

Existence results for nonconvex equilibrium problems*

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Abstract. In this paper we establish sufficient conditions for the existence of solutions of equilibrium problems in a metric space that do not involve any convexity assumption neither for the domain nor for the function. To prove these results a weak notion of semicontinuity is considered. Furthermore, some existence results for systems of equilibrium problems are provided.

Keywords: Equilibrium problems; generalized semicontinuity; coercivity conditions.

1 Introduction

Let D be a subset of a metric space (X, d) and $f : D \times D \rightarrow \mathbb{R}$ be a given bifunction. By equilibrium problem we understand the problem of finding

$$\bar{x} \in D \text{ such that } f(\bar{x}, y) \geq 0 \text{ for all } y \in D. \quad (\text{EP})$$

Classical examples of equilibrium problems are scalar and vector optimization problems, variational inequalities, Nash equilibria problems, fixed point problems, and saddle points problems.

Problems like (EP) were initially studied by Ky Fan [9] and by Brezis, Nirenberg, and Stampacchia [7]. After the paper of Blum and Oettli [6], where the term *equilibrium problem* was used for the first time, many mathematicians studied in depth equilibrium problems. Convexity and closure of the set D and generalized convexity and monotonicity, together with some continuity for f , were the most used conditions in dealing with the existence of the solutions of equilibrium problems (see [10, 12] and references therein).

In order to avoid any assumption of convexity both for the domain D and for the bifunction f , some authors (see for instance [1, 3, 6, 14] and references therein) proposed a different approach in which the existence of solutions for (EP) is obtained assuming the following triangle inequality property:

$$f(x, y) \leq f(x, z) + f(z, y), \quad \forall x, y, z \in D. \quad (1)$$

In this paper, we focus on the properties of the class of bifunctions

$$\mathcal{A} = \{f : D \times D \rightarrow \mathbb{R} : \text{property (1) holds}\},$$

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and we derive new existence theorems for (EP) assuming $f \in \mathcal{A}$. Such results extend and generalize several results recently appeared in the literature. Furthermore, we prove existence of solutions for systems of equilibrium problems by exploiting the results obtained for (EP).

2 Properties of class \mathcal{A}

Some authors recently considered particular classes of bifunctions contained in \mathcal{A} , e.g. τ - functions [13], fitting functions [14], and Q -functions [1]. Exploiting the properties of these functions they established various generalizations of the Ekeland's variational principle and they deduced existence results for nonconvex equilibrium problems.

Before to extend such results, we deepen the study of the class \mathcal{A} that is obviously not empty since every distance function belongs to \mathcal{A} . But the class \mathcal{A} is quite wide as the following results show.

Proposition 2.1. (i) Let $\varphi_1, \varphi_2 : D \rightarrow \mathbb{R}$ such that $\varphi_1(x) + \varphi_2(x) \geq 0$ for all $x \in D$, then $f(x, y) = \varphi_1(x) + \varphi_2(y)$ belongs to \mathcal{A} .

(ii) Let $\varphi : D \rightarrow \mathbb{R}$ be subadditive i.e. $\varphi(x + y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in D$, then $f(x, y) = \varphi(x - y)$ belongs to \mathcal{A} .

(iii) If $f \in \mathcal{A}$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing and subadditive, then $\varphi \circ f \in \mathcal{A}$.

(iv) Let $g : D \times D \rightarrow \mathbb{R}$ be a bounded bifunction, that is there exist $L, U \in \mathbb{R}$ such that

$$L \leq g(x, y) \leq U \quad \forall (x, y) \in D \times D,$$

then $f(x, y) = g(x, y) + \alpha$ belongs to \mathcal{A} for all $\alpha \geq U - 2L$.

Proof. The proof of the first three statements is straightforward and it is omitted; we prove only (iv). Fixed $x, y, z \in D$ we have

$$\begin{aligned} f(x, z) + f(z, y) &= g(x, z) + g(z, y) + 2\alpha \\ &\geq g(x, z) + g(z, y) - 2L + U + \alpha \\ &\geq U + \alpha \geq g(x, y) + \alpha = f(x, y) \end{aligned}$$

and the proof is completed. □

We now show two well-known special cases of equilibrium problems with the bifunction $f \in \mathcal{A}$. It is known that the problem of finding a minimizer of a function $\varphi : D \rightarrow \mathbb{R}$ is equivalent to (EP) setting $f(x, y) = \varphi(y) - \varphi(x)$. In such a case it is easy to check that $f \in -\mathcal{A} \cap \mathcal{A}$.

Given a function $\psi : D \rightarrow \mathbb{R}^m$ and a closed convex cone $K \subset \mathbb{R}^m$ with nonempty interior $\text{int}(K)$, the vector minimization problem defined as

$$\text{find } \bar{x} \in D \text{ such that } \psi(\bar{x}) - \psi(y) \notin \text{int}(K) \text{ for all } y \in D,$$

is equivalent to (EP) setting

$$f(x, y) = \max_{z \in K^+, \|z\|=1} \langle z, \psi(y) - \psi(x) \rangle,$$

where K^+ is the positive polar cone of K . Also in this case $f \in \mathcal{A}$; indeed for any $x, y, z \in D$ there exists $\bar{z} \in K^+$ with $\|\bar{z}\| = 1$ such that

$$f(x, y) = \langle \bar{z}, \psi(y) - \psi(x) \rangle = \langle \bar{z}, \psi(z) - \psi(x) \rangle + \langle \bar{z}, \psi(y) - \psi(z) \rangle \leq f(x, z) + f(z, y).$$

Notice that each bifunction $f \in \mathcal{A}$ is nonnegative on the diagonal of $D \times D$ indeed; for each $x, y \in D$, we have

$$f(x, y) \leq f(x, x) + f(x, y) \quad \Rightarrow \quad f(x, x) \geq 0.$$

As immediate consequence we have

$$f(x, y) + f(y, x) \geq f(x, x) \geq 0, \quad \forall x, y \in D;$$

therefore condition (1) implies the cyclic monotonicity of $-f$ that extends a similar condition given for variational inequalities: for every $x_1, x_2, \dots, x_k \in D$ we have

$$\sum_{i=1}^k f(x_i, x_{i+1}) \geq 0,$$

where $x_{k+1} = x_1$ (see also [3]).

Now we show two results on the algebraic structure of \mathcal{A} .

Proposition 2.2. (i) \mathcal{A} is a convex cone, i.e. if $\lambda \geq 0$ and $f \in \mathcal{A}$, then $\lambda f \in \mathcal{A}$, and if $f_1, f_2 \in \mathcal{A}$ then $f_1 + f_2 \in \mathcal{A}$.

(ii) $f \in -\mathcal{A} \cap \mathcal{A}$ if and only if there exists $\varphi : D \rightarrow \mathbb{R}$ such that $f(x, y) = \varphi(y) - \varphi(x)$.

Proof. The proof of the first statement is trivial and we prove the second one only. As mentioned above, every bifunction $f(x, y) = \varphi(y) - \varphi(x)$, with $\varphi : D \rightarrow \mathbb{R}$, belongs to $-\mathcal{A} \cap \mathcal{A}$. For the converse, if $f \in -\mathcal{A} \cap \mathcal{A}$ then

$$f(x, z) = f(x, y) + f(y, z), \quad \forall x, y, z \in D.$$

Hence, fixed $\bar{z} \in D$ and put $\varphi = -f(\cdot, \bar{z})$ we have

$$f(x, y) = f(x, \bar{z}) - f(y, \bar{z}) = \varphi(y) - \varphi(x), \quad \forall x, y \in D,$$

which concludes the proof. □

Remark 2.1. Proposition 2.2 shows that the class of equilibrium problems with f belonging to the lineality space of \mathcal{A} coincides with the class of optimization problems.

Proposition 2.3. If I is a finite set of indices and $\{f_i\}_{i \in I} \subseteq \mathcal{A}$, then $f(x, y) = \max_{i \in I} f_i(x, y)$ belongs to \mathcal{A} .

Proof. Let $x, y, z \in D$ be fixed; then there exists $i \in I$ such that

$$f(x, y) = f_i(x, y) \leq f_i(x, z) + f_i(z, y) \leq f(x, z) + f(z, y),$$

and the proof is completed. □

The next result concerns the semicontinuity of the elements of \mathcal{A} .

Proposition 2.4. *Let $f \in \mathcal{A}$ be zero on the diagonal of $D \times D$ i.e. $f(z, z) = 0$ for each $z \in D$. If $f(\cdot, y)$ is upper semicontinuous at $y \in D$, then $f(x, \cdot)$ is lower semicontinuous at y for every $x \in D$.*

Proof. Let $x, y \in D$ and $\varepsilon > 0$ be fixed. From the upper semicontinuity of the function $f(\cdot, y)$ at y , we deduce the existence of $r > 0$ such that

$$f(z, y) \leq f(y, y) + \varepsilon = \varepsilon, \quad \forall z \in B(y, r),$$

where $B(y, r)$ is the open ball with center y and radius r . Hence, since f belongs to \mathcal{A} we achieve

$$f(x, y) \leq f(x, z) + f(z, y) \leq f(x, z) + \varepsilon, \quad \forall z \in B(y, r),$$

that completes the proof. \square

Remark 2.2. The assumption that f is zero on the diagonal can not be deleted. Indeed, if we choose two nonnegative upper semicontinuous functions φ_1 and φ_2 defined on D , then the bifunction $f(x, y) = \varphi_1(x) + \varphi_2(y)$ belongs to \mathcal{A} and $f(\cdot, y)$ is upper semicontinuous, but $f(x, \cdot)$ is not necessarily lower semicontinuous. Furthermore, we remark that the reverse implication in Proposition 2.4 is not true. Indeed, let us consider the sequence of functions $\varphi_n : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$\varphi_n(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ nx & \text{if } 0 < x < 1/n, \\ 1 & \text{if } x \geq 1/n. \end{cases}$$

Then the bifunction $f(x, y) = \sup_{n \in \mathbb{N}} [\varphi_n(x) - \varphi_n(y)]$ belongs to \mathcal{A} , it is zero on the diagonal, and $f(x, \cdot)$ is continuous for any x . Indeed, if $x \leq 0$ then

$$f(x, y) = \begin{cases} 0 & \text{if } y \leq 0, \\ -y & \text{if } 0 < y < 1, \\ -1 & \text{if } y \geq 1; \end{cases}$$

if $x \in [1/(m+1), 1/m)$ for some $m \in \mathbb{N}$, then

$$f(x, y) = \begin{cases} 1 & \text{if } y \leq 0, \\ 1 - (m+1)y & \text{if } 0 < y \leq 1 - mx, \\ m(x-y) & \text{if } 1 - mx < y < x, \\ 0 & \text{if } y \geq x; \end{cases}$$

finally, if $x \geq 1$ then

$$f(x, y) = \begin{cases} 1 & \text{if } y \leq 0, \\ 1 - y & \text{if } 0 < y < 1, \\ 0 & \text{if } y \geq 1. \end{cases}$$

However, since

$$f(x, 0) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0, \end{cases}$$

it follows that $f(\cdot, 0)$ is not upper semicontinuous at 0.

3 Existence of equilibria

In this section we prove some new existence results for (EP) with the bifunction f belonging to \mathcal{A} . First we recall the following continuity definitions which will be useful in the following.

Definition 3.1. [8, 13] A function $\varphi : X \rightarrow \mathbb{R}$ is said to be

- *lower semicontinuous from above at $x_0 \in X$* if, for any sequence $\{x_n\} \subset X$ converging to x_0 and satisfying $\varphi(x_{n+1}) \leq \varphi(x_n)$, for all $n \in \mathbb{N}$, we have $\varphi(x_0) \leq \lim_{n \rightarrow +\infty} \varphi(x_n)$;
- *upper semicontinuous from below at $x_0 \in X$* if, for any sequence $\{x_n\} \subset X$ converging to x_0 and satisfying $\varphi(x_{n+1}) \geq \varphi(x_n)$, for all $n \in \mathbb{N}$, we have $\varphi(x_0) \geq \lim_{n \rightarrow +\infty} \varphi(x_n)$.

The function φ is said to be lower semicontinuous from above (resp. upper semicontinuous from below) on X if it is lower semicontinuous from above (resp. upper semicontinuous from below) at every point of X .

It is clear that the lower (resp. upper) semicontinuity implies the lower semicontinuity from above (resp. upper semicontinuity from below). The following example shows that the reverse implications do not hold.

Example 3.1. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be defined as follows

$$\varphi(x) = \begin{cases} x - 1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ x + 1 & \text{if } x > 0. \end{cases}$$

Observe that φ is lower semicontinuous from above and upper semicontinuous from below at 0, but it is neither lower nor upper semicontinuous at 0.

We start considering (EP) in which the set D is compact but not necessarily convex. In [8] the authors showed that the Weierstrass' Theorem for minimum holds also changing the lower semicontinuity with the weaker assumption of lower semicontinuity from above. Using this fact we are able to prove the following result.

Theorem 3.1. *Suppose that $f \in \mathcal{A}$ and D is compact. If there exists $\bar{z} \in D$ such that $f(\bar{z}, \cdot)$ is lower semicontinuous from above, then the set of solutions of (EP) is nonempty.*

Proof. Fixed $\bar{z} \in D$, consider the optimization problem

$$\min_{y \in D} f(\bar{z}, y). \tag{2}$$

Since D is compact, the lower semicontinuity from above of $f(\bar{z}, \cdot)$ guarantees the existence of a minimum point $\bar{x} \in D$. Since $f \in \mathcal{A}$ and \bar{x} solves (2) then

$$f(\bar{x}, y) \geq f(\bar{z}, y) - f(\bar{z}, \bar{x}) \geq 0, \quad \forall y \in D,$$

i.e. \bar{x} is a solution of (EP). □

Remark 3.1. Theorem 3.1 generalizes Theorem 5.1 in [1] in the following ways.

- (a) We do not require the completeness of the metric space X .
- (b) We do not require $f(x, \cdot)$ is lower semicontinuous from above and lower bounded for all $x \in D$.
- (c) We do not require the upper semicontinuity of $f(\cdot, y)$ for each fixed $y \in D$. However, it is easy to prove that this additional assumption guarantees the set of solutions of (EP) is closed.

In the following result we show that the assumption of lower semicontinuity from above of $f(\bar{z}, \cdot)$ can be replaced by the upper semicontinuity from below of $f(\cdot, \bar{z})$.

Theorem 3.2. *Suppose that $f \in \mathcal{A}$ and D is compact; if there exists $\bar{z} \in D$ such that $f(\cdot, \bar{z})$ is upper semicontinuous from below, then the set of solutions of (EP) is nonempty.*

Proof. It follows as to the previous one applying Weierstrass' Theorem to the optimization problem

$$\max_{x \in D} f(x, \bar{z}).$$

If $\bar{x} \in D$ is an optimal solution, then

$$f(\bar{x}, y) \geq f(\bar{x}, \bar{z}) - f(y, \bar{z}) \geq 0, \quad \forall y \in D,$$

and the proof is completed. □

We now consider the case when D is not compact. If X is a finite dimensional Euclidean space and D is closed and convex, interesting results were presented in [10]. Instead, in order to deduce existence results in the infinite dimensional case, we need to restrict our setting and, in what follows, we suppose that the space X admits a topology τ such that the following two assumptions are satisfied:

- (A1) the closed balls are compact with respect to τ ;
- (A2) for every $y \in X$ the distance function $d(\cdot, y)$ is lower semicontinuous with respect to τ .

For instance conditions (A1) and (A2) are satisfied if X is a reflexive Banach space and τ is the weak topology. From now on, all the topological concepts will be considered with respect to the topology τ .

The study of the existence of solutions of the equilibrium problems on noncompact domains D usually involves some coercivity conditions. In [5] some coercivity conditions were introduced to prove existence results for (EP) under generalized monotonicity properties of f and the weakest condition was used in [3, 4] in order to deduce existence results. We recall this definition:

$$\begin{aligned} & \text{there exist } x_0 \in X \text{ and a compact set } C \subseteq D \text{ such that} \\ & \forall x \in D \setminus C, \exists y \in D \text{ such that } d(y, x_0) < d(x, x_0) \text{ and } f(x, y) \leq 0. \end{aligned} \tag{3}$$

We now show the existence of solutions of (EP) under coercivity condition (3) exploiting Theorem 3.1.

Theorem 3.3. *Suppose that $f \in \mathcal{A}$ and D is closed. If coercivity condition (3) holds and $f(x, \cdot)$ is lower semicontinuous from above for all $x \in C$, then the set of solutions of (EP) is nonempty.*

Proof. Applying Theorem 3.1 to the equilibrium problem with domain C we deduce that there exists a point $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in C. \quad (4)$$

We now prove that \bar{x} is a solution of (EP). Let us consider any fixed $x \in D \setminus C$ and define the set

$$L = \{y \in D : d(y, x_0) \leq d(x, x_0)\}.$$

The set L is compact by Assumption (A1), thus the set

$$M = \arg \min_{y \in L} f(\bar{x}, y)$$

is nonempty and compact since $f(\bar{x}, \cdot)$ is lower semicontinuous from above. From Assumption (A2) it follows that there exists $\bar{y} \in M$ solving the optimization problem

$$\min_{y \in M} d(y, x_0). \quad (5)$$

We now prove that $\bar{y} \in C$. Suppose, by contradiction, that $\bar{y} \in D \setminus C$. Then coercivity condition (3) guarantees that there exists $z \in D$ such that $d(z, x_0) < d(\bar{y}, x_0)$ and $f(\bar{y}, z) \leq 0$. Therefore we obtain that $z \in L$ and

$$f(\bar{x}, z) \leq f(\bar{x}, \bar{y}) + f(\bar{y}, z) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y) \quad \forall y \in L.$$

Thus $z \in M$ and $d(z, x_0) < d(\bar{y}, x_0)$, but this is impossible because \bar{y} solves (5). Hence $\bar{y} \in C$. Finally, since $x \in L$, $\bar{y} \in M$, and \bar{x} satisfies (4), we obtain

$$f(\bar{x}, x) \geq f(\bar{x}, \bar{y}) \geq 0,$$

and the proof is complete. \square

Remark 3.2. When $X = \mathbb{R}^N$ assumptions (A1) and (A2) are trivially satisfied from a metric topology and coercivity condition (3) assumes the following form:

$$\exists r > 0 : \forall x \in D \text{ with } \|x\| > r, \exists y \in D \text{ with } \|y\| < \|x\| : f(x, y) \leq 0. \quad (6)$$

In this setting, we note that Theorem 3.3 generalizes Theorem 4.1 in [3] in the following directions.

- (a) We do not need the assumption that f is zero on the diagonal of $D \times D$.
- (b) We assume that $f(x, \cdot)$ is lower semicontinuous from above (instead of lower semicontinuous) for any $x \in C$ (instead of for any $x \in D$).
- (c) We do not require $f(x, \cdot)$ is lower bounded.
- (d) The upper semicontinuity of $f(\cdot, y)$ is not required. As observed in the previous remark, this assumption guarantees the set of solutions of (EP) is closed.

Furthermore, it is easy to check that if we strengthen coercivity condition (3) replacing $f(x, y) \leq 0$ with $f(x, y) < 0$, then the set of solutions of (EP) turns out to be bounded as well.

4 Existence of solutions for systems of equilibrium problems

Systems of equilibrium problems were introduced in [2]. Later, existence results for systems of equilibrium problems in Euclidean spaces were obtained in [3]. In this section we show new existence results for systems of equilibrium problems in Euclidean spaces as consequence of the results for (EP) proved in Section 3.

Given a finite set $I = \{1, \dots, m\}$, by a system of equilibrium problems we understand the problem of finding

$$\bar{x} = (\bar{x}_1, \dots, \bar{x}_m) \in D \text{ such that } f_i(\bar{x}, y_i) \geq 0 \text{ for all } i \in I \text{ and } y_i \in D_i, \quad (\text{SEP})$$

where $f_i : D \times D_i \rightarrow \mathbb{R}$, $D = \prod_{i \in I} D_i$, $D_i \subset \mathbb{R}^{n_i}$, and $N = n_1 + \dots + n_m$. We shall consider the space \mathbb{R}^N endowed with the Chebyshev norm, i.e. $\|x\| = \max_{i \in I} \|x_i\|$ for any $x = (x_1, \dots, x_m) \in \mathbb{R}^N$.

The following result is crucial to relate (SEP) with (EP).

Lemma 4.1. *Consider the bifunction $f : D \times D \rightarrow \mathbb{R}$ defined as*

$$f(x, y) = \sum_{i \in I} f_i(x, y_i).$$

Then the following hold:

(i) *If $f_i(x, x_i) = 0$ for every $i \in I$ and $x \in D$, then \bar{x} solves (SEP) if and only if \bar{x} solves (EP).*

(ii) *If for every $i \in I$ we have*

$$f_i(x, y_i) \leq f_i(x, z_i) + f_i(z, y_i), \quad \forall x, y, z \in D, \quad (7)$$

then $f \in \mathcal{A}$.

Proof. (i) If \bar{x} solves (SEP) then it is trivial to prove that \bar{x} solves (EP). Vice versa, assume that \bar{x} solves (EP). If $i \in I$ and $y_i \in D_i$ are fixed, then the vector $z \in D$ defined as

$$z_j = \begin{cases} y_i & \text{if } j = i, \\ \bar{x}_j & \text{otherwise,} \end{cases}$$

is such that $0 \leq f(\bar{x}, z) = f_i(\bar{x}, y_i)$, thus \bar{x} solves (SEP).

(ii) For all $x, y, z \in D$ we have

$$f(x, y) = \sum_{i \in I} f_i(x, y_i) \leq \sum_{i \in I} f_i(x, z_i) + \sum_{i \in I} f_i(z, y_i) = f(x, z) + f(z, y),$$

i.e. $f \in \mathcal{A}$. □

From Lemma 4.1 and existence theorems for (EP) proved in section 3, we can derive the following results for the existence of solutions of (SEP) on both compact and noncompact sets. First, we consider the case in which the sets D_i are compact but not necessarily convex.

Theorem 4.1. *Suppose that for every $i \in I$ the set D_i is compact, the bifunction f_i satisfies (7), and $f_i(x, x_i) = 0$ for every $x \in D$. If there exists $\bar{z} \in D$ such that $f_i(\bar{z}, \cdot)$ is lower semicontinuous for every $i \in I$, then the set of solutions of (SEP) is nonempty.*

Proof. It is a direct consequence of Lemma 4.1 and Theorem 3.1. \square

Remark 4.1. Theorem 4.1 generalizes Proposition 3.3 in [3] in the following directions.

- (a) We do not require that $f_i(x, \cdot)$ is lower semicontinuous for every $i \in I$ and $x \in D$.
- (b) We do not require that $f_i(\cdot, y_i)$ is upper semicontinuous for every $i \in I$ and $y_i \in D_i$. Note that, adding this assumption in Theorem 4.1, the set of solutions of (SEP) turns out to be closed, as well.
- (c) Note that, in order to exploit Theorem 3.1, we assume that $f_i(\bar{z}, \cdot)$ is lower semicontinuous instead of lower semicontinuous from above, since the sum of lower semicontinuous from above functions is not necessarily lower semicontinuous from above. Indeed, the functions

$$f_1(x) = \begin{cases} x - 1 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ x + 1 & \text{if } x > 0 \end{cases} \quad f_2(x) = \begin{cases} -x + 1 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ -x - 1 & \text{if } x > 0 \end{cases}$$

are lower semicontinuous from above at 0, but $f_1 + f_2$ is not.

Theorem 4.2. *Suppose that for every $i \in I$ the set D_i is compact, the bifunction f_i satisfies (7), and $f_i(x, x_i) = 0$ for every $x \in D$. If there exists $\bar{z} \in D$ such that $f_i(\cdot, \bar{z}_i)$ is upper semicontinuous for every $i \in I$, then the set of solutions of (SEP) is nonempty.*

Proof. It is a direct consequence of Lemma 4.1 and Theorem 3.2. \square

When the sets D_i are not necessarily compact, we consider the following coercivity condition introduced in [3]:

$$\begin{aligned} \exists r > 0 : \forall x \in D \text{ such that } \|x_i\| > r \text{ for some } i \in I, \\ \exists y_i \in D_i \text{ such that } \|y_i\| < \|x_i\| \text{ and } f_i(x, y_i) \leq 0. \end{aligned} \tag{8}$$

Theorem 4.3. *Suppose that for every $i \in I$ the set D_i is closed, the bifunction f_i satisfies (7), $f_i(x, x_i) = 0$ for every $x \in D$, and $f_i(x, \cdot)$ is lower semicontinuous for every $x \in D$ with $\|x\| \leq r$. If coercivity condition (8) holds, then the set of solutions of (SEP) is nonempty.*

Proof. The result follows from Lemma 4.1 and Theorem 3.3, because coercivity condition (8) implies coercivity condition (3) in the form (6). Indeed, if $x \in D$ and $\|x\| > r$, then the vector $z \in D$ defined as

$$z_i = \begin{cases} y_i \text{ given by (8)} & \text{if } \|x_i\| > r, \\ x_i & \text{otherwise,} \end{cases}$$

is such that $\|z\| < \|x\|$ and $f(x, z) = \sum_{i: \|x_i\| > r} f_i(x, y_i) \leq 0$. \square

Remark 4.2. Theorem 4.3 generalizes Theorem 4.2 in [3] in the following directions.

- (a) For every $i \in I$ we assume the lower semicontinuity of $f_i(x, \cdot)$ only for every $x \in D$ with $\|x\| \leq r$.
- (b) We do not require $f_i(x, \cdot)$ is lower bounded.
- (c) We do not require the upper semicontinuity of $f_i(\cdot, y_i)$ for every $i \in I$ and $y_i \in D_i$. As mentioned above, adding this assumption in Theorem 4.3 the set of solutions of (SEP) turns out to be closed, as well.

Finally, if we strengthen coercivity condition (8) replacing $f_i(x, y) \leq 0$ with $f_i(x, y) < 0$, then the set of solutions of (SEP) turns out to be bounded, as well.

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