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A Liouville-type result for quasi-linear elliptic equations on complete Riemannian manifolds

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Abstract

We extend a Liouville-type result of D. G. Aronson and H. F. Weinberger and E.N. Dancer and Y. Du concerning solutions to the equation $\Delta_p u = b(x) f(u)$ to the case of a class of singular elliptic operators on Riemannian manifolds, which include the φ-Laplacian and are the natural generalization to manifolds of the operators studied by J. Serrin and collaborators in Euclidean setting. In the process, we obtain an a priori lower bound for positive solutions of the equation in consideration, which complements an upper bound previously obtained by the authors in the same context.
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0. Introduction

Let f be a continuous function on $\mathbb{R}$ satisfying the conditions

$$\text{(i) } f(0) = f(a) = 0, \ \text{(ii) } f(s) > 0 \text{ in } (0, a), \ \text{(iii) } f(s) < 0 \text{ in } (a, +\infty). \tag{0.1}$$

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In recent years, the study of the semilinear diffusion equation

$$u_t = \Delta u + f(u) \quad \text{on } [0, +\infty) \times \mathbb{R}^m, \tag{0.2}$$

which arises in population biology and chemical reaction theory, has attracted the attention of many researchers in the field, see [AW1] and [AW2] for references and background.

In [AW2], Aronson and Weinberger showed that if f is C^1 and

$$\liminf_{s\to 0+} \frac{f(s)}{s^{1+2/m}} > 0, \tag{0.3}$$

then a "hair trigger" effect takes place, and any non-identically zero solution $u(x,t)$ of (0.2) with values in $[0,a]$ is such that

$$\lim_{t\to+\infty} u(x,t) = a,$$

uniformly in $x \in \mathbb{R}^m$. Moreover, the exponent $1+2/m$ in (0.3) is sharp in the sense that the hair trigger effect fails if $1+2/m$ in (0.3) is replaced by any larger σ.

As a consequence of the hair trigger effect one deduces a Liouville result for the elliptic problem associated to (0.4), namely, any solution u of

$$\Delta u + f(u) = 0 \tag{0.4}$$

with values in $[0,a]$ is constant and identically equal to either 0 or a.

It should be noted that the assumption $f(s) < 0$ for $s > a$ implies that any non-negative, globally bounded solution u of (0.4) satisfies $0 \leqslant u \leqslant a$, and that if f is superlinear at $+\infty$, then any non-negative solution is in fact globally bounded (see [DM,PRS1]).

As for the sharpness of the exponent $1+2/m$ in (0.3) in order that this kind of Liouville-type result hold, it was shown by Dancer [D], that if $m > 2$ and $\sigma > m/(m-2)$ one can find a function $f \in C^1(\mathbb{R})$ satisfying (0.1) and $f(s) \geqslant cs^\sigma$ for $s \to 0+$, such that (0.4) has a positive solution u with $0 < u < a$ which tends to zero at infinity.

In subsequent work, Du and Guo, [DG], extended the investigation to the case of the p-Laplace operator, and conjectured that if $m > p$ then the sharp exponent should be given by Serrin's exponent $\sigma = m(p-1)/(m-p)$ (which reduces to $\sigma = m/(m-2)$ in the case of the Laplacian).

The conjecture has been recently established by Dancer and Du [DD], using results due to Bidaut-Veron and Pohozaev [BVP], and to Serrin and Zou [SZ]. For the sake of comparison, we report here their result.

Theorem (Dancer and Du [DD]). *Let f be continuous on $[0,+\infty)$ and locally quasi-monotone (in the sense that for any bounded interval $[s_1, s_2]$ contained in $[0,+\infty)$*

there exists a continuous increasing function h such that $f(s)+h(s)$ *is non-decreasing in* $[s_1, s_2]$), *and assume that f satisfy* (0.1) *for some* $a > 0$. *Let* $p > 1$ *and, if* $m \geqslant p$, *assume furthermore that there exist* $\varepsilon > 0$ *and* $c > 0$ *such that*

$$f(s) \geqslant cs^{\xi} \quad \forall s \in (0, \varepsilon), \tag{0.5}$$

where

$$\xi \in (0, +\infty) \text{ if } p = m \text{ and } \xi \in (0, m(p-1)/(m-p)] \quad \text{if } m > p.$$

Let $b(x) \in C^0(\mathbb{R}^m)$ *satisfy* $0 < c_1 \leqslant b(x) \leqslant c_2 < +\infty$ *on* $\mathbb{R}^m$. *Then any solution of*

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + b(x)f(u) = 0 \quad \text{on } \mathbb{R}^m \tag{0.6}$$

satisfying $0 \leqslant u \leqslant a$ *is constant* (*and identically equal to either* 0 *or* a).

As remarked in [DD], the range of values of ξ in (0.5) is sharp. Furthermore, it follows from the condition that $f(s) < 0$ for $s > a$ that any globally bounded non-negative solution u of (0.6) satisfies $0 \leqslant u \leqslant a$, and if f in addition satisfies a condition of the type

$$\liminf_{s \to +\infty} -\frac{f(s)}{s^{\sigma}} > 0$$

for some $\sigma > p-1$, then any non-negative solution of (0.6) is in fact, globally bounded (see [DG]).

The purpose of this paper is to extend this result in various directions. First of all, we consider the equation in the setting of Riemannian manifolds, where the techniques used in the Euclidean setting are no longer applicable. We also consider a class of operators which is substantially more general than the standard p-Laplacian, and variations thereof studied elsewhere (see, e.g. [Ho,HeKM]), where the operators are assumed to satisfy a suitable homogeneity condition. The lack of this homogeneity again requires the introduction of entirely new methods. Finally, we relax the condition on $b(x)$. In this respect, we note that the conditions on $b(x)$ assumed in Dancer and Du's result, essentially amount to the constancy of $b(x)$. On the other hand, we are able to deal with the case where $b(x)$ tends to zero, and we relate its rate of decay to the global geometry of the ambient manifold.

Towards this aim, let $(M, \langle\, , \rangle)$ be a complete Riemannian manifold, and let o be a fixed reference point on M. We denote by $r(x)$ the Riemannian distance from x to o, and by B_r the geodesic ball of radius r centered at o.

Let $\varphi \in C^1((0, +\infty)) \cap C^0([0, +\infty))$ satisfy the following structural conditions:

$$\text{(i) } \varphi(0) = 0; \quad \text{(ii) } \varphi(t) > 0 \quad \forall t > 0; \quad \text{(iii) } \varphi(t) \leqslant At^{\delta} \quad \forall t \geqslant 0 \tag{0.7}$$

for some positive constants A and δ. For $u \in C^1(M)$, we consider the differential operator defined (in the appropriate weak sense) by

$$\mathcal{L}_\varphi u = \operatorname{div}\left(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u\right)$$

and which, from now on, we will refer to as the φ-Laplacian.

Different choices of φ lead to well-known operators such as

- the Laplace–Beltrami operator, corresponding to $\varphi(t) = t$;
- the p-Laplacian, $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ corresponding to $\varphi(t) = t^{p-1}$, $p > 1$;
- the mean curvature operator $\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$, corresponding to $\varphi(t) = t(1+t^2)^{-1/2}$,

and so on.

The extension of Dancer and Du result mentioned above is the following:

Theorem A. *Let φ be a function satisfying the conditions listed in* (0.7) (i)–(iii), *and*

$$\text{(iv) } \varphi'(t) > 0 \quad \forall t > 0. \tag{0.7}$$

Let $f \in C^0([0, +\infty))$ satisfy (0.1) *for some $a > 0$, and*

$$\liminf_{s \to +\infty} -\frac{f(s)}{s^\sigma} > 0 \tag{0.8}$$

for some $\sigma > \max\{1, \delta\}$; let also $b(x) \in C^0(M)$, and suppose that

$$b(x) \geqslant \frac{C}{(1 + r(x))^\mu} \quad \text{on } M \tag{0.9}$$

for some $C > 0$ and $0 \leqslant \mu < 1 + \delta$. Let u be a non-negative solution of

$$\mathcal{L}_\varphi u = -b(x) f(u) \quad \text{on } M. \tag{0.10}$$

Assume that

$$\liminf_{r \to +\infty} \frac{\log \operatorname{vol} B_r}{r^{1+\delta-\mu}} < +\infty \tag{0.11}$$

and, if

$$\left(\operatorname{vol}(\partial B_r)\right)^{-1/\delta} \in L^1(+\infty), \tag{0.12}$$

assume furthermore that

$$f(t) \geqslant ct^{\xi} \quad 0 < t \ll 1 \tag{0.13}$$

for some $\xi > 0$ *and* $c > 0$. *Finally, if*

$$\xi \geqslant \delta, \tag{0.14}$$

suppose also that

$$u(x) \geqslant Cr(x)^{-\theta}, \quad r(x) \gg 1 \tag{0.15}$$

for some $\theta \geqslant 0$, $C > 0$, *and that*

$$\liminf_{r\to+\infty} \frac{\log \operatorname{vol} B_r}{r^{1+\delta-\theta(\xi-\delta+\varepsilon)-\mu}} < +\infty \tag{0.16}$$

for some $\varepsilon > 0$. *Then* u *is constant and identically equal to* 0 *or* a.

Remark 0.1. With respect to assumptions (0.11) and (0.12), it is quite easy to see that the former may hold independently of the validity of the latter. Rather more elaborate arguments allows to construct models manifolds such that

$$(\operatorname{vol} \partial B_r)^{-1/\delta} \notin L^1(+\infty), \tag{0.17}$$

and yet $\operatorname{vol} B_r$ grows arbitrarily fast. In particular, (0.17) does not imply that (0.11) or (0.16), hold.

As for condition (0.15), it seems to have no counterpart in [DD]. We will show at the end of Section 1 below, that it is automatically satisfied in the situation considered in [DD], but becomes necessary in our more general setting.

We point out that our methods allow us to obtain the following L^q version of (the first part of) Theorem A. The second part can be generalized in a similar way.

Theorem A′. *Let* $f \in C^0([0, +\infty)$ *and* $b(x)$ *satisfy the conditions listed in the statement of Theorem* A, *and let* u *be a non-negative solution of* (0.10). *Suppose that*

$$\liminf_{r\to+\infty} \frac{\log \int_{B_r} u^q}{r^{1+\delta-\mu}} < +\infty \tag{0.18}$$

for some $q > 0$, *and if* (0.12) *holds, and assume furthermore that* (0.13) *holds for some* ξ *with* $0 < \xi < \delta$. *Then either* $u \equiv 0$ *or* a.

In fact our methods allow to obtain a version of Theorem A for the more general class of divergence form operators defined as follows: let h be a symmetric tensor field defined on M, that is a section of S^2T^*M, and let φ be a real-valued function with the properties described in (0.7) (i)–(iii). For $u \in C^1(M)$ we consider the differential operator defined (in weak sense) by

$$\mathcal{L}_{\varphi,h}u = \operatorname{div}\left(|\nabla u|^{-1}\varphi(|\nabla u|)h(\nabla u, \cdot)^\sharp\right), \tag{0.19}$$

where $^\sharp : T^*M \to TM$ denotes the musical isomorphism, so that $h(|\nabla u|, \cdot)^\sharp$ is the vector field on M defined by

$$\langle h(\nabla u, \cdot)^\sharp, X\rangle = h(\nabla u, X) \quad \forall X \in T_xM.$$

Note that the φ-Laplacian is obtained by choosing as h the metric of M. The above operators may be viewed as the natural, intrinsic generalization to Riemannian manifolds of the fully quasi-linear singular elliptic operators considered by Pucci Serrin and Zou (see [PuSZ,PuS]). They also generalize the $\mathcal{A}$-Laplace operators as defined in [HeKM] in the setting of nonlinear potential theory. For the latter class of operators we refer to work by Holopainen [Ho], who obtains interesting Liouville-type results. From a somewhat different point of view, see also the recent paper by Coulhon, Holopainen and Saloff-Coste [CHSC].

The paper is organized as follows: in Section 1 we will outline a proof of Theorem A, describe some consequences and examples, and, in particular, show how Dancer and Du's result compares with Theorem A. This depends on an a priori estimate for positive solutions of (0.10) under curvature assumption, Proposition B, which we prove at the end of the Section. It should be stressed that in order that the results of Proposition B be applicable in Theorem A, the bounds obtained must be polynomial in $r(x)$. It turns out that this is, in some sense, a Euclidean phenomenon. In most of the genuinely non-Euclidean settings, the bounds are in fact exponential, and the conclusion of the second part of Theorem A fails. In this sense, the geometrically significant part of the Theorem A is that up to (0.13) inclusive. The second part of the Theorem, as well as the full strength of Dancer and Du's result, depends upon very specific properties of $\mathbb{R}^m$.

In Section 2, we consider the more general operators defined in (0.19), and we will describe how to extend to such a class of operators some of the results obtained in [RS,PRS1]. In particular, we will describe conditions ensuring that the analog of the φ-parabolicity holds on M, and that solutions of differential inequalities of the form

$$\mathcal{L}_{\varphi,h}u \geqslant b(x)g(u)$$

are necessarily bounded above. We will then state a version of Theorem A valid in this context.

1. Outline of the proof of Theorem A and Dancer and Du's result

To better appreciate the content of Theorem A and compare it with Dancer and Du's result, we begin by illustrating an outline of the proof, referring to Section 2 below for the complete proofs of many of the facts we are going to use. We set $u^* = \sup_M u$ and $u_* = \inf_M u$.

Step 1. The assumptions $-f(s) > Cs^\sigma$ for $s \gg 1$, with $\sigma > \max\{1, \delta\}$, $b(x) \geqslant C(1+r(x))^{-\mu}$, $0 \leqslant \mu < 1+\delta$, and the volume growth condition (0.11), $\liminf_{r\to+\infty} \log \mathrm{vol}\, B_r/r^{1+\delta-\mu} < +\infty$, imply that $u^* < +\infty$ (see [PRS1, Theorem B, Remark 1.6b], and Theorem 2.3 below). Note that the same conclusion holds if we assume that condition (0.11) is replaced by condition (0.18) in the statement of Theorem A′ (see [PRS1, Remark 1.6e]).

Step 2. Since u is bounded above, and (0.11) holds, Theorem A in [PRS1] (see also Theorem 2.1 below) implies that $-f(u^*) \leqslant 0$, so that, by (0.1), $u^* \in [0, a]$, and $0 \leqslant u \leqslant a$ on M. It follows that:

$$\mathcal{L}_\varphi u \leqslant 0 \quad \text{on } M. \tag{1.1}$$

Again, the same conclusion holds if we assume condition (0.18) instead of (0.11).

Step 3. If

$$\left(\mathrm{vol}\, \partial B_r\right)^{-1/\delta} \notin L^1(+\infty),$$

then, by Theorem A in [RS], $(M, \langle\, , \rangle)$ is φ-parabolic, and therefore $u \geqslant 0$ together with (1.1) imply that u is constant. Since $b(x)$ is positive on M and f vanishes only in 0 and a, it follows form (0.10) that either $u \equiv 0$ or a.

Step 4. If

$$\left(\mathrm{vol}\, \partial B_r\right)^{-1/\delta} \in L^1(+\infty),$$

then $(M, \langle\, , \rangle)$ is not necessarily φ-parabolic, and further analysis is required. First, we note that $0 \leqslant u_* \leqslant a$, and that, applying Theorem A′ in [PRS1] (see also Remark 2.2 after the proof of Theorem 2.1), we have $f(u_*) \leqslant 0$. Thus, u_* is either 0 or a. In the latter case we have $u_* = u^* = a$, so that $u \equiv a$; if $u_* = u^* = 0$, again u is constant. Thus, the only case to consider is $u_* = 0$ and $0 < u^* \leqslant a$. To show that this cannot occur, it suffices to prove that under appropriate assumptions $u_* > 0$.

Now, since u satisfies (1.1) and it does not vanish identically, by the strong minimum principle, [PuSZ,PuS], u is strictly positive on M. If (0.13) holds with $\xi < \delta$, then (0.11) and Theorem 2.5 below, imply that $u_* > 0$, and we are done. If (0.13) holds and $\xi \geqslant \delta$, then we observe that u is a solution of

$$\mathcal{L}_\varphi u = -\tilde{b}(x)\tilde{f}(u)$$

with

$$\tilde{f}(u) = f(u)u^{-(\xi-\delta+\varepsilon)} \quad \text{and} \quad \tilde{b}(x) = b(x)u^{\xi-\delta+\varepsilon}.$$

According to (0.8) and (0.15) we have

$$\tilde{b}(x) \geqslant C(1+r(x))^{-\mu-\theta(\xi-\delta+\varepsilon)}$$

and the required conclusion follows from (0.16) and a further application of Theorem 2.5.

To compare the conclusion of Theorem A with Dancer and Du's result, we need to obtain a priori lower bounds for non-negative solutions of (0.10). As mentioned in the introduction, these can be obtained by a comparison argument under curvature conditions.

We begin with a lemma, whose content is to describe some properties of solutions of a suitably radialized version of the inequality $\mathcal{L}_\varphi v \geqslant 0$.

Lemma 1.1. *Let $\varphi \in C^0([0,+\infty)) \cap C^1((0,+\infty))$, satisfy conditions (0.7) (i) and (ii), and assume that φ is strictly increasing on $[0,\varepsilon)$ and that*

$$\varphi(t) \sim C_0 t^\zeta \quad \textit{as } t \to 0+ \tag{1.2}$$

for some C_0, $\zeta > 0$. Let $g \in C^1([0,+\infty))$ be such that $g(0)=0$, $g(t)>0$, if $t>0$, $g'(t)>0$ for $t \gg 1$, and suppose that, for some $m>1$,

$$g(t)^{-(m-1)/\zeta} \in L^1(+\infty). \tag{1.3}$$

Fix $H>0$ and $R>0$; then there exists $B>0$ such that, having denoted with $\psi : [0,\varphi(\varepsilon)) \to [0,\varepsilon)$ the local inverse of φ, the function α defined by

$$\alpha(r) = \int_r^{+\infty} \psi\left(Bg(t)^{1-m}\right) dt, \tag{1.4}$$

is defined and C^2 on $[R,+\infty)$ and satisfies

$$\begin{cases} \varphi(|\alpha'|)' + (m-1)\frac{g'}{g}\varphi(|\alpha'|) = 0, \\ \alpha(r) \leqslant \alpha(R) = D < H, \ \alpha'(r) < 0 \quad \forall r \geqslant R. \end{cases} \tag{1.5}$$

Furthermore,

$$\alpha(r) \sim (B/C_0)^{1/\zeta} \int_r^{+\infty} g(t)^{-(m-1)/\zeta}\, dt \quad \textit{as } r \to +\infty. \tag{1.6}$$

In particular, if $\limsup_{r\to+\infty} g'/g < +\infty$, *then there exists* $C > 0$ *such that*

$$\alpha(r) \geqslant C g(r)^{-(m-1)/\zeta} \quad \textit{for } r \geqslant R \tag{1.7}$$

and if g'/g *is eventually decreasing,*

$$\alpha(r) \geqslant C \frac{g(r)}{g'(r)} g(r)^{-(m-1)/\zeta} \quad \textit{for } r \geqslant R. \tag{1.8}$$

Proof. Note that since g is eventually increasing and $g(t)^{-(m-1)/\zeta}$ is integrable at infinity, $g(t) \to +\infty$ as $t \to +\infty$. In particular, if B is sufficiently small, $Bg(t)^{-(m-1)} < \varphi(\varepsilon)$ for every $t \geqslant R$. Furthermore, it follows from (1.2) that $\psi(s) \sim (s/C_0)^{1/\zeta}$, so that

$$\psi\left(Bg(t)^{-(m-1)}\right) \sim (B/C_0)^{1/\zeta} g(t)^{-(m-1)/\zeta} \tag{1.9}$$

and the integral in (1.4) is finite for every $r \geqslant R$. It is clear that α is C^2, decreasing, and that, by choosing a smaller B if necessary, it can be arranged that $\alpha(r) < H$ on $[R, +\infty)$. A computation shows that α satisfies (1.5). It follows from (1.9) that α satisfies (1.6). Finally, if $g'/g \leqslant \eta$ for $t \geqslant R$, the integrand in (1.6) is bounded from below by

$$\frac{1}{\eta} g(t)^{-(m-1)/\zeta - 1} g'(t),$$

and (1.7) follows integrating, and recalling that $g(t) \to +\infty$ as $t \to +\infty$. A similar argument shows that if g/g' is eventually decreasing then (1.8) holds. $\square$

Proposition B. *Let* φ *and* g *satisfy the conditions listed in the statement of Lemma* 1.1, *and assume that, having denoted by* $r(x)$ *the distance function from* $o \in M$, *we have*

$$\Delta r \leqslant (m-1) \frac{g'}{g}(r(x)) \tag{1.10}$$

pointwise in the complement of the cut locus of o. Let u be a non-negative C^1 *solution of*

$$\mathcal{L}_\varphi u \leqslant 0. \tag{1.11}$$

Then there exist constants C and $R > 0$ such that

$$u(x) \geqslant C \int_{r(x)}^{+\infty} g(t)^{-(m-1)/\zeta} \, dt \quad \text{on } M \setminus B_R. \tag{1.12}$$

Furthermore, if $\limsup_{r \to +\infty} g'/g < +\infty$, *then there exists* $C > 0$ *such that*

$$u(x) \geqslant C g(r(x))^{-(m-1)/\zeta} \quad \text{if } x \in M \setminus B_R, \tag{1.13}$$

and if g'/g is eventually decreasing,

$$u(x) \geqslant C \frac{g(r(x))}{g'(r(x))} g(r(x))^{-(m-1)/\zeta} \quad \text{if } x \in M \setminus B_R. \tag{1.14}$$

Proof. Fix $R > 0$ such that $g'(t) > 0$ for $t > R$, choose B small enough that the function α defined in (1.4) satisfies the conditions in the statement of Lemma 1.1 with $H = \inf_{\partial B_R} u$, and set $v(x) = \alpha(r(x))$. It follows from (1.5) and (1.10), that the inequality

$$\begin{aligned} \mathcal{L}_\varphi v &= -\varphi\left(|\alpha'|\right)' - \varphi\left(|\alpha'|\right)\Delta r \\ &\geqslant -\varphi\left(|\alpha'|\right)' - (m-1)\frac{g'}{g}\varphi\left(|\alpha'|\right) = 0 \end{aligned} \tag{1.15}$$

holds pointwise in the complement of the cut locus of o, and, by adapting an argument of Yau [Y], weakly on M. Thus

$$\begin{aligned} \mathcal{L}_\varphi v &\geqslant \mathcal{L}_\varphi u \text{on } M \setminus B_R \\ v &< u \text{on } \partial B_R. \end{aligned} \tag{1.16}$$

We claim that $u \geqslant v$ on $M \setminus B_R$. Indeed, if this were not the case, there would exist $\eta > 0$ and $x_0 \in M \setminus \overline{B_R}$ such that $u(x_0) < v(x_0) - \eta$. Thus the set

$$A_\eta = \{x \in M \setminus B_R \,:\, u(x) < v(x) - \eta\}$$

would be open, non-empty, and $x_0 \in A_\eta \subseteq \overline{A}_\eta \subseteq M \setminus \overline{B}_R$. Moreover, since $v(x) \to 0$ as $r(x) \to +\infty$, while u is positive on M, $\overline{A}_\eta$ is bounded, and since M is complete, compact. Since $u = v - \eta$ on ∂A_η, by the weak comparison principle (see e.g., [PuSZ, Lemma 2] or [RS, Proposition 2.5]) $u \geqslant v - \eta$ on A_η, and therefore $u(x_0) \geqslant v(x_0) - \eta$, contradicting the definition of η and x_0.

Now the required lower estimates follows from Lemma 1.1. □

As mentioned above, the upper estimate (1.10) in the statement of the Proposition can be deduced from suitable curvature bounds. The following corollary illustrates a typical result.

Corollary 1.2. *Let $(M, \langle , \rangle)$ be an m-dimensional complete Riemannian manifold, let $o \in M$ be a fixed reference point in M, and let $r(x)$ be the distance function from o. Assume that the radial Ricci curvature of M satisfy*

$$\mathrm{Ricc}_{(M,\langle , \rangle)}(\nabla r, \nabla r) \geqslant -(m-1)G(r), \tag{1.17}$$

for some positive function $G \in C^1([0, +\infty)$ such that

(i) $\inf_{r>0} \frac{G'}{G^{3/2}} > -\infty$,

(ii) $\limsup_{r \to +\infty} G(r) < +\infty$,

(iii) $G(r)^{1/2} \notin L^1(+\infty)$,

(vi) $\exp\left(-\zeta^{-1}(m-1)D_0 \int_0^r G(s)^{1/2}\, ds\right) \in L^1(+\infty)$ (1.18)

for some $D_0 > 0$. Let φ be as in the statement of the proposition, and let u be a non-negative, non-identically zero solution of

$$\mathcal{L}_\varphi u \leqslant 0 \quad \text{on } M.$$

Then there exist constants $C > 0$ and $D \geqslant D_0$ such that

$$u(x) \geqslant C \exp\left(-\zeta^{-1}(m-1)D \int_0^{r(x)} G(s)^{1/2}\, ds\right). \tag{1.19}$$

If G is assumed to be non-increasing then

$$u(x) \geqslant C G(r(x))^{-1/2} \exp\left(-\zeta^{-1}(m-1)D \int_0^{r(x)} G(s)^{1/2}\, ds\right). \tag{1.20}$$

Proof. Set

$$g(r) = \frac{1}{DG(0)^{1/2}} \left[\exp\left(D \int_0^r G(s)^{1/2}\, ds\right) - 1\right].$$

It follows from the Laplacian comparison theorem (see [RRS, Lemma 2.4], or [PRS2, Lemma 2.1] for a more analytically flavored approach) that if D is sufficiently large, then

$$\Delta r \leqslant (m-1)\frac{g'(r)}{g(r)}$$

pointwise in the complement of the cut locus of o and weakly on M. Note that since $g(r) \to +\infty$ as $r \to +\infty$, by (1.18) (iii), we have

$$\frac{g'(r)}{g(r)} \sim DG(r)^{1/2} \quad \text{as } r \to +\infty.$$

We choose $D \geqslant D_0$, so that, by (1.18) (iv), condition (1.3) in Lemma 1.1 holds, and applying Proposition B we deduce that, for some $H > 0$,

$$u(x) \geqslant Hg(r(x))^{-\frac{m-1}{\zeta}} \geqslant C \exp\left(-(m-1)\frac{D}{\zeta}\int_0^{r(x)} G(s)^{1/2}\,ds\right),$$

which can be improved to

$$u(x) \geqslant CG(r(x))^{-1/2} \exp\left(-(m-1)\frac{D}{\zeta}\int_0^{r(x)} G(s)^{1/2}\,ds\right),$$

if G is non-increasing. $\square$

To illustrate the kind of lower bounds that can be obtained applying Corollary 1.2, assume that (1.17) holds with

$$G(r) = \frac{B^2}{1+r^2},$$

which corresponds to a geometric behavior borderline between the Euclidean and non-Euclidean case. Indeed, a manifold with a pole, whose radial Ricci curvature is non-positive, and tends to zero faster than $(1+r(x)^2)^{-1}$ is quasi-isometric to Euclidean space (see, e.g. [GW]).

It can be shown that, if g is defined as in the proof of Corollary 1.2, then the inequality

$$\Delta r \leqslant (m-1)\frac{g'}{g}$$

holds if

$$D \geqslant B' = \frac{1+\sqrt{1+4B^2}}{2}$$

and, in this case,

$$g(r) \asymp r^D \quad \text{as,} \quad r \to +\infty$$

(see, [RRS, Lemma 2.4] or [BRS, Lemma 5.1]). Thus, $g(r)^{-(m-1)/\zeta} \asymp r^{-(m-1)D/\zeta} \in L^1(+\infty)$ provided $D > \zeta/(m-1)$, and then non-negative solutions of

$$\mathcal{L}_\varphi u \leqslant 0$$

satisfy the bound

$$u(x) \geqslant C r(x)^{1-D(m-1)/\zeta}.$$

In particular, in the case of the p-Laplacian, for which $\zeta = p-1$, if $B' > (p-1)/(m-1)$ then

$$u(x) \geqslant C r(x)^{1-B'(m-1)/(p-1)} \quad \text{if } B' > (p-1)/(m-1) \tag{1.21}$$

while, if $B' \leqslant (p-1)/(m-1)$, then, for every $\eta > 0$ there exists $C = C(\eta) > 0$ such that

$$u(x) \geqslant C r(x)^{-\eta}. \tag{1.22}$$

Similarly, if $(M, \langle , \rangle) = (\mathbb{R}^m, \text{can})$ we have $\Delta r = (m-1)/r$, so that the inequality $\Delta r \leqslant (m-1)g'/g$ holds if $g(r) = r^D$ with $D \geqslant 1$, and we deduce that non-negative solutions of

$$\Delta_p u \leqslant 0$$

satisfy the bound

$$u(x) \geqslant C r^{1-(m-1)/(p-1)} \quad \text{if } m > p \tag{1.23}$$

while, if $m \leqslant p$, for every $\eta > 0$ there exists $C = C(\eta) > 0$ such that

$$u(x) \geqslant C r(x)^{-\eta}. \tag{1.24}$$

We note in passing that these estimates agree with the bounds (1.21) and (1.22) letting $B \to 0$ and therefore $B' \to 1$.

Inserting (1.23) in the statement of Theorem A, with $\delta = p - 1$ and $\mu = 0$, we see that condition (0.16) becomes

$$\liminf \frac{m \log r}{r^{p - \frac{m-p}{p-1}(\xi - p + 1 + \varepsilon)}} < +\infty$$

for some $\varepsilon > 0$. It follows that in this case Theorem A is applicable provided

$$0 < \xi < \frac{m(p-1)}{m-p},$$

which should be compared with the range

$$0 < \xi \leqslant \frac{m(p-1)}{m-p},$$

obtained by Dancer and Du.

On the other hand, if $m = p$, using (1.24) we see that condition (0.16) holds provided $p - \eta(\xi - p + 1 + \varepsilon) > 0$ for some $\eta, \varepsilon > 0$, and this clearly holds for every $\xi > 0$.

We remark that if $p = m$, then $\mathbb{R}^m$ is p-parabolic, and in this case the conclusion of Theorem A actually holds without having to assume any further condition like (0.13) on the behavior of f near 0.

As noted in the Introduction, while Proposition B is applicable under fairly weak geometric assumptions, the bounds it provides can be used in Theorem A only if they are polynomial in $r(x)$, and this imposes rather strict restrictions on the geometry of the manifold.

Indeed, if we assume that the Ricci curvature satisfies (1.17) with $G(r) = (1+r^2)^{-\mu/2}$ with $0 \leqslant \mu < 2$ then the lower bound given by Corollary 1.2 is no longer polynomial in $r(x)$, and it turns out that the conclusion of Theorem A fails.

To see this, let $M = \mathbb{R}^m$ and let ds^2 be the metric on $\mathbb{R}^m \setminus \{0\}$ given, in polar coordinates, by the formula $ds^2 = dr^2 + g(r)^2 d\theta^2$, where g solves the differential equation

$$\begin{cases} g'' = B^2(1+r^2)^{-\mu/2} g, \\ g(0) = 0, \, g'(0) = 1. \end{cases} \tag{1.25}$$

Then g is smooth and even at the origin, and therefore ds^2 extends to a metric on M^m with radial sectional curvature given by $K(x) = B^2(1 + r(x)^2)^{-\mu/2}$.

We claim that if $0 \leqslant \mu < 2$, and $a(r) = \frac{m-1}{2} K(r)$, then the differential equation

$$\Delta u + a(r) u = 0 \tag{1.26}$$

has a positive radial solution on M^m, with values in $(0,1]$, and with the further property that

$$C^{-1}r^{1-(m-1)\mu/8}\exp\left[-\frac{m-1}{2-\mu}Br^{1-\mu/2}\right] \leqslant u(x) \leqslant Cr\exp\left[-\frac{m-1}{2-\mu}Br^{1-\mu/2}\right] \tag{1.27}$$

for $r \gg 1$, for some $C > 0$. Thus, if $f \in C^0(\mathbb{R})$ is such that $f(t) > 0$ in $(0,2)$, $f(t) = t$ if $t \in [0,1]$, $f(t) < 0$ in $(2,+\infty)$, $f(t) = -t^\sigma$ if $t \in [4,+\infty)$, for some $\sigma > 1$, then u is a positive, bounded, non-constant solution of

$$\Delta u + a(r)f(u) = 0.$$

Note that, according to [BRS] Proposition 5.1, we have

$$g(r) \leqslant Cr^{\mu/4}\exp\left[\frac{2B}{2-\mu}r^{1-\mu/2}\right] \tag{1.28}$$

for $r \gg 1$, so that

$$\log\operatorname{vol} B_r \leqslant Cr^{1-\mu/2} \quad \text{for } r \gg 1$$

and therefore

$$\frac{\log\operatorname{vol} B_r}{r^{2-\mu}} \leqslant Cr^{-1+\mu/2} \to 0 \quad \text{as } r \to +\infty,$$

and (0.11) holds. Also (0.12) holds with $\alpha = 1 = \delta$. On the other hand, (0.14) does not hold for any θ and consequently (0.15) does not make sense.

To prove the claim one proceeds as [BR]. Let β be a solution of the differential initial value problem

$$\begin{cases} \beta'' + (m-1)\frac{g'(r)}{g(r)}\beta' + a(r)\beta = 0, \\ \beta(0) = 1, \ \ \beta'(0) = 0. \end{cases} \tag{1.29}$$

Then β is defined on $[0,+\infty)$ and $u(x) = \beta(r(x))$ is a radial solution of (1.26) on M. To show that β is positive on $(0,+\infty)$ and has the required asymptotic behavior, we note that for every $s > 1$ the function $\phi_s(t)$ defined by

$$\phi_s(t) = (t-s)\,[g(s)g(t)]^{-(m-1)/2}$$

is a subsolution of (1.29) on $(0, s]$, and satisfies the conditions listed in [BR] Lemma 1.2. The lemma implies that β is everywhere positive, and in fact there exists $C_1 > 0$, such that, for every $0 < \delta \leqslant \frac{1}{2}$, $1 \leqslant s$,

$$\beta(s) \geqslant C_1 \phi_s'(\delta)\delta^{m-1}.$$

Since

$$\phi_s'(\delta) = -\left[\frac{1}{s-\delta} + \frac{m-1}{2}\frac{g'(\delta)}{g(\delta)}\right]\phi_s(\delta),$$

the left-hand side of (1.27) follows from the definition of ϕ_s, the upper bound (1.28) for g, and the asymptotic relations $g(\delta) \sim \delta$, $g'(\delta) \sim 1$ as $\delta \to 0$.

As for the upper estimate, one observes that the function defined by

$$v(r) = (r-b)g(r)^{-(m-1)/2}$$

is a positive radial subsolution of (1.29) defined on $(b, +\infty)$, such that $v'(b_1) > 0$ if $b_1 > b$ is sufficiently close to b. According to Lemma 1.3 in [BR], there exists $C > 0$ such that

$$\beta \leqslant Cv \quad \text{on } [b_1, +\infty).$$

The right-hand side of (1.27) now follows from this, and the lower bound

$$g(r) \geqslant C \exp\left[-\frac{2B}{2-\mu} r^{1-\mu/2}\right]$$

obtained in [BRS], Proposition 5.2.

2. More general operators

In this section we turn our attention to the more general operators $\mathcal{L}_{\varphi,h}$ defined in the introduction. We will prove versions of our results for this class of operators, and deduce the results stated for the φ-Laplacian as special cases.

We begin by introducing some terminology. Let h be the symmetric tensor field which enters in the definition (0.19) of the operator $\mathcal{L}_{\varphi,h}$. We will assume throughout that h satisfies the following bounds

$$h_-(r) \leqslant h(X, X) \leqslant h_+(r) \quad \forall X \in T_x M, |X| = 1, x \in \partial B_r \tag{2.1}$$

for some positive continuous functions $h_\pm$ defined on $[0, +\infty)$, and define

$$h_\delta(r) = \begin{cases} h_+(s) & \text{if } \delta \leqslant 1, \\ h_-(s)^{(1-\delta)/2} h_+(s)^{(1+\delta)/2} & \text{if } \delta > 1, \end{cases} \tag{2.2}$$

and

$$H(r) = \sup_{s \leqslant r} h_\delta(s). \tag{2.3}$$

Now we are ready to state our first result, which extends to the operator $\mathcal{L}_{\varphi,h}$ Theorem A in [PRS1], valid for the φ-Laplacian.

Theorem 2.1. *Let $(M, \langle , \rangle)$ be a complete Riemannian manifold, let o be a reference point in M, and $r(x)$ be the distance function from o. Let φ satisfy the structural conditions* (0.7) (i)–(iii), *let h be a symmetric covariant two tensor field such that $h_-(r) > 0$ for every $r > 0$, and let H be defined in* (2.2). *Suppose that $b(x)$ is a continuous function on M satisfying*

$$b(x) \geqslant \frac{1}{Q(r(x))} \quad \text{on } M, \tag{2.4}$$

where Q is a positive, continuous, non-increasing function.

Given $f \in C^0(\mathbb{R})$, assume that $u \in C^1(M)$ satisfies $u^* = \sup_M u < +\infty$ and

$$\mathcal{L}_{\varphi,h} u \geqslant b(x) f(u) \tag{2.5}$$

on the set

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\} \tag{2.6}$$

for some $\gamma < u^*$. If

$$\lim_{r \to +\infty} \frac{H(r)Q(r)}{r^{1+\delta}} = 0 \tag{2.7}$$

and, either

$$\liminf_{r \to +\infty} \frac{Q(r)H(r) \log \operatorname{vol} \mathrm{B_r}}{r^{1+\delta}} < +\infty \tag{2.8}$$

or

$$\liminf_{r \to +\infty} \frac{Q(r)H(r)}{r^{1+\delta}} \int_{B_r} |u|^p < +\infty \quad \text{for some } p > 0, \tag{2.9}$$

then $f(u^*) \leqslant 0$.

Proof. The proof is a modification of that of Theorem A in [PRS1]. First of all we note that if (2.5) holds on Ω_γ, then it holds on $\Omega_{\gamma'}$ for every $\gamma \leqslant \gamma' < u^*$.

Next, we assume by contradiction that $f(u^*) > 0$. By increasing γ if necessary, we may suppose that $f(u) \geqslant C > 0$ in Ω_γ, and that u satisfies

$$\mathcal{L}_{\varphi,h} u \geqslant \frac{B}{Q(r(x))} \quad \text{on } \Omega_\gamma.$$

for some $B > 0$. Fix $0 < \eta < 1$. By choosing γ sufficiently close to u^* we may suppose that $\Gamma := \gamma - u^* + \eta \geqslant \eta/2 > 0$, so that, having defined $v = u - u^* + \eta$, we have $\sup v = \eta$, $\Omega^v_\Gamma = \Omega^u_\gamma$ and

$$\mathcal{L}_{\varphi,h} v \geqslant \frac{B}{Q(r(x))} \quad \text{on } \Omega^v_\Gamma. \tag{2.10}$$

Pick $R > 0$ large enough that $B_R \cap \Omega^v_\Gamma \neq \emptyset$, fix $\zeta > 1$ to be determined later, and let $\psi : M \to [0, 1]$ be a smooth cutoff function such that

$$\text{(i) } \psi \equiv 1 \text{ on } B_r; \quad \text{(ii) } \psi \equiv 0 \text{ on } M \setminus B_{2r}; \quad \text{(iii) } |\nabla\psi| \leqslant \frac{C_0}{r}\psi^{1/\zeta}, \tag{2.11}$$

for some constant $C_0 = C_0(\zeta) > 0$. Note that this is possible since $\zeta > 1$. Next, let $\lambda : \mathbb{R} \to [0, +\infty)$ be a C^1 function such that

$$\lambda(t) = 0 \quad \text{for } t \leqslant \Gamma, \quad \lambda'(t) \geqslant 0 \quad \forall t, \tag{2.12}$$

fix $\alpha > 2$ to be determined later, and consider the vector field W which is defined by

$$W = \psi^{2\alpha}\lambda(v)v^{\alpha-1}|\nabla v|^{-1}\varphi(|\nabla v|)h(\nabla v, \cdot)^\sharp$$

on Ω^v_Γ and vanishes elsewhere. Note that, in fact, W is zero off $B_{2r} \cap \Omega^v_\Gamma$.

Setting for ease of notation $h_v = h(\nabla v, \nabla v)/|\nabla v|^2$, a computation that uses (2.10), $h_v > 0$, $\lambda' \geqslant 0$, $|h(\nabla v, \nabla\psi)| \leqslant h_v^{1/2}h_+^{1/2}|\nabla v||\nabla\psi|$ and $|\nabla v|\varphi(|\nabla v|) \geqslant A^{-1/\delta}\varphi(|\nabla v|)^{1+1/\delta}$ yields

$$\operatorname{div} W \geqslant \psi^{2\alpha}\lambda(v)v^{\alpha-1}\frac{B}{Q(r(x))} + \frac{\alpha-1}{A^{1/\delta}}\psi^{2\alpha}\lambda(v)v^{\alpha-2}\varphi(|\nabla v|)^{1+1/\delta}h_v$$
$$-2\alpha\psi^{2\alpha-1}\lambda(v)v^{\alpha-1}\varphi(|\nabla v|)|\nabla\psi|h_v^{1/2}h_+^{1/2}.$$

Since W is compactly supported, integrating, and applying the divergence theorem, we obtain

$$\int \psi^{2\alpha}\lambda(v)v^{\alpha-1}Q(r(x))^{-1} \leqslant -\frac{\alpha-1}{A^{1/\delta}}\int \psi^{2\alpha}\lambda(v)v^{\alpha-2}\varphi(|\nabla v|)^{1+1/\delta}h_v + 2\alpha\int \psi^{2\alpha-1}\lambda(v)v^{\alpha-1}\varphi(|\nabla v|)|\nabla\psi|h_v^{1/2}h_+^{1/2}. \tag{2.13}$$

We apply to the second integral on the right-hand side the inequality $ab \leqslant \sigma^p a^p/p + b^q/(q\sigma^q)$, with $p = 1+\delta$, $q = (1+\delta)/\delta$, and with $\sigma > 0$ chosen in such a way that the first integral cancels out, and obtain, for $\alpha > (1+\delta)/2$

$$\int \psi^{2\alpha}\lambda(v)v^{\alpha-1}Q(r(x))^{-1} \leqslant \frac{2^{\delta+1}\delta^\delta A}{(1+\delta)^{1+\delta}}\frac{\alpha^\delta}{(\alpha-1)^\delta}\alpha \times\int \psi^{2\alpha-(1+\delta)}|\nabla\psi|^{1+\delta}\lambda(v)v^{\alpha-1+\delta} \times h_v^{(1-\delta)/2}h_+^{(1+\delta)/2}. \tag{2.14}$$

Now, since ψ is supported on B_{2r}, and Q is non-decreasing, $Q(r(x)) \leqslant Q(2r)$ on the support of ψ, and the left-hand side of (2.14) is bounded from below by

$$Q(2r)^{-1}\int \psi^{2\alpha}v^{\alpha-1}\lambda(v). \tag{2.15}$$

On the other hand, since $[\alpha/(\alpha-1)]^\delta \leqslant 2^\delta$ for $\alpha \geqslant 2$, the constant on the right-hand side of (2.14) is estimated by $C(A,\delta)\alpha$ with $C(A,\delta)$ independent of $\alpha \geqslant 2$. Further, using (2.11) (iii), we may write

$$\psi^{2\alpha-(1+\delta)}|\nabla\psi|^{1+\delta} = \psi^{2\alpha-(1+\delta)(1-1/\zeta)}(\psi^{-1/\zeta}|\nabla\psi|)^{1+\delta} \leqslant \psi^{2\alpha-(1+\delta)(1-1/\zeta)}\frac{C_0}{r^{1+\delta}}.$$

Finally, recalling that $h_- \leqslant h_v \leqslant h_+$, we see that

$$h_v^{(1-\delta)/2} \leqslant \begin{cases} h_+^{(1-\delta)/2} & \text{if } \delta \leqslant 1, \\ h_-^{(1-\delta)/2} & \text{if } \delta \geqslant 1 \end{cases}$$

and therefore

$$h_v^{(1-\delta)/2}h_+^{(1+\delta)/2} \leqslant h_\delta \leqslant H(2r) \tag{2.16}$$

on B_{2r}. Thus, the right-hand side of (2.14) is estimated from above by

$$\frac{C(C_0, A, \delta)\alpha}{r^{1+\delta}} H(2r) \int \psi^{2\alpha-(1+\delta)(1-1/\zeta)} \eta^{\alpha-1+\delta} \lambda(v).$$

Now, we choose $\zeta > 1$ close enough to 1 that $2-(1+\delta)(1-1/\zeta) > 0$, and apply Hölder inequality with conjugate exponents α and $\alpha/(\alpha-1)$, to estimate the last expression with

$$\frac{C(C_0, A, \delta)\alpha}{r^{1+\delta}} \eta^{\alpha-1+\delta} H(2r) \left(\int \psi^{2\alpha} v^{\alpha-1} \lambda(v) \right)^{(\alpha-1)/\alpha} \times \left(\int \psi^{2\alpha-(1+\delta)(1-1/\zeta)\alpha} v^{\alpha-1+\delta\alpha} \lambda(v) \right)^{1/\alpha}. \tag{2.17}$$

Using (2.15) and (2.17) in (2.14), simplifying and rearranging, and recalling that $\psi = 1$ on B_r and $\psi = 0$ off B_{2r}, and that $\eta/2 \leqslant v \leqslant \eta$ on the set Ω^v_Γ where $\lambda(v) > 0$, we deduce that, if $\alpha > \max\{2, (1+\delta)\}$,

$$\int_{B_r} \lambda(v) \leqslant \left\{ C_1 \frac{Q(2r)H(2r)}{r^{1+\delta}} \eta^\delta \alpha \right\}^\alpha \int_{B_{2r}} \lambda(v) \tag{2.18}$$

with $C_1 = 2^{1-1/\alpha} C(C_0, A, \delta)$. We now set

$$\alpha = \alpha(r) = \frac{r^{1+\delta}}{4C_1 Q(2r)H(2r)\eta^\delta},$$

so that we may rewrite (2.18) as

$$\int_{B_r} \lambda(v) \leqslant (1/2)^{B\eta^{-\delta} r^{1+\delta}/(Q(2r)H(2r))} \int_{B_{2r}} \lambda(v) \quad \forall r \geqslant R,$$

where $B = 1/(4C_1)$. We remark that B depends on A, δ and ζ, but is independent of R and therefore of η.

Applying Lemma 1.1 in [PRS1] with $G(r) = \int_{B_r} \lambda(v)$, we deduce that there exists a constant S which depends only on δ such that, for every $r \geqslant 2R$,

$$\frac{Q(r)H(r)}{r^{1+\delta}} \log \int_{B_r} \lambda(v) \geqslant \frac{Q(r)H(r)}{r^{1+\delta}} \log \int_{B_R} \lambda(v) + SB\eta^{-\delta} \log 2. \tag{2.19}$$

Now choose λ in such a way that $\sup \lambda = 1$. Then the integral on the left-hand side is bounded above by a multiple of $\log \operatorname{vol} B_r$, while, as $r \to +\infty$, the first term on the right-hand side tends to zero. Thus, we conclude that

$$\liminf_{r\to+\infty} \frac{Q(r)H(r)}{r^{1+\delta}} \log \operatorname{vol} B_r \geqslant SB\eta^{-\delta} \log 2,$$

with S and B independent of η. Letting η tend to zero we contradict (2.8). Similarly, choosing a function λ such that $0 \leqslant \lambda(t) \leqslant (t + u^* - \eta)^q$ for $t \geqslant \Gamma$ we see that the integral on the left-hand side of (2.19) is bounded from above by

$$\int_{B_r} |u|^p$$

and, arguing as above, and letting $\eta \to 0$ we contradict (2.9). □

Remark 2.2. Let φ, f, b, Q, h and H be as in Theorem 2.1, and assume that $u \in C^1(M)$ is such that $u_* = \sup u > -\infty$ and satisfies

$$-\mathcal{L}_{\varphi,h} u \geqslant b(x) f(u) \tag{2.20}$$

on the set

$$\Omega_\gamma = \{x \in M : u(x) < \gamma\}, \tag{2.21}$$

for some $\gamma < u_*$. If either (2.8) or (2.9) hold, then $f(u_*) \leqslant 0$.

Indeed, it suffices to note that the function $v = -u$ is bounded above, $v^* = -u_*$ and v satisfies

$$\mathcal{L}_{\varphi,h} v \geqslant b(x) g(v)$$

with $g(t) = f(-t)$. In the assumptions of Theorem 2.1, $g(v^*) = f(u_*) \leqslant 0$.

Our next task is to prove that, under appropriate assumptions, solutions of $\mathcal{L}_{\varphi,h} u \geqslant b(x) f(u)$ are necessarily bounded above.

Theorem 2.3. *Let φ, b, Q, h and H be as in Theorem 2.1, and assume that $u \in C^1(M)$ satisfies*

$$\mathcal{L}_{\varphi,h} u \geqslant b(x) f(u), \tag{2.22}$$

on the set $\Omega_\gamma = \{x : u(x) > \gamma\}$ *for some* $\gamma < u^*$, *where* f *is a continuous function on* $\mathbb{R}$ *such that*

$$\liminf_{t\to+\infty} \frac{f(t)}{t^\sigma} > 0 \tag{2.23}$$

for some $\sigma > \delta$. *If either* (2.8) *or* (2.9) *holds then* u *is bounded above*.

Proof. Again the proof follows adapting the arguments used in the proof of Lemma 1.5 in [PRS1]. Assume by contradiction that u is unbounded, so that Ω_γ is non-empty for every $\gamma > 0$. By increasing γ if necessary we may assume that $f(t) \geqslant Bt^\sigma$ if $t \geqslant \gamma$. For ease of notation, we assume that $B = 1$, so that, on Ω_γ,

$$\mathcal{L}_{\varphi,h} u \geqslant b(x)u^\sigma.$$

Clearly, we may also assume that $b(x)$ is bounded above

Let $R > 0$ be large enough that $\Omega_\gamma \cap B_R$ is non-empty. Let $\lambda : \mathbb{R} \to [0, +\infty)$ be a C^1, non-decreasing function such that $\lambda(t) = 0$ for $t \leqslant \gamma$, fix $\zeta > 1$ such that

$$1 - \frac{1+\delta}{\sigma-\delta}\left(1 - 1/\zeta\right) > 0 \tag{2.24}$$

and, as in the proof of Theorem 2.1, choose a C^∞ cutoff function $\psi = \psi_r : M \to [0, 1]$ such that, for $r \geqslant R$,

$$\text{(i) } \psi \equiv 1 \text{ on } B_r; \quad \text{(ii) } \psi \equiv 0 \text{ on } M \setminus B_{2r}; \quad \text{(iii) } |\nabla\psi| \leqslant \frac{C_0}{r}\psi^{1/\zeta} \tag{2.25}$$

for some constant $C_0 = C_0(\zeta) > 0$. Finally, fix $\alpha > \max\{1 + \delta, 2\sigma\}$ and $\beta > 0$ to be determined later, and consider the vector field W defined by

$$W = \psi^\alpha \lambda(u) u^\beta |\nabla u|^{-1} \varphi(|\nabla u|) h(\nabla u, \cdot)^\sharp.$$

Note that the properties of λ and ψ imply that W vanishes off $B_{2r} \cap \Omega_\gamma$. Proceeding as in the proof of Theorem 2.1, we estimate

$$\begin{aligned} \operatorname{div} W \geqslant{}& \psi^\alpha \lambda(u) b(x) u^{\sigma+\beta} + \frac{\beta}{A^{1/\delta}} \psi^\alpha \lambda(u) u^{\beta-1} \varphi\left(|\nabla u|\right)^{1+1/\delta} h_u \\ & - \alpha\psi^{\alpha-1}\lambda(u)u^\beta \varphi(|\nabla u|) h_u^{1/2} h_+^{1/2} |\nabla\psi|, \end{aligned}$$

where $h_\pm$ is defined in (2.1) and $h_u = h(\nabla u, \nabla u)/|\nabla u|^2$. Next we apply to the second term on the right-hand side the inequality $ab \leqslant \sigma^p a^p/p + b^q/(q\sigma^q)$, with $p = 1 + \delta$,

$q = (1+\delta)/\delta$, and with $\sigma > 0$ chosen in such a way that the first term on the right-hand side cancels out, namely,

$$\sigma^{(1+\delta)/\delta} = \frac{1+\delta}{\delta A^{1/\delta}} \frac{\beta}{\alpha}$$

and insert the resulting inequality in the above estimate, to obtain

$$\begin{aligned} \operatorname{div} W \geqslant & \psi^{\alpha}\lambda(u)b(x)u^{\sigma+\beta} \\ & -C(A,\delta)\frac{\alpha^{1+\delta}}{\beta^{\delta}}\psi^{\alpha-(1+\delta)}\lambda(u)u^{\beta+\delta}h_u^{(1-\delta)/2}h_+^{(1+\delta)/2}|\nabla\psi|^{1+\delta}, \end{aligned}$$

where $C(A,\delta) > 0$ depends only upon A and δ. Now, we integrate, apply the divergence theorem, recall that W is compactly supported, and obtain

$$\begin{aligned} & \int \psi^{\alpha}\lambda(u)b(x)u^{\sigma+\beta} \\ & \quad \leqslant C(A,\delta)\frac{\alpha^{1+\delta}}{\beta^{\delta}} \int \psi^{\alpha-(1+\delta)}\lambda(u)u^{\beta+\delta}h_u^{(1-\delta)/2}h_+^{(1+\delta)/2}|\nabla\psi|^{1+\delta}. \end{aligned} \tag{2.26}$$

Multiplying and dividing by $b(x)^{1/p}$ in the integral on the right-hand side, and applying Hölder inequality with conjugate exponents p and q yield

$$\begin{aligned} & \int \psi^{\alpha-(1+\delta)}\lambda(u)u^{\beta+\delta}h_u^{(1-\delta)/2}h_+^{(1+\delta)/2}|\nabla\psi|^{1+\delta} \\ & \quad \leqslant \left(\int \psi^{\alpha}b(x)\lambda(u)u^{(\beta+\delta)p}\right)^{1/p} \\ & \quad \times \left(\int \psi^{\alpha-(1+\delta)(1-1/\zeta)q}\lambda(u)b(x)^{1-q}h_u^{(1-\delta)q/2}h_+^{(1+\delta)q/2}\left(\frac{|\nabla\psi|}{\psi^{1/\zeta}}\right)^{(1+\delta)q}\right)^{1/q}, \end{aligned}$$

provided $\alpha-(1+\delta)(1-1/\zeta)q > 0$. Choosing $p = (\beta+\sigma)/(\beta+\delta)$ (which is greater than 1 by the condition $\sigma > \delta$), the first integral on the right-hand side is equal to the integral on the left-hand side of (2.26). Thus inserting into (2.26), and simplifying, we obtain

$$\begin{aligned} & \int \psi^{\alpha}\lambda(u)b(x)u^{\sigma+\beta} \\ & \quad \leqslant \left(C(A,\delta)\frac{\alpha^{1+\delta}}{\beta^{\delta}}\right)^{q} \int \psi^{\alpha-(1+\delta)q}\lambda(u)b(x)^{1-q}h_u^{(1-\delta)q/2} \\ & \qquad \times h_+^{(1+\delta)q/2}|\nabla\psi|^{(1+\delta)q}. \end{aligned} \tag{2.27}$$

Since $u > \gamma$ on Ω_γ and $\psi = 1$ on B_r the integral on the left-hand side is bounded from below by

$$\gamma^{\beta+\sigma} \int_{B_r} b(x)\lambda(u). \tag{2.28}$$

On the other hand, using (2.25) (ii) and (iii), and the fact that ψ is supported on B_{2r}, we show that the right-hand side of (2.27) is bounded from above by

$$\left\{ C(A,\delta) \frac{C_0}{r^{1+\delta}} \frac{\alpha^{1+\delta}}{\beta^\delta} \sup_{B_{2r}} \left(\frac{h_u^{(1-\delta)/2} h_+^{(1+\delta)/2}}{b(x)} \right) \right\}^q \int_{B_{2r}} \lambda(u) b(x). \tag{2.29}$$

We insert (2.28) and (2.29) into (2.27), use $b(x) \geqslant Q(r(x))^{-1}$, with Q non-decreasing, apply the reasoning that led to (2.16), and recall the definition of H and the expression of q, to get

$$\int_{B_r} \lambda(u) b(x) \leqslant \left\{ \frac{C}{\gamma^{\sigma-\delta}} \frac{H(2r)Q(2r)}{r^{1+\delta}} \frac{\alpha^{1+\delta}}{\beta^\delta} \right\}^{\frac{\beta+\sigma}{\sigma-\delta}} \int_{B_{2r}} \lambda(u) b(x), \tag{2.30}$$

where $C > 0$ depends only on A, δ and C_0.

Now we choose

$$\beta + \sigma = \alpha = \frac{1}{4C} \gamma^{\sigma-\delta} \frac{r^{1+\delta}}{H(2r)Q(2r)}.$$

so that, (2.24), implies that the condition $\alpha - (1+\delta)(1-1/\zeta)q > 0$ holds. Moreover, by (2.7), $\alpha \to +\infty$ as $r \to +\infty$, and, therefore, $\alpha/\beta \leqslant 2$ holds for sufficiently large r. Thus, for such values of r, (2.30) yields

$$\int_{B_r} \lambda(u) b(x) \leqslant \left(\frac{1}{2} \right)^{\frac{\gamma^{\sigma-\delta}}{4C(\sigma-\delta)} \frac{r^{1+\delta}}{H(2r)Q(2r)}} \int_{B_{2r}} \lambda(u) b(x), \tag{2.31}$$

At this point, arguing as in the final part of the proof of Theorem 2.1, one verifies that, suitable choices of λ, allow to contradict assumptions (2.8) or (2.9), respectively. □

Remark 2.4. Arguing as above, one verifies that it is possible to obtain a version for the $\mathcal{L}_{\varphi,h}$ operator of Theorem B in [PRS1]. Again the volume growth conditions (0.7) and (0.8) in Theorem B are replaced by (2.8) and (2.9), respectively.

We next prove a counterpart of the above result, stating that under appropriate conditions, non-negative, non-identically zero solutions of the inequality

$$-\mathcal{L}_{\varphi,h}u \geqslant b(x)f(u) \tag{2.32}$$

are necessarily bounded away from 0.

Theorem 2.5. *Let φ satisfy the conditions* (0.7) (i)–(iv), *and let $b(x)$, Q, h and H satisfy the assumptions of Theorem* 2.1. *Let $f \in C^0(\mathbb{R})$, and assume that $u \in C^1(M)$ is non-negative and non-identically zero, and satisfies* (2.32) *on the set*

$$\Omega_{\gamma_0} = \{x : u(x) < \gamma_0\} \tag{2.33}$$

for some $\gamma_0 > u_* = \inf u$. If

$$f(t) \geqslant Ct^{\xi}, \quad \text{as } t \to 0+ \quad \text{for some } \xi < \delta, \tag{2.34}$$

and either (2.8) *or* (2.9) *hold, then $u_* > 0$.*

Proof. Observe that, by the strong minimum principle, u is strictly positive on M. We assume by contradiction that $u_* = 0$, so that u satisfies (2.32) on Ω_γ for every $\gamma \leqslant \gamma_0$.

We fix $\gamma \in (0, \gamma_0]$ in such a way that

$$f(t) \geqslant Bt^{\xi}, \quad \text{for some constant } B > 0 \text{ and } t \in (0, \gamma), \tag{2.35}$$

so that

$$-\mathcal{L}_{\varphi,h}u \geqslant Bu^{\xi} \tag{2.36}$$

on Ω_γ. For ease of notation, we may suppose that $B = 1$.

Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $\lambda(t) = 0$ if $t \geqslant \gamma$, $\lambda(t) > 0$ if $t < \gamma$, and $\lambda' \leqslant 0$. Choose $R > 0$ large enough that $B_R \cap \Omega_\gamma \neq \emptyset$, and, for $r \geqslant R$, let $\psi = \psi_r$ be a smooth cutoff function with $\psi = 1$ on B_r, $\psi = 0$ off B_{2r} and $|\nabla\psi| \leqslant (C_0/r)\psi^{\zeta}$ for some C_0 and $\zeta > 1$ independent of r.

Finally, let W be the vector field defined by

$$W = -\psi^{\alpha}\lambda(u)u^{-\beta}|\nabla u|^{-1}\varphi(|\nabla u|)h(\nabla u, \cdot)^{\sharp}, \tag{2.37}$$

where $\alpha, \beta > 0$ are constants to be determined later.

Using, as in the proofs of Theorems 2.1 and 2.3, $\lambda' \geqslant 0$, $|\nabla\varphi|(|\nabla u|) \geqslant A^{-1/\delta}\varphi(|\nabla u|)^{1+1/\delta}$, $h_u = h(\nabla u, \nabla u)/|\nabla u|^2 > 0$ and $|h(\nabla v, \nabla\psi)| \leqslant h_v^{1/2} h_+^{1/2} |\nabla v||\nabla\psi|$, we estimate

$$\operatorname{div} W \geqslant \psi^\alpha b(x) u^{\xi-\beta} + \frac{\beta}{A^{1/\delta}} \psi^\alpha u^{-\beta-1} \lambda(u) \varphi(|\nabla u|)^{1+1/\delta} h_u - \alpha \psi^{\alpha-1} \lambda(u) u^{-\beta} \varphi(|\nabla u|) |\nabla\psi| h_u^{1/2} h_+^{1/2}. \tag{2.38}$$

Now we argue as in the previous proofs, and estimate the last term on the right-hand side using the inequality $ab \leqslant \frac{\sigma^p a^p}{p} + \frac{b^q}{\sigma^q q}$, with $p = 1 + 1/\delta$, $q = 1 + \delta$, and with $\sigma = [\beta(1+\delta)/(A^{1/\delta}\alpha\delta)]^{\delta/(1+\delta)}$, chosen in such a way as to cancel the second term.

Integrating the resulting inequality, applying the divergence theorem, observing that W is compactly supported, and using the properties of the cut off function ψ yield

$$\int \psi^\alpha \lambda(u) b(x) u^{\xi-\beta} \leqslant C_1 \left(\frac{\alpha}{\beta}\right)^\delta \frac{\alpha}{r^{1+\delta}} \times \int \psi^{\alpha-(1+\delta)(1-1/\xi)} \lambda(u) u^{\delta-\beta} h_u^{(1-\delta)/2} h_+^{(1+\delta)/2}, \tag{2.39}$$

provided $\alpha - (1+\delta)(1-1/\zeta) > 0$, where the constant $C_1 = C_1(C_0, A, \delta)$ is independent of α, β, and r.

Multiplying and dividing by $b(x)^{1/p}$, and applying Hölder inequality with conjugate exponents p and q to be determined later subject to the condition $\alpha - (1+\delta)(1-1/\zeta) q > 0$, we estimate from above the integral on right-hand side of (2.39) by

$$\left(\int \psi^\alpha b(x) \lambda(u) u^{(\delta-\beta)p}\right)^{1/p} \times \left(\int \psi^{\alpha-(1+\delta)(1-1/\xi)q} \lambda(u) b(x)^{1-q} h_u^{(1-\delta)q/2} h_+^{(1+\delta)q/2}\right)^{1/q}. \tag{2.40}$$

Choosing p in such a way that $(\delta-\beta)p = \xi - \beta$, namely $p = (\beta-\xi)/(\beta-\delta)$, so that $p > 1$ by the assumption that $\xi < \delta$, the first integral in (2.40) is equal to the integral on the left-hand side of (2.39). Thus inserting, simplifying, and using the definition of H, the lower bound $b(x) \geqslant Q(r(x))^{-1}$, and $1_{B_r} \leqslant \psi \leqslant 1_{B_{2r}}$, we obtain

$$\int_{B_r} \lambda(u) b(x) u^{\xi-\beta} \leqslant \left(C_1 \left(\frac{\alpha}{\beta}\right)^\delta \frac{H(2r)Q(2r)}{r^{1+\delta}} \alpha\right)^q \int_{B_{2r}} \lambda(u) b(x),$$

provided $\alpha > (1+\delta)(1-1/\zeta)q$. Since $q = (\beta-\xi)/(\delta-\xi)$, if we choose $\beta = \alpha + \xi$, the condition becomes $1 > (1+\delta)(1-1/\zeta)/(\delta-\xi)$, which holds provided ζ is sufficiently close to 1. Now, since $u < \gamma$ on Ω_γ, $u^{\xi-\beta} > \gamma^{\xi-\beta}$ for $\beta > \xi$, and $\alpha/\beta < 1$, we deduce that

$$\int_{B_r} \lambda(u)b(x) \leqslant \left(C_1 \gamma^{\delta-\xi} \frac{H(2r)Q(2r)}{r^{1+\delta}} \alpha \right)^{\alpha/(\delta-\xi)} \int_{B_{2r}} \lambda(u)b(x),$$

whence, choosing

$$\alpha = \frac{1}{2C_1} \gamma^{-(\delta-\xi)} \frac{r^{1+\delta}}{H(2r)Q(2r)},$$

and assuming that either (2.8) or (2.9) hold, a contradiction is reached arguing as in the last part of the proof of Theorem 2.1. $\square$

In order to obtain a version of Theorem A for the $\mathcal{L}_{\varphi,h}$ operator, the only missing ingredients are the analog of the volume growth conditions that imply the φ-parabolicity. We will say that a manifold is (φ, h)-parabolic if the only bounded above C^1 solutions of the inequality

$$\mathcal{L}_{\varphi,h} u \geqslant 0 \tag{2.41}$$

are constant.

It is a relatively straightforward matter to check that the proofs in [RS] can be adapted to treat the case at hand. We state the analogue of Theorem A therein.

Theorem 2.6. *Let $(M, \langle\, , \rangle)$ be a complete manifold, assume that φ satisfies the structural conditions* (0.7) (i)–(iii), *and let h_δ be defined, as in* (2.2), *by*

$$h_\delta(r) = \begin{cases} h_+(r) & \text{if } \delta \leqslant 1 \\ h_-(r)^{(1-\delta)/2} h_+(r)^{(1+\delta)/2} & \text{if } \delta > 1. \end{cases}$$

If

$$\left(h_\delta(r) \operatorname{vol}(\partial B_r)^{1/\delta} \right)^{-1} \notin L^1(+\infty).$$

Then M is (φ, h)-parabolic.

The proof of the theorem follows as in [RS] and depends on a version of Lemma 1.1 in [RS], which in the present case reads as follows:

Lemma 2.7. *Let $f \in C^0(\mathbb{R})$, and let u be a non-constant C^1 solution of the differential inequality*

$$\mathcal{L}_{\varphi,h} u \geqslant |\nabla u|^{-1} \varphi(|\nabla u|) h(\nabla u, \nabla u) f(u). \tag{2.42}$$

Assume that there are functions $\alpha \in C^1(I)$ *and* $\beta \in C^0(I)$ *defined on an interval* $I \supset u(M)$ *such that*

$$\alpha(u) \geqslant 0, \tag{2.43}$$

$$\alpha'(u) + f(u)\alpha(u) \geqslant \beta(u) > 0 \tag{2.44}$$

on M. Then there exist R_0 *depending only on u and a constant* $C > 0$ *independent of* α *and* β, *such that, for every* $r > R \geqslant R_0$,

$$\left\{ \int_{B_R} \beta(u)\varphi\left(|\nabla u|\right)|\nabla u| \right\}^{-1} \geqslant C \left\{ \int_R^r \left(\int_{\partial B_t} \frac{\alpha(u)^{1+\delta}}{\beta(u)^\delta} h_\delta \right)^{-1/\delta} \right\}^\delta. \tag{2.45}$$

Proof. The proof follows the lines of that of Lemma 1.1 in [RS]. The vector field considered there is replaced by

$$Z = \alpha(u)|\nabla u|^{-1}\varphi(|\nabla u|)h(\nabla u, \cdot)^\sharp.$$

Then, applying the divergence theorem, Hölder inequality, and arguing as in the original proof one arrives at the differential inequality, valid for $r > R \geqslant R_0$,

$$G(r)^{-(1+1/\delta)}G'(r) \geqslant C \left(\int_{\partial B_r} \frac{\alpha(u)^{1+\delta}}{\beta(u)^\delta} h_u^{(1-\delta)/2} h_+^{(1+\delta)/2} \right)^{-1/\delta},$$

where we have set

$$G(r) = \int_{B_r} \beta(u)\varphi(|\nabla u|)^{1+1/\delta} h_u.$$

Recalling that

$$h_u^{(1-\delta)/2} h_+^{(1+\delta)/2} \leqslant h_\delta,$$

one concludes the proof as in [RS]. □

Proceeding as in Section 1 yields the following version of Theorem A of the Introduction for the $\mathcal{L}_{\varphi,h}$ operator.

Theorem 2.8. *Let* φ, h, h_δ *and* H *be as in the statement of Theorem* 2.5, *and suppose that*

$$h_\delta(r) \leqslant H(r) \leqslant Cr^\nu,$$

for some $\nu > 0$. *Let* $u \in C^1(M)$ *be a non-negative solution of*

$$\mathcal{L}_{\varphi,h} u = -b(x) f(u) \quad \text{on } M.$$

where $b \in C^0(M)$ *is such that*

$$b(x) \geqslant \frac{C}{(1 + r(x))^\mu} \quad \text{on } M$$

for some $C > 0$ *and* $0 < \mu < 1 + \delta$, *and* $f \in C^0([0, +\infty))$ *satisfies* $f(0) = f(a) = 0$, $f(t) > 0$ $(0, a)$, $f(t) < 0$ *in* $(a, +\infty)$, *for some* $a > 0$, *and*

$$\liminf_{s \to +\infty} -\frac{f(s)}{t^\sigma} > 0$$

for some $\sigma > \max\{1, \delta\}$. *Assume that*

$$\liminf_{r \to +\infty} \frac{\log \operatorname{vol} B_r}{r^{1+\delta-(\mu+\nu)}} < +\infty$$

and, if

$$\left(h_\delta \ \operatorname{vol}(\partial B_r)\right)^{-1/\delta} \in L^1(+\infty),$$

assume furthermore that

$$f(t) \geqslant ct^\xi \quad 0 < t \ll 1,$$

for some $\xi > 0$ *and* $c > 0$. *Finally, if*

$$\xi \geqslant \delta,$$

suppose also that

$$u(x) \geqslant Cr(x)^{-\theta}, \quad r(x) \gg 1$$

for some $\theta \geqslant 0$, $C > 0$, *and that*

$$\liminf_{r\to+\infty} \frac{\log \operatorname{vol} B_r}{r^{1+\delta-\theta(\xi-\delta+\varepsilon)-(\mu+\nu)}} < +\infty$$

for some $\varepsilon > 0$. *Then* u *is constant and identically equal to* 0 *or* a.

In order to apply the methods described in Section 1 to obtain “a priori” lower bounds for solutions of

$$\mathcal{L}_{\varphi,h} u \leqslant 0,$$

one needs to describe the action of the operator $\mathcal{L}_{\varphi,h}$ on radial functions. This is the content of the following lemma.

Lemma 2.9. *Let* α *be a strictly monotonic* C^2 *function on* $[R, +\infty)$ *and set* $v(x) = \alpha(r(x))$. *Then, on* $M \setminus B_R$, *we have*

$$\begin{aligned}\mathcal{L}_{\varphi,h} v(x) =& (\operatorname{sgn} \alpha')[\varphi(|\alpha'|)]' h(\nabla r, \nabla r)\\ &+ (\operatorname{sgn} \alpha')\varphi(|\alpha'|)\{\operatorname{div} h(\nabla r) + \langle h, \operatorname{Hess} r\rangle_{S^2T^*M}\}.\end{aligned} \tag{2.46}$$

Proof. Recalling the definition of $\mathcal{L}_{\varphi,h}$, we compute

$$\begin{aligned}\mathcal{L}_{\varphi,h} v(x) =& (\operatorname{sgn} \alpha')\varphi(|\alpha'|)\operatorname{div}[h(\nabla r, \cdot)^\sharp]\\ &+ \langle \nabla[(\operatorname{sgn} \alpha')\varphi(|\alpha'|)], h(\nabla r, \cdot)^\sharp\rangle.\end{aligned} \tag{2.47}$$

Now,

$$\langle \nabla[(\operatorname{sgn} \alpha')\varphi(|\alpha'|)], h(\nabla r, \cdot)^\sharp\rangle = (\operatorname{sgn} \alpha')[\varphi(|\alpha'|)]' h(\nabla r, \nabla r).$$

On the other hand, if E_i is a local orthonormal frame,

$$\begin{aligned}\operatorname{div}[h(\nabla r, \cdot)^\sharp] =& \sum_i \langle D_{E_i} h(\nabla r, \cdot)^\sharp, E_i\rangle\\ =& \sum_i E_i h(\nabla r, e_i) - h(\nabla r, D_{E_i} E_i)\\ =& \sum_i (D_{E_i} h)(\nabla r, E_i) + h(D_{E_i} \nabla r, E_i)\\ =& (\operatorname{div} h)(\nabla r) + \sum_{i,j} h(E_i, E_j)\operatorname{Hess}(E_i, E_j)\end{aligned}$$

$$= (\operatorname{div} h)(\nabla r) + \langle h, \operatorname{Hess} r\rangle_{S^2T^*M},$$

whence the required conclusion follows upon inserting the above identities into (2.47). □

Now, if we assume that

$$h_- \leqslant h \leqslant h_+, \quad |\operatorname{div} h| \leqslant \beta(r), \quad |\operatorname{Hess} r| \leqslant \frac{g'}{g}\left(\langle\ ,\ \rangle - dr \otimes dr\right),$$

we have

$$h(\nabla r, \nabla r) \geqslant h_- \quad \text{and} \quad \langle h, \operatorname{Hess} r\rangle_{S^2T^*M} \leqslant (m-1)h_+ \frac{g'}{g}.$$

If, in addition, $\alpha' < 0$, it follows that

$$\mathcal{L}_{\varphi,h} v(x) \geqslant -h(\nabla r, \nabla r)\left\{[\varphi(|\alpha'|)]' + \frac{1}{h_-(r)}\varphi(|\alpha'|)\left[\beta(r) + (m-1)h_+(r)\frac{g'}{g}\right]\right\}. \tag{2.48}$$

This proves

Lemma 2.10. *Maintaining the notation introduced above, assume that α is a solution of the problem*

$$\begin{cases} [\varphi(|\alpha'|)]' + \frac{1}{h_-(r)}\varphi(|\alpha'|)[\beta(r) + (m-1)h_+(r)\frac{g'}{g}] = 0 \quad \text{on } [R, +\infty), \\ \alpha(R) = D, \alpha' < 0, \end{cases} \tag{2.49}$$

and let $v(x) = \alpha(r(x))$, then

$$\mathcal{L}_{\varphi,h} v \geqslant 0 \quad \text{on } M \setminus B_R.$$

As in the proof of Proposition B above, a comparison argument shows that, if u is a non-negative solution of

$$\mathcal{L}_{\varphi,h} u \leqslant 0 \quad \text{on } M \tag{2.50}$$

and α satisfies (2.49) with $D \leqslant \min_{\partial B_R} u$, then

$$u \geqslant v \quad \text{on } M \setminus B_R. \tag{2.51}$$

To find a solution of (2.49) we write

$$\frac{1}{h_-(r)}\left[\beta(r)+(m-1)h_+(r)\frac{g'}{g}\right]=(m-1)\frac{\tilde{g}'}{\tilde{g}} \tag{2.52}$$

so that the equation satisfied by α becomes

$$[\varphi(|\alpha'|)]'+(m-1)\frac{\tilde{g}'}{\tilde{g}}\varphi(|\alpha'|)=0, \tag{2.53}$$

which can be analyzed as in Section 1.

To illustrate the kind of bounds that can be obtained in the manner described above, we consider the case where $(M,\langle,\rangle)$ is $\mathbb{R}^m$ with its canonical Euclidean metric, so that the Hessian condition holds with $g(r)=r$. We assume further that $h_-=C_1\leqslant C_2=h_+$, and that $\beta(r)=C_3/r$ for $r\gg 1$, so that

$$\frac{\tilde{g}'}{\tilde{g}}=\frac{\Lambda}{r},\ \text{and}\ \tilde{g}(r)=Cr^{\Lambda},\quad r\gg 1$$

with $\Lambda C_1^{-1}[C_3+(m-1)C_2]\,(m-1)^{-1}$.

Assuming that the function φ satisfies the condition stated in Lemma 1.1, condition (1.3), namely $\tilde{g}^{-(m-1)/\zeta}\in L^1(+\infty)$ amounts to

$$(m-1)\Lambda>\zeta.$$

If this is the case, Lemma 1.1 applies, and we conclude that any positive solution u of (2.50) satisfies the a priori lower estimate

$$u(x)\geqslant Cr^{-(m-1)\Lambda/\zeta}\quad\text{for } r\gg 1.$$

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