SOBOLEV-TYPE REGULARITY AND POHOZAEV-TYPE IDENTITIES FOR SOME DEGENERATE AND SINGULAR PROBLEMS

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Abstract. Sobolev-type regularity results are proved for solutions to a class of second order elliptic equations with a singular or degenerate weight, under non-homogeneous Neumann conditions. As an application a Pohozaev-type identity for weak solutions is derived.

Keywords. Degenerate and singular elliptic equations, Sobolev-type regularity, Pohozaev-type identities.

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1. Introduction

This note is concerned with the following class of second order elliptic equations

$$-\operatorname{div}(t^{1-2s}A(x,t)\nabla U(x,t)) + t^{1-2s}c(x,t) = 0, \quad x \in \mathbb{R}^N,\ t \in (0,+\infty), \tag{1.1}$$

with the weight t^{1-2s} (being $s \in (0,1)$) which belongs to the second Muckenhoupt class and is singular if $s > 1/2$ and degenerate if $s < 1/2$; we couple (1.1) with non-homogeneous Neumann conditions

$$\lim_{t\to 0^+} t^{1-2s}A(x,t)\nabla U(x,t)\cdot \nu = hU(x,0) + g(x) \tag{1.2}$$

on the bottom of a half $(N+1)$-dimensional ball.

The interest in such a type of equations and related regularity issues has developed starting from the pioneering paper [9], proving local Hölder continuity results and Harnack's inequalities, and has grown significantly in recent years stimulated by the study of the fractional Laplacian in its realization as a Dirichlet-to-Neumann map [3].

In this context, among recent regularity results for problems of type (1.1)–(1.2), we mention [2] and [14] for Schauder and gradient estimates with A being the identity matrix and $c \equiv 0$. More general degenerate/singular equations of type (1.1), admitting a varying coefficient matrix A, are considered in [22, 23]. In [22], under suitable regularity assumptions on A and c, Hölder continuity and $C^{1,\alpha}$-regularity are established for solutions to (1.1)–(1.2) in the case $h \equiv g \equiv 0$, which, up to a reflection through the hyperspace $t = 0$, corresponds to the study

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of solutions to the equation $-\operatorname{div}(|t|^{1-2s}A\nabla U) + |t|^{1-2s}c = 0$ which are even with respect to the t-variable; Hölder continuity of solutions which are odd in t is instead investigated in [23]. In addition, in [22] $C^{0,\alpha}$ and $C^{1,\alpha}$ bounds are derived for some inhomogeneous Neumann boundary problems (i.e. for $g \not\equiv 0$) in the case $c \equiv 0$. We also mention [7, 8] for regularity results in weighted Sobolev spaces and mixed-norm weighted Sobolev spaces for a class of singular or degenerate parabolic and elliptic equations in the upper half space.

The goal of the present note is to derive Sobolev-type regularity results for solutions to (1.1)–(1.2). Under suitable assumptions on c, h, g, the presence of the singular/degenerate homogenous weight, involving only the $(N+1)$-th variable t, makes the solutions to have derivates with respect to the first N variables $x_1, x_2, \dots, x_N$ belonging to a weighted H^1-space (with the same weight t^{1-2s}); concerning the regularity of the derivative with respect to t, we obtain instead that the weighted derivative $t^{1-2s}\frac{\partial U}{\partial t}$ belongs to a H^1-space with the dual weight t^{2s-1}, confirming what has already been observed in [22, Lemma 7.1] for even solutions of the reflected problem corresponding to (1.1)–(1.2) with $h \equiv g \equiv 0$.

Our motivation for studying this question lies in the search for the minimal regularity needed to prove Pohozaev-type identities for solutions of the extended problem, resulting from the Caffarelli-Silvestre extension for the fractional Laplacian; Pohozaev-type identities can in turn be used to obtain Almgren-type monotonicity formulas in the spirit of [10]. Indeed, the Sobolev-type regularity results obtained in Theorem 2.1 allow us to directly obtain a Pohozaev-type identity (Proposition 2.3), without requiring C^1-regularity for the potential h as in [10] and without approximating potentials in Sobolev spaces with smooth ones as done in [5]. Furthermore, the presence of the matrix A makes our results applicable even to the problem modified by a diffeomorphic deformation of the domain, which straightens a $C^{1,1}$-boundary and produces the appearance of a variable coefficient matrix A, satisfying conditions (2.4), (2.5), and (2.6); such a procedure is useful to study the behaviour of solutions at the boundary, see e.g. [5]. In the forthcoming paper [4] the Pohozaev-type identity established in Proposition 2.3 will be used to derive Almgren type monotonicity formulas and unique continuation from the boundary for solutions of an extended problem associated to the spectral fractional Laplacian.

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2. Statement of the main results

Let $s \in (0,1)$, $N \in \mathbb{N}$, $N > 2s$ and $z = (x,t) \in \mathbb{R}^N \times [0,\infty)$. Let

$$\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times (0,\infty),$$

and, for any $r > 0$,

$$B_r^+ := \{z \in \mathbb{R}^{N+1}_+ : |z| < r\}, \quad B'_r = \{x \in \mathbb{R}^N : |x| < r\},$$
$$S_r^+ := \{z \in \mathbb{R}^{N+1}_+ : |z| = r\}, \quad S'_r := \{x \in \mathbb{R}^N : |x| = r\}.$$

For all $r > 0$ and $\phi \in C^\infty(\overline{B_r^+})$ we define

$$\|\phi\|_{H^1(B_r^+, t^{1-2s})} := \left(\int_{B_r^+} t^{1-2s}(\phi^2 + |\nabla\phi|^2)\, dz\right)^{\frac{1}{2}} \tag{2.1}$$

and $H^1(B_r^+, t^{1-2s})$ as the completion of $C^\infty(\overline{B_r^+})$ with respect to the norm defined in (2.1). Thanks to [16, Theorem 11.11, Theorem 11.2, 11.12 Remarks (iii)], for any $r > 0$, the space $H^1(B_r^+, t^{1-2s})$ can be explicitly characterized as

$$H^1(B_r^+, t^{1-2s}) = \left\{ w \in W^{1,1}_{\rm loc}(B_r^+) : \int_{B_r^+} t^{1-2s}(w^2 + |\nabla w|^2)\, dz < +\infty \right\}.$$

We observe that $H^1(B_r^+, t^{1-2s}) \subset W^{1,1}(B_r^+)$, hence, denoting as Tr the classical trace operator from $W^{1,1}(B_r^+)$ to $L^1(B'_r)$, we may consider its restriction to $H^1(B_r^+, t^{1-2s})$; furthermore, for any $r > 0$, such a restriction (still denoted as Tr) turns out to be a linear, continuous trace operator

$$\mathrm{Tr} : H^1(B_r^+, t^{1-2s}) \to H^s(B'_r) \tag{2.2}$$

which is onto, see [1, 18], [14, Proposition 2.1], and [19, Theorem 2.8], where $H^s(B'_r)$ denotes the usual fractional Sobolev space.

Let $R > 0$ and let ν be the outer normal unit vector to B_R^+ on B'_R, that is $\nu(x) = (0, \dots, 0, -1)$ for any $x \in B'_R$. We are interested in proving Sobolev-type regularity results for a weak solution $U \in H^1(B_R^+, t^{1-2s})$ of the problem

$$\begin{cases} -\operatorname{div}(t^{1-2s} A\nabla U) + t^{1-2s} c = 0, & \text{on } B_R^+, \\ \lim_{t\to 0^+} t^{1-2s} A\nabla U \cdot \nu = h\,\mathrm{Tr}(U) + g, & \text{on } B'_R, \end{cases} \tag{2.3}$$

under suitable regularity hypotheses on the matrix-valued function A and the functions c, h, g. More precisely we make the following assumptions:

$$A(z) = \left(\begin{array}{c|c} B(z) & 0 \\ \hline 0 & \alpha(z) \end{array}\right) \quad \text{for any } z \in \overline{B_R^+}, \tag{2.4}$$

$$B \in W^{1,\infty}(B_R^+, \mathbb{R}^{N\times N}) \text{ is symmetric}, \quad \alpha \in W^{1,\infty}(B_R^+, \mathbb{R}), \tag{2.5}$$

$$\text{there exist } \lambda_1, \lambda_2 > 0 \text{ s.t. } \lambda_1|y|^2 \le A(z)y\cdot y \le \lambda_2|y|^2 \tag{2.6}$$
$$\text{for all } z \in \overline{B_R^+} \text{ and } y \in \mathbb{R}^{N+1},$$

$$g \in W^{1,\frac{2N}{N+2s}}(B'_R), \quad h \in W^{1,\frac{N}{2s}}(B'_R), \tag{2.7}$$

$$c \in L^2(B_R^+, t^{1-2s}). \tag{2.8}$$

To state the weak formulation of (2.3), let, for any $r > 0$,

$$H^1_{0,S_r^+}(B_r^+, t^{1-2s}) := \overline{\{\phi \in C^\infty(\overline{B_r^+}) : \phi = 0 \text{ on } S_r^+\}}^{\|\cdot\|_{H^1(B_r^+, t^{1-2s})}}. \tag{2.9}$$

Under this conditions, a weak solution of (2.3) is a function $U \in H^1(B_R^+, t^{1-2s})$ such that

$$\int_{B_R^+} t^{1-2s} A\nabla U \cdot \nabla \phi \, dz + \int_{B_R^+} t^{1-2s} c\phi \, dz = \int_{B_R'} [g + h\,\mathrm{Tr}(U)]\,\mathrm{Tr}(\phi)\, dx, \tag{2.10}$$

for any $\phi \in H^1_{0,S_R^+}(B_R^+, t^{1-2s})$.

The above definition is well posed since each term in (2.10) is finite, thanks to inequality (3.2), see Remark 3.4.

Our main result is the following theorem.

Theorem 2.1. *Let U be a weak solution of* (2.3) *in the sense of* (2.10). *If assumptions* (2.4), (2.5), (2.6), (2.7), (2.8) *are satisfied, then*

$$\nabla_x U \in H^1(B_r^+, t^{1-2s}) \quad \text{and} \quad t^{1-2s}\frac{\partial U}{\partial t} \in H^1(B_r^+, t^{2s-1}) \tag{2.11}$$

for all $r \in (0, R)$. Furthermore

$$\|\nabla_x U\|_{H^1(B_r^+, t^{1-2s})} + \left\| t^{1-2s}\frac{\partial U}{\partial t}\right\|_{H^1(B_r^+, t^{2s-1})} \leq C\left(\|U\|_{H^1(B_R^+, t^{1-2s})} + \|c\|_{L^2(B_R^+, t^{1-2s})} + \|g\|_{W^{1,\frac{2N}{N+2s}}(B_R')}\right) \tag{2.12}$$

for a positive constant $C > 0$ depending only on N, s, r, R, $\|h\|_{W^{1,\frac{N}{2s}}(B_R')}$, λ_1, $\|A\|_{W^{1,\infty}(B_R^+, \mathbb{R}^{(N+1)^2})}$ (but independent of U).

The proof of Theorem 2.1 is based on the classical Nirenberg difference quotient method [20].

As an application of Theorem 2.1 we prove a Pohozaev-type identity for weak solutions of (2.3). To this aim we require that the matrix-valued function A satisfies, besides assumptions (2.4), (2.5), and (2.6), also the condition

$$A(0) = \mathrm{Id}_{N+1} \tag{2.13}$$

where Id_{N+1} is the identity $(N+1) \times (N+1)$ matrix.

We first introduce some notation. Let

$$\mu(z) := \frac{A(z)z \cdot z}{|z|^2} \quad \text{and} \quad \beta(z) := \frac{A(z)z}{\mu(z)} \quad \text{for any } z \in \overline{B_R^+} \setminus \{0\}, \tag{2.14}$$
$$\beta'(x) := \beta(x, 0) \quad \text{for any } x \in \overline{B_R'} \setminus \{0\}.$$

We also define $dA(z)yy$, for every $y = (y_1, \dots, y_{N+1}) \in \mathbb{R}^{N+1}$ and $z \in \overline{B_R^+}$, as the vector of $\mathbb{R}^{N+1}$ with i-th component given by

$$(dA(z)yy)_i = \sum_{h,k=1}^{N+1} \frac{\partial a_{kh}}{\partial z_i}(z) y_h y_k, \quad i = 1, \dots, N+1, \tag{2.15}$$

where we have defined the matrix $A = (a_{kh})_{k,h=1,\dots,N+1}$ in (2.4).

Remark 2.2. From (2.5), (2.6), and (2.13) it easily follows that

$$\mu \in C^{0,1}(\overline{B_R^+}), \quad \frac{1}{\mu} \in C^{0,1}(\overline{B_R^+}), \quad \beta \in C^{0,1}(\overline{B_R^+}, \mathbb{R}^{N+1}),$$
$$J_\beta \in L^\infty(B_R^+, \mathbb{R}^{(N+1)^2}), \quad \mathrm{div}(\beta) \in L^\infty(B_R^+),$$
$$\beta' \in L^\infty(B_R', \mathbb{R}^N), \quad \mathrm{div}(\beta') \in L^\infty(B_R'), \tag{2.16}$$

where J_β is the Jacobian matrix of β.

Proposition 2.3. *Under assumptions* (2.4), (2.5), (2.6), (2.7), (2.8), *and* (2.13), *let U be a solution of* (2.10). *Then for a.e.* $r \in (0, R)$

$$\begin{aligned} \frac{r}{2}\int_{S_r^+} t^{1-2s} A\nabla U \cdot \nabla U \, dS - r\int_{S_r^+} t^{1-2s}\frac{|A\nabla U\cdot \nu|^2}{\mu}\, dS \\ + \frac{1}{2}\int_{B_r'} (\mathrm{div}_x(\beta')h + \beta'\cdot\nabla h)|\,\mathrm{Tr}(U)|^2\,dx - \frac{r}{2}\int_{S_r'} h|\,\mathrm{Tr}(U)|^2\,dS' \\ + \int_{B_r'} (\mathrm{div}_x(\beta')g + \beta'\cdot\nabla g)\,\mathrm{Tr}(U)\,dx - r\int_{S_r'} g\,\mathrm{Tr}(U)\,dS' \\ = \frac{1}{2}\int_{B_r^+} t^{1-2s} A\nabla U\cdot\nabla U\,\mathrm{div}(\beta)\,dz - \int_{B_r^+} t^{1-2s} c(\nabla U\cdot\beta)\,dz \\ - \int_{B_r^+} t^{1-2s} J_\beta(A\nabla U)\cdot\nabla U\,dz + \frac{1}{2}\int_{B_r^+} t^{1-2s}(dA\nabla U\nabla U)\cdot\beta\,dz \\ + \frac{1-2s}{2}\int_{B_r^+} t^{1-2s}\frac{\alpha}{\mu}A\nabla U\cdot\nabla U\,dz, \end{aligned} \tag{2.17}$$

where ν is the outer normal vector to B_r^+ on S_r^+, that is $\nu(z) = \frac{z}{|z|}$.

Remark 2.4. The two integrals in the first line of (2.17) must be understood for a.e. $r \in (0, R)$ as explained in Remark 3.8.

The integrals over S_r' in (2.17) can be instead understood in the classical trace sense. Indeed, $h \in W^{1,\frac{N}{2s}}(B_r')$ by (2.7) and $(\mathrm{Tr}(U))^2 \in W^{1,\frac{N}{N-2s}}(B_r')$ thanks to (2.11) and (3.5); then h has a trace on S_r' belonging to $L^{\frac{N}{2s}}(S_r')$ and $(\mathrm{Tr}(U))^2$ has a trace on S_r' belonging to $L^{\frac{N}{N-2s}}(S_r')$, so that $h(\mathrm{Tr}(U))^2$ has a trace on S_r' belonging to $L^1(S_r')$ for all $r \in (0, R)$. Moreover $g \in W^{1,\frac{2N}{N+2s}}(B_r')$ by (2.7) and $\mathrm{Tr}(U) \in W^{1,\frac{2N}{N-2s}}(B_r')$ thanks to (2.11) and (3.5); then, on S_r', g has a trace in $L^{\frac{2N}{N+2s}}(S_r')$ and $\mathrm{Tr}(U)$ has a trace in $L^{\frac{2N}{N-2s}}(S_r')$, so that $g\,\mathrm{Tr}(U)$ has a trace on S_r' belonging to $L^1(S_r')$ for all $r \in (0, R)$.

3. Preliminaries: traces and inequalities

In this section we collect some preliminary results that will be used throughout the paper.

For any $i = 1, \dots, N+1$, let $e_i = (\delta_{i,j})_{j=1,\dots,N+1} \in \mathbb{R}^{N+1}$ be the vector with i-th component equal to 1 and all the remaining components equal to 0.

It is well know that if $w \in W^{1,p}(\Omega)$ with $\Omega \subset \mathbb{R}^{N+1}$ open and $p \in [1,\infty)$, then, for any $i = 1, \dots, N+1$ and $k \in \mathbb{R}$,

$$\int_{\Omega_{k,i}} \frac{|u(x+ke_i) - u(x)|^p}{|k|^p} \le \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx < +\infty,$$

where $\Omega_{k,i} := \{x \in \Omega : x + \tau k e_i \in \Omega \text{ for any } \tau \in [0,1]\}$, see e.g. [17, Theorem 10.55]. We prove below an analogous result for the weighted space $H^1(B_r^+, t^{1-2s})$.

Lemma 3.1. *For any $r > 0$, $w \in H^1(B_r^+, t^{1-2s})$, $i = 1, \dots, N$, and $k \in \mathbb{R}$*

$$\int_{B^+_{r,k,i}} t^{1-2s} \frac{|w(z+ke_i) - w(z)|^2}{|k|^2} dz \le \int_{B_r^+} t^{1-2s} \left|\frac{\partial w}{\partial x_i}\right|^2 dz, \tag{3.1}$$

where $B^+_{r,k,i} := \{z \in B_r^+ : z + \tau k e_i \in B_r^+ \text{ for any } \tau \in [0,1]\}$.

Proof. For a.e. $z \in B^+_{r,k,i}$, by the absolute continuity of Sobolev functions on lines,

$$|w(z+ke_i) - w(z)| = \left|\int_0^1 \frac{d}{d\tau} w(z+\tau k e_i) d\tau\right| \le \int_0^1 \left|\frac{\partial w}{\partial x_i}(z+\tau k e_i)\right| |k|\, d\tau.$$

Multiplying by t^{1-2s} and integrating on $B^+_{r,k}$ we obtain, by Cauchy-Schwarz's inequality and Fubini-Tonelli's Theorem,

$$\int_{B^+_{r,k,i}} t^{1-2s} \frac{|w(z+ke_i) - w(z)|^2}{|k|^2} dz$$
$$\le \int_{B^+_{r,k,i}} t^{1-2s} \left(\int_0^1 \left|\frac{\partial w}{\partial x_i}(z+\tau k e_i)\right|^2 d\tau\right) dz \le \int_{B_r^+} t^{1-2s} \left|\frac{\partial w}{\partial x_i}\right|^2 dz$$

which proves (3.1). □

We refer to [10] for the following result, which can be deduced from [21, Theorem 19.7].

Proposition 3.2. *For any $r > 0$ there exists a linear, continuous, compact trace operator*

$$\mathrm{Tr}_1 : H^1(B_r^+, t^{1-2s}) \to L^2(S_r^+, t^{1-2s}).$$

For the sake of simplicity we will always denote $\mathrm{Tr}_1(w)$ with w for any function $w \in H^1(B_r^+, t^{1-2s})$.

Lemma 3.3. [10, Lemma 2.6] *There exists a constant $\mathcal{S}_{N,s} > 0$ such that, for any $r > 0$ and $w \in H^1(B_r^+, t^{1-2s})$,*

$$\left(\int_{B_r'} |w|^{2_s^*} dx\right)^{\frac{2}{2_s^*}} \le \mathcal{S}_{N,s} \left(\int_{B_r^+} t^{1-2s} |\nabla w|^2 dz + \frac{N-2s}{2r} \int_{S_r^+} t^{1-2s} w^2 dS\right), \tag{3.2}$$

where $2_s^* = \frac{2N}{N-2s}$.

Remark 3.4. By (3.2) and Hölder's inequality, the definition of weak solution given in (2.10) is well posed.

Lemma 3.5. *Let* Tr *be the trace operator introduced in* (2.2)*. Then*

(i) *For any* $r > 0$*,* $f \in C^{0,1}(\overline{B_r^+})$ *and* $w \in H^1(B_r^+, t^{1-2s})$*,*

$$\mathrm{Tr}(fw) = f(\cdot, 0)\,\mathrm{Tr}(w). \tag{3.3}$$

(ii) *For any* $r > 0$*,* $u \in H^1(B_r^+, t^{1-2s})$ *and* $v \in H^1(B_r^+, t^{2s-1})$*, we have that* $uv \in W^{1,1}(B_r^+)$ *and*

$$\mathrm{Tr}(uv) = \mathrm{Tr}(u)\,\mathrm{Tr}(v). \tag{3.4}$$

Proof. Let us first prove (i). If $w \in C^\infty(\overline{B_r^+})$ then (3.3) is trivial; if w belongs to $H^1(B_r^+, t^{1-2s})$ there exists $\{\phi_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{B_r^+})$ such that $\phi_n \to w$ in $H^1(B_r^+, t^{1-2s})$ as $n \to \infty$. Furthermore, for any $f \in C^{0,1}(\overline{B_r^+})$, it is easy to see that $\phi_n f \to wf$ in $H^1(B_r^+, t^{1-2s})$ as $n \to \infty$. Then (3.3) follows from the continuity of the operator Tr.

We now prove (ii). If $u \in H^1(B_r^+, t^{1-2s})$ and $v \in H^1(B_r^+, t^{2s-1})$, the fact that $uv \in W^{1,1}(B_r^+)$ follows easily from Hölder's inequality. Moreover there exist $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{B_r^+})$ such that $u_n \to u$ in $H^1(B_r^+, t^{1-2s})$ and $\{v_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{B_r^+})$ such that $v_n \to v$ in $H^1(B_r^+, t^{2s-1})$. One can easily verify that $u_n v_n \to uv$ in $W^{1,1}(B_r^+)$, so that $\mathrm{Tr}(u_n v_n) \to \mathrm{Tr}(uv)$ in $L^1(B_r')$. On the other hand, since by continuity of the operator (2.2) $\mathrm{Tr}(u_n) \to \mathrm{Tr}(u)$ and $\mathrm{Tr}(v_n) \to \mathrm{Tr}(v)$ in $L^2(B_r')$, we have also that $\mathrm{Tr}(u_n v_n) = \mathrm{Tr}(u_n)\,\mathrm{Tr}(v_n) \to \mathrm{Tr}(u)\,\mathrm{Tr}(v)$ in $L^1(B_r')$, so that necessarily $\mathrm{Tr}(uv) = \mathrm{Tr}(u)\,\mathrm{Tr}(v)$. □

For any $r > 0$, let

$$H^{1+s}(B_r') := \left\{ w \in H^1(B_r') : \frac{\partial w}{\partial x_i} \in H^s(B_r') \text{ for any } i = 1, \dots, N \right\},$$

see [6] for details on this class of fractional Sobolev spaces. We also consider the space

$$H^2_x(B_r^+, t^{1-2s}) := \left\{ w \in H^1(B_r^+, t^{1-2s}) : \frac{\partial w}{\partial x_i} \in H^1(B_r^+, t^{1-2s}) \text{ for any } i = 1, \dots, N \right\}.$$

Proposition 3.6. *Let* Tr *be the trace operator introduced in* (2.2)*. For any* $r > 0$

$$\mathrm{Tr}(H^2_x(B_r^+, t^{1-2s})) \subseteq H^{1+s}(B_r'). \tag{3.5}$$

Furthermore, for any $w \in H^2_x(B_r^+, t^{1-2s})$*,*

$$\mathrm{Tr}(\nabla_x w) = \nabla\,\mathrm{Tr}(w), \tag{3.6}$$

where $\nabla_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}\right)$ *denotes the gradient with respect to the first* N *variables.*

Proof. Let $w \in H^2_x(B_r^+, t^{1-2s})$. Let us fix $\phi \in C^\infty_c(B_r')$; then there exists a function $\tilde\phi \in C^\infty_c(B_r^+ \cup B_r')$ such that $\tilde\phi(x, 0) = \phi(x)$ for all $x \in B_r'$. Let $\eta \in C^\infty_c(B_r)$ be a smooth cut-off function such that $\eta \equiv 1$ on $\operatorname{supp}\tilde\phi$. Then, denoting as $\hat w$ the even reflection of w through the hyperplane $t = 0$, $\tilde{w} = \eta\hat{w} \in H^1(\mathbb{R}^{N+1}, |t|^{1-2s})$ and $\frac{\partial \tilde{w}}{\partial x_i} \in H^1(\mathbb{R}^{N+1}, |t|^{1-2s})$ for all $i \in \{1, \dots, N\}$. Then, letting $\{\rho_n\}$ be a sequence of mollifiers and $w_n = \rho_n * \tilde{w}$, from [15, Lemma 1.5] it follows that $w_n \in C^\infty(\mathbb{R}^{N+1})$ and, for all $i \in \{1, \dots, N\}$,

$$w_n \to \tilde{w} \quad \text{and} \quad \frac{\partial w_n}{\partial x_i} = \rho_n * \frac{\partial \tilde{w}}{\partial x_i} \to \frac{\partial \tilde{w}}{\partial x_i} \quad \text{in } H^1(\mathbb{R}^{N+1}, |t|^{1-2s}).$$

Then, for any $i = 1, \dots, N$,

$$\begin{aligned}\int_{B'_r} \operatorname{Tr}(w) \frac{\partial \phi}{\partial x_i}\, dx &= \int_{B'_r} \operatorname{Tr}(\tilde{w}) \frac{\partial \phi}{\partial x_i}\, dx = \lim_{n\to\infty} \int_{B'_r} w_n(x,0) \frac{\partial \phi}{\partial x_i}(x,0)\, dx \\ &= -\lim_{n\to\infty} \int_{B'_r} \frac{\partial w_n}{\partial x_i}(x,0)\phi(x,0)\, dx = -\int_{B'_r} \operatorname{Tr}\left(\frac{\partial \tilde{w}}{\partial x_i}\right) \phi\, dx \\ &= -\int_{B'_r} \operatorname{Tr}\left(\frac{\partial w}{\partial x_i}\right) \phi\, dx,\end{aligned}$$

so that the distributional derivative in B'_r of $\operatorname{Tr}(w)$ with respect to x_i is $\operatorname{Tr}\left(\frac{\partial w}{\partial y_i}\right)$ which belongs to $H^s(B'_r)$. Therefore we have proved (3.6), which directly implies (3.5) in view of (2.2). ☐

Proposition 3.7. *Let U be a solution of* (2.10)*. For a.e. $r \in (0, R)$ and for all $\phi \in H^1(B_r^+, t^{1-2s})$*

$$\int_{B_r^+} t^{1-2s}[A\nabla U \cdot \nabla \phi + c\phi]dz = \frac{1}{r}\int_{S_r^+} t^{1-2s} A\nabla U \cdot z\, \phi\, dS + \int_{B'_r} [h \operatorname{Tr}(U) + g] \operatorname{Tr}(\phi)\, dx. \tag{3.7}$$

Remark 3.8. By Coarea Formula

$$\int_{B_R^+} \left| t^{1-2s} A\nabla U \cdot \frac{z}{|z|}\phi \right| dz = \int_0^R \left(\int_{S_r^+} \left| t^{1-2s} A\nabla U \cdot \frac{z}{r}\, \phi \right| dS \right) dr. \tag{3.8}$$

It follows that the function $f(r) := \int_{S_r^+} t^{1-2s} A\nabla U \cdot \frac{z}{r}\phi\, dS$ is well-defined as an element of $L^1(0, R)$ and hence a.e. $r \in (0, R)$ is a Lebesgue point of f.

Proof. By density it is enough to prove (3.7) for any $\phi \in C^\infty(\overline{B_r^+})$. Let us consider the following sequence of radial cut-off functions

$$\eta_n(|z|) := \begin{cases} 1, & \text{if } \quad 0 \le |z| \le r - \frac{1}{n}, \\ n(r - |z|), & \text{if } \quad r - \frac{1}{n} \le |z| \le r, \\ 0, & \text{if } \quad |z| \ge r. \end{cases}$$

Testing (2.10) with $\phi\eta_n$ and passing to the limit as $n \to \infty$ we obtain (3.7) thanks to the Dominated Convergence Theorem, (3.3) and Remark 3.8. ☐

4. Regularity of weak solutions: proof of Theorem 2.1

For any $r > 0$ and $\delta \in (0, r)$, we define

$$B^+_{r,\delta} := \{(x,t) \in B^+_r : t > \delta\}, \quad S^+_{r,\delta} := \{(x,t) \in S^+_r : t > \delta\}. \tag{4.1}$$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. For any $r > 0$ we denote

$$C_r := B'_r \times (0, r).$$

Let us fix $0 < r_3 < r_2 < r_1 < R$ with r_2 small enough so that $\overline{C}_{r_2} \subset B^+_{r_1} \cup B'_{r_1}$. We will show that $U \in H^2(B^+_{r_3}, t^{1-2s})$, eventually choosing a smaller r_1.

We start by defining a suitable cut-off function $\eta \in C^\infty_c(B'_{r_2} \times [0, r_2))$. We choose a cut-off function $\rho \in C^\infty_c(B'_{r_2})$ such that $0 \leq \rho(x) \leq 1$ for any $x \in \mathbb{R}^N$ and $\rho(x) \equiv 1$ on B'_{r_3} and a function $\sigma \in C^\infty_c([0, r_2))$ such that $0 \leq \sigma(t) \leq 1$ for any $t \in \mathbb{R}$ and $\sigma(t) = 1$ if $t \in [0, r_3]$. Then we define

$$\eta(z) = \eta(x,t) := \rho(x)\sigma(t). \tag{4.2}$$

Then $\eta \in C^\infty_c(B'_{r_2} \times [0, r_2))$ and $0 \leq \eta \leq 1$. For any $\phi \in H^1(B^+_{r_1}, t^{1-2s})$ we can test (2.10) with $\eta\phi$ obtaining

$$\begin{aligned} \int_{B^+_{r_1}} [t^{1-2s}\eta A\nabla U \cdot \nabla\phi + t^{1-2s} A\nabla U \cdot \nabla\eta\, \phi]\, dz + \int_{B^+_{r_1}} t^{1-2s} c\eta\phi\, dz \\ = \int_{B'_{r_1}} [h \operatorname{Tr}(U) + g]\rho \operatorname{Tr}(\phi)\, dx, \end{aligned} \tag{4.3}$$

thanks to (3.3) and (4.2). We would like to rewrite (4.3) as an equation for $U_1 := \eta U$. To this end we observe that

$$\begin{aligned} \operatorname{div}(t^{1-2s} U\phi\, A\nabla\eta) = U\phi\, \operatorname{div}(t^{1-2s} A\nabla\eta) \\ + t^{1-2s}\phi\, A\nabla\eta \cdot \nabla U + t^{1-2s} U\, A\nabla\eta \cdot \nabla\phi \in L^1(B^+_{r_1}). \end{aligned} \tag{4.4}$$

Letting $B^+_{r_1,\delta}$ be as in (4.1), the Divergence Theorem yields

$$\int_{B^+_{r_1,\delta}} \operatorname{div}(t^{1-2s} U\phi\, A\nabla\eta) dz = -\delta^{1-2s} \int_{B'_{r_1}} U(x,\delta)\phi(x,\delta)\alpha(x,\delta)\frac{\partial\eta}{\partial t}(x,\delta)\, dx,$$

where α has been defined in (2.4). Since $\frac{\partial\eta}{\partial t}(x,\delta) = 0$ for any $(x,\delta) \in \mathbb{R}^N \times [0, r_3]$, passing to the limit as $\delta \to 0^+$ we conclude that

$$\int_{B^+_{r_1}} \operatorname{div}(t^{1-2s} U\phi\, A\nabla\eta) dz = 0, \tag{4.5}$$

thanks to the Dominated Convergence Theorem and the fact that

$$\operatorname{div}(t^{1-2s} U\phi\, A\nabla\eta) \in L^1(B^+_{r_1})$$

by (4.4). Furthermore

$$\operatorname{div}(t^{1-2s}A\nabla\eta) = t^{1-2s}\left[\operatorname{div}(A\nabla\eta) + \frac{(1-2s)}{t}\alpha\frac{\partial\eta}{\partial t}\right], \tag{4.6}$$

and so, thanks to (4.2) and (2.8),

$$f := U\operatorname{div}(A\nabla\eta) + U\frac{(1-2s)}{t}\alpha\frac{\partial\eta}{\partial t} + 2A\nabla U\cdot\nabla\eta + \eta c \in L^2(B^+_{r_1}, t^{1-2s}). \tag{4.7}$$

In conclusion, combining (4.4), (4.5), and (4.6) we can rewrite (4.3) as

$$\int_{B^+_{r_1}} t^{1-2s}A\nabla U_1\cdot\nabla\phi\,dz + \int_{B^+_{r_1}} t^{1-2s}f\phi\,dz = \int_{B'_{r_1}}[h\operatorname{Tr}(U_1)+\rho g]\operatorname{Tr}(\phi)\,dx \tag{4.8}$$

for any $\phi\in H^1(B^+_{r_1}, t^{1-2s})$, in view of (3.3) and (4.7).

If we show that $\nabla_x U_1 \in H^1(C_{r_2}, t^{1-2s})$ and $t^{1-2s}\frac{\partial U_1}{\partial t}\in H^1(C_{r_2}, t^{2s-1})$, then we obtain that $\nabla_x U\in H^1(B^+_{r_3}, t^{1-2s})$ and $t^{1-2s}\frac{\partial U}{\partial t}\in H^1(B^+_{r_3}, t^{2s-1})$, since $\eta\equiv 1$ on C_{r_3}. To this end we use Nirenberg's tangential difference quotient method [20], proving that the family of the second incremental ratios is L^2-bounded; see also [13] for the difference quotient method for classical elliptic equations.

For any $i=1,\dots,N$ and $k\in\mathbb{R}\setminus\{0\}$ and for any measurable function w on $\mathbb{R}^{N+1}_+$, we define

$$(\tau_{i,k}w)(x,t) = w(x+ke_i,t) \quad\text{and}\quad (\zeta_{i,k}w)(x,t) = \frac{(\tau_{i,k}w)(x,t)-w(x,t)}{k}.$$

If $\overline{w}=(w_1,\dots,w_{N+1})$ is a vector of measurable functions we set

$$\tau_{i,k}(\overline{w}) := (\tau_{i,k}w_1,\dots,\tau_{i,k}w_{N+1}).$$

We can define $\tau_{i,k}$ similarly for a matrix of measurable functions.

It is easy to see that $\tau_{i,k}: L^2(\mathbb{R}^{N+1}_+, t^{1-2s})\to L^2(\mathbb{R}^{N+1}_+, t^{1-2s})$ is a well-defined, continuous, linear operator, and the adjoint operator of $\tau_{i,k}$ with respect to the $L^2(\mathbb{R}^{N+1}_+, t^{1-2s})$-scalar product is $\tau_{i,-k}$.

Furthermore $\tau_{i,k}: H^1(\mathbb{R}^{N+1}_+, t^{1-2s})\to H^1(\mathbb{R}^{N+1}_+, t^{1-2s})$ is a well-defined, continuous, linear operator and, for any $i=1,\dots,N$ and any $w\in H^1(B^+_r, t^{1-2s})$,

$$\frac{\partial\tau_{i,k}(w)}{\partial x_i} = \tau_{i,k}\left(\frac{\partial w}{\partial x_i}\right),$$

that is, the operator commutes with tangential derivatives. With a slight abuse of notation, for any $i=1,\dots,N$ and $k\in\mathbb{R}\setminus\{0\}$ we denote as $\tau_{i,k}$, respectively $\zeta_{i,k}$, also the operator $\tau_{i,k}v(x)=v(x+ke_i)$, respectively $\zeta_{i,k}v=\frac{1}{k}(\tau_{i,k}v-v)$, acting on measurable functions $v:\mathbb{R}^N\to\mathbb{R}$ and observe that $\tau_{i,k},\zeta_{i,k}: W^{1,p}(\mathbb{R}^N)\to W^{1,p}(\mathbb{R}^N)$ are linear and continuous for any $p\in[1,\infty)$; furthermore, the adjoint operator of $\tau_{i,k}$ is $\tau_{i,-k}$.

It is easy to see that, for all measurable functions v,w,

$$\zeta_{i,k}(vw) = \zeta_{i,k}(v)\tau_{i,k}w + v\zeta_{i,k}(w) \tag{4.9}$$

and

$$\frac{w(x+ke_i,t)-2w(x,t)+w(x-ke_i,t)}{k^2}=(\zeta_{i,k}\circ\zeta_{i,-k})(w)(x,t).$$

We note that $U_1\equiv 0$ on $B_{r_1}^+\setminus C_{r_2}$, so that its trivial extension to $\mathbb{R}^{N+1}_+$ belongs to $H^1(\mathbb{R}^{N+1}_+,t^{1-2s})$; with a slight abuse of notation we will still indicate this extension with U_1.

Let $|k|<\sqrt{r_1^2-r_2^2}-r_2$ (we note that $\sqrt{r_1^2-r_2^2}-r_2>0$ since $C_{r_2}\subset B_{r_1}^+$). The function $\tilde{\phi}:=(\zeta_{i,k}\circ\zeta_{i,-k})(U_1)$ belongs to $H^1_{0,S^+_{r_1}}(B^+_{r_1},t^{1-2s})$ thanks to (4.2) and so its trivial extension, still denoted as $\tilde{\phi}$, belongs to $H^1(\mathbb{R}^{N+1}_+,t^{1-2s})$. Moreover by (2.9) we have that, for any $i=1,\dots,N$,

$$\operatorname{Tr}(\zeta_{i,k}(\zeta_{i,-k}(\tilde{\phi})))=\zeta_{i,k}(\operatorname{Tr}(\zeta_{i,-k}(\tilde{\phi})))=\zeta_{i,k}(\zeta_{i,-k}(\operatorname{Tr}(\tilde{\phi}))). \tag{4.10}$$

Therefore testing (4.8) with $\tilde{\phi}$ we obtain

$$\begin{aligned}\int_{B^+_{r_1}}&t^{1-2s}\zeta_{i,-k}(A\nabla U_1)\cdot\nabla(\zeta_{i,-k}(U_1))\,dz+\int_{B^+_{r_1}}t^{1-2s}f\;(\zeta_{i,k}\circ\zeta_{i,-k})(U_1)\,dz\\&=\int_{B'_{r_1}}\zeta_{i,-k}(\rho g)\operatorname{Tr}(\zeta_{i,-k}(U_1))\,dx+\int_{B'_{r_1}}\zeta_{i,-k}(h\operatorname{Tr}(U_1))\operatorname{Tr}(\zeta_{i,-k}(U_1))\,dx,\end{aligned}\tag{4.11}$$

thanks to (4.10). From (4.11) it follows that, for any $i=1,\dots,N$,

$$\begin{aligned}\int_{B^+_{r_1}}&t^{1-2s}A\nabla(\zeta_{i,-k}(U_1))\cdot\nabla(\zeta_{i,-k}(U_1))\,dz\\&\le\int_{B^+_{r_1}}t^{1-2s}|\zeta_{i,-k}(A)\nabla(\tau_{i,-k}(U_1))\cdot\nabla(\zeta_{i,-k}(U_1))|\,dz\\&\quad+\int_{B^+_{r_1}}t^{1-2s}|f\;(\zeta_{i,k}\circ\zeta_{i,-k})(U_1)|\,dz+\int_{B'_{r_1}}|\zeta_{i,-k}(\rho g)\operatorname{Tr}(\zeta_{i,-k}(U_1))|\,dx\\&\quad+\int_{B'_{r_1}}|\zeta_{i,-k}(h)\operatorname{Tr}(\tau_{i,-k}(U_1))\operatorname{Tr}(\zeta_{i,-k}(U_1))|\,dx\\&\quad+\int_{B'_{r_1}}|h||\operatorname{Tr}(\zeta_{i,-k}(U_1))|^2\,dx,\end{aligned}\tag{4.12}$$

thanks to (4.9) and (4.10). Now we estimate each term of the right hand side of (4.12). We start by noticing that, thanks to (2.5), there exists a constant $\Lambda>0$ (depending only on the Lipschitz constants of the entries of A) such that

$$\begin{aligned}\|\zeta_{i,-k}(A)(z)\|_{\mathcal{L}(\mathbb{R}^{N+1},\mathbb{R}^{N+1})}\le\Lambda\quad&\text{for all } i=1,\dots,N,\ z\in B^+_{r_1},\\&\text{and } k\in\left(r_2-\sqrt{r_1^2-r_2^2},\sqrt{r_1^2-r_2^2}-r_2\right),\end{aligned}\tag{4.13}$$

where $\|\zeta_{i,-k}(A)(z)\|_{\mathcal{L}(\mathbb{R}^{N+1},\mathbb{R}^{N+1})}$ is the norm of $\zeta_{i,-k}(A)(z)$ as a linear operator from $\mathbb{R}^{N+1}$ to $\mathbb{R}^{N+1}$. Then by (4.13), Hölder's inequality and Cauchy-Schwarz's inequality in $\mathbb{R}^{N+1}$,

$$\int_{B_{r_1}^+} t^{1-2s}|\zeta_{i,-k}(A)\nabla(\tau_{i,-k}(U_1))\cdot\nabla(\zeta_{i,-k}(U_1))|\,dz \\ \leq \Lambda \left\|\nabla(\tau_{i,-k}(U_1))\right\|_{L^2(B_{r_1}^+,t^{1-2s})}\left\|\nabla(\zeta_{i,-k}(U_1))\right\|_{L^2(B_{r_1}^+,t^{1-2s})}. \tag{4.14}$$

By Hölder's inequality and (3.1),

$$\int_{B_{r_1}^+} t^{1-2s}|f\ (\zeta_{i,k}\circ\zeta_{i,-k})(U_1)|\,dz \\ \leq \left\|f\right\|_{L^2(B_{r_1}^+,t^{1-2s})}\left\|\nabla(\zeta_{i,-k}(U_1))\right\|_{L^2(B_{r_1}^+,t^{1-2s})}. \tag{4.15}$$

Furthermore by (3.2) and Hölder's inequality

$$\int_{B'_{r_1}} |\zeta_{i,-k}(\rho g)\,\mathrm{Tr}(\zeta_{i,-k}(U_1))|\,dx \\ \leq \mathcal{S}_{N,s}^{\frac{1}{2}}\left\|\rho g\right\|_{W^{1,\frac{2N}{N+2s}}(B'_{r_1})}\left\|\nabla(\zeta_{i,-k}(U_1))\right\|_{L^2(B_{r_1}^+,t^{1-2s})}, \tag{4.16}$$

$$\int_{B'_{r_1}} |\zeta_{i,-k}(h)\,\mathrm{Tr}(\tau_{i,-k}(U_1))\,\mathrm{Tr}(\zeta_{i,-k}(U_1))|\,dx \\ \leq \mathcal{S}_{N,s}\left\|h\right\|_{W^{1,\frac{N}{2s}}(B'_{r_1})}\left\|\nabla(\tau_{i,-k}(U_1))\right\|_{L^2(B_{r_1}^+,t^{1-2s})}\left\|\nabla(\zeta_{i,-k}(U_1))\right\|_{L^2(B_{r_1}^+,t^{1-2s})}, \tag{4.17}$$

and

$$\int_{B'_{r_1}} |h||\,\mathrm{Tr}(\zeta_{i,-k}(U_1))|^2\,dx \\ \leq \mathcal{S}_{N,s}\left\|h\right\|_{W^{1,\frac{N}{2s}}(B'_{r_1})}\left\|\nabla(\zeta_{i,-k}(U_1))\right\|^2_{L^2(B_{r_1}^+,t^{1-2s})}. \tag{4.18}$$

Putting together (4.12), (4.14), (4.15), (4.16), (4.17), (4.18) and (2.6) we obtain that

$$\begin{aligned} &\left(\lambda_1-\mathcal{S}_{N,s}\left\|h\right\|_{W^{1,\frac{N}{2s}}(B'_{r_1})}\right)\left\|\nabla(\zeta_{i,-k}(U_1))\right\|_{L^2(B_{r_1}^+,t^{1-2s})} \\ &\leq \Lambda\left\|\nabla(\tau_{i,-k}(U_1))\right\|_{L^2(B_{r_1}^+,t^{1-2s})}+\left\|f\right\|_{L^2(B_{r_1}^+,t^{1-2s})} \\ &\quad+\mathcal{S}_{N,s}^{\frac{1}{2}}\left\|\rho g\right\|_{W^{1,\frac{2N}{N+2s}}(B'_{r_1})} \\ &\quad+\mathcal{S}_{N,s}\left\|h\right\|_{W^{1,\frac{N}{2s}}(B'_{r_1})}\left\|\nabla(\tau_{i,-k}(U_1))\right\|_{L^2(B_{r_1}^+,t^{1-2s})} \\ &=(\Lambda+\mathcal{S}_{N,s}\left\|h\right\|_{W^{1,\frac{N}{2s}}(B'_{r_1})})\left\|\nabla U_1\right\|_{L^2(B_{r_1}^+,t^{1-2s})} \\ &\quad+\left\|f\right\|_{L^2(B_{r_1}^+,t^{1-2s})}+\mathcal{S}_{N,s}^{\frac{1}{2}}C_\rho\left\|g\right\|_{W^{1,\frac{2N}{N+2s}}(B'_{r_1})} \end{aligned} \tag{4.19}$$

for some positive constant $C_\rho > 0$ depending only on $\|\nabla\rho\|_{L^\infty(B'_{r_1})}$, where we have used the fact that $\|\nabla(\tau_{i,-k}(U_1))\|_{L^2(B^+_{r_1},t^{1-2s})} = \|\nabla U_1\|_{L^2(B^+_{r_1},t^{1-2s})}$ since $\operatorname{supp}\tau_{i,-k}(U_1) \subset B^+_{r_1} \cup B'_{r_1}$ for all $|k| < \sqrt{r_1^2 - r_2^2} - r_2$.

Eventually choosing r_1 smaller form the beginning, we may suppose that

$$\lambda_1 - \mathcal{S}_{N,s} \|h\|_{W^{1,\frac{N}{2s}}(B'_{r_1})} > 0,$$

by the absolute continuity of the integral. We conclude that for any $i = 1, \dots, N$ and any $j = 1, \dots, N+1$

$$\left\{ \frac{\partial(\zeta_{i,-k}(U_1))}{\partial z_j} : |k| < \sqrt{r_1^2 - r_2^2} - r_2 \right\} \qquad \text{is bounded in } L^2(B^+_{r_1}, t^{1-2s}).$$

It follows that, for any $i = 1, \dots, N$ and $j = 1, \dots, N+1$, there exist a function $\psi_{i,j} \in L^2(B^+_{r_1}, t^{1-2s})$ and a sequence $k_n \to 0$ such that $\frac{\partial(\zeta_{i,-k_n}(U_1))}{\partial z_j} \rightharpoonup \psi_{i,j}$ weakly in $L^2(B^+_{r_1}, t^{1-2s})$ as $n \to \infty$. Furthermore, by (3.1), the family of functions $\{(\zeta_{i,-k_n}(U_1)) : n \in \mathbb{N}\}$ is bounded in $L^2(B^+_{r_1}, t^{1-2s})$ and so there exists a function $\varphi_i \in L^2(B^+_{r_1}, t^{1-2s})$ such that $\zeta_{i,-k_n}(U_1) \rightharpoonup \varphi_i$ weakly in $L^2(B^+_{r_1}, t^{1-2s})$ for any $i = 1, \dots, N$, up to a subsequence. For any test function $\phi \in C^\infty_c(B^+_{r_1})$, thanks to the Dominated Converge Theorem,

$$\begin{aligned}\int_{B^+_{r_1}} \varphi_i \phi \, dz &= \lim_{n\to\infty} \int_{B^+_{r_1}} \zeta_{i,-k_n}(U_1)\phi \, dz \\ &= -\lim_{n\to\infty} \int_{B^+_{r_1}} U_1 \zeta_{i,k_n}(\phi) \, dz = -\int_{B^+_{r_1}} U_1 \frac{\partial \phi}{\partial z_i} \, dz = \int_{B^+_{r_1}} \frac{\partial U_1}{\partial z_i} \phi \, dz\end{aligned}$$

and hence $\varphi_i = \frac{\partial U_1}{\partial z_i}$, i.e. $\zeta_{i,-k_n}(U_1) \rightharpoonup \frac{\partial U_1}{\partial z_i}$ weakly in $L^2(B^+_{r_1}, t^{1-2s})$, up to a subsequence. Furthermore, for any $\phi \in C^\infty_c(B^+_{r_1})$, $i = 1, \dots, N$, and $j = 1, \dots, N+1$, we have that

$$\begin{aligned}\int_{B^+_{r_1}} \psi_{i,j} \phi \, dz &= \lim_{n\to\infty} \int_{B^+_{r_1}} \frac{\partial(\zeta_{i,-k_n}(U_1))}{\partial z_j} \phi \, dz \\ &= -\lim_{n\to\infty} \int_{B^+_{r_1}} \zeta_{i,-k_n}(U_1) \frac{\partial \phi}{\partial z_j} dz = -\int_{B^+_{r_1}} \frac{\partial U_1}{\partial z_i} \frac{\partial \phi}{\partial z_j} dz,\end{aligned}$$

that is, $\psi_{i,j} = \frac{\partial}{\partial z_j} \frac{\partial U_1}{\partial z_i}$. Therefore the distributional derivative of $\frac{\partial U_1}{\partial z_i}$ respect to the variable z_j belongs to $L^2(B^+_{r_1}, t^{1-2s})$ for any $j = 1, \dots, N+1$, and $i = 1, \dots, N$, i.e.

$$\nabla_x U_1 \in H^1(B^+_{r_1}, t^{1-2s}). \tag{4.20}$$

Furthermore, estimate (4.19) and weak lower semi-continuity of the $L^2(B^+_{r_1}, t^{1-2s})$-norm imply that

$$\begin{aligned}&\|\nabla_x U_1\|_{H^1(B^+_{r_1},t^{1-2s})} \\ &\quad \leq C \left(\|\nabla U_1\|_{L^2(B^+_{r_1},t^{1-2s})} + \|f\|_{L^2(B^+_{r_1},t^{1-2s})} + \|g\|_{W^{1,\frac{2N}{N+2s}}(B'_{r_1})} \right)\end{aligned} \tag{4.21}$$

for a positive constant $C = C(N, s, \|h\|_{W^{1,\frac{N}{2s}}(B'_{r_1})}, \lambda_1, \Lambda, \|\nabla\rho\|_{L^\infty(B^+_{r_1})}) > 0$.

This also implies that $\nabla_x(t^{1-2s}\frac{\partial U_1}{\partial t}) \in L^2(B^+_{r_1}, t^{2s-1})$ with norm estimated as above. To conclude, it remains to show that $\frac{\partial}{\partial t}(t^{1-2s}\frac{\partial U_1}{\partial t}) \in L^2(B^+_{r_1}, t^{2s-1})$.

To this aim we observe that, for any $\phi \in C^\infty_c(B^+_{r_1})$, (4.8), the Divergence Theorem, (2.4), and (2.6) imply that

$$\begin{aligned}\int_{B^+_{r_1}} t^{1-2s}\frac{\partial U_1}{\partial t}\frac{\partial \phi}{\partial t}\,dz &= \int_{B^+_{r_1}} t^{1-2s}\alpha\frac{\partial U_1}{\partial t}\frac{\partial}{\partial t}\left(\frac{\phi}{\alpha}\right)dz + \int_{B^+_{r_1}} t^{1-2s}\frac{\partial \alpha}{\partial t}\frac{\partial U_1}{\partial t}\frac{\phi}{\alpha}\,dz \\ &= -\int_{B^+_{r_1}} t^{1-2s}B\nabla_x U_1\cdot\nabla_x\left(\frac{\phi}{\alpha}\right)dz - \int_{B^+_{r_1}} t^{1-2s} f\frac{\phi}{\alpha}\,dz \\ &\quad + \int_{B^+_{r_1}} t^{1-2s}\frac{\partial \alpha}{\partial t}\frac{\partial U_1}{\partial t}\frac{\phi}{\alpha}\,dz \\ &= -\int_{B^+_{r_1}} t^{1-2s}\frac{1}{\alpha}\left(-\operatorname{div}_x(B\nabla_x U_1) + f - \frac{\partial \alpha}{\partial t}\frac{\partial U_1}{\partial t}\right)\phi\,dz.\end{aligned}$$

Thanks to (2.5), (2.6), (2.8), (4.7), (4.20), and Hölder's inequality, we then conclude that

$$t^{2s-1}\frac{\partial}{\partial t}\left(t^{1-2s}\frac{\partial U_1}{\partial t}\right) = \frac{1}{\alpha}\left(-\operatorname{div}(B\nabla_x U_1) + f - \frac{\partial \alpha}{\partial t}\frac{\partial U_1}{\partial t}\right) \in L^2(B^+_{r_1}, t^{1-2s})$$

which implies that $\frac{\partial}{\partial t}(t^{1-2s}\frac{\partial U_1}{\partial t}) \in L^2(B^+_{r_1}, t^{2s-1})$ and hence

$$t^{1-2s}\frac{\partial U_1}{\partial t} \in H^1(B^+_{r_1}, t^{2s-1}),$$

with $H^1(B^+_{r_1}, t^{2s-1})$-norm estimated as in (4.21).

Since $\eta \equiv 1$ on $B^+_{r_3}$ we have thereby proved that

$$\nabla_x U \in H^1(B^+_{r_3}, t^{1-2s}) \quad \text{and} \quad t^{1-2s}\frac{\partial U}{\partial t} \in H^1(B^+_{r_3}, t^{2s-1})$$

and, in view of (4.7),

$$\begin{aligned}&\|\nabla_x U\|_{H^1(B^+_{r_3}, t^{1-2s})} + \left\|t^{1-2s}\frac{\partial U}{\partial t}\right\|_{H^1(B^+_{r_3}, t^{2s-1})} \\ &\qquad \le C\left(\|U\|_{H^1(B^+_R, t^{1-2s})} + \|c\|_{L^2(B^+_R, t^{1-2s})} + \|g\|_{W^{1,\frac{2N}{N+2s}}(B'_R)}\right)\end{aligned} \tag{4.22}$$

for a positive constant C depending only on N, s, r_1, r_3, $\|h\|_{W^{1,\frac{N}{2s}}(B'_R)}$, λ_1, $\|A\|_{W^{1,\infty}(B^+_R, \mathbb{R}^{(N+1)^2})}$.

Reasoning in a similar way we can show that, for any $r \in (0, R)$ and any $x \in \overline{B'_r}$, there exists $r_x > 0$ such that $B^+_{r_x}(x) \subset B^+_R$, $\nabla_x U \in H^1(B^+_{r_x}(x), t^{1-2s})$, and $t^{1-2s}\frac{\partial U}{\partial t} \in H^1(B^+_{r_x}(x), t^{2s-1})$, where

$$B^+_{r_x}(x) := \{z = (y, t) \in \mathbb{R}^{N+1}_+ : |(x, 0) - (y, t)| < r_x\}.$$

Then we can cover $\overline{B_r'}$ with a finite family of open sets $\{B^+_{r_{x_i}}(x_i)\}_{i\in I}$ such that

$$\nabla_x U \in H^1(B^+_{r_{x_i}}(x_i), t^{1-2s}) \text{ and } t^{1-2s}\frac{\partial U}{\partial t} \in H^1(B^+_{r_{x_i}}(x_i), t^{2s-1}) \quad \text{for all } i \in I$$

and an estimate of type (4.22) is satisfied. Furthermore, letting $B^+_{R,\delta}$ be as in (4.1), it is easy to verify that $t^{1-2s}A \in C^{0,1}(\overline{B^+_{R,\delta}})$ and $t^{1-2s}c \in L^2(B^+_{R,\delta})$ for any $\delta \in (0,R)$, since the weight t^{1-2s} is Lipschitz continuous on $\overline{B^+_{R,\delta}}$. Then we may conclude that $U \in H^2(B^+_{r,\delta}, t^{1-2s})$ for any $r \in (0,R)$ and $\delta \in (0,R)$ by classical elliptic regularity theory (see e.g. [12, Theorem 8.8]).

Combining the above information we obtain (2.11) and (2.12). □

Remark 4.1. The regularity result of Theorem 2.1 applies also to problems of the form

$$\begin{cases} -\operatorname{div}(t^{1-2s}A\nabla U) + t^{1-2s}bU + t^{1-2s}c = 0, & \text{on } B^+_R, \\ \lim_{t\to 0^+} t^{1-2s}A\nabla U \cdot \nu = h\operatorname{Tr}(U) + g, & \text{on } B'_R, \end{cases}$$

with c, h, g as in assumptions (2.7) and (2.8), and a potential $b \in L^{q_{N,s}}(B^+_R, t^{1-2s})$, where

$$q_{N,s} := \begin{cases} N+2-2s, & \text{if } s \in \left(0, \frac{1}{2}\right), \\ N+1, & \text{if } s \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Indeed if $b \in L^{q_{N,s}}(B^+_R, t^{1-2s})$ and $U \in H^1(B^+_R, t^{1-2s})$, then $bU \in L^2(B^+_R, t^{1-2s})$ in view of Hölder's inequality and the following Sobolev-type embedding result.

Lemma 4.2. *For any $r > 0$, $H^1(B^+_r, t^{1-2s}) \subset L^{2^{**}_s}(B^+_r, t^{1-2s})$, where*

$$2^{**}_s := \min\left\{2\frac{N+2-2s}{N-2s}, 2\frac{N+1}{N-1}\right\} = \begin{cases} 2\frac{N+2-2s}{N-2s}, & \text{if } s \in \left(0, \frac{1}{2}\right), \\ 2\frac{N+1}{N-1}, & \text{if } s \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Furthermore, there exists a constant $K_{N,s} > 0$ such that, for any $r > 0$ and any $w \in H^1(B^+_r, t^{1-2s})$,

$$\left(\int_{B^+_r} t^{1-2s}|w|^{2^{**}_s}\,dz\right)^{\frac{2}{2^{**}_s}} \le K_{N,s,r}\left(\frac{1}{r^2}\int_{B^+_r} t^{1-2s}w^2\,dz + \int_{B^+_r} t^{1-2s}|\nabla w|^2\,dz\right),$$

where

$$K_{N,s,r} := \begin{cases} K_{N,s}, & \text{if } s \in \left(0, \frac{1}{2}\right), \\ K_{N,s}(2s-1)r^{\frac{2}{N+1}}, & \text{if } s \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Proof. The claim follows from a scaling argument, [11, Appendix A.1] and [21, Theorem 19.20], see also [22, Theorem 2.4]. □

5. Proof of the Pohozaev-type identity (2.17)

Proof of Proposition 2.3. The following Rellich-Nečas type identity

$$\operatorname{div}\left(t^{1-2s}(A\nabla U\cdot\nabla U)\beta-2t^{1-2s}(\nabla U\cdot\beta)A\nabla U\right)=t^{1-2s}A\nabla U\cdot\nabla U\operatorname{div}(\beta)$$
$$-2\beta\cdot\nabla U\operatorname{div}\left(t^{1-2s}A\nabla U\right)+(d(t^{1-2s}A)\nabla U\nabla U)\cdot\beta-2J_\beta(t^{1-2s}A\nabla U)\cdot\nabla U$$

holds in a distributional sense in B_R^+. In view of (2.4) and (2.3) the above equation can be rewritten as

$$\begin{aligned}\operatorname{div}&\left(t^{1-2s}(A\nabla U\cdot\nabla U)\beta-2t^{1-2s}(\nabla U\cdot\beta)A\nabla U\right)\\&=t^{1-2s}A\nabla U\cdot\nabla U\operatorname{div}(\beta)-2t^{1-2s}c(\beta\cdot\nabla U)+t^{1-2s}dA\nabla U\nabla U\cdot\beta\\&\qquad+(1-2s)t^{1-2s}\frac{\alpha}{\mu}A\nabla U\cdot\nabla U-2J_\beta(t^{1-2s}A\nabla U)\cdot\nabla U\end{aligned}\tag{5.1}$$

with dA as in (2.15).

Let $r\in(0,R)$. By Theorem 2.1 an Remark 2.2, letting $\beta=(\beta_1,\dots,\beta_N,\alpha/\mu)$ (see (2.4) and (2.14)), we have that

$$\nabla U\cdot\beta=\nabla_x U\cdot(\beta_1,\dots,\beta_N)+\frac{\alpha}{\mu}tU_t\in H^1(B_r^+,t^{1-2s}).\tag{5.2}$$

In particular, to prove that $\frac{\partial}{\partial t}(tU_t)\in L^2(B_r^+,t^{1-2s})$, it is useful to observe that

$$\frac{\partial}{\partial t}(tU_t)=t^{2s}\frac{\partial}{\partial t}(t^{1-2s}U_t)+2sU_t$$

and recall that $\frac{\partial}{\partial t}\left(t^{1-2s}\frac{\partial U}{\partial t}\right)\in L^2(B_r^+,t^{2s-1})$ by (2.11).

We observe that $tU_t=t^{2s}(t^{1-2s}U_t)$, with

$$t^{2s}\in H^1(B_r^+,t^{1-2s})\quad\text{and}\quad t^{1-2s}U_t\in H^1(B_r^+,t^{2s-1})$$

by (2.11); hence (3.4) implies that $\operatorname{Tr}(tU_t)=\operatorname{Tr}(t^{2s})\operatorname{Tr}(t^{1-2s}U_t)=0$, so that from (5.2), (3.3), and (3.6) we deduce that

$$\operatorname{Tr}(\nabla U\cdot\beta)=\operatorname{Tr}(\nabla_x U\cdot(\beta_1,\dots,\beta_N))+\operatorname{Tr}\left(\frac{\alpha}{\mu}tU_t\right)=\nabla_x\operatorname{Tr}(U)\cdot\beta'.\tag{5.3}$$

From (2.3), (2.8), and (5.2) it follows that

$$\operatorname{div}(t^{1-2s}(\nabla U\cdot\beta)A\nabla U)=t^{1-2s}c(\nabla U\cdot\beta)+t^{1-2s}A\nabla U\cdot\nabla(\nabla U\cdot\beta)\in L^1(B_r^+)\tag{5.4}$$

so that, in view of (5.1), (2.16), (2.8), and (5.2) we obtain also that

$$\operatorname{div}\left(t^{1-2s}(A\nabla U\cdot\nabla U)\beta\right)\in L^1(B_r^+).\tag{5.5}$$

Applying the Divergence Theorem on the set $B_{r,\delta}^+$ defined in (4.1) (and recalling from Theorem 2.1 or classical elliptic regularity theory that $U\in H^2(B_{r,\delta}^+)$), we have that

$$\int_{B^+_{r,\delta}} \operatorname{div}(t^{1-2s}(A\nabla U\cdot\nabla U)\beta)\,dz = r\int_{S^+_{r,\delta}} t^{1-2s}A\nabla U\cdot\nabla U\,dS - \delta^{2-2s}\int_{B'_{\sqrt{r^2-\delta^2}}} \frac{\alpha(x,\delta)}{\mu(x,\delta)}(A\nabla U\cdot\nabla U)(x,\delta)\,dx \tag{5.6}$$

with $S^+_{r,\delta}$ as in (4.1). We claim that there exists a sequence $\delta_n\to 0^+$ such that

$$\lim_{n\to\infty}\delta_n^{2-2s}\int_{B'_{\sqrt{r^2-\delta_n^2}}} \frac{\alpha(x,\delta_n)}{\mu(x,\delta_n)}(A\nabla U\cdot\nabla U)(x,\delta_n)\,dx = 0. \tag{5.7}$$

To prove (5.7) we argue by contradiction. If the claim does not hold, then there exist a constant $C>0$ and $\bar r\in(0,r)$ such that

$$\delta^{1-2s}\int_{B'_r}\frac{\alpha(x,\delta)}{\mu(x,\delta)}(A\nabla U\cdot\nabla U)(x,\delta)\,dx \ge \frac{C}{\delta}\quad\text{for any }\delta\in(0,\bar r). \tag{5.8}$$

We may suppose that $B'_{\bar r}\times(0,\bar r)\subset B^+_r$ and integrating (5.8) in $(0,\bar r)$ we obtain

$$\begin{aligned}\int_{B^+_R} t^{1-2s}\frac{\alpha}{\mu}A\nabla U\cdot\nabla U\,dz &\ge \int_0^{\bar r} t^{1-2s}\left(\int_{B'_r}\frac{\alpha(x,t)}{\mu(x,t)}(A\nabla U\cdot\nabla U)(x,t)\,dx\right)dt\\ &\ge C\int_0^{\bar r}\frac{1}{t}\,dt = +\infty,\end{aligned}$$

which is a contradiction since $\frac{\alpha}{\mu}A\nabla U\cdot\nabla U\in L^1(B^+_R,t^{1-2s})$ thanks to (2.16) and Hölder's inequality. Therefore passing to the limit as $n\to\infty$ and $\delta=\delta_n$ in (5.6) and taking into account (5.5) we conclude that

$$\int_{B^+_r}\operatorname{div}(t^{1-2s}(A\nabla U\cdot\nabla U)\beta)\,dz = r\int_{S^+_r} t^{1-2s}A\nabla U\cdot\nabla U\,dS \tag{5.9}$$

for a.e. $r\in(0,R)$. From (5.4) and (3.7) it follows that

$$\begin{aligned}&\int_{B^+_r}\operatorname{div}(t^{1-2s}(\nabla U\cdot\beta)A\nabla U)\,dz\\ &= \int_{B^+_r} t^{1-2s}c(\nabla U\cdot\beta)\,dz + \int_{B^+_r} t^{1-2s}A\nabla U\cdot\nabla(\nabla U\cdot\beta)\,dz\\ &= \frac{1}{r}\int_{S^+_r} t^{1-2s}(A\nabla U\cdot z)(\nabla U\cdot\beta)\,dS + \int_{B'_r}[h\operatorname{Tr}(U)+g]\operatorname{Tr}(\nabla U\cdot\beta)\,dx\\ &= r\int_{S^+_r} t^{1-2s}\frac{|A\nabla U\cdot\nu|^2}{\mu}\,dS + \int_{B'_r}[h\operatorname{Tr}(U)+g](\nabla_x\operatorname{Tr}(U)\cdot\beta')\,dx,\end{aligned} \tag{5.10}$$

thanks to (2.5), (2.14), and (5.3). We observe that $\beta' h\in W^{1,\frac{N}{2s}}(B'_r,\mathbb{R}^N)$ in view of (2.7) and (2.16) and $(\operatorname{Tr}(U))^2\in W^{1,\frac{N}{N-2s}}(B'_r)$ thanks to (2.11) and (3.5); then an integration by parts on B'_r yields

$$\int_{B'_r} h\,\mathrm{Tr}(U)(\nabla_x \mathrm{Tr}(U)\cdot\beta')\,dx = \frac{1}{2}\int_{B'_r} \nabla_x(\mathrm{Tr}(U))^2\cdot(h\beta')\,dx$$
$$= \frac{r}{2}\int_{S'_r} h|\mathrm{Tr}(U)|^2\,dS' - \frac{1}{2}\int_{B'_r}(\mathrm{div}_x(\beta')h+\beta'\cdot\nabla h)|\mathrm{Tr}(U)|^2\,dx. \tag{5.11}$$

Moreover $\beta' g \in W^{1,\frac{2N}{N+2s}}(B'_r,\mathbb{R}^N)$ by (2.7) and $\mathrm{Tr}(U)\in W^{1,\frac{2N}{N-2s}}(B'_r)$ by (2.11) and (3.5), hence, integrating by parts, we obtain that

$$\int_{B'_r}\nabla_x \mathrm{Tr}(U)\cdot(\beta' g)\,dx = r\int_{S'_r} g\,\mathrm{Tr}(U)\,dS'$$
$$-\int_{B'_r}(\mathrm{div}_x(\beta')g+\beta'\cdot\nabla g)\,\mathrm{Tr}(U)\,dx. \tag{5.12}$$

Putting together (5.1), (5.9), (5.10), (5.11), and (5.12), we obtain (2.17). □

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