Balance Laws with Singular Source Term and Applications to Fluid Dynamics

Rinaldo M. Colombo, Graziano Guerra, and Yannick Holle

Abstract Consider a balance law where the flux may depend explicitly on the space variable. At jump discontinuities, modeling considerations may impose the defect in the conservation of some quantities, thus leading to non conservative products. Below, we deduce the evolution in the smooth case from the jump conditions at discontinuities. Moreover, the resulting framework enjoys well posedness and solutions are uniquely characterized. These results apply, for instance, to the flow of water in a canal with varying width and depth, as well as to the inviscid Euler equations in pipes with varying geometry.

1 Motivation

When a gas flows through a straight pipe, its dynamics can be described by the 2×2 system of isentropic gas dynamics, left system in (1), or, taking into account the heat flow, by the full Euler 3×3 system of conservation laws, right system in (1). In (1), ρ is the density of the fluid, v is the velocity, e is the energy density (per unit mass) and p is the pressure.

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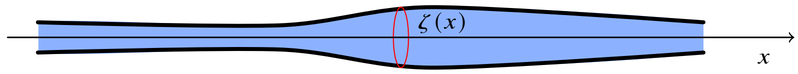

Fig. 1 Pipe with cross section ζ that changes smoothly with respect to the spatial variable x

$$\begin{cases}\partial_t \rho + \partial_x (\rho v) = 0,\\ \partial_t (\rho v) + \partial_x \left(p + \rho v^2\right) = 0,\end{cases} \qquad \begin{cases}\partial_t \rho + \partial_x (\rho v) = 0,\\ \partial_t (\rho v) + \partial_x \left[p + \rho v^2\right] = 0,\\ \partial_t \left[\frac{1}{2}\rho v^2 + \rho e\right] + \partial_x \left[v\left(\rho e + p + \frac{1}{2}\rho v^2\right)\right] = 0\,.\end{cases} \tag{1}$$

Both systems fit into the framework of $n\times n$ one dimensional systems of conservation laws

$$\partial_t u + \partial_x f(u) = 0, \qquad u(t,x) \in \mathbb{R}^n, \quad t \geq 0, \quad x \in \mathbb{R},$$

where u is the unknown vector of densities of conserved quantities and it depends on the two scalar independent variables (t,x), which represent respectively time and the one dimensional spatial variable. If the cross section ζ of the pipe changes smoothly with respect to the spatial variable, see Figure 1, then the physical system can be modeled introducing a non zero source term in the right hand side of the equations. The source term depends on the cross section ζ and on its derivative ζ'. The 2×2 system of one dimensional isentropic gas dynamics with varying cross section is given by:

$$\begin{cases}\partial_t \rho + \partial_x (\rho v) = -\frac{\zeta'}{\zeta}\rho v,\\ \partial_t (\rho v) + \partial_x \left(p + \rho v^2\right) = -\frac{\zeta'}{\zeta}\rho v^2\,.\end{cases} \tag{2}$$

This system is included in the general form for $n\times n$ systems of **balance** laws

$$\partial_t u + \partial_x f(u) = g(\zeta(x), u)\,, \qquad u(t,x) \in \mathbb{R}^n, \quad t \geq 0, \quad x \in \mathbb{R},$$

which have been extensively studied, see for instance [1, 2, 5, 13, 20, 21].

Remark that (2) admits also the following form, where we introduce the linear mass density $\zeta\rho$ and the linear momentum density $\zeta\rho v$:

$$\begin{cases}\partial_t (\zeta\rho) + \partial_x (\zeta\rho v) = 0,\\ \partial_t (\zeta\rho v) + \partial_x \left(\zeta p + \zeta\rho v^2\right) = \zeta' p\,.\end{cases}$$

The mass is (obviously) conserved but the linear momentum is not. This lack of conservation is due to the pressure of the fluid against the non straight pipe walls. Therefore, the product in the source term is non conservative and needs to be appropriately defined, see [14], as soon as ζ' is a measure and p is discontinuous.

If there is a discontinuity in the geometry of the pipe, for instance a discontinuous change in the cross section as in Figure 2, then the dynamics can be modeled by two systems of conservation laws restricted to $x < 0$ and to $x > 0$. At $x = 0$ a further condition is necessary to prescribe the possible defect in the conservation of the otherwise conserved quantities:

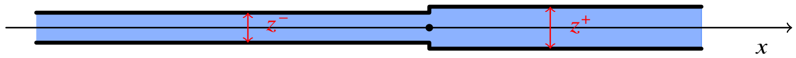

Fig. 2 Pipe with a discontinuity in the cross section, z^- being the cross section of the pipe to the left of the discontinuity and z^+ the cross section to the right.

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & \text{for } x < 0 \\ \partial_t u + \partial_x f(u) = 0, & \text{for } x > 0 \\ \Psi\left(z^+, u\left(t, 0+\right), z^-, u\left(t, 0-\right)\right) = 0 & \text{for } x = 0, \quad t \geq 0\,. \end{cases}$$

Under suitable hypotheses, see [7, Lemma 4.1 and 4.2], this three equations are equivalent to a system of balance laws with a suitable weighted Dirac delta centered at the origin as a source term:

$$\partial_t u + \partial_x f\left(u\right) = \Xi\left(z^+, z^-, u\left(t, 0-\right)\right)\delta_0\,. \tag{3}$$

Here and in what follows, δ_x is the Dirac delta centered at x.

The weight Ξ describes the defect in the conservation and depends on the cross sections at the sides of the discontinuity z^- and z^+ and on the value of the unknown on the left of the discontinuity $u\left(t, 0-\right)$.

More generally, one could have several discontinuities in the pipe profile, say at the points $x_1, \ldots, x_N$ with cross section described by $\zeta = \zeta^N(x)$, see Figure 3. Therefore, we have a sum of Dirac deltas in the source term in the corresponding

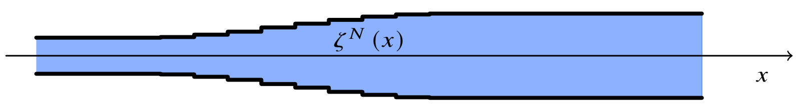

Fig. 3 Pipe with several discontinuities

modeling equations:

$$\partial_t u^N + \partial_x f\left(u^N\right) = \sum_{i=1}^{N} \Xi\left(\zeta^N\left(x_i+\right), \zeta^N\left(x_i-\right), u^N\left(t, x_i-\right)\right)\delta_{x_i}. \tag{4}$$

When, as $N \to +\infty$, this finite sum of Dirac deltas tends, in some strong sense specified in [7, Proof of Theorem 2.2, Step 1], to a measure g, [7, Theorem 2.2] shows that the limit of the solutions u^N is a solution of a system of balance laws with g as source term. Moreover, in the same Theorem, the connection between the jump condition described by Ξ and the limit source term g is given. Specific sample cases of this limit were studied, for instance, in [10, 11, 16].

A different geometrical feature in one dimensional fluid dynamics is a kink in the pipe profile, see Figure 4. In [9, 17] the kink was modeled by the 2×2 system of isentropic gas dynamics provided with a measure source term given by a Dirac delta times a weight dependent on the change in the orientation of the pipe:

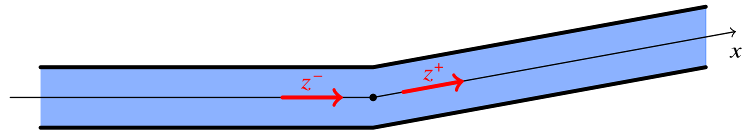

Fig. 4 Pipe with a kink at a point, resulting in a discontinuity in the vector z describing the pipe orientation.

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0, \\ \partial_t (\rho v) + \partial_x \left(p + \rho v^2\right) = -K \left\| z^+ - z^- \right\| \rho v \, \delta_0 . \end{cases}$$

This equation is included in the general formulation (3) choosing

$$\Xi \left(z^+, z^-, (\rho, v)\right) = \begin{pmatrix} 0 \\ -K \left\| z^+ - z^- \right\| \rho v \end{pmatrix}.$$

Note that Ξ is not differentiable with respect to the geometric parameters of the system z^- and z^+. On the other hand, it has one sided directional derivatives. To allow these kind of geometrical structures, in Theorem 1 below, we do not require differentiability of Ξ but only the existence of one sided directional derivatives (see Definition (13) and hypothesis **(Ξ.4)**).

A pipe with N kinks, see Figure 5, can again be modeled by (4), ζ^N being the

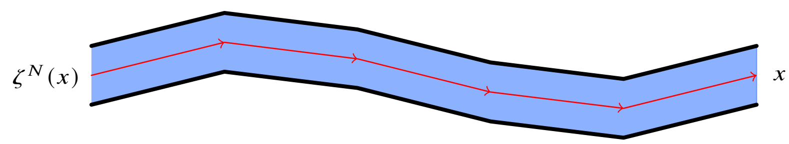

Fig. 5 A pipe with a finite number of kinks

piecewise constant pipe's orientation. In [7, Theorem 3.1] it is shown that when $N \to +\infty$ and, in this limit, we have a smoothly curved pipe, the limit measure source term g depends on the curvature of the pipe. Indeed, we have

$$g = -K \left\| \zeta' \right\| \mathcal{L} = -K \left\| \Gamma'' \right\| \mathcal{L},$$

$\mathcal{L}$ being the Lebesgue measure and $s \mapsto \Gamma(s)$ being the curve describing the pipe profile parameterized by arc length.

As a final example we consider the flow of water in a canal of smoothly varying width and smoothly varying bed elevation. Its dynamics can be described by the following balance law

$$\begin{cases} \partial_t a + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\dfrac{q^2}{a} + \dfrac{1}{2} g \dfrac{a^2}{\sigma} \right) = \dfrac{1}{2} g \dfrac{a^2}{\sigma^2} \partial_x \sigma - g \, a \, \partial_x b , \end{cases} \tag{5}$$

see [22, Formula (1.1)]. Here g is gravity, t is time, x is the longitudinal coordinate along the canal, $a = a(t, x)$ is the wetted cross sectional area, $q = q(t, x)$ is the water flow, $\sigma = \sigma(x)$ is the canal width and $b = b(x)$ is the height of the bottom.

The presence of discontinuities in the channel width σ or in the bed elevation b prevents the application of standard theorems to (5). Indeed, discontinuities arise in the flux and non conservative products appear in the source term. As is well known, these products lack a unique way to be defined. As a reference to non conservative products, we refer to [15, 19].

In our framework, system (5) is meaningful and is well posed too, requiring σ and b to be merely of bounded variation. Indeed, our results comprise also the case of balance laws with a space dependent flux and a non conservative product in the source term of the type

$$\partial_t u + \partial_x f(\zeta, u) = D_\zeta G(\zeta, u)\ D\zeta \tag{6}$$

see [7, § 3.4]. Setting $u = (a, q)$, $p = 2$, and

$$\begin{aligned} f\,(a, q) &= \begin{bmatrix} q \\ \frac{q^2}{a} + \frac{1}{2} g \zeta_1 a^2 \end{bmatrix}, \quad \zeta(x) = \begin{bmatrix} 1/\sigma(x) \\ b(x) \end{bmatrix}, \\ G\,(z, (a, q)) &= \begin{bmatrix} 0 \\ -\frac{1}{2}\, g\, a^2\, z_1 - g\, a\, z_2 \end{bmatrix}, \end{aligned} \tag{7}$$

we see that (5) fits into (6):

$$\begin{cases} \partial_t a + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\dfrac{q^2}{a} + \dfrac{1}{2}\, g\, \zeta_1\, a^2 \right) = -\dfrac{1}{2}\, g\, a^2\, \partial_x \zeta_1 - g\, a\, \partial_x \zeta_2 \end{cases} \tag{8}$$

and hence our main result, Theorem 1, applies setting, for instance,

$$\Xi(z^+, z^-, u^-) = G(z^+, u^-) - G(z^-, u^-)\,. \tag{9}$$

As noted in [7, Section 3], different choices of Ξ may yield different solutions emanating from discontinuities in ζ while giving the same solutions wherever ζ is smooth, see [6] for a related result.

The choice (9) actually singles out the source term in (11) below, which accounts both for the smooth changes as well as for the points of jump in ζ. In the case of (5), this amounts to show that a careful choice of Ξ allows to extend (5) to the case of σ and b in **BV**.

2 Main Results

Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{Z} \subset \mathbb{R}^p$ be open and convex. Consider a left continuous function $\zeta \in \mathbf{BV}(\mathbb{R}; \mathcal{Z})$ and denote by $\mathcal{I}(\zeta)$ the set of jump discontinuities in ζ.

Consider a piecewise constant approximation ζ^h of ζ and the following balance law with measure-valued source term

$$\begin{cases} \partial_t u + \partial_x f(\zeta^h, u) = \sum_{\bar{x} \in \mathcal{I}(\zeta^h)} \Xi\left(\zeta^h(\bar{x}+), \zeta^h(\bar{x}-), u(\cdot, \bar{x}-)\right) \; \delta_{\bar{x}} \\ u(0, x) = u_o(x) \, . \end{cases} \tag{10}$$

In the general - non characteristic - setting established below, solutions to (10) are shown to converge as ζ^h converges to ζ in a suitable - strong - sense described in [7, Proof of Theorem 2.2, Step 1], to solutions to

$$\begin{cases} \partial_t u + \partial_x f(\zeta, u) = \sum_{\bar{x} \in \mathcal{I}(\zeta)} \Xi\left(\zeta(\bar{x}+), \zeta(\bar{x}-), u(\cdot, \bar{x}-)\right) \delta_{\bar{x}} + D_v^+ \Xi(\zeta, \zeta, u) \; \|\mu\| \\ u(0, x) = u_o(x) \, . \end{cases} \tag{11}$$

The terms in the singular source above are defined as follows. Since $\zeta \in \mathbf{BV}(\mathbb{R}; \mathbb{R}^p)$, the right and left limits $\zeta(\bar{x}+)$ and $\zeta(\bar{x}-)$ are well defined and the distributional derivative $D\zeta$ can be split in a discrete part and a non discrete one, which may contain a Cantor part:

$$D\zeta = \sum_{\bar{x} \in \mathcal{I}(\zeta)} \left(\zeta(\bar{x}+) - \zeta(\bar{x}-)\right) \; \delta_{\bar{x}} + v \; \|\mu\| \; , \tag{12}$$

where the function v is Borel measurable with norm 1 and μ is the non atomic part of $D\zeta$. In (11) we also used the (one sided) directional Dini derivative

$$D_v^+ \Xi(z, z, u) = \lim_{t \to 0+} \frac{\Xi(z + t\, v, z, u) - \Xi(z, z, u)}{t} \, . \tag{13}$$

We require the following assumptions on f, Ξ and ζ:

(f.1) $f \in \mathbf{C}^4(\mathcal{Z} \times \Omega; \mathbb{R}^n)$;
(f.2) the Jacobian matrix $D_u f(z, u)$ is strictly hyperbolic for every $z \in \mathcal{Z}$ and $u \in \Omega$;
(f.3) each characteristic field is either genuinely nonlinear or linearly degenerate for all $z \in \mathcal{Z}$.

In the latter assumption we refer to the definitions by Lax [18], see also [12, § 7.5].

By **(f.1)** and **(f.2)** we know that, possibly restricting Ω, the eigenvalues $\lambda_1(z, u)$, $\ldots$,$\lambda_n(z, u)$ of $D_u f(z, u)$ depend smoothly on z and can be indexed so that, for all $u \in \Omega$ and $z \in \mathcal{Z}$, $\lambda_1(z, u) < \lambda_2(z, u) < \cdots < \lambda_n(z, u)$. We thus require the usual non resonance condition

(f.4) there exists $i_o \in \{1, \ldots, n-1\}$ such that $\lambda_{i_o}(z, u) < 0 < \lambda_{i_o+1}(z, u)$ for all $z \in \mathcal{Z}$ and all $u \in \Omega$.

Note that both the cases of characteristic speeds being either all positive or all negative are simpler.

(Ξ.1) $\Xi\colon \mathcal{Z} \times \mathcal{Z} \to \mathbf{C}^1(\overline{\Omega}; \mathbb{R}^n)$ is a Lipschitz continuous map and $\Xi\colon \mathcal{Z} \times \mathcal{Z} \to \mathbf{C}^2(\overline{\Omega}; \mathbb{R}^n)$;

(Ξ.2) $\sup_{z^+,z^-\in\mathcal{Z}} \|\Xi(z^+,z^-,\cdot)\|_{\mathbf{C}^2(\Omega;\mathbb{R})} < +\infty$;
(Ξ.3) $\Xi(z,z,u) = 0$ for every $z \in \mathcal{Z}$ and $u \in \Omega$;
(Ξ.4) there exists a non decreasing map $\sigma\colon [0,\bar t[\to \mathbb{R}$ with $\lim_{t\to 0}\sigma(t) = 0$ such that for all $(z,v,u) \in \mathcal{Z} \times \overline{B(0;1)} \times \Omega$, $\left\|\Xi(z+t\,v,z,u) - D_v^+\Xi(z,z,u)\,t\right\| \le \sigma(t)\,t$ and moreover the map $(z,v,u) \to D_v^+\Xi(z,z,u)$ is Lipschitz continuous.

We now precisely state what we mean by *solution* to (11).

Definition 1 Let $u_o \in \mathbf{L}^1_{loc}(\mathbb{R};\mathbb{R}^n)$. A map $u \in \mathbf{C}^0([0,+\infty[;\mathbf{L}^1_{loc}(\mathbb{R};\mathbb{R}^n))$ with $u(t) \in \mathbf{BV}(\mathbb{R};\mathbb{R}^n)$ and left continuous for all $t \in \mathbb{R}_+$, is a *solution* to (11) if for all test functions $\phi \in \mathbf{C}^1_c(]0,+\infty[\times\mathbb{R};\mathbb{R})$,

$$\begin{aligned} &-\int_0^{+\infty}\int_{\mathbb{R}} \left(u(t,x)\,\partial_t\phi(t,x) + f\left(\zeta(x),u(t,x)\right)\,\partial_x\phi(t,x)\right)\mathrm{d}x\,\mathrm{d}t \\ &= \sum_{\bar x\in I(\zeta)}\int_0^{+\infty} \Xi\left(\zeta(\bar x+),\zeta(\bar x),u(t,\bar x)\right)\phi(t,\bar x)\,\mathrm{d}t \\ &\quad+\int_0^{+\infty}\int_{\mathbb{R}} D^+_{v(x)}\Xi\left(\zeta(x),\zeta(x),u(t,x)\right)\phi(t,x)\,\mathrm{d}\|\mu\|(x)\,\mathrm{d}t \end{aligned} \tag{14}$$

where v, μ are as in (12), and moreover $u(0) = u_o$.

In the last integral in (14), the integrand is Borel measurable in (t,x) since, for instance, by the above assumptions on u, we have at *every* $(t,x) \in \mathbb{R}_+\times\mathbb{R}$

$$u(t,x) = \lim_{h\to 0}\frac{1}{h}\int_{x-h}^{x} u(t,y)\,\mathrm{d}y\,.$$

Moreover, Borel measurability on $\mathbb{R}^2$ ensures measurability with respect to the product measure.

Note that the value of the integrand in the first line in (14) is independent of changes of the integrand on sets of *Lebesgue* measure 0 in $\mathbb{R}^2$, while the latter integrand is integrated with respect to the product measure $\|\mu\|\otimes\mathrm{d}t$. Nevertheless, (14) is meaningful, since u is prescribed *pointwise*, at *every* point and not merely almost everywhere.

The above definition is known not to guarantee uniqueness. On the contrary, Theorem 1 below does guarantee uniqueness, relying on an extension to the case of (11) of the precise characterization originally provided in [3] for homogeneous systems of conservation laws.

Definition 2 By *Generalized Riemann Problem* we mean the Cauchy Problem (11) with ζ and the initial datum u_o as follows:

$$\zeta(x) = z^-\,\chi_{\mathbb{R}_-}(x) + z^+\,\chi_{\mathbb{R}_+}(x) \quad\text{and}\quad u_o(x) = u^\ell\,\chi_{\mathbb{R}_-}(x) + u^r\,\chi_{\mathbb{R}_+}(x)\,. \tag{15}$$

For $z \in \mathcal{Z}$ and $u \in \Omega$, call $\sigma_i \to H_i(z,\sigma_i)(u)$ the Lax curve of the i–th family relative to $f(z,\cdot)$ exiting u, see [4, § 5.2] or [12, § 9.3]. Introduce recursively the states $w_0,\ldots,w_{n+1} \in \Omega$ with $w_0 = u^\ell$, $w_{n+1} = u^r$ and

$$\begin{cases} w_{i+1} = H_{i+1}(z^+, \sigma_{i+1})(w_i) & \text{if } i = 0, \dots, i_o - 1, \\ f(z^+, w_{i_o+1}) - f(z^-, w_{i_o}) = \Xi(z^+, z^-, w_{i_o}) & \\ w_{i+1} = H_i(z^-, \sigma_i)(w_i) & \text{if } i = i_o + 1, \dots, n\,. \end{cases}$$

We thus define as *Admissible Solution* to the Generalized Riemann Problem (11)–(15) the gluing along $x = 0$ of the Lax solutions to the (standard) Riemann Problems

$$\begin{cases} \partial_t u + \partial_x f(z^-, u) = 0 \\ u(0,x) = u^\ell \chi_{\mathbb{R}_-}(x) + w_{i_o} \chi_{\mathbb{R}_+}(x), \end{cases} \qquad \begin{cases} \partial_t u + \partial_x f(z^+, u) = 0 \\ u(0,x) = w_{i_o+1} \chi_{\mathbb{R}_-}(x) + u^r \chi_{\mathbb{R}_+}(x). \end{cases}$$

Throughout, we refer to the stationary jump discontinuities due to jumps in z as to *zero waves*. [8, Lemma 3.3] ensures that, with the above definition, the Generalized Riemann Problem (11)–(15) turns out to be well posed.

Aiming at the characterization of solutions to (11), we now extend to the present case the general definitions introduced in [3], see also [4, Chapter 9]. Fix $\zeta \in \mathbf{BV}(\mathbb{R}; \mathcal{Z})$, a function $u = u(t,x)$ with $u(t) \in \mathbf{BV}(\mathbb{R}; \Omega)$ for all t and a point $(\tau, \xi) \in [0, +\infty[\, \times \mathbb{R}$. Define the function $U^\sharp_{(u;\tau,\xi)}$ as the solution to the generalized Riemann Problem

$$\begin{cases} \partial_t U + \partial_x f\left(\zeta(\xi), U\right) = \Xi\left(\zeta(\xi+), \zeta(\xi), u(t, \xi-)\right)\, \delta_\xi \\ U(0,x) = \begin{cases} u(\tau, \xi-) & x < \xi\,; \\ u(\tau, \xi+) & x > \xi\,. \end{cases} \end{cases} \tag{16}$$

Note that if $\xi \notin \mathcal{I}(\zeta)$, then the right hand side in (16) vanishes due to **(Ξ.3)** and the above definition of $U^\sharp_{(u;\tau,\xi)}$ reduces to the classical one in [3, Chapter 9] related to the homogeneous flow $u \to f\left(\zeta(\xi), u\right)$.

We define the function $U^\flat_{(u;\tau,\xi)}$ as the solution to the following linear hyperbolic problem with constant coefficients and measure-valued source term

$$\begin{cases} \partial_t U + A\, \partial_x U = g \\ U(0,x) = u(\tau, x) \end{cases} \tag{17}$$

with $A = D_u f\left(\zeta(\xi), u(\tau, \xi)\right)$ and for any Borel subset E of $\mathbb{R}$,

$$\begin{aligned} g(E) = \sum_{\bar{x} \in \mathcal{I}(\zeta)} \Big(& \Xi\left(\zeta(\bar{x}+), \zeta(\bar{x}), u(\tau, \xi)\right) \\ & - f\left(\zeta(\bar{x}+), u(\tau, \xi)\right) + f\left(\zeta(\bar{x}), u(\tau, \xi)\right) \Big) \delta_{\bar{x}}(E) \\ & + \int_E \Big(D^+_{v(x)} \Xi\left(\zeta(x), \zeta(x), u(\tau, \xi)\right) - D_z f\left(\zeta(x), u(\tau, \xi)\right)\, v(x) \Big)\, \mathrm{d}\|\mu\|\,(x). \end{aligned} \tag{18}$$

where we used the same notation as in (12) and (13).

We are now ready to state the main result.

Theorem 1 ([8, Theorem 2.3])

Let f satisfy **(f.1)–(f.4)***, Ξ satisfy* **(Ξ.1)–(Ξ.4)***. Fix $\bar z \in \mathcal{Z}$, $\bar u \in \Omega$. Then, there exist positive δ and L such that for any $\zeta \in \mathbf{BV}(\mathbb{R};\mathcal{Z})$ with $\mathbf{TV}(\zeta) < \delta$ and $\|\zeta(x) - \bar z\| < \delta$ there exist a domain $\mathcal{D}^\zeta \subseteq \bar u + \mathbf{L}^1(\mathbb{R};\Omega)$ containing all functions u in $\bar u + \mathbf{L}^1(\mathbb{R};\Omega)$ with $\mathbf{TV}(u) < \delta$ and a semigroup $S^\zeta : \mathbb{R}_+ \times \mathcal{D}^\zeta \to \mathcal{D}^\zeta$ such that*

1. *For all $u_o \in \mathcal{D}^\zeta$, the orbit $t \to S_t^\zeta u_o$ solves* (11) *in the sense of Definition 1.*
2. *S^ζ is $\mathbf{L}^1$–Lipschitz continuous, i.e. for all $u_o, u_o^1, u_o^2 \in \mathcal{D}^\zeta$ and for all $t, t_1, t_2 \in \mathbb{R}_+$*

$$\left\|S_t^\zeta u_o^1 - S_t^\zeta u_o^2\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \le L \, \left\|u_o^1 - u_o^2\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} ;$$
$$\left\|S_{t_1}^\zeta u_o - S_{t_2}^\zeta u_o\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \le L \, |t_1 - t_2| \, .$$

3. *If $\zeta \in \mathbf{PC}(\mathbb{R};\mathcal{Z})$ and $u_o \in \mathbf{PC}(\mathbb{R};\Omega)$, then for t sufficiently small, the map $(t,x) \to (S_t^\zeta u_o)(x)$ coincides with the gluing of Admissible Solutions, in the sense of Definition 2, to Generalized Riemann Problems at the points of jumps of u_o and of ζ.*

Moreover, let $\hat\lambda$ be an upper bound for the (moduli of) characteristic speeds and define $u(t,x) = (S_t^\zeta u_o)(x)$. Then, for every $(\tau,\xi) \in \mathbb{R}_+ \times \mathbb{R}$,

(i)

$$\lim_{\theta\to 0} \frac{1}{\theta} \int_{\xi-\theta\hat\lambda}^{\xi+\theta\hat\lambda} \left| u(\tau+\theta,x) - U^\sharp_{(u;\tau,\xi)}(\theta,x) \right| \mathrm{d}x = 0 \, .$$

(ii) There exists a constant C such that for every $a, b \in \mathbb{R}$ with $a < \xi < b$ and for every $\theta \in \,]0, (b-a)/(2\hat\lambda)[$,

$$\begin{aligned} &\frac{1}{\theta} \int_{a+\theta\hat\lambda}^{b-\theta\hat\lambda} \left| u(\tau+\theta,x) - U^\flat_{(u;\tau,\xi)}(\theta,x) \right| \mathrm{d}x \\ &\le C \left[\mathbf{TV}\left(u(\tau),]a,b[\right) + \mathbf{TV}\left(\zeta,]a,b[\right)\right]^2 . \end{aligned}$$

If $u \colon [0,T] \to \mathcal{D}^\zeta$ is $\mathbf{L}^1$–Lipschitz continuous and satisfies (i) and (ii) for almost every time τ and for all $\xi \in \mathbb{R}$, then $t \to u(t,\cdot)$ coincides with an orbit of the semigroup S^ζ.

Note that whenever ζ is piecewise constant, the properties 1., 2. and 3. above uniquely characterize the semigroup S^ζ, see [8, Lemma 3.14].

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