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ORIGINAL RESEARCH

On a general class of portfolio diversification measures induced by risk measures

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Abstract
This paper introduces a novel and effective methodology for constructing portfolio diversification measures derived from any reference risk measure. The central contribution lies in leveraging the extensive theoretical developments in risk measurement to systematically inform and enhance the design of diversification metrics. The link between risk and diversification is exploited through an optimization framework in which the objective function is defined as a weighted Euclidean distance dependent on risk. We prove that the resulting objective function satisfies key axiomatic properties typically required to diversification measures, and that the corresponding optimization problem admits a unique solution that is inherently related to the intuitive concept of geometric diversification, thereby providing theoretical support for it. The key economic interpretation relies on determining the point in the allocation space that is equally distant – under a risk-sensitive metric – from the vertices of the simplex, i.e. the fully concentrated portfolios. An important economic insight of our approach is its applicability within a general long-short investment framework–a significant advancement, given that most classical diversification measures are restricted to long-only portfolios. Finally, to support the robustness of our findings, we present a comprehensive empirical analysis across multiple real-world financial datasets, highlighting meaningful comparisons between our proposed measure and several widely used diversification metrics.

Keywords Portfolio Diversification Measures · Risk Measures · Coherent Risk Measures · Risk-Adjusted Geometric Diversified Portfolio

Mathematics Subject Classification G10 · G11 · D81

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1 Introduction

The concept of diversification was first introduced by Markowitz (1952), laying the foundation for what is now known as modern portfolio theory. The main idea is that the optimal investment should reduce portfolio risk by exploiting low or negative correlations among asset returns. However, it is well known that in long-only settings, the classical Markowitz model frequently yields highly concentrated portfolios, allocating the wealth to a small subset of the available assets with a few large positions. In response to this limitation, a significant body of literature over the past two decades has reformulated the portfolio allocation problem from the perspective of diversification. Among others, Choueifaty and Coignard (2008) and Choueifaty et al. (2013) propose an allocation strategy based on maximizing the so-called diversification ratio, which quantifies the degree of risk spreading. DeMiguel et al. (2009) examine the equally weighted portfolio as an example of naive diversification and assess its out-of-sample performance relative to more sophisticated approaches. Several studies–including Clarke et al. (2013), Maillard et al. (2010), Qian (2006), and Roncalli and Weisang (2016)–advance the equal risk contribution strategy, which aims to balance risk exposure equally across all assets in the portfolio. Moreover, Meucci (2009) proposes the use of principal component analysis (PCA) to extract uncorrelated risk factors and promote diversification at the factor level. The aforementioned studies approach diversification from a strictly quantitative perspective, focusing on factors such as the number of assets held in a portfolio, individual asset exposures, and the contribution of each asset to overall portfolio risk. However, diversification can also be interpreted more broadly, with attention to its economic and strategic implications. For instance, diversification based on environmental, social, and governance (ESG) criteria has been analyzed in relation to both short- and long-term performance outcomes (see Bertelli & Torricelli, 2024). Similarly, Hendriks et al. (2024) examine the role of country-level diversification from a behavioral finance standpoint. Additional studies explore other dimensions of diversification, including its macroeconomic effects and implications for investor decision-making (see Hunjra et al. (2021), Hunjra et al. (2020), Mehmood et al. (2019)).

Despite the simplicity of the concept of portfolio diversification, there is, to the best of our knowledge, no universally accepted definition currently established in the literature. Recent efforts to review and consolidate the existing body of work reflect a growing interest in systematizing and unifying the theoretical and practical approaches to diversification. Over the past decade, three notable academic contributions–an unpublished working paper Fragiskos (2013), two peer-reviewed articles Flint et al. (2020); Koumou and Dionne (2022), and a comprehensive volume by Lhabitant (2017)–have attempted to put order into the existing measures. Fragiskos (2013) focus on asset diversification, providing a non-technical overview of the most commonly employed diversification strategies. Flint et al. (2020) propose a classification of diversification measures into five distinct categories: cardinality-based, weight-based, return-based, risk-based, and higher-moment-based measures. The monograph by Lhabitant (2017) offers an in-depth and technical exposition of existing diversification measures. Finally, Koumou and Dionne (2022) present a structured review of the foundational concepts of portfolio diversification.

In contrast, the axiomatic theory of risk measures is a well-established area of the literature. Starting from the seminal contribution by Artzner et al. (1999), which introduced the notion of coherent risk measures, numerous axiomatic frameworks have been developed to formalize the desired properties of risk measures. Among others, we recall the deviation risk measures proposed by Rockafellar et al. (2006), the convex risk measures introduced in Föllmer and

Schied (2002), the spectral risk measures presented in Acerbi (2002), the downside risk measures examined in Sortino and Van der Meer (1991), and the dynamic risk measures discussed in Acciaio and Penner (2011). Each of these axiomatic frameworks encompasses an infinite variety of risk measures, resulting in a highly complex and interdependent landscape of theoretical models, as documented in Frittelli and Rosazza Gianin (2002). Given that many risk measures were employed in practice prior to the development of formal axiomatic foundations, a substantial branch of the literature has been dedicated to classifying widely used measures within the corresponding theoretical frameworks. For example, Acerbi and Tasche (2002) demonstrate that Conditional Value-at-Risk (CVaR) satisfies the properties of a coherent risk measure, while Inui and Kijima (2005) examine the conditions under which Expected Shortfall adheres to coherency axioms.

There exists a fundamental connection between portfolio diversification and risk measurement. In particular, the sub-additivity property–commonly required in the axiomatic definitions of risk measures–captures the economic principle of diversification by ensuring that the risk of a portfolio does not exceed the aggregate risk of its individual components. This property reflects the intuitive notion that combining assets should not increase overall risk and may, in fact, reduce it. For comprehensive reviews of diversification measures within portfolio theory, we refer the reader to Flint et al. (2020) and Lhabitant (2017).

In this paper, we contribute to the literature by proposing a general theoretical framework that defines a diversification measure induced by a given risk measure. This approach leverages the extensive variety of risk measures available in the literature to construct a broad class of portfolio diversification measures. A key strength of the framework lies in its generality: it operates directly from the axiomatic properties of risk measures, without requiring explicit functional forms. In particular, deviation-based and coherent risk measures–such as those introduced by Rockafellar et al. (2006) and Artzner et al. (1999), respectively–give rise to induced diversification measures that differ primarily in their treatment of deterministic translations, thus preserving the intrinsic features of the original risk measure. By incorporating risk explicitly, the proposed diversification measure goes beyond simple functions of portfolio weights; it reflects how asset allocation interacts with the underlying risk structure. Within the taxonomy proposed by Lhabitant (2017), the measure we develop can be classified as one that depends jointly on both asset weights and the portfolio's risk characteristics. From a technical standpoint, we reformulate the conditions defining the Risk-Adjusted Geometric Diversified Portfolio (defined in Torrente & Uberti, 2023) as an equivalent optimization problem. We show that the associated objective function–called Geometric Portfolio Diversification Measure (GPDM)–satisfies several desirable properties for a portfolio diversification measure. These properties are discussed with attention to their economic relevance. The intuition behind geometric diversification is to identify a portfolio allocation that is equidistant, in a risk-adjusted sense, from the vertices of the unit simplex–each representing a portfolio fully concentrated in a single asset and thus maximally concentrated. When conventional distances are used, the solution to this problem is trivially the equally weighted portfolio. However, when the distance is adjusted to account for risk, the optimization yields nontrivial and economically meaningful allocations. The interpretation of the allocation is straightforward when the dependence structure among the assets is not considered. Yet an important strength of our approach is that it also allows for closed-form solutions even when the dependence structure within the risks of the asset classes is taken into account. Technically, the optimization problem can be solved in a closed form whether the matrix defining the risk-adjusted distance is diagonal or not. Taken together, these results provide a novel contribution to the growing literature on portfolio diversification assessed through risk-based metrics. This is a rapidly developing area of research, with notable contributions including Embrechts et al. (2009) and

Tasche (2006). More recently, the diversification quotient, introduced by Han et al. (2023a, b) within an axiomatic framework, represents a significant advancement in this domain. Finally, unlike most classical diversification measures, which are restricted to the long-only setting, our proposed measures are defined in a general long-short framework, enhancing their applicability in more flexible and realistic investment environments. Considering that the main contribution of this paper is methodological, the empirical application–although based on real financial data–is designed to compare our proposed diversification measures with those already present in the literature and to support the robustness of our approach. A separate study focusing on the economic implications and potential applications of the proposal is presented in Torrente and Uberti (2025), where the maximum diversification paradigm is systematically compared with the minimum risk approach for a given risk measure. That study reports particularly interesting results in terms of effective risk reduction in out-of-sample experiments.

The paper is organized as follows. Section 2 contains the notation together with preliminary and background results useful for the comprehension of the paper. In Section 3 the GPDM is introduced, its main theoretical properties are proved and analyzed by providing their useful economic interpretation; further, as a corollary result, it is shown that the GPDMs induced by the deviation risk measures fulfill all the stated properties, and the ones induced by the coherent and the convex risk measures do satisfy all but one of them. Section 4 proposes some theoretical directions to provide a generalization of the notions of RAD, RAGDP and GPDM aimed at incorporating dependence structure between assets. Section 5 provides a comprehensive empirical study to evaluate the performance of some GPDMs, defined using different well-known risk measures (standard deviation, Value-At-Risk computed at a significance level of 5%, mean absolute deviation, maximum drawdown), in comparison with various classical diversification measures and concentration indexes, listed in Section 5.1. The numerical experiments are performed on the datasets S&P500, DAX, ESX, FTSE, Nikkei, S&P500 constituents (see also Appendix 8). Finally, Section 6 contains some final remarks and concludes the paper. All proofs can be found in the Appendix 7.

2 Background material

In this section, we introduce the notation used throughout the paper (see Section 2.1), enumerate some general properties required for the definition of axiomatic risk measures (see Section 2.2), and formally recall the concept of risk-adjusted geometric diversified portfolios (see Section 2.3).

2.1 Preliminaries and notation

Let $m \geq n \geq 2$, let $\mathrm{Mat}_{m\times n}(\mathbb{R})$ denote the set of $m \times n$ real matrices and $\mathcal{M}_{m\times n}(\mathbb{R})$ be the subset of $\mathrm{Mat}_{m\times n}(\mathbb{R})$ consisting of all full-rank matrices, i.e., those matrices $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ such that $\mathrm{rank}(A) = n$. We denote by $\mathbf{0}_{m\times n}$ and $\mathbf{1}_{m\times n}$ the $m \times n$ matrices whose elements are all equal to zero and one, respectively.

Throughout the paper, we interpret the columns A^j, with $j = 1, \ldots, n$, of any matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ as the historical returns of the j-th asset of the portfolio. Thus, A represents the standard matrix of portfolio returns, with m observations across n risky assets. We define $\Gamma_n = \{w = (w_1, \ldots, w_n) \in \mathbb{R}^n \,:\, \sum_{j=1}^n w_j = 1\}$ and $\mathbb{W}_n = \{w = (w_1, \ldots, w_n) \in \mathbb{R}^n_{\geq 0} \,:\, \sum_{j=1}^n w_j = 1\}$ as the sets of long-short and long-only portfolios, respectively. A portfolio is uniquely identified by the vector w, where the j-th component w_j is the weight of the j-th asset. We denote by $\partial\mathbb{W}_n$ the subset of $\mathbb{W}_n$ (and of Γ_n) containing portfolios with at least one null component. The single asset portfolio fully allocated to the j-th asset is denoted by $e^j \in \partial\mathbb{W}_n$, where $e^1, \ldots, e^n$ are the standard basis vectors of $\mathbb{R}^n$.

2.2 Axiomatic risk measures

We recall the setting used to provide a general axiomatic characterization of *risk measures*, see among others Acerbi and Tasche (2002), Artzner et al. (1999), Rachev et al. (2008), Rockafellar et al. (2006). We consider a standard probability space $(\Omega, \mathcal{A}, P)$, and the set of functions $X : \Omega \rightarrow \mathbb{R}$, treated as random variables, belonging to the linear space $\mathcal{L}^2(\Omega, \mathcal{A}, P)$, where $\mathcal{L}^2(\Omega, \mathcal{A}, P)$ is the space of all real-valued random variables $X : \Omega \rightarrow \mathbb{R}$ that are square-integrable with respect to P. Note that this space contains all constant random variables $X \equiv \alpha$, with $\alpha \in \mathbb{R}$. In what follows, with a slight abuse of notation, we will use the real number α to denote the constant random variable taking value α almost surely. Similarly, inequalities of the form $X \leq Y$ or $X \geq Y$, where $X, Y \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$, are to be understood as holding almost surely.

Definition 1 A function $\rho : \mathcal{L}^2(\Omega, \mathcal{A}, P) \rightarrow \mathbb{R} \cup \{+\infty\}$ is:

- *normalized* if $\rho(0) = 0$;
- *strictly positive* if $\rho(X) > 0$ for all non-constant X and $\rho(X) = 0$ for constant X;
- *monotone* if for each X_1, X_2 with $X_1 \leq X_2$ it follows that $\rho(X_1) \geq \rho(X_2)$;
- *sub-additive* if $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$ for each X_1, X_2;
- *shift-invariant* if $\rho(X + \alpha) = \rho(X)$ for each X and $\alpha \in \mathbb{R}$;
- *translation-invariant* if $\rho(X + \alpha) = \rho(X) - \alpha$ for each X and $\alpha \in \mathbb{R}$;
- *positively homogeneous* if $\rho(\alpha X) = \alpha\rho(X)$ for each X and $\alpha \geq 0$;
- *convex* if $\rho(\alpha X_1 + (1-\alpha)X_2) \leq \alpha\rho(X_1) + (1-\alpha)\rho(X_2)$ for each X_1, X_2 and $\alpha \in [0, 1]$.

Definition 2 A *coherent risk measure* is a mapping $\rho : \mathcal{L}^2(\Omega, \mathcal{A}, P) \rightarrow \mathbb{R} \cup \{+\infty\}$ that satisfies the properties of monotonicity, sub-additivity, translation invariance, and positive homogeneity (see Artzner et al., 1999). If ρ is normalized, strictly positive, sub-additive, shift-invariant and positively homogeneous then it is called a *deviation risk measure* (see Rockafellar et al., 2006). Alternatively, if the conditions of subadditivity and positive homogeneity are replaced by the weaker property of convexity, the risk measure is said *convex* (see Föllmer & Schied, 2002).

A more detailed account of the various axiomatic definitions of risk measures proposed in the literature is beyond the scope of the present paper.

Remark 1 Note that the properties of translation invariance and shift invariance, as stated in Definition 1, are mutually incompatible. Consequently, an axiomatic risk measure can satisfy at most one of these two alternative properties. Moreover, if the risk measure is absolute–in the sense that its value carries intrinsic economic meaning (as in the case of CVaR, where the value represents the expected average loss at a given confidence level)–then translation invariance plays a significant role. Conversely, in a relative context, where the risk measure serves only to induce a ranking among alternatives, translation invariance and shift invariance are effectively equivalent, as neither property affects the ordering of risky alternatives determined by the risk measure.

The previous remark is useful to anticipate the specific behavior of the portfolio diversification measures proposed in this paper with respect to deterministic translations (see Section 3), which directly depends on the properties of the underlying risk measure.

2.3 Risk-adjusted geometric diversified portfolio

In this section, we recall the notions of *Risk-Adjusted Distance* and *Risk-Adjusted Geometric Diversified Portfolio*, as introduced in Torrente and Uberti (2023).

Let $n \geq 2$ be the number of assets, and let X_j, for each $j = 1, \dots, n$, denote the return of the the j-th asset over a given period. Consider the tuple of n possibly correlated random variables $X = (X_1, \dots, X_n)^t$. Using the notations defined above, we assume that each $X_j \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$, and let $\rho : \mathcal{L}^2(\Omega, \mathcal{A}, P) \to \mathbb{R} \cup \{+\infty\}$ be a chosen risk measure. Throughout the paper, we make the standing assumption that $\rho(X_j) > 0$ for each $j = 1, \dots, n$. To simplify notation, we denote by $\rho(X) = (\rho(X_1), \dots, \rho(X_n))^t$.

Definition 3 *(Risk-Adjusted Distance)* The *Risk-Adjusted Distance (RAD)* $d_{\rho(X)} : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ is defined by

$$d_{\rho(X)}(x, y) = \left(\sum_{j=1}^{n} \frac{1}{\rho(X_j)} (x_j - y_j)^2 \right)^{\frac{1}{2}}, \quad \forall x, y \in \mathbb{R}^n.$$

Definition 4 *(Risk-Adjusted Geometric Diversified Portfolio)* Let $d_{\rho(X)}$ be a RAD; the *Risk-Adjusted Geometric Diversified Portfolio (RAGDP)* with respect to $d_{\rho(X)}$ is represented by the portfolio $w^* = (w_1^*, \dots, w_n^*) \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n} w_j^* = 1 \quad \text{and} \quad d_{\rho(X)}(w^*, e^i) = d_{\rho(X)}(w^*, e^k), \quad \forall i, k \in \{1, \dots, n\}.$$

Equivalently the coordinates of the RAGDP satisfy

$$w_j^* = \frac{1}{2} \left(1 - \frac{n-2}{n M_{\rho(X)}} \rho(X_j) \right), \quad j = 1, \dots, n, \tag{1}$$

where $M_{\rho(X)}$ is the arithmetic mean of $\rho(X_1), \dots, \rho(X_n)$.

Remark 2 The RAGDP w^* is a long-only portfolio, that is $w^* \in \mathbb{W}_n$ if and only if $n \leq 3$ or $n > 3$ and $M_{\rho(X)} \geq \frac{n-2}{n} \rho_{\max}$, where $\rho_{\max}$ is the maximum of $\rho(X_1), \dots, \rho(X_n)$ (see Torrente & Uberti, 2023, Proposition 3.3). Further, notice that the more restrictive condition $w_j^* > 0$ for each $j = 1, \dots, n$ is equivalent to the cases $n \leq 3$ or $n > 3$ and $M_{\rho(X)} > \frac{n-2}{n} \rho_{\max}$.

3 Geometric portfolio diversification measures

In this section, we introduce the Geometric Portfolio Diversification Measures (GPDMs), a class of functions that, starting from suitable risk measures, aim to quantify the portfolio diversification. By construction, GPDMs are induced by risk measures and rely on the geometric diversification strategy, recalled in Section 2.3, which serves as a pure technical tool for their definition. We adopt the notation and assumptions introduced in Section 2.

Definition 5 *(Geometric Portfolio Diversification Measure)* The *Geometric Portfolio Diversification Measure (GPDM)* with respect to the risk measure ρ is the map $\Phi_{\rho(X)} : \Gamma_n \times \mathrm{Mat}_{m\times n}(\mathbb{R}) \setminus \{\mathbf{0}_{m\times n}\} \to \mathbb{R}_{\geq 0}$ defined by

$$\Phi_{\rho(X)}(w, A) := (\mathrm{rank}(A) - 1)\left(1 - \frac{f_{\rho(X)}(w)}{\max_{j=1,\ldots,n} f_{\rho(X)}(e^j)}\right) \tag{2}$$

for each $w \in \Gamma_n$ and $A \in \mathrm{Mat}_{m\times n} \setminus \{\mathbf{0}_{m\times n}\}$, where $f_{\rho(X)} : \Gamma_n \to \mathbb{R}_{\geq 0}$ is:

$$f_{\rho(X)}(w) := \sum_{j=1}^{n} \sum_{k=j+1}^{n} \left(d^2_{\rho(X)}(w, e^j) - d^2_{\rho(X)}(w, e^k)\right)^2 \tag{3}$$

for each $w \in \Gamma_n$.

The GPDM depends on both the portfolio return matrix A and the individual risk measure values involved in the function $f_{\rho(X)}$. The multiplicative factor $(\mathrm{rank}(A)-1)$ accounts for the dependence structure among the stochastic returns and thus reflects the number of effectively different assets. The second multiplicative factor, involving the function $f_{\rho(X)}$, quantifies the diversification of the portfolio w based on the specified risk measure. In the following, we enumerate and prove several important theoretical properties that characterize $\Phi_{\rho(X)}$ as a valid diversification measure. We then discuss their economic interpretation. It is useful to precede these results with Lemma 1, which addresses some technical aspects of the function $f_{\rho(X)}$. For the reader's convenience, we briefly recall that a permutation matrix is a square matrix containing exactly one entry equal to 1 in each row and each column, with all other entries equal to zero; such a matrix represents a reordering of the components of a vector or the rows/columns of a matrix.

Lemma 1 *Let $f_{\rho(X)}$ be the map introduced in Definition 5 and w^* be the RAGDP with respect to $d_{\rho(X)}$. Then the following properties hold true:*

(i) $f_{\rho(X)}(e^i) > 0$, *for each* $i = 1, \ldots, n$;
(ii) $f_{\rho(X)}(e^i) \leq f_{\rho(X)}(e^j)$ *if and only if* $\rho(X_i) \geq \rho(X_j)$ *for each* $i, j = 1, \ldots, n$;
(iii) $f_{\rho(X)}$ *is strictly convex;*
(iv) *for each permutation matrix* $\Pi \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ *we have* $f_{\rho(X\Pi)}(\Pi^t w) = f_{\rho(X)}(w)$ *for each* $w \in \Gamma_n$;
(v) w^* *is the unique global minimum of* $f_{\rho(X)}$ *over* Γ_n *and* $f_{\rho(X)}(w^*) = 0$.

If ρ is shift-invariant then

(vi) $f_{\rho(X+\alpha)} = f_{\rho(X)}$ *for each* $\alpha \in \mathbb{R}$.

If ρ is positively homogeneous then

(vii) $f_{\rho(\alpha X)} = \frac{1}{\alpha^2} f_{\rho(X)}$ *for each* $\alpha > 0$.

Remark 3 Note that the function $\Phi_{\rho(X)}$ is well-defined for any choice of the risk measure ρ: this is an immediate consequence of Lemma 1, item (i), which implies that $\max_{j=1,\ldots,n} f_{\rho(X)}(e^j) > 0$. Furthermore, from the convexity of $f_{\rho(X)}$ (see Lemma 1, item (iii)) it follows that $f_{\rho(X)}(w) \leq \max_{j=1,\ldots,n} f_{\rho(X)}(e^j)$, hence $\Phi_{\rho(X)}(w, A) \geq 0$ for any $(w, A) \in \Gamma_n \times \mathrm{Mat}_{m\times n}(\mathbb{R}) \setminus \{\mathbf{0}_{m\times n}\}$.

In Proposition 2, we compute the maximum of the GPDM and prove that the corresponding optimization problem admits a unique solution, which is attained at the RAGDP, as defined in Definition 4.

Proposition 2 *Let $\Phi_{\rho(X)}$ be the GPDM with respect to $d_{\rho(X)}$ and $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ be a nonzero matrix. The optimization problem $\max_{\omega\in\Gamma_n} \Phi_{\rho(X)}(w, A)$ admits a unique solution that coincides with w^*, the RAGDP with respect to $d_{\rho(X)}$, and satisfies $\Phi_{\rho(X)}(w^*, A) = \mathrm{rank}(A) - 1$.*

As a consequence of Proposition 2 and formula (1), it is worth noting that the point w^*, which attains the maximum of the GPDM, never coincides with any vector of the standard basis e^j, $j = 1, \dots, n$. This result underscores the model's inherent preference for diversification over concentration. Furthermore, item (ii) of Lemma 1 implies the existence of an ordering among the evaluations $f_{\rho(X)}(e^j)$, which depends on the values $\rho(X_j)$, $j = 1, \dots, n$. This induces an ordering of the diversification measures associated with single-asset portfolios, which, in general, assume different values. This outcome reflects the fact that the GPDM is designed to address diversification in a long-short setting, where fully concentrated portfolios do not necessarily represent the worst-case scenario, as is typically the case in the long-only framework.

The main properties of $\Phi_{\rho(X)}$ are collected in Propositions 3, 4 and 5.

Proposition 3 *The GPDM $\Phi_{\rho(X)}$ satisfies the following properties.*

P1. (Quasi-concavity) *Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ be a non-zero matrix; for any $w, z \in \Gamma_n$ and $\alpha \in [0, 1]$ it holds that*

$$\Phi_{\rho(X)}(\alpha w + (1-\alpha)z, A) \geq \min\{\Phi_{\rho(X)}(w, A), \Phi_{\rho(X)}(z, A)\},$$

with strict inequality for at least one value of α.

P2. (Generalized Risk Degeneracy) *Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ be a non-zero matrix such that A^i and A^j are linearly dependent for each $i, j = 1, \dots, n$; then $\Phi_{\rho(X)}(w, A) = 0$ for each $w \in \Gamma_n$.*

P3. (Reverse Risk Degeneracy) *For each $w \in \mathbb{W}_n \setminus \partial\mathbb{W}_n$ the equation $\Phi_{\rho(X)}(w, A) = 0$ in the variable $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ admits a solution A^* which is lower comonotonic.*

P4. (Symmetry) *Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ be a non-zero matrix and $\Pi \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a permutation matrix; then $\Phi_{\rho(X)}(\Pi^t w, A\Pi) = \Phi_{\rho(X)}(w, A)$ for each $w \in \Gamma_n$.*

We provide the economic interpretation of the properties stated in Proposition 3. *Quasi-concavity*, implied by concavity, although a less restrictive condition, captures the economic intuition that a diversified portfolio is preferable to holding its individual constituents in isolation. *Generalized Risk Degeneracy* ensures that the diversification measure does not depend solely on the number of assets in the portfolio, but rather on the number of effectively different assets. In particular, it prevents the possibility of improving diversification by adding redundant assets, such as those constructed through linear combinations of existing ones. *Reverse Risk Degeneracy* prevents an undesirable edge case in which a portfolio composed of linearly independent assets is assigned a diversification level equivalent to that of a completely undiversified portfolio. *Symmetry* is straightforward and ensures that the degree of diversification is invariant under any reordering of the assets in the portfolio.

Proposition 4 *Let $\Phi_{\rho(X)}$ be the GPDM with respect to the risk measure ρ.*
If ρ is shift-invariant then $\Phi_{\rho(X)}$ satisfies the following property.

P5. (Shift-invariance) *Let* $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ *be a non-zero matrix such that* $\mathbf{1}_{m\times 1}$ *and* $\mathbf{1}_{1\times n}$ *do not belong to the linear span of* A *and* A^t *respectively. Let* $\alpha \in \mathbb{R}$*; then* $\Phi_{\rho(X)}(w, A+\alpha) = \Phi_{\rho(X)}(w, A)$ *for each* $w \in \Gamma_n$.

If ρ *is positively homogeneous then* $\Phi_{\rho(X)}$ *satisfies the following property.*

P6. (Homogeneity) *Let* $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ *be a non-zero matrix and* $\alpha > 0$*; then* $\Phi_{\rho(X)}(w, \alpha A) = \Phi_{\rho(X)}(w, A)$ *for each* $w \in \Gamma_n$.

Shift-invariance expresses the idea that any deterministic translation of the returns in matrix A–including the addition of a risk-free asset–does not improve the degree of diversification. *Homogeneity* states that the diversification of a portfolio remains unchanged under any scale transformation of the input data.

Proposition 5 *Let* $\Phi_{\rho(X)}$ *be the GPDM with respect to* $d_{\rho(X)}$. *Let* $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ *be a nonzero matrix,* $A^{n+1} \in Mat_{m\times 1}(\mathbb{R})$ *be the return of a further asset and* $A^+ = (A : A^{n+1}) \in \mathrm{Mat}_{m\times(n+1)}(\mathbb{R})$. *Let* w^* *and* w_+^* *be the unique solutions of the optimization problems* $\max_{w\in\Gamma_n} \Phi_{\rho(X)}(w, A)$ *and* $\max_{w_+\in\Gamma_{n+1}} \Phi_{\rho(X)}(w_+, A^+)$ *respectively. Then the GPDM* $\Phi_{\rho(X)}$ *satisfies the following properties.*

P7. (Generalized Duplication Invariance) *If* A^{n+1} *is linearly dependent on the columns of* A *then* $\Phi_{\rho(X)}(w_+^*, A^+) = \Phi_{\rho(X)}(w^*, A)$.

P8. (Generalized Size Monotonicity) *If* A^{n+1} *is linearly independent of the columns of* A *then* $\Phi_{\rho(X)}(w_+^*, A^+) > \Phi_{\rho(X)}(w^*, A)$.

The two properties in Proposition 5 describe the behavior of the GPDM as the number n of portfolio assets increases. In particular, *Generalized Duplication Invariance* states that diversification remains unchanged when a linearly redundant asset is added to the portfolio. In contrast, *Generalized Size Monotonicity* asserts that the inclusion of an effectively different asset strictly improves the level of diversification.

There are analogies between the properties stated in Propositions 3, 4, and 5 and the axioms that characterize a coherent portfolio diversification measure, as introduced in Koumou and Dionne (2022). In particular, each of our properties can be seen as an analogue or a generalization of those axioms. Regarding *Generalized Risk Degeneracy*, if multiple copies of the same asset are included, the resulting diversification remains null. As for *Generalized Duplication Invariance* and *Generalized Size Monotonicity*, adding an asset that is either identical to one of the existing portfolio constituents or entirely distinct from all of them results in a diversification value that is either unchanged or strictly greater than that of the original portfolio.

Remark 4 We emphasize that the proposed approach is well-suited to transform a given risk measure into a GPDM. As highlighted in the introduction, this choice allows us to leverage the vast number of risk measures available in the literature to define induced diversification measures. Moreover, the approach is fully developed within a long-short framework, indicating that the resulting class of diversification measures remains well-defined and meaningful even when some portfolio weights are negative.

In the following corollary, we summarize the results of Propositions 3, 4, and 5, stating the properties satisfied by GPDMs induced by several well-known classes of risk measures, as introduced in Definition 2.

Corollary 6 *If ρ is a:*

- *deviation risk measure, then $\Phi_{\rho(X)}$ satisfies all the properties* **P1-P8***;*
- *coherent risk measure, then $\Phi_{\rho(X)}$ satisfies all the above properties except* **P5***;*
- *a convex risk measure, then $\Phi_{\rho(X)}$ satisfies all the above properties except* **P6***.*

4 Extending RAGDP to multivariate asset dependence

The current formulation of RAD, RAGDP and GPDM (see Definitions 3, 4 and 5) does not adequately capture the dependence structure among assets, except in the special case where one asset is a linear combination of others. However, in realistic financial contexts, assets often exhibit complex multivariate dependencies that extend beyond linear relationships. In this section, we propose a generalization of the notions of RAD, RAGDP and GPDM to incorporate such dependence structures. We outline several theoretical directions for this generalization, leaving their implementation and empirical validation as promising topics for future research.

To this end, we begin Section 4.1 by recalling the *Mahalanobis distance* which, by accounting for both the variances of individual assets and the covariances between them, provides a natural foundation for introducing the *Mahalanobis strategy* (see Definition 7), an asset allocation strategy that explicitly incorporates dependencies among assets. This construction extends the RAGDP to the case of correlated assets, with variance taken as the default risk measure. A more general framework capable of accommodating arbitrary risk measures is then presented in Section 4.2. There, we introduce a broader class of distances, proper generalizations of the Mahalanobis distance, defined through a symmetric positive definite *risk matrix* that encodes the information associated with any chosen risk measure. Such generalization allows us to define both the *general RAGDP strategy* and the *general GPDM* (see Definition 10 and 11).

We use the notation introduced in Sections 2 and 3, and introduce additional notation here. For real matrices, when $m = n$, we denote by $\mathrm{Mat}_n(\mathbb{R})$ the set of all real square matrices and, similarly, by $\mathcal{M}_{m\times n}(\mathbb{R}) \subset \mathrm{Mat}_{m\times n}(\mathbb{R})$ the subset of full-rank matrices. Let $\mathcal{S}_n^+(\mathbb{R})$ denote the set of all $n \times n$ real symmetric positive definite matrices, and let $\mathcal{D}_n^+(\mathbb{R}) \subset \mathcal{S}_n^+(\mathbb{R})$ be the subset consisting of diagonal matrices with strictly positive diagonal entries. We indicate with $\mathbf{1}_n$ the $n \times n$ matrix whose elements are all equal to one. For each $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ we denote by $A_i \in \mathrm{Mat}_{1\times n}(\mathbb{R})$ the ith row of A, and by $A^{(i,j)} \in \mathrm{Mat}_{(m-1)\times(n-1)}(\mathbb{R})$ the submatrix of A obtained by deleting its ith row and its jth column. For each $A \in \mathrm{Mat}_n(\mathbb{R})$ we denote by $\mathrm{diag}(A) \in \mathrm{Mat}_{n\times 1}(\mathbb{R})$ the column vector of the diagonal elements of A.

4.1 The Mahalanobis strategy

We begin by recalling the definition of the Mahalanobis distance. To this end, we assume from now on that $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, and let $\Sigma \in \mathcal{S}_n^+(\mathbb{R})$ denote the covariance matrix of A.

Definition 6 Let $\Sigma \in \mathrm{Mat}_n(\mathbb{R})$ be the covariance matrix of portfolio returns $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$. The *Mahalanobis distance* $d_{\Sigma^{-1}} : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ is

$$d_{\Sigma^{-1}}(x, y) := (x - y)^t \Sigma^{-1}(x - y), \quad \forall x, y \in \mathbb{R}^n.$$

Note that the Mahalanobis distance accounts for both the variances of individual assets and the covariances between them. It reduces to the classical Euclidean distance in the special case of uncorrelated assets with unit variance. From a geometric perspective, the Mahalanobis distance first transforms the data into standardized, uncorrelated returns, and then computes the standard Euclidean distance in the transformed space. Mathematically, it can be viewed as a special case of a broader class of distances defined through a symmetric positive definite matrix (see Definition 8).

In the following, we introduce an asset allocation strategy, called *Mahalanobis strategy*, based on the Mahalanobis distance.

Definition 7 The *Mahalanobis strategy* is represented by the portfolio $w^* = (w_1^*, \dots, w_n^*) \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n} w_j^* = 1 \quad \text{and} \quad d_{\Sigma^{-1}}(w^*, e^i) = d_{\Sigma^{-1}}(w^*, e^k), \quad \forall i, k \in \{1, \dots, n\}.$$

Remark 5 The Mahalanobis strategy can equivalently be defined as the unique solution to the following convex constrained minimization problem:

$$\min_{w \in \Gamma_n} f_{\Sigma^{-1}}(w), \tag{4}$$

where

$$f_{\Sigma^{-1}}(w) := \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(d^2_{\Sigma^{-1}}(w, e^i) - d^2_{\Sigma^{-1}}(w, e^j) \right)^2, \quad \text{for each } w \in \Gamma_n. \tag{5}$$

Further, by some simple computations an explicit closed-form representation of the Mahalanobis strategy can be derived, as established in the following proposition.

Proposition 7 *Let $S = (s_{ij}) \in \mathcal{S}_n^+(\mathbb{R})$ be the inverse of the covariance matrix Σ, that is $S = \Sigma^{-1}$. The coordinates of the Mahalanobis strategy $w^* = (w_1^*, \dots, w_n^*) \in \Gamma_n$ are given by*

$$\begin{array}{ll} (w_1^*, \dots, w_{n-1}^*)^t & = E_n^{-1} b_n \\ \quad w_n^* & = 1 - \sum_{j=1}^{n-1} w_j^*, \end{array} \tag{6}$$

where $E_n \in \mathcal{S}_{n-1}^+(\mathbb{R})$ and $b_n \in Mat_{(n-1)\times 1}(\mathbb{R})$ are defined by

$$E_n := S^{(n,n)} + s_{nn} \mathbf{1}_{n-1} - (D_n + D_n^t) \tag{7}$$

$$b_n := \frac{1}{2} \left(\operatorname{diag}(S)^{(n)} + s_{nn} \mathbf{1}_{(n-1)\times 1} \right) \tag{8}$$

with

$$D_n := (S_n^t)^{(n)} \mathbf{1}_{1\times(n-1)} \in Mat_{n-1}(\mathbb{R}). \tag{9}$$

In the case of uncorrelated assets, the Mahalanobis strategy simplifies to the RAGDP where the variance is chosen as given risk, as shown in the following proposition.

Proposition 8 *In the case of uncorrelated assets, the Mahalanobis strategy coincides with the RAGDP when the variance is used as the reference risk measure.*

In the following, we provide a toy example to describe, in a stylized framework, the impact on the allocation when considering the dependence structure among the assets.

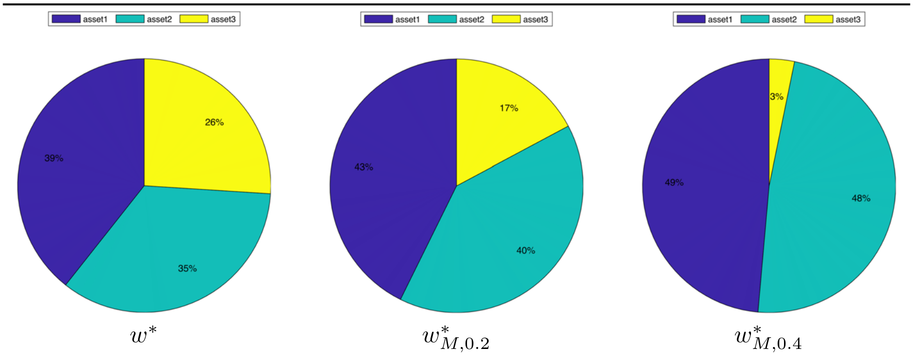

Fig. 1 Compositions of the optimal RAGDP portfolios when the dependence structure among assets is either ignored (w^*) or incorporated. In the latter case, $w^*_{M,0.2}$ and $w^*_{M,0.4}$ correspond to scenarios where the correlation between assets 1 and 2 is 0.2 and 0.4, respectively

Example 1 Let us consider $n = 3$ risky assets, and assume that their covariance matrix is given by

$$\Sigma = \begin{bmatrix} 1 & 0.24 & 0 \\ 0.24 & 1.44 & 0 \\ 0 & 0 & 2.25 \end{bmatrix}.$$

The correlation between the first two assets is 0.2. In this case, the RAGDP is $w^* = (0.3934, 0.3465, 0.2601)$. As expected, when the dependence structure is ignored, the allocation favors the assets with lower risk, while the riskiest asset receives the smallest weight. Once the dependence is taken into account, as encoded in Σ, the portfolio identified by what we called the Mahalanobis strategy is $w^*_{M,0.2} = (0.4269, 0.4010, 0.1721)$ (see Figure 1). In this case, the allocation to the third asset (the riskiest one) decreases, whereas the weights of the first two correlated assets increase (albeit at different rates).

If the correlation between the first two assets increases to 0.4, the optimal portfolio becomes $w^*_{M,0.4} = (0.4860, 0.4818, 0.0322)$, further highlighting the reduction in the allocation to the riskiest uncorrelated asset and the corresponding increase in the weights of the two positively correlated assets (see again Figure 1).

4.2 Generalization to different risk measures

The Mahalanobis strategy extends the RAGDP to the case of correlated assets by adopting variance as the default risk measure (see Proposition 8). To introduce a more general framework that accommodates arbitrary risk measures, we begin by recalling a broader class of distances – generalizations of the Mahalanobis distance defined through a symmetric positive definite matrix.

Definition 8 Let $W \in \mathcal{S}_n^+(\mathbb{R})$. The map $\langle \cdot, \cdot \rangle_W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $\langle x, y \rangle_W := x^t W y$, for each $x, y \in \mathbb{R}^n$, defines an inner product on $\mathbb{R}^n$ and induces the norm $\| \cdot \|_W : \mathbb{R}^n \to \mathbb{R}$, where $\|x\|_W := \sqrt{\langle x, x \rangle_W} = \sqrt{x^t W x}$, for each $x \in \mathbb{R}^n$, and the *distance function* (*metric*) $d_W : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ defined by

$$d_W(x, y) := \|x - y\|_W = \sqrt{(x - y)^t W (x - y)}, \qquad \forall x, y \in \mathbb{R}^n.$$

In the special case $W \in \mathcal{D}_n^+(\mathbb{R})$, the function d_W, called *W-weighted Euclidean distance function (metric)*, becomes

$$d_W(x, y) = \left(\sum_{i=1}^{n} w_i (x_i - y_i)^2 \right)^{\frac{1}{2}}, \quad \forall x, y \in \mathbb{R}^n,$$

where $w_i > 0$, $i = 1, \ldots, n$, are the diagonal entries of W.

Risk measures are incorporated into the framework through the definition of a *risk matrix* (see Definition 9), which, under the general assumptions stated in Section 2 (in particular, we recall the standing assumption that $\rho(X_j) > 0$ for each $j = 1, \ldots, n$, see Section 2.3), can be shown to be symmetric and positive definite (see Proposition 9).

Definition 9 The *risk matrix* $\mathcal{R}_{\rho,X}$ *of* X *associated to the risk* ρ is the $n \times n$ matrix whose elements r_{ij} are defined by:

$$r_{ij} = \operatorname{corr}(X_i, X_j)\sqrt{\rho(X_i)\rho(X_j)}, \qquad \forall i, j \in \{1, \ldots, n\},$$

where $\operatorname{corr}(X_i, X_j)$ denotes the Pearson correlation coefficient of the pair (X_i, X_j).

Proposition 9 *The risk matrix* $\mathcal{R}_{\rho,X}$ *is symmetric and positive definite.*

We consider the distance $d_{\mathcal{R}^{-1}_{\rho,X}}$ defined by the positive definite matrix $\mathcal{R}^{-1}_{\rho,X}$ (see Definition 8). Now, we can generalize the notion of Mahalanobis strategy to any risk measure ρ.

Definition 10 Let ρ be any risk measure and $\mathcal{R}_{\rho,X}$ be the risk matrix of X associated to the risk ρ. The *general Risk-Adjusted Geometric Diversified Portfolio (RAGDP) strategy* with respect to the distance $d_{\mathcal{R}^{-1}_{\rho,X}}$ is represented by the portfolio $w^* = (w_1^*, \ldots, w_n^*) \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n} w_j^* = 1 \quad \text{and} \quad d_{\mathcal{R}^{-1}_{\rho,X}}(w^*, e^i) = d_{\mathcal{R}^{-1}_{\rho,X}}(w^*, e^k), \quad \forall i, k \in \{1, \ldots, n\}.$$

In the special case in which the risk measure ρ is represented by the variance of X then $\mathcal{R}_{\mathrm{Var},X}$ is the covariance matrix of X, $d_{\mathcal{R}^{-1}_{\mathrm{Var},X}}$ is the Mahalanobis distance (see Definition 6) and the general RAGDP coincides with the Mahalanobis strategy (see Definition 7).

We now state a proposition that yields an explicit characterization of the general RAGDP strategy.

Proposition 10 *Assumptions as in Definition 10. Let* $S \in \mathcal{S}_n^+(\mathbb{R})$ *be the inverse of the risk matrix* $\mathcal{R}_{\rho,X}$*, that is* $S = \mathcal{R}^{-1}_{\rho,X}$*. The coordinates of the general RAGDP strategy* $w^* = (w_1^*, \ldots, w_n^*) \in \Gamma_n$ *are given by* (6) *where* $E_n \in \mathcal{S}^+_{n-1}(\mathbb{R})$ *and* $b_n \in Mat_{(n-1)\times 1}(\mathbb{R})$ *are defined by* (7) *and* (8).

In the following, we introduce the *general Geometric Portfolio Diversification Measures*, which quantify the level of portfolio diversification by incorporating information on the dependence structure among assets.

Definition 11 The *general Geometric Portfolio Diversification Measure (GPDM)* with respect to the risk measure ρ is the map $\Phi_{\mathcal{R}^{-1}_{\rho,X}} : \Gamma_n \times \mathrm{Mat}_{m\times n}(\mathbb{R}) \setminus \{\mathbf{0}_{m\times n}\} \to \mathbb{R}_{\geq 0}$ defined by

$$\Phi_{\mathcal{R}^{-1}_{\rho,X}}(w, A) := (\mathrm{rank}(A) - 1)\left(1 - \frac{f_{\mathcal{R}^{-1}_{\rho,X}}(w)}{\max_{j=1,\ldots,n} f_{\mathcal{R}^{-1}_{\rho,X}}(e^j)}\right)$$

for each $w \in \Gamma_n$ and $A \in \mathrm{Mat}_{m\times n} \setminus \{\mathbf{0}_{m\times n}\}$, where $f_{\mathcal{R}^{-1}_{\rho,X}} : \Gamma_n \to \mathbb{R}_{\geq 0}$ is:

$$f_{\mathcal{R}^{-1}_{\rho,X}}(w) := \sum_{j=1}^{n}\sum_{k=j+1}^{n}\left(d^2_{\mathcal{R}^{-1}_{\rho,X}}(w, e^j) - d^2_{\mathcal{R}^{-1}_{\rho,X}}(w, e^k)\right)^2$$

for each $w \in \Gamma_n$.

5 Empirical results

In this section, we present a comprehensive empirical analysis to evaluate the performance of the proposed GPDMs[1] (see Definition 5). In this section, we specifically focus on selected instances of the GPDMs and, due to space limitations, we do not conduct the empirical analysis for the general GPDMs introduced in Section 4; nevertheless, such an analysis can be carried out by exploiting the theoretical framework developed therein. We then conduct a direct comparison between several GPDMs–constructed using different well-established risk measures–and a variety of classical diversification measures and concentration indices, as listed in Section 5.1. The numerical experiments are conducted on multiple datasets (S&P500, DAX, ESX, FTSE, Nikkei, and S&P500 constituents), which differ in the number of assets; a detailed description is provided in Section 5.2. As most diversification measures proposed in the literature are defined for the long-only context, a comprehensive comparison with GPDMs is feasible only within this context. The results of this comparison are reported in Section 5.3.1. Conversely, the performance of the GPDMs in the long-short setting is analyzed in Section 5.3.2, where the study is exclusively focused on the GPDMs.

5.1 Diversification measures and concentration indexes

Following the classification proposed by Lhabitant (2017), we consider four distinct classes of diversification measures: weight-based measures, entropy-based measures, correlation-based measures, and risk-based portfolio measures. To introduce each class, we begin by briefly outlining the corresponding general framework.

Let us assume a single-period framework: the allocation is decided at time $t = 0$, and the return is realized at time $t = T$. The investment universe consists of $n \geq 2$ risky, non-redundant assets. The returns of these assets over the single period are collected in the $n \times 1$ column vector $A = (A_1, \ldots, A_n)^t$, where A_i denotes the (random) return of asset i over the interval $[0, T]$. Let μ be the vector of expected returns of A, i.e.,

$$\mu = \mathbb{E}(A) = (\mu_1, \ldots, \mu_n)^t,$$

[1] The algorithm used to compute the GPDMs has been implemented in MATLAB. The full source code is publicly available on GitHub at: https://github.com/mltorrente/GeometricPortfolioDiversificationMeasure.git

where $\mu_i := \mathbb{E}(A_i)$. Let Σ be the $n \times n$ covariance matrix of A, defined as

$$\Sigma = \mathrm{cov}(A) = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_n^2 \end{pmatrix},$$

where the diagonal elements $\sigma_i^2 := \mathbb{E}[(A_i - \mu_i)^2]$ are the variances of the asset returns, and the off-diagonal elements $\sigma_{ij} := \mathbb{E}[(A_i - \mu_i)(A_j - \mu_j)]$ represent the covariances between assets i and j. Since the n risky assets are non-redundant–that is, no asset return can be expressed as a linear combination of the others–Σ is non-singular.
We also introduce the notation for the $n \times 1$ vector $\sigma = (\sigma_1, \ldots, \sigma_n)$ which represents the volatilities of the assets and, for the $n \times n$ correlation matrix

$$C = \begin{pmatrix} 1 & \rho_{12} & \ldots & \rho_{1n} \\ \rho_{12} & 1 & \ldots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \ldots & 1 \end{pmatrix},$$

where ρ_{ij} is the correlation between assets i and j. Finally, for any long-only portfolio $w \in \mathbb{W}_n$, we denote by $\widetilde{w} \in \mathbb{W}_n$ the vector of weights sorted in ascending order, that is $\widetilde{w} = (\widetilde{w}_1, \ldots, \widetilde{w}_n)$, where $\widetilde{w}_i \leq \widetilde{w}_j$ for all $i < j$.

5.1.1 Weight-based diversification measures

Weight-based diversification measures encompass metrics that, by applying various functions to the portfolio weights, quantify the concentration of a given long-only portfolio. Within this framework, the most diversified portfolio is the one that minimizes the concentration index. For any long-only portfolio $w \in \mathbb{W}_n$, we consider the following measures.

1. The *concentration ratio of order* k, denoted by CR_k, measures the concentration of a portfolio by computing the cumulative weight of its k-largest positions, where k is an exogenous parameter; it is defined by:

$$\mathrm{CR}_k(w) := \sum_{i=1}^{k} \widetilde{w}_{n-i+1}.$$

   It is bounded between 0 and 1. The main issue of CR_k is related to the arbitrariness of the parameter k: for small values of k only a negligible part of the exposures is considered, while as k gets larger CR_k gathers less and less information, till the case $k = n$ for which $\mathrm{CR}_n \equiv 1$, bringing no information at all.
2. The *Herfindal-Hirschman index*, denoted by HHI and introduced in Herfindal (1950) and in Hirschman (1964), is a classical indicator of the amount of competition in a market; it is defined by:

$$\mathrm{HHI}(w) := \sum_{i=1}^{n} w_i^2.$$

   It is bounded between $1/n$ (in the case of the equally weighted portfolio) and 1 (when the portfolio contains one single asset).

3. The *Gini index of concentration*, denoted by GIC and introduced in Gini (1921), measures the degree of inequality in a distribution and is computed as the mean expected absolute deviation between all pairs of components (weights) scaled by their mean; it is defined by:
$$\mathrm{GIC}(w) := \frac{n+1-2\sum_{i=1}^{n}(n+1-i)\widetilde{w}_i}{n}.$$
It is bounded between 0 (in the case of the equally weighted portfolio) and 1 (when the portfolio contains one single asset).
4. The *Hall-Tideman index*, denoted by HT and introduced in Hall and Tideman (1967), aims at evaluating and quantifying the concentration risk; it is defined by
$$\mathrm{HT}(w) := \frac{1}{\sum_{i=1}^{n} i\widetilde{w}_i - 1}.$$
It is bounded between $1/n$ (in the case of the equally weighted portfolio) and 1 (when the portfolio contains one single asset) and it is related to the Gini index of concentration through the formula
$$HT(w) = \frac{1}{(1-\mathrm{GIC}(w))n}.$$
5. The *reciprocal of Hannah-Kay index*, denoted by RHK and introduced in Hannah and Kay (1977), is a generalization of the HHI; it is defined by:
$$\mathrm{RHK}(w) := \left(\sum_{i=1}^{n} w_i^{\alpha}\right)^{\frac{1}{\alpha-1}}, \text{ with } \alpha \geq 0, \alpha \neq 1.$$
It is bounded between $1/n$ (in the case of the equally weighted portfolio) and 1 (when the portfolio is composed by one single asset).
6. The *comprehensive concentration index*, denoted by CCI and introduced in Horvath (1970), takes into consideration the relative dispersion and the absolute magnitude of the weights. It is defined as the sum of the largest weight and the square of the other assets scaled by a coefficient which is proportional to the size of the rest of the position:
$$\mathrm{CCI}(w) := \widetilde{w}_n + \sum_{i=1}^{n-1} \widetilde{w}_i^2(1+(1-\widetilde{w}_i)).$$
It is bounded between the positive value $(3n^2-3n+1)/n^3$ (in the case of the equally weighted portfolio) and 1 (when the portfolio contains one single asset).
7. The L_p *norm*, denoted by Y_α and introduced in Bouchaud et al. (1997), proposes to use the p-norm as an indicator of portfolio concentration; it is defined by
$$Y_\alpha(w) := \sum_{i=1}^{n} w_i^{\alpha}, \text{ with } \alpha \geq 0.$$
It is a generalization of Shannon's entropy; special cases are $\alpha = 1$, which leads to the uninteresting case $Y_1 \equiv 1$, and $\alpha = 2$, which returns HHI, that is $Y_2 \equiv \mathrm{HHI}$.

5.1.2 Correlation-based diversification measures

Correlation-based diversification measures rely on the idea that portfolio diversification can be evaluated using the average correlation between all the pairs of assets. Obviously, in this approach weights need to be considered: indeed, if an asset has zero weight, then its correlation should not play a role in the portfolio diversification; on the contrary, if it has a large weight, its correlation should be highly considered. Correlation-based diversification measures are weighted-average correlation functions constructed using different approaches. In this context, the most diversified strategies are realized by portfolios with minimum measure. For any long-only portfolio $w \in \mathbb{W}_n$ we list the following measures (see Lhabitant, 2017).

1. The *weighted average of all pairwise correlations*, denoted by ρ_{AVG1}, is defined by:

$$\rho_{AVG1}(w) := \frac{w^t C w - w^t w}{1 - w^t w}.$$

   It is bounded between -1 and 1.

2. The *implied average correlation*, denoted by ρ_{AVG2}, is obtained from ρ_{AVG1} by replacing the $n(n-1)$ correlations ρ_{ij} with a constant average cross-sectional coefficient; it is defined by:

$$\rho_{AVG2}(w) := \frac{w^t \Sigma w - w^t \mathrm{diag}(\Sigma) w}{(w^t \sigma)^2 - w^t \mathrm{diag}(\Sigma) w}.$$

3. The *volatility-based correlation proxy*, denoted by ρ_{AVG3}, is obtained from ρ_{AVG2} by deleting the term $w^t \mathrm{diag}(\Sigma) w$ which can be neglected when the number of assets gets large; it is defined by:

$$\rho_{AVG3}(w) := \frac{w^t \Sigma w}{(w^t \sigma)^2}.$$

4. The *variance-based correlation proxy*, denoted by ρ_{AVG4} and introduced in Bossu and Gu (2004), is defined by:

$$\rho_{AVG4}(w) := \frac{w^t \Sigma w}{w^t \mathrm{diag}(\Sigma)}.$$

5.1.3 Entropy-based diversification measures

The most popular function used to measure information is represented by the Shannon entropy, which has been easily applied in the context of long-only portfolios as an inverse measure of concentration. Entropy-based diversification measures include Shannon entropy and other entropy functions. In this context, the most diversified strategies are realized by portfolios with maximum diversification measure. For any long-only portfolio $w \in \mathbb{W}_n$ we list the following measures.

1. The *Shannon entropy*, denoted by SE and introduced in Shannon (1948), is defined by:

$$\mathrm{SE}(w) := -\sum_{i=1}^{n} w_i \ln w_i.$$

   It is related to RHK concentration index since

$$\lim_{\alpha\to\infty} \mathrm{RHK}(w) = e^{-\mathrm{SE}(w)} \qquad \text{for each } w \in \mathbb{W}_n.$$

It is bounded between 0 (for a single-asset portfolio) and $\ln n$ (in the case of the equally weighted portfolio).

2. The *Renyi entropy*, denoted by H_α and introduced in Renyi (1961), is a generalization of Shannon entropy; it is defined by:

$$H_\alpha(w) := \frac{1}{1-\alpha} \ln\left(\sum_{i=1}^{n} w_i^\alpha\right), \text{ with } \alpha \geq 0$$

When $\alpha = 0$ and $\alpha = 2$ the Renyi entropy leads to the *Hartley entropy* and the *collision entropy* respectively; when $\alpha \to 1$ then H_α converges to the Shannon entropy.

3. The *Tsallis entropy*, denoted by HC_q and introduced in Tsallis (1988), is a new form of non-additive entropy defined by:

$$\mathrm{HC}_q(w) := \begin{cases} \dfrac{1}{q-1}\left(1 - \displaystyle\sum_{i=1}^{n} w_i^q\right) & \text{if } q \neq 1 \\ -\sum_{i=1}^{n} w_i \ln w_i & \text{if } q = 1. \end{cases}$$

4. The *Hill's effective numbers*, denoted by D_q and introduced in Hill (1973), are defined by:

$$D_q(w) := \left(\sum_{i=1}^{n} w_i^q\right)^{\frac{1}{1-q}}, \text{ with } q \in \mathbb{R}.$$

The parameter q, which in the biological context is called the "order of the diversity" of the species, can assume any real value, but positive values are usually considered the most interesting ones. When $q = 0$, the diversity counts the total number of nonzero weights, regardless their magnitude; for $q = 1$, only the relevant weights are considered; for $q = 2$, only the very relevant weights are taken into consideration, till the case $q \to \infty$ where $D_q(w)$ returns the reciprocal of the maximum component of w. It corresponds to the exponential of the Renyi entropy; it is a bounded quantity that reaches its maximum for the equally weighted portfolio.

5. The *Rao quadratic entropy*, denoted by RQE and introduced in Rao (1982), computes the average difference between two individuals that would be randomly selected (with replacements) in a population; it is defined by:

$$\mathrm{RQE}(w) := \sum_{i=1}^{n-1} \sum_{j=i+1}^{n-1} d_{ij} w_i w_j,$$

where d_{ij} is the pairwise distance between the ith and jth individuals, with the requirements that $d_{ij} = d_{ji}$ and $d_{ii} = 0$, for each i, j. In the case $d_{ij} = 1$, for each $i \neq j$, RQE is the *Simpson diversity index*, which is equal to $1 - \mathrm{HHI}$.

5.1.4 Risk-based diversification measures

Risk-based diversification measures gather metrics that, based exclusively on risk, compute the level of concentration or diversification of a given long-only portfolio. For any $w \in \mathbb{W}_n$ we list the following measures.

1. The *Equal Risk Contribution*, denoted by ERC, has been firstly defined using volatility (see Maillard et al., 2010), but other measures have been considered in the recent literature: Conditional Value-At-Risk (see Boudt et al., 2013, Cesarone & Colucci, 2018, Mausser & Romanko, 2018), expectiles (see Bellini et al., 2021) and mean absolute deviation (see Ararat et al., 2024). When volatility is used as a risk measure ERC computes the sum of the difference of each assets effective risk contribution as follows:

$$\mathrm{ERC}(w) := \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\mathrm{TRC}_i(w) - \mathrm{TRC}_j(w)\right)^2$$

where

$$\mathrm{TRC}_i(w) = w_i \frac{(\Sigma w)_i}{\sqrt{w^t \Sigma w}}$$

denotes the ith Total Contribution to portfolio Risk. The most diversified strategy is realized by the portfolio with minimum ERC, that is, by the portfolio where each asset equally contributes to the total risk.

2. The *diversification ratio*, denoted by DR and introduced in Choueifaty and Coignard (2008), computes the portfolio diversification as the ratio between the hypothetical volatility of the portfolio if all assets are perfectly correlated and the effective volatility of the portfolio:

$$\mathrm{DR}(w) := \frac{w^t \sigma}{\sqrt{w^t \Sigma w}}.$$

It is bounded from below by 1, when considering the single asset portfolio or the portfolio of perfectly correlated assets, and it is unbounded from above, with higher values corresponding to portfolios with higher diversification.

3. The *extension of diversification ratio*, denoted by EDR and introduced in Carmichael et al. (2015) and Carmichael et al. (2023), is a generalization of DR based on Rao's quadratic entropy:

$$\mathrm{EDR}(w) := \frac{w^t D w}{w^t \Sigma w},$$

where $D = (d_{ij})_{i,j=1,\ldots,N}$ is a dissimilarity matrix whose generic element d_{ij} measures the degree of diversity between each couple of assets, with the requirements that $d_{ij} \geq 0$, $d_{ij} = d_{ji}$ and $d_{ii} = 0$ for each i, j. The most diversified strategy is realized by the portfolio with maximum EDR.

5.2 Datasets

The empirical analysis is conducted on six different financial indices. All datasets consist of daily asset returns spanning the period from 01/03/2000 to 09/17/2020. The logarithmic returns used in the analysis are computed from daily closing prices obtained from Bloomberg. For consistency, assets lacking a complete time series over the selected time window have been excluded. As a result, the number n of effective constituents in each index is smaller than its nominal size. The number of assets ranges from 10, in the case of the S&P500, to 391 for the S&P500 constituents dataset, thereby enabling an assessment of the diversification measures across a wide spectrum of portfolio sizes. The dataset used in the empirical analysis is available upon request from the authors.

The datasets are:

1. S&P500 ($n = 10$) is composed by the ten sectors portfolios of the S&P500 index obtained using the Global Industry Classification Standard (GICS): Material (MATR), Information-Technology (INFT), Healthcare (HLTH), Energy (ENRS), Utilities (UTIL), Financials (FINL), Consumer-Staples (CONS), Consumer-Discretionary (COND), Industrials (INDU), Telecommunications (TELS).
2. DAX ($n = 23$) is the stock index of the major German blue chip companies trading on the Frankfurt Stock Exchange.
3. ESX ($n = 38$), EURO STOXX 50, is a stock index of the Euro zone composed of 50 European blue-chip companies considered as leaders in their respective sectors.
4. FTSE ($n = 69$) is the Financial Times Stock Exchange 100 Index composed of the 100 companies listed on the London Stock Exchange with the highest market capitalization.
5. Nikkei ($n = 180$) is the Nikkei 225, a stock market index for the Tokyo Stock Exchange composed of 225 large publicly owned companies in Japan.
6. S&P500 constituents ($n = 391$) is The Standard and Poor's 500 index composed of 500 large companies listed on stock exchanges in the United States.

5.3 Numerical tests

We consider the diversification measures and concentration indexes described in Section 5.1, with the following parameter settings: $k = n-1$ for CR_k; $\alpha = 3$ for RHK, Y_α, and H_α; $q = 2$ for HC_q and D_q; and the matrix $D = C - I_n$, where I_n is the identity matrix of order n, for RQE and EDR. To provide a thorough evaluation of the proposed GPDMs, we consider the following commonly used risk measures: standard deviation (StDev), Value-at-Risk computed at a 5% significance level (V@R), mean absolute deviation (MAD), and maximum drawdown (MDD). The corresponding GPDMs are denoted by $\mathrm{GPDM}^{\mathrm{StDev}}$, $\mathrm{GPDM}^{\mathrm{V@R}}$, $\mathrm{GPDM}^{\mathrm{MAD}}$, and $\mathrm{GPDM}^{\mathrm{MDD}}$, respectively.

5.3.1 The long-only case

For each dataset described in Section 5.2, we construct the mean-variance efficient frontier in the long-only case and consider the associated sets of portfolios: $\mathcal{P}\mathrm{LO}^{\mathrm{S\&P500}}$, $\mathcal{P}\mathrm{LO}^{\mathrm{DAX}}$, $\mathcal{P}\mathrm{LO}^{\mathrm{ESX}}$, $\mathcal{P}\mathrm{LO}^{\mathrm{FTSE}}$, $\mathcal{P}\mathrm{LO}^{\mathrm{Nikkei}}$, and $\mathcal{P}\mathrm{LO}^{\mathrm{S\&Pconst.}}$. This section is devoted to a detailed and comprehensive description of the numerical tests conducted on the S&P500 index. For the reader's convenience, the results and discussion related to the remaining datasets are deferred to Appendix 8.1.

Figure 2 displays the daily return level of each portfolio on the x-axis and the corresponding values of the diversification measures and concentration indexes on the y-axis, computed for the portfolios in $\mathcal{P}\mathrm{LO}^{\mathrm{S\&P500}}$. For each measure, the most diversified portfolio is identified and represented on the mean-variance efficient frontier (see Figure 4, left panel); its composition is shown in Figure 3. Next, we restrict the analysis to the dominant part of the mean-variance efficient frontier and again determine the most diversified portfolio for each of the considered measures. The resulting optimal portfolios are depicted in Figure 4, right panel. We observe that the most diversified portfolios differ in the following cases: CR_k, ρ_{AVG1}, ρ_{AVG2}, ρ_{AVG3}, ρ_{AVG4}, SE, and DR. For these measures, Figure 5 reports the composition of the corresponding most diversified portfolios.

A first general remark is that the efficient frontier does not account for diversification. Therefore, it may be of interest for an investor to select, within the set $\mathcal{P}_{\mathrm{LO}}^{\mathrm{S\&P500}}$, the most

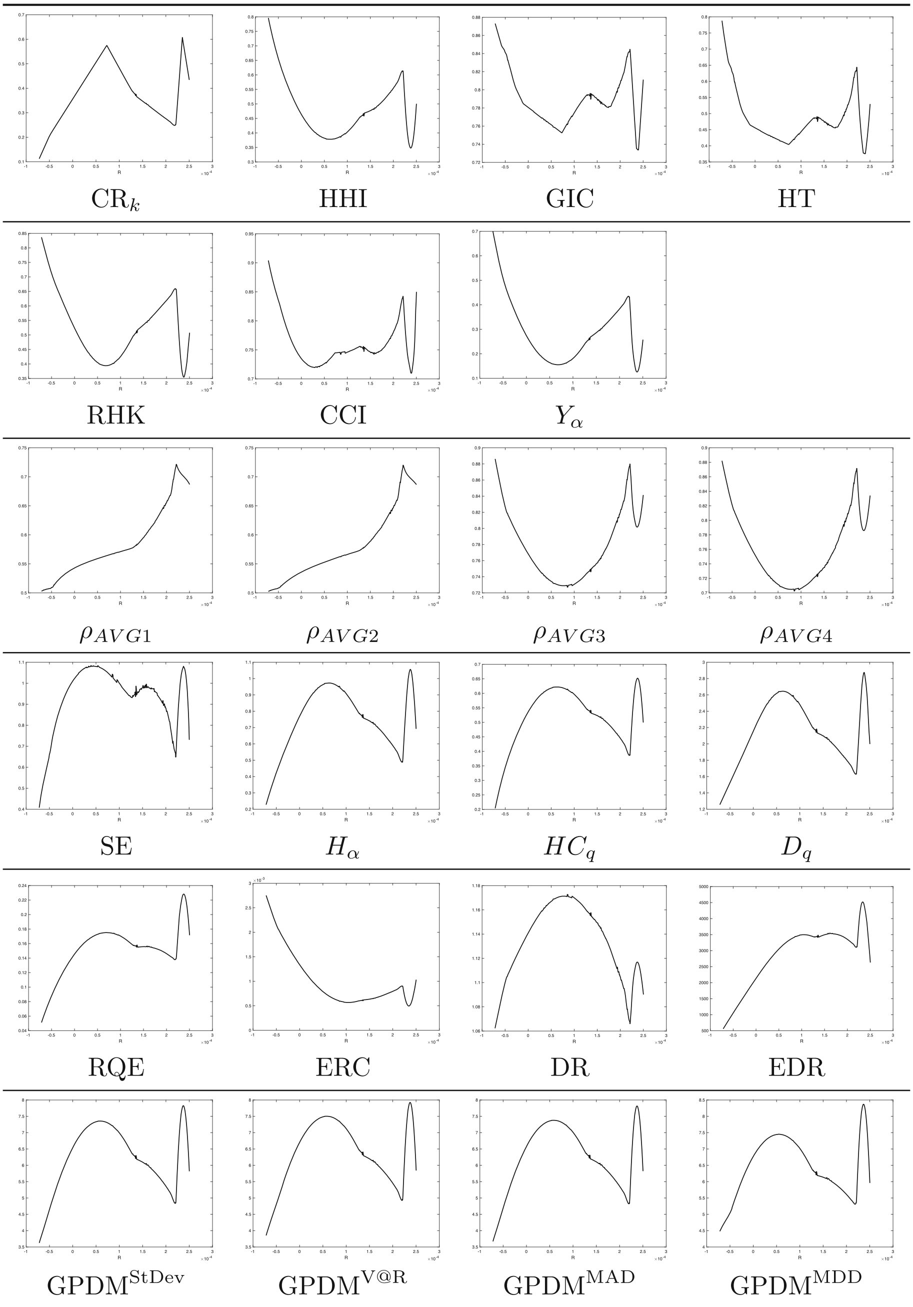

Fig. 2 The values of the various diversification measures and concentration indexes computed for the portfolios in $\mathcal{P}_{\mathrm{LO}}^{\mathrm{S\&P500}}$

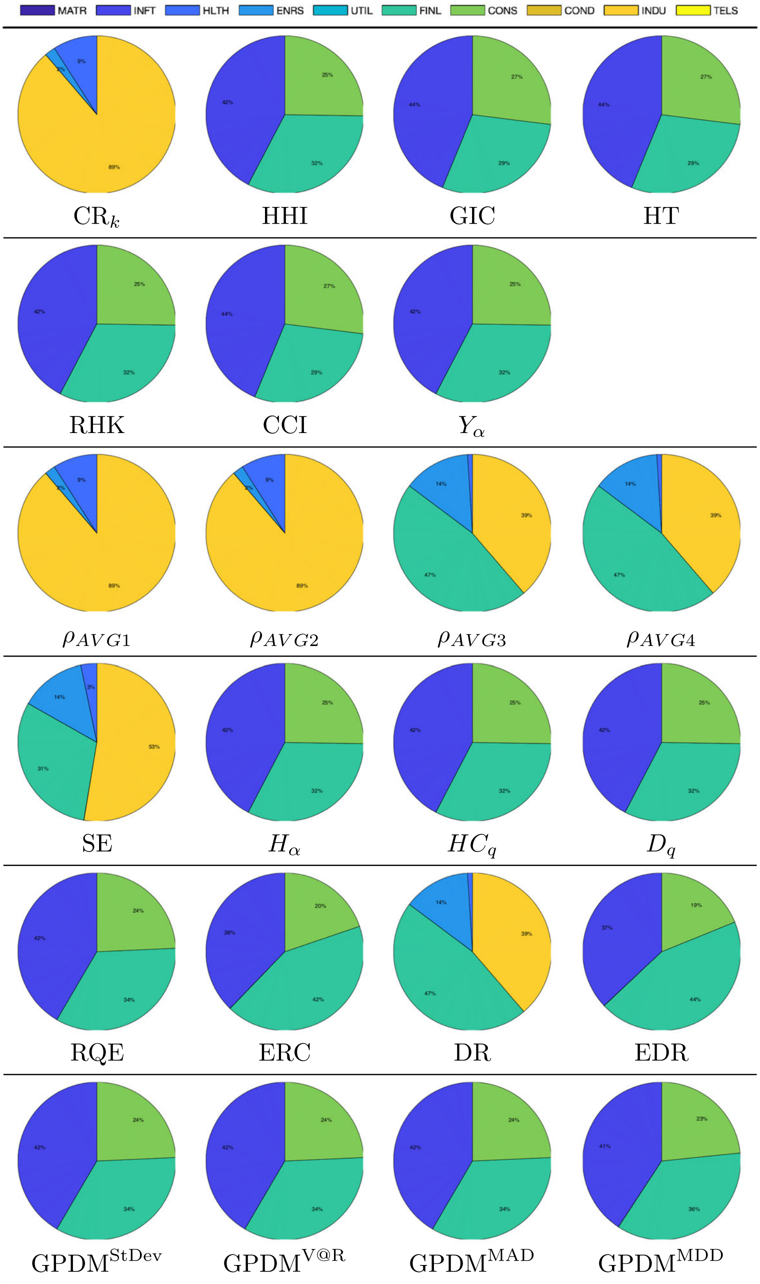

Fig. 3 The compositions of the optimal portfolios of $\mathcal{P}_{\mathrm{LO}}^{\mathrm{S\&P500}}$ for various diversification measures and concentration indexes

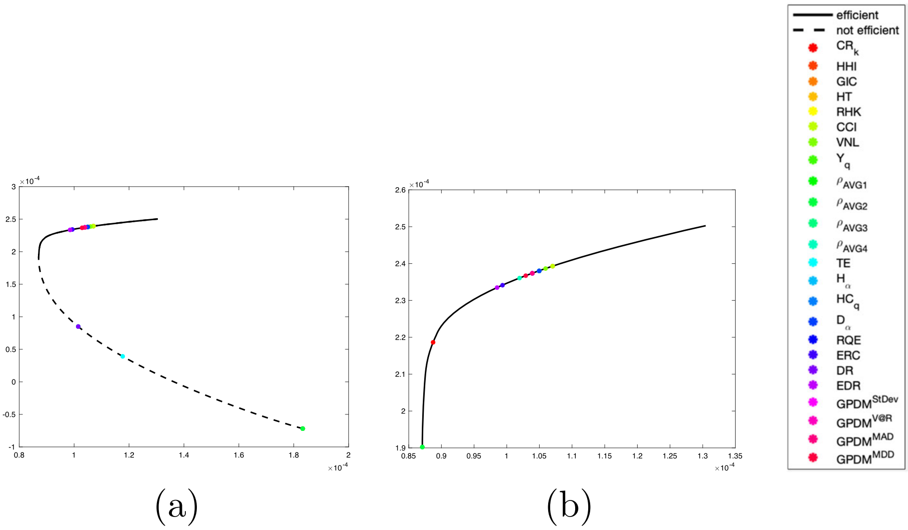

Fig. 4 Representation of the most diversified portfolios of the dataset S&P500 for various diversification measures and concentration indexes on the mean-variance efficient frontier (left panel) and when restricted over the dominant part of the mean-variance efficient frontier (right panel)

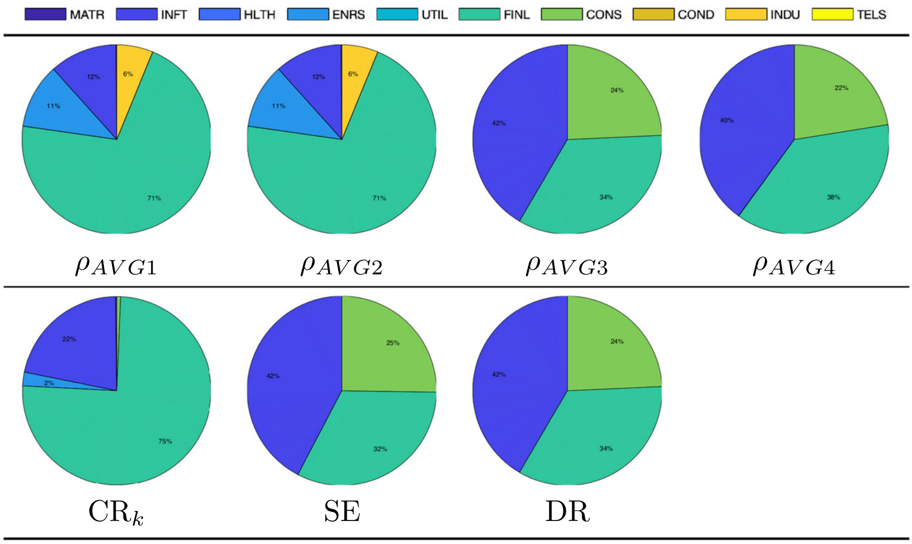

Fig. 5 The compositions of some optimal portfolios of $\mathcal{P}_{\text{LO}}^{\text{S\&P500}}$ over the dominant part of the mean-variance efficient frontier

diversified portfolio according to a given diversification measure. As expected, the most diversified portfolios do not necessarily lie on the dominant segment of the efficient frontier. This observation may partially explain why many empirical out-of-sample studies have found that simple diversification-based investment strategies can outperform the optimal mean-variance portfolios; see DeMiguel et al. (2009). From Figure 3 and Figure 4, left panel, we observe that most of the considered measures yield very similar results, identifying the most diversified portfolios with comparable positions on the efficient frontier and with similar asset compositions. In particular, we emphasize that the GPDMs–despite being defined using different underlying risk measures–closely align with the majority of other measures in identifying the most diversified portfolios.

From Figure 4, we observe that the measures identifying the most diversified portfolios on the dominated part of the efficient frontier are CR_k, ρ_{AVG1}, ρ_{AVG2}, ρ_{AVG3}, ρ_{AVG4}, SE, and DR. When restricting to the dominant part of the frontier, the measures ρ_{AVG3}, ρ_{AVG4}, SE, and DR return most diversified portfolios with compositions that are consistent with the allocations selected by the majority of other measures. In contrast, the remaining measures–CR_k, ρ_{AVG1}, and ρ_{AVG2}–exhibit distinctive behavior compared to all others. The values of CR_k change rapidly (see Figure 2), which may primarily depend on the arbitrariness in the choice of the parameter k and the associated challenges. In this example, with $n = 10$, the portfolios on the efficient frontier allocate wealth to no more than 4 assets, which forces the choice of a relatively high value of k to ensure a mathematically meaningful value for the measure. However, as previously noted, a high k negatively impacts the informational content of the measure. Furthermore, Figure 2 clearly shows that the values of both ρ_{AVG1} and ρ_{AVG2} exhibit trends that diverge significantly from those of the other measures. In these cases, the most diversified portfolios correspond to extreme allocations located at the bottom-right end of the efficient frontier.[2]

5.3.2 The long-short case

In the long-short case, we can compute only the GPDMs, as the classical diversification measures proposed in the literature (see Section 5.1) are defined exclusively for long-only portfolios. In particular, most of the considered measures are well defined only when portfolio weights are strictly positive–for instance, entropy-based diversification measures (see Section 5.1.3). Other measures, such as the Herfindahl-Hirschman Index (HHI), can technically be computed in the presence of negative weights, though their meaningfulness in such settings is not guaranteed. In the absence of the possibility for a direct comparison with alternative measures, we focus below on highlighting a few key features of the GPDMs in the long-short setting.

For each dataset described in Section 5.2, we construct the mean-variance efficient frontier in the long-short framework and consider the associated sets of portfolios: $\mathcal{P}_{\mathrm{LS}}^{\mathrm{S\&P500}}$, $\mathcal{P}_{\mathrm{LS}}^{\mathrm{DAX}}$, $\mathcal{P}_{\mathrm{LS}}^{\mathrm{ESX}}$, $\mathcal{P}_{\mathrm{LS}}^{\mathrm{FTSE}}$, $\mathcal{P}_{\mathrm{LS}}^{\mathrm{Nikkei}}$, and $\mathcal{P}_{\mathrm{LS}}^{\mathrm{S\&Pconst.}}$ This section provides a complete and detailed discussion of the numerical results for the S&P500 index. For convenience, the results for the remaining datasets, along with the corresponding analysis, are deferred to Appendix 8.2.

Figure 6 reports the values of the four considered GPDMs, computed for the portfolios in $\mathcal{P}_{\mathrm{LS}}^{\mathrm{S\&P500}}$. For each measure, the most diversified portfolio is identified and represented on the mean-variance efficient frontier (see Figure 7, left panel). Next, we restrict our analysis to the dominant part of the mean-variance efficient frontier and, for each GPDM, determine the corresponding most diversified portfolio. The optimal portfolios are shown in Figure 7, right panel. From the left panel of Figure 7, we observe that, as in the long-only case, the most diversified portfolios may lie on the dominated portion of the efficient frontier.

All the GPDMs under analysis consistently identify the same portfolio as the most diversified. This optimal portfolio, whether considering the entire efficient frontier or restricting

[2] Note that, in theory, the portfolio at the bottom-right of the efficient frontier would correspond to a full allocation of wealth in the asset with the lowest expected return. In our implementation, however, this situation does not arise, since we constrain the portfolio's expected return to lie within the interval $(\min \mu + \varepsilon, \max \mu - \varepsilon)$, with $\varepsilon = 10^{-5}$, to avoid numerical issues in the computation of the frontier.

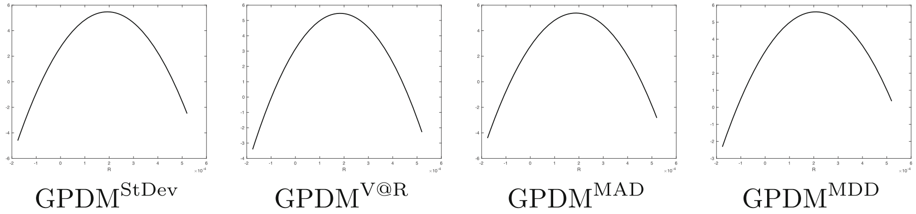

Fig. 6 The values of the various GPDMs computed for the portfolios in $\mathcal{P}_{\mathrm{LS}}^{\mathrm{S\&P500}}$

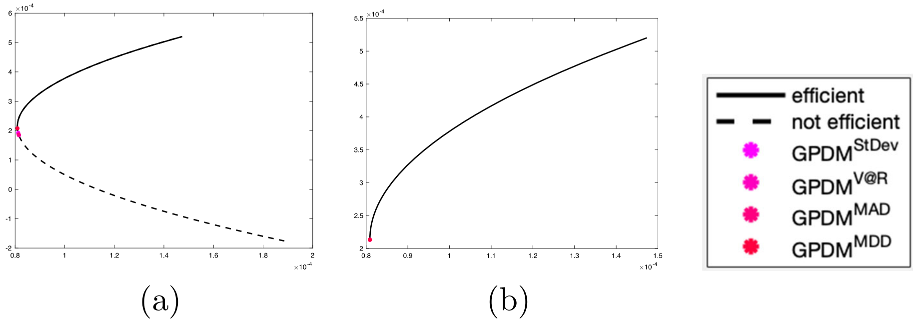

Fig. 7 Representation of the most diversified portfolios of the dataset S&P500 for various GPDMs on the mean-variance efficient frontier (left panel) and when restricted over the dominant part of the mean-variance efficient frontier (right panel)

to its dominant part, lies very close to the global minimum variance portfolio. This finding holds across all datasets under examination (see Appendix 8.2). An intuitive explanation for this observation is that, in the long-short framework, portfolios that deviate significantly from the global minimum variance portfolio tend to exhibit higher leverage–i.e., they involve larger absolute exposures to individual assets.

6 Conclusions

In this paper, we highlight the intimate relationship between risk measures and portfolio diversification measures, and we propose an effective method to construct a portfolio diversification measure starting from a given risk measure. The key tool linking risk and diversification is the Risk-Geometric Portfolio Diversification Measure (GPDM). This connection allows the incorporation of the information content of a given risk measure into the calculation of diversification, moving beyond the common practice of relying solely on portfolio weights as the relevant variable for assessing diversification, as is the case in the majority of existing diversification measures. In the mathematical formalization of our approach, we prove that GDPMs satisfy a set of significant theoretical properties. Furthermore, empirical comparisons between GDPMs and many standard diversification measures support the validity and relevance of our proposal. This comparison is necessarily limited to the long-only case due to the restrictions on applying most standard measures in other settings. Indeed, a further distinctive feature of GDPMs is that they are well-defined in a long-short framework, allowing their calculation even when short selling is permitted. While most of the literature on diversification focuses on the long-only case, we advocate for evaluating diversification in the presence of short positions as well. Our empirical application reveals interesting findings in this broader context. Nonetheless, we believe that the generalization and implementation of portfolio diversification measures within the long-short framework warrant deeper investigation in future research. Considering that the main contribution of this research is methodological, the empirical application–although conducted on real financial data–is primarily devoted to comparing our proposed GDPMs with diversification measures from the existing literature, thereby supporting the robustness of GDPMs. As pointed put in the introduction, a comprehensive study focused on the economic interpretation and impact of GDPMs in the portfolio optimization context is provided in Torrente and Uberti (2025). That research investigates the out-of-sample performance of portfolios constructed by maximizing diversification compared to those obtained by minimizing a risk measure. As a final, marginal byproduct, this paper supports the validity of the geometric diversification strategy in portfolio theory by showing that the Risk-Adjusted Geometric Diversified Portfolio can equivalently be obtained as the solution to a maximization problem whose objective function is a GDPM.

7 Appendix - Proofs

Proof of Lemma 1

Proof We let $\rho_j = \rho(X_j) > 0$ for each $j = 1, \dots, n$.

(i) By some computations, using the expression of $f_{\rho(X)}$, see formula (3), and Definition 3, we get

$$f_{\rho(X)}(w) = \sum_{j=1}^{n} \sum_{k=j+1}^{n} \left[\left(\frac{1}{\rho_j} - \frac{1}{\rho_k} \right) - 2 \left(\frac{w_j}{\rho_j} - \frac{w_k}{\rho_k} \right) \right]^2. \tag{10}$$

By evaluating the above expression at $w = e^i$ we have:

$$\begin{aligned} f_{\rho(X)}(e^i) = &\sum_{j=1}^{i-1} \sum_{k=j+1, k \neq i}^{n} \left(\frac{1}{\rho_j} - \frac{1}{\rho_k} \right)^2 + \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} \left(\frac{1}{\rho_j} - \frac{1}{\rho_k} \right)^2 \\ &+ \sum_{j=1, j \neq i}^{n} \left(\frac{1}{\rho_j} + \frac{1}{\rho_i} \right)^2 > 0. \end{aligned} \tag{11}$$

(ii) We consider the nontrivial case $i \neq j$; by some computations, using expression (11), we get:

$$f_{\rho(X)}(e^j) - f_{\rho(X)}(e^i) = \left(\frac{1}{\rho_j} - \frac{1}{\rho_i} \right) \sum_{k=1, k \neq i, k \neq j}^{n} \frac{4}{\rho_k}$$

from which the result immediately follows.

(iii) Let $w, z \in \Gamma_n$, with $w \neq z$ and $\alpha \in (0, 1)$. Recalling that $g(x) = x^2$ is a strictly convex function we obtain

$$\begin{aligned}
&f_{\rho(X)}(\alpha w + (1-\alpha)z) = \\
&= \sum_{j=1}^{n} \sum_{k=j+1}^{n} \left[\left(\frac{1}{\rho_j} - \frac{1}{\rho_k} \right) - 2\alpha \left(\frac{w_j}{\rho_j} - \frac{w_k}{\rho_k} \right) - 2(1-\alpha) \left(\frac{z_j}{\rho_j} - \frac{z_k}{\rho_k} \right) \right]^2 \\
&= \sum_{j=1}^{n} \sum_{k=j+1}^{n} \left[\alpha \left(\left(\frac{1}{\rho_j} - \frac{1}{\rho_k} \right) - 2 \left(\frac{w_j}{\rho_j} - \frac{w_k}{\rho_k} \right) \right) \right. \\
&\qquad\qquad \left. + (1-\alpha) \left(\left(\frac{1}{\rho_j} - \frac{1}{\rho_k} \right) - 2 \left(\frac{z_j}{\rho_j} - \frac{z_k}{\rho_k} \right) \right) \right]^2 \\
&< \alpha \sum_{j=1}^{n} \sum_{k=j+1}^{n} \left(\left(\frac{1}{\rho_j} - \frac{1}{\rho_k} \right) - 2 \left(\frac{w_j}{\rho_j} - \frac{w_k}{\rho_k} \right) \right)^2 \\
&\quad + (1-\alpha) \sum_{j=1}^{n} \sum_{k=j+1}^{n} \left(\left(\frac{1}{\rho_j} - \frac{1}{\rho_k} \right) - 2 \left(\frac{z_k}{\rho_j} - \frac{z_k}{\rho_k} \right) \right)^2 \\
&= \alpha f_{\rho(X)}(w) + (1-\alpha) f_{\rho(X)}(z),
\end{aligned}$$

which proves that $f_{\rho(X)}$ is strictly convex.

(iv) Let π be the permutation of $\{1, \ldots, n\}$ associated to Π, that is $\pi(j)$ satisfies $e^{\pi(j)} = \Pi^t e^j$, for each $j = 1, \ldots, n$. Note that $f_{\rho(X)}(w)$ can also be expressed as follows:

$$f_{\rho(X)}(w) = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(d^2_{\rho(X)}(w, e^j) - d^2_{\rho(X)}(w, e^k) \right)^2.$$

Using formula (10) and the fact that π is a bijection we get

$$\begin{aligned}
f_{\rho(X\Pi)}(\Pi^t w) &= \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \left[\left(\frac{1}{\rho_{\pi(j)}} - \frac{1}{\rho_{\pi(k)}} \right) - 2 \left(\frac{w_{\pi(j)}}{\rho_{\pi(j)}} - \frac{w_{\pi(k)}}{\rho_{\pi(k)}} \right) \right]^2 \\
&= f_{\rho(X)}(w),
\end{aligned}$$

for each $w \in \Gamma_n$, so that the item is proved.

(v) The uniqueness of the global minimum of $f_{\rho(X)}$ over Γ_n directly follows from item (i). Further, using formula (1) and (10), we get $f_{\rho(X)}(w^*) = 0$; consequently, since $f_{\rho(X)}(w) \geq 0$ for each $w \in \Gamma_n$, the result follows.

(vi) By the shift-invariance of ρ and the definition of $d_{\rho(X)}$ (see Definition 3) it immediately follows that $d_{\rho(X+\alpha)}(w, e^j) = d_{\rho(X)}(w, e^j)$ and $d_{\rho(X+\alpha)}(w, e^k) = d_{\rho(X)}(w, e^k)$, hence $f_{\rho(X+\alpha)}(w) = f_{\rho(X)}(w)$, for each $w \in \Gamma_n$, so that the item is proved.

(vii) Using formula (10) and the positive homogeneity of ρ we get

$$f_{\rho(\alpha X)}(w) = \sum_{j=1}^{n} \sum_{k=j+1}^{n} \left[\left(\frac{1}{\alpha\rho_j} - \frac{1}{\alpha\rho_k} \right) - 2 \left(\frac{w_j}{\alpha\rho_j} - \frac{w_k}{\alpha\rho_k} \right) \right]^2 = \frac{1}{\alpha^2} f_{\rho(X)}(w),$$

for each $w \in \Gamma_n$, so that the item is proved.

□

Proof of Proposition 2

Proof By Lemma 1, item (i), it follows that $\max_{j=1,\ldots,n} f_{\rho(X)}(e^j) > 0$, consequently

$$\max_{w\in\Gamma_n} \Phi_{\rho(X)}(w, A) = (\text{rank}(A) - 1)\left(1 - \frac{\min_{w\in\Gamma_n} f_{\rho(X)}(w)}{\max_{j=1,\ldots,n} f_{\rho(X)}(e^j)}\right).$$

Since w^* is the unique global minimum of $f_{\rho(X)}$ over Γ_n and $f_{\rho(X)}(w^*) = 0$ (see Lemma 1, item (v)), the result immediately follows. □

Proof of Proposition 3

***Proof* P1.** We start considering some trivial cases: if $\text{rank}(A) = 1$ the property is obviously verified, so we assume rank(A) > 1. Further, in the cases $\alpha = 0$ and $\alpha = 1$, the property immediately holds, so we let $\alpha \in (0, 1)$. Denote $\xi(\alpha) = \alpha w + (1-\alpha)z \in \Gamma_n$; by Lemma 1, item (iii), $f_{\rho(X)}(\xi(\alpha)) < \alpha f_{\rho(X)}(w) + (1-\alpha) f_{\rho(X)}(z)$. Consequently

$$\begin{aligned}
\Phi_{\rho(X)}(\xi(\alpha), A) &= (\text{rank}(A) - 1))\left(1 - \frac{f_{\rho(X)}(\xi(\alpha))}{\max_{j=1,\ldots,n} f_{\rho(X)}(e^j)}\right)\\
&> (\text{rank}(A) - 1)\left(1 - \frac{\alpha f_{\rho(X)}(w) + (1-\alpha) f_{\rho(X)}(z)}{\max_{j=1,\ldots,n} f_{\rho(X)}(e^j)}\right)\\
&= (\text{rank}(A) - 1)\left[\alpha\left(1 - \frac{f_{\rho(X)}(w)}{\max_{j=1,\ldots,n} f_{\rho(X)}(e^j)}\right)\right.\\
&\quad \left.+ (1-\alpha)\left(1 - \frac{f_{\rho(X)}(z)}{\max_{j=1,\ldots,n} f_{\rho(X)}(e^j)}\right)\right]\\
&\geq \min\{\Phi_{\rho(X)}(w, A), \Phi_{\rho(X)}(z, A)\},
\end{aligned}$$

thus property **P1** is proved.

P2. From the hypothesis it follows that $\text{rank}(A) = 1$, consequently $\Phi_{\rho(X)}(w, A) = 0$, so that property **P2** is proved.

P3. Since by Lemma 1, item (iii), $f_{\rho(X)}$ is a strictly convex function then $f_{\rho(X)}(w) < \max_{j=1,\ldots,n} f_\rho(e^j)$ for each $w \in \mathbb{W}_n \setminus \partial\mathbb{W}_n$; consequently, A^* is a solution of the equation $\Phi_{\rho(X)}(w, A) = 0$ if and only if $\text{rank}(A^*) = 1$, so that property **P3** is proved.

P4. From Lemma 1, item (iv), and equality $\text{rank}(A\Pi) = \text{rank}(A)$, it immediately follows that $\Phi_{\rho(X)}(\Pi^t w, A\Pi) = \Phi_{\rho(X)}(w, A)$, so that property **P4** is proved. □

Proof of Proposition 4

***Proof* P5.** Since $\text{rank}(\alpha\mathbf{1}_{m\times n}) = 1$, then from the rank-sum inequality (see Horn & Johnson, 2012, Section 0.4.5) yields $\text{rank}(A + \alpha) = \text{rank}(A)$ which, combined with the result of Proposition 1, item (vi), yields $\Phi_{\rho(X)}(w, A + \alpha) = \Phi_{\rho(X)}(w, A)$, so that property **P5** is proved.

P6. From equality $\text{rank}(\alpha A) = \text{rank}(A)$ and Proposition 1, item (vii), it immediately follows that $\Phi_{\rho(X)}(w, \alpha A) = \Phi_{\rho(X)}(w, A)$, so that property **P6** is proved. □

Proof of Proposition 5

Proof Proposition 2 yields $\Phi_{\rho(X)}(w^*, A) = \text{rank}(A) - 1$ and $\Phi_{\rho(X)}(w^*_+, A^+) = \text{rank}(A^+) - 1$. Since by the hypothesis of **P7** and **P8** we have either $\text{rank}(A^+) = \text{rank}(A)$ or $\text{rank}(A^+) > \text{rank}(A)$, the results are immediately proved. □

Proof of Proposition 7

For the following proofs (see also the proof of Proposition 8), we recall some of the notation already introduced in the main text of the paper, and we also introduce some additional notation required for clarity. We indicate with $\mathbf{0}_{m\times n}$ and $\mathbf{1}_{m\times n}$ the $m \times n$ matrices whose elements are all equal to zero and one respectively (if $m = n$ we simply use $\mathbf{0}_n$ and $\mathbf{1}_n$). For each $A \in \text{Mat}_{m\times n}(\mathbb{R})$ we denote by $A_i \in \text{Mat}_{1\times n}(\mathbb{R})$ the ith row of A, by $A^{(i)} \in \text{Mat}_{(m-1)\times n}(\mathbb{R})$ the submatrix of A obtained by deleting its ith row and by $A^{(i,j)} \in \text{Mat}_{(m-1)\times(n-1)}(\mathbb{R})$ the submatrix of A obtained by deleting its ith row and its jth column. For each $A \in \text{Mat}_n(\mathbb{R})$ we denote by $\text{diag}(A) \in \text{Mat}_{n\times 1}(\mathbb{R})$ the column vector of the diagonal elements of A.

Proof From Remark 5, Problem (4) is a convex constrained minimization problem that admits unique solution $w^* = (w^*_1, \dots, w^*_n) \in \Gamma_n$. We consider system

$$\begin{cases} (w - e^j)^t\, S(w - e^j) = (w - e^n)^t\, S(w - e^n), \quad \forall j = 1, \dots, n-1 \\ \sum_{k=1}^n w_k = 1. \end{cases} \tag{12}$$

If system (12) admits a solution w^*, then by the positivity of $f_{\Sigma^{-1}}$ (defined in formula (5)) it follows that w^* is the minimizer of $f_{\Sigma^{-1}}$ over Γ_n and $f_{\Sigma^{-1}}(w^*) = 0$. System (12) can be equivalently rewritten as follows:

$$\begin{cases} (S_j - S_n)w = \frac{1}{2}(s_{jj} - s_{nn}), \quad \forall j = 1, \dots, n-1 \\ w_n = 1 - \sum_{j=1}^{n-1} w_j. \end{cases}$$

Hence

$$w_n = 1 - \mathbf{1}_{1\times(n-1)} w^{(n)}, \tag{13}$$

and

$$\begin{pmatrix} S_1 - S_n \\ \vdots \\ S_{n-1} - S_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_{n-1} \\ 1 - \mathbf{1}_{1\times(n-1)} w^{(n)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} s_{11} - s_{nn} \\ \vdots \\ s_{n-1,n-1} - s_{nn} \end{pmatrix}. \tag{14}$$

We rewrite the left-hand side of (14) as follows:

$$\begin{pmatrix} S_1 - S_n \\ \vdots \\ S_{n-1} - S_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_{n-1} \\ 1 - \mathbf{1}_{1\times(n-1)} w^{(n)} \end{pmatrix} =$$
$$= \left(S^{(n,n)} - \mathbf{1}_{(n-1)\times 1} S_n^{(n)} \right) w^{(n)} - (S_n^t)^{(n)} \mathbf{1}_{1\times(n-1)} w^{(n)} - s_{nn} \mathbf{1}_{(n-1)\times 1} + s_{nn} \mathbf{1}_{n-1} w^{(n)}$$
$$= \left(S^{(n,n)} - \mathbf{1}_{(n-1)\times 1} S_i^{(n)} - (S_n^t)^{(n)} \mathbf{1}_{1\times(n-1)} + s_{nn} \mathbf{1}_{n-1} \right) w^{(n)} - s_{nn} \mathbf{1}_{(n-1)\times 1}$$
$$= \left(S^{(n,n)} - (D_n + D_n^t) + s_{nn} \mathbf{1}_{n-1} \right) w^{(n)} - s_{nn} \mathbf{1}_{(n-1)\times 1}$$
$$= E_n w^{(n)} - s_{nn} \mathbf{1}_{(n-1)\times 1}.$$

We also rewrite the right-hand side of (14):

$$\frac{1}{2} \begin{pmatrix} s_{11} - s_{nn} \\ \vdots \\ s_{n-1,n-1} - s_{nn} \end{pmatrix} = \frac{1}{2} \operatorname{diag}(S)^{(n)} - \frac{1}{2} s_{nn} \mathbf{1}_{(n-1)\times 1}$$

Hence, system (14) becomes

$$E_n w^{(n)} = b_n. \tag{15}$$

Now, we prove that $E_n \in \mathcal{S}_{n-1}^{+}(\mathbb{R})$. First, we consider the following $n \times n$ symmetric matrices

$$B := S + s_{nn} \mathbf{1}_n, \qquad D := \mathbf{1}_{n\times 1} S_n + (\mathbf{1}_{n\times 1} S_n)^t, \qquad E := B - D,$$

and denote their real eigenvalues by $\lambda_1(B) \leq \ldots \leq \lambda_n(B)$, $\lambda_1(D) \leq \ldots \leq \lambda_n(D)$ and $\lambda_1(E) \leq \ldots \leq \lambda_n(E)$, respectively.

We analyze the matrices B, We observe that B is positive definite since it is the sum of a symmetric positive definite matrix and a symmetric positive semidefinite matrix (see Horn & Johnson, 2012, Corollary 4.3.12); therefore, $\lambda_i(B) > 0$ for each $i \in \{1, \ldots, n\}$.

Now, we consider the matrix D. Since $S \in \mathcal{S}_n^{+}(\mathbb{R})$, then S_n is not the zero vector, that is $\operatorname{rank}(S_n) = 1$. By the rank-sum inequality (see Horn & Johnson, 2012, formula 0.4.5.1) it holds

$$\operatorname{rank}(D) \leq 2 \cdot \operatorname{rank}(\mathbf{1}_{n\times 1} S_n) = 2 \cdot \operatorname{rank}(S_n) = 2,$$

with $\operatorname{rank}(D) = 2$ if and only if

$$\{0\} = \operatorname{range}(\mathbf{1}_{n\times 1} S_n) \cap \operatorname{range}((\mathbf{1}_{n\times 1} S_n)^t) = \{k \cdot \mathbf{1}_{n\times 1} \mid k \in \mathbb{R}\} \cap \{k \cdot S_n^t \mid k \in \mathbb{R}\}.$$

Hence, the case $\operatorname{rank}(D) = 1$ occurs if and only if there exists $\delta > 0$ such that $S_n = \delta \times \mathbf{1}_{1\times n}(\mathbb{R})$; consequently, the eigenvalues of D are $0 = \lambda_1(D) = \ldots = \lambda_{n-1}(D) < \lambda_n(D) = n\delta$. Otherwise, $\operatorname{rank}(D) = 2$, and the characteristic polynomial of D is

$$p_D(\lambda) = \det(\lambda I_n - D) = \lambda^{n-2}(\lambda^2 + a_{n-1}\lambda + a_{n-2}), \tag{16}$$

where, according to (Horn & Johnson, 2012, formula (1.2.13)),

$$
\begin{aligned}
a_{n-2} &= \sum_{1\le k<j\le n} \det \begin{pmatrix} 2s_{nk} & s_{nk}+s_{nj} \\ s_{nj}+s_{nk} & 2s_{nj} \end{pmatrix} \\
&= \sum_{1\le k<j\le n} \left(4s_{nk}s_{nj} - (s_{nk}+s_{nj})^2\right) \\
&= -\sum_{1\le k<j\le n} \left(s_{nj}-s_{nk}\right)^2 < 0,
\end{aligned}
$$

where the last inequality follows from the fact that there exist $j, k \in \{1, \ldots, n\}$, with $j \neq k$, such that $s_{nj} \neq s_{nk}$. Consequently, from (16) the characteristic equation $p_D(\lambda) = 0$ has exactly 2 nonzero solutions with opposite signs, that is the eigenvalues of D are $\lambda_1(D) < 0 = \lambda_2(D) = \ldots = \lambda_{n-1}(D) < \lambda_n(D)$.

We consider the matrix E. We note that by construction the nth row and the nth column of E are equal to the zero vector, hence the characteristic polynomial of E is $p_E(\lambda) = \det(\lambda I_n - E) = \lambda g(\lambda)$, where $g(\lambda)$ is a real polynomial in λ of degree $n-1$, and there is at least one eigenvalue of E which is equal to zero. Further, recalling that $E = B - D$ and exploiting classical Weyl inequalities (see Horn et al., 1998, Theorem 3.1) we have

$$\lambda_2(E) \ge \lambda_1(B) + \lambda_2(-D.)$$

Since the eigenvalues of the matrix $-D$ are $\lambda_i(-D) = -\lambda_{n-i+1}(D)$, for each $i \in \{1, \ldots, n\}$, the previous inequality becomes:

$$\lambda_2(E) \ge \lambda_1(B) - \lambda_{n-1}(D) = \lambda_1(B) > 0,$$

implying that $\lambda_1(E) = 0$ and that $0 < \lambda_2(E) \le \ldots \le \lambda_n(E)$ are the zeros of the equation $g(\lambda) = 0$.

Finally we consider the symmetric matrix $E_n \in \mathrm{Mat}_{n-1}(\mathbb{R})$. Since E_n is the principal submatrix of E obtained by deleting its nth row and nth column (which are both equal to the zero vector), then the characteristic polynomial of E_n is $p_{E_n}(\lambda) = g(\lambda)$, so the real eigenvalues of E_n are exactly $0 < \lambda_2(E) \le \ldots \le \lambda_n(E)$. Consequently, $E_n \in \mathcal{S}^+_{n-1}(\mathbb{R})$ and so E_n is invertible. Then, by using (13) and (15), formula (6) is proved. □

Proof of Proposition 8

Proof Let σ_i be the variance of asset i, for each $i \in \{1, \ldots, n\}$. Then, it holds $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, $S = \mathrm{diag}(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_n})$, $D_n = \mathbf{0}_{n-1}$ (see formula (9)) and

$$
\begin{aligned}
E_n &= (\Sigma^{-1})^{(n,n)} + \frac{1}{\sigma_n}\mathbf{1}_{n-1} \\
b_n &= \frac{1}{2}\left(\mathrm{diag}(\Sigma^{-1})^{(n)} + \frac{1}{\sigma_n}\mathbf{1}_{(n-1)\times 1}\right)
\end{aligned}
$$

(see (7) and (8)). In order to compute E_n^{-1} we apply the Sherman-Morrison-Woodbury formula, see Hager (1989) and Miller (1981), to E_n:

$$
\begin{aligned}
E_n^{-1} &= \left((\Sigma^{-1})^{(n,n)} + \frac{1}{\sigma_n}\mathbf{1}_{n-1}\right)^{-1} \\
&= \left(I_{n-1} - \frac{1}{1+\operatorname{tr}(\frac{1}{\sigma_n}\mathbf{1}_{n-1}\Sigma^{(n,n)})}\Sigma^{(n,n)}\frac{1}{\sigma_n}\mathbf{1}_{n-1}\right)\Sigma^{(n,n)} \\
&= \left(I_{n-1} - \frac{1}{1+\frac{1}{\sigma_n}\sum_{j=1}^{n-1}\sigma_j}\Sigma^{(n,n)}\frac{1}{\sigma_n}\mathbf{1}_{n-1}\right)\Sigma^{(n,n)} \\
&= \left(I_{n-1} - \frac{\sigma_n}{\sum_{j=1}^{n}\sigma_j}\Sigma^{(i,i)}\frac{1}{\sigma_n}\mathbf{1}_{n-1}\right)\Sigma^{(n,n)} \\
&= \left(I_{n-1} - \frac{1}{\sum_{j=1}^{n}\sigma_j}\Sigma^{(n,n)}\mathbf{1}_{n-1}\right)\Sigma^{(n,n)}.
\end{aligned}
$$

Then applying, Proposition 7, formula (6), we compute

$$
\begin{aligned}
(w^*)^{(n)} &= \frac{1}{2}\left(I_{n-1} - \frac{1}{\sum_{j=1}^{n}\sigma_j}\Sigma^{(n,n)}\mathbf{1}_{n-1}\right)\Sigma^{(n,n)}\left(\operatorname{diag}(\Sigma^{-1})^{(n)} + \frac{1}{\sigma_n}\mathbf{1}_{(n-1)\times 1}\right) \\
&= \frac{1}{2}\Bigg(\mathbf{1}_{(n-1)\times 1} - \frac{n-1}{\sum_{j=1}^{n}\sigma_j}\Sigma^{(n,n)}\mathbf{1}_{(n-1)\times 1} \\
&\quad + \frac{1}{\sigma_n}\left(\Sigma^{(n,n)}\mathbf{1}_{(n-1)\times 1} - \frac{\sum_{j=1}^{n-1}\sigma_j}{\sum_{j=1}^{n}\sigma_j}\Sigma^{(n,n)}\mathbf{1}_{n-1}\right)\Bigg) \\
&= \frac{1}{2}\left(\mathbf{1}_{(n-1)\times 1} - \left(\frac{n-1}{\sum_{j=1}^{n}\sigma_j} - \frac{1}{\sigma_n} + \frac{\sum_{j=1}^{n-1}\sigma_j}{\sigma_i\sum_{j=1}^{n}\sigma_j}\right)\operatorname{diag}(\Sigma)^{(n)}\right) \\
&= \frac{1}{2}\left(\mathbf{1}_{(n-1)\times 1} - \frac{1}{\sigma_n\sum_{j=1}^{n}\sigma_j}\left((n-1)\sigma_n - \sum_{j=1}^{n}\sigma_j + \sum_{j=1}^{n-1}\sigma_j\right)\operatorname{diag}(\Sigma)^{(n)}\right) \\
&= \frac{1}{2}\left(\mathbf{1}_{(n-1)\times 1} - \frac{n-2}{\sum_{j=1}^{n}\sigma_j}\operatorname{diag}(\Sigma)^{(n)}\right)
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
w_n^* &= 1 - \frac{1}{2}\mathbf{1}_{1\times(n-1)}\left(\mathbf{1}_{(n-1)\times 1} - \frac{n-2}{\sum_{j=1}^{n}\sigma_j}\operatorname{diag}(\Sigma)^{(n)}\right) \\
&= 1 - \frac{1}{2}\left((n-1) - \frac{n-2}{\sum_{j=1}^{n}\sigma_j}\sum_{j=1}^{n-1}\sigma_j\right) \\
&= 1 - \frac{1}{2}\left(\frac{(n-2)}{\sum_{j=1}^{n}\sigma_j}\sigma_n + 1\right) = \frac{1}{2}\left(1 - \frac{(n-2)}{\sum_{j=1}^{n}\sigma_j}\sigma_n\right).
\end{aligned}
\tag{18}
$$

Finally, from (17) and (18) formula (1) immediately follows. □

Proof of Proposition 9

Proof By construction, the matrix $\mathcal{R}_{\rho,X}$ is symmetric and can be rewritten as follows:

$$\mathcal{R}_{\rho,X} = DC_X D, \tag{19}$$

where $C_X \in \mathrm{Mat}_n(\mathbb{R})$ is the correlation matrix of $X_1, \ldots, X_n$ and $D \in \mathrm{Mat}_n(\mathbb{R})$ is the diagonal matrix with diagonal elements $\sqrt{\rho(X_1)}, \ldots, \sqrt{\rho(X_n)}$. From the hypothesis that X has full-rank, it follows that C_X is symmetric and positive definite. Further, the assumption $\rho(X_j) > 0$, for each $j \in \{1, \ldots, n\}$, yields $\mathrm{rank}(D) = n$. By applying (Horn & Johnson, 2012, Observation 7.1.8) to (19), we conclude that $\mathcal{R}_{\rho,X}$ is positive definite. □

Proof of Proposition 10

Proof It is enough to repeat the proof of Proposition 7 with $S = \mathcal{R}_{\rho,X}^{-1}$. □

8 Appendix

We consider the datasets DAX, ESX, FTSE, Nikkei, S&P500 constituents as described in Section 5.2.

8.1 The long-only case

Figures 8-12 report the values of the diversification measures and concentration indexes defined in Section 5.3 and computed for the portfolios in $\mathcal{P}_{\mathrm{LO}}^{\mathrm{DAX}}$, $\mathcal{P}_{\mathrm{LO}}^{\mathrm{ESX}}$, $\mathcal{P}_{\mathrm{LO}}^{\mathrm{FTSE}}$, $\mathcal{P}_{\mathrm{LO}}^{\mathrm{Nikkei}}$, $\mathcal{P}_{\mathrm{LO}}^{\mathrm{S\&Pconst.}}$ (see Section 5.3.1) for the datasets DAX, ESX, FTSE, Nikkei and S&P500 constituents (see Section 5.2), respectively. In each case, the most diversified portfolios are computed and represented on the mean-variance efficient frontier (see Figure 13, left panel). When the problem is restricted to the dominant part of the mean-variance efficient frontier, the most diversified portfolios are represented in Figure 13, right panel.

From a qualitative point of view, the present experiments confirm the features already observed for the S&P500 dataset (see Section 5.3.1): Figure 13 shows that, in some cases, the most diversified portfolios do not belong to the dominant part of the frontier. Consequently, restricting to the dominant part ends up with a sub-optimal portfolio. Furthermore, we observe some clusters of points almost overlapping on the frontiers, meaning that many measures identify very similar most diversified portfolios. Figure 8, 9, 10, 11 and 12 show that the measures CR_k, ρ_{AVG1}, ρ_{AVG2}, ρ_{AVG3}, ρ_{AVG4}, SE and DR have peculiar behaviors, significantly differing from the majority of the other measures; such phenomenon has already been highlighted and discussed in the application on the S&P500. We notice that, as n grows, the most diversified portfolios with respect to the majority of the measures tend to concentrate very close to the global minimum variance portfolio. The GDPMs behave coherently with the other measures.

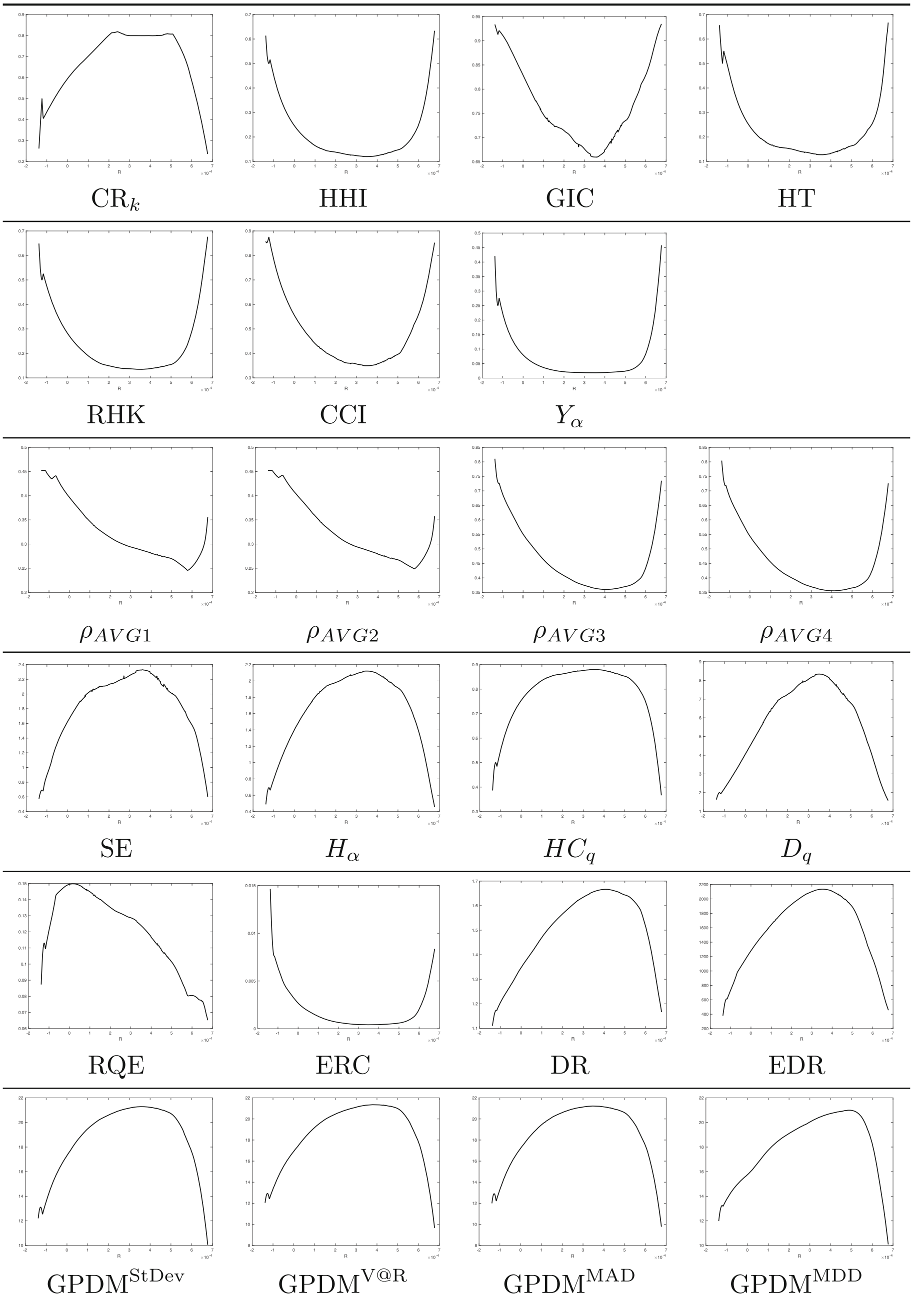

Fig. 8 The values of the various diversification measures and concentration indexes computed for the portfolios in $\mathcal{P}_{\mathrm{LO}}^{\mathrm{DAX}}$

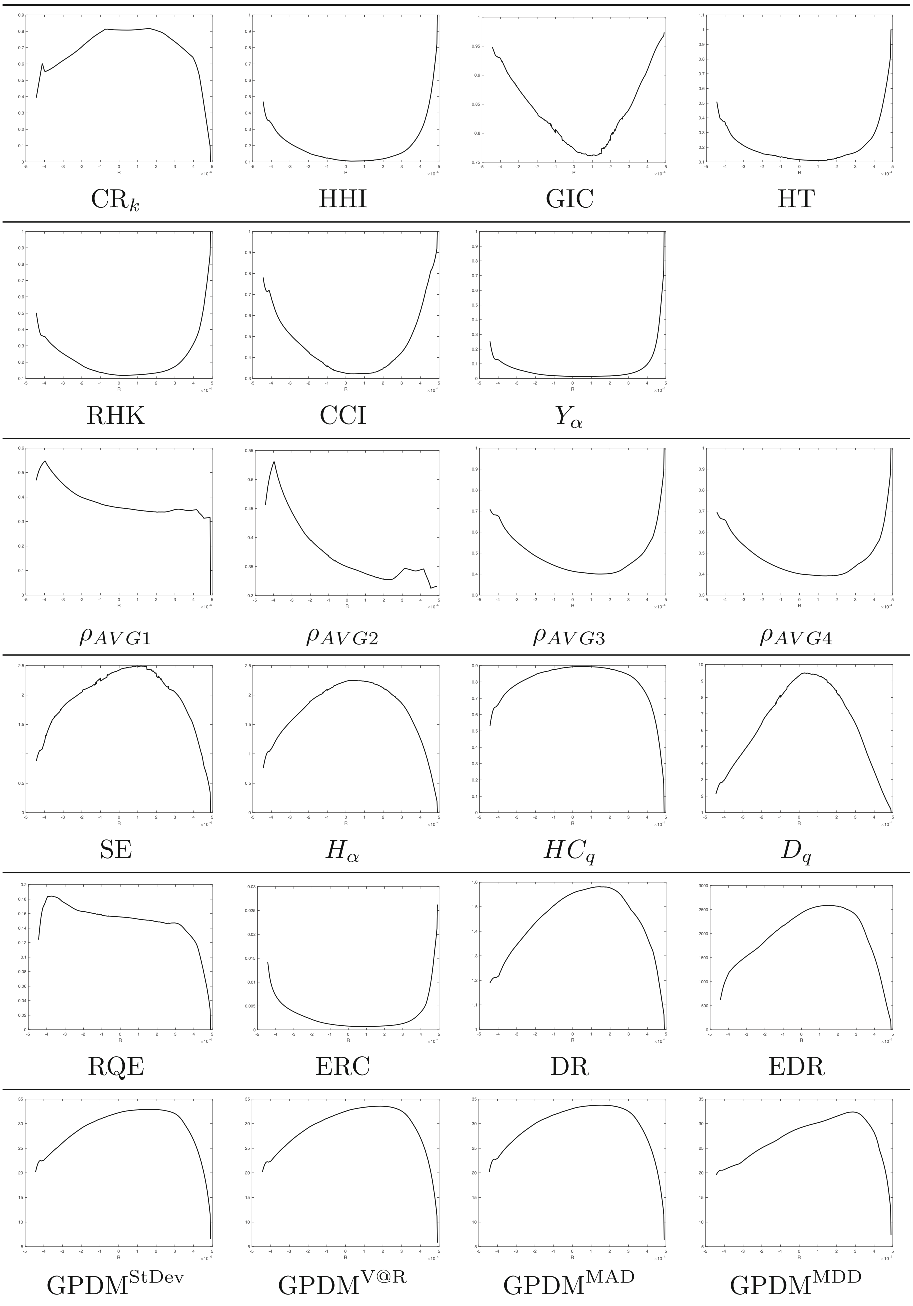

Fig. 9 The values of the various diversification measures and concentration indexes computed for the portfolios in $\mathcal{P}_{\mathrm{LO}}^{\mathrm{ESX}}$

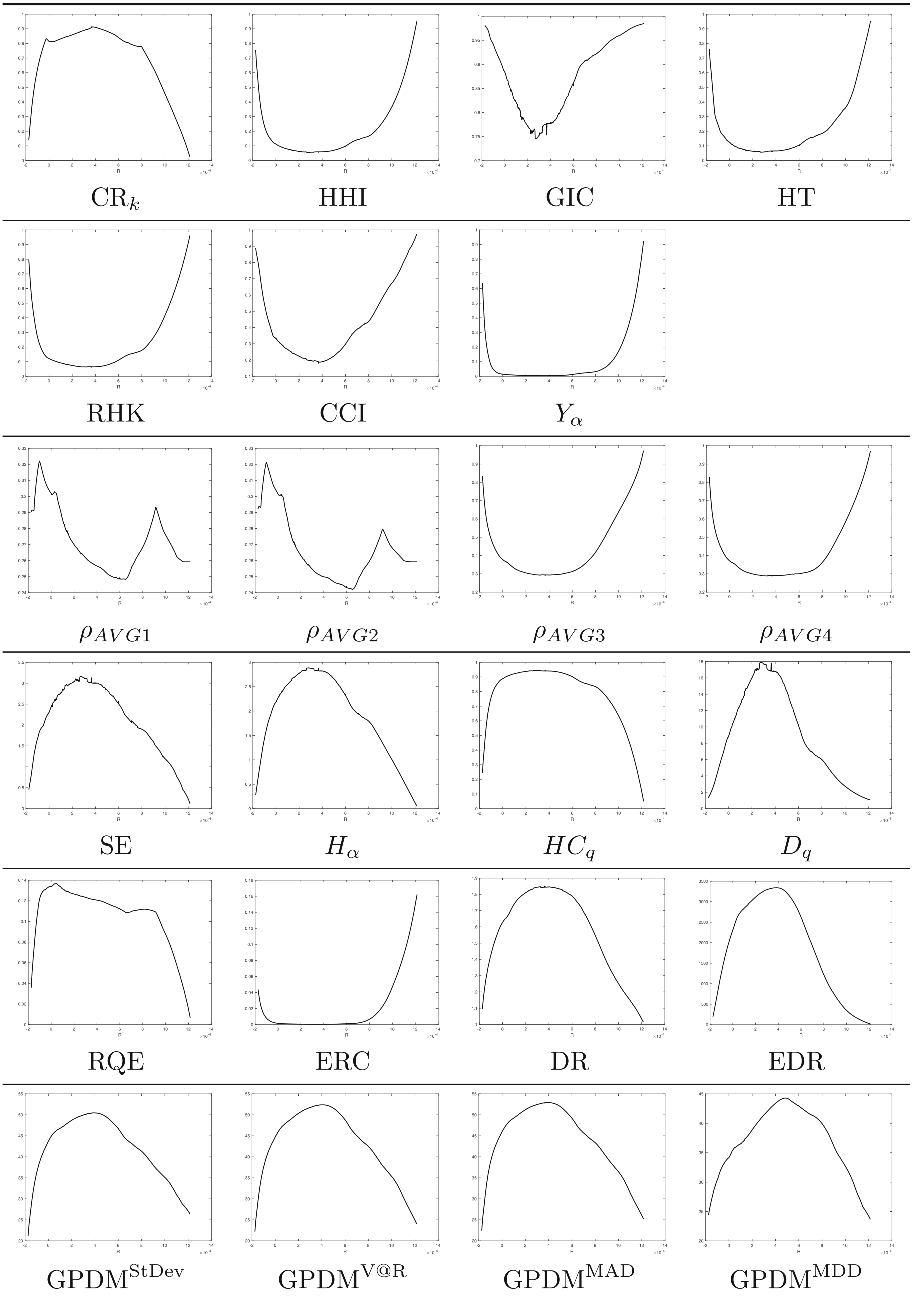

Fig. 10 The values of the various diversification measures and concentration indexes computed for the portfolios in $\mathcal{P}_{\mathrm{LO}}^{\mathrm{FTSE}}$

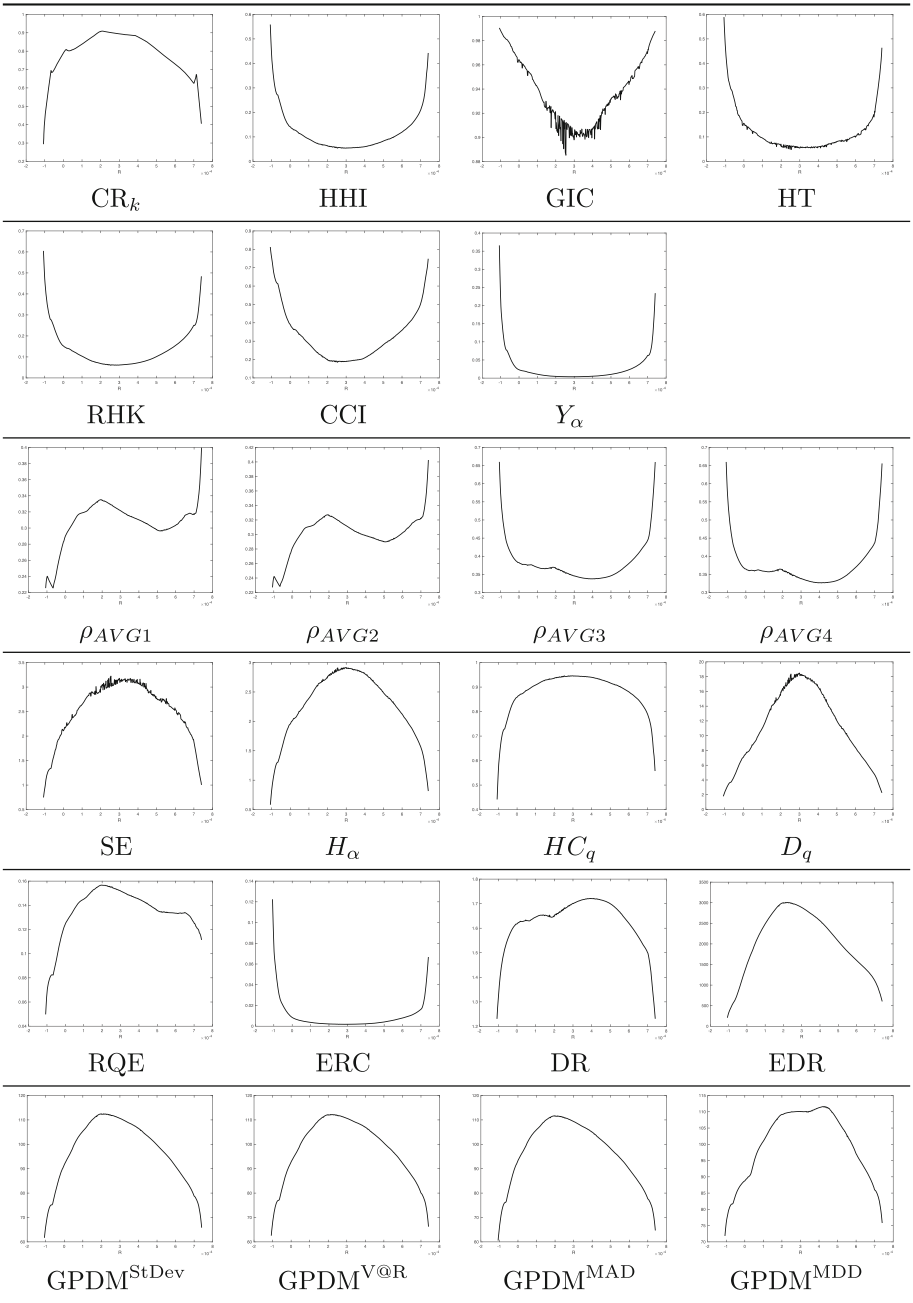

Fig. 11 The values of the various diversification measures and concentration indexes computed for the portfolios in $\mathcal{P}_{\mathrm{LO}}^{\mathrm{Nikkei}}$

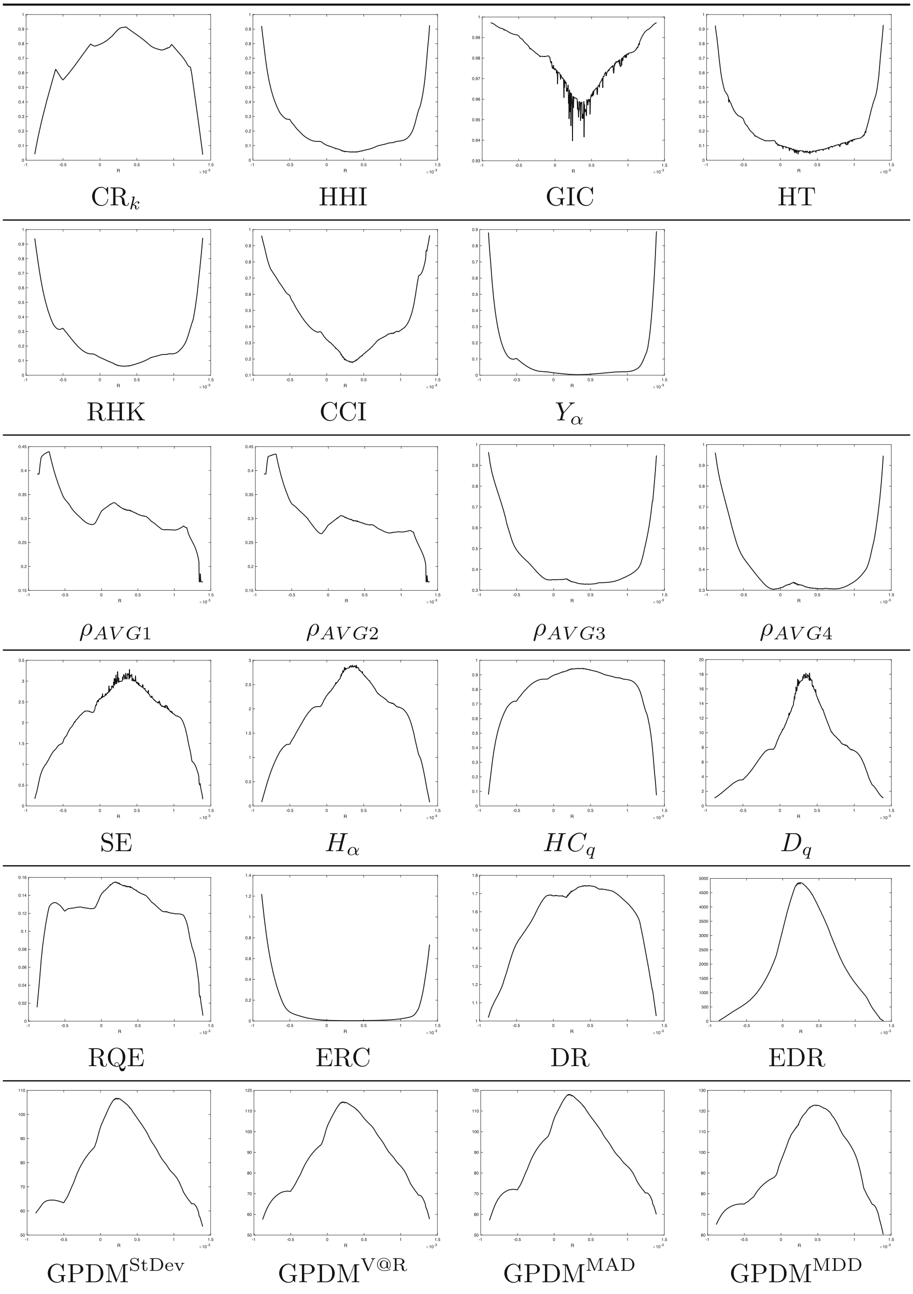

Fig. 12 The values of the various diversification measures and concentration indexes computed for the portfolios in $\mathcal{P}_{\mathrm{LO}}^{\mathrm{S\&Pconst.}}$

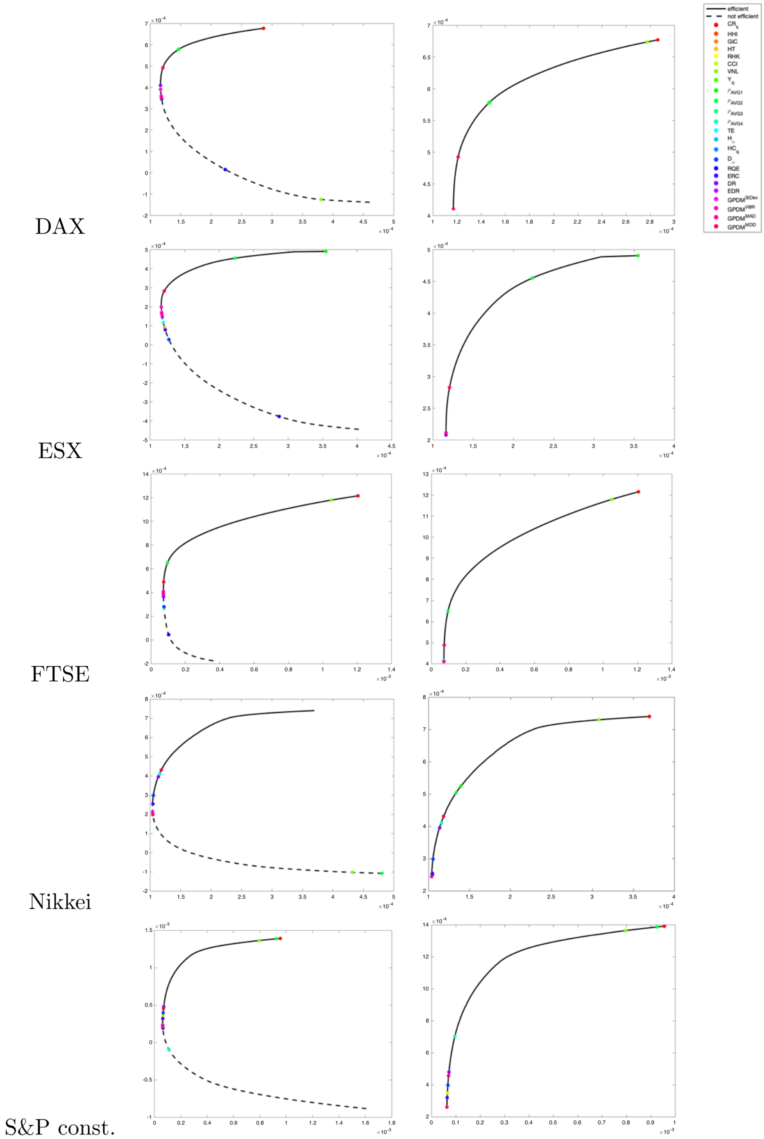

Fig. 13 Representation of the most diversified portfolios of the datasets DAX, ESX, FTSE, Nikkei, S&P const. for various diversification measures and concentration indexes on the relative mean-variance efficient frontier (left panel) and when restricted over the dominant part of the mean-variance efficient frontier (right panel)

8.2 The long-short case

Figure 14 reports the values of the GDPMs defined in Section 5.3 and computed for the portfolios in $\mathcal{P}_{\mathrm{LS}}^{\mathrm{DAX}}$, $\mathcal{P}_{\mathrm{LS}}^{\mathrm{ESX}}$, $\mathcal{P}_{\mathrm{LS}}^{\mathrm{FTSE}}$, $\mathcal{P}_{\mathrm{LS}}^{\mathrm{Nikkei}}$, $\mathcal{P}_{\mathrm{LS}}^{\mathrm{S\&Pconst.}}$ (see Section 5.3.2) for the datasets DAX, ESX, FTSE, Nikkei and S&P500 constituents (see Section 5.2). For each case, the most diversified portfolios are represented on the mean-variance efficient frontier (see Figure 15, left panel). When the problem is restricted to the dominant part of the mean-variance

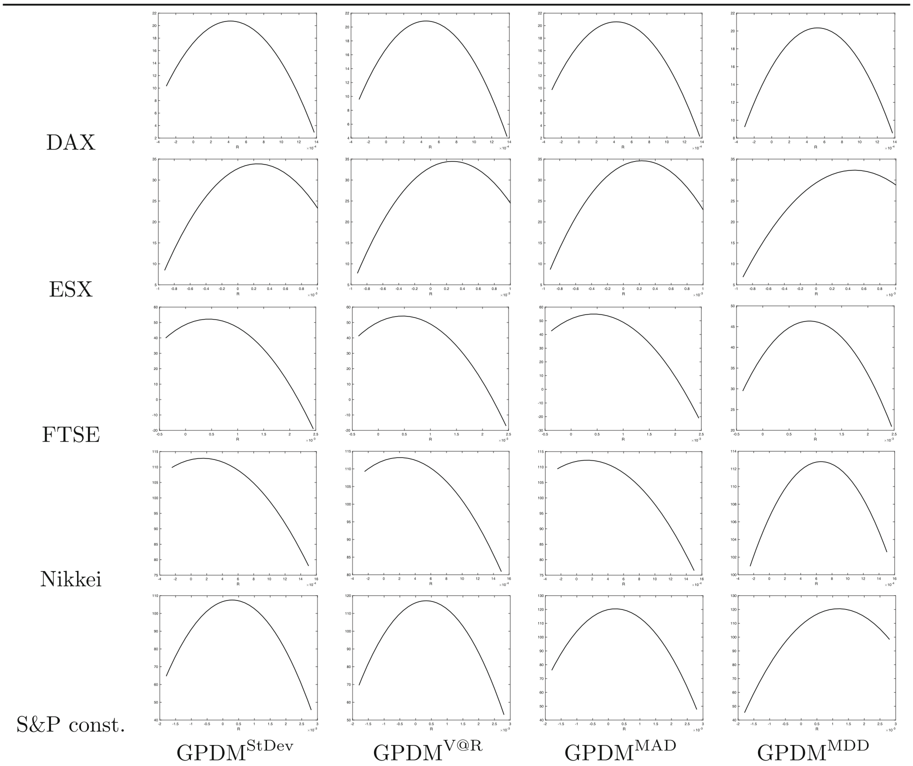

Fig. 14 The values of the various GDPMs computed for the portfolios in $\mathcal{P}_{\mathrm{LS}}^{\mathrm{DAX}}$, $\mathcal{P}_{\mathrm{LS}}^{\mathrm{ESX}}$, $\mathcal{P}_{\mathrm{LS}}^{\mathrm{FTSE}}$, $\mathcal{P}_{\mathrm{LS}}^{\mathrm{Nikkei}}$, $\mathcal{P}_{\mathrm{LS}}^{\mathrm{S\&Pconst}}$

efficient frontier, the most diversified portfolios are represented in Figure 15, right panel. The experiments confirm the tendency, already pointed out for the S&P500 index, of the most diversified portfolios to concentrate in a small area close to the global minimum variance portfolio. A different behavior characterizes GDPM$^{\mathrm{MDD}}$ since in this case the most diversified portfolio significantly differs from the global minimum variance and such a difference seems to grow as n increases. We think that this phenomenon could be related to a not negligible impact of portfolio's size n on the calculation of the maximum drawdown.

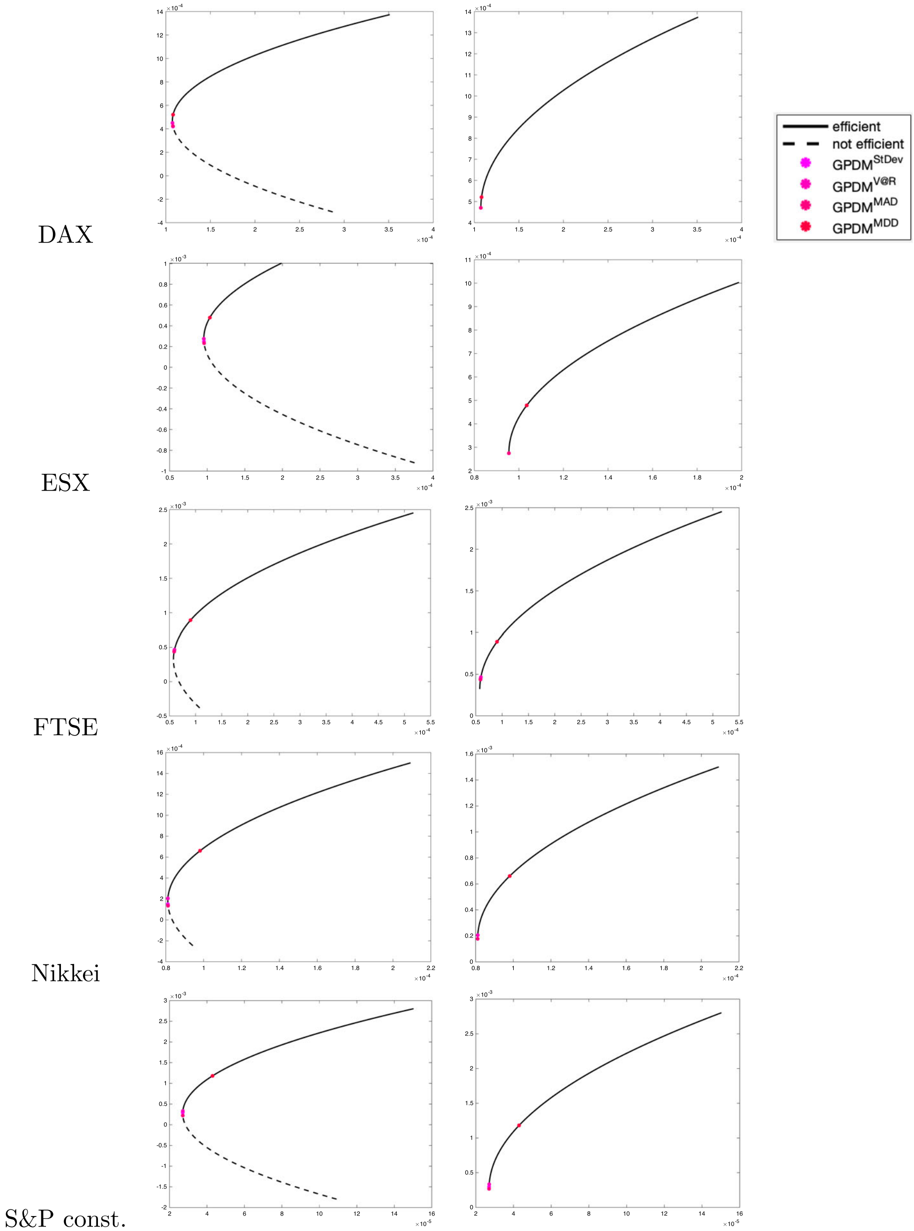

Fig. 15 Representation of the most diversified portfolios of the datasets DAX, ESX, FTSE, Nikkei, S&P constituents for various GPDMs on the mean-variance efficient frontier (left panel) and when restricted over the dominant part of the mean-variance efficient frontier (right panel)

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Data Availability The datasets analysed in the current study are available from the corresponding author on reasonable request.

Declarations

Competing interests The authors have no relevant financial or non-financial interests to disclose.

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