Maximum Likelihood Estimation of Multivariate Regime Switching Student-*t* Copula Models

Fulvia Pennoni
joint work with
F. Bartolucci, F. Cortese

Department of Statistics and Quantitative Methods - University of Milano-Bicocca (Italy)
fulvia.pennoni@unimib.it

2nd International Conference on Computational Finance and Business Analytics - *CFBA* -
April 6th, 2024

Motivation

- In financial applications we are interested in *Market Analysis*: detection of bull-bear market states; *Portfolio Management*: diversification, allocation, and *Risk Management*: hedging, loss coverage
- How can we track the dynamics of *complex systems* such as financial markets?
- How can we correctly specify the *multivariate distribution* of financial time-series?
- A suitable statistical model should account for stylized facts of financial assets log-returns: *non-normality, heavy tails, cross-correlation, auto-correlation*

Regime Switching Copula Models

- We propose a multivariate regime switching model based on a *Student-t copula* function with parameters governed by a *latent Markov process* (Cortese et al., 2024, https://doi.org/10.1111/insr.12562)

- In such a way *tails* of the joint distribution are accounted by the number of degrees of freedom and *pair-specific dependence parameters* model the correlation structure among assets

- We apply the proposal to jointly analyze daily log-returns of the *five* cryptocurrencies: *Bitcoin*, *Ethereum*, *Ripple*, *Litecoin*, and *Bitcoin Cash* (Pennoni et al., 2021)

UNIVERSITA' DEGLI STUDI DI MILANO BICOCCA

Model Formulation

- We consider *multivariate log-returns* collected in vectors $\boldsymbol{y}_1, \dots, \boldsymbol{y}_T$, with elements y_{tj}, $t = 1, \dots, T$, $j = 1, \dots, r$, and with joint distribution F

- By adopting *inference for margins* method (Joe and Xu, 1996) we split the modeling process into two steps:

    - First, fit *the marginal distribution* of each univariate time-series $F_1, \dots, F_r$, then compute *pseudo-observations*

$$e_{tj} = F_j(y_{tj})$$

    - Second, estimate their *joint distribution* described by a regime switching copula model

- The uniform pseudo-observations $\boldsymbol{e}_t$, $t = 1, \dots, T$, follow a *regime switching copula model* if:

  - There exists a latent process u_t, $t = 1, \dots, T$ assumed to follow a *homogeneous Markov chain* with $\boldsymbol{k}$ *states* defined by a *transition* matrix $\boldsymbol{\Pi}$ with elements

    $$\pi_{v|u} = P(u_t = v \mid u_{t-1} = u), \quad u, v = 1, \dots, k$$

    and *initial* probabilities $\pi_u = P(u_1 = u)$, $u = 1, \dots, k$

  - *Local independence assumption*: given $u_1, \dots, u_T$, the vectors of uniform pseudo-observations $\boldsymbol{e}_1, \dots, \boldsymbol{e}_T$, are independent and distributed with copula densities

    $$c(\cdot; \boldsymbol{R}_{u_1}, \nu_{u_1}), \dots, c(\cdot; \boldsymbol{R}_{u_T}, \nu_{u_T})$$

    being ν_u the *number of degrees of freedom* and $\boldsymbol{R}_u$ the matrix of dependence parameters of the *Student-t* copula function

Expectation-Maximization Algorithm

- Let $\boldsymbol{\theta}$ be the *full vector of parameters* corresponding to the non-redundant elements of $\boldsymbol{R}_u$, together with ν_u and π_u, $u = 1, \ldots, k$, and $\pi_{v|u}$, $u, v = 1, \ldots, k$
- The EM algorithm is based on the *complete data log-likelihood* $\ell^*(\boldsymbol{\theta})$
- It alternates *two steps until convergence*:
- **E-step**: compute the *posterior expected value* given the observed data
- **M-step**: *maximize* the expected complete data log-likelihood with respect to the model parameters

Application

- Application based on the market of *five cryptocurrencies* Bitcoin (BTC), Ethereum (ETH), Ripple (XRP), Litecoin (LTC), and Bitcoin Cash (BCH), for the period August 2, 2017 - October, 2, 2022

- Data are provided by the *Crypto Asset Lab*, which is an independent academic lab established at the University of Milano-Bicocca, Italy

- The selection of the *cryptocurrencies* is based on the criteria underlying the Crypto Asset Lab Index concerning some aspects (reliability, liquidity, low manipulation)

- We present results for the *2-state RSStC model*: selection based on *Integrated Composite Likelihood* criteria (ICL)

- An *ARMA(1,1)-GARCH(1,1)* model (Engle and Bollerslev, 1986) for the marginals, with *Skewed Generalized Error* Distribution for the residual terms

- Results show that the model identifies **bull** (profitable) and **bear** market *states* based on the intensity of correlation and fatness of the tail of the joint distribution

- We predict the latent regimes considering *global decoding* (Viterbi, 1967)

- Sample *unconditional means* and standard deviations of the log-returns

	Cryptocurrency				
	BTC	ETH	XRP	LTC	BCH
Mean (%)	0.090	0.087	0.049	0.002	-0.073
S.D. (%)	4.166	5.271	6.451	5.659	6.585

- Linear *correlations*: a positive association is present for each pair

	BTC	ETH	XRP	LTC	BCH
BTC	1.000	-	-	-	-
ETH	0.787	1.000	-	-	-
XRP	0.560	0.653	1.000	-	-
LTC	0.765	0.823	0.644	1.000	-
BCH	0.678	0.742	0.583	0.732	1.000

Results

- Estimated *dependences parameters* $\hat{\rho}_u^{(ij)}$ for the 2-states RSStC model

State 1	BTC	ETH	XRP	LTC	BCH
BTC	1.000	-	-	-	-
ETH	0.911	1.000	-	-	-
XRP	0.902	0.910	1.000	-	-
LTC	0.875	0.902	0.910	1.000	-
BCH	0.902	0.907	0.927	0.901	1.000
State 2	BTC	ETH	XRP	LTC	BCH
BTC	1.000	-	-	-	-
ETH	0.652	1.000	-	-	-
XRP	0.667	0.728	1.000	-	-
LTC	0.487	0.613	0.592	1.000	-
BCH	0.569	0.645	0.672	0.517	1.000

- Estimated *Kendall's tau*: $\hat{\tau}_u^{(ij)} = \frac{2}{\pi} \arcsin \hat{\rho}_u^{(ij)}$ under the 2-state RSStC model with estimated *number of degrees of freedom*: $\nu_1 = 6.231$, $\nu_2 = 9.416$
- First regime shows *high interdependence*

State 1	BTC	ETH	XRP	LTC	BCH
BTC	1.000	-	-	-	-
ETH	0.730	1.000	-	-	-
XRP	0.715	0.728	1.000	-	-
LTC	0.678	0.715	0.728	1.000	-
BCH	0.716	0.724	0.756	0.714	1.000
State 2	BTC	ETH	XRP	LTC	BCH
BTC	1.000	-	-	-	-
ETH	0.452	1.000	-	-	-
XRP	0.465	0.519	1.000	-	-
LTC	0.324	0.420	0.403	1.000	-
BCH	0.385	0.446	0.469	0.346	1.000

- Estimated *transition probabilities* for the 2-state RSStC model

State	*1*	2
1	**0.912**	0.088
2	0.121	**0.879**

- *High persistence*, that is high estimated self-transition probabilities, is observed in both specifications

- Useful property for the construction of efficacy *trading strategy*: frequent changes in the inferred regimes might result in excessive trading costs and inferior portfolio performance

UNIVERSITA' DEGLI STUDI DI MILANO BICOCCA

State-Conditional Mean/S.D.

- State allocation obtained through the *Viterbi algorithm*
- These estimates charaterize each state *bearish and bullish* market regimes, respectively
- Standard deviations indicate *high volatility* in both regimes

State 1	Mean (%)	S.D. (%)	State 2	Mean (%)	S.D. (%)
BTC	-0.363	4.121	BTC	0.744	4.147
ETH	-0.509	5.400	ETH	0.948	4.957
XRP	-0.699	5.377	XRP	1.131	7.619
LTC	-0.725	5.454	LTC	1.052	5.788
BCH	-0.902	6.013	BCH	1.125	7.169

Concluding remarks

- New proposal of a *multivariate* regime switching Student-t copula model
- To the best of our knowledge, there are no previous works studying cryptocurrencies with regime switching copulas
- *Bear* market regimes are characterized by *high cross-correlation*, *fat-tailed* joint distribution
- RSStC models can account for observed *persistence* of market regimes
- The model can be easily estimated and it is suitable to be used for simulating portfolio scenarios and to compute *risk-measures* (VaR, backtesting)

UNIVERSITA' DEGLI STUDI DI MILANO BICOCCA

Main References I

F. Cortese, F. Pennoni, and F. Bartolucci. Maximum likelihood estimation of multivariate regime switching student-t copula models. *International Statistical Review*, pages 1–28, 2024, https://doi.org/10.1111/insr.12562.

R. F. Engle and T. Bollerslev. Modelling the persistence of conditional variances. *Econometric Reviews*, 5:1–50, 1986.

H. Joe and J. J. Xu. The estimation method of inference functions for margins for multivariate models. *University of British Columbia, Department of Statistics, Technical Report*, 166, 1996.

F. Pennoni, F. Bartolucci, G. Forte, and F. Ametrano. Exploring the dependencies among main cryptocurrency log-returns: A hidden Markov model. *Economic Notes*, 51:e12193, 2021.

A. Viterbi. Error bounds for convolutional codes and an asymptotically optimum decoding algorithm. *IEEE Transactions on Information Theory*, 13:260–269, 1967.

Expectation-Maximization Algorithm

- Let $\boldsymbol{\theta}$ be the *full vector of parameters* corresponding to the non-redundant elements of $\boldsymbol{R}_u$, together with ν_u and π_u, $u = 1, \ldots, k$, and $\pi_{v|u}$, $u, v = 1, \ldots, k$
- The EM algorithm is based on the *complete data log-likelihood*

$$\begin{aligned}\ell^*(\boldsymbol{\theta}) = \sum_{t=1}^{T} \sum_{u=1}^{k} w_{tu} \log c(\boldsymbol{e}_t; \boldsymbol{R}_u, \nu_u) + \\ \sum_{u=1}^{k} w_{1u} \log \pi_u + \sum_{t=2}^{T} \sum_{u=1}^{k} \sum_{v=1}^{k} z_{tuv} \log \pi_{v|u}\end{aligned} \tag{1}$$

where $w_{tu} = I(u_t = u)$ is an indicator variable equal to 1 if the process is in state u at time t and 0 otherwise, while $z_{tuv} = w_{t-1,u} w_{tv}$ denotes the transition at time t from state u to v

- The EM algorithm alternates *two steps until convergence*:
  - **E-step**: compute the *posterior expected value* of each indicator variable w_{tu}, $t = 1, \ldots, T$, $u = 1, \ldots, k$, and z_{tuv}, $t = 2, \ldots, T$, $u, v = 1, \ldots, k$, given the observed data
  - **M-step**: *maximize* the expected complete data log-likelihood with respect to the model parameters. The estimates at the $(m+1)$-th step are given by

$$\hat{\pi}_u^{(m+1)} = \frac{\hat{w}_{1u}}{\sum_{v=1}^{k} \hat{w}_{1v}}, \quad u = 1, \ldots, k$$

$$\hat{\pi}_{v|u}^{(m+1)} = \frac{\sum_{t \geqslant 2} \hat{z}_{tuv}}{\sum_{t \geqslant 2} \hat{w}_{t-1u}}, \quad u, v = 1, \ldots, k$$

$$\left(\hat{\boldsymbol{R}}_u^{(m+1)}, \hat{\nu}_u^{(m+1)}\right) = \underset{\boldsymbol{R}_u, \nu_u}{\operatorname{argmax}} \sum_{t=1}^{T} \sum_{u=1}^{k} \hat{w}_{tu} \log c(\boldsymbol{e}_t; \boldsymbol{R}_u, \nu_u), \quad u = 1, \ldots, k$$

but this optimization problem is not feasible in higher dimension