Euler–MacLaurin Summation Formula on Polytopes and Expansions in Multivariate Bernoulli Polynomials

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Abstract

We provide a multidimensional weighted Euler–MacLaurin summation formula on polytopes and a multidimensional generalization of a result due to L. J. Mordell on the series expansion in Bernoulli polynomials. These results are consequences of a more general series expansion; namely, if $\chi_{\tau\mathcal{P}}$ denotes the characteristic function of a dilated integer convex polytope $\mathcal{P}$ and q is a function with suitable regularity, we prove that the periodization of $q\chi_{\tau\mathcal{P}}$ admits an expansion in terms of multivariate Bernoulli polynomials. These multivariate polynomials are related to the Lerch Zeta function. In order to prove our results we need to carefully study the asymptotic expansion of $\widehat{q\chi_{\tau\mathcal{P}}}$, the Fourier transform of $q\chi_{\tau\mathcal{P}}$.

Keywords Euler–MacLaurin summation formula · Bernoulli polynomials · Fourier transform

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1 Introduction

Our goal is to generalize to higher dimensions a result due to L. J. Mordell and to deduce from this generalization a multidimensional weighted version of the classical

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Euler–MacLaurin formula and an associated quadrature rule. We first recall these results in the one dimensional setting. In order to do so we introduce the classical Bernoulli polynomials; there are two possible normalizations that differ by a $n!$ factor and we use the following one.

Definition 1 The periodized Bernoulli polynomials $\{B_n\}_{n \in \mathbb{N}}$ are the periodic functions that in the interval $(0, 1)$ are defined recursively by the conditions

$$B_0(x) = 1, \qquad \frac{d}{dx} B_{n+1}(x) = B_n(x), \qquad \int_0^1 B_{n+1}(x)dx = 0.$$

The value of these periodized functions when x is an integer is given by

$$B_n(x) = \lim_{\varepsilon \to 0+} \frac{B_n(x + \varepsilon) + B_n(x - \varepsilon)}{2}.$$

Mordell's theorem reads as follows.

Theorem 2 (Mordell 1966 [27, 28]) *Let $a < b$ and set*

$$\omega_{[a,b]}(x) = \begin{cases} 0 & x < a \text{ or } x > b, \\ 1 & a < x < b, \\ 1/2 & x = a \text{ or } x = b. \end{cases}$$

(i) *If $q \in C^{w+1}(\mathbb{R})$ $n \in \mathbb{N}$, then, for every $x \in \mathbb{R}$,*

$$\sum_{n=-\infty}^{+\infty} \omega_{[a,b]}(x+n)q(x+n) = \int_a^b q(y)dy + \sum_{j=0}^{+w} \left(\frac{d^j q}{dx^j}(b) B_{j+1}(x-b) \right.$$
$$\left. - \frac{d^j q}{dx^j}(a) B_{j+1}(x-a) \right)$$
$$- \int_a^b \frac{d^{w+1}q}{dy^{w+1}}(y) B_{w+1}(x-y)dy.$$

(ii) *If $q \in C^\infty(\mathbb{R})$ and*

$$\lim_{w \to +\infty} \left(\frac{1}{2\pi} \right)^w \int_a^b \left| \frac{d^{w+1}q}{dy^{w+1}}(y) \right| dy = 0,$$

then, for every $x \in \mathbb{R}$,

$$\sum_{n=-\infty}^{+\infty} \omega_{[a,b]}(x+n)q(x+n) = \int_a^b q(y)dy + \sum_{j=0}^{+\infty} \left(\frac{d^j q}{dx^j}(b) B_{j+1}(x-b) \right.$$
$$\left. - \frac{d^j q}{dx^j}(a) B_{j+1}(x-a) \right).$$

Mordell's original result is essentially (ii) above with $a = 0$ and $b = 1$. Observe that the assumption on the growth of the derivatives of $q(x)$ implies that this function can be analytically extended to the entire complex plane. The example $q(x) = \cos(2\pi x)$, which is 1-periodic and has expansion zero, shows that this assumption is sharp. Variants of this theorem seem to be prior to Mordell's work (see e.g. [8]).

An immediate application of Theorem 2 is the classical Euler–MacLaurin summation formula. Indeed, from (i), when $a, b \in \mathbb{Z}$ and $x = 0$, since $B_{j+1}(0) = 0$ for even values of j, we obtain, for $q \in C^{w+1}(\mathbb{R})$,

$$
\left| \frac{1}{2} q(a) + q(a+1) + \cdots + q(b-1) + \frac{1}{2} q(b) \right.
$$
$$
\left. - \int_a^b q(y) dy - \sum_{j=1}^{\lfloor (w+1)/2 \rfloor} \left(\frac{d^{2j-1} q}{dx^{2j-1}}(b) - \frac{d^{2j-1} q}{dx^{2j-1}}(a) \right) B_{2j}(0) \right|
$$
$$
\leq \frac{\pi}{6} \left(\frac{1}{2\pi} \right)^w \int_a^b \left| \frac{d^{w+1} q}{dy^{w+1}}(y) \right| dy.
$$

It is well known that the above Euler–MacLaurin formula provides a quadrature rule. Indeed, setting $a = 0, b = N \in \mathbb{Z}^+$ and $q(x) = f(x/N)/N$ with $f \in C^{w+1}(\mathbb{R})$, one obtains

$$
\int_0^1 f(y) dy = \frac{1}{N} \left(\frac{1}{2} f(0) + f\left(\frac{1}{N} \right) + \cdots + f\left(\frac{N-1}{N} \right) + \frac{1}{2} f(1) \right)
$$
$$
+ \sum_{j=1}^{\lfloor w/2 \rfloor} \frac{1}{N^{2j}} \left(\frac{d^{2j-1} f}{dx^{2j-1}}(0) - \frac{d^{2j-1} f}{dx^{2j-1}}(1) \right) B_{2j}(0) + O(N^{-w-1}).
$$

Notice that only even powers of N appear in the remainder terms.

To state our results in the multidimensional setting we need to introduce a number of definitions.

Definition 3 Let $\mathcal{P}$ be a measurable subset in $\mathbb{R}^d$. For every $x \in \mathbb{R}^d$ the normalized solid angle at x is given by

$$
\omega_{\mathcal{P}}(x) = \lim_{\varepsilon \to 0+} \frac{1}{|\{|y| \leqslant 1\}|} \int_{|y| \leqslant 1} \chi_{\mathcal{P}}(x - \varepsilon y) dy.
$$

Assuming that the above limit exists for every $x \in \mathbb{R}^d$, then, for every continuous function $f(x)$ and for every positive integer N, we set

$$
S_N(f, \mathcal{P}) = N^{-d} \sum_{n \in \mathbb{Z}^d} \omega_{\mathcal{P}}(N^{-1} n) f(N^{-1} n).
$$

When $\mathcal{P}$ is a convex polytope (the convex hull of a finite number of points) the weight $\omega_{\mathcal{P}}(x)$ is well defined for every $x \in \mathbb{R}^d$. When $d = 3$ the value of $\omega_{\mathcal{P}}(x)$

can be computed explicitly from the coordinates of the vertices of the polytope using standard formulas of spherical trigonometry. See e.g. [17]. When $d > 3$ see [2], [5] and [29].

These weights $\omega_{\mathcal{P}}(x)$ and weighted sums $S_N(q, \mathcal{P})$ are not new in the literature; for example MacDonald showed that if $\mathcal{P}$ is a convex integer polytope (that is a convex polytope with integer vertices) and τ is an integer dilation, then

$$\sum_{n \in \mathbb{Z}^d} \omega_{\tau \mathcal{P}}(n) = (\mathrm{vol} \mathcal{P})\tau^d + a_{d-2}\tau^{d-2} + \cdots + \begin{cases} a_1 \tau & \text{if } d \text{ is odd}, \\ a_2 \tau^2 & \text{if } d \text{ is even}. \end{cases}$$

See e.g. [4] and [15].

An important property of these weights is that they are additive with respect to $\mathcal{P}$. More precisely, if $\mathcal{P}_1$ and $\mathcal{P}_2$ have disjoint interior then

$$\omega_{\mathcal{P}_1 \cup \mathcal{P}_2}(x) = \omega_{\mathcal{P}_1}(x) + \omega_{\mathcal{P}_2}(x).$$

This implies that also the weighted Riemann sums $S_N(f, \mathcal{P})$ are additive,

$$S_N(f, \mathcal{P}_1 \cup \mathcal{P}_2) = S_N(f, \mathcal{P}_1) + S_N(f, \mathcal{P}_2).$$

On the contrary, a different choice of weights may not guarantee the additivity.

Definition 4 For every multi-index of non-negative integers $J = (j_1, j_2, \ldots, j_d)$ and every $x = (x_1, x_2, \ldots, x_d)$ in $\mathbb{R}^d$, define the multivariate Bernoulli polynomials

$$B_J(x) = \begin{cases} B_{j_1}(x_1)B_{j_2}(x_2) \cdots B_{j_d}(x_d) & \text{if } 0 \leq x_k < 1, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, for $L \in GL(d, \mathbb{Z})$, define

$$B_{J,L}(x) = |L|^{-1} B_J \left((L^{-1})^t x \right).$$

Finally, define the periodized Bernoulli polynomials

$$\mathfrak{B}_{J,L}(x) = \sum_{n \in \mathbb{Z}^d} B_{J,L}(x + n).$$

At the points of discontinuity we assume the periodized Bernoulli polynomials to be regularized so that

$$\mathfrak{B}_{J,L}(x) = \lim_{\varepsilon \to 0+} \frac{1}{|\{|y| \leq 1\}|} \int_{|y| \leq 1} \mathfrak{B}_{J,L}(x - \varepsilon y) dy.$$

We refer the reader to Sect. 1 (Appendix B) for more comments on the construction of the periodized multivariate Bernoulli polynomials and their connections with the Lerch Zeta functions.

The next definitions are more technical and will be needed to describe the asymptotic behavior along different directions of the Fourier transform of functions supported on a simplex.

Definition 5 For every dimension $d \geqslant 1$, $\mathcal{F}^{(d)}$ is a collection of 2^{d-1} bases of $\mathbb{R}^d$,

$$\mathcal{F}^{(d)} = \left\{ \mathcal{B}_1^{(d)}, \ldots, \mathcal{B}_{2^{d-1}}^{(d)} \right\}.$$

Each basis $\mathcal{B}_j^{(d)}$ consists of the vectors $b_{j,k}^{(d)}$,

$$\mathcal{B}_j^{(d)} = \left\{ b_{j,1}^{(d)}, \ldots, b_{j,d}^{(d)} \right\}.$$

The vectors $b_{j,k}^{(d)}$ are defined recursively as follows. If $d = 1$, set $b_{1,1}^{(1)} = 1$. If $d = 2$, set

$$b_{1,1}^{(2)} = (1, 0), \quad b_{1,2}^{(2)} = (0, 1)$$

and

$$b_{2,1}^{(2)} = (1, -1), \quad b_{2,2}^{(2)} = (0, 1).$$

More generally, for $d \geqslant 2$,

$$\mathcal{F}^{(d)} = \mathcal{F}_1^{(d)} \cup \mathcal{F}_2^{(d)}$$

where

$$\mathcal{F}_1^{(d)} = \left\{ \mathcal{B}_1^{(d)}, \ldots, \mathcal{B}_{2^{d-2}}^{(d)} \right\},$$
$$\mathcal{F}_2^{(d)} = \left\{ \mathcal{B}_{2^{d-2}+1}^{(d)}, \ldots, \mathcal{B}_{2^{d-1}}^{(d)} \right\}$$

and for $1 \leqslant j \leqslant 2^{d-2}$ we set

$$b_{j,k}^{(d)} = \left(b_{j,k}^{(d-1)}, 0 \right), \quad k = 1, \ldots, d-1,$$
$$b_{j,d}^{(d)} = (0, \ldots, 0, 1),$$

and for $2^{d-2} + 1 \leqslant j \leqslant 2^{d-1}$ we set

$$b_{j,k}^{(d)} = \left(b_{j-2^{d-2},k}^{(d-1)}, -b_{j-2^{d-2},k}^{(d-1)} \cdot \mathbf{1}_{d-1} \right), \quad k = 1, \ldots, d-1,$$

$$b_{j,d}^{(d)} = (0, \ldots, 0, 1),$$

where $\mathbf{1}_{d-1} = (1, \ldots, 1) \in \mathbb{R}^{d-1}$. We will also associate to every basis $\mathcal{B} \in \mathcal{F}^{(d)}$ with $d > 1$ a $(d-1)$-dimensional multi-index $(v_2, \ldots, v_d) \in \{1, 2\}^{d-1}$ in the following way: $v_d = \ell$ if and only if $\mathcal{B} \in \mathcal{F}_\ell^{(d)}$ and, if $d > 2$, $(v_2, \ldots, v_{d-1})$ is the multi-index associated with the $(d-1)$-dimensional basis $\mathcal{B}'$ used to define $\mathcal{B}$ recursively. Observe that there is a one to one correspondence between the bases in $\mathcal{F}_\ell^{(d)}$ and the multi-indices in $\{1, 2\}^{d-1}$. Therefore, given a multi-index $V = (v_1, v_2, \ldots, v_d) \in \{1, 2\}^d$, we will denote also by $\mathcal{B}_V$ the basis corresponding to the vector $(v_2, \ldots, v_d)$.

The role of v_1, the first component of the vector V, will be made clear in what follows.

Definition 6 For every multi-index $V = (v_1, \ldots, v_d) \in \{1, 2\}^d$ we define the vectors $\lambda_V \in \mathbb{R}^d$ recursively as follows. For $d = 1$

$$\lambda_1 = 0,$$
$$\lambda_2 = 1.$$

If $d = 2$,

$$\lambda_{(1,1)} = (0, 0),$$
$$\lambda_{(2,1)} = (1, 0),$$

and

$$\lambda_{(1,2)} = (0, 1),$$
$$\lambda_{(2,2)} = (1, 0).$$

In general, for all $d \geqslant 2$, if $v_d = 1$ we set

$$\lambda_{(v_1, v_2, \ldots, v_d)} = \left(\lambda_{(v_1, v_2, \ldots, v_{d-1})}, 0 \right),$$

if $v_d = 2$ we set

$$\lambda_{(v_1, v_2, \ldots, v_d)} = \left(\lambda_{(v_1, v_2, \ldots, v_{d-1})}, 1 - \lambda_{(v_1, v_2, \ldots, v_{d-1})} \cdot \mathbf{1}_{d-1} \right).$$

We state our main result. Let $S_d \subseteq \mathbb{R}^d$ be the standard simplex given by

$$S_d = \left\{ x \in \mathbb{R}^d : x_j \geqslant 0, \ \sum_{j=1}^d x_j \leqslant 1 \right\}.$$

Table 1 The various bases and multi-indices of Definitions 5 and 6, for the dimensions $d = 1, 2, 3$

d	V	$(v_2, \ldots, v_d)$	$\mathcal{F}_d$	λ_V
1	1	/	$\mathcal{B}_1^{(1)} = \{1\}$	0
	2			1
2	$(1, 1)$	1	$\mathcal{B}_1^{(2)} = \{(1, 0), (0, 1)\}$	$(0, 0)$
	$(2,1)$			$(1, 0)$
	$(1, 2)$	2	$\mathcal{B}_2^{(2)} = \{(1, -1), (0, 1)\}$	$(0, 1)$
	$(2, 2)$			$(1, 0)$
3	$(1, 1, 1)$	$(1, 1)$	$\mathcal{B}_1^{(3)} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$	$(0, 0, 0)$
	$(2, 1, 1)$			$(1, 0, 0)$
	$(1, 2, 1)$	$(2, 1)$	$\mathcal{B}_2^{(3)} = \{(1, -1, 0), (0, 1, 0), (0, 0, 1)\}$	$(0, 1, 0)$
	$(2, 2, 1)$			$(1, 0, 0)$
	$(1, 1, 2)$	$(1, 2)$	$\mathcal{B}_3^{(3)} = \{(1, 0, -1), (0, 1, -1), (0, 0, 1)\}$	$(0, 0, 1)$
	$(2, 1, 2)$			$(1, 0, 0)$
	$(1, 2, 2)$	$(2, 2)$	$\mathcal{B}_4^{(3)} = \{(1, -1, 0), (0, 1, -1), (0, 0, 1)\}$	$(0, 1, 0)$
	$(2, 2, 2)$			$(1, 0, 0)$

Theorem 7 *Let $\mathcal{P}$ be a simplex in $\mathbb{R}^d$ with vertices $\mathbf{0}, \mathbf{m}_1, \ldots, \mathbf{m}_d \in \mathbb{Z}^d$, and let $M \in GL(d, \mathbb{Z})$ be the $d \times d$ matrix with columns $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_d$, which maps the standard simplex onto $\mathcal{P}$. Let $q \in C^{w+1}(\mathbb{R}^d)$ with $w \in \mathbb{N}$ and for $\tau > 0$, let $q_{\tau, M}(x) = q(\tau M x)$. Then, for every $x \in \mathbb{R}^d$ and for every $\tau > 0$,*

$$\sum_{n \in \mathbb{Z}^d} \omega_{\tau \mathcal{P}}(x + n) q(x + n)$$
$$= \det(M) \sum_{V \in \{1,2\}^d} \sum_{I \in \{0,1\}^d} \sum_{|J| \leqslant w, J \sqsubseteq I} \tau^{d - |I| - |J|} \langle \mu(V, I, J), q_{\tau, M} \rangle \mathfrak{B}_{J + I, (MD_V)^t}$$
$$\times (x - \tau M \lambda_V) + \mathcal{R}_w(x).$$

Here $\mu(V, I, J)$ are certain integro-differential functionals that will be introduced in Definition 28, $D_V = [b_1|b_2|\cdots|b_d]$ where $\{b_1, b_2, \ldots, b_d\}$ is the basis $\mathcal{B}_V$ and $J \sqsubseteq I$ means that $j_k = 0$ if $i_k = 0$. Moreover, for every $\delta > 0$ and every $\tau_0 > 0$ there exists a constant c depending on δ and τ_0 but independent of q, M and w, such that for every $\tau > \tau_0$

$$|\mathcal{R}_w(x)| \leqslant c \det(M) \tau^{d - w - 1} (2^{d-2} \pi^{-1} + \delta)^{w+1} \sup_{w - d + 2 \leqslant |\alpha| \leqslant w + 1} \sup_{x \in S_d} \left| \frac{\partial^\alpha q_{\tau, M}}{\partial x^\alpha}(x) \right|.$$

For $d = 2$ a similar formula is contained in [9]. An immediate consequence is the following corollary.

Corollary 8 *With the above notation, assume that $q \in C^\infty(\mathbb{R}^d)$ and that there exist $c, \delta > 0$ such that for every positive integer w*

$$\sup_{|\alpha|=w} \sup_{x \in S_d} \left| \frac{\partial^\alpha q_{\tau,M}}{\partial x^\alpha}(x) \right| \leqslant c\tau^w (2^{d-2}\pi^{-1} + \delta)^{-w}.$$

Then $\sum_{n \in \mathbb{Z}^d} \omega_{\tau\mathcal{P}}(x+n)q(x+n)$ can be expanded in a uniformly convergent series of Bernoulli polynomials

$$\sum_{n \in \mathbb{Z}^d} \omega_{\tau\mathcal{P}}(x+n)q(x+n)$$

$$= \det(M) \sum_{V \in \{1,2\}^d} \sum_{I \in \{0,1\}^d} \sum_{|J| \geq 0, J \sqsubseteq I} \tau^{d-|I|-|J|} \langle \mu(V,I,J), q_{\tau,M} \rangle \mathfrak{B}_{J+I,(MD_V)^t}$$

$$\times (x - \tau M\lambda_V).$$

The uniform convergence in the above corollary seems paradoxical, since the periodized function in the left-hand side is a priori discontinuous, but observe that also in the right-hand side there are a priori infinitely many Bernoulli polynomials that are discontinuous.

Taking $x = 0$ and $\tau \in \mathbb{Z}$, since the functions $\mathfrak{B}_{J,L}(x)$ are periodic, from Theorem 7 one immediately obtains an Euler–MacLaurin formula.

Theorem 9 *Let $\mathcal{P}$ be a simplex in $\mathbb{R}^d$ with vertices $\mathbf{0}, \mathbf{m}_1, \ldots, \mathbf{m}_d \in \mathbb{Z}^d$, and let $M \in GL(d, \mathbb{Z})$ be the $d \times d$ matrix with columns $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_d$, which maps the standard simplex onto $\mathcal{P}$. Let $q \in C^{w+1}(\mathbb{R}^d)$ with $w \in \mathbb{N}$ and for $\tau > 0$, let $q_{\tau,M}(x) = q(\tau M x)$. Then, for every positive integer $\tau > 0$,*

$$\sum_{n \in \mathbb{Z}^d} \omega_{\tau\mathcal{P}}(n)q(n)$$

$$= \det(M) \sum_{V \in \{1,2\}^d} \sum_{I \in \{0,1\}^d} \sum_{|J| \leqslant w, J \sqsubseteq I} \tau^{d-|I|-|J|} \langle \mu(V,I,J), q_{\tau,M} \rangle \mathfrak{B}_{J+I,(MD_V)^t}(0)$$

$$+ \mathcal{R}_w.$$

Moreover, for every $\delta > 0$ and every $\tau_0 > 0$ there exists a constant c depending on δ and τ_0 but independent of q, M and w, such that for every $\tau > \tau_0$,

$$|\mathcal{R}_w| \leqslant c \det(M)\tau^{d-w-1}(2^{d-2}\pi^{-1} + \delta)^{w+1} \sup_{w-d+2 \leqslant |\alpha| \leqslant w+1} \sup_{x \in S_d} \left| \frac{\partial^\alpha q_{\tau,M}}{\partial x^\alpha}(x) \right|.$$

We will see that when $I = (0, \ldots, 0)$ the only non-vanishing term in the above sum corresponds to $V = (1, \ldots, 1)$ and is the integral of q over $\tau\mathcal{P}$.

Similarly to the one dimensional case, Theorem 9 applied to the function $N^{-d} f(x/N)$ gives a quadrature formula for simplices. Then, the additivity of the

weighted Riemann sums allows to extend this quadrature formula to more general settings. Let us recall that a homogeneous simplicial d-complex is a simplicial complex where every simplex of dimension less than d is a face of some simplex of dimension d. It is known that every (bounded) convex polytope can be decomposed into simplices without additional vertices. Hence, one can associate to a convex polytope a homogeneous simplicial complex with the same vertices. This is obvious in dimension $d = 2$, less obvious in higher dimensions (see [16], see also Proposition 5.2 and Theorem 5.3 in [32, Chapter 5]).

Theorem 10 *Let $\mathcal{P}$ be a homogeneous simplicial d-complex with integer vertices in $\mathbb{R}^d$. Let w be a non-negative integer and let $f \in C^{w+1}(\mathbb{R}^d)$. Then, there exists a numerical sequence $\{\gamma_k\}_{0 < k \leqslant w/2}$ such that for every positive integer N we have*

$$S_N(f, \mathcal{P}) = \int_{\mathcal{P}} f(x)dx + \sum_{0 < k \leqslant w/2} \gamma_k N^{-2k} + O(N^{-w-1}).$$

For $d = 2$ a similar formula is contained in [9]. A simple consequence of Theorem 10 is the following.

Theorem 11 *Let $\mathcal{P}$ be a homogeneous simplicial d-complex with integer vertices in $\mathbb{R}^d$. Let w be a non-negative integer and let $f \in C^{w+1}(\mathbb{R}^d)$. Finally, let $\{c_j\}_{0 \leqslant j \leqslant w/2}$ be the solution of the Vandermonde system*

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2^{-2} & (2^{-2})^2 & \cdots & (2^{-2})^{\lfloor w/2 \rfloor} \\ 1 & 2^{-4} & (2^{-4})^2 & \cdots & (2^{-4})^{\lfloor w/2 \rfloor} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{-2\lfloor w/2 \rfloor} & (2^{-2\lfloor w/2 \rfloor})^2 & \cdots & (2^{-2\lfloor w/2 \rfloor})^{\lfloor w/2 \rfloor} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{\lfloor w/2 \rfloor} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then,

$$\int_{\mathcal{P}} f(x)dx = \sum_{0 \leqslant j \leqslant w/2} c_j S_{2^j N}(f, \mathcal{P}) + O(N^{-w-1}).$$

The coefficients γ_k in Theorem 10 are integro-differential functionals applied to the function $f(x)$. In Theorem 11 these cumbersome coefficients have disappeared and only weighted Riemann sums are present.

We have not found a multidimensional analog of Mordell's theorem in the literature. On the contrary the literature on multidimensional Euler–MacLaurin summation formulas is vast and in continuous growth and to have a comprehensive list of references is a challenging task. Here we recall a few of these results and we try to compare them with ours, apologizing in advance with all the authors that we do not explicitly mention.

If $\mathcal{P}$ denotes a regular integral convex polytope, Karshon, Sternberg and Weitsman obtained in [22] the weighted formula

$$\sum_{n \in \mathbb{Z}^d} \sigma_{\mathcal{P}}(n)q(n) = \prod_{i=1}^{\ell} L^{2k}(D_i) \int_{\mathcal{P}(h_1,\dots,h_\ell)} q(x)\,dx + R_{\mathcal{P}}^{2k+1}(q)$$

where $q \in C^{2k+1}$ is compactly supported, $R_{\mathcal{P}}^{2k+1}(f)$ is a remainder explicitly given, ℓ is the number of facets, i.e., faces of $\mathcal{P}$ of codimension 1 and $\mathcal{P}(h_1,\dots,h_\ell)$ is a perturbation of the original polytope obtained expanding outward at distance h_i in the direction of th i-th facet. The weight function $\sigma_{\mathcal{P}}$ is defined to be 0 in the exterior of $\mathcal{P}$, 1 in the interior of $\mathcal{P}$ and $\sigma_{\mathcal{P}}(x) = 2^{-c(x)}$ if x is on the boundary of $\mathcal{P}$ and where $c(x)$ is the codimension of the smallest face containing x. The operators $L^{2k}(D_i)$ are the differential operators defined by the operators $D_i = \partial/\partial h_i$, $i = 1, \dots, d$ and the functions

$$L^{2k}(x) = 1 + \sum_{j=1}^{k} \frac{1}{(2j)!} b_{2j} x^{2j}$$

where the b_{2j}'s are Bernoulli numbers. A similar formula is proved for simple polytopes in [23]. Such Euler–MacLaurin formula is quite close to our formula in the spirit, but we highlight a main difference. On one hand the weight function $\sigma_{\mathcal{P}}$ is immediate to compute, since it only depends on the codimension of a face at a given point. On the other hand $\sigma_{\mathcal{P}}$ is not additive, whereas $\omega_{\mathcal{P}}$ is, and this allows to apply Theorem 9 to polytopes by glueing simplices together.

We also refer the reader to the paper [24] and the references therein; in this work the authors review and discuss the results in [22, 23] together with previous results by several different authors [11–13, 25]. See also [1].

Another result we recall is the Euler–MacLaurin summation formula in [7]. Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a semi-rational convex polyhedron of dimension $\ell \leq d$. Semi-rational means that the facets of $\mathcal{P}$ are affine hyperplanes parallel to rational ones. Then, the authors provide the asymptotic expansion, as $N \to +\infty$,

$$\frac{1}{N^\ell} \sum_{n \in N\mathcal{P} \cap \mathbb{Z}^d} f(N^{-1}n) \sim \int_{\mathcal{P}} f(x)dx + \sum_{k \geq 1} a_k(N)N^{-k}.$$

The authors also discuss their results in comparison with other previous results [26, 33].

Finally, we also recall the works [3, 6, 19–21].

Our proofs exploit harmonic analysis techniques with classical tools such as the Poisson summation formula. Recall that if f is an integrable function on $\mathbb{R}^d$ its Fourier transform $\widehat{f}$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i \xi \cdot x}dx.$$

The following lemmas are well known.

Lemma 12 *Let $\varphi(x)$ be a non-negative, radial, smooth, function in $\mathbb{R}^d$, with compact support and integral one, and for every $\varepsilon > 0$ set $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(\varepsilon^{-1}x)$. Let $\mathcal{P}$ be a convex polytope in $\mathbb{R}^d$, let $q(x)$ be a smooth function in $\mathbb{R}^d$ and let $Q(x) = q(x)\chi_\mathcal{P}(x)$. Then*

$$\lim_{\varepsilon \to 0+} \varphi_\varepsilon * Q(x) = \omega_\mathcal{P}(x)q(x).$$

Proof Integrate in polar coordinates. □

Lemma 13 *With the notation of the above lemma, for every $\varepsilon > 0$ and every $x \in \mathbb{R}^d$ one has*

$$\sum_{k \in \mathbb{Z}^d} \varphi_\varepsilon * Q(x+k) = \sum_{k \in \mathbb{Z}^d} \widehat{\varphi}(\varepsilon k)\widehat{Q}(k)e^{2\pi i k \cdot x}.$$

Moreover

$$\sum_{k \in \mathbb{Z}^d} \omega_\mathcal{P}(x+k)q(x+k) = \lim_{\varepsilon \to 0+} \sum_{k \in \mathbb{Z}^d} \widehat{\varphi}(\varepsilon k)\widehat{Q}(k)e^{2\pi i k \cdot x}.$$

The first series is a finite sum of smooth functions. The second series converges absolutely and uniformly.

Proof This is the Poisson summation formula. □

To prove our results we need an explicit formula for the asymptotic expansion of $\widehat{q\chi_\mathcal{P}}$ when $\mathcal{P}$ is a simplex, which requires a non-trivial effort to be proved (Lemmas 20 and 33). The 2-dimensional case was dealt with in [9, Lemma 5]. Very elegant expansion formulas for $\widehat{\chi_\mathcal{P}}$ (that is when $q \equiv 1$) in any dimension d already appeared in [10, 15]. See also [30] and the references therein.

The paper is organized as follows. In Sect. 2 we present a Fourier analytic proof of Mordell's theorem both for sake of completeness and for illustrating the proof strategy that we will use in the multivariate setting. In Sect. 3 we study the Fourier transform of a function supported on a simplex. In Sect. 4 we prove our main result on the expansion of

$$\sum_{n \in \mathbb{Z}^d} \omega_{\tau\mathcal{P}}(x+n)q(x+n)$$

in terms of our multivariate Bernoulli polynomials, that is Theorem 7 and Corollary 8, whereas in Sect. 5 we prove Theorems 10 and 11. We also include Appendix A (Sect. 1), where we collect some well-known results in harmonic analysis on groups that we use, and Appendix B (Sect. 1), where a further description of the periodized multivariate Bernoulli polynomials is given.

2 Bernoulli Polynomials and a Theorem of Mordell

The classical Bernoulli polynomials have elegant trigonometric expansions, which predate Fourier. Recall that if f is an integrable function on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, its Fourier series at a point ξ is given by

$$\sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n \xi}$$

where the Fourier coefficient $\widehat{f}(n)$ is defined as

$$\widehat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} \, dx.$$

Theorem 14 (L. Euler, 1752) *If $n \geqslant 1$, then, for every x,*

$$B_n(x) = - \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi i k x}}{(2\pi i k)^n}.$$

Proof If $n = 1$ and $0 < x < 1$, then

$$B_1(x) = x - 1/2 = \sum_{k=-\infty}^{+\infty} \left(\int_0^1 (y - 1/2) e^{-2\pi i k y} \, dy \right) e^{2\pi i k x} = - \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi i k x}}{2\pi i k}.$$

The symmetric partial sums with $-K \leqslant k \leqslant K$ of the above series converge pointwise for every $0 < x < 1$, and by symmetry they also converge to zero for $x = 0$ and for $x = 1$. Since $\frac{d}{dx} B_{n+1}(x) = B_n(x)$, the Fourier expansion of $B_{n+1}(x)$ follows by integrating term by term the series of $B_n(x)$. Since $\int_0^1 B_{n+1}(x) dx = 0$ the constant of integration is zero. Observe that for $n > 1$ the Fourier series of $B_n(x)$ converges absolutely and uniformly. $\square$

The original proof of Euler is different and very interesting, see [18]. The following bounds are a consequence of the above trigonometric expansions.

Corollary 15 *The periodic Bernoulli polynomials $B_n(x)$ with $n \geqslant 0$ are bounded by $(\pi^2/3)(2\pi)^{-n}$. More precisely,*

$$(2\pi)^{-n} \leqslant \sup_{x \in [0,1]} |B_n(x)| \leqslant (\pi^2/3)(2\pi)^{-n}.$$

Proof If $n = 0$ then $B_0(x) = 1$ and the lemma holds. If $n = 1$ and $\lfloor x \rfloor$ denotes the integer part of x, then

$$B_1(x) = x - \lfloor x \rfloor - 1/2,$$

so that

$$\sup_{x\in[0,1]} |B_1(x)| = 1/2$$

and again the lemma holds. Finally, if $n > 1$,

$$|B_n(x)| = \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{2\pi ikx}}{(2\pi ik)^n} \right| \leqslant 2(2\pi)^{-n} \sum_{k=1}^{+\infty} k^{-n} = 2(2\pi)^{-n}\zeta(n).$$

Observe that $\zeta(n) \leqslant \zeta(2) \leqslant \pi^2/6$. Also observe that the Fourier coefficient with $k = 1$ is $(2\pi i)^{-n}$, so that

$$(2\pi)^{-n} \leqslant \int_0^1 |B_n(x)|dz \leqslant \sup_{x\in[0,1]} |B_n(x)|.$$

$\square$

The following lemma provides an asymptotic expansion of the Fourier transform of a piecewise smooth function.

Lemma 16 *Let $w \geqslant 0$. If the function $q(x)$ has $w + 1$ integrable derivatives in $[a, b]$, then for every $\xi \neq 0$*

$$\int_a^b q(x)e^{-2\pi ix\xi}dx = \sum_{j=0}^{w}(2\pi i\xi)^{-j-1}\left(e^{-2\pi ia\xi}\frac{d^j q}{dx^j}(a) - e^{-2\pi ib\xi}\frac{d^j q}{dx^j}(b)\right)$$

$$+ (2\pi i\xi)^{-w-1}\int_a^b \frac{d^{w+1}q}{dx^{w+1}}(x)e^{-2\pi ix\xi}dx.$$

Proof Integrate by parts.

$\square$

With the above results one easily obtains the following.

Proof of Theorem 2 By Lemma 16 and with the notation of Lemmas 12 and 13, for every $x \in \mathbb{R}$ we have the chain of equalities

$$\sum_{n=-\infty}^{+\infty} \omega_{[a,b]}(x+n)q(x+n) = \lim_{\varepsilon\to0+} \sum_{k=-\infty}^{+\infty} \widehat{\varphi}(\varepsilon k)\widehat{Q}(k)e^{2\pi ikx}$$

$$= \lim_{\varepsilon\to0+} \sum_{k=-\infty}^{+\infty} \widehat{\varphi}(\varepsilon k)\left(\int_a^b q(y)e^{-2\pi iky}dy\right)e^{2\pi ikx}$$

$$= \int_a^b q(y)dy + \lim_{\varepsilon\to0+} \sum_{k\neq0} \widehat{\varphi}(\varepsilon k)\left(\sum_{j=0}^{w}(2\pi ik)^{-j-1}\left(e^{-2\pi iak}\frac{d^j q}{dx^j}(a)\right.\right.$$

$$- e^{-2\pi i b k} \frac{d^j q}{dx^j}(b) \bigg) \bigg) e^{2\pi i k x}$$

$$+ \lim_{\varepsilon \to 0+} \sum_{k \neq 0} \widehat{\varphi}(\varepsilon k) \left((2\pi i k)^{-w-1} \int_a^b \frac{d^{w+1} q}{dy^{w+1}}(y) e^{-2\pi i y k} dy \right) e^{2\pi i k x}$$

$$= \int_a^b q(y) dy + \sum_{j=0}^w \frac{d^j q}{dx^j}(a) \left(\lim_{\varepsilon \to 0+} \sum_{k \neq 0} \widehat{\varphi}(\varepsilon k)(2\pi i k)^{-j-1} e^{2\pi i k(x-a)} \right)$$

$$- \sum_{j=0}^w \frac{d^j q}{dx^j}(b) \left(\lim_{\varepsilon \to 0+} \sum_{k \neq 0} \widehat{\varphi}(\varepsilon k)(2\pi i k)^{-j-1} e^{2\pi i k(x-b)} \right)$$

$$+ \int_a^b \frac{d^{w+1} q}{dy^{w+1}}(y) \left(\lim_{\varepsilon \to 0+} \sum_{k \neq 0} \widehat{\varphi}(\varepsilon k)(2\pi i k)^{-w-1} e^{2\pi i k(x-y)} \right) dy$$

$$= \int_a^b q(y) dy - \sum_{j=0}^w \frac{d^j q}{dx^j}(a) B_{j+1}(x-a) + \sum_{j=0}^w \frac{d^j q}{dx^j}(b) B_{j+1}(x-b)$$

$$- \int_a^b \frac{d^{w+1} q}{dy^{w+1}}(y) B_{w+1}(x-y) dy$$

and (i) is proved. The second part follows from Corollary 15.   □

The above proof is not the original one of Mordell but it is inspired by [14].

3 The Fourier Transform of a Function Supported on a Simplex

A key ingredient for the proofs of our main results is a precise estimate of the Fourier transform of a function restricted to a simplex. We first consider the standard simplex.

3.1 The Standard Simplex

Let

$$S_d = \left\{ x \in \mathbb{R}^d : x_j \geqslant 0, \quad \sum_{j=1}^d x_j \leqslant 1 \right\}$$

be the standard simplex. We want to give an asymptotic expansion of the Fourier transform of the function $G(x) = g(x)\chi_{S_d}(x)$ where $g \in C^{w+1}(\mathbb{R}^d)$. In [10, 15], see also [30] and the references therein, there are elegant symmetric formulas for $\widehat{\chi_{S_d}}(\xi)$. The formulas we obtain are less elegant but somehow more explicit and, in particular, we provide a formula when ξ belongs to a singular direction as well. Since the asymptotic behaviour of $\widehat{G}(\xi)$ depends on the faces of S_d that are orthogonal to ξ,

it is natural to have different formulas in different regions. Hence, we need to partition the space of frequencies into a finite number of cones $\mathcal{Q}(\theta)$.

Definition 17 Let Θ_d be the class of all subspaces of $\mathbb{R}^d$ generated by any possible choice of vectors in all the bases of $\mathcal{F}^{(d)}$. Then, Θ_d induces a partition of $\mathbb{R}^d$ into a finite number of (possibly disconnected) conical regions $\mathcal{Q}(\theta)$, $\theta \in \Theta_d$, defined as follows: $\xi \in \mathcal{Q}(\theta)$ if and only if ξ is orthogonal to all the vectors in θ, but it is not orthogonal to any other vector in the bases of $\mathcal{F}^{(d)}$ which is not in θ. Namely,

$$
\mathcal{Q}(\theta) = \left\{ \xi \in \mathbb{R}^d : \text{for all } b \in \bigcup_{\mathcal{B} \in \mathcal{F}^{(d)}} \mathcal{B}, \, \xi \cdot b = 0 \text{ iff } b \in \theta \right\}
$$

$$
= \left\{ \xi \in \theta^\perp : \prod_{b \in \mathcal{B} \setminus \theta} (b \cdot \xi) \neq 0, \text{ for all } \mathcal{B} \in \mathcal{F}^{(d)} \right\}.
$$

We explicitly assume that the zero dimensional space belongs to Θ_d and in this case the associated cone has nonempty interior. In the other cases such cones have empty interior.

Lemma 18 (i) $\{\mathcal{Q}(\theta)\}_{\theta \in \Theta_d}$ is a partition of $\mathbb{R}^d$.
(ii) Let $F(x)$ be a bounded function on $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ and let θ be in Θ_d. Then

$$
\sum_{m \in \mathcal{Q}(\theta) \cap \mathbb{Z}^d} \widehat{F}(m) e^{2\pi i m x}
$$

is a bounded function and there exists $c(\theta)$ such that

$$
\sup_x \left| \sum_{m \in \mathcal{Q}(\theta) \cap \mathbb{Z}^d} \widehat{F}(m) e^{2\pi i m x} \right| \leqslant c(\theta) \sup_x |F(x)|.
$$

Proof (i) For any $\xi \in \mathbb{R}^d$, $\xi \in \mathcal{Q}(\theta_\xi)$ where $\theta_\xi = \left\langle b \in \bigcup_{\mathcal{B} \in \mathcal{F}^{(d)}} \mathcal{B} : b \cdot \xi = 0 \right\rangle$. On the other hand if $\theta_1 \neq \theta_2$, that is if there exists, say, $v \in \theta_1 \setminus \theta_2$, then there exists $\bar{b} \in \bigcup_{\mathcal{B} \in \mathcal{F}^{(d)}} \mathcal{B}$ such that $\bar{b} \in \theta_1 \setminus \theta_2$. Now if $\xi \in \mathcal{Q}(\theta_1)$ then $\xi \cdot \bar{b} = 0$. This implies that $\xi \notin \mathcal{Q}(\theta_2)$. Hence $\mathcal{Q}(\theta_1) \cap \mathcal{Q}(\theta_2) = \emptyset$ and it follows that $\{\mathcal{Q}(\theta)\}_{\theta \in \Theta_d}$ is a partition of $\mathbb{R}^d$.
(ii) Let $\theta \in \Theta_d$, let $\theta^\perp = \{\xi \in \mathbb{R}^d : \xi \cdot b = 0 \text{ for every } b \in \theta\}$ and for $b \notin \theta$ let

$$
\theta_b^\perp = \theta^\perp \cap \langle b \rangle^\perp.
$$

Also $\{b_1, \ldots, b_N\} = \left\{ b \in \bigcup_{\mathcal{B} \in \mathcal{F}^{(d)}} \mathcal{B} : b \notin \theta \right\}$. Then

$$Q(\theta) = \theta^\perp \setminus \bigcup_{j=1}^N \theta_{b_j}^\perp$$

and therefore

$$\sum_{m \in Q(\theta) \cap \mathbb{Z}^d} \widehat{F}(m) e^{2\pi i m \cdot t} = \sum_{m \in \theta^\perp \cap \mathbb{Z}^d} \widehat{F}(m) e^{2\pi i m \cdot t} - \sum_{m \in \bigcup_{j=1}^N L_j} \widehat{F}(m) e^{2\pi i m \cdot t}$$

where $L_j = \theta_{b_j}^\perp \cap \mathbb{Z}^d$. By the inclusion–exclusion principle

$$\chi_{\bigcup_{j=1}^N L_j}(m) = \sum_{k=1}^N (-1)^{k-1} \sum_{\substack{I \subseteq \{1,2,\ldots,N\} \\ |I|=k}} \chi_{L_I}(m)$$

where $L_I = \bigcap_{j \in I} L_j$. Therefore

$$\sum_{m \in Q(\theta) \cap \mathbb{Z}^d} \widehat{F}(m) e^{2\pi i m \cdot t} = \sum_{m \in \theta^\perp \cap \mathbb{Z}^d} \widehat{F}(m) e^{2\pi i m \cdot t}$$
$$+ \sum_{k=1}^N (-1)^k \sum_{\substack{I \subseteq \{1,2,\ldots,N\} \\ |I|=k}} \sum_{m \in L_I} \widehat{F}(m) e^{2\pi i m \cdot t}.$$

Observe now that $\theta^\perp \cap \mathbb{Z}^d$ and L_I are subgroups of $\mathbb{Z}^d$. To conclude the proof then it suffices to recall that the restriction operator to a subgroup $\mathcal{H}$

$$R_\mathcal{H} F(t) = \sum_{m \in \mathcal{H}} \widehat{F}(m) e^{2\pi i m \cdot t}$$

is a bounded operator on $L^\infty(\mathbb{T}^d)$. See Lemma 40 in Appendix A. $\square$

We also need the following elementary lemma.

Lemma 19 (i) *Set* $n = (n', n_d) \in \mathbb{Z}^{d-1} \times \mathbb{Z}$ *and* $(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$. *Assume that* $H(n')$ *are the Fourier coefficients of a periodic function* $h(x')$ *and* $K(n_d)$ *are the Fourier coefficients of a periodic function* $k(x_d)$. *Then* $H(n')K(n_d)$ *are the Fourier coefficients of* $h(x')k(x_d)$.

(ii) *Assume that* $H(n)$ *with* $n \in \mathbb{Z}^d$ *are the Fourier coefficients of a periodic function bounded by* A *and assume that* $K(n_d)$ *are the Fourier coefficients of a periodic function*

bounded by B. Then $H(n)K(n_d)$ are the Fourier coefficients of a periodic function bounded by AB.

(iii) *Let $T \in SL(d, \mathbb{Z})$, $y \in \mathbb{R}^d$, and let $H(n)$ be the Fourier coefficients of a periodic function $h(x)$. Then $e^{2\pi i y \cdot n} H(Tn)$ are the Fourier coefficients of the periodic function $h\left((T^{-1})^t(x+y)\right)$.*

Proof The first part (i) is trivial. To prove (ii) let $h(x)$ be the periodic function on $\mathbb{T}^d$ with Fourier coefficients $H(n)$ and $k(x_d)$ be the periodic function on $\mathbb{T}$ with Fourier coefficients $K(n_d)$. Also, let μ be the product on the torus $\mathbb{T}^d$ of the Dirac delta centered at the origin in the variables x' and $k(x_d)$. Then $\widehat{\mu}(n) = K(n_d)$ and the total variation $\|\mu\|$ of μ is bounded by B. Finally observe that $H(n)K(n_d)$ are the Fourier coefficients of $h * \mu(x)$ and

$$|h * \mu(x)| \leqslant \sup |h(x)| \, \|\mu\|.$$

Finally, the proof of (iii) is very simple. If suffices to observe that

$$\sum_{n \in \mathbb{Z}^d} e^{2\pi i y \cdot n} H(Tn) e^{2\pi i n \cdot x} = \sum_{n \in \mathbb{Z}^d} H(Tn) e^{2\pi i n \cdot (x+y)} = \sum_{m \in \mathbb{Z}^d} H(m) e^{2\pi i m \cdot (T^{-1})^t(x+y)}.$$

$\square$

With the notation introduced in Sect. 1, we have the following crucial lemma.

Lemma 20 *Let S_d be the standard simplex in $\mathbb{R}^d$. There exist linear functionals $\{\alpha(\theta, V, J)\}$ indexed by $\theta \in \Theta_d$, $V \in \{1, 2\}^d$, $J \in \mathbb{N}^d$ with the following properties.*

(i) *For any integer $w \geqslant 1$, for any $g \in C^{w+1}(\mathbb{R}^d)$, for every $\theta \in \Theta_d$ and for every $\xi \in \mathcal{Q}(\theta)$,*

$$\widehat{g \chi_{S_d}}(\xi) = \sum_{V \in \{1,2\}^d} \sum_{|J| \leqslant w} \frac{\langle \alpha(\theta, V, J), g \rangle \, e^{-2\pi i \lambda_V \cdot \xi}}{\prod_{b_k \in \mathcal{B}_V \setminus \theta} (2\pi i b_k \cdot \xi)^{j_k+1}} + \mathcal{R}_{\theta, w}(g, \xi).$$

In the above formula we adopt the following convention: $\mathcal{B}_V = \{b_1, \ldots, b_d\}$ is the basis associated to the multi-index V and $J = (j_1, \ldots, j_d)$ with $j_k = 0$ whenever $b_k \in \theta$.

(ii) *The coefficients $\langle \alpha(\theta, V, J), g \rangle$ satisfy the estimates*

$$|\langle \alpha(\theta, V, J), g \rangle| \leqslant c \, 2^{(d-1)|J|} \sup_{|\alpha| \leqslant |J|} \sup_{x \in S_d} \left| \frac{\partial^\alpha g}{\partial x^\alpha}(x) \right|.$$

(iii) *The remainder $\mathcal{R}_{\theta, w}(g, \xi)$ has the property that for every $\Omega > 1/(2\pi)$ and every $\tau_0 > 0$ there exists $U = U(d) = U(d, \Omega, \tau_0) > 0$ such that for every $\tau > \tau_0$ and $w \geqslant 1$,*

$$\left\{ \chi_{\mathcal{Q}(\theta)}(n) \mathcal{R}_{\theta, w}(g, \tau n) \right\}_{n \in \mathbb{Z}^d}$$

are the Fourier coefficients of a function on the torus $\mathbb{T}^d$ bounded by

$$U \, (2^{d-1} \Omega \tau^{-1})^{w+1} \quad \sup_{w-d+2 \leqslant |\alpha| \leqslant w+1} \quad \sup_{x \in S_d} \left| \frac{\partial^\alpha g}{\partial x^\alpha}(x) \right|.$$

Proof The proof is by induction on the dimension d. Let $G(x) = g(x) \, \chi_{S_d}(x)$.
Case $d = 1$. This case is covered by Lemma 16. Here we reinterpret the result using the formalism of the cones $\mathcal{Q}(\theta)$. We have $\mathcal{F}^{(1)} = \{\mathcal{B}\}$ where $\mathcal{B} = \{1\}$ and $\Theta_1 = \{\{0\}, \, \mathbb{R}\}$.
   • $\theta = \mathbb{R}$. We have $\mathcal{Q}(\theta) = \{0\}$, and

$$\widehat{G}(0) = \int_0^1 g(x) dx,$$

so that

$$\langle \alpha \, (\mathbb{R}, V, j), g \rangle = \begin{cases} \int_0^1 g(x) dx & j = 0, \, V = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and the remainder $\mathcal{R}_{\mathbb{R}, w}(g, 0) = 0$ for every w.
   • $\theta = \{0\}$. We have $\mathcal{Q}(\theta) = \{\xi \in \mathbb{R}, \, \xi \neq 0\}$. In this case, by Lemma 16,

$$\int_0^1 g(x) e^{-2\pi i x \xi} dx = \sum_{j=0}^w \frac{\dfrac{d^j g}{dx^j}(0) - e^{-2\pi i \xi} \dfrac{d^j g}{dx^j}(1)}{(2\pi i \xi)^{j+1}}$$

$$+ (2\pi i \xi)^{-w-1} \int_0^1 \frac{d^{w+1} g}{dx^{w+1}}(x) e^{-2\pi i x \xi} dx.$$

It follows that

$$\langle \alpha \, (\{0\}, 1, j), g \rangle = \frac{d^j g}{dx^j}(0),$$

$$\langle \alpha \, (\{0\}, 2, j), g \rangle = -\frac{d^j g}{dx^j}(1).$$

Moreover the remainder evaluated at the lattice points $\xi = \tau n$ is

$$\mathcal{R}_{\{0\}, w}(g, \tau n) = (2\pi i \tau n)^{-w-1} \int_0^1 \frac{d^{w+1} g}{dx^{w+1}}(x) e^{-2\pi i \tau n x} dx$$

Hence for every $w \geq 1$,

$$
\left| \sum_{n \in \mathbb{Z}} \mathcal{R}_{\{0\},w}(g, \tau n) \chi_{\mathcal{Q}(\{0\})}(\tau n) e^{2\pi i n x} \right| = \left| \sum_{n \neq 0} \mathcal{R}_{\{0\},w}(g, \tau n) e^{2\pi i n x} \right|
$$

$$
\leq (2\pi\tau)^{-w-1} \left\{ \sum_{n \neq 0} |n|^{-w-1} \right\} \sup_{x \in [0,1]} \left| \frac{d^{w+1} g}{dx^{w+1}}(x) \right|
$$

$$
\leq (2\pi\tau)^{-w-1} \frac{\pi^2}{3} \sup_{x \in [0,1]} \left| \frac{d^{w+1} g}{dx^{w+1}}(x) \right|.
$$

Case $d \geq 2$. Assume that the theorem holds in dimension $d - 1$. Fix $\theta \in \Theta_d$. Observe that the vector $e_d = (0, \dots, 0, 1)$ belongs to at least one (actually all) bases in $\mathcal{F}^{(d)}$.

- $e_d \in \theta$. For all $\xi \in \mathcal{Q}(\theta)$ one has $\xi_d = 0$. For this choice of θ and for all $\xi = (\xi', 0) \in \mathcal{Q}(\theta)$ we have the following formula

$$
\widehat{G}(\xi) = \int_{S_d} g(x) e^{-2\pi i x \cdot \xi} dx = \int_{S_{d-1}} e^{-2\pi i x' \cdot \xi'} \left[\int_0^{1-(x_1+x_2+\cdots+x_{d-1})} g(x', x_d) dx_d \right] dx'
$$

$$
= \int_{S_{d-1}} e^{-2\pi i x' \cdot \xi'} F(x') dx'.
$$

Observe that $\xi \in \mathcal{Q}(\theta)$ if and only if $\xi' \in \mathcal{Q}(\theta')$ where θ' is the space of the vectors b' such that $(b', b_d) \in \theta$ for some b_d. Indeed, since $\xi \cdot b = \xi' \cdot b'$, ξ' is orthogonal to θ' if and only if ξ is orthogonal to θ. Since $\theta' \in \Theta_{d-1}$ (see Lemma 22 for details), we may therefore apply the $(d-1)$-dimensional formula corresponding to θ' to the function $F(x')$ so that

$$
\widehat{G}(\xi) = \sum_{V' \in \{1,2\}^{d-1}} \sum_{|J'| \leq w} \frac{\langle \alpha(\theta', V', J'), F \rangle e^{-2\pi i \lambda_{V'} \cdot \xi'}}{\prod_{b'_k \in \mathcal{B}_{V'} \backslash \theta'} (2\pi i b'_k \cdot \xi')^{j'_k + 1}} + \mathcal{R}_{\theta', w}(F, \xi'),
$$

where $j'_k = 0$ if $b'_k \in \theta'$. Observe that this expression can be written in the form

$$
\widehat{G}(\xi) = \sum_{V = (V', 1) \in \{1,2\}^d} \sum_{J = (J', 0), |J| \leq w} \frac{\langle \alpha(\theta', V', J'), F \rangle e^{-2\pi i \lambda_V \cdot \xi}}{\prod_{b_k \in \mathcal{B}_V \backslash \theta} (2\pi i b_k \cdot \xi)^{j_k + 1}} + \mathcal{R}_{\theta', w}(F, \xi')
$$

where $j_k = 0$ if $b_k \in \theta$. The coefficients $\langle \alpha(\theta', V', J'), F \rangle$ satisfy the estimates

$$
\left| \langle \alpha(\theta', V', J'), F \rangle \right| \leq c \, 2^{(d-2)|J'|} \sup_{|\alpha| \leq |J'|} \sup_{x' \in S_{d-1}} \left| \left(\frac{\partial}{\partial x'} \right)^{\alpha} F(x') \right|
$$

$$
\leq c \, 2^{(d-1)|J|} \sup_{|\alpha| \leq |J|} \sup_{x \in S_d} \left| \frac{\partial^{\alpha} g}{\partial^{\alpha} x}(x) \right|.
$$

Indeed, for every $1 \leqslant j, k, \ldots \leqslant d - 1$, we have

$$
\frac{\partial F}{\partial x_j}(x') = \frac{\partial}{\partial x_j} \int_0^{1 - x' \cdot \mathbf{1}_{d-1}} g(x', x_d) dx_d
$$

$$
= -g(x', 1 - x' \cdot \mathbf{1}_{d-1}) + \int_0^{1 - x' \cdot \mathbf{1}_{d-1}} \frac{\partial g}{\partial x_j}(x', x_d) dx_d,
$$

$$
\frac{\partial^2 F}{\partial x_k \partial x_j}(x') = -\frac{\partial g}{\partial x_k}(x', 1 - x' \cdot \mathbf{1}_{d-1}) + \frac{\partial g}{\partial x_d}(x', 1 - x' \cdot \mathbf{1}_{d-1})
$$

$$
- \frac{\partial g}{\partial x_j}(x', 1 - x' \cdot \mathbf{1}_{d-1}) + \int_0^{1 - x' \cdot \mathbf{1}_{d-1}} \frac{\partial^2 g}{\partial x_k \partial x_j}(x', x_d) dx_d,
$$

and so on. Thus

$$
\langle \alpha \left(\theta, V, J \right), g \rangle
$$
$$
= \begin{cases} \left\langle \alpha \left(\theta', V', J' \right), \int_0^{1 - x' \cdot \mathbf{1}_{d-1}} g(x', x_d) dx_d \right\rangle & \text{if } J = \left(J', 0 \right) \text{ and } V = (V, 1), \\ 0 & \text{otherwise.} \end{cases}
$$

For $\xi \in \mathcal{Q}(\theta)$ set $\mathcal{R}_{\theta, w}(g, \xi) = \mathcal{R}_{\theta', w}(F, \xi')$ (recall that $\xi = \left(\xi', 0 \right)$) and observe that

$$
\sum_{n \in \mathbb{Z}^d} \chi_{\mathcal{Q}(\theta)}(n) \mathcal{R}_{\theta, w}(g, \tau n) e^{2\pi i n \cdot x} = \sum_{n' \in \mathbb{Z}^{d-1}} \chi_{\mathcal{Q}(\theta)}(n', 0) \mathcal{R}_{\theta, w} \left(g, (\tau n', 0) \right) e^{2\pi i n' \cdot x'}
$$
$$
= \sum_{n' \in \mathbb{Z}^{d-1}} \chi_{\mathcal{Q}(\theta')}(n') \mathcal{R}_{\theta', w}(F, \tau n') e^{2\pi i n' \cdot x'}.
$$

By induction, $\left\{ \chi_{\mathcal{Q}(\theta')}(n') \mathcal{R}_{\theta', w}(F, \tau n') \right\}_{n' \in \mathbb{Z}^{d-1}}$ are the Fourier coefficients of a function on the $(d-1)$-dimensional torus bounded by

$$
U(d-1)(2^{d-2} \Omega \tau^{-1})^{w+1} \sup_{w-d+3 \leqslant |\alpha| \leqslant w+1} \sup_{x' \in S_{d-1}} \left| \left(\frac{\partial}{\partial x'} \right)^\alpha F(x') \right|
$$

$$
\leqslant U(d-1)(2^{d-2} \Omega \tau^{-1})^{w+1} \sup_{w-d+2 \leqslant |\alpha| \leqslant w+1} 2^{|\alpha|} \sup_{x \in S_d} \left| \left(\frac{\partial}{\partial x} \right)^\alpha g(x) \right|
$$

$$
\leqslant U(d-1)(2^{d-1} \Omega \tau^{-1})^{w+1} \sup_{w-d+2 \leqslant |\alpha| \leqslant w+1} \sup_{x \in S_d} \left| \left(\frac{\partial}{\partial x} \right)^\alpha g(x) \right|.
$$

Hence, by (i) in Lemma 19 with $K(n_d) = 0$ if $n_d \neq 0$ and $K(n_d) = 1$ if $n_d = 0$ so that $k(x_d) = 1$, and $H(n') = \mathcal{R}_w(F, \tau n')$ it follows that

$$
H(n') K(n_d) = \begin{cases} \chi_{\mathcal{Q}(\theta')}(n') \mathcal{R}_{\theta', w}(F, \tau n') & n_d = 0, \\ 0 & n_d \neq 0 \end{cases}
$$

are the Fourier coefficients of a function on the d-dimensional torus bounded by

$$U(d-1)(2^{d-1}\Omega\tau^{-1})^{w+1} \sup_{w-d+2\leqslant|\alpha|\leqslant w+1} \sup_{x\in S_d} \left|\left(\frac{\partial}{\partial x}\right)^\alpha g(x)\right|.$$

- $e_d \notin \theta$. Hence $\xi_d \neq 0$ for all $\xi \in \mathcal{Q}(\theta)$. Then, by Lemma 16,

$$\widehat{G}(\xi) = \int_{S_{d-1}} e^{-2\pi ix'\cdot\xi'} \left[\int_0^{1-x'\cdot\mathbf{1}_{d-1}} g(x', x_d)e^{-2\pi ix_d\xi_d}\,dx_d\right]dx'$$

$$= \sum_{j_d=0}^{w} (2\pi i\xi_d)^{-j_d-1} \int_{S_{d-1}} e^{-2\pi ix'\cdot\xi'}\frac{\partial^{j_d}g}{\partial x_d^{j_d}}(x', 0)\,dx'$$

$$- \sum_{j_d=0}^{w} (2\pi i\xi_d)^{-j_d-1} \int_{S_{d-1}} e^{-2\pi ix'\cdot\xi'} e^{-2\pi i(1-x'\cdot\mathbf{1}_{d-1})\xi_d}\frac{\partial^{j_d}g}{\partial x_d^{j_d}}(x', 1-x'\cdot\mathbf{1}_{d-1})\,dx'$$

$$+ (2\pi i\xi_d)^{-w-1} \int_{S_d} \frac{\partial^{w+1}g}{\partial x_d^{w+1}}(x)e^{-2\pi ix\cdot\xi}\,dx$$

$$= I + II + III.$$

$\boxed{III}$ The term III is part of the remainder. In view of Lemma 18 and since $n \in \mathcal{Q}(\theta) \cap \mathbb{Z}^d$ implies $n_d \neq 0$ it suffices to show that

$$\sum_{n\in\mathbb{Z}^d, n_d\neq 0} \left[(2\pi i\tau n_d)^{-w-1} \int_{S_d} \frac{\partial^{w+1}g}{\partial x_d^{w+1}}(x)e^{-2\pi ix\cdot\tau n}\,dx\right] e^{2\pi inx}$$

is the Fourier series of a bounded function. Observe first that the integrals in the above sum are the Fourier coefficients of a bounded function on the torus,

$$\int_{S_d} \frac{\partial^{w+1}g}{\partial x_d^{w+1}}(x)e^{-2\pi ix\cdot\tau n}\,dx$$

$$= \tau^{-d} \int_{\mathbb{R}^d} \chi_{S_d}(\tau^{-1}y)\frac{\partial^{w+1}g}{\partial x_d^{w+1}}(\tau^{-1}y)e^{-2\pi iy\cdot n}\,dy$$

$$= \tau^{-d} \int_{\mathbb{T}^d} \left[\sum_{k\in\mathbb{Z}^d} \chi_{S_d}\left(\tau^{-1}(y+k)\right)\frac{\partial^{w+1}g}{\partial x_d^{w+1}}\left(\tau^{-1}(y+k)\right)\right]e^{-2\pi iy\cdot n}\,dy.$$

The inner sum is finite and consists of at most $c(1+\tau)^d$ terms. By (ii) in Lemma 19 and Corollary 15, multiplying by $(2\pi i\tau n_d)^{-w-1}$, we obtain again the Fourier coefficients of a periodic function bounded by

$$(2\pi\tau)^{-w-1}\tau^{-d}\frac{\pi^2}{3}\sup_{y\in\mathbb{R}^d}\left|\sum_{k\in\mathbb{Z}^d}\chi_{S_d}\left(\tau^{-1}(y+k)\right)\frac{\partial^{w+1}g}{\partial x_d^{w+1}}\left(\tau^{-1}(y+k)\right)\right|$$

$$\leqslant c\tau^{-d}\left(1+\tau\right)^d(2\pi\tau)^{-w-1}\sup_{x\in S_d}\left|\frac{\partial^{w+1}g}{\partial x_d^{w+1}}(x)\right|$$

$$\leqslant c(1+\tau_0^{-1})^d(2^{d-1}\Omega\tau^{-1})^{w+1}\sup_{x\in S_d}\left|\frac{\partial^{w+1}g}{\partial x_d^{w+1}}(x)\right|$$

with c independent of w and g and $\tau\geqslant\tau_0>0$.

$\boxed{\text{I}}$ Let us consider the term I. Let $\theta_1=\{b':(b',0)\in\theta\}\in\Theta_{d-1}$ (see Lemma 22 for details). We claim that

$$\xi=(\xi',\xi_d)\in\mathcal{Q}(\theta)\implies\xi'\in\mathcal{Q}(\theta_1).$$

Indeed, let $\xi\in\mathcal{Q}(\theta)$ and let $b'\in\theta_1\cap\left(\bigcup_{\mathcal{B}\in\mathcal{F}^{(d-1)}}\mathcal{B}\right)$. Then $(b',0)\in\theta$ and therefore

$$0=\xi\cdot(b',0)=\xi'\cdot b'.$$

Now, let $b'\in\bigcup_{\mathcal{B}\in\mathcal{F}^{(d-1)}}\mathcal{B}$, $b'\notin\theta_1$, then $(b',0)\notin\theta$. Since $(b',0)\in\bigcup_{\mathcal{B}\in\mathcal{F}^{(d)}}\mathcal{B}$ we have

$$0\neq\xi\cdot(b',0)=\xi'\cdot b'.$$

Applying the $d-1$ dimensional formula corresponding to θ_1 to the function $\frac{\partial^{j_d}g}{\partial x_d^{j_d}}(x',0)$ we obtain

$$I=\sum_{j_d=0}^w(2\pi i\xi_d)^{-j_d-1}\int_{S_{d-1}}e^{-2\pi ix'\cdot\xi'}\frac{\partial^{j_d}g}{\partial x_d^{j_d}}(x',0)dx'$$

$$=\sum_{j_d=0}^w(2\pi i\xi_d)^{-j_d-1}\sum_{V'\in\{1,2\}^{d-1}}\sum_{|J'|\leqslant w-j_d}\frac{\left\langle\alpha(\theta_1,V',J'),(\partial/\partial x_d)^{j_d}g(\cdot,0)\right\rangle e^{-2\pi i\lambda_{V'}\cdot\xi'}}{\prod_{b'_k\in\mathcal{B}_{V'}\setminus\theta_1}(2\pi ib'_k\cdot\xi')^{j'_k+1}}$$

$$+\sum_{j_d=0}^w(2\pi i\xi_d)^{-j_d-1}\mathcal{R}_{\theta_1,w-j_d}\left(\frac{\partial^{j_d}g}{\partial x_d^{j_d}}(\cdot,0),\xi'\right)$$

$$=\sum_{V=(V',1)\in\{1,2\}^d}\sum_{|J|\leqslant w}\frac{\left\langle\alpha(\theta_1,V',J'),(\partial/\partial x_d)^{j_d}g(\cdot,0)\right\rangle e^{-2\pi i\lambda_V\cdot\xi}}{\prod_{b_k\in\mathcal{B}_V\setminus\theta}(2\pi ib_k\cdot\xi)^{j_k+1}}$$

$$+\sum_{j_d=0}^w(2\pi i\xi_d)^{-j_d-1}\mathcal{R}_{\theta_1,w-j_d}\left(\frac{\partial^{j_d}g}{\partial x_d^{j_d}}(\cdot,0),\xi'\right).$$

The double sum is part of the main term in the asymptotic expansion and the last sum is part of the remainder. By induction, since $J = (J', j_d)$,

$$\left| \left\langle \alpha(\theta_1, V', J'), (\partial/\partial x_d)^{j_d} g(\cdot, 0) \right\rangle \right| \leqslant c2^{(d-2)|J'|} \sup_{|\alpha| \leqslant |J'|} \sup_{x \in S_d} \left| \left(\frac{\partial}{\partial x'} \right)^{\alpha} \left(\frac{\partial}{\partial x_d} \right)^{j_d} g(x) \right|$$

$$\leqslant c2^{(d-2)|J|} \sup_{|\alpha| \leqslant |J|} \sup_{x \in S_d} \left| \frac{\partial^{\alpha} g}{\partial x^{\alpha}}(x) \right|.$$

We now deal with the remainder in I as follows. By the induction assumption

$$\sum_{n' \in \mathcal{Q}(\theta') \cap \mathbb{Z}^{d-1}} \mathcal{R}_{\theta_1, w - j_d} \left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(\cdot, 0), \tau n' \right) e^{2\pi i n' \cdot x'}$$

is the Fourier expansion of a function $F(x')$ on $\mathbb{T}^{d-1}$ bounded by

$$U(d-1)(2^{d-2}\Omega\tau^{-1})^{w+1-j_d} \sup_{w - j_d - (d-1) + 2 \leqslant |\alpha| \leqslant w - j_d + 1} \sup_{x' \in S_{d-1}} \left| \left(\frac{\partial}{\partial x'} \right)^{\alpha} \frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 0) \right|$$

$$\leqslant U(d-1)(2^{d-2}\Omega\tau^{-1})^{w+1-j_d} \sup_{w - d + 3 \leqslant |\alpha| \leqslant w + 1} \sup_{x \in S_d} \left| \frac{\partial^{\alpha} g}{\partial x^{\alpha}}(x) \right|.$$

Hence, $-\tau^{-j_d - 1} B_{j_d + 1}(x_d) F(x')$ is a function on $\mathbb{T}^d$ bounded by

$$\tau^{-j_d - 1}(2\pi)^{-j_d - 1} \frac{\pi^2}{3} U(d-1)(2^{d-2}\Omega\tau^{-1})^{w+1-j_d} \sup_{w - d + 3 \leqslant |\alpha| \leqslant w + 1} \sup_{x \in S_d} \left| \frac{\partial^{\alpha} g}{\partial x^{\alpha}}(x) \right|$$

$$= (2^{d-2}\Omega 2\pi)^{-j_d - 1} \frac{\pi^2}{3} U(d-1)(2^{d-2}\Omega\tau^{-1})^{w+2} \sup_{w - d + 3 \leqslant |\alpha| \leqslant w + 1} \sup_{x \in S_d} \left| \frac{\partial^{\alpha} g}{\partial x^{\alpha}}(x) \right|,$$

and with Fourier expansion

$$\sum_{n_d \neq 0, n' \in \mathcal{Q}(\theta_1) \cap \mathbb{Z}^{d-1}} \left[(2\pi i \tau n_d)^{-j_d - 1} \mathcal{R}_{\theta_1, w - j_d} \left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(\cdot, 0), \tau n' \right) \right] e^{2\pi i n \cdot x}$$

$$= \sum_{n \in \mathbb{Z}^d} \chi_{\{n_d \neq 0, n' \in \mathcal{Q}(\theta_1) \cap \mathbb{Z}^{d-1}\}}(n) \left[(2\pi i \tau n_d)^{-j_d - 1} \mathcal{R}_{\theta_1, w - j_d} \left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(\cdot, 0), \tau n' \right) \right] e^{2\pi i n \cdot x}.$$

Since $\mathcal{Q}(\theta) \cap \mathbb{Z}^d \subset \left(\mathcal{Q}(\theta_1) \cap \mathbb{Z}^{d-1} \right) \times (\mathbb{Z} \setminus \{0\})$, by Lemma 18,

$$\sum_{n \in \mathcal{Q}(\theta) \cap \mathbb{Z}^d} \chi_{\{n_d \neq 0, n' \in \mathcal{Q}(\theta_1) \cap \mathbb{Z}^{d-1}\}}(n) \left[(2\pi i \tau n_d)^{-j-1} \mathcal{R}_{\theta_1, w - j} \left(\frac{\partial^j g}{\partial x_d^j}(\cdot, 0), \tau n' \right) \right] e^{2\pi i n \cdot x}$$

$$= \sum_{n \in \mathcal{Q}(\theta) \cap \mathbb{Z}^d} \left[(2\pi i \tau n_d)^{-j-1} \mathcal{R}_{\theta_1, w - j} \left(\frac{\partial^j g}{\partial x_d^j}(\cdot, 0), \tau n' \right) \right] e^{2\pi i n \cdot x}.$$

is the Fourier series of a function on $\mathbb{T}^d$ bounded by

$$c\,(\theta)\,(2^{d-2}\Omega 2\pi)^{-j_d-1}\frac{\pi^2}{3}U(d-1)(2^{d-2}\Omega\tau^{-1})^{w+2}\sup_{w-d+3\leqslant|\alpha|\leqslant w+1}\sup_{x\in S_d}\left|\frac{\partial^\alpha g}{\partial x^\alpha}(x)\right|.$$

Adding up on j_d we obtain that

$$\left\{\chi_{\mathcal{Q}(\theta)}(n)\sum_{j_d=0}^{w}(2\pi i\tau n_d)^{-j_d-1}\mathcal{R}_{\theta_1,w-j_d}\left(\frac{\partial^{j_d}g}{\partial x_d^{j_d}}(\cdot,0),\tau n'\right)\right\}$$

are the Fourier coefficients of a function on the d-dimensional torus bounded by

$$\sum_{j_d=0}^{w}c\,(\theta)\,(2^{d-2}\Omega 2\pi)^{-j_d-1}\frac{\pi^2}{3}U(d-1)(2^{d-2}\Omega\tau^{-1})^{w+2}$$

$$\sup_{w-d+3\leqslant|\alpha|\leqslant w+1}\sup_{x\in S_d}\left|\frac{\partial^\alpha g}{\partial x^\alpha}(x)\right|$$

$$=c\,(\theta)\frac{\pi^2}{3}U(d-1)(2^{d-2}\Omega\tau^{-1})^{w+2}\sum_{j_d=0}^{w}(2^{d-2}\Omega 2\pi)^{-j_d-1}$$

$$\sup_{w-d+3\leqslant|\alpha|\leqslant w+1}\sup_{x\in S_d}\left|\frac{\partial^\alpha g}{\partial x^\alpha}(x)\right|$$

$$\leqslant c\,(\theta)\frac{\pi^2}{3}\frac{U(d-1)}{2^{d-1}\Omega\pi-1}(2^{d-2}\Omega\tau^{-1})^{w+2}$$

$$\sup_{w-d+3\leqslant|\alpha|\leqslant w+1}\sup_{x\in S_d}\left|\frac{\partial^\alpha g}{\partial x^\alpha}(x)\right|$$

$$\leqslant\left(c\,(\theta)\frac{\pi^2}{3}\frac{U(d-1)}{2^{d-1}\Omega\pi-1}2^{d-2}\Omega\tau_0^{-1}\right)(2^{d-1}\Omega\tau^{-1})^{w+1}$$

$$\sup_{w-d+3\leqslant|\alpha|\leqslant w+1}\sup_{x\in S_d}\left|\frac{\partial^\alpha g}{\partial x^\alpha}(x)\right|.$$

Here we used the assumption $\Omega>(2\pi)^{-1}$.

$\boxed{II}$ The term II is similar to I, but to estimate the remainder we need (iii) in Lemma 19. Let $\theta_2=\{b':(b',-b'\cdot\mathbf{1}_{d-1})\in\theta\}\in\Theta_{d-1}$ (see Lemma 22 for details). We claim that

$$\xi=(\xi',\xi_d)\in\mathcal{Q}(\theta)\implies\xi'-\xi_d\mathbf{1}_{d-1}\in\mathcal{Q}(\theta_2).$$

Indeed, let $\xi\in\mathcal{Q}(\theta)$ and let $b'\in\theta_2\cap\left(\bigcup_{\mathcal{B}\in\mathcal{F}^{(d-1)}}\mathcal{B}\right)$. Then $(b',-b'\cdot\mathbf{1}_{d-1})\in\theta$ and therefore

$$0=\xi\cdot(b',-b'\cdot\mathbf{1}_{d-1})=\xi'\cdot b'-\xi_d\mathbf{1}_{d-1}\cdot b'=(\xi'-\xi_d\mathbf{1}_{d-1})\cdot b'.$$

Similarly if $b' \in \left(\bigcup_{\mathcal{B} \in \mathcal{F}^{(d-1)}} \mathcal{B} \right) \backslash \theta_2$, then $(b', -b' \cdot \mathbf{1}_{d-1}) \notin \theta$. Since $(b', -b' \cdot \mathbf{1}_{d-1}) \in \bigcup_{\mathcal{B} \in \mathcal{F}^{(d)}} \mathcal{B}$, we have

$$0 \neq \xi \cdot (b', -b' \cdot \mathbf{1}_{d-1}) = (\xi' - \xi_d \mathbf{1}_{d-1}) \cdot b'.$$

By applying the $(d-1)$-dimensional formula corresponding to θ_2 to the function $\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1})$ we get

$$
\begin{aligned}
II &= -\sum_{j_d=0}^{w} (2\pi i \xi_d)^{-j_d-1} \int_{S_{d-1}} e^{-2\pi i x' \cdot \xi'} e^{-2\pi i (1 - x' \cdot \mathbf{1}_{d-1}) \xi_d} \frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1}) dx' \\
&= -\sum_{j_d=0}^{w} (2\pi i \xi_d)^{-j_d-1} e^{-2\pi i \xi_d} \int_{S_{d-1}} e^{-2\pi i x' \cdot (\xi' - \xi_d \mathbf{1}_{d-1})} \frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1}) dx' \\
&= -\sum_{j_d=0}^{w} (2\pi i \xi_d)^{-j_d-1} e^{-2\pi i \xi_d} \\
&\quad \times \sum_{V' \in \{1,2\}^{d-1}} \sum_{|J'| \leqslant w - j_d} \frac{\left\langle \alpha(\theta_2, V', J'), \left((\partial/\partial x_d)^{j_d} g\right)(x', 1 - x' \cdot \mathbf{1}_{d-1}) \right\rangle e^{-2\pi i \lambda_{V'} \cdot (\xi' - \xi_d \mathbf{1}_{d-1})}}{\prod_{b'_k \in \mathcal{B}_{V'} \backslash \theta_2} (2\pi i b'_k \cdot (\xi' - \xi_d \mathbf{1}_{d-1}))^{j_k+1}} \\
&\quad - \sum_{j_d=0}^{w} (2\pi i \xi_d)^{-j_d-1} e^{-2\pi i \xi_d} \mathcal{R}_{\theta_2, w - j_d}\left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1}), \xi' - \xi_d \mathbf{1}_{d-1} \right) \\
&= \sum_{V = (V', 2) \in \{1,2\}^d} \sum_{|J| \leqslant w} \frac{-\left\langle \alpha(\theta_2, V', J'), \left((\partial/\partial x_d)^{j_d} g\right)(x', 1 - x' \cdot \mathbf{1}_{d-1}) \right\rangle e^{-2\pi i \lambda_V \cdot \xi}}{\prod_{b_k \in \mathcal{B}_V \backslash \theta} (2\pi i b_k \cdot \xi)^{j_k+1}} \\
&\quad - \sum_{j_d=0}^{w} (2\pi i \xi_d)^{-j_d-1} e^{-2\pi i \xi_d} \mathcal{R}_{\theta_2, w - j_d}\left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1}), \xi' - \xi_d \mathbf{1}_{d-1} \right).
\end{aligned}
$$

By induction

$$
\begin{aligned}
&\left| \left\langle \alpha(\theta_2, V', J'), \left((\partial/\partial x_d)^{j_d} g\right)(x', 1 - x' \cdot \mathbf{1}_{d-1}) \right\rangle \right| \\
&\leqslant c 2^{(d-2)|J'|} \sup_{|\alpha| \leqslant |J'|} \sup_{x' \in S_{d-1}} \left| \left(\frac{\partial}{\partial x'} \right)^\alpha \left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1}) \right) \right|.
\end{aligned}
$$

Observe that for every $1 \leqslant k \leqslant d - 1$

$$
\begin{aligned}
\frac{\partial}{\partial x_k} \left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1}) \right) &= \frac{\partial^{j_d+1} g}{\partial x_k \partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1}) \\
&\quad - \frac{\partial^{j_d+1} g}{\partial x_d^{j_d+1}}(x', 1 - x' \cdot \mathbf{1}_{d-1}).
\end{aligned}
$$

Hence

$$\left| \left(\frac{\partial}{\partial x'}\right)^{\alpha} \left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1})\right) \right| \leqslant 2^{|\alpha|} \sup_{|\beta|=|\alpha|+j_d} \sup_{x \in S_d} \left|\frac{\partial^{\beta} g}{\partial x^{\beta}}(x)\right|,$$

so that, with the notation $J = (J', j_d)$,

$$\left| \left\langle \alpha(\theta_2, V', J'), \left((\partial/\partial x_d)^{j_d} g\right)(x', 1 - x' \cdot \mathbf{1}_{d-1}) \right\rangle \right|$$

$$\leqslant c 2^{(d-2)|J'|} 2^{|J'|} \sup_{|\beta|\leqslant|J'|+j_d} \sup_{x \in S_d} \left|\frac{\partial^{\beta} g}{\partial x^{\beta}}(x)\right| \leqslant c 2^{(d-1)|J|} \sup_{|\beta|\leqslant|J|} \sup_{x \in S_d} \left|\frac{\partial^{\beta} g}{\partial x^{\beta}}(x)\right|.$$

Let us consider the remainder and set $T\xi = (\xi' - \xi_d \mathbf{1}_{d-1}, \xi_d)$. Then $T \in SL(d, \mathbb{Z})$ and, by (iii) in Lemma 19,

$$- \sum_{j_d=0}^{w} \chi_{\mathcal{Q}(\theta)}(n)(2\pi i \tau n_d)^{-j_d-1} e^{-2\pi i \tau n_d} \mathcal{R}_{\theta_2, w-j_d}$$

$$\left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1}), \tau n' - \tau n_d \mathbf{1}_{d-1}\right) \tag{1}$$

are the Fourier coefficients of a bounded function if

$$- \sum_{j_d=0}^{w} \chi_{T(\mathcal{Q}(\theta))}(n)(2\pi i \tau n_d)^{-j_d-1} e^{-2\pi i \tau n_d} \mathcal{R}_{\theta_2, w-j_d}$$

$$\left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1}), \tau n'\right) \tag{2}$$

are the Fourier coefficients of a bounded function, and the bound is the same. By induction

$$\left\{ \chi_{\mathcal{Q}(\theta_2)}(n') \mathcal{R}_{\theta_2, w-j_d}\left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1}), \tau n'\right) \right\}$$

are the Fourier coefficients of a function on the $d - 1$ dimensional torus bounded by

$$U(d - 1)(2^{d-2}\Omega\tau^{-1})^{w+1-j_d} \sup_{\substack{w-j_d-(d-1)+2\leqslant|\alpha| \\ \leqslant w-j_d+1}} \sup_{x' \in S_{d-1}}$$

$$\left| \left(\frac{\partial}{\partial x'}\right)^{\alpha} \left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1 - x' \cdot \mathbf{1}_{d-1})\right) \right|$$

$$\leqslant U(d - 1)(2^{d-2}\Omega\tau^{-1})^{w+1-j_d} \sup_{w-j_d-(d-1)+2\leqslant|\alpha|\leqslant w-j_d+1} 2^{|\alpha|}$$

$$\sup_{|\beta|=|\alpha|+j_d} \sup_{x\in S_d} \left|\frac{\partial^\beta g}{\partial x^\beta}(x)\right|$$

$$\leqslant U(d-1)(2^{d-1}\Omega\tau^{-1})^{w+1-j_d} \sup_{w-j_d-(d-1)+2\leqslant|\alpha|\leqslant w-j_d+1}$$

$$\sup_{|\beta|=|\alpha|+j_d} \sup_{x\in S_d} \left|\frac{\partial^\beta g}{\partial x^\beta}(x)\right|$$

$$\leqslant U(d-1)(2^{d-1}\Omega\tau^{-1})^{w+1-j_d} \sup_{w-d+3\leqslant|\beta|\leqslant w+1} \sup_{x\in S_d} \left|\frac{\partial^\beta g}{\partial x^\beta}(x)\right|.$$

Hence, by (i) in Lemma 19 and Corollary 15,

$$-\sum_{n_d\neq 0, n'\in\mathcal{Q}(\theta_2)} \left[\sum_{j_d=0}^{w} (2\pi i\tau n_d)^{-j_d-1} e^{-2\pi i\tau n_d} \mathcal{R}_{\theta_2,w-j_d}\right.$$

$$\left.\left(\frac{\partial^{j_d} g}{\partial x_d^{j_d}}(x', 1-x'\cdot\mathbf{1}_{d-1}), \tau n'\right)\right] e^{2\pi in\cdot x}$$

is the Fourier series of a function on $\mathbb{T}^d$ bounded by

$$\sum_{j_d=0}^{w} \left(\frac{\pi^2}{3}(2\pi\tau)^{-j_d-1} U(d-1)(2^{d-1}\Omega\tau^{-1})^{w+1-j_d} \sup_{w-d+3\leqslant|\beta|\leqslant w+1} \sup_{x\in S_d} \left|\frac{\partial^\beta g}{\partial x^\beta}(x)\right|\right)$$

$$\leqslant \frac{\pi^2}{3} U(d-1)(2^{d-1}\Omega\tau^{-1})^{w+2} \sum_{j_d=0}^{w} (2^{d-1}2\pi\Omega)^{-j_d-1} \sup_{w-d+3\leqslant|\beta|\leqslant w+1} \sup_{x\in S_d} \left|\frac{\partial^\beta g}{\partial x^\beta}(x)\right|$$

$$\leqslant \frac{\pi^2}{3} U(d-1)(2^{d-1}\Omega\tau^{-1})^{w+2} \frac{1}{2^{d-1}2\pi\Omega-1} \sup_{w-d+3\leqslant|\beta|\leqslant w+1} \sup_{x\in S_d} \left|\frac{\partial^\beta g}{\partial x^\beta}(x)\right|.$$

Since $T\mathcal{Q}(\theta)\cap\mathbb{Z}^d \subset \left(\mathcal{Q}(\theta_2)\cap\mathbb{Z}^{d-1}\right)\times(\mathbb{Z}\setminus\{0\})$ (notice that $\xi\in\mathcal{Q}(\theta)$ implies $\xi'-\xi_d\mathbf{1}_{d-1}\in\mathcal{Q}(\theta_2)$), by Lemma 18, the remainder (2) and therefore (1) is bounded by

$$c\frac{\pi^2}{3}\frac{U(d-1)}{2^{d-1}2\pi\Omega-1}(2^{d-1}\Omega\tau^{-1})^{w+2} \sup_{w-d+3\leqslant|\beta|\leqslant w+1} \sup_{x\in S_d} \left|\frac{\partial^\beta g}{\partial x^\beta}(x)\right|$$

$$\leqslant c\left(\frac{\pi^2}{3}\frac{U(d-1)}{2^{d-1}2\pi\Omega-1}(2^{d-1}\Omega\tau_0^{-1})\right)(2^{d-1}\Omega\tau^{-1})^{w+1} \sup_{w-d+3\leqslant|\beta|\leqslant w+1} \sup_{x\in S_d} \left|\frac{\partial^\beta g}{\partial x^\beta}(x)\right|.$$

This proves the formula when $e_d\notin\theta$. In particular,

$$\langle \alpha\,(\theta, V, J)\,, g \rangle = \begin{cases} \left\langle \alpha\,(\theta_1, V', J')\,, \left(\dfrac{\partial}{\partial x_d}\right)^{jd} g\,(x', 0) \right\rangle & V = (V', 1)\,, \\[2ex] -\left\langle \alpha\,(\theta_2, V', J')\,, \left(\dfrac{\partial}{\partial x_d}\right)^{jd} g\,(x', 1 - x' \cdot \mathbf{1}_{d-1}) \right\rangle & V = (V', 2)\,. \end{cases}$$

□

As mentioned, our formulas are not symmetric since they depend both on the faces of S_d that are orthogonal to the considered point ξ and on the way we iterate the integration in our computation. However one can obtain more symmetric formulas by averaging on all different ways of computing the Fourier transform $\widehat{g\chi_{S_d}}\,(\xi)$ as an iterated integral.

In the following we obtain a precise expression for the functionals $\alpha(\theta, V, J)$ that makes explicit the dependence of the coefficients $\langle \alpha(\theta, V, J), g \rangle$ on the function g and on θ, V and J.

Definition 21 For every $h = 2, \ldots, d$, let $U_h : \mathbb{R}^h \to \mathbb{R}^{h-1}$ be the operator that removes the last coordinate. For every $\theta \in \Theta_h$ and for every $\mathcal{B} \in \mathcal{F}^{(h)}$, let $P_{h,\mathcal{B}}\theta$ be the subspace of $\mathbb{R}^{h-1}$ defined as follows:

$$P_{h,\mathcal{B}}\theta = \begin{cases} U_h\theta & \text{if } e_h \in \theta, \\ U_h\,(\theta \cap \{x \cdot e_h = 0\}) & \text{if } e_h \notin \theta \text{ and } \mathcal{B} \in \mathcal{F}_1^{(h)}, \\ U_h\,(\theta \cap \{x \cdot \mathbf{1}_h = 0\}) & \text{if } e_h \notin \theta \text{ and } \mathcal{B} \in \mathcal{F}_2^{(h)}. \end{cases}$$

Here $e_h = (0, \ldots, 0, 1)$ is the last vector in the canonical basis of $\mathbb{R}^h$.

Observe that $P_{d,\mathcal{B}}\theta$ is just the subspace θ', θ_1, or θ_2 used in the proof of Lemma 20.

Lemma 22 *With the above notation we have*

$$P_{h,\mathcal{B}}\theta = \begin{cases} U_h\,(\theta \cap \{x \cdot e_h = 0\}) & \text{if } \mathcal{B} \in \mathcal{F}_1^{(h)}, \\ U_h\,(\theta \cap \{x \cdot \mathbf{1}_h = 0\}) & \text{if } \mathcal{B} \in \mathcal{F}_2^{(h)}. \end{cases}$$

Moreover, $P_{h,\mathcal{B}}\theta \in \Theta_{h-1}$.

Proof Let $\Pi_h : \mathbb{R}^h \to \mathbb{R}^h$ be the orthogonal projection

$$\Pi_h(x_1, \ldots, x_h) = (x_1, \ldots, x_{h-1}, 0)$$

so that $U_h = U_h \Pi_h$. It suffices to observe that if $e_h \in \theta$, then

$$\Pi_h\theta = \Pi_h\,(\theta \cap \{x \cdot e_h = 0\}) = \Pi_h\,(\theta \cap \{x \cdot \mathbf{1}_h = 0\})\,.$$

Indeed, if $x \in \Pi_h\theta$ then $x = \Pi_h y$ for some $y \in \theta$, so that $x = \Pi_h(\Pi_h y)$. Since $\Pi_h y \in \theta \cap \{x \cdot e_h = 0\}$ we have $\Pi_h\theta \subseteq \Pi_h\,(\theta \cap \{x \cdot e_h = 0\})$. Similarly $x =$

$\Pi_h(\Pi_h y - (y \cdot \mathbf{1}_h)e_h)$ and $\Pi_h y - (y \cdot \mathbf{1}_h)e_h \in \theta \cap \{x \cdot \mathbf{1}_h = 0\}$ so that $\Pi_h \theta \subseteq \Pi_h(\theta \cap \{x \cdot \mathbf{1}_h = 0\})$. The reverse inclusions are trivial.

Concerning the last point of the lemma, observe first that it follows easily from the recurrence definition of the bases of $\mathcal{F}^{(h)}$ that

$$\bigcup_{\mathcal{B} \in \mathcal{F}^{(h)}} \mathcal{B} = \{e_j, e_\ell - e_k : 1 \leq j \leq h, 1 \leq \ell < k \leq h\}.$$

Assume now that $\theta = \mathrm{span}\{b_1, \ldots, b_N\}$ with $b_j \in \bigcup_{\mathcal{B} \in \mathcal{F}^{(h)}} \mathcal{B}$. If $e_h \in \theta$, then $P_{h,\mathcal{B}}\theta = U_h \theta = \mathrm{span}\{U_h b_1, \ldots, U_h b_N\}$, and by construction each vector $U_h b_j$ belongs to $\bigcup_{\mathcal{B} \in \mathcal{F}^{(h-1)}} \mathcal{B}$, so that $P_{h,\mathcal{B}}\theta \in \Theta_{h-1}$. Assume $\mathcal{B} \in \mathcal{F}_1^{(h)}$ (the case $\mathcal{B} \in \mathcal{F}_2^{(h)}$ is treated similarly). If $e_h \notin \theta$ and $b_j \cdot e_h = 0$ for all $j = 1, \ldots, N$, then again $P_{h,\mathcal{B}}\theta = U_h \theta = \mathrm{span}\{U_h b_1, \ldots, U_h b_N\} \in \Theta_{h-1}$. If instead $e_h \notin \theta$ but, say, $b_N \cdot e_h \neq 0$, then in particular $b_N \cdot e_h = -1$ and

$$P_{h,\mathcal{B}}\theta = U_h\left(\theta \cap \{x \cdot e_h = 0\}\right) = U_h \left\{ \sum_{j=1}^N c_j b_j : \sum_{j=1}^N c_j b_j \cdot e_h = 0 \right\}$$

$$= U_h \left\{ \sum_{j=1}^N c_j b_j : c_N = \sum_{j=1}^{N-1} c_j b_j \cdot e_h \right\} = \left\{ \sum_{j=1}^{N-1} c_j U_h \left(b_j + (b_j \cdot e_h) b_N\right) \right\}.$$

Now if $b_j \cdot e_h = 0$ then $U_h\left(b_j + (b_j \cdot e_h) b_N\right) = U_h b_j \in \bigcup_{\mathcal{B} \in \mathcal{F}^{(h-1)}} \mathcal{B}$. If on the contrary $b_j \cdot e_h \neq 0$, then $b_j \cdot e_h = -1$ and either $b_j + (b_j \cdot e_h) b_N = b_j - b_N$ or its opposite belong to $\bigcup_{\mathcal{B} \in \mathcal{F}^{(h)}} \mathcal{B}$. It follows that $U_h\left(b_j + (b_j \cdot e_h) b_N\right)$ or $-U_h\left(b_j + (b_j \cdot e_h) b_N\right)$ belong to $\bigcup_{\mathcal{B} \in \mathcal{F}^{(h-1)}} \mathcal{B}$. Thus, $P_{h,\mathcal{B}}\theta \in \Theta_{h-1}$. $\qquad\square$

Definition 23 For every $h = 2, \ldots, d$, and for every $\mathcal{B} = \{b_1, \ldots, b_h\} \in \mathcal{F}^{(h)}$, define $P_h\mathcal{B} \in \mathcal{F}^{(h-1)}$ as $P_h\mathcal{B} = \{U_h b_1, \ldots, U_h b_{h-1}\}$. Observe that by Definition 5, $P_h\mathcal{B}$ is the basis used to construct $\mathcal{B}$.

Notation. In the next lemmas and definitions for a given basis $\mathcal{B} \in \mathcal{F}^{(d)}$, we shall call $\mathcal{B}_d = \mathcal{B}$, $\mathcal{B}_{d-1} = P_d\mathcal{B}$, $\mathcal{B}_{d-2} = P_{d-1}P_d\mathcal{B}$, ..., $\mathcal{B}_2 = P_3 P_4 \ldots P_d\mathcal{B}$, and $\mathcal{B}_1 = P_2 P_3 \ldots P_d\mathcal{B} = \{1\}$. Also, we denote by $b_k^{(h)}$ the k-th vector of the basis $\mathcal{B}_h$. Similarly, $e_k^{(h)}$ denotes the k-th vector of the canonical basis of $\mathbb{R}^h$.

Lemma 24 *For all $h \geqslant 2$ and for all $k = 1, \ldots, h-1$, $b_k^{(h-1)} = U_h b_k^{(h)} \in P_{h,\mathcal{B}}\theta$ if and only if $b_k^{(h)} \in \theta$.*

Proof If $\mathcal{B} \in \mathcal{F}_1^{(h)}$ then $b_k^{(h)} \cdot e_h^{(h)} = 0$. Thus if $b_k^{(h)} \in \theta$ then, by Lemma 22, $U_h b_k^{(h)} \in P_{h,\mathcal{B}}\theta$. Conversely, if $U_h b_k^{(h)} \in P_{h,\mathcal{B}}\theta$ then there exists $y \in \theta \cap \{x \cdot e_h^{(h)} = 0\}$ such that $U_h b_k^{(h)} = U_h y$, so that $U_h(b_k^{(h)} - y) = 0$. This implies that the first $h-1$ coordinates of $b_k^{(h)}$ and y coincide. The last coordinate of y is 0 by construction, and since $\mathcal{B} \in \mathcal{F}_1^{(h)}$ and $k < h$, the last coordinate of $b_k^{(h)}$ also is 0. Thus $b_k^{(h)} = y$ so that $b_k^{(h)} \in \theta$.

Similarly, if $\mathcal{B} \in \mathcal{F}_2^{(h)}$ then $b_k^{(h)} \cdot \mathbf{1}_h = 0$. Thus if $b_k^{(h)} \in \theta$ then $U_h b_k^{(h)} \in P_{h,\mathcal{B}}\theta$. Conversely, if $U_h b_k^{(h)} \in P_{h,\mathcal{B}}\theta$ then there exists $y \in \theta \cap \{x \cdot \mathbf{1}_h = 0\}$ such that $U_h b_k^{(h)} = U_h y$, so that $U_h(b_k^{(h)} - y) = 0$. This implies that the first $h-1$ coordinates of $b_k^{(h)}$ and y coincide. The fact that $b_k^{(h)} \cdot \mathbf{1}_h = 0$ and $y \cdot \mathbf{1}_h = 0$ implies that also the last coordinate of $b_k^{(h)}$ and y coincide. Thus $b_k^{(h)} = y$ so that $b_k^{(h)} \in \theta$.   □

Definition 25 For any given $\theta \in \Theta_d$ and for every $\mathcal{B} \in \mathcal{F}^{(d)}$ the multi-index $Z = Z_{\mathcal{B},\theta} = (z_1, \ldots, z_d) \in \{0, 1\}^d$ is defined recursively as follows:

$$z_d = 0 \text{ iff } e_d^{(d)} \in \theta$$
$$z_{d-1} = 0 \text{ iff } e_{d-1}^{(d-1)} \in P_{d,\mathcal{B}_d}\theta$$
$$z_{d-2} = 0 \text{ iff } e_{d-2}^{(d-2)} \in P_{d-1,\mathcal{B}_{d-1}}P_{d,\mathcal{B}_d}\theta$$
$$z_{d-3} = 0 \text{ iff } e_{d-3}^{(d-3)} \in P_{d-2,\mathcal{B}_{d-2}}P_{d-1,\mathcal{B}_d}P_{d,\mathcal{B}_d}\theta$$
$$\cdots$$
$$z_1 = 0 \text{ iff } 1 \in P_{2,\mathcal{B}_2} \ldots P_{d-2,\mathcal{B}_{d-2}}P_{d-1,\mathcal{B}_{d-1}}P_{d,\mathcal{B}_d}\theta$$

Definition 26 For any given $\theta \in \Theta_d$ and any multi-index $V \in \{1, 2\}^d$ define the multi-index $I = I_{V,\theta} = (i_1, \ldots, i_d) \in \{0, 1\}^d$ by

$$i_k = 0 \iff b_k^{(d)} \in \theta$$

where $\mathcal{B}_V = \left\{ b_1^{(d)}, \ldots, b_d^{(d)} \right\} \in \mathcal{F}^{(d)}$ is the basis associated with V.

Lemma 27 *For any given $\theta \in \Theta_d$ and for every $\mathcal{B}_V \in \mathcal{F}^{(d)}$, $Z_{\mathcal{B}_V,\theta} = I_{V,\theta}$.*

Proof By definition, it suffices to observe that $z_h = 0$ if and only if $e_h^{(h)}$ belongs to $P_{h+1,\mathcal{B}_{h+1}}(\ldots P_{d-1,\mathcal{B}_{d-1}}(P_{d,\mathcal{B}_d}\theta))$. But $e_h^{(h)} = b_h^{(h)}$ is the last vector of the basis $\mathcal{B}_h$ so that, by Lemma 24,

$$e_h^{(h)} = b_h^{(h)} \in P_{h+1,\mathcal{B}_{h+1}}\left(\ldots P_{d-1,\mathcal{B}_{d-1}}(P_{d,\mathcal{B}_d}\theta)\right),$$

which is equivalent to

$$b_h^{(h+1)} \in P_{h+2,\mathcal{B}_{h+2}}\left(\ldots \left(P_{d-1,\mathcal{B}_{d-1}}(P_{d,\mathcal{B}_d}\theta)\right)\right).$$

Proceeding iteratively, this is equivalent to

$$b_h^{(d-1)} \in P_{d,\mathcal{B}_d}\theta,$$

which, in turn, is equivalent to $b_h^{(d)} \in \theta$ and therefore to $i_h = 0$.   □

Definition 28 Fix $V = (v_1, v_2, \ldots, v_d) \in \{1, 2\}^d$, $I = (i_1, \ldots, i_d) \in \{0, 1\}^d$, $J = (j_1, \ldots, j_d)$ a non-negative multi-index such that $j_h = 0$ if $i_h = 0$, that is $J \sqsubseteq I$. For each $h = 1, \ldots, d$ and for $N \geq 1$, define the operators

$$T_h^{v_h, i_h, j_h} : \mathcal{C}^N(\mathbb{R}^h) \to \mathcal{C}^{N-1}(\mathbb{R}^{h-1})$$

(if $h = 1$ then $\mathcal{C}^{N-1}(\mathbb{R}^{h-1}) = \mathbb{C}$) as follows: if $h = 1$, set

$$T_1^{1,0,0} g = \int_0^1 g(x_1) dx_1,$$

$$T_1^{2,0,0} g = 0,$$

$$T_1^{1,1,j_1} g = -\frac{d^{j_1} g}{dx^{j_1}}(0),$$

$$T_1^{2,1,j_1} g = \frac{d^{j_1} g}{dx^{j_1}}(1).$$

If $2 \leq h \leq d$, for all $x' \in \mathbb{R}^{h-1}$, set

$$T_h^{1,0,0} g(x') = \int_0^{1 - x' \cdot \mathbf{1}} g(x', x_h) dx_h,$$

$$T_h^{2,0,0} g(x') = 0,$$

$$T_h^{1,1,j_h} g(x') = -\frac{\partial^{j_h} g}{\partial x_h^{j_h}}(x', 0),$$

$$T_h^{2,1,j_h} g(x') = \frac{\partial^{j_h} g}{\partial x_h^{j_h}}(x', 1 - x' \cdot \mathbf{1}).$$

Let us define the integro-differential functionals

$$\mu(V, I, J) = T_1^{v_1, i_1, j_1} T_2^{v_2, i_2, j_2} \ldots T_d^{v_d, i_d, j_d}.$$

Lemma 29 *Fix* $V = (v_1, v_2, \ldots, v_d) \in \{1, 2\}^d$, $I = (i_1, \ldots, i_d) \in \{0, 1\}^d$ *and let* $J = (j_1, \ldots, j_d)$ *be a non-negative multi-index such that* $j_h = 0$ *if* $i_h = 0$, *that is* $J \sqsubseteq I$. *Let* $\mathcal{B} = \mathcal{B}_V \in \mathcal{F}^{(d)}$ *be the basis associated to the multi-index V, and let* $\theta \in \Theta_d$ *be such that* $I_{V, \subseteq} = I$. *Then*

$$\alpha(\theta, V, J) = (-1)^{|I|} \mu(V, I, J).$$

Proof One has to go through the proof of Lemma 20 and notice that θ', θ_1 and θ_2 are all simply $P_{d, \mathcal{B}_d} \theta$, and that $\mathcal{B}' = P_d \mathcal{B} = \mathcal{B}_{d-1}$. The conclusion follows proceeding recursively and recalling that $Z_{\mathcal{B}, \theta} = I_{V, \theta}$. □

The above functional $\alpha(\theta, V, J)$ is a compactly supported distribution, with support contained in the simplex S_d. In particular, the dependence of $\alpha(\theta, V, J)$ on V, θ and

J is condensed in the multi-indices V and I. Recall that each $V \in \{1, 2\}^d$ determines a unique basis in $\mathcal{F}^{(d)}$, precisely $\mathcal{B}_V$. On the other hand, given a basis $\mathcal{B} \in \mathcal{F}^{(d)}$ associated with the multi-index V, for any $I \in \{0, 1\}^d$ there might be several subspaces $\theta \in \Theta_d$ such that $I_{V,\theta} = I$. By the above lemma, all these subspaces therefore produce identical coefficients $\alpha(\theta, V, J)$.

Notice that if $v_1 = 2$, then $\mu(V, I, J)$ reduces to a linear combination of derivatives of the Dirac delta centered at $(1, 0, \ldots, 0)$ of order at most $|J|$. This follows easily from the fact that the only point in the simplex S_d with first coordinate equal to 1 is $(1, 0, \ldots, 0)$. If furthermore $i_1 = 0$, then $\mu(V, I, J) = 0$.

Assume $v_1 = 1$. We already mentioned that the support of $\mu(V, I, J)$ is contained in the simplex S_d. Furthermore, for any $h \geq 2$,

(1) if $(v_h, i_h) = (1, 1)$ then the support of $\mu(V, I, J)$ is contained in the hyperplane $x_h = 0$.
(2) If $(v_h, i_h) = (2, 1)$ then the support of $\mu(V, I, J)$ is contained in the hyperplane $x_h = 1 - (x_1 + \ldots + x_{h-1})$.
(3) If $(v_h, i_h) = (2, 0)$ then $\mu(V, I, J) = 0$.
(4) The couple $(v_h, i_h) = (1, 0)$ gives no restrictions on the support of $\mu(V, I, J)$.

Similarly, in the case $h = 1$,

(5) if $(v_1, i_1) = (1, 1)$ then the support of $\mu(V, I, J)$ is contained in the hyperplane $x_1 = 0$.
(6) The couple $(v_1, i_1) = (1, 0)$ gives no restrictions on the support of $\mu(V, I, J)$.

Remark 30 If g is smooth with compact support in S_d, then $\widehat{g\chi_{S_d}}(\xi) = \widehat{g}(\xi)$ has fast decay at infinity. Observe that this does not contradict the above theorem. Indeed, by the previous remarks all coefficients $\langle \mu(V, I, J), g \rangle$ vanish except when $V = (1, \ldots, 1)$ and $I = (0, \ldots, 0)$ which implies that $J = (0, \ldots, 0)$. This choice of V and I forces $\theta = \mathbb{R}^d$ and $\mathcal{Q}(\theta) = \{0\}$. In this case we have

$$\widehat{g\chi_{S_d}}(0) = \int_{S_d} g(x)\,dx.$$

For $\xi \neq 0$ all the coefficients $\langle \mu(V, I, J), g \rangle$ vanish so that

$$\widehat{g\chi_{S_d}}(\xi) = \mathcal{R}_{\theta, w}(g, \xi).$$

3.2 General Simplex

With an affine change of variables Lemma 20 for the standard simplex can be transferred to a general simplex.

Definition 31 Let $M \in GL(d, \mathbb{Z})$ and let $\mathcal{B} = \{b_1, \ldots, b_d\} \in \mathcal{F}^{(d)}$. Then we shall denote by $M\mathcal{B}$ the basis $\{Mb_1, \ldots, Mb_d\}$ and by $M\mathcal{F}^{(d)}$ the collection of the bases $M\mathcal{B}$ with $\mathcal{B} \in \mathcal{F}^{(d)}$. Similarly $M\Theta_d$ is the collection of all the spaces $M\theta$ with $\theta \in \Theta_d$. Clearly $M\Theta_d$ consists of all subspaces generated by any possible choice of vectors in

$M\mathcal{F}^{(d)}$. For every $\eta \in M\Theta_d$ we set

$$\mathcal{Q}_M(\eta) = \left\{ \xi \in \mathbb{R}^d : \text{for all } v \in \bigcup_{\mathcal{B} \in \mathcal{F}^{(d)}} M\mathcal{B}, \xi \cdot v = 0 \text{ iff } v \in \eta \right\}.$$

Lemma 32 *Let* $M \in GL(d, \mathbb{Z})$. *For every* $\theta \in \Theta_d$

$$\mathcal{Q}_M(M\theta) = (M^t)^{-1}\mathcal{Q}(\theta).$$

Proof This follows immediately from the definitions. Observe that $\xi \in (M^t)^{-1}\mathcal{Q}(\theta)$ if and only if $M^t\xi \in \mathcal{Q}(\theta)$ if and only if, for all $b \in \bigcup_{\mathcal{B} \in \mathcal{F}^{(d)}} \mathcal{B}$,

$$M^t\xi \cdot b = \xi \cdot Mb = 0 \text{ iff } b \in \theta,$$

if and only if, for every $v \in \bigcup_{\mathcal{B} \in \mathcal{F}^{(d)},} M\mathcal{B}$,

$$\xi \cdot v = 0 \text{ iff } v \in M\theta,$$

if and only if $\xi \in \mathcal{Q}_M(M\theta)$. □

Lemma 33 *Let* $\mathcal{P}$ *be a simplex in* $\mathbb{R}^d$ *with vertices* $\mathbf{0}, \mathbf{m}_1, \ldots, \mathbf{m}_d \in \mathbb{Z}^d$ *and let* $M \in GL(d, \mathbb{Z})$ *be the* $d \times d$ *matrix with columns* $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_d$, *which maps the standard simplex* S_d *onto* $\mathcal{P}$. *Let* $q \in C^{w+1}(\mathbb{R}^d)$ *with* $w \in \mathbb{N}$, *let* $Q(x) = q(x)\chi_{\tau\mathcal{P}}(x)$ *with* $\tau > 0$, *and let* $q_{\tau,M}(x) = q(\tau Mx)$. *Then, following the definitions and notations of the previous section, for every* $\theta \in \Theta_d$ *and* $\xi \in \mathcal{Q}_M(M\theta)$,

$$\begin{aligned}
\widehat{Q}(\xi) &= \int_{\tau\mathcal{P}} q(x)e^{-2\pi i x \cdot \xi} dx \\
&= \tau^d \det(M) \sum_{V \in \{1,2\}^d} \sum_{|J| \leqslant w, J \sqsubseteq I_{V,\theta}} \frac{(-1)^{|I_{V,\theta}|} \langle \mu(V, I_{V,\theta}, J), q_{\tau,M} \rangle e^{-2\pi i \tau M \lambda_V \cdot \xi}}{\prod_{b_k \in \mathcal{B}_V \backslash \theta} (2\pi i \tau M b_k \cdot \xi)^{j_k+1}} \\
&\quad + \tau^d \det(M)\mathcal{R}_{\theta,w}(q_{\tau,M}, \tau M^t \xi).
\end{aligned}$$

In the above formula we adopt the convention: $\mathcal{B}_V = \{b_1, \ldots, b_d\}$ is the basis associated with the multi-index $V = (v_1, v_2, \ldots, v_d)$, $I_{V,\theta} = (i_1, \ldots, i_d) \in \{0, 1\}^d$ is the multi-index such that $i_k = 0$ if and only if $b_k \in \theta$, $J \sqsubseteq I_{V,\theta}$ means that $j_k = 0$ if $i_k = 0$. The coefficients $\langle \mu(V, I_{V,\theta}, J), q_{\tau,M} \rangle$ and the remainder $\mathcal{R}_{\theta,w}(q_{\tau,M}, \tau M^t \xi)$ are the ones defined in Lemma 20 and Definition 28. In particular they satisfy the following:

(i) the coefficients $\langle \mu(V, I_{V,\theta}, J), q_{\tau,M} \rangle$ satisfy the estimate

$$\left| \langle \mu(V, I_{V,\theta}, J), q_{\tau,M} \rangle \right| \leqslant c2^{(d-1)|J|} \sup_{|\alpha| \leqslant |J|} \sup_{x \in S_d} \left| \frac{\partial^\alpha q_{\tau,M}}{\partial x^\alpha}(x) \right|.$$

(ii) *The remainder $\mathcal{R}_{\theta,w}(q_{\tau,M}, \tau M^t \xi)$ has the following property: for every $\Omega > 1/(2\pi)$ and every $\tau_0 > 0$ there exists a constant $c = c(d, \Omega, \tau_0) > 0$ independent of $q_{\tau,M}$ and w such that for every $\tau > \tau_0$ the coefficients $\{\chi_{\mathcal{Q}_M(M\theta)}(n)\mathcal{R}_{\theta,w}(q_{\tau,M}, \tau M^t n)\}_{n \in \mathbb{Z}^d}$ are the Fourier coefficients of a function on the torus $\mathbb{T}^d$ bounded by*

$$c(2^{d-1}\Omega\tau^{-1})^{w+1} \sup_{w-d+2\leqslant|\alpha|\leqslant w+1} \sup_{x\in S_d} \left|\frac{\partial^\alpha q_{\tau,M}}{\partial x^\alpha}(x)\right|.$$

Proof This lemma follows from Lemma 20 via an affine change of variables. Define

$$G(x) = Q(\tau M x) = q(\tau M x)\chi_{\tau \mathcal{P}}(\tau M x) = q_{\tau,M}(x)\chi_{S_d}(x).$$

Then,

$$\begin{aligned}
\widehat{Q}(\xi) &= \int_{\mathbb{R}^d} Q(x)e^{-2\pi i \xi \cdot x}dx = \tau^d \det(M) \int_{\mathbb{R}^d} Q(\tau M x)e^{-2\pi i \xi \cdot \tau M x}dx \\
&= \tau^d \det(M) \int_{\mathbb{R}^d} Q(\tau M x)e^{-2\pi i \tau M^t \xi \cdot x}dx \\
&= \tau^d \det(M)\widehat{G}(\tau M^t \xi).
\end{aligned}$$

Applying Lemma 20 to the function $G(x)$ we obtain the desired expansion. The same lemma also shows that $\{\chi_{\mathcal{Q}(\theta)}(n)\mathcal{R}_{\theta,w}(q_{\tau,M}, \tau n)\}_{n \in \mathbb{Z}^d}$ are the Fourier coefficients of a function on the torus bounded by

$$U(2^{d-1}\Omega\tau^{-1})^{w+1} \sup_{w-d+2\leqslant|\alpha|\leqslant w+1} \sup_{x\in S_d} \left|\frac{\partial^\alpha q_{\tau,M}}{\partial x^\alpha}(x)\right|$$

where U is the same constant that appears in Lemma 20. By Lemma 40 in Appendix A, $\{\succ_{\mathcal{Q}(\theta)}(M^t n)\mathcal{R}_{\theta,w}(q_{\tau,M}, \tau M^t n)\}_{n \in \mathbb{Z}^d}$ are the Fourier coefficients of a function on the torus satisfying the same bound.   □

4 Expansion in Multivariate Bernoulli Polynomials

In this section we shall prove our Theorem 7. Let us start with a lemma on the Fourier expansion of the multivariate Bernoulli polynomials.

Lemma 34 *Let φ be a non-negative, radial, smooth function in $\mathbb{R}^d$, with compact support and integral one. Let $J = (j_1, j_2, \ldots, j_d)$ be a multi-index of non-negative integers and let $L \in GL(d, \mathbb{Z})$. If $\mathfrak{B}_{J,L}(x)$ are as in Definition 4, then, for every $x \in \mathbb{R}^d$,*

$$\mathfrak{B}_{J,L}(x) = \lim_{\varepsilon\to0+}\left\{(-1)^{|I|}\sum_{n\in\Delta(I,L)}\widehat{\varphi}(\varepsilon n)\frac{e^{2\pi i n\cdot x}}{(2\pi i L n)^J}\right\}.$$

Here, $I = (i_1, \ldots, i_d)$ with $i_k = 0$ if $j_k = 0$ and $i_k = 1$ if $j_k > 0$, and the set $\Delta(I, L)$ is the subset of frequencies in $\mathbb{Z}^d$ defined by

$$\Delta(I, L) = \left\{ n \in \mathbb{Z}^d : (Ln)_k = 0 \text{ iff } i_k = 0 \right\}.$$

Finally, in the denominators

$$(2\pi i\, Ln)^J = (2\pi i\, (Ln)_1)^{j_1} (2\pi i\, (Ln)_2)^{j_2} \cdots (2\pi i\, (Ln)_d)^{j_d}$$

we adopt the convention that $0^0 = 1$. In particular, all the denominators in the Fourier expansion of $\mathfrak{B}_{J,L}(x)$ are different from zero.

Proof Recall that if $f(x)$ is an integrable function with Fourier transform $\widehat{f}(\xi)$ and L is a non-singular matrix, then $\widehat{f}(L\xi)$ is the Fourier transform of $|\det L|^{-1} f\left((L^{-1})^t x\right)$. Moreover, if $f(x)$ is a function with bounded support, the Poisson summation formula gives

$$|\det L|^{-1} \sum_{n \in \mathbb{Z}^d} \varphi_\varepsilon * f\left((L^{-1})^t (x + n)\right) = \sum_{n \in \mathbb{Z}^d} \widehat{\varphi}(\varepsilon n)\, \widehat{f}(Ln) e^{2\pi i n x}.$$

Observe that the series on the left is finite and the one on the right is absolutely convergent and that the application of the summation formula is legitimate (see Lemma 13). Then the lemma follows by choosing $f(x) = B_J(x) = B_{j_1}(x_1) \cdots B_{j_d}(x_d)$. Indeed, for every n in $\mathbb{Z}^d$, one has

$$\int_{\mathbb{R}^d} B_J(x) e^{-2\pi i n \cdot x} dx = \prod_{k=1}^d \int_0^1 B_{j_k}(x_k) e^{-2\pi i n_k \cdot x_k} dx_k$$

$$= \prod_{k=1}^d \begin{cases} -1/(2\pi i n_k)^{j_k} & \text{if } j_k \neq 0 \text{ and } n_k \neq 0, \\ 0 & \text{if } j_k \neq 0 \text{ and } n_k = 0, \\ 0 & \text{if } j_k = 0 \text{ and } n_k \neq 0, \\ 1 & \text{if } j_k = 0 \text{ and } n_k = 0. \end{cases}$$

Hence, by the definition of $\Delta(I, L)$,

$$\int_{\mathbb{R}^d} B_J(x) e^{-2\pi i Ln \cdot x} dx = \begin{cases} (-1)^{|I|}/(2\pi i\, Ln)^J & \text{if } n \in \Delta(I, L), \\ 0 & \text{if } n \notin \Delta(I, L). \end{cases}$$

$\square$

We shall also need the following lemma.

Lemma 35 *For a fixed $V \in \{1, 2\}^d$ and for every $I \in \{0, 1\}^d$ we have*

$$\bigcup_{\theta \in \Theta_d : I_{V,\theta} = I} \left[\mathbb{Z}^d \cap \mathcal{Q}_M(M\theta) \right] = \Delta\left(I, (M D_V)^t\right).$$

Here if, as usual, $\mathcal{B}_V = \{b_1, \ldots, b_d\}$ is the basis associated with the multi-index V, then $I_{V,\theta} = (i_1, \ldots, i_d)$ where $i_k = 0$ if and only if $b_k \in \theta$, and D_V is the matrix with columns $b_1, \ldots, b_d$.

Proof Assume that

$$m \in \bigcup_{\theta \in \Theta_d : I_{V,\theta} = I} \left[\mathbb{Z}^d \cap \mathcal{Q}_M(M\theta) \right]$$

Then $m \in \mathcal{Q}_M(M\theta)$ for some θ such that $I_{V,\theta} = I$. Thus, if $b_k \in \mathcal{B}_V$, then $m \cdot Mb_k = 0$ if and only if $b_k \in \theta$, but since $I_{V,\theta} = I$, then $b_k \in \theta$ if and only if $i_k = 0$. Thus $m \cdot Mb_k = 0$ if and only if $i_k = 0$, which implies $m \in \Delta\left(I, (MD_V)^t\right)$, since

$$
\begin{aligned}
\Delta\left(I, (MD_V)^t\right) &= \left\{ m \in \mathbb{Z}^d : (MD_V)^t m \cdot e_k = 0 \text{ iff } i_k = 0 \right\} \\
&= \left\{ m \in \mathbb{Z}^d : \left((MD_V)^t m\right)^t e_k = 0 \text{ iff } i_k = 0 \right\} \\
&= \left\{ m \in \mathbb{Z}^d : m^t MD_V e_k = 0 \text{ iff } i_k = 0 \right\} \\
&= \left\{ m \in \mathbb{Z}^d : m^t Mb_k = 0 \text{ iff } i_k = 0 \right\} \\
&= \left\{ m \in \mathbb{Z}^d : m \cdot Mb_k = 0 \text{ iff } i_k = 0 \right\}.
\end{aligned}
$$

Conversely, if $m \in \Delta\left(I, (MD_V)^t\right)$, that is if $m \in \mathbb{Z}^d$ is such that $Mb_k \cdot m = 0$ if and only if $i_k = 0$, then, calling

$$\theta_m = \langle b \text{ in the bases} : Mb \cdot m = 0 \rangle,$$

we have $I_{V,\theta_m} = I$, (indeed, setting $I_{V,\theta_m} = (r_1, \ldots, r_d)$, we have $r_k = 0$ if and only if $b_k \in \theta_m$ if and only if $Mb_k \cdot m = 0$ if and only if $i_k = 0$). Finally, obviously, $m \in \mathcal{Q}_M(M\theta_m)$. □

We are ready to prove our main result.

Proof of Theorem 7 Let $Q(x) = \chi_{\tau\mathcal{P}}(x)q(x)$. We have

$$
\begin{aligned}
\sum_{n \in \mathbb{Z}^d} \varphi_\varepsilon * Q(x+n) &= \sum_{n \in \mathbb{Z}^d} \widehat{\varphi}(\varepsilon n)\widehat{Q}(n)e^{2\pi i n \cdot x} \\
&= \sum_{\theta \in \Theta_d} \sum_{n \in \mathbb{Z}^d \cap \mathcal{Q}_M(M\theta)} \widehat{\varphi}(\varepsilon n)\widehat{Q}(n)e^{2\pi i n \cdot x}.
\end{aligned}
$$

Since the multiplier $\widehat{\varphi}(\varepsilon n)$ is rapidly decreasing the series converge absolutely and the rearrangements of the terms of the series are allowed. Then, by Lemma 33,

$$\sum_{n\in\mathbb{Z}^d} \varphi_\varepsilon * Q(x+n)$$

$$= \sum_{\substack{\theta\in\Theta_d\ n\in\mathbb{Z}^d\cap\\ \mathcal{Q}_M(M\theta)}} \widehat{\varphi}(\varepsilon n)\tau^d \det(M) \sum_{V\in\{1,2\}^d} \sum_{|J|\leqslant w} \frac{(-1)^{|I_{V,\theta}|}\big\langle \mu(V, I_{V,\theta}, J), q_{\tau,M}\big\rangle e^{-2\pi i(\tau M\lambda_V)\cdot n} e^{2\pi in\cdot x}}{\prod_{b_k\in\mathcal{B}_V\setminus\theta}(2\pi i\tau Mb_k\cdot n)^{j_k+1}}$$

$$+ \sum_{\theta\in\Theta_d} \sum_{n\in\mathbb{Z}^d\cap\mathcal{Q}_M(M\theta)} \widehat{\varphi}(\varepsilon n)\tau^d \det(M)\mathcal{R}_{\theta,w}(q_{\tau,M}, \tau M^t n)e^{2\pi in\cdot x}$$

$$= \sum_{V\in\{1,2\}^d}\sum_{\theta\in\Theta_d} \Phi(V,\theta) + \sum_{\theta\in\Theta_d}\Psi(\theta)$$

where

$$\Phi(V,\theta) = \tau^d \det(M) \sum_{|J|\leqslant w} \big\langle \mu(V, I_{V,\theta}, J), q_{\tau,M}\big\rangle$$

$$\times (-1)^{|I_{V,\theta}|} \sum_{n\in\mathbb{Z}^d\cap\mathcal{Q}_M(M\theta)} \widehat{\varphi}(\varepsilon n)\frac{e^{2\pi i(x-\tau M\lambda_V)\cdot n}}{\prod_{b_k\in\mathcal{B}_V\setminus\theta}(2\pi i\tau Mb_k\cdot n)^{j_k+1}}.$$

Rearranging the sum we have

$$\sum_{V\in\{1,2\}^d}\sum_{\theta\in\Theta_d}\Phi(V,\theta) = \sum_{V\in\{1,2\}^d}\sum_{I\in\{0,1\}^d}\sum_{\theta\in\Theta_d:I_{V,\theta}=I}\Phi(V,\theta)$$

$$= \sum_{V\in\{1,2\}^d}\sum_{I\in\{0,1\}^d}\sum_{\theta\in\Theta_d:I_{V,\theta}=I}\tau^d\det(M)$$

$$\sum_{|J|\leqslant w}\big\langle\mu(V, I, J), q_{\tau,M}\big\rangle$$

$$\times(-1)^{|I|}\sum_{n\in\mathbb{Z}^d\cap\mathcal{Q}_M(M\theta)}\widehat{\varphi}(\varepsilon n)\frac{e^{2\pi i(x-\tau M\lambda_V)\cdot n}}{\prod_{b_k\in\mathcal{B}_V\setminus\theta}(2\pi i\tau Mb_k\cdot n)^{j_k+1}}$$

$$= \det(M)\sum_{V\in\{1,2\}^d}\sum_{I\in\{0,1\}^d}\sum_{|J|\leqslant w}\tau^d\big\langle\mu(V, I, J), q_{\tau,M}\big\rangle$$

$$\times(-1)^{|I|}\sum_{\theta\in\Theta_d:I_{V,\theta}=I}\sum_{n\in\mathbb{Z}^d\cap\mathcal{Q}_M(M\theta)}\widehat{\varphi}(\varepsilon n)$$

$$\frac{e^{2\pi i(x-\tau M\lambda_V)\cdot n}}{\prod_{b_k\in\mathcal{B}_V\setminus\theta}(2\pi i\tau Mb_k\cdot n)^{j_k+1}}$$

$$= \det(M)\sum_{V\in\{1,2\}^d}\sum_{I\in\{0,1\}^d}\sum_{|J|\leqslant w}\tau^{d-|J|-|I|}\big\langle\mu(V, I, J), q_{\tau,M}\big\rangle$$

$$\times(-1)^{|I|}\sum_{\theta\in\Theta_d:I_{V,\theta}=I}\sum_{n\in\mathbb{Z}^d\cap\mathcal{Q}_M(M\theta)}\widehat{\varphi}(\varepsilon n)$$

$$\frac{e^{2\pi i(x-\tau M\lambda_V)\cdot n}}{\prod_{b_k\in\mathcal{B}_V}(2\pi iMb_k\cdot n)^{j_k+i_k}}$$

with the usual convention that in the denominators $0^0 = 1$. Now, since $\mathbb{Z}^d \cap \left(\bigcup_{\theta : I_{V,\theta} = I} \mathcal{Q}_M(M\theta) \right) = \Delta \left(I, (MD_V)^t \right)$ by Lemma 35, we have

$$
\sum_{\theta \in \Theta_d : I_{V,\theta} = I} \sum_{n \in \mathbb{Z}^d \cap \mathcal{Q}_M(M\theta)} \widehat{\varphi}(\varepsilon n) \frac{e^{2\pi i (x - \tau M\lambda_V) \cdot n}}{\prod_{b_k \in \mathcal{B}_V} (2\pi i M b_k \cdot n)^{j_k + i_k}}
$$
$$
= \sum_{n \in \Delta(I, (MD_V)^t)} \widehat{\varphi}(\varepsilon n) \frac{e^{2\pi i n \cdot (x - \tau M\lambda_V)}}{\prod_{b_k \in \mathcal{B}_V} (2\pi i M b_k \cdot n)^{j_k + i_k}}.
$$

Observe that $M b_k \cdot n = (b_k^t M^t)n = (D_V^t M^t)_k n = \left((MD_V)^t n \right)_k = \left((MD_V)^t n \right) \cdot e_k$.
Hence,

$$
(-1)^{|I|} \sum_{n \in \Delta(I, (MD_V)^t)} \widehat{\varphi}(\varepsilon n) \frac{e^{2\pi i n \cdot (x - \tau M\lambda_V)}}{\prod_{b_k \in \mathcal{B}_V} (2\pi i M b_k \cdot n)^{j_k + i_k}}
$$
$$
= (-1)^{|I|} \sum_{n \in \Delta(I, (MD_V)^t)} \widehat{\varphi}(\varepsilon n) \frac{e^{2\pi i n \cdot (x - \tau M\lambda_V)}}{\prod_{b_k \in \mathcal{B}_V} \left(2\pi i \left((MD_V)^t n \right) \cdot e_k \right)^{j_k + i_k}}
$$
$$
= (-1)^{|I|} \sum_{n \in \Delta(I, (MD_V)^t)} \widehat{\varphi}(\varepsilon n) \frac{e^{2\pi i n \cdot (x - \tau M\lambda_V)}}{(2\pi i (MD_V)^t n)^{J+I}}
$$
$$
= \varphi_\varepsilon * \mathfrak{B}_{J+I, (MD_V)^t}(x - \tau M\lambda_V).
$$

Since $J \sqsubseteq I$, the vanishing components of $J + I$ appear in the same spots as those of I, so that $\Delta \left(I, (MD_V)^t \right) = \Delta \left(I + J, (MD_V)^t \right)$. Hence, the principal part becomes

$$
\sum_{V \in \{1,2\}^d} \sum_{\theta \in \Theta_d} \Phi(V, \theta)
$$
$$
= \det(M) \sum_{V \in \{1,2\}^d} \sum_{I \in \{0,1\}^d} \sum_{|J| \leqslant w} \tau^{d - |J| - |I|} \left\langle \mu(V, I, J), q_{\tau, M} \right\rangle
$$
$$
\varphi_\varepsilon * \mathfrak{B}_{J+I, (MD_V)^t}(x - \tau M\lambda_V).
$$

Let us consider now the remainder

$$
\Psi(\theta) = \tau^d \det(M) \sum_{n \in \mathbb{Z}^d \cap \mathcal{Q}_M(M\theta)} \widehat{\varphi}(\varepsilon n) \mathcal{R}_{\theta, w}(q_{\tau, M}, \tau M^t n) e^{2\pi i n \cdot x}.
$$

For every $\theta \in \Theta_d$, by Lemma 33, $\Psi(\theta)$ is a function bounded by

$$
c \tau^{d - w - 1} \det(M) (2^{d-2} \pi^{-1} + \delta)^{w+1} \sup_{w - d + 2 \leqslant |\alpha| \leqslant w + 1} \sup_{x \in S_d} \left| \frac{\partial^\alpha q_{\tau, M}}{\partial x^\alpha}(x) \right|
$$

It follows that $\sum_{\theta \in \Theta_d} \Psi(\theta)$ is a bounded function, with the same bound. Letting $\varepsilon \to 0$ gives the desired result. $\square$

5 Proofs of Theorems 10 and 11

Theorems 10 and 11 are corollaries of Theorem 7. In particular, Theorem 10 follows applying the next result to a decomposition of the given polytope into simplices.

Theorem 36 *Let S_d be the standard simplex in $\mathbb{R}^d$, let $\mathcal{P} = M S_d$ with $M \in GL(d, \mathbb{Z})$. Let $p \in \mathbb{Z}^d$, let w be a non-negative integer and let $f \in C^{w+1}(\mathbb{R}^d)$. Then, there exists a numerical sequence $\{\gamma_k\}_{0 < k \leqslant w/2}$ such that for every positive integer N we have*

$$
N^{-d} \sum_{n \in \mathbb{Z}^d} \omega_{p+\mathcal{P}}(N^{-1}n) f(N^{-1}n) = \int_{p+\mathcal{P}} f(x) dx + \sum_{0 < k \leqslant w/2} \gamma_k N^{-2k} + O(N^{-w-1}).
$$

More precisely, with the notation in Theorem 7,

$$
\gamma_k = \det(M) \sum_{V \in \{1,2\}^d} \sum_{I \in \{0,1\}^d} \sum_{J \sqsubseteq I, \, |I+J|=2k} \langle \mu(V, I, J), f(p + M\cdot) \rangle \, \mathfrak{B}_{J+I,(MD_V)^t}(0).
$$

Proof Assume first $p = 0$. Since

$$
N^{-d} \sum_{n \in \mathbb{Z}^d} \omega_{\mathcal{P}}(N^{-1}n) f(N^{-1}n) = N^{-d} \sum_{n \in \mathbb{Z}^d} \omega_{N\mathcal{P}}(n) f(N^{-1}n),
$$

we apply Theorem 9 with $\tau = N$ to the function $q_N(x) = f(N^{-1}x)$. Let $g_{N,M}(x) = q_N(NMx)$, then

$$
N^{-d} \sum_{n \in \mathbb{Z}^d} \omega_{N\mathcal{P}}(n) f(N^{-1}n)
$$
$$
= \det(M) \sum_{V \in \{1,2\}^d} \sum_{I \in \{0,1\}^d} \sum_{|J| \leqslant w, J \sqsubseteq I} N^{-|J|-|I|} \langle \mu(V, I, J), g_{N,M} \rangle \, \mathfrak{B}_{J+I,(MD_V)^t}(0)
$$
$$
+ N^{-d} \mathcal{R}_w(0)
$$

with

$$
\left| N^{-d} \mathcal{R}_w(x) \right| \leqslant c N^{-w-1} \det(M) (2^{d-2} \pi^{-1} + \delta)^{w+1} \sup_{w-d+2 \leqslant |\alpha| \leqslant w+1} \sup_{x \in S_d} \left| \frac{\partial^\alpha g_{N,M}}{\partial x^\alpha}(x) \right|.
$$

Since $g_{N,M}(x) = f(Mx)$, then

$$
N^{-d} \sum_{n \in \mathbb{Z}^d} \omega_{M S_d}(N^{-1}n) f(N^{-1}n)
$$
$$
= \det(M) \sum_{V \in \{1,2\}^d} \sum_{I \in \{0,1\}^d} \sum_{|J| \leqslant w, J \sqsubseteq I} N^{-|J|-|I|} \langle \mu(V, I, J), f(M\cdot) \rangle \, \mathfrak{B}_{J+I,(MD_V)^t}(0)
$$
$$
+ N^{-d} \mathcal{R}_w(0).
$$

Now observe that when $I = (0, \ldots, 0)$ and therefore $J = (0, \ldots, 0)$ we have

$$\mu(V, I, J) = 0$$

if $V \neq (1, \ldots, 1)$, whereas when $V = (1, \ldots, 1)$ we have

$$\langle \mu(V, I, J), f(M\cdot) \rangle = \int_{S_d} f(Mx)dx = \det(M)^{-1} \int_{MS_d} f(x)dx.$$

Also observe that

$$\mathfrak{B}_{J+I,(MD_V)^t}(0) = 0$$

whenever $|J| + |I|$ is odd. Indeed, since $\Delta(I + J, (MD_V)^t)$ is a cone and φ is radial, by Lemma 34 we have

$$
\begin{aligned}
\mathfrak{B}_{I+J,(MD_V)^t}(0) &= \lim_{\varepsilon \to 0+} \left\{ (-1)^{|I|} \sum_{n \in \Delta(I+J,(MD_V)^t)} \frac{\widehat{\varphi}(\varepsilon n)}{(2\pi i (MD_V)^t n)^{I+J}} \right\} \\
&= \lim_{\varepsilon \to 0+} \left\{ (-1)^{|I|} \sum_{n \in \Delta(I+J,(MD_V)^t)} \frac{\widehat{\varphi}(\varepsilon n)}{(-2\pi i (MD_V)^t n)^{I+J}} \right\} \\
&= (-1)^{|I+J|} \mathfrak{B}_{I+J,(MD_V)^t}(0).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
N^{-d} &\sum_{n \in \mathbb{Z}^d} f(N^{-1}n) \omega_{MS_d}(N^{-1}n) = \int_{MS_d} f(x)dx \\
&+ \sum_{k \geq 1} N^{-2k} \left(\det(M) \sum_{V \in \{1,2\}^d} \sum_{I \in \{0,1\}^d} \sum_{\substack{|J| \leq w, J \sqsubseteq I, \\ |I+J|=2k}} \langle \mu(V, I, J), f(M\cdot) \rangle \mathfrak{B}_{J+I,(MD_V)^t}(0) \right) \\
&+ N^{-d} \mathcal{R}_w(0) \\
&= \int_{MS_d} f(x)dx + \sum_{0 < 2k \leq w} \gamma_k N^{-2k} + O(N^{-w-1})
\end{aligned}
$$

where

$$\gamma_k = \det(M) \sum_{V \in \{1,2\}^d} \sum_{I \in \{0,1\}^d} \sum_{J \sqsubseteq I, \, |I+J|=2k} \langle \mu(V, I, J), f(M\cdot) \rangle \mathfrak{B}_{J+I,(MD_V)^t}(0).$$

Now assume $p \neq 0$. Then

$$N^{-d} \sum_{n \in \mathbb{Z}^d} \omega_{p+\mathcal{P}}(N^{-1}n) f(N^{-1}n) = N^{-d} \sum_{n \in \mathbb{Z}^d} \omega_{\mathcal{P}}(N^{-1}n - p) f(N^{-1}n)$$

$$= N^{-d} \sum_{n \in \mathbb{Z}^d} \omega_{\mathcal{P}}(N^{-1}n) f(N^{-1}n + p).$$

Hence, the case $p \neq 0$ follows from the case $p = 0$ applied to the function $f(x + p)$. □

Theorems 10 and 11 are now easily deduced.

Proof of Theorem 10 We use Theorem 36 and the additivity of $S_N(f, \mathcal{P})$ with respect to $\mathcal{P}$. □

Proof of Theorem 11 By Theorem 10,

$$S_N(f, \mathcal{P}) = \int_{\mathcal{P}} f(x)\, dx + \sum_{0 < k \leqslant w/2} \gamma_k N^{-2k} + O\left(N^{-w-1}\right).$$

Then

$$\sum_{0 \leqslant j \leqslant w/2} c_j S_{2^j N}(f, \mathcal{P})$$

$$= \left(\sum_{0 \leqslant j \leqslant w/2} c_j \right) \int_{\mathcal{P}} f(x)dx + \sum_{0 < k \leqslant w/2} \gamma_k N^{-2k} \left(\sum_{0 \leqslant j \leqslant w/2} c_j 2^{-2kj} \right) + O(N^{-w-1})$$

and the conclusion follows since the Vandermonde system is solvable. □

Appendix A: Some Basic Facts on the Harmonic Analysis on Commutative Groups

The following results on the harmonic analysis on groups, subgroups and quotient spaces are well known (see [31, Sect. 2.7]). We include the case of the torus for the reader's convenience.

Definition 37 Let $\mathcal{H}$ be a subgroup of $\mathbb{Z}^d$. The annihilator of $\mathcal{H}$ is the compact subgroup $\mathcal{H}^\perp$ of $\mathbb{T}^d$ given by

$$\mathcal{H}^\perp = \left\{ t \in \mathbb{T}^d : \forall h \in \mathcal{H}, \ t \cdot h \in \mathbb{Z} \right\} = \left\{ h \in \mathbb{Z}^d : \forall t \in \mathcal{H}^\perp, \ e^{2\pi i t \cdot h} = 1 \right\}.$$

Lemma 38 *We have*

$$\mathcal{H} = \left\{ h \in \mathbb{Z}^d : \forall t \in \mathcal{H}^\perp, \ t \cdot h \in \mathbb{Z} \right\}.$$

This is a particular case of Lemma 2.1.3 in [31]. The following is a direct elementary proof.

Proof Let $h \in \mathcal{H}$, then by definition $t \cdot h \in \mathbb{Z}$ for every $t \in \mathcal{H}^\perp$. To show the converse observe that since every subgroup of $\mathbb{Z}^d$ is a lattice, there exists an integer $d \times q$ matrix B, with $q \leqslant d$, of maximal rank such that

$$\mathcal{H} = \left\{ Bz : z \in \mathbb{Z}^q \right\}.$$

Then

$$\mathcal{H}^\perp = \left\{ t \in \mathbb{T}^d : \forall z \in \mathbb{Z}^q, \ Bz \cdot t \in \mathbb{Z} \right\} = \left\{ t \in \mathbb{T}^d : \forall z \in \mathbb{Z}^q, \ z \cdot B^t t \in \mathbb{Z} \right\}$$
$$= \left\{ t \in \mathbb{T}^d : B^t t \in \mathbb{Z}^q \right\}.$$

Since B has rank q we can assume without loss of generality that there exists a $q \times q$ invertible matrix C and a $q \times (d - q)$ matrix D such that $B^t = \begin{bmatrix} C & D \end{bmatrix}$. Hence for $t = (t_1, t_2)$,

$$B^t t = C t_1 + D t_2 = z \in \mathbb{Z}^q.$$

It follows that

$$\mathcal{H}^\perp = \left\{ \left(C^{-1}(z - D t_2), t_2 \right) : z \in \mathbb{Z}^q, t_2 \in \mathbb{T}^{d-q} \right\}.$$

Now, let $m \in \mathbb{Z}^d$ such that for every $t \in \mathcal{H}^\perp$ we have $m \cdot t \in \mathbb{Z}$. Then, if $m = (m_1, m_2)$, for every $z \in \mathbb{Z}^q, t_2 \in \mathbb{T}^{d-q}$ we have

$$(m_1, m_2) \cdot \left(C^{-1}(z - D t_2), t_2 \right) = m_1 \cdot C^{-1} z - m_1 \cdot C^{-1} D t_2 + t_2 \cdot m_2 \in \mathbb{Z}. \quad (3)$$

Let $t_2 = 0$. Then $m_1 \cdot C^{-1} z \in \mathbb{Z}$ and hence $\left(C^{-1} \right)^t m_1 \cdot z \in \mathbb{Z}$ for every $z \in \mathbb{Z}^q$. Therefore $\left(C^{-1} \right)^t m_1 \in \mathbb{Z}^q$. It follows that $m_1 = C^t h$ for some $h \in \mathbb{Z}^q$. From (3) we obtain that

$$C^t h \cdot C^{-1} z - C^t h \cdot C^{-1} D t_2 + t_2 \cdot m_2$$

$$= h \cdot z - h \cdot Dt_2 + t_2 \cdot m_2$$

is an integer for every $t_2 \in \mathbb{T}^{d-q}$. It follows that for every $t_2 \in \mathbb{T}^{d-q}$ we have

$$(m_2 - D^t h) \cdot t_2 \in \mathbb{Z},$$

this implies that $m_2 = D^t h$ and therefore $m = Bh \in \mathcal{H}$. $\square$

Let $d\mu$ be the Haar measure on $\mathcal{H}^\perp$. Since $\mathcal{H}^\perp$ is compact we can assume that $|d\mu| = 1$.

Lemma 39 *With the normalized Haar measure $d\mu$ on $\mathcal{H}^\perp$, for every $m \in \mathbb{Z}^d$ we have*

$$\int_{\mathcal{H}^\perp} e^{2\pi i m \cdot t} d\mu(t) = \begin{cases} 1 & m \in \mathcal{H}, \\ 0 & m \notin \mathcal{H}. \end{cases}$$

Proof The case $m \in \mathcal{H}$ is immediate since $e^{2\pi i m \cdot t} = 1$ for every $t \in \mathcal{H}^\perp$. Let $m \notin \mathcal{H}$. By Lemma 38 there exists $t_0 \in \mathcal{H}^\perp$ such that $e^{2\pi i m \cdot t_0} \neq 1$. By the invariance of the Haar measure we have

$$\int_{\mathcal{H}^\perp} e^{2\pi i m \cdot t} d\mu(t) = \int_{\mathcal{H}^\perp} e^{2\pi i m \cdot (t_0 + t)} d\mu(t) = e^{2\pi i m \cdot t_0} \int_{\mathcal{H}^\perp} e^{2\pi i m \cdot t} d\mu(t).$$

Hence

$$\int_{\mathcal{H}^\perp} e^{2\pi i m \cdot t} d\mu(t) = 0.$$

     $\square$

Lemma 40 *Let $\mathcal{H}$ be a subgroup of $\mathbb{Z}^d$ and let $d\mu$ be the normalized Haar measure on the annihilator $\mathcal{H}^\perp$. In particular $d\mu$ is a probability measure on $\mathbb{T}^d$. Let $f \in L^1(\mathbb{T}^d)$ and let $g(s) = \mu * f(s)$, that is*

$$g(s) = \int_{\mathcal{H}^\perp} f(s - t) d\mu(t).$$

Then,
(i) $\|g\|_\infty \leqslant \|f\|_\infty$;
(ii)

$$\widehat{g}(m) = \begin{cases} \widehat{f}(m) & m \in \mathcal{H}, \\ 0 & m \notin \mathcal{H}. \end{cases}$$

Proof (i) follows from the fact that the convolution with a probability measure is an operator with norm 1 on $L^\infty\left(\mathbb{T}^d\right)$. (ii) follows from the fact that $\widehat{\mu * f}(m) = \widehat{\mu}(m)\widehat{f}(m)$ and

$$\widehat{\mu}(m) = \int_{\mathcal{H}^\perp} e^{-2\pi i m \cdot t}\, d\mu(t) = \begin{cases} 1 & m \in \mathcal{H}, \\ 0 & m \notin \mathcal{H}. \end{cases}$$

$\square$

Appendix B: Bernoulli Polynomials and Lerch Zeta Functions

Here we give a different description of the functions $\mathfrak{B}_{J,L}(x)$. Such functions were defined (Definition 4) starting with a product of Bernoulli polynomials, restricting this product to the unit cube, composing it with an affine transformation and finally periodizing. One may ask if these operations commute and if these functions can be obtained as a linear combination of affine transformation of the periodic multivariate Bernoulli polynomials $\mathfrak{B}_{J,\mathrm{Id}}$ (here Id denotes the identity matrix).

We start from the Fourier expansion

$$\mathfrak{B}_{J,L}(x) = \lim_{\varepsilon \to 0+} (-1)^{|I|} \sum_{n \in \Delta(I,L)} \widehat{\varphi}(\varepsilon n) \frac{e^{2\pi i n x}}{(2\pi i L n)^J}$$

with the usual conventions on the notation (Lemma 34). In particular, the multi-index $I = (i_1, \ldots, i_d)$ is such that $i_k = 0$ if $j_k = 0$ and $i_k = 1$ if $j_k > 0$. Recall that in the points of discontinuity the definition of $\mathfrak{B}_{J,L}$ is by regularization and that $L \in GL(d, \mathbb{Z})$. Assume now that x is a point of continuity, so that the mollifier φ may not be taken to be necessarily radial. More precisely, we may set $\varphi(x) = |\det L|^{-1}\psi\left((L^T)^{-1}x\right)$ where

$$\psi(x) = \eta(x_1)\ldots\eta(x_d)$$

and η is a non-negative smooth function with compact support and integral one. In particular

$$\widehat{\varphi}(\xi) = \widehat{\psi}(L\xi).$$

Since L has integer entries, L has a unique (column) Hermite normal form H, that is, $L = HU$, where H is a lower triangular matrix with positive coefficients on the diagonal and such that all the other coefficients are nonnegative and smaller than the diagonal coefficient in the same row, whereas U is a unimodular integer matrix. The invertibility of the linear map $n \mapsto Un$ in $\mathbb{Z}^d$ immediately implies that the lattice $L\mathbb{Z}^d$ coincides with the lattice $H\mathbb{Z}^d$, and more specifically

$$\{Ln : n \in \Delta(I, L)\} = \{HUn : n \in \Delta(I, L)\} = \{Hm : m \in \Delta(I, H)\}.$$

Thus, setting $y = (L^{-1})^T x$, one obtains

$$
\begin{aligned}
\mathfrak{B}_{J,L}(x) &= \lim_{\varepsilon \to 0+} (-1)^{|I|} \sum_{n \in \Delta(I,L)} \widehat{\varphi}(\varepsilon n) \frac{e^{2\pi i n \cdot x}}{(2\pi i L n)^J} \\
&= \lim_{\varepsilon \to 0+} (-1)^{|I|} \sum_{n \in \Delta(I,L)} \widehat{\varphi}(\varepsilon L^{-1} L n) \frac{e^{2\pi i L n \cdot y}}{(2\pi i L n)^J} \\
&= \lim_{\varepsilon \to 0+} (-1)^{|I|} \sum_{m \in \Delta(I,H)} \widehat{\varphi}(\varepsilon L^{-1} H m) \frac{e^{2\pi i H m \cdot y}}{(2\pi i H m)^J} \\
&= \lim_{\varepsilon \to 0+} (-1)^{|I|} \sum_{m \in \Delta(I,H)} \widehat{\psi}(\varepsilon H m) \frac{e^{2\pi i H m \cdot y}}{(2\pi i H m)^J}.
\end{aligned}
$$

Let now $H = (h_{j,k})$, $k_j = \prod_{s=j}^{d} h_{s,s}$ for all $j = 1, \ldots, d$, and set $K = \mathrm{diag}(k_1, \ldots, k_d)$ to be the corresponding diagonal matrix. We claim that $K\mathbb{Z}^d \subseteq H\mathbb{Z}^d$. Indeed, it suffices to show that all vectors Ke_j belong to $H\mathbb{Z}^d$. Obviously, $Ke_d = h_{d,d}e_d = He_d$. By induction, assuming that $Ke_s \in H\mathbb{Z}^d$ for all $s = j+1, j+2, \ldots, d$, let us show that $Ke_j \in H\mathbb{Z}^d$. We have

$$
\begin{aligned}
Ke_j &= h_{j,j} \cdots h_{d,d}e_j \\
&= \sum_{s=j}^{d} h_{s,j} h_{j+1,j+1} \ldots h_{d,d}e_s - \sum_{s=j+1}^{d} h_{s,j} h_{j+1,j+1} \ldots h_{d,d}e_s \\
&= H(h_{j+1,j+1} \ldots h_{d,d}e_j) - \sum_{s=j+1}^{d} h_{s,j} h_{j+1,j+1}\ldots \left(h_{s,s} \ldots h_{d,d}e_s \right) \\
&= H(h_{j+1,j+1} \ldots h_{d,d}e_j) - \sum_{s=j+1}^{d} h_{s,j} h_{j+1,j+1}\ldots h_{s-1,s-1} Ke_s \in H\mathbb{Z}^d
\end{aligned}
$$

and the claim is proved. Observe that there is a finite number of different integer translates of $K\mathbb{Z}^d$ (precisely $k_1 k_2 \ldots k_d$). Take any point of $H\mathbb{Z}^d$ which is not in $K\mathbb{Z}^d$, say $v^{(1)}$. By linearity it follows that $v^{(1)} + K\mathbb{Z}^d$ is contained in $H\mathbb{Z}^d$ and is disjoint from $K\mathbb{Z}^d$. Take again a second vector in $H\mathbb{Z}^d$ which is not in $K\mathbb{Z}^d \cup (v^{(1)}+K\mathbb{Z}^d)$, say $v^{(2)}$. Then $v^{(2)} + K\mathbb{Z}^d$ is contained in $H\mathbb{Z}^d$ (and is disjoint from $K\mathbb{Z}^d \cup (v^{(1)}+K\mathbb{Z}^d)$). We can iterate this procedure until we exhaust all of $H\mathbb{Z}^d$. In other words, we have

$$
L\mathbb{Z}^d = H\mathbb{Z}^d = \bigcup_{\ell=1}^{\mathcal{L}} (v^{(\ell)} + K\mathbb{Z}^d)
$$

where the union is a disjoint union. Thus, recalling that $y = (L^{-1})^T x$,

$$
\mathcal{B}_{J,L}(x) = \lim_{\varepsilon \to 0+} (-1)^{|I|} \sum_{m \in \Delta(I,H)} \widehat{\psi}(\varepsilon H m) \frac{e^{2\pi i H m y}}{(2\pi i H m)^J}
$$

$$
= \lim_{\varepsilon \to 0+} (-1)^{|I|} \sum_{\ell=1}^{\mathcal{L}} \sum_{\substack{m \in \mathbb{Z}^d \\ v_s^{(\ell)} + (Km)_s = 0 \text{ iff } i_s = 0}} \widehat{\psi}\left(\varepsilon(v^{(\ell)} + Km)\right) \frac{e^{2\pi i (v^{(\ell)} + Km) y}}{\left(2\pi i (v^{(\ell)} + Km)\right)^J}
$$

$$
= \lim_{\varepsilon \to 0+} (-1)^{|I|} \sum_{\ell=1}^{\mathcal{L}} \prod_{\substack{s=1 \\ \text{s.t. } i_s=1}}^{d} e^{2\pi i v_s^{(\ell)} y_s}
$$

$$
\sum_{\substack{m_s \in \mathbb{Z} \\ m_s \neq -\frac{v_s^{(\ell)}}{k_s}}} \widehat{\eta}\left(\varepsilon(v_s^{(\ell)} + k_s m_s)\right) \frac{e^{2\pi i k_s m_s y_s}}{\left(2\pi i (v_s^{(\ell)} + k_s m_s)\right)^{j_s}}
$$

$$
= \sum_{\ell=1}^{\mathcal{L}} \prod_{\substack{s=1 \\ \text{s.t. } i_s=1}}^{d} (2\pi i k_s)^{-j_s} e^{2\pi i v_s^{(\ell)} y_s} \times
$$

$$
\times \lim_{\varepsilon \to 0+} (-1) \sum_{\substack{m_s \in \mathbb{Z} \\ m_s \neq -\frac{v_s^{(\ell)}}{k_s}}} \widehat{\eta}\left(\varepsilon k_s \left(\frac{v_s^{(\ell)}}{k_s} + m_s\right)\right) \frac{e^{2\pi i m_s (k_s y_s)}}{\left(\frac{v_s^{(\ell)}}{k_s} + m_s\right)^{j_s}}
$$

$$
= \sum_{\ell=1}^{\mathcal{L}} \prod_{\substack{s=1 \\ \text{s.t. } i_s=1}}^{d} e^{2\pi i v_s^{(\ell)} y_s} (2\pi i k_s)^{-j_s} L_{j_s}\left(k_s y_s, \frac{v_s^{(\ell)}}{k_s}\right)
$$

where for $j \geq 1$ we set

$$
L_j(x, r) = - \lim_{\varepsilon \to 0+} \sum_{n \in \mathbb{Z} \setminus \{-r\}} \widehat{\eta}(\varepsilon(n+r)) \frac{e^{2\pi i n x}}{(n+r)^j}.
$$

Observe that for $r \in \mathbb{Z}$

$$
L_j(x, r) = e^{-2\pi i r x} (2\pi i)^j B_j(x)
$$

where $B_j(x)$ is the j-th Bernoulli polynomial, whereas when $r \notin \mathbb{Z}$ the function $L_j(x, r)$ is related to the Lerch Zeta function

$$
\mathfrak{L}(x, j, r) = \sum_{n=0}^{+\infty} \frac{e^{2\pi i n x}}{(n+r)^j}.
$$

Indeed, formally,

$$L_j(x, r) = - \sum_{\substack{n \in \mathbb{Z} \\ n+r \neq 0}} \frac{e^{2\pi i n x}}{(n+r)^j} = -\mathfrak{L}(x, j, r) - (-1)^j \mathfrak{L}(-x, j, -r) + r^{-j}.$$

Moreover, when $r = p/q$ is rational, it is not difficult to write $L_j(x, r)$ in terms of periodic Bernoulli polynomials. Indeed, one can verify that for every periodic integrable function $f(x)$,

$$\frac{1}{q} \sum_{a=0}^{q-1} e^{-2\pi i \frac{p}{q}(x+a)} f\left(\frac{x+a}{q}\right) = \sum_{n \in \mathbb{Z}} \widehat{f}(nq + p) e^{2\pi i n x}.$$

Therefore, with $f(x) = B_j(x)$,

$$L_j(x, p/q) = - \sum_{\substack{n \in \mathbb{Z} \\ n+p/q \neq 0}} \frac{e^{2\pi i n x}}{(n+p/q)^j} = (2\pi i q)^j \sum_{\substack{n \in \mathbb{Z} \\ nq+p \neq 0}} \frac{-1}{(2\pi i (nq+p))^j} e^{2\pi i n x}$$

$$= (2\pi i q)^j \sum_{n \in \mathbb{Z}} \widehat{B}_j(nq+p) e^{2\pi i n x} = (2\pi i q)^j e^{-2\pi i x \frac{p}{q}} \frac{1}{q}$$

$$\sum_{a=0}^{q-1} e^{-2\pi i a \frac{p}{q}} B_j\left(\frac{x+a}{q}\right).$$

Thus,

$$\mathfrak{B}_{J,L}(x) = \sum_{\ell=1}^{\mathcal{L}} \prod_{\substack{s=1 \\ \text{s.t. } i_s=1}}^{d} e^{2\pi i v_s^{(\ell)} y_s} (2\pi i k_s)^{-j_s} L_{j_s}\left(k_s y_s, \frac{v_s^{(\ell)}}{k_s}\right)$$

$$= \sum_{\ell=1}^{\mathcal{L}} \prod_{\substack{s=1 \\ \text{s.t. } i_s=1}}^{d} \frac{1}{k_s} \sum_{a=0}^{k_s-1} e^{-2\pi i \frac{v_s^{(\ell)}}{k_s} a} B_{j_s}\left(y_s + \frac{a}{k_s}\right).$$

Now, recalling that $K = \operatorname{diag}(k_1, \dots, k_d)$, if we set $K^I = \operatorname{diag}(k_1^{i_1}, \dots, k_d^{i_d})$ so that k_s is replaced with 1 whenever $i_s = 0$, then

$$\mathfrak{B}_{J,L}(x) = \sum_{\ell=1}^{\mathcal{L}} \prod_{\substack{s=1 \\ \text{s.t. } i_s=1}}^{d} \frac{1}{k_s} \sum_{a=0}^{k_s-1} e^{-2\pi i \frac{v_s^{(\ell)}}{k_s} a} B_{j_s}\left(y_s + \frac{a}{k_s}\right)$$

$$= \sum_{\ell=1}^{\mathcal{L}} \sum_{a_1=0}^{k_1^{i_1}-1} \frac{1}{k_1^{i_1}} e^{-2\pi i \frac{v_1^{(\ell)}}{k_1^{i_1}} a_1} B_{j_1}\left(y_1 + \frac{a_1}{k_1^{i_1}}\right) \cdots$$

$$\sum_{a_d=0}^{k_d^{i_d}-1} \frac{1}{k_d^{i_d}} e^{-2\pi i \frac{v_d^{(\ell)}}{k_d^{i_d}} a_d} B_{j_d}\left(y_d + \frac{a_d}{k_d^{i_d}}\right)$$

$$= \sum_{\ell=1}^{\mathcal{L}} \det\left(K^I\right)^{-1} \sum_{0 \leqslant A \leqslant K^I - 1} e^{-2\pi i ((K^I)^{-1}) A \cdot v^{(\ell)}}$$

$$\mathfrak{B}_{J,\mathrm{Id}}\left((L^{-1})^T x + (K^I)^{-1} A\right),$$

where $0 \leqslant A = (a_1, \ldots, a_d) \leqslant K^I - 1$ means $0 \leqslant a_s \leqslant k_s^{i_s} - 1$.

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