} \mathrm{d}_\mathsf{B}(x,y)$$

with $a, b \in \mathcal{A}_k$.
We note that d_B is an intrinsic metric with respect to d_Γ, and hence we know that $\mathrm{d}_\Gamma \leq \mathrm{d}_\mathsf{B}$ (see Remark 1.7), that is $\mathrm{d}_\Gamma(a,b) \leq \mathrm{d}_\mathsf{B}(a,b)$ for all $a, b \in \mathcal{A}_k$ and for all k. It follows that d_B is faster than d_Γ as a semi-metric (when defined on atoms). We are going to prove a technical lemma, which will be useful to prove that d_B represents the gluing.

Lemma 2.11. *Let $a, b \in \mathcal{A}_k$ such that $\mathrm{d}_\mathsf{F}(a,b) \leq C$. Then there exists $n \geq k$ such that there exist $\tilde{x} \in a - B_n$ and $\tilde{y} \in b - B_n$ with the following properties: $\mathrm{d}_\Gamma(\tilde{x},\tilde{y}) = \min_{\substack{x\in a-B_n\\ y\in b-B_n}} \mathrm{d}_\Gamma(x,y)$ and there exists a geodesic $[\tilde{x},\tilde{y}]_\Gamma$ (with respect to d_Γ) fully contained in*

$\Gamma - B_{k-1}$.

Proof. By hypothesis, we know that for all $n \geq k$ it holds we have $\min_{\substack{x \in a - B_n \\ y \in b - B_n}} \mathrm{d}_\Gamma(x, y) \leq C$. We take $n \geq \frac{C}{2} + k - 1$ and $\tilde{x}, \tilde{y} \in \Gamma$ such that

$$l := \mathrm{d}_\Gamma(\tilde{x}, \tilde{y}) = \min_{\substack{x \in a - B_n \\ y \in b - B_n}} \mathrm{d}_\Gamma(x, y).$$

Suppose that there exists $[\tilde{x}, \tilde{y}]_\Gamma = \{z_i\}_{i=0}^{l}$ and there exists a $j \in \{0, \ldots, l\}$ such that $z_j \in B_{k-1}$. Now

$$C \geq \mathrm{d}_\Gamma(\tilde{x}, \tilde{y}) = \mathrm{d}_\Gamma(\tilde{x}, z_j) + \mathrm{d}_\Gamma(z_j, \tilde{y}) > C$$

and the claim follows. $\square$

Combining Lemma 2.11 with Remark 1.7, in the case that $\mathrm{d}_{\mathsf{F}}(a, b) \leq C$, we get

$$\mathrm{d}_{\mathsf{B}}(a, b) \leq \mathrm{d}_{\mathsf{B}}(\tilde{x}, \tilde{y}) = \mathrm{d}_\Gamma(\tilde{x}, \tilde{y}) \leq \mathrm{d}_{\mathsf{F}}(a, b).$$

Note that $\mathrm{d}_{\mathsf{F}}(a, b) = \infty$ the inequality is true. So we can state the following

Proposition 2.12. *The semi-metric d_{B} is slower than d_{F}, namely $\mathrm{d}_{\mathsf{B}} \leq \mathrm{d}_{\mathsf{F}}$, which means $\mathrm{d}_{\mathsf{B}}(a, b) \leq \mathrm{d}_{\mathsf{F}}(a, b)$ for all $a, b \in \mathcal{A}_k$ and for all k.*

If we put together the previous Proposition and the discussion we made about intrinsic metrics, we get the following

Corollary 2.13. *The semi-metric d_{B} represents the gluing.*

2.2 Distances on tips and consequences

In this section we will combine distances on atoms and tips to answer the question on exponential behaviors of atom-codings that arose at the end of the previous section, more precisely we will find a distance with the following property:

Definition 2.14. A distance d on atom-codings has the **exponential property** if taking two horofunctions $u = (u_k)_{k=1}^{\infty}$ and $v = (v_k)_{k=1}^{\infty}$ one of the following holds

(1) $u\pi_h = v\pi_h$ and $\mathrm{d}(u_k, v_k)$ is bounded above by some constant (depending only on Γ) for all k (that is d represents the gluing);

(2) $u\pi_h \neq v\pi_h$ and $\mathrm{d}(u_k, v_k) \geq E_1 e^{E_2(k-j)}$ for some constant E_1 and E_2 and for all $k \geq j$ with j sufficiently large (we say that d **diverges exponentially**).

Note that this definition resembles the property of geodesics in hyperbolic spaces (see Proposition 1.23) and that it will be discussed again later on using more powerful tools to relate it with the Gromov product.

We will also prove that the Hausdorff distance can represent the gluing and we will bound the fibers of π_h.

We start by introducing a useful notation. Given a semi-metric d on some level of atoms $\mathcal{A}_k$, we will denote by

$$\mathrm{T}^* \mathrm{d}(a, b) := \mathrm{d}(T(a), T(b)) \quad \text{with } a, b \in \mathcal{A}_k.$$

where we recall that $T(a)$ and $T(b)$ denote the tips of a and b.

Remark 2.15. We point out that we already know something important about $\mathrm{T}^* \mathrm{d}_\Gamma$. Indeed, if we look at the proof of Theorem 2.4, we see that we actually prove that *(A) implies (B)* for $\mathrm{T}^* \mathrm{d}_\Gamma$, while for the other direction we can easily exploit the same technique in the proof to get the result.

So we have the following

Corollary 2.16. *The semi-metric* $\mathrm{T}^* \mathrm{d}_\Gamma$ *represents the gluing.*

In fact, we can prove a stronger result about the relation between d_Γ and

$\mathrm{T}^* \mathrm{d}_\Gamma$.

Proposition 2.17. *Let $a, b \in \mathcal{A}_k$. Then*

$$\mathrm{d}_\Gamma(a, b) \leq \mathrm{T}^* \mathrm{d}_\Gamma(a, b) \leq 2\, \mathrm{d}_\Gamma(a, b) + 4\delta + 2\lambda_a.$$

Proof. We only need to prove the second inequality. Let $x \in a$ and $y \in b$ such that $\mathrm{d}_\Gamma(a, b) = \mathrm{d}_\Gamma(x, y)$. By Proposition 1.47, we have that $x \in C(\overline{x})$ for some $\overline{x} \in N(a, B_k)$. The same is true for y and $\overline{y} \in N(b, B_k)$. Now we apply Proposition 1.22 to the geodesics $[x_0, x]$ and $[x_0, y]$ respectively passing through the two nearest neighbors at the time k, so that $\mathrm{d}_\Gamma(\overline{x}, \overline{y}) \leq 2(\mathrm{d}_\Gamma(x, y) + 2\delta)$. To finish the proof, we use the the Hooking Lemma together with a triangle inequality

$$\mathrm{d}_\Gamma(\widehat{x}, \widehat{y}) \leq \mathrm{d}_\Gamma(\widehat{x}, \overline{x}) + \mathrm{d}_\Gamma(\overline{x}, \overline{y}) + \mathrm{d}_\Gamma(\overline{y}, \widehat{y}) \leq 2\, \mathrm{d}_\Gamma(x, y) + 4\delta + 2\lambda_a,$$

which leads to $\mathrm{d}_\Gamma(\widehat{x}, \widehat{y}) \leq 2\, \mathrm{d}_\Gamma(x, y) + 4\delta + 2\lambda_a$ for any two vertices $\widehat{x} \in T(a)$ and $\widehat{y} \in T(b)$.

□

With a bit of work, we can say something about $\mathrm{T}^* \mathrm{d}_\mathsf{B}$ too.

Proposition 2.18. *The semi-metric $\mathrm{T}^* \mathrm{d}_\mathsf{B}$ represents the gluing.*

Proof. Since $\mathrm{d}_\Gamma \leq \mathrm{d}_\mathsf{B}$, it follows that $\mathrm{T}^* \mathrm{d}_\Gamma \leq \mathrm{T}^* \mathrm{d}_\mathsf{B}$, or more precisely $\mathrm{T}^* \mathrm{d}_\Gamma(a, b) \leq \mathrm{T}^* \mathrm{d}_\mathsf{B}(a, b)$ for all $a, b \in \mathcal{A}_k$ and for all k. It remains to prove that: if $u = (u_k)_{k=1}^\infty$ and $v = (v_k)_{k=1}^\infty$ are two horofunctions and there exists C_1 such that $\mathrm{T}^* \mathrm{d}_\Gamma(u_k, v_k) \leq C_1$ for all $k \geq 1$, then there exists C_2 such that $\mathrm{T}^* \mathrm{d}_\mathsf{B}(u_k, v_k) \leq C_2$ for all $k \geq 1$.

Let $\widehat{x} \in T(u_k)$ and $\widehat{y} \in T(v_k)$. We know that if $u_{\tilde{k}} \subseteq u_k$ and $v_{\tilde{k}} \subseteq v_k$ are such that $u_{\tilde{k}} \cup v_{\tilde{k}} \subseteq \Gamma - B_{\frac{C_1}{2}+k-1}$, then $\mathrm{T}^* \,\mathrm{d_B}^k(u_{\tilde{k}}, v_{\tilde{k}}) = \mathrm{T}^* \,\mathrm{d}_\Gamma(u_{\tilde{k}}, v_{\tilde{k}})$ (this is the same argument from Lemma 2.11).

We take $u_{\tilde{k}}$ and $v_{\tilde{k}}$ to be the first occurrences of atoms contained in $\Gamma - B_{\frac{C_1}{2}+k-1}$ (i.e. the one with minimal distance from x_0 contained in u_k and v_k respectively). We set $\tilde{x} \in T(u_{\tilde{k}})$ and $\tilde{y} \in T(v_{\tilde{k}})$ such that $\mathrm{T}^* \,\mathrm{d}_\Gamma(u_{\tilde{k}}, v_{\tilde{k}}) = \mathrm{d}_\Gamma(\tilde{x}, \tilde{y})$.

Note that by the Hooking Lemma we know that $T(u_{\tilde{k}}) \cup T(v_{\tilde{k}}) \subseteq \Gamma - B_{\frac{C_1}{2}+k-1+\lambda_a}$.

Since we want to study $\mathrm{d_B}(\widehat{x}, \widehat{y})$, we start by applying a triangle inequality and the main hyphotesis:

$$\mathrm{d_B}(\widehat{x}, \widehat{y}) \leq \mathrm{d_B}^k(\widehat{x}, \tilde{x}) + \mathrm{d_B}^k(\tilde{x}, \tilde{y}) + \mathrm{d_B}^k(\tilde{y}, \widehat{y}) \leq \mathrm{d_B}^k(\widehat{x}, \tilde{x}) + C_1 + \mathrm{d_B}^k(\tilde{y}, \widehat{y}).$$

Now $\tilde{x} \in \left(\Gamma - B_{\frac{C_1}{2}+k-1+\lambda_a}\right) \cap C(p)$ for every $p \in N(u_k, B_k)$. We have that there exists $[p, \tilde{x}]_\Gamma \subseteq \Gamma - B_{k-1}$ and its length is less or equal than $\frac{C_1}{2} - 1 + \lambda_a$. Exploiting the Hooking Lemma again, we get

$$\mathrm{d_B}^k(\widehat{x}, \tilde{x}) \leq \mathrm{d_B}^k(x, p) + \mathrm{d_B}^k(p, \tilde{x}) = \mathrm{d}_\Gamma(x, p) + \mathrm{d}_\Gamma(p, \tilde{x}) \leq \frac{C_1}{2} - 1 + 2\lambda_a.$$

In an analogous way, we get $\mathrm{d_B}^k(\widehat{y}, \tilde{y}) \leq \frac{C_1}{2} - 1 + 2\lambda_a$; and hence the claim. $\square$

It remains to prove the following

Proposition 2.19. *The semi-metric* $\mathrm{T}^* \,\mathrm{d_B}$ *diverges exponentially.*

Proof. Let $u, v \in \partial_h \Gamma$ such that $u\pi_h \neq v\pi_h$. Since $\mathrm{d}_\Gamma(u_k, v_k)$ is unbounded as k tends to infinity, we can choose j such that $\mathrm{d}_\Gamma(u_j, v_j)$ is arbitrarily large. In fact, taken $\overline{x} \in N(u_j, B_j)$ and $\overline{y} \in N(v_j, B_j)$, we know by the Hooking Lemma that

$$\mathrm{d}_\Gamma(\overline{x}, \overline{y}) \geq \mathrm{d}_\Gamma(u_j, v_j) - 2\lambda_a$$

hence we can take $\mathrm{d}_\Gamma(\overline{x}, \overline{y})$ arbitrarily large.

Now for all $k \geq j$ we have that $x_k \in u_k$ belongs to $C(\overline{x})$, and the same holds for $y_k \in v_k$ and $C(\overline{y})$. We construct a pair of geodesics $[x_0, x_k]$ and $[x_0, y_k]$ passing through $\overline{x}$ and $\overline{y}$ respectively and by Proposition 1.23, we know that their fellow travelers diverge exponentially. We consider x to be nearest neighbor of x_k such that it belongs to $[x_0, x_k] \cap S_k$ and the same for $y \in [x_0, y_k] \cap S_k$, so it holds that

$$\mathrm{d_B}^k(x, y) \geq E_1 e^{E_2(k-j)}$$

for some constants E_1 and E_2. To conclude, we take $x_k \in T(u_k)$ and $y_k \in T(v_k)$ (with $\mathrm{d_B}^k(x_k, y_k) = \mathrm{T}^* \mathrm{d_B}^k(u_k, v_k)$) and we have

$$\begin{aligned}\mathrm{T}^* \mathrm{d_B}^k(u_k, v_k) = \mathrm{d_B}^k(x_k, y_k) &\geq \mathrm{d_B}^k(x, y) - \mathrm{d_B}^k(x, x_k) - \mathrm{d_B}^k(y, y_k) \\ &\geq E_1 e^{E_2(k-j)} - \mathrm{d_B}^k(x, x_k) - \mathrm{d_B}^k(y, y_k).\end{aligned}$$

By construction, the geodesic $[x, x_k] \subseteq \Gamma - B_{k-1}$ and hence $\mathrm{d}_\Gamma(x, x_k) = \mathrm{d_B}^k(x, x_k)$. Analogously, we get $\mathrm{d}_\Gamma(y, y_k) = \mathrm{d_B}^k(y, y_k)$.

Again, by virtue of the Hooking Lemma combined with the previous inequality, we finally obtain

$$\mathrm{T}^* \mathrm{d_B}^k(u_k, v_k) \geq E_1 e^{E_2(k-j)} - 2\lambda_a.$$

□

Corollary 2.20. *The semi-metric* $\mathrm{T}^* \mathrm{d_B}$ *has the exponential property.*

There are two other important consequences that we want to briefly discuss, that comes from the tips approach. The first one involves a well-known metric.

Proposition 2.21. *The Hausdorff metric* $\mathrm{T}^* \mathrm{d}_\mathcal{H}$ *represents the gluing.*

Proof. Let $a, b \in \mathcal{A}_k$. Since $\mathrm{T}^* \mathrm{d}_\Gamma$ is the minimum over all pairs of elements $(\widehat{x}, \widehat{y})$ that belong to the tips $T(a)$ and $T(b)$ respectively, it is straightforward that

$$\mathrm{T}^* \mathrm{d}_\Gamma(a,b) \le \min_{\widehat{y} \in T(b)} \mathrm{d}_\Gamma(\widehat{x}, \widehat{y}) \le \max_{\widehat{x} \in T(a)} \min_{\widehat{y} \in T(b)} \mathrm{d}_\Gamma(\widehat{x}, \widehat{y})$$

and that the same holds in the other way (with a and b switched). So $\mathrm{T}^* \mathrm{d}_\Gamma(a,b) \le \mathrm{T}^* \mathrm{d}_{\mathcal{H}}(a,b)$.

For the other direction, we need to use Proposition 1.54, that leads to

$$\max_{\widehat{x} \in T(a)} \min_{\widehat{y} \in T(b)} \mathrm{d}_\Gamma(\widehat{x}, \widehat{y}) \le \mathrm{T}^* \mathrm{d}_\Gamma(a,b) + 2\lambda_a$$

and hence $\mathrm{T}^* \mathrm{d}_{\mathcal{H}}(a,b) \le \mathrm{T}^* \mathrm{d}_\Gamma(a,b) + 2\lambda_a$ as before. □

The last consequence is about the fibers of π_h. We have already discussed some properties, in particular Theorem 2.4 tells us when two elements belong to the same fiber. But now, we can prove the following

Theorem 2.22. *The map $\pi_h : \partial_h \Gamma \twoheadrightarrow \partial \Gamma$ is finite-to-one. Moreover, the number of elements in a fiber is bounded by a constant that depends only on λ_a and δ.*

Proof. We recall our assumption on the graph Γ: it is locally finite and hence the balls are finite. We consider the atom-coding of a horofunction $u = (u_k)_{k=1}^\infty$ and we know that the tip $T(u_k)$ has a finite diameter due to Proposition 1.54 for all k. In particular, it is bounded above by $2\lambda_a$. We choose an element in $T(u_k)$ and we consider a ball B of radius $2(\lambda_a + 2\delta) + 2\lambda_a$ centered at that element. By Theorem 2.4 and Remark 2.15, we get that two horofunctions map onto the same point in the Gromov boundary if the distance of their tips at each level is less or equal than $2(\lambda_a + \delta)$. So by definition of the ball B, the k-level tip of every horofunction that is contained in the same fiber of u must intersect B. Since it holds for all k and the constants do not depend on k, we have the claim. □

It is worth noting that this result can be achieved in a different way. In their work [CP93, CP01], Coornaert and Papadopoulos use a different notion of horofunction (this defintion was provided by Gromov in [Gro87]).

Definition 2.23. Let Γ a hyperbolic graph and let x_0 be a distinguished vertex in Γ. A map $f : |\Gamma| \to \mathbb{R}$ with $f(x_0) = 0$ is called a CP-function if it satisfies the following two conditions:

(1) There exists $\epsilon > 0$ such that

$$f(l\gamma(t)) \leq (1-t)f(l\gamma(0)) + tf(l\gamma(1)) + \epsilon$$

with $\gamma : [0, l] \to |\Gamma|$ geodesic and $l : [0, 1] \to [0, l]$ that maps $t \in [0, 1]$ to $lt \in [0, l]$;

(2) $f(x) = \tilde{t} + \mathrm{d}_\Gamma(x, f^{-1}(\tilde{t}))$ for every $x \in |\Gamma|$ and every $\tilde{t} \in]-\infty, f(x)]$.

They then managed to prove that the space of CP-functions that assume only integer values on the vertices of Γ projects onto $\partial\Gamma$ and the quotient map is finite-to-one (see [CP01, Proposition 4.5]). So all we need to conclude that the fibers are finite is the following

Proposition 2.24 ([Bel19])**.** *Let Γ be a hyperbolic graph. Then a horofunction is a CP-function.*

Note that the converse is false. Since there are hyperbolic graphs such that the two notions do not coincide (see [Bel19] for details).

Theorem 2.22 gives us a tool to bound the *topological dimension* of $\partial\Gamma$.

Theorem 2.25 (Hurewicz)**.** *Let X be a compact metrizable space that is a continuous image of a Cantor set. If the fibers of the map are bounded above by an integer $n > 0$,* *then the topological dimension of X is less than $n-1$.*

See [Kur66, Chapter XIX].

Applications of this theorem are common in geometric group theory, see e.g. the bound for limit sets of contracting self-similar groups in [Nek07, Proposition 5.7] and the bound for hyperbolic graphs in two different versions, namely Proposition 3.7 and Proposition 4.2 followed by Corollary 5.2 in [CP93].

2.3 Using geodesics and geodesic rays

Our first step is to construct a quasi-geodesic ray for each atom-coding in a very natural way. To start, we need to iterate the hooking lemma, that is

Proposition 2.26. *Let $u_k \supseteq u_{k+1}$ two atoms respectively of level k and $k+1$. Then*

$$\max_{\substack{x \in T(u_k) \\ y \in T(u_{k+1})}} \mathrm{d}_\Gamma(x, y) \leq 2\lambda_a + 1.$$

Proof. Let $\widehat{x}_{k+1} \in T(u_{k+1})$. By the Hooking Lemma, we know that $\mathrm{d}_\Gamma(x_0, \widehat{x}_{k+1})$ is less than or equal to $k+1+\lambda_a$. And so it is the length of a geodesic starting from x_0 and passing through any nearest neighbor $\overline{x} \in S_k$ of u_k (recall that we can take any of them due to Lemma 1.42(c)). We know that such a geodesic exists because $\widehat{x}_{k+1} \in u_k$ and by Proposition 1.47. But this means that $\mathrm{d}_\Gamma(\overline{x}, \widehat{x}_{k+1}) \leq \lambda_a + 1$ and again by the Hooking Lemma, we have $\mathrm{d}_\Gamma(\overline{x}, \widehat{x}_k) \leq \lambda_a$ with $\widehat{x}_k$ any element in the tip of u_k. Combining these two facts in a triangle inequality we get the claim. □

Note that if we look at the minimum, namely

$$\min_{\substack{x \in T(u_k) \\ y \in T(u_{k+1})}} \mathrm{d}_\Gamma(x, y),$$

then maybe the estimate provided is naive. Indeed, it can happen that two elements in two consecutive tips are one the successor of the other.

Example 2.27. Looking at Figure 8.(a), we get an example of two consecutive tips. Indeed, every element in the sphere of radius 2 which is an element in the tip of a 2-level atom is a successor of an element in the tip of the corresponding 1-level atom.

Despite this aspect, we are able to construct a quasi-geodesic ray

Proposition 2.28. *Let $u = (u_k)_{k=1}^{\infty}$ be an element of $\partial_h \Gamma$ described by its atom-coding. Then any sequence of points $\{\widehat{x}_k\}_{k=1}^{\infty}$ such that $\widehat{x}_k \in T(u_k)$ has a subsequence that is a quasi-geodesic ray.*

Proof. Take $\{\widehat{x}_k\}_{k=1}^{\infty}$ as in the statement. It may occur that some $\widehat{x}_k$ are equal so, for each k, we remove all the redundant copies of the same $\widehat{x}_k$ to get a new sequence $\{\widehat{x}_n\}_{n=1}^{\infty}$. We claim that $\{\widehat{x}_n\}_{n=1}^{\infty}$ is a quasi-isometric embedding of $\mathbb{N}$ in Γ.
By Proposition 2.26 we know that $\mathrm{d}_\Gamma(\widehat{x}_n, \widehat{x}_{n+1}) \leq D$ for some constant D depending only on λ_a. To conclude, we know that if $\widehat{x}_n, \widehat{x}_m \in \{\widehat{x}_n\}_{n=1}^{\infty}$ and $n \leq m$, then $\mathrm{d}_\Gamma(\widehat{x}_n, \widehat{x}_m) \leq D(m-n)$ by iterations of the triangle inequality.
On the other hand, $\widehat{x}_n \in B_{n-1+\lambda_a}$ by virtue of the Hooking Lemma and $\widehat{x}_n \in \Gamma - B_{m-1}$ by Proposition 1.38, hence $\mathrm{d}_\Gamma(\widehat{x}_n, \widehat{x}_m) \geq m - n - \lambda_a$. □

An interesting fact to remark is that if such a quasi-geodesic ray is a geodesic ray, then by definition the horofunction is a Busemann point.

Remark 2.29. Note that two horofunctions u and v are mapped into the same point in $\partial\Gamma$ if and only if the Hausdorff distance between the two geodesic rays constructed using the previous proposition is bounded (see [BH13, Lemma H.3.1] for a different, but equivalent, definition of $\partial\Gamma$ that leads to this fact).

The technique used in the following Remark will not only improve the structure of the quasi-geodesic ray, but it will also be the key ingredient for most of the incoming proofs:

Remark 2.30. Let $a_n \in \mathcal{A}_n$ and take p_n a proximal point of a_n. By Lemma 1.42(b), we can consider a combinatorial geodesic $[p_0, p_n] = (p_0, p_1, \dots, p_n)$ such that $p_0 = x_0$ and $p_i \in P(x, S_i)$ for some (and hence all) points $x \in a_n$ and for all $i \leq n$. But we can say more, we actually find a geodesic such that $p_i \in P(a_i, S_i)$ with $a_i \in \mathcal{A}_i$ and $a_i \supseteq a_n$ for all $i \leq n$, as $x \in a_i$.

Following Definition 1.31 and the discussion right after it, we look at $\partial\mathcal{A}$ as a metric space with respect to $\curlyvee_A$, which is nothing more than the standard visual metric on the boundary of a rooted tree. Explicitly, we define $\curlyvee_A(u, v) := \beta^{-k}$ where k is such that $u_k = v_k$ (this implies $u_i = v_i$ for every $i \leq k$) and $u_{k+1} \neq v_{k+1}$. As for other tree structures that are related to $\partial\Gamma$ (see e.g. the beginning of Section 1.4 and [CP93, Proposition 2.3]), we have the following

Proposition 2.31. *The map $\pi_h : (\partial\mathcal{A}, \curlyvee_A) \twoheadrightarrow (\partial\Gamma, \curlyvee)$ is Lipschitz.*

Proof. Let $u = (u_n)_{n=1}^{\infty}$ and $v = (v_n)_{n=1}^{\infty}$ be two horofunctions with $\curlyvee_A(u, v) = \beta^{-k}$. Pick $\{\widehat{x}_n\}_{n=1}^{\infty}$ and $\{\widehat{y}_n\}_{n=1}^{\infty}$ such that $\widehat{x}_n \in T(u_n)$ and $\widehat{y}_n \in T(v_n)$ for all n and with $\widehat{x}_n = \widehat{y}_n$ for all $n \leq k$, now Remark 1.29(c) leads to

$$2\tilde{\delta} + (u\pi_h \mid v\pi_h) \geq \sup(\sup_{n\geq 0} \inf_{m\geq n} (x_m \mid y_m)) \geq \sup_{n\geq 0} \inf_{m\geq n} (\widehat{x}_m \mid \widehat{y}_m)$$

where the first $\sup$ is over all the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that x_n converges to $u\pi_h$ and y_n converges to $v\pi_h$.

Since $\widehat{x}_m = \widehat{y}_m$ for all $m \leq k$, then $(\widehat{x}_m \mid \widehat{y}_m) = d(\widehat{x}_m, x_0) \geq m$, while if $m > k$ then we have $\mathrm{d}_\Gamma(\widehat{x}_m, \widehat{y}_m) \leq \mathrm{d}_\Gamma(\widehat{x}_m, \widehat{x}_k) + \mathrm{d}_\Gamma(\widehat{y}_m, \widehat{y}_k)$ and

$$\mathrm{d}_\Gamma(\widehat{x}_m, \widehat{x}_k) \leq \mathrm{d}_\Gamma(\widehat{x}_m, \overline{x}_m) + \mathrm{d}_\Gamma(\overline{x}_m, p_m) + \mathrm{d}_\Gamma(p_m, p_k) + \mathrm{d}_\Gamma(p_k, \overline{x}_k) + \mathrm{d}_\Gamma(\widehat{x}_k, \overline{x}_k)$$

with $\overline{x}_m$ and $\overline{x}_k$ nearest neighbors of $\widehat{x}_m$ and $\widehat{x}_k$ respectively; and p_m and p_k proximal points of $\widehat{x}_m$ and $\widehat{x}_k$ on the same geodesic (see Remark 2.30). So that $\mathrm{d}_\Gamma(\widehat{x}_i, \overline{x}_i) \leq \lambda_a$ by the Hooking Lemma and $\mathrm{d}_\Gamma(\overline{x}_i, p_i) \leq 4\delta + 2$ by Lemma 1.42(b) for $i = k, m$. Moreover, $\mathrm{d}_\Gamma(p_m, p_k) = m - k$ and so we can conclude that

$$\mathrm{d}_\Gamma(\widehat{x}_m, \widehat{x}_k) \leq m - k + D \text{ with } D = 2(\lambda_a + 4\delta + 2).$$

The same holds for $\widehat{y}_m$ and $\widehat{y}_k$.

Returning to the Gromov product, we can say that

$$\mathrm{d}_\Gamma(\widehat{x}_m, x_0) + \mathrm{d}_\Gamma(\widehat{y}_m, x_0) - \mathrm{d}_\Gamma(\widehat{x}_m, \widehat{y}_m) \geq 2m - 2(m - k + D) = 2(k - D)$$

with D a constant only depending on λ_a and δ. Hence $\Upsilon(u\pi_h, v\pi_h) \leq \tilde{D}\beta^{-k}$ with $\tilde{D} = \beta^{D+2\tilde{\delta}}$ as desired. □

Open Question. *Is there a connection between the visual metric and the uniform metric on $\partial_h\Gamma$ so that we can say the map $(\partial_h\Gamma, |\cdot|_\infty) \twoheadrightarrow (\partial\Gamma, \Upsilon)$ is Lipschitz?*

Starting from an atom-coding, we want to find a geodesic ray that represents the same element of $\partial\Gamma$ as the horofunction associated to the atom-coding, more formally

Proposition 2.32. *Let u be a horofunction. Then there exists a geodesic ray γ_u such that $\gamma_u(0) = x_0$ and $\gamma_u(k)$ is proximal to u_k for all $k \in \mathbb{N}$. Furthermore $[\gamma_u]_{\partial\Gamma} = u\pi_h$.*

Note that we are not saying that every horofunction is a Busemann point, this is false in general for hyperbolic graphs, as we already mentioned. What is true is that the horofunction u is a Busemann point if and only if it coincides with the horofunction defined by γ_u.

The proof of the Proposition involves the following well-known result, together with the proximal points technique mentioned before.

Theorem 2.33 (Arzelà-Ascoli)**.** *Let X be a proper geodesic space with a distinguished point x_0. Let $\{\gamma_k\}_{k\in\mathbb{N}}$ be a sequence of functions $\gamma_k : [0,\infty[\to X$ such that*

(a) $\gamma_k(0) = x_0$ *for all* $k \in \mathbb{N}$*,*

(b) γ_k *is a geodesic on* $[0, k]$*.*

Then there exists a subsequence $\{\tilde{\gamma}_n\}_{n\in\mathbb{N}}$ that converges uniformly on compacts to a geodesic ray $\gamma : [0,\infty[\to X$.

Proof(Proposition 2.32). Let $(u_k)_{k=1}^{\infty}$ be the sequence of atoms representing u. For each k, we exploit Remark 2.30 to get a geodesic γ_k such that $\gamma_k(n) \in P(u_n, S_n)$ for all $n \leq k$. Then we apply Theorem 2.33 to the sequence $\{\gamma_k\}_{k\in\mathbb{N}}^{\infty}$ by saying that $\gamma_k(t) := \gamma_k(k)$ for all $t \geq k$ and we obtain a geodesic ray γ. We claim that every n-vertex of γ (i.e. $\gamma(n)$ with $n \in \mathbb{N}$) is a proximal point of u_k. Indeed, we are dealing with uniform convergence on compacts, that is

$$\lim_k \max_{t\in[0,n]} \mathrm{d}_\Gamma(\tilde{\gamma}_k(t), \gamma(t)) = 0$$

or in other words

$$\forall \epsilon > 0 \ \ \exists j \text{ s.t. } \forall k \geq j \text{ and } \forall t \in [0, n], \ \ \mathrm{d}_\Gamma(\tilde{\gamma}_k(t), \gamma(t)) < \epsilon.$$

This means that any vertex $\gamma(n)$ is arbitrarily close to a proximal point, but since we are in a graph, taking ϵ small enough means that the proximal point and $\gamma(n)$

are in fact the same vertex.

To prove that γ represents the same point of $u\pi_h$, we argue as before: we use the Hooking Lemma to bound the distance between an element $\widehat{x}_k \in \mathrm{T}^*(u_k)$ and a nearest neighbor $\overline{x}_k$ of u_k; then we apply Lemma 1.42(b) to say that $\overline{x}_k$ and $\gamma(k)$ are $(4\delta + 2)$-near. And we conclude with a triangle inequality that yields $\mathrm{d}_\Gamma(\gamma(k), \widehat{x}_k) \leq 4\delta + 2 + \lambda_a$. □

Definition 2.34. Let u be a horofunction. We call the geodesic ray γ_u defined by Proposition 2.32 a **proximal ray** with respect to u.

We know that, in some sense, the Gromov product of two points in $\partial\Gamma$ measures how long two geodesic rays representing these points fellow travel. In the following result, we will provide an atom-coding version of this fact.

Theorem 2.35. *Let $u = (u_k)_{k=1}^{\infty}$ and $v = (v_k)_{k=1}^{\infty}$ be two horofunctions coded by atoms. Then the following holds.*

(a) If $\mathrm{T}^* \mathrm{d}_\Gamma(u_i, v_i) \leq C$ for all $i \leq k$, then $(u\pi_h \mid v\pi_h) \geq k - C'$,

(b) if $(u\pi_h \mid v\pi_h) \geq k$, then $\mathrm{T}^* \mathrm{d}_\Gamma(u_i, v_i) \leq C''$ for all $i \leq k$,

where C, C' and C'' are constants. Furthermore, if C depends only on λ_a and δ, so do C' and C''.

In particular, we obtain a new proof of Theorem 2.4 when the hypotheses hold for each k.

Proof. For the first assertion, we proceed as in the proof that π_h is Lipschitz (see Proposition 2.31). When $l \leq k$, we have $(u_l \mid v_l) \geq l - C$ by hyphotesis. If $l > k$, we consider the following triangle inequality

$$\mathrm{d}_\Gamma(\widehat{x}_l, \widehat{y}_l) \leq \mathrm{d}_\Gamma(\widehat{x}_l, \widehat{x}_k) + \mathrm{d}_\Gamma(\widehat{x}_k, \widehat{y}_k) + \mathrm{d}_\Gamma(\widehat{y}_k, \widehat{y}_l)$$

with $\widehat{x}_i \in T(u_i)$ and $\widehat{y}_i \in T(v_i)$ for $i = l, k$.

Then we use the argument in Remark 2.30 to get two geodesics of proximal points and to give an estimate of $\mathrm{d}_\Gamma(\widehat{x}_l, \widehat{x}_k)$ and $\mathrm{d}_\Gamma(\widehat{y}_l, \widehat{y}_k)$, so that we have

$$\mathrm{T}^* \, \mathrm{d}_\Gamma(u_l, v_l) \leq 2(l-k) + 4(\lambda_a + 4\delta + 2) + \mathrm{d}_\Gamma(\widehat{x}_k, \widehat{y}_k).$$

To conclude, we use again the hypothesis that $\mathrm{d}_\Gamma(\widehat{x}_k, \widehat{y}_k) \leq C$ and hence $\mathrm{T}^* \, \mathrm{d}_\Gamma(u_l, v_l) \leq 2(l-k) + 4(\lambda_a + 4\delta + 2) + C$. In this way, we found a lower bound for each Gromov product of the type $(u_l \mid v_l)$ (we implicitly use the Hooking Lemma and the claim follows).

What is left to point out is that $(u \mid v) \leq (u\pi_h \mid v\pi_h) + 2\tilde{\delta}$ due to Remark 1.29(c).

For the second assertion, we set γ_u and γ_v to be to proximal rays with respect to u and v (see Definition 2.34 and Proposition 2.32). We consider

$$\mathrm{T}^* \, \mathrm{d}_\Gamma(u_k, v_k) \leq \mathrm{d}_\Gamma(T(u_k), \gamma_u(k)) + \mathrm{d}_\Gamma(\gamma_u(k), \gamma_v(k)) + \mathrm{d}_\Gamma(\gamma_v(k), T(v_k)).$$

The distance $\mathrm{d}_\Gamma(T(u_k), \gamma_u(k))$ is bounded by $4\delta + 2 + \lambda_a$ for the same argument as before, that is taking a nearest neighbor and exploiting Lemma 1.42(b) together with the Hooking Lemma. The same occurs to $\mathrm{d}_\Gamma(T(v_k), \gamma_v(k))$. The distance $\mathrm{d}_\Gamma(\gamma_u(k), \gamma_v(k))$ is bounded by virtue of Lemma 1.30. More clearly if $(u\pi_h \mid v\pi_h) \geq k$ it means that the distance between $\mathrm{d}_\Gamma(\gamma_u(i), \gamma_v(i))$ for all $i \leq k$ is bounded by 5δ due to the quasi-isometry. □

This version of the theorem gives a chance of characterizing the Gromov boundary via horofunctions through a metric viewpoint.

Corollary 2.36. *The function $\beta^{-(\cdot|\cdot)}$ on $\partial_h\Gamma$ is a distance and if $\curlyvee_h$ is the pseudo-metric computed using the First Move, then the quotient $\partial_h\Gamma/\curlyvee_h$ defined by the Second Move is $\partial\Gamma$.*

We recall that the two moves are explained after the Definition 1.2 at the very beginning of the dissertation.

Proof. We start by proving that the function is a distance. Symmetry is obvious. Taken $u \in \partial_h\Gamma$, then

$$\max_{x_k \in T(u_k), y_k \in T(v_k)} \liminf_k (x_k|y_k) = \infty$$

and so is $(u \mid u)$, by virtue of the proof of Lemma 1.59. Hence $\beta^{-(u,u)} = 0$. Using the First Move, we get the pseudo-metric $\curlyvee$. It remains to prove that the metric quotient is the Gromov boundary. We point out that two horofunctions $u \neq v$ can satisfy $\beta^{-(u,v)} = 0$ and indeed this happens if they glue (they are in the same fiber of π_h). We need to show that this is the only possible case, which means that if $\beta^{-(u,v)} = 0$, then u and v glue. But now $(u \mid v) = \infty$, and so there exists a couple of Gromov sequences converging to u and v such that their Gromov product is infinite, hence these two sequences are the same element in $\partial\Gamma$. $\square$

In literature, there are many examples of metric spaces (or similar structures) that in some way converge to the Gromov boundary of a hyperbolic group. We cite as an example, the work of Pawlik [Paw15] and Lemma 3.8 of [GMS19] which says that spheres with center in a distinguished point x_0 and endowed with the visual metric *weakly converge to $\partial\Gamma$ in the sense of Gromov-Hausdorff*. As mentioned before, we can look at the tips as a coarse version of spheres and so our aim now is to provide a tip-version of this convergence.
In the following discussion, we will adopt the notation $\curlyvee_k(u_k, v_k) = \beta^{-(u_k|v_k)}$ in which u_k and v_k are atoms of the same level. We recall that the Gromov product

is the one defined right before Lemma 1.59 and that Υ_k is not a metric (not even a distance), but still plays an important role in the theory. With this in mind, we will consider the weak Gromov-Hausdorff limit of $(\mathcal{A}_k, \Upsilon_k)$ as k goes to infinity even if they are not metric spaces.

A formal definition for the limit we discussed is the following

Definition 2.37. Let $\{\Gamma_i, \mathrm{d}_i\}_{i\in\mathbb{N}}$ be a sequence of graphs endowed with the standard metric. We say that the graph (Γ, d) is the **weak Gromov-Hausdorff limit**, or that the sequence weakly converges in the sense of Gromov-Hausdorff, if for all $i \in \mathbb{N}$ there exists a quasi-isometry $\phi_i : \Gamma \to \Gamma_i$ with L_1^i not depending on i and L_2^i that goes to zero as i tends to infinity, where L_1^i is the multiplicative constant and L_2^i is the additive constant of the quasi-isometric embedding.

Note that this is a coarse version of the standard notion of Gromov-Hausdorff convergence in metric geometry (see e.g. [BBI01]).

Before proving the result, we need a technical lemma that links the Gromov product between two atoms with the one between two proximal points:

Lemma 2.38. *Let u_k and v_k be two k-level atoms. Let $p_k \in P(u_k, S_k)$ and $q_k \in P(v_k, S_k)$. Then*

$$|\,\mathrm{T}^* \mathrm{d}_\Gamma(u_k, v_k) - \mathrm{d}_\Gamma(p_k, q_k)| \leq 8\delta + 4 + 2\lambda_a \textit{ and } |(u_k|v_k) - (p_k|q_k)| \leq 4\delta + 2 + 2\lambda_a.$$

Proof. The first part is the usual consequence of the Hooking Lemma together with Lemma 1.42(b) applied to

$$\mathrm{T}^* \mathrm{d}_\Gamma(u_k, v_k) \leq \mathrm{d}_\Gamma(T(u_k), p_k) + \mathrm{d}_\Gamma(p_k, q_k) + \mathrm{d}_\Gamma(q_k, T(v_k))$$

and the triangle inequality where proximal points and atoms are switched.

For the second part, we use the first part as follows

$$2(u_k \mid v_k) \geq 2k - \mathrm{d}_\Gamma(p_k, q_k) - 8\delta - 4 - 2\lambda_a,$$

and

$$2(u_k \mid v_k) \leq 2k + 2\lambda_a - \mathrm{d}_\Gamma(p_k, q_k) + 8\delta + 4 + 2\lambda_a.$$

All is left is to notice that $2k - \mathrm{d}_\Gamma(p_k, q_k) = 2(p_k \mid q_k)$. $\square$

Proposition 2.39. *The metric space $(\partial\Gamma, \curlyvee)$ is the weak Gromov-Hausdorff limit of the sequence $(\mathcal{A}_k, \curlyvee_k)$ as $k \to \infty$.*

Proof. First, we introduce the map we claim induces the quasi-isometry.

$$\Phi : \partial\Gamma \to T(\mathcal{A}_k)$$

$$x_\infty \mapsto u \mapsto T(u_k)$$

We consider an element $x_\infty \in \partial\Gamma$ and a section $\mathcal{S}_h : \partial\Gamma \to \partial_h\Gamma$ of the projection π_h. Now $x_\infty\mathcal{S}_h = u$ and u_k is the k-level atom of the atom-coding. So $x_\infty\Phi := T(u_k)$.

Quasi-dense image. Let $a \in \mathcal{A}_k$. Take $x_\infty \in \partial a\pi_h$ and evaluate $T^* \mathrm{d}_\Gamma(x_\infty\Phi, a)$. Since there exists a horofunction that passes through a and projects onto x_∞, we know that such a horofunction and $x_\infty\mathcal{S}_h$ identify on $\partial\Gamma$. Hence by Theorem 2.4, the tips of their k-th terms of the atom-codings (which are a and $x_\infty\Phi$) stay within $2(\lambda_a + \delta)$.

Quasi-isometric embedding. We will proceed by cases.

If $(x_\infty \mid y_\infty) \geq k$, we apply Theorem 2.35(b) and we have $\mathrm{T}^* \, \mathrm{d}_\Gamma(u_i, v_i) \leq C$ for all $i \leq k$ and hence $(u_k|v_k) \geq k - C$ by applying Theorem 2.35(a). To conclude then that $| \curlyvee_k (u_k, v_k) - \curlyvee(x_\infty, y_\infty)| \leq \curlyvee_k(u_k, v_k) \leq \beta^{-k+C}$.

If $(x_\infty \mid y_\infty) \leq k$, then we consider two proximal rays $\gamma \in x_\infty$ and $\eta \in y_\infty$ and by virtue of Lemma 1.30, we have

$$|(\gamma(k)|\eta(k)) - (x_\infty|y_\infty)| \leq \frac{5}{2}\delta.$$

All that is left to do is combine it with Lemma 2.38 and get

$$\begin{aligned} |(u_k|v_k) - (x_\infty|y_\infty)| \leq & |(u_k|v_k) - (\gamma(k)|\eta(k))| + \\ & |(\gamma(k)|\eta(k)) - (x_\infty|y_\infty)| \leq 4\delta + 2 + 2\lambda_a + \frac{5}{2}\delta. \end{aligned}$$

Hence $D^{-1} \curlyvee (x_\infty, y_\infty) \leq \curlyvee_k(u_k, v_k) \leq D \curlyvee (x_\infty, y_\infty)$ for a suitable constant D as desired. $\square$

CHAPTER 3

QUASI-ISOMETRIES

3.1 The set of tips and the graph of atoms

We now continue our parallelism between the graph Γ, its spheres, its geodesic rays and the atom-coding tree $\mathcal{A}$, the tips and the horofunctions; we will now present a couple of quasi-isometries between the set of tips and the graph Γ.

Note that for the other tree structure introduced in Section 1.1.4 (see Figure 7 for an example and the discussion at the very beginning of the section for a general setting), a quasi-isometry between the vertices of the trees and the elements in the graph is in some sense trivial: vertices of the tree correspond to vertices in the graph and the distance between two of them in the tree is actually the one on the original graph.

The following result will help us restricting our attention to elements with infinite cones in both the quasi-isometries we are going to describe.

Lemma 3.1. *Let Γ be a hyperbolic graph quasi-isometric to the Cayley graph of some hyperbolic group. Then there exists a constant λ_∞ such that every element $x \in \Gamma$ with a finite cone is in a ball of radius λ_∞ centered at an element with an infinite cone.*

From now on λ_∞ will be such a constant.

Proof. Let us start by determining λ_∞. Since two finite cones with the same type have the same number of points, we can consider λ_∞ to be the maximum of the cardinalities among all types of finite cones (they are finite by Proposition 1.45). This allows us to find a predecessor x^c of x (i.e. an element that belongs to a geodesic between x_0 and x), such that $\mathrm{d}_\Gamma(x^c, x) \leq \lambda_\infty$ that has an infinite cone. Indeed, suppose that every element that belongs to a geodesic $[x^c, x]$ with x^c a predecessor and $\mathrm{d}_\Gamma(x^c, x) = \lambda_\infty$ has a finite cone. This means that the geodesic is fully contained in the cone $C(x^c)$ but exceeds the number of possible elements in the cone. Hence we have a contradiction. This implies that the cone $C(x^c)$ has to be infinite. □

The following is useful for proving the quasi-density in both cases.

Lemma 3.2. *Let Γ be a hyperbolic graph. If S_n^∞ is the subset of all elements in S_n such that their cones are infinite, then $S_n^\infty \subseteq \bigcup_{a \in \mathcal{A}_n} V(a, B_n)$.*

Proof. Let x' be an element of S_n^∞. Now $C(x')$ is infinite and there are finitely many atoms of level n, so there exists an atom $a \in \mathcal{A}_n$ such that $C(x') \cap a \neq \emptyset$. We take $y \in C(x') \cap a$ so that $[x', y] \cap B_n = x'$ and by definition $x' \in V(y, B_n)$. By the property of visible points (see Lemma 1.42(c)), we have $V(y, B_n) = V(a, B_n)$. □

Throughout Chapter 2, we were dealing with many distances (almost all of them were not metrics) and we studied the connections between them. We then proved two quasi-isometry like results (Proposition 2.17 and Proposition 2.21). As a first step, we now want to formally prove what these results naturally suggest.

Proposition 3.3. *Let $(\mathbf{T}, \mathrm{d}_\mathcal{H})$ be the set of tips without repetitions (that means that if two atoms share the same tip, we count it once) endowed with the usual Hausdorff metric. Then $\mathbf{T}$ is quasi-isometric to Γ.*

We recall the discussion made in Remark 1.57 to better understand what "without repetitions" means.

Proof. The map $\mathcal{S}_{\mathbf{T}} : \mathbf{T} \to \Gamma$ we want to show is a quasi isometry is defined as $T(a)\mathcal{S}_{\mathbf{T}} := \widehat{x}$ with $\widehat{x}$ some fixed element in $T(a)$.

Quasi-dense image. Let $x \in \Gamma$. We consider n such that $x \in B_{n-1}$.
First, suppose that $x \in S^{\infty}_{n-1}$, then there exists at least one of its successor x' that belongs to S^{∞}_{n}. By Lemma 3.2, we know that x' is a visible point for some atom $a \in \mathcal{A}_n$. Now we take $\overline{x} \in N(a, B_n)$ and we know that $\mathrm{d}_\Gamma(x', \overline{x}) \leq 4\delta + 2$ by a property of visible points (by combining Lemma 1.42(a) with Lemma 1.42(b)). So $\mathrm{d}_\Gamma(x, \overline{x}) \leq 4\delta + 3$ and by the Hooking Lemma we get $\mathrm{d}_\Gamma(x, \widehat{x}) \leq 4\delta + 3 + \lambda_a$ for any point $\widehat{x} \in T(a)$.
Now, suppose that the cone of x is finite. Combining Lemma 3.1 with the first case, we get $\mathrm{d}_\Gamma(x, T(a)) \leq \lambda_\infty + 4\delta + 3\lambda_a$.

Quasi-isometric embedding. We argue as in the proof of Proposition 2.21. Indeed, we already know that $\mathrm{d}_{\mathcal{H}}(T(a), T(b)) \leq \min_{\widehat{x} \in T(a), \widehat{y} \in T(b)} \mathrm{d}_\Gamma(\widehat{x}, \widehat{y}) + 2\lambda_a$ and by Proposition 1.54 we get

$$\mathrm{d}_{\mathcal{H}}(T(a), T(b)) \leq \mathrm{d}_\Gamma(T(a)\mathcal{S}_{\mathbf{T}}, T(b)\mathcal{S}_{\mathbf{T}}) + 4\lambda_a.$$

On the other side, Proposition 2.21 provides

$$\min_{\widehat{x} \in T(a), \widehat{y} \in T(b)} \mathrm{d}_\Gamma(\widehat{x}, \widehat{y}) \leq \mathrm{d}_{\mathcal{H}}(T(a), T(b)).$$

All we have to do is combine it with

$$\mathrm{d}_\Gamma(T(a)\mathcal{S}_{\mathbf{T}}, T(b)\mathcal{S}_{\mathbf{T}}) \leq \min_{\widehat{x} \in T(a), \widehat{y} \in T(b)} \mathrm{d}_\Gamma(\widehat{x}, \widehat{y}) + \lambda_a$$

coming from Proposition 1.54. □

The second step of the chapter is inspired by the work of Kaimanovich ([Kai03]) on fractals and of Nekrashevych ([Nek03], [BGN03]) on limit spaces of contracting self-similar groups. See also [LW09] for an application in dynamical systems.

The main object of the discussion is:

Definition 3.4. Let Γ be a hyperbolic graph and $\mathcal{A}$ its tree of atoms. We define $\Gamma_{\mathcal{A}}$ and we call it the **graph of atoms** in the following way:

Vertices: all elements of $\mathcal{A}$;

Vertical Edges: given two vertices $a_n \in \mathcal{A}_n$ and $a_{n+1} \in \mathcal{A}_{n+1}$, there exists an edge if and only if $a_n \supseteq a_{n+1}$;

Horizontal Edges: given two vertices $a_n, b_n \in \mathcal{A}_n$ there exists an edge if and only if $\mathrm{d}_\Gamma(a_n, b_n) \leq 2(\lambda_\infty + 4\delta + \lambda_a) + 7$ and define this as λ_e.

As before, λ_e will be such a constant. Its peculiar definition will be clarified during the proof of the quasi-isometry result.

Definition 3.5. We denote by $T(\Gamma_{\mathcal{A}})$ the same construction as before, but using the distance $\mathrm{T}^* \mathrm{d}_\Gamma$ and the constant $2\lambda_e + 4\delta + 2\lambda_a$ for horizontal edges. We call it the **graph of tips**. Please note that the vertices of the graph are still atoms.

Despite the choice of the constant looking strange, it is related to the fact that $\mathrm{T}^* \mathrm{d}_\Gamma \leq 2 \mathrm{d}_\Gamma + 4\delta + 2\lambda_a$ (see Proposition 2.17) and will be fully explained in the following.

Remark 3.6. Some straightforward properties of $\Gamma_{\mathcal{A}}$ are the following

(a) the tree $\mathcal{A}$ is a spanning tree for the graph;

(b) the vertices of the n-sphere $(\Gamma_{\mathcal{A}})_n$ are in bijection with $\mathcal{A}_n$;

(c) the projection $\pi_n : (\Gamma_{\mathcal{A}})_n \twoheadrightarrow (\Gamma_{\mathcal{A}})_{n-1}$ is well-defined;

(d) the graph is locally finite, indeed $\mathcal{A}$ is locally finite and horizontal edges starting from a vertex are finite due to the same argument that proves the fibers of π_h are finite (see Theorem 2.22).

(e) the graph of atoms is a subgraph of the graph of tips, since $\mathrm{d}_\Gamma(a,b) \le \lambda_e$ implies $\mathrm{T}^* \, \mathrm{d}_\Gamma(a,b) \le 2\lambda_e + 4\delta + 2\lambda_a$. In particular, we have $\mathrm{d}_{T(\Gamma_{\mathcal{A}})}(a,b) \le \mathrm{d}_{\Gamma_{\mathcal{A}}}(a,b)$.

(f) $\Gamma_{\mathcal{A}}$ is an *augmented tree* in the sense of [Kai03]. Roughly speaking, an augmented tree is a graph constructed starting from a tree where we add edges between some vertices on the same level with the condition that if x and y are two vertices that share such an edge and $\tilde{x} \in [x_0, x]$, $\tilde{y} \in [x_0, y]$ are at the same level, then $\tilde{x} = \tilde{y}$ or they share an edge (recall that x_0 is the root).

Theorem 3.7. *Let Γ be a hyperbolic graph. Then the graph of atoms $\Gamma_{\mathcal{A}}$ is quasi-isometric to Γ. In particular, it is hyperbolic and its boundary is homeomorphic to $\partial\Gamma$.*

Proof. Our quasi-isometry candidate map $\mathcal{S}_{\mathcal{A}} : \Gamma_{\mathcal{A}} \to \Gamma$ is defined as $a\mathcal{S}_{\mathcal{A}} := \widehat{x}$ with $\widehat{x}$ a fixed element in $T(a)$.

Quasi-dense image. This argument is exactly the same as Proposition 3.3 as both maps are defined from tips to element in Γ.

Quasi-isometric embedding. Let $a\mathcal{S}_{\mathcal{A}} = \widehat{x}$ and $b\mathcal{S}_{\mathcal{A}} = \widehat{y}$.
We start by proving that $\mathrm{d}_\Gamma(\widehat{x}, \widehat{y}) \le M \, \mathrm{d}_{\Gamma_{\mathcal{A}}}(a,b)$ for some constant M.

Let $a = a_0, a_1, \dots, a_l = b$ be a geodesic between a and b in $T(\Gamma_{\mathcal{A}})$. If $\{a_i, a_{i+1}\}$

is vertical, then we recall the bound on consecutive tips (see Proposition 2.26). If $\{a_i, a_{i+1}\}$ is horizontal, by definition we have a bound between two of their elements. We need to pay attention: we have a bound for every distance outside the atoms, but we also need a bound for what happens inside the tips, so that an element involved in the bound for the edge $\{a_{i-1}, a_i\}$ is at a reasonable distance from an element involved in $\{a_i, a_{i+1}\}$. This internal bound follows from the fact that a tip has diameter at most $2\lambda_a$ (see Proposition 1.54). Let D be the maximum between the two external bounds, more explicitly if D' is the bound provided for vertical edges by Proposition 2.26 and D'' is the bound coming from the definition of horizontal edge, then $D = \max\{D', D''\}$. If we put all together, we have

$$\mathrm{d}_\Gamma(\widehat{x}, \widehat{y}) \leq \sum_{i=0}^{l-1} \mathrm{d}_\Gamma(\widehat{x}_i, \widehat{x}_{i+1}) + \sum_{i=1}^{l-1} \operatorname{diam} T(a_i) \leq Dl + 2\lambda_a(l-1) \leq M \, \mathrm{d}_{T(\Gamma_{\mathcal{A}})}(a, b)$$

with $\widehat{x}_0 = \widehat{x}$, $\widehat{x}_l = \widehat{y}$ and $\widehat{x}_i \in T(a_i)$ one of the two elements of the tips involved in the bound for the left and for the right edges. And $M = D + 2\lambda_a$.

By part (e) of Remark 3.6, we get the claim.

We now prove the other part, namely $\mathrm{d}_{\Gamma_{\mathcal{A}}}(a, b) \leq W \, \mathrm{d}_\Gamma(\widehat{x}, \widehat{y})$ for some constant W. We proceed in the same way as before. We take a geodesic in Γ, explicitly $y_0 = \widehat{x}, y_1, y_2, \ldots, \widehat{y} = y_l$, between $\widehat{x}$ and $\widehat{y}$. By the quasi-density, we know that for every point y_i there exists a n_i-atom a_i such that $\mathrm{d}_\Gamma(y_i, a_i) \leq \lambda_\infty + 4\delta + 3 + \lambda_a$ and $\max\{0, \mathrm{d}_\Gamma(x_0, y_i) - \lambda_\infty\} \leq n_i \leq \mathrm{d}_\Gamma(x_0, y_i)$. This is due to the fact that either y_i has a infinite cone, hence it is a visible point and the atom a_i is at level $\mathrm{d}_\Gamma(x_0, y_i)$ (see Lemma 3.2) or y_i has a finite cone, but there exists another element $y \in [x_0, y_i]$ at a distance at most λ_∞ (see Lemma 3.1) that has an infinite cone and the associated atom a_i is at level $\mathrm{d}_\Gamma(x_0, y)$. Note that, in this way, two consecutive atoms are at a distance d_Γ less than λ_e.

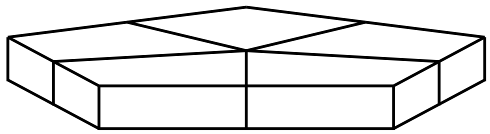
We want to prove that two consecutive atoms a_i and a_{i+1} have a distance in $\Gamma_{\mathcal{A}}$ bounded by some constant. So if they are at the same level, they are adjacent by the definition of horizontal edges. If they are on two different levels n and m, then $|n-m| \leq \lambda_\infty + 1$. Indeed, we combine the fact that two consecutive vertices y_i and y_{i+1} are such that $|\,\mathrm{d}_\Gamma(x_0, y_i) - \mathrm{d}_\Gamma(x_0, y_{i+1})| \leq 1$ (they are two consecutive points of a geodesic) and $\max\{0, \mathrm{d}_\Gamma(x_0, y_i) - \lambda_\infty\} \leq n_i \leq \mathrm{d}_\Gamma(x_0, y_i)$. Now, we can assume without loss of generality that $m < n$. We denote with a_i^m the m-atom such that $a_i \subseteq a_i^m$ and we have

$$\mathrm{d}_\Gamma(a_i^m, a_{i+1}) = \min_{x \in a_i^m, y \in a_{i+1}} \mathrm{d}_\Gamma(x, y) \leq \min_{z \in a_i, y \in a_{i+1}} \mathrm{d}_\Gamma(z, y) \leq \lambda_e.$$

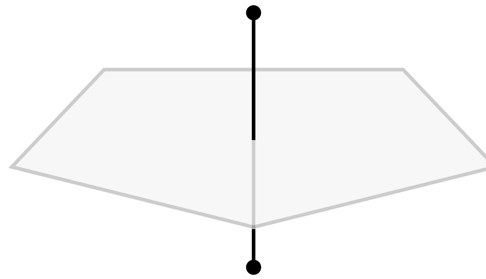
This means that a_i^m and a_{i+1} are adjacent and so $\mathrm{d}_{\Gamma_{\mathcal{A}}}(a_i, a_{i+1}) \leq \mathrm{d}_\Gamma(a_i, a_i^m) + \mathrm{d}_\Gamma(a_i^m, a_{i+1}) \leq \lambda_\infty + 1 + 1 = \lambda_\infty + 2$. To conclude, for each edge of the geodesic in Γ, we have constructed a geodesic in $\Gamma_{\mathcal{A}}$ of length at most $\lambda_\infty + 2$, hence $\mathrm{d}_{\Gamma_{\mathcal{A}}}(a, b) \leq W\, \mathrm{d}_\Gamma(\widehat{x}, \widehat{y})$ with $W = \lambda_\infty + 2$. $\square$

3.2 Example: uniform tiling of the hyperbolic space.

In this section, we want to study the 1-skeleton of a uniform tiling for the hyperbolic space. The tiling is made by cubes and it resembles the one in Figure 1.(b), which, in some sense, is its 2-dimensional version. More precisely, there are five cubes meeting in each edge (see Figure 12.(a)) and twenty meeting in each vertex. We have twelve edges starting from a given vertex x_0 arranged in a way such that the dual tiling is made by dodecahedra, which means: each edge of the original graph starting from x_0 intersects one of the twelve pentagons (see Figure 12.(b)), each square face of a cube of the original graph intersects an edge of the dodecahedron of the dual and at the center of each cube there is one of



(a) Five cubes meeting in a edge.



(b) Each edge in the tiling corresponds to a face of a dodecahedron in the dual tiling.

Figure 12: The tiling and the dual point of view.

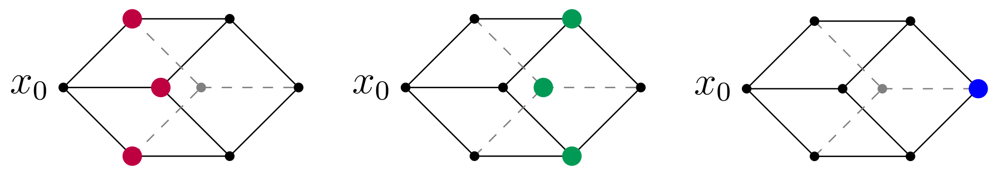
the twenty vertices of the dodecahedron of the dual.

This dual point of view, will be helpful to visualize atoms.

We start our description of the atoms by saying that we will count them using isometric equivalence classes (in this case we are not specifying a group action, so it is improper to speak about *types*). We take our distinguished vertex x_0 and we notice that, beside the trivial 0-level atom that we will denote again by A, there are three different isometric equivalence classes: B, C and D.

Class B. These atoms are the ones that correspond to the vertices that share an edge with x_0. More formally, they are the atoms such that their tips consist in exactly one element of the sphere of radius 1. So, we need to count the number of elements in the sphere, or equivalently the number of pentagons, to get how many children of this class we have. Informally, they resemble the type B in the plane. See Figure 13.(a).

Class C. In this case, the tip of an atom is exactly one of the vertices opposite to x_0 in one of the squares. In order to count them, we just need to look at the number of edges of a dodecahedron. They resemble the type C in the plane. See



(a) Three tips of class B. (b) Three tips of class C. (c) The only tip of class D.

Figure 13: The three classes of atoms displayed on one of the twelve cubes sharing x_0 as a vertex. In fact, every atom corresponds to its tip.

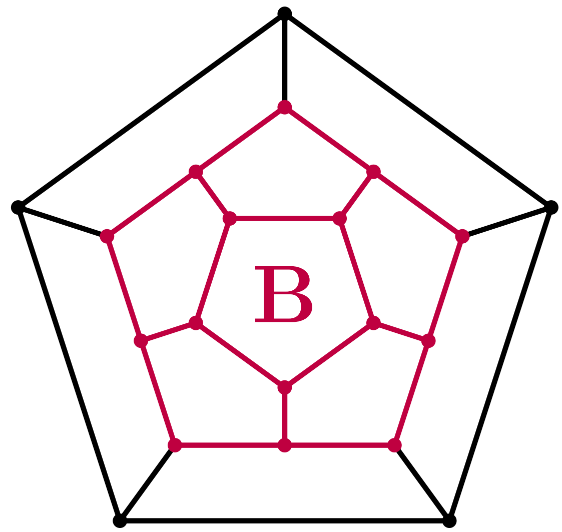
Figure 13.(b).

Class D. These atoms too can be defined by looking at their tips: we need to consider the vertices opposite to x_0 in each cube. This means that they correspond to the number of vertices of a dodecahedron. See Figure 13.(c).

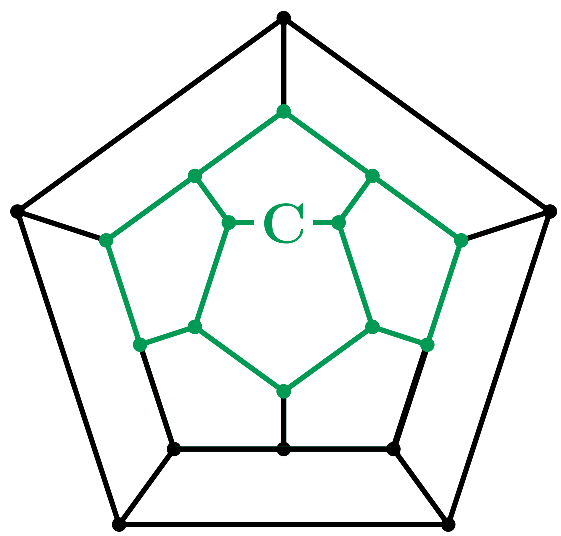
Now that we discussed the first level of the tree of atoms, we want to investigate the children of each class. As we did before, we will use the dual perspective to know how many children for each class we have: we associate to each tip its dodecahedron and then we associate faces to class B, edges to class C and vertices to class D. We only need to pay attention to the pentagons (we mean a face with its five edges and five vertices) that we already use in the previous levels.

Children of B. In this case the tip has one pentagon that corresponds to the edge from which we come. But this also means that all the five pentagon that share an edge with it are also used to define B, C and D classes of the previous level. So, what is left can be collected in what follows: 6 faces, 10 edges and 5 vertices (see Figure 14.(a)).

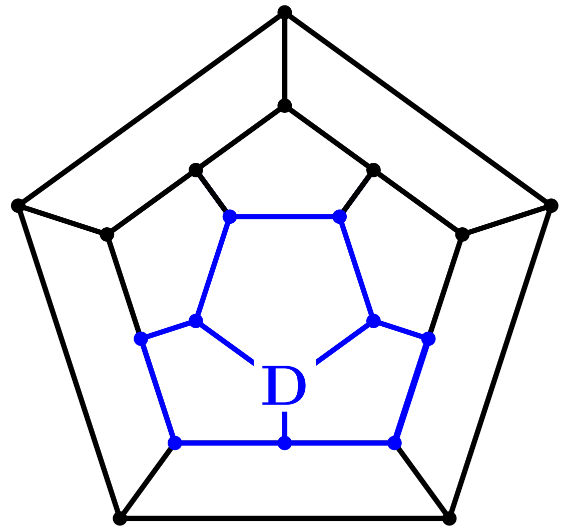
Children of C. For this class, we recall that the tip is the opposite vertex of a square. Here, two pentagons correspond to such square and their common edge



(a) For the class B, we enter the dodecahedron from the central face. All the pentagons around it describe atoms of the previous level.



(b) In the class C case, we have two edges entering the dodecahedron: one from the central face and the other from the face above it. The other two pentagons sharing a vertex with the C-edge are the one that complete the cube we are coming from.



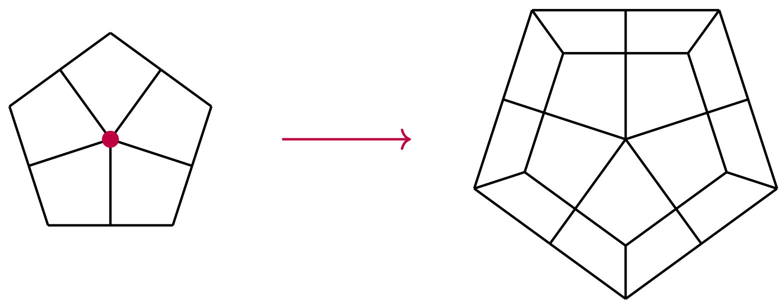
(c) Class D is associated to a vertex of the dodecahedron and has three edges entering: they complete the cube and so, no other pentagon, apart from the central one and the two below it, is involved in the previous level.

Figure 14: Every tip of an atom is the center of a dodecahedron (from the dual point of view). The Schlegel diagrams give us the perspective from the previous level. Each vertex, edge or face that is not colored is a child for our class.

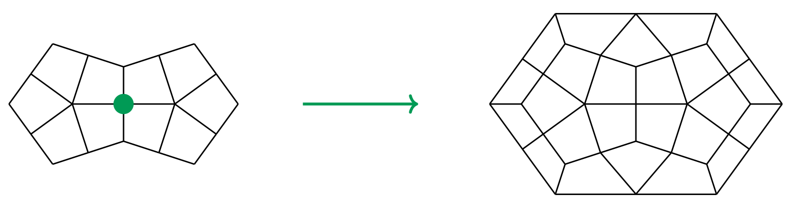
is the representative of the class C atom in the previous level. Hence, we also remove the other two pentagons that share a vertex with the edge. Counting what remains, we get: 8 class B children, 15 atoms of class C and 8 of class D (see Figure 14.(b)).

Children of D. In this case, we rely on a cube that means a vertex in the dodecahedron. So we need to remove the three pentagons that touch this vertex. We are left with 9 class B atoms, 18 class C atoms and 10 atoms of class D (see Figure 14.(c)).

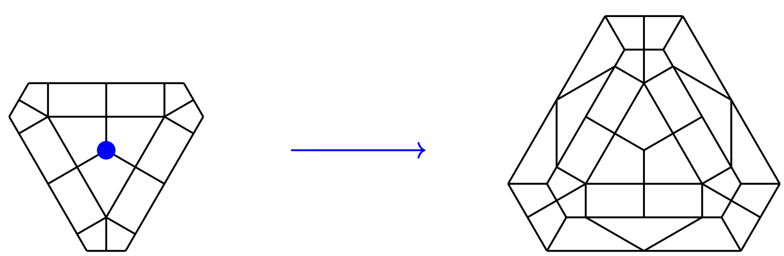
We are now ready to introduce the horizontal edges in the graph of atoms. We assume λ_e to be 1 to simplify the pictures. One can easily observe that considering the *n-horizontal graph* at each level, i.e. the graph made just by horizontal edges between atoms of the n-level, we get a tiling of the 2-dimensional sphere. Instead of showing the whole graph, we exhibit a rule to replace each vertex of the $(n-1)$-horizontal graph with the portion of the n-horizontal graph made just by children of such vertex. There is a specific rule for each class (see Figure 15). And we explain how to glue all these portions to get the result (see Figure 16). This way of describing the horizontal graph resembles the approach of *subdivision rules* (see e.g. [Rus14, Rus17]) and it may be possible that further and deeper analogies between the graph of atoms and the history graph of such rules exist.



(a) Each class B atom can only be found in the middle of a pentagon made by squares as in the picture on the left. On the right the way that its children are displaced in the horizontal graph of the next step.



(b) In this case atoms of class C can be only be found between two square-pentagons. On the right we see their expansion.



(c) Class D atoms correspond to vertices shared by three square-pentagons. On the right we see the children in the horizontal graph of their level.

Figure 15: On the left we have a portion of the current horizontal graph. The "center" of each portion is a vertex of a class we want to expand. On the right we have the corresponding expansion in the next horizontal graph of such vertex, that means the way the children of the vertex are displaced. How to replace the edge will be described in the next figure.

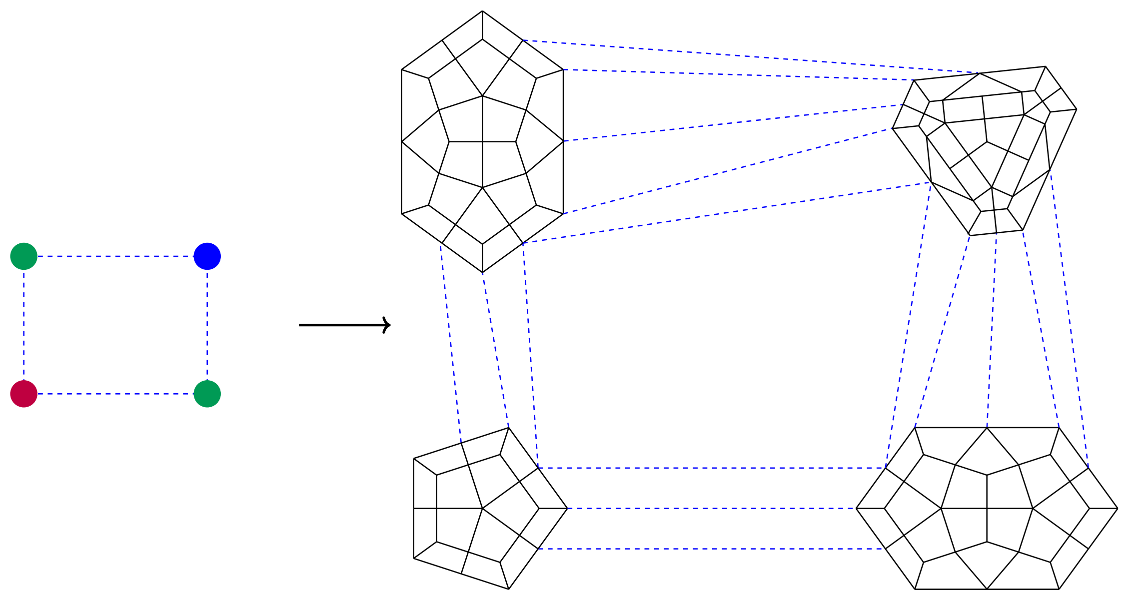


Figure 16: Any square in a horizontal graph is as in the picture (left), where the red atom is of Class B, the blue one is of class D and the other two in green are of class C. After replacing each of them, we also need to replace edges, that means we need a rule for gluing each subgraph to get the next horizontal graph. Gluing rules are depicted on the right.

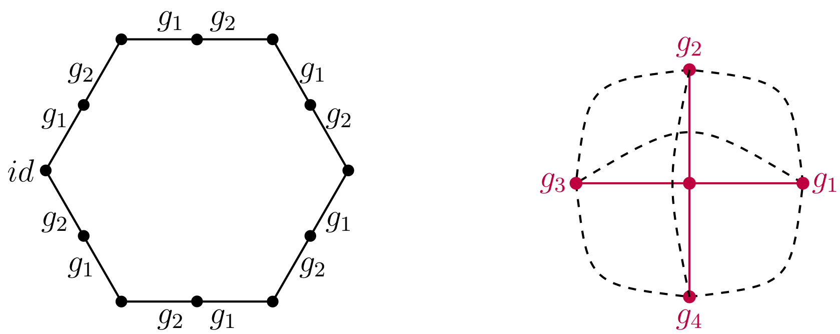
3.3 Example: fractal.

In this section, we will deal with the group

$$\langle g_1, g_2, g_3, g_4 \mid g_i^2,\ \ (g_i g_j)^6,\ \ i \in \{1, 2, 3, 4\},\ j > i \rangle.$$

Geometrically, we can represent each of the relations with an hexagon of edge $g_i g_j$ (see Figure 17.(a)). Since we have four generators and all of them are involutions, we can imagine the situation depicted in Figure 17.(b), that is the vertex g_i coincides with its inverse and we have six "hexagonal" relations.

Now, we can consider two types of atoms at the first level: a wide one (in Figure 18 there are two of them, outlined by blue lines) and a narrow one (there



(a) The relation $(g_1g_2)^6$.

(b) In red the four generators, the dashed lines are part of the six hexagons.

Figure 17: Generators and relations of the group.

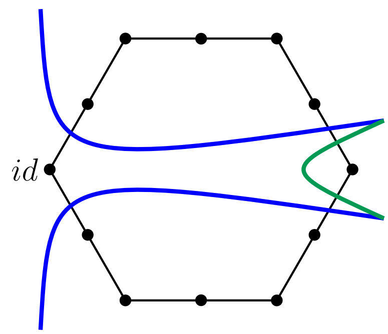


Figure 18: The two types of atom at the first level seen on a single relation $(g_ig_j)^6$.

is one of these in Figure 18 and it is the green one). The narrow type, unlike the wide, will not split for the next four levels, this is due to the fact that we have to wait until $B_6(id)$ to intersect the atom. So, the only child has a different type at each step, but all of them are homeomorphic.

(a) (b)

Figure 19: The first horizontal graph and a portion of it.

To construct the horizontal graph (again here we suppose $\lambda_e = 1$ for the sake of simplicity, as in the previous example), we have to imagine the four wide type atoms as vertices of a tetrahedron, while the six narrow type atoms are the middle point of the edges (see Figure 19.(a)).

Due to the self-similar nature of the atoms, we can focus only to one portion of the tetrahedron: we consider a wide atom and its three adjacent narrow atoms (see Figure 19.(b)). Then we just focus on this portion and it can be seen that the sequence of horizontal graphs is the one depicted in Figure 20. In particular, in these four steps the narrow type atom remains a vertex, while the expansion is made by the wide type atom. This process leads to an Apollonian gasket.

Remark 3.8. This is the last of three examples, after Figure 1.(b) and the example developed in Section 3.2, for which the containment of Proposition 1.47 is in fact an equality.

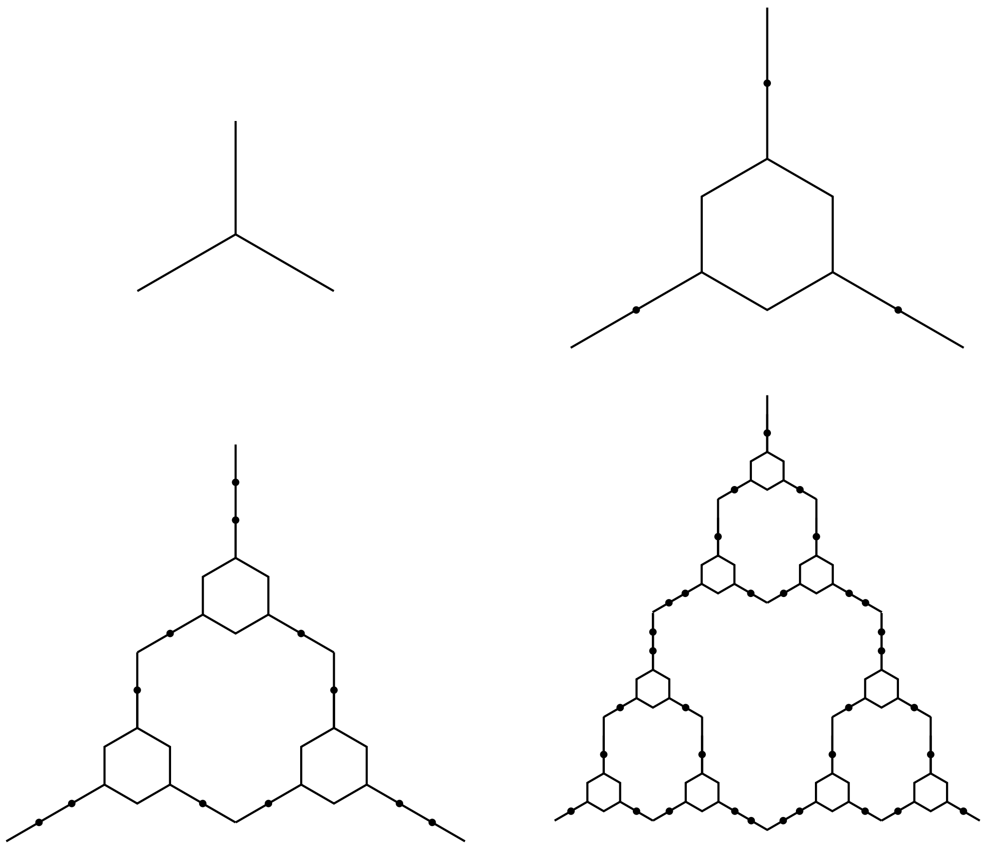


Figure 20: The first four portions of horizontal graphs.

CHAPTER 4

RATIONAL GLUING OF HOROFUNCTIONS

4.1 The gluing automaton

The goal of this chapter is to construct a machine that can tell if two elements of $\partial_h G$, represented by their atom-codings, are in the same π_h-fiber.

In order to do that, we give the following

Definition 4.1. Let Σ be a finite alphabet and Σ^ω its associate Cantor set of infinite strings. We say that an equivalence relation $\mathcal{G}$ on Σ^ω is **rational** if it is a rational subset of $\Sigma^\omega \times \Sigma^\omega$ in the sense of Definition 1.61.

Before proving that the gluing relation on atoms is rational, we want to present a few examples of the property and point out that the definition is in some way well-posed.

Example 4.2. Gromov boundaries of hyperbolic groups can be seen as quotients of the Cantor set given by geodesic rays (see beginning of Section 1.4). The fact that the relation is rational is due to the fact that Gromov boundaries are *semi-Markovian*. See [CP93] for definitions and for a proof in the torsion free case, and see [Paw15] for the connection between semi-Markovian and rational and for the groups with torsion.

Example 4.3. Limit spaces of contracting self similar groups (the gluing relation is given by the orbits of the action). See [Nek07] for definitions and, in particular, Proposition 5.6 for the rationality.

Example 4.4. Limit spaces of rearrangement groups of fractals (see [BF19]) seem to be natural candidates. We bring attention on the work of Donoven on the more general topic of invariant factors ([Don16]). We are interested in Section 4.3, which is devoted to replacement systems. Even if his approach is similar, the question is still open.

Proposition 4.5. *Let $\mathcal{G}$ be a rational equivalence relation on Σ^ω. Then $\mathcal{G}$ is preserved by rational homeomorphisms of Σ^ω.*

What we are going to do is to show that this is a direct consequence of Proposition 1.65.

Proof. To start, we observe that $\Sigma^\omega \times \Sigma^\omega = (\Sigma \times \Sigma)^\omega$. Setting $\Xi = \Sigma \times \Sigma$, by virtue of Proposition 1.65 we have a rational map $\phi : \overline{\Xi}^\omega \to \Xi^\omega$with image $\mathcal{G}$. Let $\psi : \Sigma^\omega \to \tilde{\Sigma}^\omega$ be a rational homeomorphism. We denote by $\psi \times \psi : \Xi^\omega \to \tilde{\Xi}^\omega$ with $\tilde{\Xi} = \tilde{\Sigma} \times \tilde{\Sigma}$ the map that is ψ on each component. Since the composition of two rational maps is a rational map, it remains to prove that $\psi \times \psi$ is rational, hence the composition $\phi(\psi \times \psi)$ is rational as well.

$$
\begin{tikzcd}
\Xi^\omega \arrow[r, "\psi\times\psi"] & \tilde{\Xi}^\omega \\
\overline{\Xi}^\omega \arrow[u, "\phi"] \arrow[ur, dashed, "\phi(\psi\times\psi)"'] &
\end{tikzcd}
$$

If $(\Sigma, \tilde{\Sigma}, \Theta, \to, out)$ is the defining transducer for ψ, we just create

$$(\Xi, \tilde{\Xi}, \Theta \times \Theta, \to \times \to, out \times out)$$

such that if $(\sigma_j, \theta_j^i) \to \theta_j^o$ with $j = 1, 2$, then $((\sigma_1, \sigma_2), (\theta_1^i, \theta_2^i)) \to (\theta_1^o, \theta_2^o)$ and an adjusted condition holds for the output function. $\square$

Now that we have set a connection with the literature, we can start working on our case. But before introducing the machine, we need to fix some notation. First of all, we define a slightly more rigid version of a self-similar structure.

Definition 4.6. Let $\mathcal{T}$ be a self-similar rooted tree. We define a **rigid structure** as the subcollection of rooted tree isomorphims of the given self-similar structure that satisfy the following

(a) for each pair of vertices x and y of the same class, there exists a unique isomorphism $\varphi_{x,y}$ that maps $\mathcal{T}_x$ on $\mathcal{T}_y$;

(b) it is closed under composition, that means $\varphi_{x,y}\varphi_{y,z} = \varphi_{x,z}$ for x, y and z of the same class;

(c) if x' is a child of x and $y' = x'\varphi_{x,y}$, then $\varphi_{x',y'}$ is the restriction of $\varphi_{x,y}$ onto $\mathcal{T}_{x'}$.

It is always possible to retrieve a rigid structure starting from a self-similar one

Proposition 4.7. *Every self-similar tree has a rigid structure.*

We are interested in the technique involved in the proof of Proposition 4.7 (see [BBM21, Proposition 2.18] for the complete version). More specifically, we need to define *markings*. We take a set of vertices that contains exactly one element for each class and the root. We denote it with $\widehat{\Omega}$. Let Ω be the set of vertices which are children of elements in $\widehat{\Omega}$. For each $o \in \Omega$ we choose τ_o to be a rooted tree isomorphism between o and the only vertex in $\widehat{\Omega}$ that belongs to the same class and we call it an **elementary marking**. Now, we take any vertex x and we define its **marking** ψ_x as follows:

(1) if x is the root, then ψ_x is the identity isomorphism of $\mathcal{T}$;

(2) if x is a vertex of $\mathcal{T}$ with marking ψ_x and y is a child of x, denote by $o = y\psi_x$ the corresponding child of $x\psi_x$ inside $\widehat{\Omega}$ and define $\psi_y = \psi_x\tau_o$.

Note that the composition is *partial*, which means that actually we are considering $\psi_x\tau_o$ with ψ'_x the restriction of ψ_x onto $\mathcal{T}_y$.

$$\begin{tikzcd}
\mathcal{T}_x \arrow[rr, "\psi_x"] & & \mathcal{T}_{x\psi} \\
\mathcal{T}_y \arrow[u, hook] \arrow[r, "\psi'_x"] & \mathcal{T}_o \arrow[ur, "\tau_o"'] &
\end{tikzcd}$$

One can see that the rooted tree isomorphism defined as $\varphi_{x,y} := \psi_x\psi_y^{-1}$ with x and y two vertices of the same class are in fact a rigid structure.

As already briefly stated before Theorem 1.67, we observe that the tree of atoms is self-similar with respect to the structure given by morphisms and types. Hence, for each atom a we associate a marking ψ_a and we have a collection $\{\tau_i \mid i \in I\}$ for some finite set of indices I. These help us defining our coding in the following sense: we take an alphabet R and we call it the set of **rigid types** that is in a 1 to 1 correspondence with the set of elementary markings. We will usually have $r_i \leftrightarrow \tau_i$. An atom $\overline{a}$ is of rigid type r_i if its marking is of the form $\psi_{\overline{a}} = \psi_a\tau_i$ with $\overline{a}$ child of a. Finally, given a horofunction $(u_n)_{n=1}^{\infty}$, we get a corresponding string based on R.

A first consequence of this coding is that we can construct the **type automaton** (see Example 2.5 in [BBM21] for a general treatment on the subject): the set of states are the types of atoms and the number of transitions between two types are the number of children that an atom of the first type has of the second type. It is easy to see that this number does not depend on the choice of the atoms and that can be labeled by the rigid types.

Example 4.8. We consider the usual hyperbolic tiling and its 1-skeleton (see Figure 1.(b)). And the groups of isometries given by

$$G = \langle g, h \mid g^5 = 1,\ h^2 = 1,\ (gh)^4 = 1 \rangle$$

with g the rotation by 72° around the center and h the 180° rotation around the middle point of one of the edges starting from the center. By looking at the first levels (see Figure 8.(a)), we can argue that there are four types of atoms.

- The root, that is the only 0-level atom; we call it A.
- The first type, we call it B, there is one of them for each edge starting from x_0, so at the first level there are five.
- The second type, we call it C, these are the intersections between two red regions; again we have five of them at the first level.

So, we will need ten letters $\{0, 1, \ldots, 9\}$ to codify the first level. At this point, we notice that every type B has three children, two of type B and one of type C. Hence, we will use the letters B_0, B_1, B_2. While for the type C, there are three children, two of type C and

- The fourth type, we call it D, it is the middle child of a type C and it has just one child of type B.

To conclude, we need three letters C_0, C_1, C_2 and a letter D_0 to fully encode the elementary markings. See Figure 21, for the type automaton.

Definition 4.9. We define the automaton $\mathcal{M} = (\Delta_{/\sim}, R^2, \to, (a_0, a_0)_\sim)$ in the following way:

States The set $\Delta := \{(a, b) \mid a, b \in \mathcal{A}_n \text{ for some } n \in \mathbb{N},\ \mathrm{d}_\Gamma(a, b) \leq \lambda\}$ with λ the gluing constant and the quotient is on the relation $(a, b) \sim (c, d)$ there

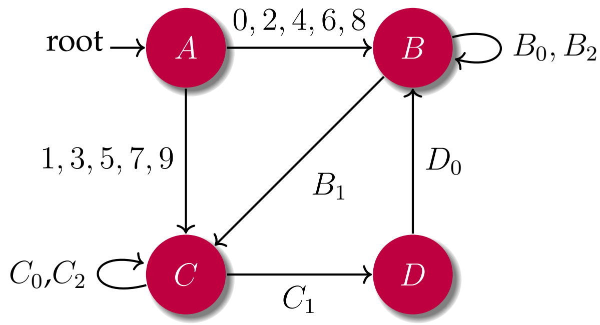


Figure 21: The type automaton for the 1-skeleton of the hyperbolic tiling.

exists $g \in G$ such that $g = \psi_a \psi_c^{-1} = \psi_b \psi_d^{-1}$. The elements of the quotient are denoted by $(a, b)_\sim$ for some representative $(a, b) \in \Delta$.

Alphabet We consider the cartesian product of the set R of rigid types with itself, that correspond to the cartesian product of the set of elementary markings.

Transition function We put an arrow $(r_1, r_2) \in R^2$ from $(a, b)_\sim$ to $(c, d)_\sim$ whenever there exist two elements $(\overline{a}, \overline{b}) \in (a, b)_\sim$ and $(\overline{c}, \overline{d}) \in (c, d)_\sim$ such that $\overline{c}$ is a child of $\overline{a}$ and in particular $\psi_{\overline{c}} = \psi_{\overline{a}} \tau_1$ with τ_1 the elementary marking associated to r_1. The same holds for $\overline{d}, \overline{c}$ and r_2.

Initial state We denote with a_0 the atom of level 0. The notation follows from the description of the states.

Remark 4.10. Note that the rigid structure does not define new types, so the requirement in the definition of $\sim$ is the same as asking for a $g \in G$ that is a morphism between a and c, but also between b and d.

Remark 4.11. The automaton contains a copy of the type automaton. Indeed, given a type there is a state $(a, a)_\sim$ which collects all a of that type paired with themselves. And the transitions between two of these states are labeled precisely by (r, r) with r ranging in the set of all transition labels of the type automaton.

We now proceed in the following way: first we need to verify that the machine is actually doing what we expect on horofunctions and then we will prove that it is finite-state.

Proposition 4.12. *Let $\mathcal{M}$ be the automaton described above and let $(u_n)_{n=1}^{\infty}$ and $(v_n)_{n=1}^{\infty}$ be two horofunctions described by their codings. We have that the horofunctions are the same element in ∂G if and only if there exists an infinite transition through the states $\{(a_n, b_n)_{\sim}\}_{n=1}^{\infty}$ in $\mathcal{M}$.*

Proof. We first suppose that two horofunctions glue together. Then by Theorem 2.4, we know that $\mathrm{d}_\Gamma(u_n, v_n) \leq \lambda$ holds for every n. Hence, by definition of the automaton, we can take $a_n = u_n$ and $b_n = v_n$. We then know that, by passing through these states, we can read the string that is the sequence of double elementary markings associated to the horofunctions.

On the other hand, suppose we can read the two horofunctions on $\mathcal{M}$. Suppose, also, that for $k \leq n$ we have $(u_k, v_k) \in (a_k, b_k)_{\sim}$. In particular, $\mathrm{d}_\Gamma(u_k, v_k) = \mathrm{d}_\Gamma(u_k g, v_k g) = \mathrm{d}_\Gamma(a_k, b_k) \leq \lambda$. We want to show that $(u_{n+1}, v_{n+1}) \in (a_{n+1}, b_{n+1})_{\sim}$. By hypothesis we know that $g = \psi_{u_n}\psi_{a_n}^{-1} = \psi_{v_n}\psi_{b_n}^{-1}$. Since we read (r_1, r_2) to reach the state $(a_{n+1}, b_{n+1})_{\sim}$ from the state $(a_n, b_n)_{\sim}$, we have

$$\psi_{u_{n+1}} = \psi_{u_n}\tau_1 \text{ and } \psi_{v_{n+1}} = \psi_{v_n}\tau_2,$$

with r_i the digit associated to the elementary marking τ_i for $i = 1, 2$.
By the same token, we can choose the representative (a_{n+1}, b_{n+1}) such that

$$\psi_{a_{n+1}} = \psi_{a_n}\tau_1 \text{ and } \psi_{b_{n+1}} = \psi_{b_n}\tau_2.$$

To conclude, a_{n+1} and u_{n+1} are of the same type (they have the same marking), the same holds for b_{n+1} and v_{n+1}. So there exist $h_1, h_2 \in G$ such that

$$h_1 = \psi_{u_{n+1}} \psi_{a_{n+1}}^{-1} \text{ and } h_2 = \psi_{v_{n+1}} \psi_{b_{n+1}}^{-1}.$$

If we put everything together, we get

$$h_1 = \psi_{u_{n+1}} \psi_{a_{n+1}}^{-1} = \psi_{u_n} \tau_1 \tau_1^{-1} \psi_{a_n}^{-1} = g = \psi_{v_n} \tau_2 \tau_2^{-1} \psi_{b_n}^{-1} = \psi_{v_{n+1}} \psi_{b_{n+1}}^{-1} = h_2.$$

This means that $\mathrm{d}_\Gamma(u_{n+1}, v_{n+1}) = \mathrm{d}_\Gamma(a_{n+1}g, b_{n+1}g) = \mathrm{d}_\Gamma(a_{n+1}, b_{n+1})$. In particular, (u_{n+1}, v_{n+1}) belongs to Δ and to $(a_{n+1}, b_{n+1})_\sim$ as desired. $\square$

A couple of remarks about efficiency and geometric interpretation of the machine are needed.

Remark 4.13.

(1) Notice that the number of steps before stopping is not optimal. This occurs since we rely on a gluing constant and so we may create some states without any possible transition from there, in other words the automaton is not reduced.

(2) The automaton roughly gives an estimate about the distance between two points in the Gromov boundary by looking at their codings. Indeed, by Theorem 2.35 and by Definition 4.9 the (discrete) amount of time at which the machine stops is not far from the Gromov product between the two elements in input.

In order to leave the previous proposition as clean as possible, we collect here the properties of the automaton needed to get the rationality.

Corollary 4.14. *The machine $\mathcal{M}$ is deterministic and recognizes the horofunctions.*

Proof.

Determinism. Suppose that the following situation occurs in the automaton:

$$\begin{tikzcd} & (a_1, b_1)_\sim \\ (a, b)_\sim \arrow[ur, bend left, "{(r,s)}"] \arrow[dr, bend right, "{(r,s)}"'] & \\ & (a_2, b_2)_\sim \end{tikzcd}$$

we want to prove that $(a_1, b_1)_\sim = (a_2, b_2)_\sim$. This is a consequence of the rigid structure, namely there exist

$$(a, b) \in (a, b)_\sim \text{ and } (a_1, b_1) \in (a_1, b_1)_\sim \text{ with } \psi_{a_1} = \tau_r \psi_a,\ \psi_{b_1} = \tau_s \psi_b$$

and

$$(\overline{a}, \overline{b}) \in (a, b)_\sim \text{ and } (a_2, b_2) \in (a_2, b_2)_\sim \text{ with } \psi_{a_2} = \psi_{\overline{a}} \tau_r,\ \psi_{b_2} = \psi_{\overline{b}} \tau_s;$$

moreover there exists $g \in G$ such that $g = \psi_{\overline{a}} \psi_a^{-1} = \psi_{\overline{b}} \psi_b^{-1}$. So we have

$$\psi_{a_2} \psi_{a_1}^{-1} = \psi_{\overline{a}} \tau_r \tau_r^{-1} \psi_a^{-1} = g = \psi_{\overline{b}} \tau_s \tau_s^{-1} \psi_b^{-1} = \psi_{b_2} \psi_{b_1}^{-1},$$

that yields the claim.

Recognizer. Suppose

$$(a_0, a_0)_\sim \to (a_1, b_1)_\sim \to \ldots \to (a_n, b_n)_\sim \to \ldots$$

is a transition of states on $\mathcal{M}$. We recall $(a_0, a_0)_\sim$ is the initial state, related to the word

$$(r_1, s_1)(r_2, s_2) \ldots (r_n, s_n) \ldots$$

This means that there exist $a_{n-1} \xrightarrow{r_n} a_n$ and $\overline{a}_n \xrightarrow{r_{n+1}} \overline{a}_{n+1}$, but a_n and $\overline{a}_n$ have the same type and so r_{n+1} must be an allowed rigid type for a_n too, hence we can provide a_{n+1} such that $a_n \xrightarrow{r_{n+1}} a_{n+1}$. □

This last part is devoted to showing that $\mathcal{M}$ is a finite state machine, namely we want a bound for the cardinality of $\Delta_{/\sim}$. For this purpose, we need the key definition introduced in [BBM21] to prove that the number of types are finite.

Before that, we recall that if two elements x and y belong to the same k-level atom a, then $\overline{d}_x = \overline{d}_y$ over B_k. Hence, $\overline{d}_a$ is well defined over B_k.
If Γ_0 is a subset of vertices of Γ, f is a function from Γ_0 to $\mathbb{Z}$ and $g \in G$, then we define fg to be the function $yfg := yg^{-1}f$ for all $y \in \Gamma_0 g$. Note also that if f_1 and f_2 differ by a constant, that $f_1 g$ and $f_1 g$ also differ by a constant. Putting these two facts together leads to the definition of $\overline{d}_a g$ on $B_k g$.

Definition 4.15. Let $a \in \mathcal{A}_m$ and $b \in \mathcal{A}_n$. We say that an element $g \in G$ induces a **geometric equivalence** between a and b if

(1) $P(a, S_m)g = P(b, S_n)$;

(2) $\overline{d}_a g = \overline{d}_b$ over $P(b, S_n)$;

(3) $C(p)g = C(pg)$ for all $p \in P(a, S_m)$.

The main result concerning this definition is the following

Proposition 4.16. *If $g \in G$ induces a geometric equivalence between two atoms a and b, then it induces a morphism. Hence a and b are of the same type.*

Most of Section 3.5 of [BBM21] consists of a proof for this Proposition. The following proof is taken from Corollary 3.28 in [BBM21] and we show it here because it will be useful to understand our case.

Proof(Theorem 1.51). By virtue of Proposition 4.16, it suffices to prove that geometric equivalence classes are finite.
Since the action of G onto Γ is cocompact, there exists a compact, hence finite, subset K of vertices such that

$$KG = \{Kg \mid g \in G\}$$

is the whole graph. Now take $p \in P(a, S_m)$, then there exists an element $h \in G$ such that $ph \in KG$; exploiting Proposition 1.42(b) we have that $P(a, S_n)$ is contained in a $8\delta+2$-neighborhood of K. This means that there are finitely many possibilities for $P(a, S_m)$ modulo the action of G. Moreover, since the action of G is properly discontinuous and by Proposition 1.45 there are finitely many cone types, we have only finitely many choices for $C(p)$ for each $p \in P(a, S_m)$, and there are only finitely many choices for the restriction of $\overline{d}_a$ to $P(a, S_m)$. □

We are going to study a slight refinement of geometric equivalences and types. In order to do that, we consider a λ-neighborhood of an atom with respect to d_Γ. The fact that the collection of atoms in the neighborhood of an atom is finite is due to the fact that $\mathrm{d}_\Gamma \leq \mathrm{T}^* \mathrm{d}_\Gamma \leq \mathrm{d}_\Gamma + \lambda$ with λ the gluing constant together with the fact that the tips are finite (see Proposition 1.54).
We denote the set of all n-level atoms within a distance λ to an n-level atom a with $\mathcal{A}_\lambda(a)$ and we call a the **center** of the neighborhood.

Definition 4.17. Two atoms $a, b \in \mathcal{A}$ have the same **λ-type** if there exists an element $g \in G$ that induces a bijection between $\mathcal{A}_\lambda(a)$ and $\mathcal{A}_\lambda(b)$ and such that $g_{|a_\lambda}$ is a morphism of types between a_λ and b_λ for all $a_\lambda \in \mathcal{A}_\lambda(a)$.

Before giving the corresponding definition of geometric equivalence, we want to notice the following.

Remark 4.18.

(1) There can be two atoms in a λ-neighborhood with the same type.

(2) Since g is an isometry, we have that the distance between two atoms in a λ-neighborhood depends just on the λ-type of its center.

Definition 4.19. Two atoms $a, b \in \mathcal{A}$ are said to be **geometric λ-equivalent** if there exists $g \in G$ such that for all $a_\lambda \in \mathcal{A}_\lambda(a)$ and $b_\lambda \in \mathcal{A}_\lambda(b)$ the following hold

(1) $P(a_\lambda, S_n)g = P(b_\lambda, S_m)$;

(2) $\overline{d}_{a_\lambda} g$ agrees with $\overline{d}_{b_\lambda}$ on $P(b_\lambda, S_m)$;

(3) $C(p)g = C(pg)$ for all $p \in P(a_\lambda, S_n)$;

for suitable positive integers n and m.

We have the following version of Proposition 4.16.

Lemma 4.20. *If two atoms are geometric λ-equivalent, than they have the same λ-type.*

Proof. By definition, the element g that induces the geometric λ-equivalence also induces a geometric equivalence on each atom that belongs to $\mathcal{A}_\lambda(a)$. By Proposition 4.16, we have that g induces a morphism on each atom. Hence the claim. $\square$

All that is left to do is prove that the number of geometric λ-equivalence classes is finite. But again this follow almost immediately by [BBM21].

Lemma 4.21. *The number of equivalence classes with respect to the geometric λ-equivalence is finite.*

Proof. One can argue as in the proof of Theorem 1.51 and by noticing that the union of all the proximal sets of atoms in $\mathcal{A}_\lambda(a)$ has a finite diameter by virtue of the Hooking Lemma. $\square$

We are now ready to prove the following

Proposition 4.22. *The set $\Delta_{/\sim}$ is finite.*

Proof. The key idea is that there exists a way to cover Δ by $\mathcal{A}_\lambda(a)$ as a ranges in $\mathcal{A}$, or, more explicitly, for each $(a,b) \in \Delta$, we have that $b \in \mathcal{A}_\lambda(a)$.
By combining Lemma 4.20 and Lemma 4.21, we have that there are finitely many λ-types. Set C_λ to be the finite number of λ-types. We also know that $|\mathcal{A}_\lambda(a)|$ is finite, and in particular there are finitely many pairs (a,b) as $b \in \mathcal{A}_\lambda(a)$. Finally, if $(a,b) \sim (c,d)$ then there exists $g \in G$ that induces a map $g : \mathcal{A}_\lambda(a) \to \mathcal{A}_\lambda(c)$ and such that $(a,b)g = (c,d)$. So there are at most $C_\lambda(|\mathcal{A}_\lambda(a)| - 1)$ elements in $\Delta_{/\sim}$. $\square$

To summarize what we achieved in this Section, we explicit give this

Theorem 4.23. *The quotient map $\pi_h : \partial_h\Gamma \twoheadrightarrow \partial\Gamma$ defines a rational equivalence relation.*

As a final remark, we point out that since two atoms in a λ-neighborhood may have the same rigid type, we cannot conclude that the gluing relation is semi-Markovian as for other tree structures on the Gromov boundary. But something more can be said about λ-types.

Proposition 4.24. *The λ-types are a self-similar structure for the tree of atoms.*

Proof. The claim follows easily from the fact that λ-types are finite and the following argument: the element $g \in G$ that maps one λ-neighborhood into another is an isometry and it is a morphism on each atom of the neighborhood. Hence, it preserves the λ-neighborhoods of chidren and the types of the atoms contained in them. $\square$

4.2 Example: uniform tiling of the hyperbolic plane.

In this section we give the automaton $\mathcal{M}$ for the 1-skeleton of the hyperbolic tiling depicted in Figure 1.(b). We already discussed the number and the types of atoms in Example 4.8 and we provided the type automaton (see Figure 21). In order to get a simplified, but still correct, version of the machine we set $\lambda = 1$, so that two atoms define an element in Δ if and only if they are adjacent.

We also recall that $\mathcal{M}$ contains a copy of the type automaton: in Figure 22 the states of such a copy are in red and the labels on the transitions are omitted, since they are the ones already discussed in Figure 1.(b). We said that Δ is composed by pairs (a, b) of atoms that are adjacent: this can happen when a is of type B and b is of type C or a is of type C, but in this case b can either be of type B or D. Obviously we also need to consider (b, a). This leads to eight equivalence classes. To understand the transitions between these classes, we consider two codings that are equal up to level n, this means that for k steps they are read by the copy of the type automaton. To enter into the blue part (see the figure), we want two children to differ just by 1 bit, this means that on the first level only the transitions $(k, k+1)$ with $k \in \{0, 1, \dots, 8\}$ and $(9, 0)$ are allowed and also their symmetric versions. When in a type B, we only read (B_0, B_1) and (B_1, B_2), and the same holds for type C. In order to get two horofunctions that glue, we need to continue with two periodic strings where the pairs are (B_0, C_2) or

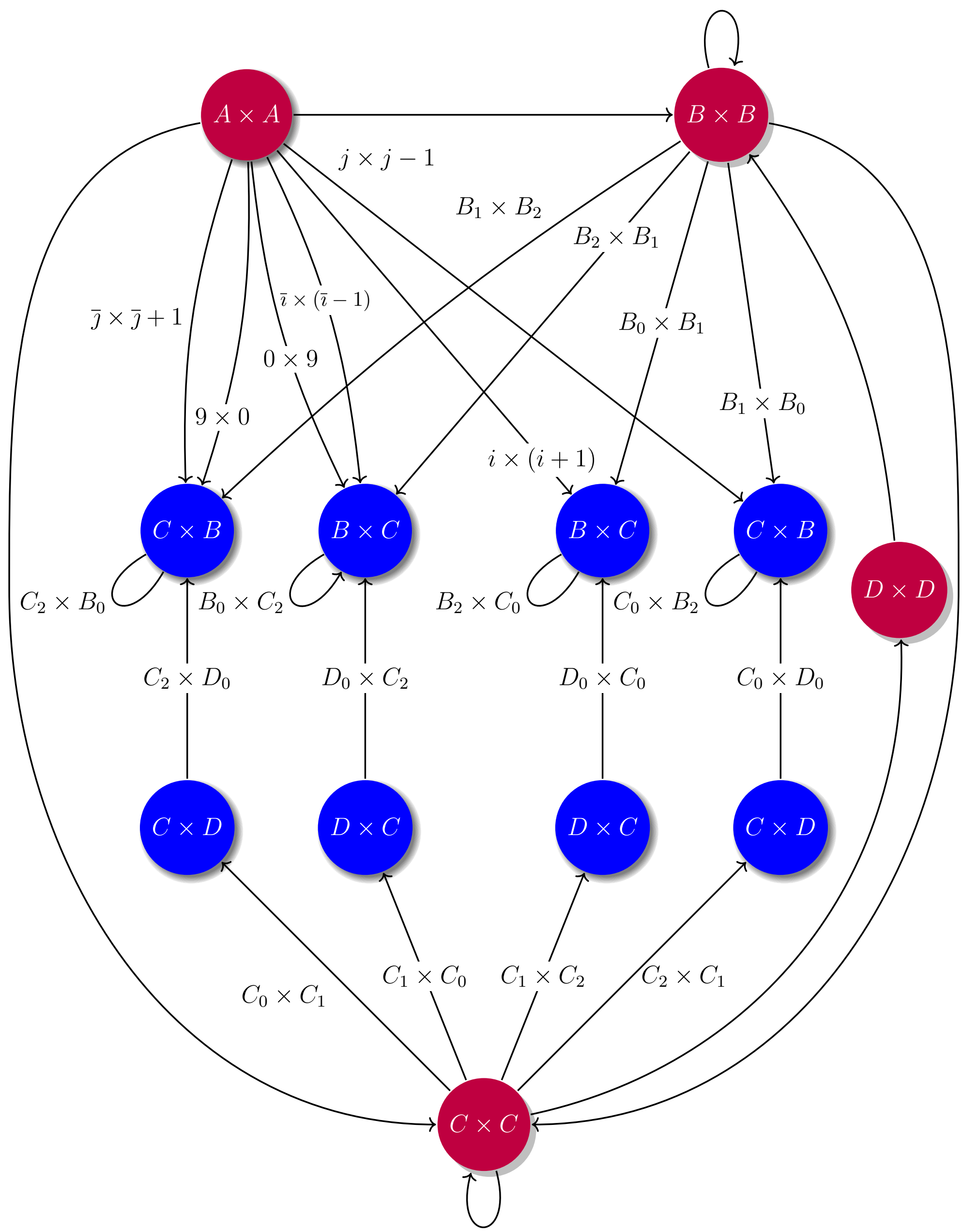


Figure 22: The gluing automaton for Figure 1.(b). Here λ is set to 1. And $i \in \{0, 2, 4, 6, 8\}$, $\bar{\imath} \in \{2, 4, 6, 8\}$, $j \in \{1, 3, 5, 7, 9\}$, $\bar{\jmath} \in \{1, 3, 5, 7\}$.

(B_2, C_0) (except at most one auxiliary step in the C case to read out the D type).

4.3 Example: fractal.

We want to discuss the construction of the gluing automaton for the fractal example of Section 3.3. We do not exhibit the full automaton, but we provide a sketch of how to build it.

We start by saying that the type automaton is depicted in Figure 23, where λ is set to 1 as for the other examples and we highlight the blue and green states corresponding respectively to the wide and the narrow type at the first level (the initial state is A as always). Names for the other states are given by following a geometric intuition that we prefer to omit since it is not useful for this description.

We proceed by levels and we only show the first one. We need to list all the elements in Δ at the current level (we can look at the tetrahedron in Figure 19.(a)). So, in this case we have

	w_1	w_2	w_3	w_4
n_1	1	0	0	1
n_2	1	0	1	0
n_3	1	1	0	0
n_4	0	1	0	1
n_5	0	1	1	0
n_6	0	0	1	1

and note that each pair need to be counted twice (w_i, n_j) and (n_j, w_i).

In order to compute the states of $\mathcal{M}$, we should provide elements of the group that send pairs to pairs. Note that this does not mean that whenever we

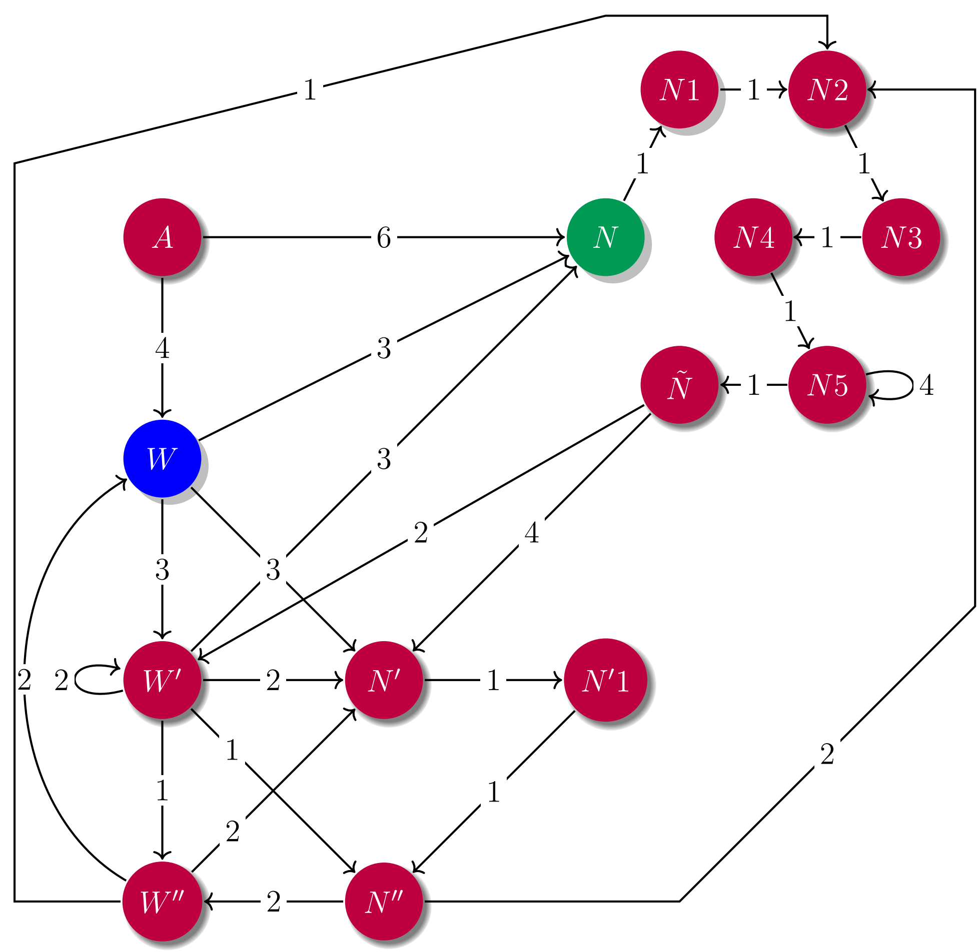


Figure 23: The type automaton: labels denote the number of arrows of that kind, the blue state and the letter W mean the wide type from Section 3.3 and the same holds for the green state and the letter N (narrow).

have a pair of atoms (a, b) and (a', b') such that a' shares the same type with a and b' with b, there exists such element. Once we have completed the first level, we pass to the second having in mind that now we already have some states in $\mathcal{M}$ and hence new elements in Δ may be in the same equivalence class of an element of the previous level. Moreover, we have to add all the possible transitions according to the rigid structure. The procedure ends when we are sure that all possible rigid types have been processed and this can be done by looking at the type automaton.

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