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A Note on Statistical Arbitrage and Long Term market Efficiency

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ABSTRACT

Market efficiency is a central topic in finance. The notion of statistical arbitrage is a suitable instrument to investigate market efficiency without the need to specify an equilibrium model. We introduce a new definition of statistical arbitrage (named Strong Statistical Arbitrage, SSA in the following) modifying the original definition in an apparently infinitesimal way. We show that some simple investment strategies, recognized as statistical arbitrages by the standard definition, do not test positive for SSA. We discuss the relations between the proposed definition and common definitions of arbitrage and prove that SSA is compatible with deviations from market efficiency in a "short term frame." The idea is that if market anomalies are small, the markets do not deviate significantly from efficiency, while an SSA requires time persistent anomalies on asset prices.

Keywords: Statistical Arbitrage, Market Efficiency

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Introduction

The notion of efficiency is central in financial economics. Asset prices are considered to be efficient when they discount relevant information[1,2]. The "efficiency property" defined on financial asset prices is then associated to the markets where the assets are traded. If prices are efficient we refer to *efficient markets*.

In an efficient market, no trading strategy can outperform the risk-adjusted market return. This argument is quite standard and can be found in a large number of papers and textbooks[3,4,5]. On the other hand, a branch of empirical literature provides several results against market efficiency[5,6]. Abnormal returns cyclically appear on financial markets (steep trends, bubbles, crisis,…), giving the chance to implement high profitable trading strategies. The evidence in this direction is strong. It is impossible to simply ascribe these phenomena as shortcomings in the underlying asset prices model.

Despite the empirical evidence of market deviations from efficiency, investors directly experience the difficulty of implementing profitable trading strategies with long run positive profits. The notion of statistical arbitrage (SA) plays a central role in this framework. An SA is a strategy allowing an extra-return and a vanishing variance; negative returns are allowed, but over a long time horizon they shall become negligible. On one hand, the SA notion represents an useful tool to investigate market efficiency. On the other hand, the goal of many investment strategies is to test positive for SA.

These facts seem to support the following idea: market deviations from efficiency in a short term period are frequent and possibly severe. It is easy to support the evidence on past financial data. Obviously, it is impossible to predict future market deviations from efficiency. Short time deviations from equilibrium have been also analyzed from the empirical point of view, among others, by Balke and Fomby (1997)[7].

There are different definitions of SA. The notion originates from econometrics works on cointegration and pair trading, see for example Gatev et al. (2006)[8] and, for a more recent review Avellaneda and Lee (2010)[12]. Some ways to formalize the SA are provided, for example, by Bondarenko (2003)[9], Hogan et al. (2004)[11], Jarrow et al. (2012)[10]. In this paper, we define the *strong statistical arbitrage* (SSA), as a correction of the definition of statistical arbitrage by Hogan et al. (2004)[11], Jarrow et al. (2012)[10]. We show how the absence of SSA can be compatible with a countable infinite number of standard arbitrage opportunities (short term horizon), being still valid asymptotically.

The paper is organized as follows. In Section 2 we introduce the definition of SSA. In Section 3 we investigate the relations between SSA and standard definitions of arbitrage and statistical arbitrage. Section 4 proposes a model describing a market and provides some conditions under which standard arbitrage opportunities are compatible with SSA. Section 5 concludes.

2 Strong Statistical Arbitrage

Define a strategy (current) value $V(t)$ as the cumulate of a profit/loss process π_t, as follows

$$V(t) = V(t-1) + \pi_t.$$

The present value of the strategy

$$v(t) = \frac{V(t)}{u(t)} \qquad (1)$$

is computed using the discount factor $\frac{1}{u(t)}$, with

$$\begin{aligned} u(t) &= \textstyle\prod_{k=0}^{t-1}(1 + r_k^f + \delta), \quad \delta \geq 0, \text{ or} \\ u(t) &= e^{\sum_{k=0}^{t-1}(r_k^f + \delta)}, \quad \delta \geq 0, \text{ or} \\ u(t) &= e^{\int_0^t (r^f(s)+\delta)\,\mathrm{d}s}, \quad \delta \geq 0, \end{aligned} \qquad (2)$$

where r_k^f is the risk free rate for the k^{th} period and δ is a spread applied to the risk free rate[a]. It worth nothing to note that the three versions of (2) just allow for different conventions on interest rate compounding. Remark that, if $u(t)$ grows

[a] Without loss of generality, δ is assumed to be constant, but it is possible to consider time varying and stochastic δ with some weak additional assumptions.

definitely faster than $V(t)$, then $\lim_{t\to+\infty} v(t) = 0$.

Definition 1 (SSA) *A strong statistical arbitrage is a zero initial cost, self-financing trading strategy* $(x(t) : t \geq 0)$ *with cumulative discounted value* $v(t)$, *computed applying (1) and (2) with* $\delta > 0$, *such that:*

1. $v(0) = 0$
2. $\lim_{t\to\infty} E[v(t)] > 0$
3. $\lim_{t\to\infty} P(v(t) < 0) = 0$, *or*
4. $\lim_{t\to\infty} \frac{\mathrm{Var}[v(t)]}{t} = 0 \quad if \quad P(v(t) < 0) > 0 \quad \forall t < \infty$

The SSA is a stronger version of the Statistical Arbitrage (SA) as proposed by Hogan et al. (2004)[11], Jarrow et al. (2012)[10]. In particular, with $\delta = 0$ the above definition boils down to the one of SA given by Hogan et al. (2004)[11]. The assumption of $\delta > 0$ implies that a strategy is a SSA only when its return is asymptotically larger than the risk free return. On the other hand, the choice of $\delta > 0$ avoids some trivial strategies to be detected as an SSA: for example, the investment into the money account of the proceeds a successful bet[2] [b](i.e. a take profit) or a standard arbitrage. To be more precise, consider the occurring of an arbitrage opportunity and a simple strategy which invests at the risk free rate r^f the proceeds of that arbitrage and hold this position. It is easy to see that this strategy does not verify condition 2 of Definition 1. From a practical point of view, we propose to interpret the δ parameter as the bid-ask spread on the risk free rate. This proposal allows to link the notion of efficiency to the one of liquidity of a market. The higher the bid-ask spread (low liquidity) the more difficult is to set up an SSA. This is only a possibility, other choices may be considered. However, we think that the connection between the SSA opportunities and a liquidity measure can add a useful interpretation of the results shown in this paper.

A market without SSA opportunities can be defined as "Long Term Efficient." The assumption $\delta > 0$ avoids that the presence of standard arbitrage opportunities in a short term period implies the existence of an SSA strategy. In other words, a market showing short term deviations from efficiency can asymptotically tend to "Long Term Efficiency."

3 Relations Between Arbitrage Definitions

In this section we investigate the relations between the proposed notion of SSA and the ones of SA and standard arbitrage (A) [c]. Moreover, we provide an example of a market where a countable number of A opportunities can be consistent with the absence of SSA.

The following relations hold for A, SA and SSA :

1. $A \underset{\not\Leftarrow}{\overset{\Rightarrow}{}} SA$
   For more details see Hogan et al. (2004)[11].

2. $SSA \underset{\not\Leftarrow}{\overset{\Rightarrow}{}} SA$
   By definition, if the conditions of Definition 1 hold for $\delta > 0$, then they still hold for $\delta = 0$. The opposite is obviously not true.

3. $A \underset{\not\Leftarrow}{\overset{\not\Rightarrow}{}} SSA$
   The investment into the money account of the proceeds of a standard arbitrage does not satisfy condition 2 of Definition 1. The second relation can be proven using the same argument by Hogan et al. (2004, p. 533)[11].

4 A model

To show with more detail the meaning and the implications of the definition of SSA and of Long Term Efficiency, in this section we propose the example of a theoretical market. Let us model a market where standard arbitrage opportunities appear with a given regularity. Let $N(t)$ be a counting process describing the arrival times of

[b] In fact, this kind of strategies can be detected as SA, being (ex-post) non distinguishable from an SA. Note that a standard arbitrage is a special case of SA. Instead, (ex-ante) a successful bet should not be classified as an SA.

[c] As usual, we denote a standard arbitrage as (A) a single-period strategy such that the initial value is null, the expected final value is positive and the probability of negative final value is null.

the standard arbitrage opportunities. Therefore, we assume that in this market a countable infinite number of standard arbitrages can take place.

Let $N(t)$ be a counting process such that $\lim_{t\to+\infty} \frac{E[N(t)]}{t} = l < \infty$, $\{M_i\}$ is a sequence of IID non negative random variables, with finite first moment. In addition, $N(t)$ and M_i are assumed to be independent for all t and i.

In our case, the process $N(t)$ models the arrival time of arbitrages whereas M_i de-scribes the size in terms of (log) return in excess with respect to the risk-free rate r^f of the one-period arbitrage occurring at t_i and yielding a profit at time $t_i + 1$. (intuitively, $N(t)$ models a frequency and M_i a magnitude). In other words, the considered arbitrages can be described by risk free assets that yield for one period (i.e. from t_i to $t_i + 1$) a return larger than r^f.

We propose the following investment strategy W: invest in the risk free asset and take profit from every standard arbitrage; hold a short position in the risk free asset. By construction, this strategy is self financing: i.e. the initial value is null.

In this settings, the value of the proposed strategy at time t and its discounted value are respectively

$$W(t) = e^{r^f t + A(t)} - e^{-r^f t} \qquad \text{and} \qquad w(t) = \frac{W(t)}{e^{(r^f+\delta)t}},$$

where $A(t) = \sum_{i=1}^{N(t)} M_i$ is the compound Poisson process accounting for the additional profits given by the standard arbitrages.

Proposition 1 *If* $(E[M_i]l - \delta) \geq 0$ *, then the strategy W defines an SSA.*

Proof. According to Definition 1: property 1 is satisfied by construction ($W(0) = w(0) = 0$); property 3 directly follows from the non-negativity of M_i.

Thanks to Jensen's Inequality,

$$\lim_{t\to+\infty} E[w(t)] = \lim_{t\to+\infty} E\left[e^{A(t)-\delta t} - e^{-\delta t}\right] \geq \lim_{t\to+\infty} e^{E[A(t)]-\delta t} - \lim_{t\to+\infty} e^{-\delta t} =$$
$$= \lim_{t\to+\infty} e^{\left(E[M_i]\frac{E[N(t)]}{t}-\delta\right)t} = \lim_{t\to+\infty} e^{(E[M_i]l-\delta)t}$$

which is positive if $E[M_i]l - \delta \geq 0$, so that property 2 of Definition 1 is satisfied.

This result highlights that the existence of even an infinite number of standard arbitrage opportunities is not sufficient to obtain an SSA. An SSA requires a market where the anomalies are both large (in terms of average size $E[M_i]$) and persistent (in terms of arrival intensity l), so that the condition $E[M_i]l - \delta \geq 0$ is satisfied.

Let us consider a special case, where $A(t)=\alpha N^P(t)$, with $\alpha > 0$ and $N^P(t)$ is a Poisson process with intensity parameter λ. In other words, we consider a constant arbitrage single-period excess return a. The proposed strategy has now the following values

$$W(t) = e^{aN^P(t)+r^f t} - e^{r^f t} \qquad \text{and} \qquad w(t) = e^{aN^P(t)-\delta t}.$$

In this case, we obtain a necessary and sufficient condition for W representing an SSA. Restricting to a specific formalization for $A(t)$ we obtain a stronger result compared to proposition 1.

Proposition 2 *The strategy W is an SSA if and only if* $[\alpha\lambda(e - 1) - \delta] \geq 0$.

Proof. According to Definition 1: property 1 is satisfied by construction ($W(0) = w(0) = 0$); property 3 directly follows from $\alpha > 0$. Take the limit of the expected value of $w(t)$

$$\lim_{t\to+\infty} E[w(t)] = \lim_{t\to+\infty} E\left[e^{aN^P(t)-\delta t}\right] = \lim_{t\to+\infty} e^{[a\lambda(e-1)-\delta]t},$$

where the last equality follows from the property of the Poisson process

$$E\left[e^{N^P(t)}\right] = \sum_{n=0}^{+\infty} e^{n} \frac{e^{-\lambda}\lambda^n}{n!} = e^{\lambda(e-1)}$$

Property 2 of Definition 1 is then satisfied if and only if $[\alpha\lambda(e-1)-\delta] \geq 0$.

Proposition 2 shows that, with some additional assumptions it is possible to obtain a stronger relation between the strategy W and the SSA. In fact, when $[\alpha\lambda(e-1)-\delta] \geq 0$ the strategy W is an SSA, as in proposition 1. Moreover, if $[\alpha\lambda(e-1)-\delta] < 0$, cumulating the profit of an infinite number of arbitrages does not lead to an SSA.

The value of δ is central in this analysis. The assumption of risk-free discounting made by Hogan et al. (2004)[11] can be relaxed, incorporating in the valuation process a spread given by, e.g., limited liquidity and market frictions.

5 Conclusions

The introduction of the notion of Strong Statistical Arbitrage permits to analyze the financial markets' behavior in a more realistic way. According to investors experience, financial markets show significant deviations from efficiency in a short term period. On the other hand, the development and implementation of a trading strategy able to provide stable long term profits (for example what we define an SSA) is still an elusive problem. The lack in the paper of an empirical analysis based on real data is justified by the logical structure of the market efficiency puzzle: if we want to prove market inefficiency it is enough to provide one single example of inefficiency while, if we want to support market efficiency, we would need to show that none of the infinite possible trading strategies provide long term stable profits. For this reason, we related market efficiency to the absence of SSA, obtaining a more inclusive notion of market efficiency which allows for the presence of standard arbitrage opportunities. Moreover, following our proposed interpretation of the parameter δ, it is possible to deduce that the less the market is liquid, the more market anomalies are consistent with the absence of SSA.

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