J. Fixed Point Theory Appl. (2020) 22:76
https://doi.org/10.1007/s11784-020-00812-6
Published online August 27, 2020
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Journal of Fixed Point Theory and Applications

Non-local to local transition for ground states of fractional Schrödinger equations on $\mathbb{R}^N$

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Abstract. We consider the nonlinear fractional problem

$$(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N$$

We show that ground state solutions converge (along a subsequence) in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, under suitable conditions on f and V, to a ground state solution of the local problem as $s \to 1^-$.

Mathematics Subject Classification. Primary 35Q55; Secondary 35A15, 35R11.

Keywords. Variational methods, fractional Schrödinger equation, non-local to local transition, ground state, Nehari manifold.

1. Introduction

The aim of this paper is to analyse the asymptotic behavior of least-energy solutions to the fractional Schrödinger problem:

$$\begin{cases} (-\Delta)^s u + V(x)u = f(x,u) & \text{in } \mathbb{R}^N \\ u \in H^s(\mathbb{R}^N), \end{cases} \tag{1.1}$$

under suitable assumptions on the scalar potential $V\colon \mathbb{R}^N \to \mathbb{R}$ and on the nonlinearity $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$. We recall that the fractional laplacian is defined as the principal value of a singular integral via the formula:

$$(-\Delta)^s u(x) = C(N,s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \, \mathrm{d}y$$

with

$$\frac{1}{C(N,s)} = \int_{\mathbb{R}^N} \frac{1-\cos\zeta_1}{|\zeta|^{N+2s}} \, \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_N.$$

This formal definition needs of course a function space in which problem (1.1) becomes meaningful: we will come to this issue in Sect. 2.

Several models have appeared in recent years that involve the use of the fractional laplacian. We only mention elasticity, turbulence, porous media flow, image processing, wave propagation in heterogeneous high contrast media, and stochastic models: see [1,11,13,19].

Instead of *fixing* the value of the parameter $s \in (0,1)$, we will start from the well-known identity (see [10, Proposition 4.4]):

$$\lim_{s \to 1^-} (-\Delta)^s u = -\Delta u \tag{1.2}$$

valid for any $u \in C_0^\infty(\mathbb{R}^N)$, and investigate the convergence properties of solutions to (1.1) as $s \to 1^-$.

In view of (1.2), it is somehow natural to conjecture that solutions to (1.1) converge to solutions of the problem:

$$\begin{cases} -\Delta u + V(x)u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.3}$$

We do not know if this conjecture is indeed correct with this degree of generality.

In this paper, we will always assume that both V and f are $\mathbb{Z}^N$-periodic in the space variables. Hence equations (1.1) and (1.3) are invariant under $\mathbb{Z}^N$-translations, and their solutions are not unique. We will prove that—up to $\mathbb{Z}^N$-translations and along a subsequence—*least-energy* solutions of (1.1) converge to a ground state solution to the local problem (1.3). Our result is a continuation of the previous paper [5], in which we consider the equation on a bounded domain and extend the very recent analysis of Biccari *et al.* (see [2]) in the *linear* case for the Poisson problem to the semilinear case. See also [6].

We collect our assumptions.

(N) $N \geq 3$, $1/2 < s < 1$;

(V) $V \in L^\infty(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic and $\inf_{\mathbb{R}^N} V > 0$;

(F1) $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, namely $f(\cdot,u)$ is measurable for any $u \in \mathbb{R}$ and $f(x,\cdot)$ is continuous for a.e. $x \in \mathbb{R}^N$. Moreover f is $\mathbb{Z}^N$-periodic in $x \in \mathbb{R}^N$ and there are numbers $C > 0$ and $p \in \left(2, \frac{2N}{N-1}\right)$ such that

$$|f(x,u)| \leq C(1 + |u|^{p-1})$$

for $u \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^N$.

(F2) $f(x,u) = o(u)$ as $u \to 0$, uniformly with respect to $x \in \mathbb{R}^N$.

(F3) $\lim_{|u| \to +\infty} \frac{F(x,u)}{u^2} = +\infty$ uniformly with respect to $x \in \mathbb{R}^N$, where $F(x,u) = \int_0^u f(x,s)\,ds$.

(F4) The function $\mathbb{R}\setminus\{0\} \ni u \mapsto f(x,u)/u$ is strictly increasing on $(-\infty,0)$ and on $(0,\infty)$, for a.e. $x \in \mathbb{R}^N$.

Remark 1.1. It follows from (F1) and (F2) that for every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|f(x,u)| \leq \varepsilon |u| + C_\varepsilon |u|^{p-1}$$

for every $u \in \mathbb{R}$ and a.e $x \in \mathbb{R}^N$. Furthermore, assumption (F4) implies the validity of the inequality:

$$0 \leq 2F(u) \leq f(x, u)u$$

for every $u \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^N$.

We can now state our main result.

Theorem 1.2. *Suppose that assumptions* (N), (V), $(F1)$–$(F4)$ *hold. Let* $u_s \in H^s(\mathbb{R}^N)$ *be a ground state solution of problem* (1.1). *Then, there exists a sequence* $\{s_n\}_n \subset (1/2, 1)$, *such that* $s_n \to 1$ *as* $n \to +\infty$ *and there exists a sequence of translations* $\{z_n\}_n$, *such that* $u_{s_n}(\cdot - z_n)$ *converges in* $L^2_{\text{loc}}(\mathbb{R}^N)$ *to a ground state solution* $u_0 \in H^1(\mathbb{R}^N)$ *of the problem* (1.3).

2. The variational setting

In this section we collect the basic tools from the theory of fractional Sobolev spaces we will need to prove our results. For a thorough discussion, we refer to [10,14] and to the references therein.

For $0 < s < 1$, we define a Sobolev space on $\mathbb{R}^N$ as

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \mid \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y < +\infty \right\},$$

endowed with the norm:

$$\|u\|^2_{H^s(\mathbb{R}^N)} = \|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y$$

One can show that $C_0^\infty(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$. For $u \in H^s(\mathbb{R}^N)$, an equivalent norm of u is (see [14, Proposition 1.18])

$$u \mapsto \left(\|u\|^2_{L^2(\mathbb{R}^N)} + \left\| (-\Delta)^{\frac{s}{2}} u \right\|^2_{L^2(\mathbb{R}^N)} \right)^{1/2}.$$

More explicitly, for every $u \in H^s(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y = \frac{2}{C(N, s)} \left\| (-\Delta)^{s/2} u \right\|^2_{L^2(\mathbb{R}^N)},$$

where

$$C(N, s) = \frac{s(1 - s)}{A(N, s)B(s)},$$
$$A(N, s) = \int_{\mathbb{R}^{N-1}} \frac{\mathrm{d}\eta}{(1 + |\eta|^2)^{(N+2s)/2}},$$
$$B(s) = s(1 - s) \int_{\mathbb{R}} \frac{1 - \cos t}{|t|^{1+2s}} \, \mathrm{d}t.$$

Lemma 2.1. *For every* $u \in H^1(\mathbb{R}^N)$, *there results*

$$\lim_{s \to 1^-} \left\| (-\Delta)^{s/2} u \right\|^2_{L^2(\mathbb{R}^N)} = \|\nabla u\|^2_{L^2(\mathbb{R}^N)}.$$

Proof. From [10, Proposition 3.6], we know that

$$\left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^N)}^2 = \frac{C(N,s)}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

From [10, Remark 4.3], we know that

$$\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = \frac{\omega_{N-1}}{2N}\|\nabla u\|_{L^2(\mathbb{R}^N)}^2.$$

Therefore, recalling [10, Corollary 4.2],

$$\begin{aligned}\lim_{s\to 1^-}\left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^N)}^2 &= \lim_{s\to 1^-}\frac{C(N,s)}{2(1-s)}\left((1-s)\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)\\ &= \frac{1}{2}\frac{4N}{\omega_{N-1}}\frac{\omega_{N-1}}{2N}\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 = \|\nabla u\|_{L^2(\mathbb{R}^N)}^2.\end{aligned}$$

□

On $H^s(\mathbb{R}^N)$ we introduce a new norm

$$\|u\|_s^2 := \left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N}V(x)u^2\,\mathrm{d}x, \quad u\in H^s(\mathbb{R}^N), \tag{2.1}$$

which is, under (V), equivalent to $\|\cdot\|_{H^s(\mathbb{R}^N)}$. Similarly we introduce the norm on $H^1(\mathbb{R}^N)$ by putting

$$\|u\|^2 := \int_{\mathbb{R}^N}|\nabla u|^2 + V(x)u^2\,\mathrm{d}x, \quad u\in H^1(\mathbb{R}^N). \tag{2.2}$$

Corollary 2.2. *For every $u\in H^1(\mathbb{R}^N)$, we have*

$$\lim_{s\to 1^-}\|u\|_s = \|u\|.$$

The following convergence result will be used in the sequel.

Lemma 2.3. *For every $\varphi\in C_0^\infty(\mathbb{R}^N)$, there results*

$$\lim_{s\to 1^-}\|(-\Delta)^s\varphi-(-\Delta)\varphi\|_{L^2(\mathbb{R}^N)} = 0.$$

Proof. We notice that

$$\begin{aligned}\|(-\Delta)^s\varphi-(-\Delta)\varphi\|_{L^2(\mathbb{R}^N)} &= \left\|\mathcal{F}_\xi^{-1}\left(\left(|\xi|^{2s}-|\xi|^2\right)\hat{\varphi}(\xi)\right)\right\|_{L^2(\mathbb{R}^N)}\\ &\le C\left\|\left(|\cdot|^{2s}-|\cdot|^2\right)\hat{\varphi}\right\|_{L^2(\mathbb{R}^N)}\end{aligned}$$

where $C>0$ is a constant, independent of s, that depends on the definition of the Fourier transform $\mathcal{F}$. It is now easy to conclude, since the Fourier transform of a test function is a rapidly decreasing function. □

We will need some precise information on the embedding constant for fractional Sobolev spaces.

Theorem 2.4. [9] *Let $N>2s$ and $2_s^*=2N/(N-2s)$. Then*

$$\|u\|_{L^{2_s^*}(\mathbb{R}^N)}^2 \le \frac{\Gamma\left(\frac{N-2s}{2}\right)}{\Gamma\left(\frac{N+2s}{2}\right)}|\mathbb{S}|^{-\frac{2s}{N}}\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^N)}^2$$

for every $u\in H^s(\mathbb{R}^N)$, where $\mathbb{S}$ denotes the N-dimensional unit sphere and $|\mathbb{S}|$ its surface area.

The following inequality in an easy consequence of Theorem 2.4, see also [5, Lemma 2.7].

Lemma 2.5. *Let $N \geq 3$ and $q \in [2, 2N/(N-1)]$. Then there exists a constant $C = C(N,q) > 0$ such that, for every $s \in [1/2, 1]$ and every $u \in H^s(\mathbb{R}^N)$, we have*

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C(N,q)\|u\|_s.$$

Definition 2.6. A weak solution to problem (1.1) is a function $u \in H^s(\mathbb{R}^N)$, such that

$$\langle (-\Delta)^{s/2}u \mid (-\Delta)^{s/2}\varphi\rangle_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} V(x)u\varphi \,\mathrm{d}x = \int_{\mathbb{R}^N} f(x,u)\varphi \,\mathrm{d}x$$

for every $\varphi \in H^s(\mathbb{R}^N)$.

Weak solutions are therefore critical points of the associated energy functional $\mathcal{J}_s \colon H^s(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{J}_s(u) = \frac{1}{2}\left\|(-\Delta)^{s/2}u\right\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x.$$

We recall also the definition of a weak solution in the local case.

Definition 2.7. A weak solution to problem (1.3) is a function $u \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)u\varphi \,\mathrm{d}x = \int_{\mathbb{R}^N} f(x,u)\varphi \,\mathrm{d}x$$

for every $\varphi \in H^1(\mathbb{R}^N)$.

For the local problem (1.3), we put $\mathcal{J} \colon H^1(\mathbb{R}^N) \to \mathbb{R}$

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x. \tag{2.3}$$

Recalling the notation (2.1) and (2.2), we can rewrite our functionals in the form:

$$\begin{aligned}\mathcal{J}_s(u) &= \frac{1}{2}\|u\|_s^2 - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \quad u \in H^s(\mathbb{R}^N),\\ \mathcal{J}(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \quad u \in H^1(\mathbb{R}^N).\end{aligned}$$

3. Uniform Lions' concentration-compactness principle

Since the summability exponent of our space is not fixed, we need a "uniform" version of a celebrated result by Lions.

Theorem 3.1. *Let $r > 0$, $2 \leq q < \frac{2N}{N-1}$ and $N \geq 3$. Suppose moreover that $\{s_n\}_n \subset (1/2, 1)$, $u_n \in H^{s_n}(\mathbb{R}^N)$ and*

$$\|u_n\|_{s_n} \leq M,$$

where $M > 0$ does not depend on s_n. If

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |u_n|^q \, \mathrm{d}x = 0$$

then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $p \in \left(2, \frac{2N}{N-1}\right)$.

Proof. Let $t \in \left(q, \frac{2N}{N-1}\right)$. Then

$$\begin{aligned} \|u_n\|_{L^t(B(y,r))} &\le \|u_n\|^{1-\lambda}_{L^q(B(y,r))} \|u_n\|^{\lambda}_{L^{\frac{2N}{N-1}}(B(y,r))} \\ &\le C\|u_n\|^{1-\lambda}_{L^q(B(y,r))} \|u_n\|^{\lambda}_{s_n}, \end{aligned}$$

where $C > 0$ is independent of s_n and $\lambda = \frac{t-q}{\frac{2N}{N-1}-q}\frac{2N}{(N-1)t}$. Choose t such that $\lambda = \frac{2}{t}$. Then

$$\int_{B(y,r)} |u_n|^t \, \mathrm{d}x \le C^t \|u_n\|^{(1-\lambda)t}_{L^q(B(y,r))} \|u_n\|^2_{s_n}.$$

Covering space $\mathbb{R}^N$ by balls of radius r, in a way that each point is contained in at most $N+1$ balls, we get

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n|^t \, \mathrm{d}x &\le (N+1)C^t \sup_{y\in\mathbb{R}^N} \left(\int_{B(y,r)} |u_n|^q \, \mathrm{d}x \right)^{\frac{(1-\lambda)t}{q}} \|u_n\|^2_{s_n} \\ &\le (N+1)M^2 C^t \sup_{y\in\mathbb{R}^N} \left(\int_{B(y,r)} |u_n|^q \, \mathrm{d}x \right)^{\frac{(1-\lambda)t}{q}} \to 0. \end{aligned}$$

Hence $u_n \to 0$ in $L^t(\mathbb{R}^N)$. Note that

$$\|u_n\|^2_{L^2(\mathbb{R}^N)} \le D\|u_n\|^2_{s_n} \le DM^2,$$

where D does not depend on s_n and n. Similarly, from Lemma 2.5, there follows that $\{u_n\}_n$ is bounded in $L^{\frac{2N}{N-1}}(\mathbb{R}^N)$. From the interpolation inequality, since $\{u_n\}_n$ is bounded in $L^2(\mathbb{R}^N)$ and in $L^{\frac{2N}{N-1}}(\mathbb{R}^N)$, we obtain $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $p \in \left(2, \frac{2N}{N-1}\right)$. □

Finally, we extend the locally compact embedding into Lebesgue spaces in a uniform way.

Theorem 3.2. *Let $\{s_n\}_n$ be a sequence such that $1/2 < s_n < 1$ and $s_n \to 1$, and let $\{v_{s_n}\}_n \subset H^{s_n}(\mathbb{R}^N)$ be such that*

$$M = \sup_n \|v_{s_n}\|_{s_n} < \infty.$$

Then, the sequence $\{v_{s_n}\}_n$ converges, up to a subsequence, to some $v \in H^1(\mathbb{R}^N)$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for every $q \in [2, 2N/(N-1))$, and pointwise almost everywhere.

Proof. Note that $H^{s_n}(\mathbb{R}^N) \subset H^{1/2}(\mathbb{R}^N)$ and

$$\| \cdot \|_{1/2} \leq C \| \cdot \|_{s_n}$$

where $C > 0$ does not depend on s_n (and therefore also on n): see for instance [14, Proposition 1.1]. In particular, for every $n \in \mathbb{N}$, we have

$$\|v_{s_n}\|_{1/2} \leq C\|v_{s_n}\|_{s_n} \leq CM. \tag{3.1}$$

Thus, $\{v_{s_n}\}_n$ is bounded in $H^{1/2}(\mathbb{R}^N)$. Hence, passing to a subsequence, there exists a function v, such that $v_{s_n} \rightharpoonup v$ in $H^{1/2}(\mathbb{R}^N)$, $v_{s_n} \to v$ pointwise almost everywhere, and $v_{s_n} \to v$ in $L^q_{\text{loc}}(\mathbb{R}^N)$ for every $q \in [2, 2N/(N-1))$. From [7, Corollary 7], it follows that $v \in H^1_{\text{loc}}(\mathbb{R}^N)$. To complete the proof, we need to show that $v \in H^1(\mathbb{R}^N)$.

Let $\widehat{v_{s_n}}$ denote the Fourier transform of v_{s_n}, similarly for $\widehat{v}$. We may assume, without loss of generality, that $\widehat{v_{s_n}} \rightharpoonup \widehat{v}$ in $L^2(\mathbb{R}^N)$. Note that (3.1) implies that

$$\sup_n \int_{\mathbb{R}^N} (1+|\xi|^2)^{s_n} \left|\widehat{v_{s_n}}\right|^2 \, \mathrm{d}\xi \leq K$$

for some constant $K > 0$. For $1/2 < t \leq 1$, we define

$$B_t := \left\{ w \in L^2(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} (1+|\xi|^2)^t |w(\xi)|^2 \, \mathrm{d}\xi \leq K \right\}.$$

First of all, we observe that

$$\bigcap_{1/2<t<1} B_t = B_1. \tag{3.2}$$

Indeed, for any $1/2 < t < 1$ we have $(1+|\xi|^2)^t \leq 1+|\xi|^2$. Take $w \in B_1$ and note that

$$\int_{\mathbb{R}^N} (1+|\xi|^2)^t \, |w(\xi)|^2 \mathrm{d}\xi \leq \int_{\mathbb{R}^N} (1+|\xi|^2) \, |w(\xi)|^2 \mathrm{d}\xi \leq K.$$

Hence $w \in B_t$ for any $t < 1$. Thus

$$\bigcap_{1/2<t<1} B_t \supset B_1.$$

On the other hand, fix $w \in \bigcap_{1/2<t<1} B_t$. Take any sequence $t_n \to 1^-$ with $t_n > 1/2$. Then, obviously

$$\liminf_{n\to+\infty} (1+|\xi|^2)^{t_n} |w(\xi)|^2 = (1+|\xi|^2)|w(\xi)|^2$$

and Fatou's lemma yields

$$\int_{\mathbb{R}^N} (1+|\xi|^2) \, |w(\xi)|^2 \mathrm{d}\xi \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^N} (1+|\xi|^2)^{t_n} \, |w(\xi)|^2 \mathrm{d}\xi \leq K.$$

Hence $w \in B_1$, or

$$\bigcap_{1/2<t<1} B_t \subset B_1,$$

and (3.2) is proved. Fix now any $t \in (1/2, 1)$ and choose n_0 such that $s_n > t$ for all $n \geq n_0$. Then

$$(1+|\xi|^2)^t \leq (1+|\xi|^2)^{s_n} \quad \text{for every } \xi \in \mathbb{R}^N,$$

and

$$\int_{\mathbb{R}^N} (1+|\xi|^2)^t \, |\widehat{v_{s_n}}|^2 \, \mathrm{d}\xi \leq \int_{\mathbb{R}^N} (1+|\xi|^2)^{s_n} \, |\widehat{v_{s_n}}|^2 \, \mathrm{d}\xi \leq K \quad \text{for } n \geq n_0.$$

Hence, $\widehat{v_{s_n}} \in B_t$ for $n \geq n_0$. Each B_t is a closed and convex subset in $L^2(\mathbb{R}^N)$, and from [8, Theorem 3.7] it is also weakly closed. Hence, $\widehat{v} \in B_t$. Therefore, recalling (3.2),

$$\widehat{v} \in \bigcap_{\frac{1}{2}<t<1} B_t = \left\{ w \in L^2(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} (1+|\xi|^2)|w(\xi)|^2 \, \mathrm{d}\xi \leq K \right\}.$$

This implies that

$$\int_{\mathbb{R}^N} (1+|\xi|^2)|\hat{v}(\xi)|^2 \, \mathrm{d}\xi \leq K < +\infty$$

and $v \in H^1(\mathbb{R}^N)$. □

4. Existence of ground states

It is easy to check that the energy functional $\mathcal{J}$ has the mountain-pass geometry. In particular, there is radius $r > 0$, such that

$$\inf_{\|u\|=r} \mathcal{J}(u) > 0.$$

The following existence result is well-known in the literature, and has been shown in various ways, see e.g. [4,12,17,18].

Theorem 4.1. *Suppose that assumptions* (N), (V), $(F1)$–$(F4)$ *hold. Then there exists a ground state solution* $u_0 \in H^1(\mathbb{R}^N)$ *to* (1.3), *i.e., a critical point of the functional* $\mathcal{J}$ *given by* (2.3), *such that*

$$\mathcal{J}(u_0) = \inf_{\mathcal{N}} \mathcal{J} = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \sup_{t \geq 0} \mathcal{J}(tu) = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{J}(\gamma(t)),$$

where $\mathcal{N}$ *is the so-called Nehari manifold*

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^N)\setminus\{0\} \mid \mathcal{J}'(u)(u) = 0\}$$

and

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^N)) \mid \gamma(0) = 0, \ \|\gamma(1)\| > r, \ \mathcal{J}(\gamma(1)) < 0\}.$$

The same methods can be applied also in the nonlocal case, and the following existence result can be shown, see e.g. [3,15,16]. In what follows, $r_s > 0$ is the radius chosen so that

$$\inf_{\|u\|_s = r_s} \mathcal{J}_s(u) > 0.$$

Theorem 4.2. *Suppose that assumptions (N), (V), (F1)–(F4) hold and* $1/2 < s < 1$*. Then there exists a ground state solution* $u_s \in H^s(\mathbb{R}^N)$ *to* (1.1)*, i.e. a critical point of the functional* $\mathcal{J}_s$ *given by* (2.3)*, such that*

$$\mathcal{J}_s(u_s) = \inf_{\mathcal{N}_s} \mathcal{J}_s = \inf_{u \in H^s(\mathbb{R}^N)\setminus\{0\}} \sup_{t \geq 0} \mathcal{J}_s(tu) = \inf_{\gamma \in \Gamma_s} \sup_{t \in [0,1]} \mathcal{J}_s(\gamma(t)), \tag{4.1}$$

where $\mathcal{N}_s$ *is the corresponding Nehari manifold*

$$\mathcal{N}_s := \{u \in H^s(\mathbb{R}^N)\setminus\{0\} \mid \mathcal{J}_s'(u)(u) = 0\}$$

and

$$\Gamma_s := \{\gamma \in \mathcal{C}([0,1], H^s(\mathbb{R}^N)) \mid \gamma(0) = 0,\ \|\gamma(1)\| > r_s,\ \mathcal{J}_s(\gamma(1)) < 0\}.$$

5. Non-local to local transition

For any $s \in (1/2, 1)$ we define

$$c_s := \inf_{\mathcal{N}_s} \mathcal{J}_s > 0.$$

Similarly, we put also

$$c := \inf_{\mathcal{N}} \mathcal{J} > 0.$$

For any $v \in H^s(\mathbb{R}^N)\setminus\{0\}$ let $t_s(v) > 0$ be the unique positive real number such that $t_s(v) \in \mathcal{N}_s$. Then, we put $m_s(v) := t_s(v)v$.

Lemma 5.1. *There results*

$$\limsup_{s \to 1^-} c_s \leq c.$$

Proof. Take $u \in H^1(\mathbb{R}^N) \subset H^s(\mathbb{R}^N)$ as a ground state solution of (1.3), in particular $u \in \mathcal{N}$ and $\mathcal{J}(u) = c$, where $\mathcal{J}$ is given by (2.3). Consider the function $m_s(u) \in \mathcal{N}_s$. Obviously

$$c_s \leq \mathcal{J}_s(m_s(u)).$$

Hence

$$\begin{aligned}
\limsup_{s \to 1^-} c_s &\leq \limsup_{s \to 1^-} \mathcal{J}_s(m_s(u)) \\
&= \limsup_{s \to 1^-} \left\{ \mathcal{J}_s(m_s(u)) - \frac{1}{2}\mathcal{J}_s'(m_s(u))(m_s(u)) \right\} \\
&= \limsup_{s \to 1^-} \left\{ \frac{1}{2} \int_{\mathbb{R}^N} f(x, m_s(u))m_s(u) - 2F(x, m_s(u))\,\mathrm{d}x \right\}.
\end{aligned}$$

Recall that $m_s(u) = t_s u$ for some real numbers $t_s > 0$. Suppose by contradiction that $t_s \to +\infty$ as $s \to 1^-$. Then, in view of the Nehari identity

$$\|u\|_s^2 = \int_{\mathbb{R}^N} \frac{f(x, t_s u)}{t_s^2} t_s u\,\mathrm{d}x \geq 2\int_{\mathbb{R}^N} \frac{F(x, t_s u)}{t_s^2 u^2} u^2\,\mathrm{d}x \to +\infty,$$

but the left-hand side stays bounded (see Corollary 2.2). Hence $(t_s)_s$ is bounded. Take any convergent subsequence (t_{s_n}) of (t_s), i.e. $t_{s_n} \to t_0$ as

$n \to +\infty$. Obviously $t_0 \geq 0$. We will show that $t_0 \neq 0$. Indeed, suppose that $t_0 = 0$, i.e. $t_{s_n} \to 0$. Then, in view of the Nehari identity

$$\|u\|_{s_n}^2 = \int_{\mathbb{R}^N} \frac{f(x, t_{s_n} u)}{t_{s_n} u} u^2 \, \mathrm{d}x.$$

By Corollary 2.2, $\|u\|_{s_n}^2 \to \|u\|^2 > 0$. Hence, in view of (F2),

$$\|u\|^2 + o(1) = \int_{\mathbb{R}^N} \frac{f(x, t_{s_n} u)}{t_{s_n} u} u^2 \, \mathrm{d}x \to 0,$$

a contradiction. Hence $t_0 > 0$. Again, by Corollary 2.2,

$$t_{s_n}^2 \|u\|_{s_n}^2 \to t_0^2 \|u\|^2 \quad \text{as } n \to +\infty.$$

Moreover, in view of Remark 1.1,

$$|f(x, t_{s_n} u) t_{s_n} u| \leq \varepsilon t_{s_n}^2 |u|^2 + C_\varepsilon t_{s_n}^p |u|^p \leq C(|u|^2 + |u|^p)$$

for some constant $C > 0$, independent of n. In view of the Lebesgue's convergence theorem,

$$\int_{\mathbb{R}^N} f(x, t_{s_n} u) t_{s_n} u \, \mathrm{d}x \to \int_{\mathbb{R}^N} f(x, t_0 u) t_0 u \, \mathrm{d}x.$$

Thus, the limit t_0 satisfies

$$t_0^2 \|u\|^2 = \int_{\mathbb{R}^N} f(x, t_0 u) t_0 u \, \mathrm{d}x.$$

Taking the Nehari identity into account we see that $t_0 = 1$. Hence $t_s \to 1$ as $s \to 1^-$. Repeating the same argument we see that

$$\begin{aligned} &\limsup_{s \to 1^-} \left\{ \frac{1}{2} \int_{\mathbb{R}^N} f(x, m_s(u)) m_s(u) - 2F(x, m_s(u)) \, \mathrm{d}x \right\} \\ &\quad = \frac{1}{2} \int_{\mathbb{R}^N} f(x, u) u - 2F(x, u) \, \mathrm{d}x = \mathcal{J}(u) = c \end{aligned}$$

and the proof is completed. □

Lemma 5.2. *There exists a constant $M > 0$, such that*

$$\|u_s\|_{L^2(\mathbb{R}^N)} + \|u_s\|_s + \|u_s\|_{L^{\frac{2N}{N-1}}(\mathbb{R}^N)} \leq M$$

for every $s \in (1/2, 1)$.

Proof. Note that $\|u_s\|_{L^2(\mathbb{R}^N)} + \|u_s\|_{L^{\frac{2N}{N-1}}(\mathbb{R}^N)} \leq C\|u_s\|_s$, for some $C > 0$ independent of s. So it is enough to show that $\|u_s\|_s \leq M$. Suppose by contradiction that

$$\|u_s\|_s \to +\infty \quad \text{as } s \to 1^-.$$

Put $v_s := \frac{u_s}{\|u_s\|_s}$. Then $\|v_s\|_s = 1$. In particular, $\{v_s\}$ is bounded in $L^2(\mathbb{R}^N)$. Suppose that

$$\sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |v_s|^2 \, \mathrm{d}x \to 0 \tag{5.1}$$

Then $v_s \to 0$ in $L^p(\mathbb{R}^N)$. Fix any $t > 0$. By (4.1) we obtain

$$\mathcal{J}_{s_n}(u_{s_n}) \geq \mathcal{J}_{s_n}\left(\frac{t}{\|u_{s_n}\|_{s_n}} u_{s_n}\right) = \mathcal{J}_{s_n}(tv_{s_n}) = \frac{t^2}{2} - \int_{\mathbb{R}^N} F(x, tv_{s_n})\, \mathrm{d}x.$$

From Remark 1.1 we see that

$$\begin{aligned}\int_{\mathbb{R}^N} F(x, tv_{s_n})\, \mathrm{d}x &\leq \varepsilon t^2 \|v_{s_n}\|^2_{L^2(\mathbb{R}^N)} + C_\varepsilon t^p \|v_{s_n}\|^p_{L^p(\mathbb{R}^N)} \\ &\to \varepsilon t^2 \limsup_{n\to\infty} \|v_{s_n}\|^2_{L^2(\mathbb{R}^N)}\end{aligned}$$

for every $\varepsilon > 0$. Thus $\int_{\mathbb{R}^N} F(x, tv_{s_n})\, \mathrm{d}x \to 0$ and for any $t > 0$

$$\mathcal{J}_{s_n}(u_{s_n}) \geq \frac{t^2}{2} + o(1),$$

which is a contradiction with the boundedness of $\{\mathcal{J}_{s_n}(u_{s_n})\}_n$. Hence (5.1) does not hold, i.e. there is a sequence $\{z_n\} \subset \mathbb{Z}^N$, such that

$$\liminf_{n\to\infty} \int_{B(z_n, 1+\sqrt{N})} |v_n|^2\, \mathrm{d}x > 0.$$

or, equivalently

$$\liminf_{n\to\infty} \int_{B(0, 1+\sqrt{N})} |v_n(x - z_n)|^2\, \mathrm{d}x > 0.$$

From Theorem 3.2, $v_n(\cdot - z_n) \to v_0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and pointwise a.e., moreover $v_0 \neq 0$. See that, for a.e. $x \in \operatorname{supp} v_0$ we have

$$|u_{s_n}(x - z_n)| = \|u_{s_n}\|_{s_n} |v_{s_n}(x - z_n)| \to +\infty.$$

Thus

$$\begin{aligned} o(1) = \frac{\mathcal{J}_{s_n}(u_{s_n})}{\|u_{s_n}\|^2_{s_n}} &= \frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x, u_{s_n})}{u_{s_n}^2} v_{s_n}^2\, \mathrm{d}x \\ &= \frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x, u_{s_n}(x - z_n))}{u_{s_n}(x - z_n)^2} v_{s_n}(x - z_n)^2\, \mathrm{d}x \\ &\leq \frac{1}{2} - \int_{\operatorname{supp} v_0} \frac{F(x, u_{s_n}(x - z_n))}{u_{s_n}(x - z_n)^2} v_{s_n}(x - z_n)^2\, \mathrm{d}x \to -\infty, \end{aligned}$$

a contradiction. □

Lemma 5.3. *Since $u_s \in \mathcal{N}_s$ there is (independent of s) constant ρ, such that*

$$\|u_s\|_s \geq \rho > 0.$$

Proof. Since $u_s \in \mathcal{N}_s$, we can write by Remark 1.1

$$\begin{aligned}\|u_s\|_s^2 = \int_{\mathbb{R}^N} f(x, u_s) u_s\, \mathrm{d}x &\leq \varepsilon \|u_s\|^2_{L^2(\mathbb{R}^N)} + C_\varepsilon \|u_s\|^p_{L^p(\mathbb{R}^N)} \\ &\leq C\left(\varepsilon \|u_s\|_s^2 + C_\varepsilon \|u_s\|_s^p\right)\end{aligned}$$

for a constant $C > 0$ independent of s. Choosing $\varepsilon > 0$ small enough, we conclude that

$$\|u_s\|_s^{p-2} \geq \frac{1 - C\varepsilon}{C \cdot C_\varepsilon} = \rho > 0.$$

□

Corollary 5.4. *There exist* $u_0 \in H^1(\mathbb{R}^N)$*, a sequence* $\{z_n\}_n \subset \mathbb{Z}^N$ *and a sequence* $\{s_n\}_n$ *such that* $s_n \to 1^-$ *and*

$$u_{s_n}(\cdot - z_n) \to u_0 \neq 0 \quad \text{in } L^\nu_{\text{loc}}(\mathbb{R}^N) \quad \text{as } n \to +\infty$$

for all $\nu \in [2, 2N/(N-1))$.

Proof. From Lemma 5.2 and Theorem 3.2 we note that

$$u_{s_n} \to u_0 \quad \text{in } L^\nu_{\text{loc}}(\mathbb{R}^N) \quad \text{as } n \to +\infty$$

for all $\nu \in [2, 2N/(N-1))$. If $u_0 \neq 0$, we can take $z_n = 0$ and the proof is completed. Otherwise $u_{s_n} \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and therefore, $u_{s_n}(x) \to 0$ for a.e. $x \in \mathbb{R}^N$. Assume that

$$\sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |u_{s_n}|^2 \, \mathrm{d}x \to 0.$$

Then from Theorem 3.1 we know that $u_{s_n} \to 0$ in $L^\nu(\mathbb{R}^N)$ for all $\nu \in [2, 2N/(N-1))$. Then

$$\int_{\mathbb{R}^N} f(x, u_{s_n}) u_{s_n} \, \mathrm{d}x \to 0$$

and $\|u_{s_n}\|^2_{s_n} = \int_{\mathbb{R}^N} f(x, u_{s_n}) u_{s_n} \, \mathrm{d}x \to 0$, which is a contradiction with Lemma 5.3. Hence there is a sequence $\{z_n\} \subset \mathbb{Z}^N$ such that

$$\liminf_{n \to +\infty} \int_{B(0,1+\sqrt{N})} |u_{s_n}(\cdot - z_n)|^2 \, \mathrm{d}x > 0. \tag{5.2}$$

Moreover $\|u_{s_n}(\cdot - z_n)\|_{s_n} = \|u_{s_n}\|_{s_n}$, so that $\|u_{s_n}(\cdot - z_n)\|_{s_n}$ is bounded (see Lemma 5.2). Hence, in view of Theorem 3.2

$$u_{s_n(\cdot - z_n)} \to \tilde{u}_0 \quad \text{in } L^\nu_{\text{loc}}(\mathbb{R}^N) \quad \text{as } n \to +\infty$$

for some $\tilde{u}_0$. Moreover, in view of (5.2), $\tilde{u}_0 \neq 0$. □

Lemma 5.5. *The limit* $u_0 \in H^1(\mathbb{R}^N) \setminus \{0\}$ *is a weak solution for* (1.3).

Proof. Take any test function $\varphi \in C_0^\infty(\mathbb{R}^N)$ and note that by [20, Section 6] we have

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u_{s_n} (-\Delta)^{s_n/2} \varphi \, \mathrm{d}x = \int_{\mathbb{R}^N} u_{s_n} (-\Delta)^{s_n} \varphi \, \mathrm{d}x.$$

Moreover

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} u_{s_n} (-\Delta)^{s_n} \varphi \, \mathrm{d}x - \int_{\mathbb{R}^N} u_0 (-\Delta \varphi) \, \mathrm{d}x \right| \\ &\quad = \left| \int_{\mathbb{R}^N} u_{s_n} \left((-\Delta)^{s_n} \varphi - (-\Delta \varphi) \right) \, \mathrm{d}x + \int_{\operatorname{supp} \varphi} (u_{s_n} - u_0)(-\Delta \varphi) \, \mathrm{d}x \right| \\ &\quad \leq \|u_{s_n}\|_{L^2(\mathbb{R}^N)} \, \|(-\Delta)^{s_n} \varphi - (-\Delta \varphi)\|_{L^2(\mathbb{R}^N)} \\ &\qquad + \|(-\Delta \varphi)\|_{L^2(\mathbb{R}^N)} \|u_{s_n} - u_0\|_{L^2(\operatorname{supp} \varphi)} \to 0. \end{aligned}$$

Hence

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}(-\Delta)^{s_n/2}u_{s_n}(-\Delta)^{s_n/2}\varphi\,\mathrm{d}x=\int_{\mathbb{R}^N}u_0(-\Delta\varphi)\,\mathrm{d}x$$
$$=\int_{\mathbb{R}^N}\nabla u_0\cdot\nabla\varphi\,\mathrm{d}x.$$

Obviously

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}V(x)u_{s_n}\varphi\,\mathrm{d}x=\lim_{n\to+\infty}\int_{\operatorname{supp}\varphi}V(x)u_{s_n}\varphi\,\mathrm{d}x=\int_{\mathbb{R}^N}V(x)u_0\varphi\,\mathrm{d}x.$$

Take any measurable set $E\subset\operatorname{supp}\varphi$ and note that, taking into account Remark 1.1,

$$\int_E|f(x,u_{s_n})\varphi|\,\mathrm{d}x\le\varepsilon\|u_{s_n}\|_{L^2(\mathbb{R}^N)}\|\varphi\chi_E\|_{L^2(\operatorname{supp}\varphi)}$$
$$+C_\varepsilon\|u_{s_n}\|^{p-1}_{L^p(\mathbb{R}^N)}\|\varphi\chi_E\|_{L^p(\operatorname{supp}\varphi)}.$$

Hence, the family $\{f(\cdot,u_{s_n})\varphi\}_n$ is uniformly integrable on $\operatorname{supp}\varphi$ and in view of the Vitali convergence theorem

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}f(x,u_{s_n})\varphi\,\mathrm{d}x=\int_{\mathbb{R}^N}f(x,u_0)\varphi\,\mathrm{d}x.$$

Therefore u_0 satisfies

$$\int_{\mathbb{R}^N}\nabla u_0\cdot\nabla\varphi\,\mathrm{d}x+\int_{\mathbb{R}^N}V(x)u_0\varphi\,\mathrm{d}x=\int_{\mathbb{R}^N}f(x,u_0)\varphi\,\mathrm{d}x,$$

i.e. u_0 is a weak solution to (1.3). □

Proof of Theorem 1.2. Recalling Corollary 5.4 and Lemma 5.5, it is sufficient to check that u_0 is a ground state solution, i.e. $\mathcal{J}(u_0)=c$. From Lemma 5.5 it follows that $u_0\in H^1(\mathbb{R}^N)\setminus\{0\}$ is a weak solution, so that $u_0\in\mathcal{N}$. Note that, from Corollary 5.4 and Fatou's lemma,

$$\begin{aligned}\liminf_{n\to+\infty}c_{s_n}&=\liminf_{n\to+\infty}\mathcal{J}_{s_n}(u_{s_n})=\liminf_{n\to+\infty}\left\{\mathcal{J}_{s_n}(u_{s_n})-\frac{1}{2}\mathcal{J}'_{s_n}(u_{s_n})(u_{s_n})\right\}\\&=\liminf_{n\to+\infty}\left\{\frac{1}{2}\int_{\mathbb{R}^N}f(x,u_{s_n})u_{s_n}-2F(x,u_{s_n})\,\mathrm{d}x\right\}\\&=\liminf_{n\to+\infty}\left\{\frac{1}{2}\int_{\mathbb{R}^N}f(x,u_{s_n}(\cdot-z_n))u_{s_n}(\cdot-z_n)-2F(x,u_{s_n}(\cdot-z_n))\,\mathrm{d}x\right\}\\&\ge\frac{1}{2}\int_{\mathbb{R}^N}f(x,u_0)u_0-2F(x,u_0)\,\mathrm{d}x=\mathcal{J}(u_0)\ge c.\end{aligned}$$

Taking into account Lemma 5.1 we see that

$$c\le\mathcal{J}(u_0)\le\liminf_{n\to+\infty}c_{s_n}\le\limsup_{n\to+\infty}c_{s_n}\le c$$

Hence $\lim_{n\to+\infty}c_{s_n}$ exists and $\lim_{n\to+\infty}c_{s_n}=c=\mathcal{J}(u_0)$. □

Acknowledgements

The authors would like to thank an anonymous referee for several valuable comments helping to improve the original version of the manuscript. Bartosz Bieganowski was partially supported by the National Science Centre, Poland (Grant No. 2017/25/N/ST1/00531). Simone Secchi is member of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM).

Funding Open access funding provided by Università degli Studi di Milano - Bicocca within the CRUI-CARE Agreement.

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