LOCAL RIESZ TRANSFORM AND LOCAL HARDY SPACES ON RIEMANNIAN MANIFOLDS WITH BOUNDED GEOMETRY

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Abstract. We prove that if τ is a large positive number, then the atomic Goldberg-type space $\mathfrak{h}^1(N)$ and the space $\mathfrak{h}^1_{\mathscr{R}_\tau}(N)$ of all integrable functions on N whose *local Riesz transform* $\mathscr{R}_\tau$ is integrable are the same space on any complete noncompact Riemannian manifold N with Ricci curvature bounded from below and positive injectivity radius. We also relate $\mathfrak{h}^1(N)$ to a space of harmonic functions on the slice $N \times (0,\delta)$ for $\delta > 0$ small enough.

1. Introduction

The classical Hardy space $H^1(\mathbb{R}^n)$ plays an important role in Euclidean Harmonic Analysis and has been the object of a huge number of investigations. Its theory, which is available also in book form (see, for instance, [56, 32]), is well understood, and has its roots in the seminal papers [26, 58]. In the first, C. Fefferman and E.M. Stein proved, amongst other important results, that $H^1(\mathbb{R}^n)$ can be equivalently defined in terms of the Riesz transforms, of various kinds of maximal operators and square functions. In the second, Stein and G. Weiss considered a space of generalised conjugate harmonic functions that may be identified with $H^1(\mathbb{R}^n)$. Their results were complemented by R.R. Coifman [12] and R. Latter [40], who proved that $H^1(\mathbb{R}^n)$ admits an atomic decomposition. All these characterisations corroborate the idea that the space $H^1(\mathbb{R}^n)$ is central in Harmonic Analysis and illustrate its flexibility, a feature of great importance in the applications. This beautiful theory, or part of it, has been extended in various directions: see, for instance, [13, 28, 11, 50, 25, 22, 23, 21, 33, 19, 4, 62, 38, 1, 3] and the references therein. We observe in passing that on certain examples of nondoubling

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measure spaces, such as the hyperbolic disc, a perhaps surprising phenomenon occurs: the Hardy-type spaces defined in terms of the Riesz transform, the Poisson maximal operator and the heat maximal operator are different spaces [46]. For more on the attempts to define an effective Hardy-type space on noncompact symmetric spaces and generalisations thereof, see [2, 39, 42, 6, 48, 49, 44] and the references therein.

The major drawback of $H^1(\mathbb{R}^n)$ is that it is not "stable" under localisation, i.e., multiplication by smooth functions of compact support does not preserve $H^1(\mathbb{R}^n)$. This fact induced D. Goldberg [31] to introduce a variant of $H^1(\mathbb{R}^n)$, denoted by $\mathfrak{h}^1(\mathbb{R}^n)$ and quite often termed "local Hardy space". It is fair to recall that R.S. Strichartz [59] had defined a suggestive predecessor of $\mathfrak{h}^1(\mathbb{R}^n)$ on any compact Riemannian manifold. Goldberg proved several characterisations of $\mathfrak{h}^1(\mathbb{R}^n)$, which are the natural "local" counterparts of many of those known for $H^1(\mathbb{R}^n)$. They include characterisations via several different maximal operators, local Riesz transforms, and an atomic decomposition. It includes also a characterisation of $\mathfrak{h}^1(\mathbb{R}^n)$ in terms of a generalised system of conjugate harmonic functions on the slice $\mathbb{R}^n \times (0,1)$.

A careful reading of [31] reveals that most of the properties of $\mathfrak{h}^1(\mathbb{R}^n)$ depend only on the local structure of the Euclidean space and not on its geometry at infinity. Thus, it is natural to speculate whether one can define an analogue of $\mathfrak{h}^1(\mathbb{R}^n)$ on "locally Euclidean spaces".

Interesting examples of such spaces are the so-called RD-spaces, i.e., homogeneous spaces X in the sense of Coifman and Weiss with the additional property that a reverse doubling condition holds in X. Following up previous works of various authors [22, 23, 33], Dachun Yang and Yuan Zhou [65, 66] constructed on such spaces an interesting and quite complete theory of "local Hardy spaces" associated to given admissible functions. See also [36] for results concerning Triebel–Lizorkin spaces on RD-spaces and their relationships with local Hardy spaces. In particular, note that if N is an RD-space, then the local Hardy space $\mathfrak{h}^1(N)$ defined below reduces to the space $H^{1,2}_\ell(N)$ of [65].

Further important examples of "locally Euclidean spaces" are Riemannian manifolds. A subclass thereof on which a satisfactory theory of local Hardy spaces can be developed is that of manifolds N with *bounded geometry*. By this we mean that N is a complete connected noncompact Riemannian manifolds with Ricci curvature bounded from below and positive injectivity radius. Notice that the Riemannian measure on N may very well be nondoubling. In analogy with the classical case [31], one can define a number of spaces on N, including $\mathfrak{h}^1_{\max}(N)$, $\mathfrak{h}^1_{\mathscr{H}}(N)$, $\mathfrak{h}^1_{\mathscr{P}}(N)$, $\mathfrak{h}^1_{\mathrm{I}}(N)$, $\mathfrak{h}^1_{\mathrm{at}}(N)$: specifically, $\mathfrak{h}^1_{\max}(N)$, $\mathfrak{h}^1_{\mathscr{H}}(N)$, $\mathfrak{h}^1_{\mathscr{P}}(N)$ are defined in terms of maximal functions (associated to a suitable grand maximal operator, to the local heat maximal operator and to the local Poisson maximal operator, respectively), $\mathfrak{h}^1_{\mathrm{I}}(N)$ and $\mathfrak{h}^1_{\mathrm{at}}(N)$ are ionic and atomic spaces, respectively. It may be worth observing that $\mathfrak{h}^1_{\mathrm{I}}(N)$ can be equivalently defined using various kinds of ions, and similarly for $\mathfrak{h}^1_{\mathrm{at}}(N)$, but with atoms playing the role of ions. For the sake of simplicity, we do not insist on this point in the introduction.

In the inspiring paper [60], M. Taylor proved, under the additional assumption that all the derivatives of the metric tensor are bounded, that $\mathfrak{h}^1_{\max}(N) = \mathfrak{h}^1_{\mathrm{I}}(N)$. In a much wider context that includes Riemannian manifolds with bounded geometry in the sense specified above, Meda and S. Volpi [51] introduced the space $\mathfrak{h}^1_{\mathrm{at}}(N)$, and proved that $\mathfrak{h}^1_{\mathrm{at}}(N) = \mathfrak{h}^1_{\mathrm{I}}(N)$. It is fair to say that both [60] and [51] contain many additional material, including duality and interpolation results and boundedness criteria for relevant (pseudo-) differential operators on N. Quite recently, A. Martini, Meda and M. Vallarino [45], following up a profound result of A. Uchiyama [63], showed that if N has bounded geometry, then $\mathfrak{h}^1_{\max}(N) = \mathfrak{h}^1_{\mathscr{H}}(N) = \mathfrak{h}^1_{\mathscr{P}}(N) = \mathfrak{h}^1_{\mathrm{I}}(N)$ (see also [65, 66] for related results in the setting of RD-spaces). Consequently, the five spaces listed above coincide (and their norms are equivalent); for the sake of brevity, we denote simply by $\mathfrak{h}^1(N)$ the resulting space, equipped with any of the corresponding norms.

A further natural local Hardy space on N may be defined as follows. Denote by ∇ the covariant derivative on N, and by $\mathscr{L}$ (minus) the Laplace–Beltrami operator, which we think of as an unbounded nonnegative operator on $L^2(N)$. For each positive number τ, denote by $\mathscr{L}_\tau$ the translated Laplacian $\tau\mathscr{I} + \mathscr{L}$. We consider the *translated Riesz transform* $\mathscr{R}_\tau := \nabla\mathscr{L}_\tau^{-1/2}$, $\tau > 0$, and the *Riesz–Goldberg* space

$$\mathfrak{h}^1_{\mathscr{R}_\tau}(N) := \big\{ f \in L^1(N) : |\mathscr{R}_\tau f| \in L^1(N) \big\}. \tag{1.1}$$

We equip $\mathfrak{h}^1_{\mathscr{R}_\tau}(N)$ with the norm $\|f\|_{\mathfrak{h}^1_{\mathscr{R}_\tau}(N)} := \|f\|_1 + \big\||\mathscr{R}_\tau f|\big\|_1$. E. Russ [53, proof of Theorem 14] (see also [51, Theorem 8]) proved that if τ is large enough, then $\mathfrak{h}^1_{\mathscr{R}_\tau}(N) \supseteq \mathfrak{h}^1(N)$ on a class of Riemannian manifolds that include those of bounded geometry. It is then natural to speculate whether $\mathfrak{h}^1_{\mathscr{R}_\tau}(N)$ agrees with $\mathfrak{h}^1(N)$ in this generality, thereby extending the result for $\mathbb{R}^n$ proved by Goldberg via Fourier transform techniques. We remark that the Riesz transform $\nabla\mathscr{L}^{-1/2}$ (which corresponds to the limiting case where $\tau = 0$) is unbounded from $\mathfrak{h}^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

In this paper, we answer to this deceptively simple question in the affirmative. Our main result, Theorem 7.9, states that if N is a complete connected noncompact Riemannian manifold with bounded geometry, then $\mathfrak{h}^1_{\mathscr{R}_\tau}(N) = \mathfrak{h}^1(N)$ as long as τ is large enough. Our strategy of proof has its roots in an old and beautiful idea of Stein and Weiss (see, in particular, [58, Theorem A]), who realised that certain powers (slightly below 1) of the gradient of harmonic functions are subharmonic. This idea is central in the classical proof that if u is a harmonic function on $\mathbb{R}^{n+1}_+ := \big\{(x,t) \in \mathbb{R}^n \times (0,\infty) : t > 0\big\}$, then

$$\big\|\partial_t u_{|\mathbb{R}^n\times\{0\}}\big\|_{H^1(\mathbb{R}^n)} \asymp \big\||\nabla u|^*\big\|_{L^1(\mathbb{R}^n)} \asymp \sup_{t>0} \int_{\mathbb{R}^n} \big|\nabla u(x,t)\big| \,\mathrm{d}x, \tag{1.2}$$

where $\mathbb{\nabla}$ denotes the gradient on $\mathbb{R}^{n+1}$ and the superscript $*$ stands for nontangential maximal function (see, for instance, [55, Ch. VII]). This result has a natural counterpart for $\mathfrak{h}^1(\mathbb{R}^n)$ [31], where the slice $\mathbb{R}^n \times (0,1)$ plays the role of $\mathbb{R}^{n+1}_+$ in the classical case.

There is a major problem in extending the latter result to Riemannian manifolds: if the curvature of N is not nonnegative, then powers (≤ 1) of the gradient of harmonic functions on $N \times \mathbb{R}$ may not be subharmonic. M. Dindoš [19, Chapter 6] was able to overcome this problem and to work out an effective strategy (modifying significantly that of Stein and Weiss) to prove an analogue of (1.2) on bounded domains of (compact) manifolds, endowed with a possibly nonsmooth metric. His strategy hinges on the observation, derived from the Bochner–Weitzenbock formula, that if u is a harmonic function on an open set of an $(n+1)$-dimensional Riemannian manifold M with Ricci curvature bounded from below by $-\kappa^2$ and $(n-1)/n < q \leq 1$, then $\big|\nabla u\big|^q$ is $q\kappa^2$-subharmonic, i.e., it satisfies an inequality of the form $\mathscr{L}|\nabla u|^q \leq q\kappa^2\, |\nabla u|^q$.

We adapt Goldberg's approach and extend Dindoš' strategy to the case of noncompact Riemannian manifolds N of bounded geometry. We consider the slice $\Sigma := N \times (0, 2\sigma)$, and prove that if σ is small enough (see (3.2)), then a harmonic function in Σ satisfies the maximal inequality $\int_N \big|\mathbb{\nabla} u(x)\big|^* \,\mathrm{d}\nu(x) < \infty$ if and only if $\sup_{t\in(0,2\sigma)} \int_N \big|\mathbb{\nabla} u(x,t)\big| \,\mathrm{d}\nu(x) < \infty$ and $|\mathbb{\nabla} u|$ tends to 0 at infinity, uniformly in each closed subslice of Σ (see Theorem 6.3). Here ν, $\mathbb{\nabla}$ and * denote the Riemannian density, the gradient of $N \times \mathbb{R}$ and an appropriate nontangential maximal function (defined at the beginning of Section 5), respectively. Our strategy requires estimating the Poisson operator and powers of the Green operator associated to Σ. In particular, we show that if σ is small enough, then the integral kernels of such operators are "integrable at infinity in Σ" (see Sections 3 and 4). Their rate of decay at infinity is controlled by $\lambda_1 := \pi/(2\sigma)$. Notice that $-\lambda_1^2$ is the first eigenvalue of the Dirichlet Laplacian on the interval $[0, 2\sigma]$. Clearly λ_1 increases as σ decreases: this is the reason for which we choose σ small.

The last ingredient we need in the proof our main result is a careful analysis of the kernel of the translated Riesz transform $\mathscr{R}_\tau$. This technical part is confined in Section 7.

The paper is organized as follows. Section 2 contains some preliminary estimates extensively used in the sequel. In Sections 3 and 4 we establish some potential estimates on Σ. Section 5 contains some maximal estimates for certain potentials on Σ. Section 6 is devoted to the analogue on slices of Σ of certain results of Dindoš [19]. The analysis of the local Riesz transform for Riemannian manifolds with bounded geometry, together with some basic information concerning the Goldberg-type space $\mathfrak{h}^1(N)$, is contained in Section 7, where our main result concerning $H^1_{\mathscr{R}_\tau}(N)$, Theorem 7.9, is proved.

We shall use the "variable constant convention", and denote by C, possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards.

Throughout the paper, given p in $[1,\infty]$, we denote by p' the conjugate exponent of p.

2. Background material and preliminary estimates

2.1. **Standing assumptions.** In this paper N will always denote an n-dimensional complete connected noncompact Riemannian manifold with **bounded geometry**. By this we mean that the Ricci curvature of N satisfies $\mathrm{Ric}_N \geq -\kappa^2$ for some nonnegative number κ, and the injectivity radius is strictly positive. The Riemannian measure of N will be denoted by ν. The operator norm of a bounded linear operator T from $L^p(N)$ to $L^q(N)$ will be denoted by $|\!|\!| T |\!|\!|_{p;q}$.

Denote by ∇ and Δ the gradient and the (negative) Laplace–Beltrami operator on N, respectively. Set $\mathscr{L} = -\Delta$. The operator $\mathscr{L}$, initially defined on smooth functions with compact support, admits a unique self adjoint extension, still denoted by $\mathscr{L}$, in $L^2(N)$. For any nonnegative number τ denote by $\mathscr{L}_\tau$ the operator $\tau\mathscr{I}+\mathscr{L}$, where $\mathscr{I}$ denotes the identity operator. In particular, $\mathscr{L}_0 = \mathscr{L}$. Denote by $\mathscr{H}_t^N$ and h_t^N the heat semigroup $\mathrm{e}^{-t\mathscr{L}}$ and the corresponding heat kernel, respectively. The following are well known consequences of our assumptions:

(i) N is *locally Ahlfors regular*. Indeed, by Bishop-Gromov's volume comparison theorems and by a well known estimate due to C.B. Croke [16, Prop. 14], for each $R > 0$ there exist positive constants C_1 and C_2 such that

$$C_1\, r^n \leq \nu\big(B_r(x)\big) \leq C_2\, r^n \qquad \forall x \in N \quad \forall r \in (0,R]. \tag{2.1}$$

Thus, in particular, ν is *locally doubling*. Furthermore, there exist nonnegative constants α and β and C such that

$$\nu\big(B_r(x)\big) \leq C\, r^\alpha\, \mathrm{e}^{2\beta r} \qquad \forall x \in N \quad \forall r \in [1,\infty); \tag{2.2}$$

(ii) there exist positive constants c and C such that

$$h_t^N(x,y) \leq C\,\gamma(t)\,\mathrm{e}^{-cd(x,y)^2/t} \qquad \forall x,y \in N \quad \forall t > 0, \tag{2.3}$$

where $\gamma(t) := \max(t^{-n/2}, t^{-1/2})$ (see, for instance, [8, Theorem 3]). Note that (2.3) directly implies the following *ultracontractivity estimate* for $\mathscr{H}_t^N$:

$$|\!|\!| \mathscr{H}_t^N |\!|\!|_{1;\infty} \leq C\,\gamma(t) \qquad \forall t > 0. \tag{2.4}$$

Suppose now that $1 \leq q \leq r \leq \infty$. Then there exists a constant C such that

$$|\!|\!| \mathscr{H}_t^N |\!|\!|_{q;r} \leq C\,\gamma(t)^{1/q-1/r} \qquad \forall t > 0. \tag{2.5}$$

The estimate (2.5) follows from (2.4), the contractivity of $\mathscr{H}_t^N$ on $L^p(N)$ for all p in $[1,\infty]$, duality and interpolation;

(iii) *Bakry's condition* [5]

$$|\nabla \mathscr{H}_t^N f| \leq \mathrm{e}^{\kappa^2 t}\, \mathscr{H}_t^N\big(|\nabla f|\big) \qquad \forall t > 0 \tag{2.6}$$

holds.

2.2. **Ultracontractivity estimates for generalised Bessel potentials.** Propositions 2.1, 2.2 and 2.3 contain some basic estimates for certain (families of) operators that will arise frequently in the sequel. It is convenient to set $\mathscr{D} := \sqrt{\mathscr{L}}$.

Proposition 2.1. *For any pair of numbers $\tau \geq \kappa^2$ and $\rho > 0$*

$$\big|\nabla(\tau\mathscr{I} + t^2\mathscr{D}^2)^{-\rho} f\big| \leq \big((\tau-\kappa^2)\mathscr{I} + t^2\mathscr{D}^2\big)^{-\rho}\big(|\nabla f|\big) \qquad \forall t \in (0,1].$$

Proof. The subordination formula

$$(\tau\mathscr{I} + t^2\mathscr{D}^2)^{-\rho} f = \frac{1}{\Gamma(\rho)} \int_0^\infty s^\rho\, \mathrm{e}^{-\tau s}\, \mathscr{H}_{st^2} f\, \frac{\mathrm{d}s}{s} \tag{2.7}$$

and Bakry's condition (2.6) imply that

$$\begin{aligned} \big|\nabla(\tau\mathscr{I} + t^2\mathscr{D}^2)^{-\rho} f\big| &\leq \frac{1}{\Gamma(\rho)} \int_0^\infty s^\rho\, \mathrm{e}^{-\tau s}\, \big|\nabla \mathscr{H}_{st^2} f\big|\, \frac{\mathrm{d}s}{s} \\ &\leq \frac{1}{\Gamma(\rho)} \int_0^\infty s^\rho\, \mathrm{e}^{-(\tau-\kappa^2 t^2)s}\, \mathscr{H}_{st^2}\big|\nabla f\big|\, \frac{\mathrm{d}s}{s} \\ &\leq \big((\tau-\kappa^2)\mathscr{I} + t^2\mathscr{D}^2\big)^{-\rho}\big|\nabla f\big|, \end{aligned}$$

as required. □

Part of the proof of the next proposition is an adaptation of the proof of [47, Proposition 2.2 (i)]. Given a nonnegative number ρ and a function $G : [0,\infty) \to \mathbb{C}$, set

$$\|G\|_{(\rho)} := \sup_{\lambda \geq 0} (1+\lambda^2)^\rho\, |G(\lambda)| \quad \text{and} \quad \Xi_\rho(G) := \sqrt{\|G\|_{(\rho)}\, \|G\|_{(\rho+1)}}.$$

In the next proposition F and $\{F_t : t > 0\}$ will denote functions on $[0,\infty)$. It is straightforward to check that if $F(\mathscr{D})$ is bounded from $L^1(N)$ to $L^2(N)$, then $F(\mathscr{D})$ is also bounded from $L^2(N)$ to $L^\infty(N)$, and $|||F(\mathscr{D})|||_{1;2} = |||F(\mathscr{D})|||_{2;\infty}$. We shall use this observation without any further comment.

Proposition 2.2. *There exists a positive constant C such that the following hold for every $t > 0$:*

(i) *if $1 \leq q \leq r \leq \infty$, $\rho > n(1/q - 1/r)/2$ and $\tau > 0$, then*

$$|||(\tau\mathscr{I} + t^2\mathscr{D}^2)^{-\rho}|||_{q;r} \leq C\,\gamma(t)^{2(1/q-1/r)};$$

(ii) *if $\rho > n/4$, then $|||F(t\mathscr{D})|||_{1;2} = |||F(t\mathscr{D})|||_{2;\infty} \leq C\,\gamma(t)\,\|F\|_{(\rho)}$ and $|||F(t\mathscr{D})|||_{1;\infty} \leq C\,\gamma(t)^2\,\|F\|_{(2\rho)}$;*

(iii) *if $\rho > n/4$, then $|||F_t(\mathscr{D})|||_{1;2} = |||F_t(\mathscr{D})|||_{2;\infty} \leq C\,\|F_t\|_{(\rho)}$ and $|||F_t(\mathscr{D})|||_{1;\infty} \leq C\,\|F_t\|_{(2\rho)}$;*

(iv) *if* $\rho > n/4$, *then* $\big\||\nabla F(t\mathscr{D})g|\big\|_2 \leq C\, t^{-1}\, \gamma(t)\, \Xi_\rho\big(F\big)\, \big\|g\big\|_1$ *and* $\big\||\nabla F(t\mathscr{D})g|\big\|_\infty \leq C\, \Xi_\rho\big(F(t\cdot)\big)\, \big\|g\big\|_2$;

(v) *if* $\rho > n/4$, *then* $\big\||\nabla F_t(\mathscr{D})g|\big\|_2 \leq C\, \Xi_\rho\big(F_t\big)\, \big\|g\big\|_1$ *and* $\big\||\nabla F_t(\mathscr{D})g|\big\|_\infty \leq C\, \Xi_\rho\big(F_t\big)\, \big\|g\big\|_2$.

Proof. First we prove (i). By (2.7) and the ultracontractivity estimate (2.5),

$$\begin{aligned}\big|\!\big|\!\big|(\tau\mathscr{I} + t^2\mathscr{D}^2)^{-\rho}\big|\!\big|\!\big|_{q;r} &\leq \frac{1}{|\Gamma(\rho)|}\int_0^\infty s^\rho\, \mathrm{e}^{-\tau s}\, \big|\!\big|\!\big|\mathrm{e}^{-st^2\mathscr{D}^2}\big|\!\big|\!\big|_{q;r}\, \frac{\mathrm{d}s}{s} \\ &\leq C\int_0^\infty s^\rho\, \mathrm{e}^{-\tau s}\, \gamma(st^2)^{1/q-1/r}\, \frac{\mathrm{d}s}{s}.\end{aligned}$$

The last integral is convergent because of the assumption $\rho > n(1/q - 1/r)/2$. Now we write the last integral as the sum of the integrals over $(0, 1/t^2)$ and $(1/t^2, \infty)$ and observe that $\gamma(st^2) = (st^2)^{-n/2}$ on $(0, 1/t^2)$ and that $\gamma(st^2) = (st^2)^{-1/2}$ on $(1/t^2, \infty)$. It is straightforward to check that

$$\int_0^{1/t^2} s^\rho\, \mathrm{e}^{-\tau s}\, (st^2)^{-n(1/q-1/r)/2}\, \frac{\mathrm{d}s}{s} \leq C \min(t^{-n(1/q-1/r)}, t^{-2\rho})$$

and that $\displaystyle\int_{1/t^2}^\infty s^\rho\, \mathrm{e}^{-\tau s}\, (st^2)^{-(1/q-1/r)/2}\, \frac{\mathrm{d}s}{s} \leq C\, \min(\mathrm{e}^{-(\tau-\varepsilon)/t^2}, t^{-(1/q-1/r)})$ for ε small. By combining the estimates above we get the required result.

Next we prove (ii). By the spectral theorem

$$\sup_{t>0} \big|\!\big|\!\big|(\mathscr{I} + t^2\mathscr{D}^2)^\rho\, F(t\mathscr{D})\big|\!\big|\!\big|_2 = \sup_{\lambda\geq 0}\, (1+\lambda^2)^\rho\, |F(\lambda)| = \|F\|_{(\rho)} < \infty.$$

Thus, (i) (with $q = 1$ and $r = 2$) yields

$$\begin{aligned}\big|\!\big|\!\big|F(t\mathscr{D})\big|\!\big|\!\big|_{1;2} &= \big|\!\big|\!\big|(\mathscr{I} + t^2\mathscr{D}^2)^{-\rho}\, (\mathscr{I} + t^2\mathscr{D}^2)^\rho\, F(t\mathscr{D})\big|\!\big|\!\big|_{1;2} \\ &\leq \big|\!\big|\!\big|(\mathscr{I} + t^2\mathscr{D}^2)^\rho\, F(t\mathscr{D})\big|\!\big|\!\big|_2\, \big|\!\big|\!\big|(\mathscr{I} + t^2\mathscr{D}^2)^{-\rho}\big|\!\big|\!\big|_{1;2} \\ &\leq C\, \gamma(t) \qquad \forall t > 0.\end{aligned} \tag{2.8}$$

Furthermore,

$$\begin{aligned}\big|\!\big|\!\big|F(t\mathscr{D})\big|\!\big|\!\big|_{1;\infty} &= \big|\!\big|\!\big|(\mathscr{I} + t^2\mathscr{D}^2)^{-\rho}\, (\mathscr{I} + t^2\mathscr{D}^2)^{2\rho}\, F(t\mathscr{D})(\mathscr{I} + t^2\mathscr{D}^2)^{-\rho}\big|\!\big|\!\big|_{1;\infty} \\ &\leq \big|\!\big|\!\big|(\mathscr{I} + t^2\mathscr{D}^2)^{-\rho}\big|\!\big|\!\big|_{2;\infty}\, \big|\!\big|\!\big|(\mathscr{I} + t^2\mathscr{D}^2)^{2\rho}\, F(t\mathscr{D})\big|\!\big|\!\big|_2\, \big|\!\big|\!\big|(\mathscr{I} + t^2\mathscr{D}^2)^{-\rho}\big|\!\big|\!\big|_{1;2} \\ &\leq C\, \gamma(t)^2\, \|F\|_{(2\rho)} \qquad \forall t > 0.\end{aligned} \tag{2.9}$$

Next we prove (iii). We argue much as in the proof of (ii), but with a slight difference. Instead of composing $F(t\mathscr{D})$ with $\big(\mathscr{I} + t^2\mathscr{D}^2\big)^\rho$, as in (ii), we write

$$F_t(\mathscr{D}) = (\mathscr{I} + \mathscr{D}^2)^{-\rho}\, (\mathscr{I} + \mathscr{D}^2)^\rho\, F_t(\mathscr{D}),$$

and then proceed as above, using the estimate $\big|\!\big|\!\big|(\mathscr{I} + \mathscr{D}^2)^\rho\, F_t(\mathscr{D})\big|\!\big|\!\big|_2 = \|F_t\|_{(\rho)}$, which follows from the spectral theorem. We omit the details.

To prove (iv), observe that, by the Green formula (see, for instance, [34, Lemma 4.4], together with [37, Theorem 3.1]),

$$\big\||\nabla F(t\mathscr{D})g|\big\|_2^2 = \big(\mathscr{L}F(t\mathscr{D})g, F(t\mathscr{D})g\big) = \frac{1}{t^2}\big(F_1(t\mathscr{D})g, F(t\mathscr{D})g\big),$$

where $F_1(z) := z^2\,F(z)$. Schwarz's inequality then implies that

$$\big\||\nabla F(t\mathscr{D})g|\big\|_2^2 \leq \frac{1}{t^2}\,\big\|F_1(t\mathscr{D})g\big\|_2\;\big\|F(t\mathscr{D})g\big\|_2\,, \tag{2.10}$$

By (ii), applied to F, and a similar estimate applied to F_1, we see that $\big\|F(t\mathscr{D})g\big\|_2 \leq C\,\gamma(t)\,\|F\|_{(\rho)}\,\big\|g\big\|_1$ and that $\big\|F_1(t\mathscr{D})g\big\|_2 \leq C\,\gamma(t)\,\|F_1\|_{(\rho)}\,\big\|g\big\|_1$. By combining the estimates above and the trivial observation that $\|F_1\|_{(\rho)} \leq \|F\|_{(\rho+1)}$, we obtain that

$$\big\||\nabla F(t\mathscr{D})g|\big\|_2 \leq C\,t^{-1}\,\gamma(t)\,\Xi_\rho(F)\,\big\|g\big\|_1 \qquad \forall t>0,$$

as required.

It remains to prove the second gradient estimate. For $\tau > \kappa^2$ we write

$$\begin{aligned}\big|\nabla F(t\mathscr{D})g\big| &= \big|\nabla(\tau\mathscr{I}+\mathscr{D}^2)^{-\rho}\,(\tau\mathscr{I}+\mathscr{D}^2)^{\rho}\,F(t\mathscr{D})g\big| \\ &\leq C\,\big((\tau-\kappa^2)\mathscr{I}+\mathscr{D}^2\big)^{-\rho}\big|\nabla(\tau\mathscr{I}+\mathscr{D}^2)^{\rho}\,F(t\mathscr{D})g\big|;\end{aligned}$$

we have used Proposition 2.1 in the inequality above. By (i), $\big((\tau-\kappa^2)\mathscr{I}+\mathscr{D}^2\big)^{-\rho}$ is bounded from $L^2(N)$ to $L^\infty(N)$, so that

$$\big\||\nabla F(t\mathscr{D})g|\big\|_\infty \leq C\,\big\||\nabla(\tau\mathscr{I}+\mathscr{D}^2)^{\rho}\,F(t\mathscr{D})g|\big\|_2\,.$$

By arguing much as in (2.10), we see that

$$\begin{aligned}&\big\||\nabla(\tau\mathscr{I}+\mathscr{D}^2)^{\rho}\,F(t\mathscr{D})g|\big\|_2 \\ &\leq C\,\big\|\mathscr{L}(\tau\mathscr{I}+\mathscr{D}^2)^{\rho}\,F(t\mathscr{D})\big\|_2^{1/2}\big\|(\tau\mathscr{I}+\mathscr{D}^2)^{\rho}F(t\mathscr{D})\big\|_2^{1/2}\,\big\|g\big\|_2 \\ &\leq C\,\Xi_\rho\big(F(t\cdot)\big)\,\big\|g\big\|_2\,;\end{aligned} \tag{2.11}$$

the last inequality follows from the spectral theorem.

The proof of (v) is similar to that of (iv). By arguing much as in (2.10), we see that

$$\big\||\nabla F_t(\mathscr{D})g|\big\|_2^2 \leq C\,\big\|\mathscr{L}F_t(\mathscr{D})g\big\|_2\,\big\|F_t(\mathscr{D})g\big\|_2\,.$$

By (iii) and its proof, $\big\|F_t(\mathscr{D})g\big\|_2 \leq C\,\|F_t\|_{(\rho)}\,\big\|g\big\|_1$ and $\big\|\mathscr{L}F_t(\mathscr{D})g\big\|_2 \leq C\,\|F_t\|_{(\rho+1)}\,\big\|g\big\|_1$. By combining the estimates above, we obtain that

$$\big\||\nabla F_t(\mathscr{D})g|\big\|_2 \leq C\,\Xi_\rho\big(F_t\big)\,\big\|g\big\|_1 \qquad \forall t>0,$$

and the first gradient estimate in (v) is proved. In order to prove the second gradient estimate we proceed as in (iv). If $\tau > \kappa^2$, then

$$\big\||\nabla F_t(\mathscr{D})g|\big\|_\infty \leq C\,\big\||\nabla(\tau\mathscr{I}+\mathscr{D}^2)^{\rho}\,F_t(\mathscr{D})g|\big\|_2\,.$$

By (2.11) (with F_t instead of $F(t\cdot)$), $\big\||\nabla(\tau\mathscr{I}+\mathscr{D}^2)^{\rho}\,F_t(\mathscr{D})g|\big\|_2 \leq C\,\Xi_\rho\big(F_t\big)\,\big\|g\big\|_2$. This concludes the proof of (v) and of the proposition. □

2.3. **Estimates for the Poisson semigroup.** We denote by $\mathscr{P}_t^N$ the Poisson semigroup $\mathrm{e}^{-t\mathscr{D}}$. Recall the subordination formula

$$\mathscr{P}_t^N = t \int_0^\infty h_s^{\mathbb{R}}(t)\, \mathscr{H}_s^N \,\frac{\mathrm{d}s}{s}, \tag{2.12}$$

where $h_s^{\mathbb{R}}$ denotes the standard Gauss–Weierstrass kernel on the real line. Notice the estimate

$$|\!|\!| \mathscr{P}_t^N |\!|\!|_{q;r} \leq C\,\gamma(t)^{2(1/q-1/r)} \qquad \forall t > 0, \tag{2.13}$$

which is a simple consequence of the subordination formula above and the corresponding estimate (2.5) for $\mathscr{H}_t^N$; see for instance [15, Corollary 1.5]. It is sometimes convenient to write $\mathscr{P}_t^N = \mathscr{Q}_t^0 + \mathscr{Q}_t^\infty$, where

$$\mathscr{Q}_t^0 := t \int_0^1 h_s^{\mathbb{R}}(t)\, \mathscr{H}_s^N \,\frac{\mathrm{d}s}{s} \qquad \text{and} \qquad \mathscr{Q}_t^\infty := t \int_1^\infty h_s^{\mathbb{R}}(t)\, \mathscr{H}_s^N \,\frac{\mathrm{d}s}{s}. \tag{2.14}$$

Proposition 2.3. *There exists a positive constant C, independent of f, such that*

(i) $\big|\nabla \mathscr{Q}_t^0 f\big| \leq \mathrm{e}^{\kappa^2}\, \mathscr{Q}_t^0\big(|\nabla f|\big)$,
(ii) $|\nabla \mathscr{Q}_t^\infty f| \leq C\, \min(t, t^{-3/2})\, \big\|f\big\|_1$
(iii) $\big\|\mathscr{Q}_t^\infty f\big\|_\infty \leq C\, \min(t, t^{-1})\, \big\|f\big\|_1$

for every $t > 0$.

Proof. First we prove (i). By Bakry's condition (2.6),

$$|\nabla \mathscr{Q}_t^0 f| \leq t \int_0^1 h_s^{\mathbb{R}}(t)\, |\nabla \mathscr{H}_s^N f| \,\frac{\mathrm{d}s}{s} \leq t \int_0^1 h_s^{\mathbb{R}}(t)\, \mathrm{e}^{\kappa^2 s} \mathscr{H}_s^N |\nabla f| \,\frac{\mathrm{d}s}{s},$$

which is clearly dominated by $\mathrm{e}^{\kappa^2}\, t \int_0^1 h_s^{\mathbb{R}}(t)\, \mathscr{H}_s^N |\nabla f| \,\frac{\mathrm{d}s}{s} = \mathrm{e}^{\kappa^2}\, \mathscr{Q}_t^0\big(|\nabla f|\big)$, as required.

Next we prove (ii). Observe that, by Propositions 2.1 and 2.2 (i) (with $q = 2$ and $r = \infty$),

$$\big\||\nabla \mathscr{H}_s f|\big\|_\infty \leq C\, \big\||\nabla (\tau \mathscr{I} + \mathscr{D}^2)^\rho \mathscr{H}_s f|\big\|_2 \,.$$

By arguing much as in (2.10) (with $(\tau\mathscr{I} + \mathscr{D}^2)^\rho \mathscr{H}_s$ in place of $F(t\mathscr{D})$), we see that

$$\big\||\nabla (\tau \mathscr{I} + \mathscr{D}^2)^\rho \mathscr{H}_s f|\big\|_2 \leq \frac{1}{\sqrt{s}}\, \big\|(\tau \mathscr{I} + \mathscr{D}^2)^\rho s\mathscr{D}^2 \mathscr{H}_s f\big\|_2^{1/2}\, \big\|(\tau \mathscr{I} + \mathscr{D}^2)^\rho \mathscr{H}_s f\big\|_2^{1/2} \,.$$

Now, set $\omega(s) := \sup_{\lambda \geq 0} (\tau + \lambda^2)^\rho\, s\lambda^2\, \mathrm{e}^{-s\lambda^2/2}$. By the spectral theorem,

$$\begin{aligned}
\big\|(\tau \mathscr{I} + \mathscr{D}^2)^\rho s\mathscr{D}^2 \mathscr{H}_s f\big\|_2 &\leq |\!|\!|(\tau \mathscr{I} + \mathscr{D}^2)^\rho s\mathscr{D}^2 \mathscr{H}_{s/2}|\!|\!|_2\, \big\|\mathscr{H}_{s/2} f\big\|_2 \\
&\leq C\,\omega(s)\, |\!|\!|\mathscr{H}_{s/2}|\!|\!|_{1;2}\, \big\|f\big\|_1 \\
&\leq C\, \max(s^{-\rho}, 1)\, \gamma(s)^{1/2}\, \big\|f\big\|_1 \,;
\end{aligned}$$

the third inequality above follows from (2.5) and the fact that

$$\omega(s) = \sup_{v \geq 0} v^2\, (\tau + v^2 s^{-1})^\rho\, \mathrm{e}^{-v^2/2} \leq \max(s^{-\rho}, 1).$$

Similarly, $\big\|(\tau\mathscr{I}+\mathscr{D}^2)^\rho\mathscr{H}_s f\big\|_2 \leq C\,\max(s^{-\rho},1)\,\gamma(s)^{1/2}\,\big\|f\big\|_1$. By combining the estimates above, we see that $\big\||\nabla\mathscr{H}_s f|\big\|_\infty \leq \dfrac{C}{\sqrt{s}}\,\max(s^{-\rho},1)\,\gamma(s)^{1/2}\,\big\|f\big\|_1$. Therefore

$$|\nabla\mathscr{Q}_t^\infty f| \leq t\int_1^\infty h_s^{\mathbb{R}}(t)\,\big\||\nabla\mathscr{H}_s f|\big\|_\infty\,\frac{\mathrm{d}s}{s} \leq C\,\big\|f\big\|_1\,t\int_1^\infty h_s^{\mathbb{R}}(t)\,s^{-3/4}\,\frac{\mathrm{d}s}{s}.$$

The last integral above is bounded above by $C\,\min(1,t^{-5/2})$. Indeed, if $t\leq 1$, then

$$\int_1^\infty h_s^{\mathbb{R}}(t)\,s^{-3/4}\,\frac{\mathrm{d}s}{s} \leq C\int_1^\infty \mathrm{e}^{-t^2/(4s)}\,s^{-5/4}\,\frac{\mathrm{d}s}{s} \leq C\int_1^\infty s^{-9/4}\,\mathrm{d}s,$$

which is clearly finite, and, if $t\geq 1$, then

$$\int_1^\infty h_s^{\mathbb{R}}(t)\,s^{-3/4}\,\frac{\mathrm{d}s}{s} \leq C\int_1^\infty \mathrm{e}^{-t^2/(4s)}\,s^{-5/4}\,\frac{\mathrm{d}s}{s} = C\,t^{-5/2}\int_0^{t^2/4} u^{1/4}\mathrm{e}^{-u}\,\mathrm{d}u,$$

which is bounded by $C\,t^{-5/2}$. Therefore $|\nabla\mathscr{Q}_t^\infty f| \leq C\,\min(t,t^{-3/2})\,\big\|f\big\|_1$, as required.

Finally, we prove (iii). We use the ultracontractivity of the heat semigroup, and estimate

$$\big\|\mathscr{Q}_t^\infty f\big\|_\infty \leq C\,\big\|f\big\|_1\,t\int_1^\infty h_s^{\mathbb{R}}(t)\,s^{-1/2}\,\frac{\mathrm{d}s}{s} \leq C\,\big\|f\big\|_1\,t\int_1^\infty \mathrm{e}^{-t^2/(4s)}\,s^{-1}\,\frac{\mathrm{d}s}{s}.$$

Now, the change of variables $t^2/s=u$ transforms the last integral to $t^{-2}\displaystyle\int_0^{t^2}\mathrm{e}^{-u/4}\,u\,\frac{\mathrm{d}u}{u}$. This is bounded for t small, and is bounded by $C\,t^{-2}$ for t large. By combining the estimates above, we get that $\big\|\mathscr{Q}_t^\infty f\big\|_\infty \leq C\,\min(t,t^{-1})\,\big\|f\big\|_1$, as required. □

2.4. **Estimates related to the wave propagator.** Define the Fourier transform of an integrable function η on the real line by $\widehat{\eta}(s) = \displaystyle\int_{-\infty}^\infty \eta(\lambda)\,\mathrm{e}^{-is\lambda}\,\mathrm{d}\lambda$. We analyse various operators by subordinating them to the wave propagator, an idea that originates in [10, 61]. At least formally, we may write $\eta(\mathscr{D}) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^\infty \widehat{\eta}(s)\,\cos(s\mathscr{D})\,\mathrm{d}s$, whenever η is even. Occasionally we need to integrate by parts in the integral above. We do it with the aid of [47, Lemma 5.1], which we restate for the reader's convenience. Hereafter $\mathscr{O}^\ell$ denotes the differential operator $s^\ell\partial_s^\ell$, acting on functions on the real line. We set $\mathscr{J}_\nu(v) = \dfrac{J_\nu(v)}{v^\nu}$, where J_ν denotes the Bessel function of the first kind and of order ν (see, for instance, [41, Section 5.3]).

Lemma 2.4. *For every positive integer J there exists a polynomial P_J of degree J without constant term, such that*

$$\int_{-\infty}^\infty \widehat{\eta}(t)\,\cos(vt)\,\mathrm{d}t = \int_{-\infty}^\infty P_J(\mathscr{O})\widehat{\eta}(t)\,\mathscr{J}_{J-1/2}(tv)\,\mathrm{d}t,$$

for all functions η such that $\mathscr{O}^\ell\widehat{\eta}\in L^1(\mathbb{R})\cap C_0(\mathbb{R})$ for all ℓ in $\{0,1,\ldots,J\}$.

Given a "nice" function f on N, the formula above and the spectral theorem suggest to establish appropriate norm estimates of $\mathscr{J}_{J-1/2}(s\mathscr{D})f$. This is done in the next lemma.

Lemma 2.5. *Suppose that $\delta > 0$ and that J is a positive integer. There exists a constant C such that the following hold:*

(i) *if $J > n/2$, then $\big\|\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f\big\|_1 \leq C\, s^{(\alpha-1)/2}\, \mathrm{e}^{\beta s}\, \big\|f\big\|_1$ for every $s \geq \delta$;*
(ii) *if $J > n/2+2$, then $\big\||\nabla\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f|\big\|_1 \leq C\, s^{(\alpha-3)/2}\, \mathrm{e}^{\beta s}\, \big\|f\big\|_1$ for every $s \geq \delta$;*
(iii) *if $J > 2+n$, then $\big\|\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f\big\|_{W^{1,\infty}(N)} \leq C\, \big\|f\big\|_1$ for every $s \geq \delta$.*

Proof. Observe preliminarily that we can reduce the problem to the case where the support of f is contained in $B_\delta(o)$, for some point o in N. This is done considering a smooth partition of unity $\{\psi_j\}$ so that the support of ψ_j is contained in $B_\delta(x_j)$, for an appropriate sequence $\{x_j\}$ of points in N. Thus, in the rest of the proof we assume that the support of f is contained in $B_\delta(o)$, for some o in N.

The proof of (i) proceeds along the lines of the proof of (ii), and it is, in fact, simpler. We leave the details to the interested reader.

Now we prove (ii). Observe that the support of $|\nabla\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f|$ is contained in the ball $B_{\delta+s}(o)$ by finite propagation speed. By Schwarz's inequality,

$$\big\||\nabla\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f|\big\|_1 \leq \nu\big(B_{\delta+s}(o)\big)^{1/2}\, \big\||\nabla\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f|\big\|_2\,.$$

Observe that if $n/2 < 2\rho \leq J-2$, which is compatible with our assumptions, then by Proposition 2.2 (iv) there exists a constant C such that

$$\begin{aligned} \big\||\nabla\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f|\big\|_1 &\leq \frac{C}{s}\,\nu\big(B_{\delta+s}(o)\big)^{1/2}\,\gamma(s)\,\big\|f\big\|_1 \\ &\leq C\, s^{(\alpha-3)/2}\,\mathrm{e}^{\beta s}\,\big\|f\big\|_1 \qquad \forall s\in[\delta,\infty), \end{aligned} \tag{2.15}$$

as required. Notice that the last inequality is a consequence of (2.2).

Finally we prove (iii). By Proposition 2.2 (ii) (with $\rho = J/2$),

$$\big\|\big\|\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)\big\|\big\|_{1;\infty} \leq C\,\gamma(s)^2\,\|\mathscr{J}_{J-1/2}\|_{(J)} \leq C\,\|\mathscr{J}_{J-1/2}\|_{(J)} \qquad \forall s \geq \delta.$$

Next we estimate the gradient of $\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)$. For $\tau > \kappa^2$ we write

$$\nabla\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f = \nabla\big(\tau\mathscr{I}+\mathscr{D}^2\big)^{-\rho/2}\,\big(\tau\mathscr{I}+\mathscr{D}^2\big)^{\rho/2}\,\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f.$$

Then Proposition 2.1 implies that

$$\big|\nabla\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f\big| \leq C\,\big((\tau-\kappa^2)\mathscr{I}+\mathscr{D}^2\big)^{-\rho/2}\,\big|\nabla\big(\tau\mathscr{I}+\mathscr{D}^2\big)^{\rho/2}\,\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f\big|.$$

Now if $\rho > n/2$, then the operator $\big((\tau-\kappa^2)\mathscr{I}+\mathscr{D}^2\big)^{-\rho/2}$ is bounded from $L^2(N)$ to $L^\infty(N)$, by Proposition 2.2 (i), whence

$$\big\||\nabla\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f|\big\|_\infty \leq C\,\big\||\nabla\big(\tau\mathscr{I}+\mathscr{D}^2\big)^{\rho/2}\,\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f|\big\|_2\,.$$

We can apply Proposition 2.2 (v) (with $F_s(\lambda) = (\tau+\lambda^2)^{\rho/2}\,\mathscr{J}_{J-1/2}(s\lambda)$ and $\rho_1 > n/4$), and conclude that

$$\big\||\nabla\big(\tau\mathscr{I}+\mathscr{D}^2\big)^{\rho/2}\,\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f|\big\|_2 \leq C\,\Xi_{\rho_1}(F_s)\,\big\|f\big\|_1\,,$$

where $2\rho_1 + 2 + \rho < J$. Standard estimates of Bessel functions imply that

$$\|F_s\|_{(\rho_1)} = \sup_{\lambda\geq 0} (\tau+\lambda^2)^{\rho_1+\rho/2} |\mathscr{J}_{J-1/2}(s\lambda)| \leq \sup_{\lambda\geq 0} \frac{(\tau+\lambda^2)^{\rho_1+\rho/2}}{(1+s\lambda)^J}.$$

Clearly for any $s \geq \delta$ this is dominated by $\sup_{\lambda\geq 0} \dfrac{(\tau+\lambda^2)^{\rho_1+\rho/2}}{(1+\delta\lambda)^J}$ which is finite. A similar estimate is satisfied by $\|F_s\|_{(\rho_1+1)}$, and the required bound follows.

It is straightforward to check that the conditions $\rho > n/2$, $\rho_1 > n/4$ and $J \geq 2\rho_1+2+\rho$ are compatible provided that $J > n+2$. □

2.5. **Laplacian cut-off functions.** We need the following result, which will be used in Section 7. In the rest of the paper for each $R > 0$ we set

$$\Upsilon_R := \big\{(x,y) \in N \times N : d(x,y) < R\big\}. \tag{2.16}$$

Lemma 2.6. *Given $R > 0$, there exists positive constants Q and Q', depending on κ, n and R, such that:*

(i) *for every $x \in N$ there exists a function χ_x in $C_c^\infty(N)$ such that $0 \leq \chi_x \leq 1$, $\chi_x = 1$ on $B_{R/4}(x)$, $\chi_x = 0$ on $B_{R/2}(x)^c$, $\big\||\nabla\chi_x|\big\|_\infty \leq Q$ and $\big\|\Delta\chi_x\big\|_\infty \leq Q$;*
(ii) *there exists a function φ in $C_c^\infty(N\times N)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in $\Upsilon_{R/4}$ and $\varphi = 0$ in Υ_R^c, $\big\||\nabla\varphi|\big\|_\infty \leq Q'$ and $\big\|\Delta\varphi\big\|_\infty \leq Q'$.*

Proof. For the proof of (i), see [9, Theorem 6.33].

We now prove (ii). Denote by $\mathfrak{P}$ an $R/4$-discretization of N, i.e., a set of points $\{p_j : j = 1,2,3,\ldots\}$ in N that is maximal with respect to the property

$$d(p_j,p_k) > R/8 \qquad \text{when } j \neq k \text{ and} \qquad d(x,\mathfrak{P}) < R/4 \qquad \forall x \in N.$$

We write P_j instead of $p_j \times p_j$ and $Q_j^{R/2}$ instead of $B_{R/2}(p_j) \times B_{R/2}(p_j)$. The family $\big\{B_R(p_j) : j = 1,2,3,\ldots\big\}$ has the finite overlapping property (see, for instance, [37, Lemma 1.1]). Hence the same is true for $\big\{Q_j^{R/2} : j = 1,2,3,\ldots\big\}$. It is straightforward to check that

$$\Upsilon_{R/4} \subseteq \bigcup_{j=1}^\infty Q_j^{R/2} \subseteq \Upsilon_R.$$

Indeed, if (x,y) is in $\Upsilon_{R/4}$, then $d(x,y) < R/4$. Since $\mathfrak{P}$ is a $R/4$-discretization of N, there exists an integer j such that $d(x,p_j) < R/4$. The triangle inequality then implies that $d(y,p_j) < R/2$, whence (x,y) belongs to $Q_j^{R/2}$, and the left inclusion above is proved. The right inclusion follows from the trivial fact that if (x,y) belongs to $Q_j^{R/2}$, then $d(x,y) < R$.

For each integer j set $\phi_j := \chi_{p_j} \otimes \chi_{p_j}$, where the χ_{p_j} are cut-offs on N as in (i). Notice that ϕ_j is a smooth function with compact support on $N \times N$, that $\phi_j = 1$ on $Q_j^{R/4}$, $\phi_j = 0$ on $\big[Q_j^{R/2}\big]^c$, $\big\||\nabla\phi_j|\big\|_\infty \leq 2Q$ and $\big\|\Delta\phi_j\big\|_\infty \leq 2Q$, where ∇ and Δ denote here the gradient and the Laplace–Beltrami operator on $N \times N$, respectively. Set

$\varphi_j := \phi_{P_j} \Big/ \sum_{k=1}^{\infty} \phi_{P_k}$ and $\varphi := \sum_{j=1}^{\infty} \varphi_j$. It is straightforward to check that φ possesses the required properties. We omit the details. □

3. Estimates for the Poisson operator on slices

In this section we consider the Riemannian manifold $N \times \mathbb{R}$, endowed with the natural product metric. Here N satisfies our standing assumptions (see the beginning of Section 2). We shall often, but not always, denote points in $N \times \mathbb{R}$ by capital letters $X, Y, Z, \ldots$. Usually, lower case latin letters $x, y, z, \ldots$ will denote points in N. Thus, a point X in $N \times \mathbb{R}$ will be often written (x, u), where x is in N and u is a real number. Denote by D the Riemannian distance on $N \times \mathbb{R}$, i.e.,

$$D\big((x,u),(y,v)\big) := \sqrt{d(x,y)^2 + |u-v|^2} \qquad \forall x, y \in N \quad \forall u, v \in \mathbb{R}. \tag{3.1}$$

The Riemannian measure on $N \times \mathbb{R}$ will be denoted by $\mathscr{Y}$. Thus, $\mathrm{d}\mathscr{Y}(Y) = \mathrm{d}\nu(y)\,\mathrm{d}v$ when $Y = (y, v)$. We shall denote by $\mathbb{\nabla}$ and $\mathbb{\Delta}$ the gradient on $N \times \mathbb{R}$ and the (negative) Laplace–Beltrami operators on $N \times \mathbb{R}$, respectively. When we choose the natural coordinate system (x, t) on $N \times \mathbb{R}$, where x varies in an open chart of N, and t is in $\mathbb{R}$, we have that $\mathbb{\nabla} F = (\nabla F, \partial_t F)$, and $\mathbb{\Delta} F = \Delta F + \partial_t^2 F$.

Throughout the paper σ will denote a fixed positive number such that

$$\sigma < \frac{\pi}{4\beta} \min\big(1 - 1/n, \sqrt{c}\big) \tag{3.2}$$

where c and β are as in (2.3) and (2.2), respectively. Set

$$\lambda_1 := \pi/(2\sigma). \tag{3.3}$$

For any η in $[0, \sigma)$, set $\Sigma_\eta := N \times (\eta, 2\sigma - \eta)$ in $N \times \mathbb{R}$. We write Σ for Σ_0. Most of our analysis is concerned with functions defined on the open slice Σ. In particular, we shall need to consider Σ_η for some $\eta \neq 0$ only in Section 6. We shall write $\|\cdot\|_p$ and $\|\cdot\|_{L^p(\Sigma_\eta)}$ for the L^p norms on N and on Σ_η, respectively. Given a function F on Σ, we denote by $F^\flat$ the function on N, defined by

$$F^\flat(x) = \int_0^{2\sigma} F(x,t)\,\mathrm{d}t \qquad \forall x \in N, \tag{3.4}$$

whenever the latter integral makes sense. Observe that, by Hölder's inequality,

$$\big\|F^\flat\big\|_p = \Big[\int_N \mathrm{d}\nu(x) \Big|\int_0^{2\sigma} F(x,u)\,\mathrm{d}u\Big|^p\Big]^{1/p} \leq (2\sigma)^{1/p'} \,\big\|F\big\|_{L^p(\Sigma)}\,. \tag{3.5}$$

For each η in $[0, \sigma)$ and t in $(\eta, 2\sigma - \eta)$ consider the meromorphic function

$$M_t^\eta(\lambda) = \frac{\cosh(t-\sigma)\lambda}{\cosh(\sigma-\eta)\lambda}.$$

For the sake of simplicity, we write M_t instead of M_t^0. Thus, $M_t(\lambda) := \dfrac{\cosh(t-\sigma)\lambda}{\cosh(\sigma\lambda)}$. An elementary computation shows that

$$M_t^\eta(\lambda) = \mathrm{e}^{(\eta-t)\lambda} + \frac{1}{2}\left[\mathrm{e}^{(t-\sigma)\lambda} - \mathrm{e}^{(2\eta-t-\sigma)\lambda}\right] M_\sigma^\eta(\lambda).$$

We shall often work with the special case of the formula above corresponding to $\eta = 0$. Set $\mathscr{P}^\eta f(\cdot,t) := M_t^\eta(\mathscr{D})f$. In the case where $\eta < t < \sigma$ it is sometimes convenient to use the expression above for M_t^η and write

$$\mathscr{P}^\eta f(\cdot,t) = \mathscr{P}_{t-\eta}^N f + \frac{1}{2}\left[\mathscr{P}_{\sigma-t}^N - \mathscr{P}_{\sigma+t-2\eta}^N\right] M_\sigma^\eta(\mathscr{D}) f. \tag{3.6}$$

The operator $\mathscr{P}^\eta$ is called the *Poisson operator for Σ_η with periodic boundary conditions.* The following proposition partially justifies this terminology.

Proposition 3.1. *Suppose that f is in $C_0(N)$ (continuous functions on N vanishing at infinity). Then the function equal to $\mathscr{P}^\eta f$ in Σ_η and to f on $\partial\Sigma_\eta$, is smooth on Σ_η, continuous on $\overline{\Sigma}_\eta$, and solves the Dirichlet problem*

$$\mathbb{\Delta} u = 0 \quad in \quad \Sigma_\eta \qquad u(\cdot,\eta) = f = u(\cdot, 2\sigma-\eta).$$

We postpone the proof of Proposition 3.1 at the end of this section. We analyse $\mathscr{P}^\eta$ by subordinating $M_t^\eta(\mathscr{D})$ to the wave propagator. Denote by K_t^η the Fourier transform of M_t^η. It is well known (see, for instance, [52, formula 7.19, p. 34]) that

$$K_t^\eta(s) = 4\pi\delta\, \frac{\sin\pi\delta(t-\eta)\,\cosh\pi\delta s}{\cosh 2\pi\delta s - \cos 2\pi\delta(\eta - t)}, \tag{3.7}$$

where $\delta := 1/[2(\sigma-\eta)]$. We shall write K_t instead of K_t^0. Thus,

$$K_t(s) = \frac{2\pi}{\sigma}\, \frac{\sin\dfrac{\pi t}{2\sigma}\,\cosh\dfrac{\pi s}{2\sigma}}{\cosh\dfrac{\pi s}{\sigma} - \cos\dfrac{\pi t}{\sigma}}. \tag{3.8}$$

By spectral theory and Fourier inversion formula

$$\mathscr{P}^\eta f(\cdot,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_t^\eta(s)\,\cos(s\mathscr{D})f\,\mathrm{d}s$$

and, when $\eta < t < \sigma$,

$$\mathscr{P}^\eta f(\cdot,t) = \mathscr{P}_{t-\eta}^N f + \left[\mathscr{P}_{\sigma-t}^N - \mathscr{P}_{\sigma+t-2\eta}^N\right]\frac{1}{4\pi}\int_{-\infty}^{\infty} K_\sigma^\eta(s)\,\cos\big(s\mathscr{D}\big)f\,\mathrm{d}s.$$

We are led to establish certain properties of K_t^η and of their derivatives. Most of our applications will involve only K_t. Thus, we mainly concentrate on this special case. Set $S := \mathbb{R}\times(0,\sigma]$. For each δ in $(0,\sigma)$ denote by D_δ the disc in the plane with radius δ centred at the origin. Set $S_\delta := S\setminus D_\delta$.

Lemma 3.2. *Suppose that $\delta\in(0,\sigma)$, $\varepsilon\in(0,\lambda_1)$, J is a positive integer and $\ell\in\{0,1\}$. Then there exists a constant C such that $\big|\partial_t^\ell P_J(\mathscr{O})K_t(s)\big| \leq C\,\min\big(|s|, \mathrm{e}^{(\varepsilon-\lambda_1)|s|}\big)$ for every (s,t) in S_δ.*

Proof. A straightforward induction argument (using (3.8)) proves that $\big|\partial_t^\ell \partial_s^j K_t(s)\big| \leq C \min\big(1, \mathrm{e}^{-\pi|s|/2\sigma}\big)$ for every (s,t) in S_δ and every nonnegative integer $j \leq J$. The required estimate then follows from the form of the differential operator $P_J(\mathscr{O})$. In particular, the required estimate for $|s|$ small follows from the fact that P_J has no constant term. ☐

Recall that the *extended Dunford class* $\mathscr{E}(\mathbf{S}_\psi)$ is defined as follows [35, p. 28]

$$\mathscr{E}(\mathbf{S}_\psi) = H_0^\infty(\mathbf{S}_\psi) \oplus \big\langle (1+z)^{-1}\big\rangle \oplus \langle 1\rangle,$$

where $H_0^\infty(\mathbf{S}_\psi)$ denotes the class of all holomorphic functions f in the sector $\mathbf{S}_\psi := \{z \in \mathbb{C} : |\arg z| < \psi\}$ for which there exist positive constants C and s such that

$$\big|f(z)\big| \leq C\, \frac{|z|^s}{1+|z|^{2s}} \qquad \forall z \in \mathbf{S}_\psi.$$

The space $\mathscr{E}(\mathbf{S}_\psi)$ is endowed with the uniform norm.

Lemma 3.3. *Suppose that $0<\delta<\sigma$. The following hold:*

(i) *for each positive even integer J there exists a constant C such that*

$$\big\|\mathscr{L}^J M_t(\mathscr{D}) f\big\|_2 \leq C\, \big\|f\big\|_1 \qquad \forall t \in [\delta, 2\sigma-\delta] \quad \forall f \in L^1(N);$$

(ii) *$M_t(\mathscr{D})f$ is smooth, and there exists a constant C such that*

$$\big\|M_t(\mathscr{D}) f\big\|_{C_b^1(N)} \leq C\, \big\|f\big\|_1 \qquad \forall t \in [\delta, 2\sigma-\delta] \quad \forall f \in L^1(N);$$

(iii) *for every $\varepsilon>0$ and $R>0$ there exists a constant C such that*

$$\sup_{t\in(0,\sigma]} \max\big(|M_t(\mathscr{D})f(x)|, |\mathscr{L} M_t(\mathscr{D})f(x)|, |\nabla M_t(\mathscr{D})f(x)|\big) \leq C\, \mathrm{e}^{(\varepsilon-\lambda_1)d(x,o)}\, \big\|f\big\|_1$$

for every o in N, every x in $B_{2R}(o)^c$ and every f in $L^1(N)$ with support contained in the ball $B_R(o)$;

(iv) *for each φ in $(\pi/4, \pi/2)$ the function M_σ belongs to $\mathscr{E}(\mathbf{S}_\varphi)$ and there exists a constant C such that*

$$\sup_{t\in(0,2\sigma)} \big|\!\big|\!\big|M_t(\mathscr{D})\big|\!\big|\!\big|_p \leq 1 + C\, \big\|M_\sigma\big\|_{\mathscr{E}(\mathbf{S}_\varphi)}$$

for every p in $[1,\infty]$;

(v) *for each p in $(1,\infty]$ there exists a constant C such that*

$$\sup_{y\in N} \big\|k_{M_t(\mathscr{D})}(\cdot, y)\big\|_p = \big|\!\big|\!\big|M_t(\mathscr{D})\big|\!\big|\!\big|_{1;p} \leq C\, t^{-n/p'} \qquad \forall t \in (0,\sigma].$$

Proof. Part (i) follows from Proposition 2.2 (iii) and the trivial fact that for any $\rho_1>0$

$$\sup_{t\in[\delta,2\sigma-\delta]} \sup_{\lambda\geq 0} (1+\lambda^2)^{\rho_1}\, \lambda^{2J} M_t(\lambda) \leq \sup_{\lambda\geq 0} (1+\lambda^2)^{\rho_1}\, \lambda^{2J} M_\delta(\lambda) < \infty. \tag{3.9}$$

Now we prove (ii). The smoothness of $M_t(\mathscr{D})f$ follows from (i) and a local Sobolev's embedding theorem. The estimate $\big\|M_t(\mathscr{D})f\big\|_\infty \leq C\, \big\|f\big\|_1$ is a direct consequence of

Proposition 2.2 (iii) and of an estimate similar to (3.9). Finally, Proposition 2.1 (i) and Proposition 2.2 (i),(v) imply that for $\tau > \kappa^2$ and $\sigma_1 > n/4$

$$\big\||\nabla M_t(\mathscr{D})f|\big\|_\infty \leq C\,\big\||\nabla(\tau\mathscr{I}+\mathscr{D}^2)^{\sigma_1}M_t(\mathscr{D})f|\big\|_2 \leq C\,\Xi_{\sigma_1}\big(F_t\big)\,\big\|f\big\|_1\,,$$

where $F_t(\lambda) := (\tau+\lambda^2)^{\sigma_1}M_t(\lambda)$. It is straightforward to check that $\sup_{t\in[\delta,2\sigma-\delta]}\Xi_{\sigma_1}\big(F_t\big) < \infty$, thereby concluding the proof of (ii).

Next, we prove the estimate in (iii) concerning $|\mathbb{\nabla} M_t(\mathscr{D})f|$. The proofs of the estimates for $|M_t(\mathscr{D})f|$ and $|\mathscr{L}M_t(\mathscr{D})f|$ are similar, perhaps easier, and we leave the details to the interested reader. Suppose that $d(x,o)\geq 2R$ and choose $J > n+2$. By finite propagation speed and Lemmata 2.4 and 2.5 (iii),

$$\begin{aligned}\big|\nabla\mathscr{P}^0 f(x,t)\big| &\leq C\int_{|s|\geq d(x,o)-R}\big|P_J(\mathscr{O})K_t(s)\big|\,\big|\nabla\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f\big|(x)\,\mathrm{d}s\\ &\leq C\,\big\|f\big\|_1\int_{|s|\geq d(x,o)-R}\big|P_J(\mathscr{O})K_t(s)\big|\,\mathrm{d}s.\end{aligned}$$

Since (s,t) is in S_R, Lemma 3.2 ensures that there exists a constant C such that

$$\sup_{t\in(0,\sigma)}\big|P_J(\mathscr{O})K_t(s)\big| \leq C\,\mathrm{e}^{(\varepsilon/2-\lambda_1)|s|}.$$

This and the estimates above imply that

$$\big|\nabla\mathscr{P}^0 f(x,t)\big| \leq C\,\mathrm{e}^{(\varepsilon-\lambda_1)d(x,o)}\,\big\|f\big\|_1\int_{|s|\geq d(x,o)-R}\mathrm{e}^{-\varepsilon|s|/2}\,\mathrm{d}s \leq C\,\mathrm{e}^{(\varepsilon-\lambda_1)d(x,o)}\,\big\|f\big\|_1\,.$$

The right hand side does not depend on t in $(0,1)$, and the required estimate (with ∇ in place of $\mathbb{\nabla}$) follows. It remains to prove a similar estimate for $\partial_t\mathscr{P}^0 f(x,t)$. By finite propagation speed and Lemmata 2.4 and 2.5 (iii),

$$\begin{aligned}\big|\partial_t\mathscr{P}^0 f(x,t)\big| &\leq C\int_{|s|\geq d(x,o)-R}\big|\partial_t P_N(\mathscr{O})K_t(s)\big|\,\big|\mathscr{J}_{J-1/2}\big(s\mathscr{D}\big)f(x)\big|\,\mathrm{d}s\\ &\leq C\,\big\|f\big\|_1\int_{|s|\geq d(x,o)-R}\big|\partial_t P_N(\mathscr{O})K_t(s)\big|\,\mathrm{d}s.\end{aligned}$$

The required estimate follows from this by arguing much as above.

Next we prove (iv). Since $M_t = M_{2\sigma-t}$ for each t in $(0,2\sigma)$, it suffices to prove the required estimate in the case where $0 < t \leq \sigma$. The contractivity of the Poisson semigroup on $L^p(N)$ and (3.6) imply the estimate

$$\sup_{t\in(0,2\sigma)}\big\|\big|M_t(\mathscr{D})\big|\big\|_p \leq \big[1+\big\|\big|M_\sigma(\mathscr{D})\big|\big\|_p\big]. \tag{3.10}$$

Since $\mathscr{L}$ generates a contraction semigroup on $L^p(N)$, $\mathscr{L}$ is a sectorial operator of angle $\pi/2$ on $L^p(N)$, by the easy part of the Hille–Yosida theorem. By abstract nonsense, $\mathscr{D}$ is a sectorial operator of angle $\pi/4$ on $L^p(N)$ [35, Proposition 3.1.2]. It is straightforward to check that the functions $z \mapsto \dfrac{1-\mathrm{e}^{-2\sigma z}}{\mathrm{e}^{\sigma z}+\mathrm{e}^{-\sigma z}}$ and $z \mapsto \mathrm{e}^{-\sigma z} - (1+z)^{-1}$ are in $H^\infty_0(\mathbf{S}_\varphi)$. Furthermore

$$M_\sigma(z) = \frac{2}{\mathrm{e}^{\sigma z}+\mathrm{e}^{-\sigma z}} - \mathrm{e}^{-\sigma z} + \mathrm{e}^{-\sigma z} = \frac{1-\mathrm{e}^{-2\sigma z}}{\mathrm{e}^{\sigma z}+\mathrm{e}^{-\sigma z}} + \mathrm{e}^{-\sigma z} - \frac{1}{z+1} + \frac{1}{z+1},$$

whence M_σ belongs to the extended Dunford class $\mathscr{E}(\mathbf{S}_\varphi)$. Therefore $|||M_\sigma(\mathscr{D})|||_p \leq C\,\|M_\sigma\|_{\mathscr{E}(\mathbf{S}_\varphi)}$ [35, Theorem 2.3.3], which, combined with (3.10), yields the required estimate.

Finally we prove (v). The first equality follows from abstract nonsense (see, for instance, [20, Theorem VI.8.6, p. 508]). Formula (3.6), the contractivity of the Poisson semigroup on $L^p(N)$ and the estimate $|||\mathscr{P}_t|||_{1;p} \leq C\, t^{-n/p'}$ (see (2.13)) yield

$$|||M_t(\mathscr{D})|||_{1;p} \leq \big[C\, t^{-n/p'} + |||M_\sigma(\mathscr{D})|||_{1;p}\big];$$

the required bound follows from Proposition 2.2 (ii) and the fact that t is small. □

Given a function f on N, we set

$$\mathscr{N} f := \|f\|_1 + \||\nabla f|\|_1 + \|\mathscr{D} f\|_1 \,, \tag{3.11}$$

whenever the right hand side makes sense. For each p in $[1,\infty)$ we denote by $\mathscr{B}^p$ the space of all measurable functions $F : \Sigma \to \mathbb{C}$ such that

$$\|F\|_{\mathscr{B}^p} := \sup_{t\in(0,2\sigma)} \|F(\cdot,t)\|_p < \infty,$$

endowed with the "norm" $\|\cdot\|_{\mathscr{B}^p}$.

Theorem 3.4. *There exists a constant C such that $\||\boldsymbol{\nabla}\mathscr{P}^0 f|\|_{\mathscr{B}^1} \leq C\,\mathscr{N} f$ for every function f such that $\mathscr{N} f$ is finite.*

Proof. Since $\mathscr{P}^0 f(\cdot,t) = \mathscr{P}^0 f(\cdot, 2\sigma - t)$, it suffices to restrict t to $(0,\sigma]$. By (3.6) and the assumption $\mathscr{D} f \in L^1(N)$,

$$\begin{aligned}\partial_t \mathscr{P}^0 f(\cdot,t) &= -\mathscr{D}\mathscr{P}^N_t f + \frac{1}{2}\mathscr{D}\left[\mathscr{P}^N_{\sigma-t} + \mathscr{P}^N_{\sigma+t}\right] M_\sigma(\mathscr{D}) f \\ &= -\mathscr{P}^N_t \mathscr{D} f + \frac{1}{2}\left[\mathscr{P}^N_{\sigma-t} + \mathscr{P}^N_{\sigma+t}\right] M_\sigma(\mathscr{D})\mathscr{D} f.\end{aligned}$$

The estimate $\|\partial_t \mathscr{P}^0 f(\cdot,t)\|_1 \leq C\,\|\mathscr{D} f\|_1 \leq C\,\mathscr{N} f$, with C independent of f and t, follows by arguing as in the proof of Lemma 3.3 (iv)(with $\mathscr{D} f$ in place of f). Hence $\|\partial_t \mathscr{P}^0 f\|_{\mathscr{B}^1} \leq C\,\mathscr{N} f$.

We now estimate $\||\nabla \mathscr{P}^0 f(\cdot,t)|\|_1$ for t in $(0,\sigma]$. Let $\{B_j\}$ be a covering of N with geodesic balls of radius 1 and centre p_j enjoying the finite overlapping property. Denote by $\{\psi_j\}$ a partition of unity subordinated to this covering with the property that $\{\nabla\psi_j\}$ are uniformly bounded with respect to j, and write $f = \sum_j f_j$, where $f_j := \psi_j\, f$. Then

$$\||\nabla \mathscr{P}^0 f(\cdot,t)|\|_1 \leq \sum_j \||\nabla \mathscr{P}^0 f_j(\cdot,t)|\|_{L^1(2B_j)} + \sum_j \||\nabla \mathscr{P}^0 f_j|(\cdot,t)\|_{L^1(N\setminus 2B_j)}\,,$$

where $2B_j$ denotes the ball with centre p_j and radius 2. Recall that $\mathscr{P}^0 f(\cdot,t) = M_t(\mathscr{D})f$ By Lemma 3.3 (iii), there exists a constant C, independent of j and of t in $(0,\sigma]$, such that

$$\big|\nabla M_t(\mathscr{D}) f_j(x)\big| \leq C\,\mathrm{e}^{(\varepsilon-\lambda_1)d(x,p_j)}\, \big\|f_j\big\|_1 \qquad \forall x \in N\setminus 2B_j.$$

Since $\lambda_1 > 2\beta$ (see (3.2)), the function $x \mapsto \mathrm{e}^{(\varepsilon-\lambda_1)d(x,p_j)}$ is, for ε small enough, in $L^1(N)$, with norm bound independent of j, so that

$$\sup_{t\in(0,\sigma]} \sum_j \big\||\nabla M_t(\mathscr{D}) f_j|\big\|_{L^1(N\setminus 2B_j)} \leq C \sum_j \big\|f_j\big\|_1 \leq C\,\big\|f\big\|_1 .$$

Schwarz's inequality and Proposition 2.2 (v) imply that for any $\rho_1 > n/4$

$$\big\||\nabla M_t(\mathscr{D}) f_j|\big\|_{L^1(2B_j)} \leq \sqrt{\nu\big(2B_j\big)}\, \big\||\nabla M_t(\mathscr{D}) f_j|\big\|_{L^2(2B_j)} \leq C\,\sqrt{\nu\big(2B_j\big)}\, \Xi_{\rho_1}\big(M_t\big)\, \big\|f_j\big\|_1 .$$

Note that $\nu\big(2B_j\big)$ is uniformly bounded with respect to j, because N has bounded geometry. Now, fix δ in $(0,\sigma)$. It is straightforward to check that $\sup_{t\in[\delta,\sigma]} \Xi_{\rho_1}\big(M_t\big)$ is finite. Thus, there exists a constant C such that

$$\sup_{t\in[\delta,\sigma]} \sum_j \big\||\nabla M_t(\mathscr{D}) f_j|\big\|_{L^1(2B_j)} \leq C \sum_j \big\|f_j\big\|_1 \leq C\,\big\|f\big\|_1 \qquad \forall t \in [\delta,\sigma].$$

It remains to estimate $\sup_{t\in(0,\delta)} \sum_j \big\||\nabla M_t(\mathscr{D}) f_j|\big\|_{L^1(2B_j)}$. It is convenient to write $M_t(\mathscr{D})$ as in (3.6). The triangle inequality and the decomposition $\mathscr{P}_t^N = \mathscr{Q}_t^0 + \mathscr{Q}_t^\infty$ (see (2.14)) imply that

$$|\nabla M_t(\mathscr{D}) f_j| \leq |\nabla \mathscr{Q}_t^0 f_j| + |\nabla \mathscr{Q}_t^\infty f_j| + \frac{1}{2}\,\big|\nabla[\mathscr{P}_{\sigma-t}^N - \mathscr{P}_{\sigma+t}^N] M_\sigma(\mathscr{D}) f_j\big|.$$

Observe that $|\nabla \mathscr{Q}_t^0 f_j| \leq \mathrm{e}^{\kappa^2}\, \mathscr{Q}_t^0 |\nabla f_j|$ by Proposition 2.3 (i), whence

$$\big\||\nabla \mathscr{Q}_t^0 f_j|\big\|_{L^1(2B_j)} \leq \mathrm{e}^{\kappa^2}\, \big\|\mathscr{Q}_t^0 |\nabla f_j|\big\|_{L^1(2B_j)} \leq \mathrm{e}^{\kappa^2}\, \big\|\mathscr{P}_t^N |\nabla f_j|\big\|_1 \leq \mathrm{e}^{\kappa^2}\, \big\||\nabla f_j|\big\|_1 ;$$

we have used the contractivity of the Poisson semigroup on $L^1(N)$ in the last inequality. Note that $|\nabla f_j| \leq C\,|\psi_j \nabla f| + C\,\mathbf{1}_{B_j}\, f$, so that

$$\sup_{t\in(0,\delta)} \sum_j \big\||\nabla \mathscr{Q}_t^0 f_j|\big\|_{L^1(2B_j)} \leq C\Big[\sum_j \big\||\psi_j \nabla f|\big\|_1 + \sum_j \big\|\mathbf{1}_{B_j}\, f\big\|_1\Big] \leq C\,\mathscr{N} f.$$

Furthermore, we have trivially

$$\sup_{t\in(0,\delta)} \sum_j \big\||\nabla \mathscr{Q}_t^\infty f_j|\big\|_{L^1(2B_j)} \leq \sum_j \nu\big(2B_j\big) \sup_{t\in(0,\delta)} \big\||\nabla \mathscr{Q}_t^\infty f_j|\big\|_\infty \leq C \sum_j \|f_j\|_1 \leq C\|f\|_1,$$

where the second inequality above follows from Proposition 2.3 (ii) and the fact that N has bounded geometry. Finally, set $F_t(\lambda) := [\mathrm{e}^{(t-\sigma)\lambda} - \mathrm{e}^{-(\sigma+t)\lambda}]\, M_\sigma(\lambda)$. It is straightforward to check that if $\rho_1 > n/4$, then $\sup_{t\in(0,\delta)} \Xi_{\rho_1}\big(F_t\big) < \infty$. Then Schwarz's inequality and Proposition 2.2 (v) imply that

$$\begin{aligned}\sup_{t\in(0,\delta)}\sum_j \big\||\nabla[F_t(\mathscr{D})f_j]|\big\|_{L^1(2B_j)} &\leq \sum_j \sqrt{\nu(2B_j)}\sup_{t\in(0,\delta)}\big\||\nabla[F_t(\mathscr{D})f_j]|\big\|_{L^2(2B_j)}\\ &\leq C\sum_j \big\|f_j\big\|_1\\ &\leq C\,\big\|f\big\|_1\,.\end{aligned}$$

The required conclusion follows by combining the estimates above. □

We complete this section by proving Proposition 3.1.

Proof (of Proposition 3.1). The function $\mathscr{P}^\eta f$ is harmonic in Σ_η, hence smooth therein, by elliptic regularity. We prove the continuity at the boundary. Note that $\mathscr{P}^\eta f(\cdot,t) = \mathscr{P}^\eta f(\cdot,2\sigma-t)$ for every t in $(\eta,2\sigma-\eta)$. Thus, it suffices to prove the continuity at $t=\eta$. Fix x in N, and write

$$\mathscr{P}^\eta f(y,t)-f(x) = \mathscr{P}^\eta f(y,t)-f(y)+f(y)-f(x).$$

Since f is in $C_0(N)$, it is uniformly continuous on N; hence for every $\varepsilon>0$ there exists $\delta>0$ such that $\big|f(y)-f(x)\big|<\varepsilon$ whenever $d(x,y)<\delta$.

By (3.6), we can write $\mathscr{P}^\eta f(\cdot,t)-f = \mathscr{P}^N_{t-\eta}f-f+\dfrac{1}{2}\big[\mathscr{P}^N_{\sigma-t}-\mathscr{P}^N_{\sigma+t-2\eta}\big]M^\eta_\sigma(\mathscr{D})f$. The heat semigroup $\{\mathscr{H}^N_t\}$ is strongly continuous on $C_0(N)$ [17, Lemma 5.2.8]. A straightforward argument using the subordination formula (2.12) shows that the same holds for the Poisson semigroup $\{\mathscr{P}^N_t\}$. Hence $\lim_{t\downarrow\eta}\big\|\mathscr{P}^N_{t-\eta}f-f\big\|_{C_0(N)}=0$.

It remains to prove that

$$\lim_{t\downarrow\eta}\big\|\big[\mathscr{P}^N_{\sigma-t}-\mathscr{P}^N_{\sigma+t-2\eta}\big]M^\eta_\sigma(\mathscr{D})f\big\|_\infty = 0. \tag{3.12}$$

To this end, fix $\varepsilon>0$ and consider a sequence $\{\varphi_k\}\subset C^\infty_c(N)$ such that $\|\varphi_k-f\|_\infty\to 0$ as $k\to\infty$. The Poisson semigroup is contractive on $L^\infty(N)$, hence so is on $C_0(N)$. Thus, by a variant of Lemma 3.3 (iv) (with $\eta>0$),

$$\begin{aligned}\big\|\big[\mathscr{P}^N_{\sigma-t}-\mathscr{P}^N_{\sigma+t-2\eta}\big]M^\eta_\sigma(\mathscr{D})(f-\varphi_k)\big\|_\infty &\leq 2\,\big\|M^\eta_\sigma(\mathscr{D})(f-\varphi_k)\big\|_\infty\\ &\leq 2\,\big\|\!\big|M^\eta_\sigma(\mathscr{D})\big|\!\big\|_\infty\,\big\|f-\varphi_k\big\|_\infty\\ &\leq C\,\big[1+\big\|M^\eta_\sigma(\mathscr{D})\big\|_{\mathscr{E}(\mathbf{S}_\varphi)}\big]\,\big\|f-\varphi_k\big\|_\infty\,.\end{aligned}$$

In particular, we can fix k_0 large enough so that

$$\big\|\big[\mathscr{P}^N_{\sigma-t}-\mathscr{P}^N_{\sigma+t-2\eta}\big]M^\eta_\sigma(\mathscr{D})(f-\varphi_{k_0})\big\|_\infty < \varepsilon/2 \qquad \forall t\in(\eta,\sigma). \tag{3.13}$$

Furthermore

$$\begin{aligned}\big[\mathscr{P}^N_{\sigma-t}-\mathscr{P}^N_{\sigma+t-2\eta}\big]\,M^\eta_\sigma\big(\mathscr{D}\big)\varphi_{k_0} &= -\int_{\sigma-t}^{\sigma+t-2\eta}\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{P}^N_s M^\eta_\sigma\big(\mathscr{D}\big)\varphi_{k_0}\,\mathrm{d}s\\ &= \int_{\sigma-t}^{\sigma+t-2\eta}\mathscr{D}\mathscr{P}^N_s M^\eta_\sigma\big(\mathscr{D}\big)\varphi_{k_0}\,\mathrm{d}s,\\ &= \int_{\sigma-t}^{\sigma+t-2\eta}\mathscr{P}^N_s\mathscr{D}M^\eta_\sigma\big(\mathscr{D}\big)\varphi_{k_0}\,\mathrm{d}s,\end{aligned}$$

so that, using also Proposition 2.2 (iii),

$$\begin{aligned}\big\|\big[\mathscr{P}^N_{\sigma-t}-\mathscr{P}^N_{\sigma+t-2\eta}\big]\,M^\eta_\sigma\big(\mathscr{D}\big)\varphi_{k_0}\big\|_\infty &\le \int_{\sigma-t}^{\sigma+t-2\eta}\big\|\mathscr{P}^N_s\mathscr{D}M^\eta_\sigma\big(\mathscr{D}\big)\varphi_{k_0}\big\|_\infty\,\mathrm{d}s\\ &\le \int_{\sigma-t}^{\sigma+t-2\eta}\big\|\mathscr{D}M^\eta_\sigma\big(\mathscr{D}\big)\varphi_{k_0}\big\|_\infty\,\mathrm{d}s\\ &\le C\,(t-\eta)\,\big\|\varphi_{k_0}\big\|_2\,,\end{aligned}$$

which is smaller than $\varepsilon/2$ for t close enough to η. Together with (3.13), this proves (3.12) and concludes the proof of the proposition. □

4. Estimates for the Green function on slices

The *Dirichlet heat semigroup* for Σ is given by $\mathscr{H}^\Sigma_t=\mathscr{H}^N_t\otimes\mathscr{H}^{[0,2\sigma]}_t$, where $\{\mathscr{H}^{[0,2\sigma]}_t : t>0\}$ denotes the heat semigroup on $[0,2\sigma]$ with Dirichlet boundary conditions. Recall that $\lambda_1=\pi/(2\sigma)$ (see (3.3)): the number λ_1^2 is the first eigenvalue of the operator $-\mathrm{d}^2/\mathrm{d}x^2$ with Dirichlet boundary conditions on $[0,2\sigma]$. The associated eigenfunction is $\sin\lambda_1 u$. Set

$$\varrho(u)=\operatorname{dist}\big(u,\mathbb{R}\setminus[0,2\sigma]\big) \tag{4.1}$$

and observe that $\varrho(u)\asymp\sin\lambda_1 u$ in $(0,2\sigma)$. Let the family $\{h^{[0,2\sigma]}_t : t>0\}$ denote the heat kernel on $[0,2\sigma]$ with Dirichlet boundary conditions and note the following well known estimate (see, for instance, [67] and the references therein)

$$h^{[0,2\sigma]}_t(u,v)\le\begin{cases} C\,\min\Big(\dfrac{\varrho(u)\,\varrho(v)}{t},1\Big)\,t^{-1/2}\,\mathrm{e}^{-|u-v|^2/(4t)} & \forall t\in(0,1]\\ C\,\varrho(u)\,\varrho(v)\,\mathrm{e}^{-\lambda_1^2 t} & \forall t\in(1,\infty)\end{cases} \tag{4.2}$$

for every u and v in $[0,2\sigma]$.

The *Green operator* $\mathscr{G}_\Sigma$ for the slice Σ is defined by

$$\mathscr{G}_\Sigma:=\int_0^\infty\mathscr{H}^\Sigma_t\,\mathrm{d}t. \tag{4.3}$$

It is not hard to prove that given a reasonable function B on Σ (for instance $B\in C^0(\Sigma)\cap L^r(\Sigma)$ for some r in $(1,\infty)$), the function $\mathscr{G}_\Sigma B$ solves the problem

$$-\Delta u=B\quad\text{in }\Sigma\qquad u(\cdot,0)=0=u(\cdot,2\sigma)$$

in the sense of distributions. At least formally, off the diagonal of $\Sigma \times \Sigma$ the kernel of $\mathscr{G}_\Sigma$ is given by the formula

$$k_{\mathscr{G}_\Sigma}\big((x,u),(y,v)\big) := \int_0^\infty h_t^N(x,y)\, h_t^{[0,2\sigma]}(u,v)\,\mathrm{d}t; \tag{4.4}$$

here (x,u) and (y,v) are in Σ, $(x,u) \neq (y,v)$. We shall consider the operators $\mathscr{G}_\Sigma^j$, $j = 1,2,\ldots$, and their distributional kernels $k_{\mathscr{G}_\Sigma^j}$.

We *claim* that

$$k_{\mathscr{G}_\Sigma^j}\big((x,u),(y,v)\big) = \frac{1}{(j-1)!}\int_0^\infty h_t^N(x,y)\, h_t^{[0,2\sigma]}(u,v)\, t^{j-1}\,\mathrm{d}t. \tag{4.5}$$

We argue by induction. If $j=1$, then (4.5) reduces to (4.4). Assume that (4.5) holds for j, and consider $k_{\mathscr{G}_\Sigma^{j+1}}$. Clearly

$$\begin{aligned}
&k_{\mathscr{G}_\Sigma^{j+1}}\big((x,u),(z,w)\big)\\
&= \int_\Sigma k_{\mathscr{G}_\Sigma^j}\big((x,u),(y,v)\big)\, k_{\mathscr{G}_\Sigma}\big((y,v),(z,w)\big)\,\mathrm{d}\nu(y)\,\mathrm{d}v\\
&= \frac{1}{(j-1)!}\int_\Sigma \mathrm{d}\nu(y)\,\mathrm{d}v \int_0^\infty\int_0^\infty h_s^{[0,2\sigma]}(u,v)\, h_s^N(x,y)\, h_t^{[0,2\sigma]}(v,w)\, h_t^N(y,z)\, s^{j-1}\,\mathrm{d}s\,\mathrm{d}t\\
&= \frac{1}{(j-1)!}\int_0^\infty\int_0^\infty h_{s+t}^{[0,2\sigma]}(u,w)\, h_{s+t}^N(x,z)\, s^{j-1}\,\mathrm{d}s\,\mathrm{d}t;
\end{aligned}$$

we have used the inductive hypothesis in the second equality above and the semigroup property of the heat kernel in the third. Then we perform two subsequent changes of variables: we set $t = \tau s$ in the integral with respect to t and obtain that

$$k_{\mathscr{G}_\Sigma^{j+1}}\big((x,u),(z,w)\big) = \frac{1}{(j-1)!}\int_0^\infty\int_0^\infty h_{s(1+\tau)}^{[0,2\sigma]}(u,w)\, h_{s(1+\tau)}^N\big(x,z\big)\, s^j\,\mathrm{d}s\,\mathrm{d}\tau;$$

then we set $s(1+\tau) = \sigma$ in the integral with respect to s, and the right hand side of the formula above transforms to

$$\frac{1}{(j-1)!}\int_0^\infty\int_0^\infty h_\sigma^{[0,2\sigma]}(u,w)\, h_\sigma^N(x,z)\,\frac{\sigma^j}{(1+\tau)^{j+1}}\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

Integrating with respect to τ gives the required formula (4.5), and concludes the proof of the claim.

In Proposition 4.2 below we establish pointwise estimates for $k_{\mathscr{G}_\Sigma^j}$. Preliminarily, we determine the order of magnitude of

$$J(d) := \int_1^\infty t^{j-3/2}\,\mathrm{e}^{-(\lambda_1\sqrt{t}-d\sqrt{c/t})^2}\,\mathrm{d}t$$

as d tends to infinity.

Lemma 4.1. *If d tends to infinity, then for every integer $j \geq 1$ one has $J(d) \asymp d^{j-1}$ (i.e., there exist positive constants C_1 and C_2 such that $C_1\, d^{j-1} \leq J(d) \leq C_2\, d^{j-1}$).*

Proof. We write $-(\lambda_1\sqrt{t}-d\sqrt{c/t})^2=-\sqrt{c}\lambda_1 d\Big(\sqrt{\frac{\lambda_1 t}{\sqrt{c}d}}-\sqrt{\frac{\sqrt{c}d}{\lambda_1 t}}\Big)^2$, change variables in the integral ($\sqrt{\lambda_1 t/\sqrt{c}d}=\tau$), and see that

$$J(d)=2\Big(\frac{\sqrt{c}d}{\lambda_1}\Big)^{j-1/2}\int_{\sqrt{\lambda_1/(\sqrt{c}d)}}^{\infty}\tau^{2(j-1)}\,\mathrm{e}^{d\psi(\tau)}\,\mathrm{d}\tau,$$

where $\psi(\tau):=-\sqrt{c}\lambda_1(\tau-1/\tau)^2$. Note that the phase $\psi(\tau)$ has just one critical point at 1. Fix $0<\tau_1<1<\tau_2$. By applying the Laplace method (see, for instance, [24, formula (2), p. 37]), one checks that

$$\int_{\tau_1}^{\tau_2}\tau^{2(j-1)}\,\mathrm{e}^{d\psi(\tau)}\,\mathrm{d}\tau\asymp d^{-1/2}$$

as d tends to infinity. Moreover, for $\delta>0$ small enough $\psi(\tau)\leq-\delta\tau^2$ in $[\tau_2,\infty)$, so that

$$\int_{\tau_2}^{\infty}\tau^{2(j-1)}\,\mathrm{e}^{d\psi(\tau)}\,\mathrm{d}\tau\leq\mathrm{e}^{-\delta d\tau_2^2/2}\int_{\tau_2}^{\infty}\tau^{2(j-1)}\,\mathrm{e}^{-\delta d\tau^2/2}\,\mathrm{d}\tau\leq C\mathrm{e}^{-\delta d\tau_2^2/2}.$$

Similarly, for $\gamma>0$ small enough $\psi(\tau)\leq-\gamma\tau^{-2}$ in $(0,\tau_1)$, so that

$$\int_{\sqrt{\lambda_1/(\sqrt{c}d)}}^{\tau_1}\tau^{2(j-1)}\,\mathrm{e}^{d\psi(\tau)}\,\mathrm{d}\tau\leq\mathrm{e}^{-\gamma d\tau_1^{-2}/2}\int_0^{\tau_1}\tau^{2(j-1)}\,\mathrm{e}^{-\gamma d\tau_1^{-2}/2}\,\mathrm{d}\tau\leq C\mathrm{e}^{-\gamma d\tau_1^{-2}/2}.$$

By combining the estimates above, we see that J has the required asymptotic behaviour at infinity. □

Proposition 4.2. *Suppose that j is a positive integer and that $n\geq 2$. There exists a positive constant C such that the following hold:*

(i) *if $D\big((x,u),(y,v)\big)\geq 2$, then $k_{\mathscr{G}_\Sigma^j}\big((x,u),(y,v)\big)\leq C\,d(x,y)^{j-1}\,\mathrm{e}^{-2\lambda_1 d(x,y)\sqrt{c}}$;*

(ii) *if $D\big((x,u),(y,v)\big)\leq 2$, then*

$$k_{\mathscr{G}_\Sigma^j}\big((x,u),(y,v)\big)\leq\begin{cases}C\,D^{2\gamma} & \text{if } \gamma<0\\ C\,\log\dfrac{4}{D} & \text{if } \gamma=0\\ C & \text{if } \gamma>0,\end{cases}$$

where $\gamma:=j-(n+1)/2$.

Proof. We estimate $k_{\mathscr{G}_\Sigma^j}$ from above by inserting in the integral in (4.5) the estimates for $h_t^{[0,2\sigma]}$ in (4.2) and the upper bound (2.3) for h_t^N (observing that the constant c in (2.3) is smaller or equal than 1/4). Thus,

$$k_{\mathscr{G}_\Sigma^j}\big((x,u),(z,w)\big)\leq C\,\Big[I\big((x,u),(z,w)\big)+J(x,z)\Big],$$

where

$$I\big((x,u),(z,w)\big)=\int_0^1 t^{\gamma-1}\mathrm{e}^{-cD^2/t}\,\mathrm{d}t$$

and

$$J(x,z)=\int_1^\infty t^{j-3/2}\mathrm{e}^{-\lambda_1^2 t-cd(x,z)^2/t}\,\mathrm{d}t.$$

We estimate I and J separately. First, changing variables ($D^2/t = \tau$), we see that

$$I = D^{2\gamma} \int_{D^2}^{\infty} \tau^{-\gamma} \, \mathrm{e}^{-c\tau} \, \frac{\mathrm{d}\tau}{\tau} \asymp \begin{cases} C\, D^{2\gamma} & \text{if } \gamma < 0 \\ C \log \dfrac{1}{D} & \text{if } \gamma = 0 \\ C & \text{if } \gamma > 0 \end{cases}$$

as D tends to 0. Furthermore,

$$I \leq C\, D^{2\gamma} \mathrm{e}^{-cD^2/2} \leq C \big[1 + d(x,y)\big]^{j-1} \, \mathrm{e}^{-2\lambda_1 d(x,y)\sqrt{c}}$$

when $D \geq 2$. Concerning J, clearly it tends to a constant as d tends to 0. Now assume that d is large and write $-\lambda_1^2 t - cd^2/t = -\big(\lambda_1 \sqrt{t} - d\sqrt{c/t}\big)^2 - 2\lambda_1 d\sqrt{c}$. Then,

$$J = \mathrm{e}^{-2\lambda_1 d\sqrt{c}} \int_1^{\infty} t^{j-3/2} \, \mathrm{e}^{-(\lambda_1\sqrt{t} - d\sqrt{c/t})^2} \, \mathrm{d}t \leq C\, d^{j-1} \, \mathrm{e}^{-2\lambda_1 d\sqrt{c}};$$

the inequality above follows from Lemma 4.1.

The estimates in (i) and (ii) follow directly from the analysis above. □

Remark 4.3. The estimates for $k_{\mathscr{G}_\Sigma}$ in Proposition 4.2 are not best possible. In particular, they do not capture the asymptotic behaviour of $k_{\mathscr{G}_\Sigma}$ near the boundary of Σ. We do not insist on this point because such behaviour is not needed in the sequel. However, for later purposes (see the proof of Lemma 6.1), we need the following straightforward estimate: for each $\delta > 0$ there exists a positive constant C such that

$$k_{\mathscr{G}_\Sigma}(X,Y) \leq C \, \min\big(\varrho(u)\, \varrho(v), \mathrm{e}^{-2\lambda_1 d(x,y)\sqrt{c}}\big) \qquad \forall X, Y \in \Sigma : D(X,Y) \geq \delta. \tag{4.6}$$

Here $X = (x,u)$ and $Y = (y,v)$ and ϱ is defined in (4.1).

The estimates in Proposition 4.2 imply that $k_{\mathscr{G}_\Sigma}(X,Y) \leq C\, \mathrm{e}^{-2\lambda_1 d(x,y)\sqrt{c}}$ for every X and Y in Σ such that $D(X,Y) \geq \delta$.

To prove that $k_{\mathscr{G}_\Sigma}(X,Y) \leq C\, \varrho(u)\, \varrho(v)$ in the same range of X and Y, we insert in the integral in (4.5) (with $j = 1$) the estimates for $h_t^{[0,2\sigma]}$ in (4.2) and the upper bound (2.3) for h_t^N. Observe that the assumption $D(X,Y) \geq \delta$ implies that $k_{\mathscr{G}_\Sigma}(X,Y) \leq C\, \big(I_1 + I_2 + I_3\big)$, where

$$I_1 := \int_0^{\varrho(u)\varrho(v)} t^{-(n+1)/2} \, \mathrm{e}^{-c\delta^2/t} \, \mathrm{d}t, \qquad I_2 := \varrho(u)\, \varrho(v) \int_{\varrho(t)\varrho(u)}^{1} t^{-(n+3)/2} \, \mathrm{e}^{-c\delta^2/t} \, \mathrm{d}t$$

and

$$I_3 := \varrho(u)\varrho(v) \int_1^{\infty} t^{j-3/2} \mathrm{e}^{-\lambda_1^2 t - cd(x,z)^2/t} \, \mathrm{d}t.$$

The required estimate follows directly from this and a straightforward calculation.

Next we establish some mapping properties of $\mathscr{G}_\Sigma^j$. For simplicity, in the sequel we write Υ instead of Υ_2 (see (2.16)). Denote by $K_j^0 : \Sigma \times \Sigma \to [0,\infty)$ the function defined by

$$K_j^0 = \begin{cases} \mathbf{1}_\Upsilon \, D^{2j-1-n} & \text{if } j < (n+1)/2 \\ \mathbf{1}_\Upsilon \, \log \dfrac{4}{D} & \text{if } j = (n+1)/2 \\ \mathbf{1}_\Upsilon & \text{if } j > (n+1)/2, \end{cases} \tag{4.7}$$

and by $\mathscr{K}_j^0$ the integral operator with kernel K_j^0 acting on functions defined on $N \times \mathbb{R}$. For each $\delta > 0$ denote by $K_\delta^\infty : N \times N \to [0,\infty)$ the function defined by

$$K_\delta^\infty(x,y) = \mathrm{e}^{-\delta d(x,y)} \qquad \forall (x,y) \in N \times N, \tag{4.8}$$

and by $\mathscr{K}_\delta^\infty$ the integral operator with kernel K_δ^∞ acting on functions defined on N. Notice that, by Proposition 4.2, for every $\delta < 2\lambda_1\sqrt{c}$ there exists a constant C such that

$$\big|\mathscr{G}_\Sigma^j F(x,u)\big| \leq C\,\big[\mathscr{K}_j^0|F|(x,u) + \mathscr{K}_\delta^\infty|F^\flat|(x)\big] \qquad \forall (x,u) \in \Sigma, \tag{4.9}$$

where $F^\flat(x)$ is as in (3.4). This observation reduces the proof of estimates for $\mathscr{G}_\Sigma^j$ to the proof of similar estimates for $\mathscr{K}_j^0$ and $\mathscr{K}_\delta^\infty$. We study the mapping properties of these operators in the next proposition.

Proposition 4.4. *Suppose that j is a positive integer and that $n \geq 2$. The following hold:*

(i) *if $\mathscr{K}_j^0$ is bounded from $L^p(\Sigma)$ to $L^r(\Sigma)$ and $\mathscr{K}_\delta^\infty$ is bounded from $L^p(N)$ to $L^r(N)$ for some $\delta < 2\lambda_1\sqrt{c}$, then $\mathscr{G}_\Sigma^j$ is bounded from $L^p(\Sigma)$ to $L^r(\Sigma)$;*
(ii) *$\mathscr{G}_\Sigma^j$ is bounded on $L^p(\Sigma)$ for all p in $[1,\infty]$;*
(iii) *$\mathscr{G}_\Sigma$ is bounded from $L^1(\Sigma)$ to weak-$L^{(n+1)/(n-1)}(\Sigma)$ and from $L^p(\Sigma)$ to $L^r(\Sigma)$ when $1 < p < (n+1)/2$ and $1/r = 1/p - 2/(n+1)$;*
(iv) *if $r > 1$, F is in $C^0(\Sigma) \cap L^r(\Sigma)$ and $J > (n+1)/2$, then $\mathscr{G}_\Sigma^J F$ is a bounded continuous function on Σ and*
$$\big\|\mathscr{G}_\Sigma^J F\big\|_{C_b(\Sigma)} \leq C\,\big\|F\big\|_{L^r(\Sigma)}\,;$$
(v) *$\mathscr{G}_\Sigma$ is bounded on $\mathscr{B}^p$ for each p in $[1,\infty)$ and $\lim_{t\to\partial(0,2\sigma)} \big\|\mathscr{G}_\Sigma F(\cdot,t)\big\|_p = 0$ for every F in $\mathscr{B}^p$.*

Proof. First we prove (i). Formula (4.9) and the assumptions on $\mathscr{K}_j^0$ and $\mathscr{K}_\varepsilon^\infty$ imply that

$$\begin{aligned} \big\|\mathscr{G}_\Sigma^j F\big\|_{L^r(\Sigma)} &\leq C\,\big[\,\big\|\mathscr{K}_j^0 F\big\|_{L^r(\Sigma)} + \big\|\mathscr{K}_\varepsilon^\infty F^\flat\big\|_r\,\big] \\ &\leq C\,\big[\,\big\|F\big\|_{L^p(\Sigma)} + \big\|F^\flat\big\|_p\,\big] \\ &\leq C\,\big\|F\big\|_{L^p(\Sigma)}\,; \end{aligned}$$

we have used (3.5) in the last inequality.

Now we prove (ii). By interpolation, it suffices to prove that $\mathscr{G}_\Sigma^j$ is bounded on $L^1(\Sigma)$ and on $L^\infty(\Sigma)$. Since $k_{\mathscr{G}_\Sigma^j}$ is symmetric, a duality argument shows that it suffices to prove that $\mathscr{G}_\Sigma^j$ is bounded on $L^1(\Sigma)$. Now, the boundedness of $\mathscr{G}_\Sigma^j$ on $L^1(\Sigma)$ follows from (i) and the boundedness of $\mathscr{K}_j^0$ on $L^1(\Sigma)$ and of $\mathscr{K}_\delta^\infty$ on $L^1(N)$ for some δ in $(2\beta, 2\lambda_1\sqrt{c})$. To prove this, it suffices to show that for such values of δ

$$\sup_{Y\in\Sigma}\int_\Sigma \mathscr{K}_j^0(X,Y)\,\mathrm{d}\mathscr{Y}(X)<\infty \quad\text{and}\quad \sup_{y\in N}\int_N \mathscr{K}_\delta^\infty(x,y)\,\mathrm{d}\nu(x)<\infty. \tag{4.10}$$

These estimates can be obtained easily by integrating in polar co-ordinates centred at Y and at y, respectively. We omit the details.

Now (iii) follows from (i) and the boundedness of $\mathscr{K}_j^0$ from $L^p(\Sigma)$ to $L^r(\Sigma)$ and of $\mathscr{K}_\delta^\infty$ from $L^r(N)$ to $L^q(N)$ for all q in $[r,\infty]$ and δ in $(2\beta, 2\lambda_1\sqrt{c})$. Specifically, K_δ^∞ is bounded, whence $\mathscr{K}_\delta^\infty$ is bounded from $L^1(N)$ to $L^\infty(N)$. We have proved in (ii) that $\mathscr{K}_\delta^\infty$ is bounded on $L^1(N)$. Since K_δ^∞ is symmetric, $\mathscr{K}_\delta^\infty$ is also bounded on $L^\infty(N)$. By interpolation and duality, it follows that $\mathscr{K}_\delta^\infty$ maps $L^p(N)$ to $L^q(N)$ for all $1\le p\le q\le\infty$.

The proof that $\mathscr{K}_j^0$ maps $L^1(\Sigma)$ to weak-$L^{(n+1)/(n-1)}(\Sigma)$ and $L^p(\Sigma)$ to $L^r(\Sigma)$ when $1<p<(n+1)/2$ and $1/r=1/p-2/(n+1)$ can be obtained by adapting the proof of [55, Theorem 1, p. 119]. We omit the details.

Now we prove (iv). Notice that for each positive integer $k\le J$

$$\left\|\boldsymbol{\Delta}^k\mathscr{G}_\Sigma^J F\right\|_{L^r(\Sigma)} = \left\|\mathscr{G}_\Sigma^{J-k} F\right\|_{L^r(\Sigma)} \le C\left\|F\right\|_{L^r(\Sigma)};$$

the last inequality follows from (ii).

For the sake of completeness we give a proof of the continuity of $\mathscr{G}_\Sigma^J F$ on Σ, which is, we believe, quite standard. Suppose that $X\in\Sigma$. Recall that $\mathscr{G}_\Sigma^J F$ is a distributional solution of $\boldsymbol{\Delta}^J V=F$ on Σ. Choose a harmonic coordinate system (U,ϕ_U) with $U\subset\Sigma$ open set containing X and $\phi_U: U\to\mathbb{R}^{n+1}$. For any function V on U, set $\widetilde{V}:=V\circ\phi_U^{-1}$. Then $L^J(\widetilde{\mathscr{G}_\Sigma^J F})=(\boldsymbol{\Delta}^J\mathscr{G}_\Sigma^J F)^\sim=\widetilde{F}$ in the sense of distributions on U, where L is the elliptic operator defined in $\phi_U(U)$ by $\displaystyle\sum_{i,j}(g\circ\phi_U^{-1})^{ij}\partial_{ij}^2$. Since $\widetilde{F}$ is continuous, it is in $L^2_{loc}(U)$. By elliptic theory (see, for instance, [27, Theorem 6.33]), $\widetilde{\mathscr{G}_\Sigma F}\in W^{2J,2}_{loc}(U)$. The latter inclusion, with $n'=n+1$, is a consequence of local Sobolev embeddings. Now, if $2J>(n+1)/2$, then $W^{2J,2}_{loc}(U)$ is contained in $C(U)$, as required to conclude the proof of the continuity of $\mathscr{G}_\Sigma^J F$.

It remains to prove that $\mathscr{G}_\Sigma^J F$ is bounded on Σ. By (i), in order to prove that $\mathscr{G}_\Sigma^J F$ is bounded, it suffices to prove that $\mathscr{K}_J^0$ maps $L^r(\Sigma)$ to $L^\infty(\Sigma)$, and that $\mathscr{K}_\delta^\infty$ maps $L^r(N)$ to $L^\infty(N)$. In the proof of (iii), we have already shown that $\mathscr{K}_\delta^\infty$ maps $L^r(N)$ to $L^q(N)$ for all q in $[r,\infty]$ (and $\delta\in(2\beta,2\lambda_1\sqrt{c})$). Thus, it remains to consider $\mathscr{K}_J^0$. The kernel K_J^0 of $\mathscr{K}_J^0$ is supported in a neighbourhood of the diagonal in $\Sigma\times\Sigma$, and it is bounded (see (4.7)). Thus, $\mathscr{K}_J^0$ maps $L^r(\Sigma)$ to $L^q(\Sigma)$ for all q in $[p,\infty]$. This concludes the proof of (iv).

Finally we prove (v). We already know that $\left\|\mathscr{K}_\delta^\infty F^\flat\right\|_p\le C\left\|F^\flat\right\|_p$ for every δ in $(2\beta,2\lambda_1\sqrt{c})$. Trivially, $\left\|F^\flat\right\|_p\le\left\|F\right\|_{\mathscr{B}^p}$, whence $\left\|\mathscr{K}_\delta^\infty F^\flat\right\|_p\le C\left\|F\right\|_{\mathscr{B}^p}$.

Thus, arguing as in (i), it suffices to show that $\mathscr{K}^0$ is bounded on $\mathscr{B}^p$ for each p in $[1,\infty)$. Notice that

$$\begin{aligned}\big\|\mathscr{K}^0 F\big\|_{\mathscr{B}^p} &\leq \sup_{t\in(0,2\sigma)} \Big[\int_N \mathrm{d}\nu(x) \Big|\int_\Sigma (\mathbf{1}_\Upsilon D^{1-n})\big((x,t),(y,v)\big)\, F(y,v)\,\mathrm{d}\nu(y)\,\mathrm{d}v\Big|^p\Big]^{1/p} \\ &\leq C \sup_{t\in(0,2\sigma)} \Big[\int_N \mathrm{d}\nu(x) \Big|\int_{B_2(x)} d^{1-n}(x,y)\, F^\flat(y)\,\mathrm{d}\nu(y)\Big|^p\Big]^{1/p} \\ &\leq C \Big[\int_N \big|F^\flat(x)\big|^p\,\mathrm{d}\nu(x)\Big]^{1/p} \\ &\leq C\,\big\|F\big\|_{\mathscr{B}^p} :\end{aligned}$$

the penultimate inequality follows from the fact that the kernel $\mathbf{1}_V\, d^{1-n}$, where $V = \big\{(x,y)\in N\times N : d(x,y)\leq 2\big\}$ is symmetric and satisfies $\sup_{y\in N}\int_N \big[\mathbf{1}_V\, d^{1-n}\big](x,y)\,\mathrm{d}\nu(x) < \infty$, whence the corresponding integral operator is bounded on $L^p(N)$ for every p in $[1,\infty]$.

Suppose now that F is in $\mathscr{B}^p$. By (4.3),

$$\begin{aligned}\big\|\mathscr{G}_\Sigma F(\cdot,t)\big\|_p &\leq \int_0^\infty \big\|\mathscr{H}_s^\Sigma F(\cdot,t)\big\|_p\ \mathrm{d}s \\ &\leq \int_0^\infty \mathrm{d}s \int_0^{2\sigma} h_s^{[0,2\sigma]}(t,u)\ \big\|\mathscr{H}_s^N F(\cdot,u)\big\|_p\ \mathrm{d}u.\end{aligned}$$

Now, $\big\|\mathscr{H}_s^N F(\cdot,u)\big\|_p \leq \big\|F(\cdot,u)\big\|_p$, by the contractivity of $\mathscr{H}_s^N$ on $L^p(N)$, whence

$$\big\|\mathscr{G}_\Sigma F(\cdot,t)\big\|_p \leq \big\|F\big\|_{\mathscr{B}^p} \int_0^{2\sigma}\int_0^\infty h_s^{[0,2\sigma]}(t,u)\,\mathrm{d}s\,\mathrm{d}u.$$

Now, the pointwise estimates (4.2) imply that

$$\int_1^\infty h_s^{[0,2\sigma]}(t,u)\,\mathrm{d}s \leq C\,\varrho(t)\,\varrho(u)\int_1^\infty \mathrm{e}^{-\lambda_1^2 s}\,\mathrm{d}s \leq C\,\varrho(t) \qquad \forall u\in(0,2\sigma).$$

and

$$\begin{aligned}\int_0^1 h_s^{[0,2\sigma]}(t,u)\,\mathrm{d}s &\leq C\int_0^{\varrho(t)\varrho(u)} \mathrm{e}^{-|t-u|^2/(4s)}\,\frac{\mathrm{d}s}{\sqrt{s}} + C\,\varrho(t)\,\varrho(u)\int_{\varrho(t)\varrho(u)}^1 \mathrm{e}^{-|t-u|^2/(4s)}\,\frac{\mathrm{d}s}{s^{3/2}} \\ &\leq C\int_0^{\varrho(t)\varrho(u)} s^{-1/2}\,\mathrm{d}s + C\,\varrho(t)\int_{\varrho(t)\varrho(u)}^1 s^{-3/2}\,\mathrm{d}s \\ &\leq C\,\Big(\frac{\varrho(t)}{\varrho(u)}\Big)^{1/2} \qquad \forall u\in(0,2\sigma).\end{aligned}$$

By combining the estimates above, we see that $\big\|\mathscr{G}_\Sigma F(\cdot,t)\big\|_p \leq C\,\varrho(t)^{1/2}\ \big\|F\big\|_{\mathscr{B}^p}$, which tends to 0 as $\varrho(t)$ tends to 0, as required.

This concludes the proof of (v), and of the proposition. □

Remark 4.5. Using (iii), it is straightforward to see that the assumption $F\in C^0(\Sigma)$ in (iv) can be skipped up to choosing larger J.

5. Maximal inequalities

The purpose of this section is to prove Theorem 5.1, which contains an analogue for the slice Σ of certain maximal inequalities that Dindoš [19, Section 10] proved in a compact setting. We emphasize that our result, Theorem 5.1 (i), is concerned with maximal operators of *harmonic* functions on Σ, whereas Dindoš proved a similar estimate for generic functions. Our additional assumption of harmonicity allows us to use Harnack's inequality in our proof, thereby simplifying Dindoš' argument.

We need the following notation. Suppose that α is a (small) real number. For z in N, denote by $\Gamma_\alpha(z)$ the subset of Σ, symmetric with respect to the "line" $t=\sigma$ and whose restriction to the slice $N\times(0,\sigma]$ is the cone $\{(x,u)\in N\times(0,\sigma]: d(x,z)\leq\alpha\, u\}$. We say that a function F on Σ is *symmetric* if $F(\cdot,u)=F(\cdot,2\sigma-u)$ for every u in $(0,2\sigma)$. Given a nonnegative symmetric function F on Σ, denote by F^* its nontangential maximal function, defined by

$$F^*(z):=\sup_{(x,u)\in\Gamma_\alpha(z)} F(x,u) \qquad \forall z\in N. \tag{5.1}$$

Theorem 5.1. *Suppose that p is in $(1,\infty)$, α is small enough and j is a positive integer. Then there exists a constant C such that the following hold:*

(i) $\big\|\big(\mathscr{G}_\Sigma^j H\big)^*\big\|_p\leq C\,\big\|H^*\big\|_p$ *for every positive symmetric harmonic function H on Σ;*
(ii) *if $J>(n+1)/2$, then* $\big\|\big(\mathscr{G}_\Sigma^J S\big)^*\big\|_p\leq C\,\big\|S\big\|_{L^p(\Sigma)}$ *for every nonnegative symmetric function S on Σ.*

Proof. We prove (i) in the case where $j\leq\lfloor(n+1)/2\rfloor$. The modifications needed to cover the case where $j>\lfloor(n+1)/2\rfloor$ are straightforward and are left to the interested reader. Simply, one needs to use different local estimates for $k_{\mathscr{G}_\Sigma^j}$, depending on the dimension n (see Proposition 4.2 (ii)).

By (4.9) it is enough to estimate $(\mathscr{K}_j^0 H)^*$ and $\sup_{(x,u)\in\Gamma_\alpha(z)}\mathscr{K}_\delta^\infty H^\flat(x)$ when δ is in $(2\beta,2\lambda_1\sqrt{c})$. Notice that, for z in N

$$\begin{aligned}\sup_{X\in\Gamma_\alpha(z)}\mathscr{K}_\delta^\infty H^\flat(x) &\leq C\sup_{x\in B_{\alpha\sigma}(z)}\int_N \mathrm{e}^{-\delta d(x,y)}\,H^\flat(y)\,\mathrm{d}\nu(y)\\ &\leq C\int_N \mathrm{e}^{-\delta d(z,y)}\,H^\flat(y)\,\mathrm{d}\nu(y)\\ &= C\,\mathscr{K}_\delta^\infty H^\flat(z),\end{aligned}$$

where $X=(x,u)$. Since $\mathscr{K}_\delta^\infty$ is bounded on $L^p(N)$ for every p in $[1,\infty]$,

$$\big\|\sup_{X\in\Gamma_\alpha(\cdot)}\mathscr{K}_\delta^\infty H^\flat(x)\big\|_p\leq C\,\big\|H^\flat\big\|_p\leq C\,\big\|H\big\|_{L^p(\Sigma)}\,. \tag{5.2}$$

We now estimate the maximal operator $(\mathscr{K}_j^0 H)^*$. In this proof for notational convenience we shall write γ instead of $j-(n+1)/2$. For z in N, consider the set $Q(z):=B_3(z)\times$

$(0, 2\sigma)$, which is contained in Σ. By (4.7), for any $X \in \Gamma_\alpha(\cdot)$ with α small enough,

$$\mathscr{K}_0^j H(X) \le C \int_{Q(z)} D(X,Y)^{2\gamma}\, H(Y)\, \mathrm{d}\mathscr{Y}(Y). \tag{5.3}$$

It is convenient to split the integral above as the sum of the integrals over $\Gamma_{2\alpha}(z)$ (which is contained in $Q(z)$ as long as α is small enough), and $Q(z) \cap \Gamma_{2\alpha}(z)^c$.

First we estimate $\int_{\Gamma_{2\alpha}(z)} D(X,Y)^{2\gamma}\, H(Y)\, \mathrm{d}\mathscr{Y}(Y)$. Recall that X is in $\Gamma_\alpha(z)$. Since $Y = (y, v)$ belongs to $\Gamma_{2\alpha}(z)$, the point Y is in the ball with centre (z, v) and radius $3\alpha v$. We choose α so small that $B_{6\alpha v}(z, v)$ (this ball is in $N \times \mathbb{R}$) is contained in Σ. Since H is harmonic, by Harnack's inequality (apply, for instance, [54, Theorem 5.4.3] with $M = N \times \mathbb{R}$ and $\delta = 1/2$. Note that, under our assumptions, N supports a local Poincaré inequality; see for instance [43, Theorems 1.1]), there exists a constant C, independent of Y in $\Gamma_{2\alpha}(z)$ and of z in N, such that $H(Y) \le C\, H(z, v)$, whence

$$\begin{aligned}\int_{\Gamma_{2\alpha}(z)} D(X,Y)^{2\gamma}\, H(Y)\, \mathrm{d}\mathscr{Y}(Y) &\le C \int_{\Gamma_{2\alpha}(z)} D\big(X,Y\big)^{2\gamma} H(z,v)\, \mathrm{d}\nu(y)\, \mathrm{d}v \\ &\le C\, H^*(z) \int_{\Gamma_{2\alpha}(z)} D\big(X,Y\big)^{2\gamma}\, \mathrm{d}\mathscr{Y}(Y).\end{aligned}$$

Observe that $\Gamma_{2\alpha}(z)$ is contained in $B_R(X)$ for R big enough (depending on σ and α). Therefore the last integral is dominated by $\int_{B_R(X)} D\big(X,Y\big)^{2\gamma}\, \mathrm{d}\mathscr{Y}(Y)$, which is bounded with respect to X in $\Gamma_\alpha(z)$ as a straightforward integration in polar coordinates shows. This implies that

$$\sup_{X \in \Gamma_\alpha(z)} \int_{\Gamma_{2\alpha}(z)} D(X,Y)^{2\gamma}\, H(Y)\, \mathrm{d}\mathscr{Y}(Y) \le C\, H^*(z). \tag{5.4}$$

Next we estimate $\int_{Q(z)\setminus\Gamma_{2\alpha}(z)} D(X,Y)^{2\gamma}\, H(Y)\, \mathrm{d}\mathscr{Y}(Y)$, where $X = (x, u)$ is in $\Gamma_\alpha(z)$. Set $Z := (z, u)$. We *claim* that, for every $Y \in Q(z) \setminus \Gamma_{2\alpha}(z)$,

$$D(X,Y) \ge \Big(1 - \frac{\sqrt{4\alpha^2+1}}{2}\Big)\, D(Y,Z). \tag{5.5}$$

To prove the claim, first observe that, by the triangle inequality,

$$\begin{aligned}D(X,Y) &\ge D(Y,Z) - D(X,Z) \\ &= D(Y,Z)\, \Big(1 - \frac{\sqrt{4\alpha^2+1}}{2}\Big) + D(Y,Z)\, \frac{\sqrt{4\alpha^2+1}}{2} - D(X,Z).\end{aligned}$$

Thus, in order to prove the claim it suffices to show that

$$D(Y,Z) \ge \frac{2}{\sqrt{4\alpha^2+1}}\, D(X,Z).$$

Denote by $\overline{W}$ any of the points on $\partial\Gamma_{2\alpha}(z)$ that realises the distance from Z to $\Gamma_{2\alpha}(z)^c$. Elementary geometric considerations show that $D(\overline{W}, Z) = u \sin\theta'$, where θ' denotes half the aperture of $\Gamma_{2\alpha}(z)$, i.e., $\tan\theta' = 2\alpha$. It is straightforward to check that $\sin\theta' = \frac{2\alpha}{\sqrt{4\alpha^2+1}}$. By combining these formulae, we get that

$$D(Y,Z) \geq D(\overline{W},Z) = \alpha\, u\, \frac{2}{\sqrt{4\alpha^2+1}} \geq D(X,Z)\, \frac{2}{\sqrt{4\alpha^2+1}},$$

as required to complete the proof of the claim. Notice that $\frac{2}{\sqrt{4\alpha^2+1}} > 1$, provided that α is small enough ($\alpha < \sqrt{3}/2$ will do).

The claim implies that

$$\begin{aligned}\int_{Q(z)\setminus\Gamma_{2\alpha}(z)} D(X,Y)^{2\gamma}\, H(Y)\,\mathrm{d}\mathscr{Y}(Y) &\leq C \int_{Q(z)\setminus\Gamma_{2\alpha}(z)} D(Z,Y)^{2\gamma}\, H(Y)\,\mathrm{d}\mathscr{Y}(Y)\\ &\leq C \int_{B_3(z)} d(y,z)^{2\gamma}\, H^\flat(y)\,\mathrm{d}\nu(y),\end{aligned}$$

where the constant C depends on α and n. Therefore

$$\sup_{X\in\Gamma_\alpha(z)} \int_{Q(z)\setminus\Gamma_{2\alpha}(z)} D(X,Y)^{2\gamma}\, H(Y)\,\mathrm{d}\mathscr{Y}(Y) \leq C \int_{B_3(z)} d(y,z)^{2\gamma}\, H^\flat(y)\,\mathrm{d}\nu(y).$$

By combining this and (5.4), recalling also (5.3), we see that

$$\big(\mathscr{K}_0^j H\big)^*(z) \leq C\,\Big[H^*(z) + \int_{B_3(z)} d(y,z)^{2\gamma}\, H^\flat(y)\,\mathrm{d}\nu(y)\Big],$$

so that

$$\begin{aligned}\big\|\big(\mathscr{K}_0^j H\big)^*\big\|_p &\leq C\,\big\|H^*\big\|_p + C\,\Big\|\int_{B_3(\cdot)} d(\cdot,y)^{2\gamma}\, H^\flat(y)\,\mathrm{d}\nu(y)\Big\|_p\\ &\leq C\,\big[\,\big\|H^*\big\|_p + \big\|H^\flat\big\|_p\,\big]\\ &\leq C\,\big\|H^*\big\|_p\,,\end{aligned}$$

as required.

Next we prove (ii). By Proposition 4.4 (iv) and Proposition 4.2 (i)-(ii), the function $\mathscr{G}_\Sigma^J S$ is continuous and for each δ in $(2\beta, 2\lambda_1\sqrt{c})$ there exists a constant C such that $k_{\mathscr{G}_\Sigma^J}(X,Y) \leq C\,\mathrm{e}^{-\delta D(X,Y)}$ for every X and Y in Σ. It is straightforward to check that there exists a constant C such that

$$\sup_{X\in\Gamma_\alpha(z)} \mathrm{e}^{-\delta D(X,Y)} \leq C\,\mathrm{e}^{-\delta d(z,y)} \qquad \forall z\in N \quad \forall Y\in\Sigma.$$

Here y is the component in N of the point Y in Σ. Consequently, $(\mathscr{G}_\Sigma^J S)^* \leq C\,\mathscr{K}_\delta^\infty S^\flat$, whence

$$\big\|(\mathscr{G}_\Sigma^J S)^*\big\|_p \leq C\,\big\|\mathscr{K}_\delta^\infty S^\flat\big\|_p \leq C\,\big\|S^\flat\big\|_p \leq C\,\big\|S\big\|_{L^p(\Sigma)}\,,$$

as required. □

6. The function G

In this section we adapt some ideas of Dindoš to our case (see [19, Chapter 6], especially Proposition 6.4 therein). The main result of this section is Theorem 6.3 below, which is a counterpart in our setting of a classical result of Stein and Weiss. First we need a of technical lemma. Recall the space $\mathscr{B}^p$, introduced just above Proposition 4.4.

Lemma 6.1. *Suppose that p is in $(1,\infty)$, and that F is a nonnegative continuous function in $\mathscr{B}^p$ satisfying $\lim_{d(x,o)\to\infty} \sup_{t\in[\eta,2\sigma-\eta]} F(x,t) = 0$ for every $\eta \in (0,\sigma)$. Suppose further that for some constant α the function $G := F - \alpha\mathscr{G}_\Sigma F$ is subharmonic in Σ. Then the following hold:*

(i) *there exists a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0$ as k tends to infinity, and a nonnegative function h in $L^p(N)$ such that $w - \lim_{k\to\infty} G(\cdot,\varepsilon_k) = h$ (weak limit in $L^p(N)$) and $\|h\|_p \le \min\big(\|F\|_{\mathscr{B}^p}, \|G\|_{\mathscr{B}^p}\big)$;*
(ii) *$G \le \mathscr{P}^\eta\big[G(\cdot,\eta)\big]$ in Σ_η;*
(iii) *$G \le \mathscr{P}^0 h$ in Σ, where h is as in* (i)*.*

Proof. First we prove (i). By the weak compactness of the unit sphere of $L^p(N)$, there exists a sequence ε_k, which tends to 0^+ as k tends to infinity, such that $F(\cdot,\varepsilon_k)$ is weakly convergent in $L^p(N)$ to a function, h say. By abstract nonsense $\|h\|_p \le \sup_k \|F(\cdot,\varepsilon_k)\|_p$. Furthermore, h is nonnegative, because so is F by assumption.

By Proposition 4.4 (v), $\|\mathscr{G}_\Sigma F(\cdot,\varepsilon_k)\|_p$ tends to 0 as k tends to infinity. *A fortiori* $\{\mathscr{G}_\Sigma F(\cdot,\varepsilon_k)\}$ tends to 0 weakly in $L^p(N)$. Thus, w-$\lim_{k\to\infty} G(\cdot,\varepsilon_k) = h$ in $L^p(N)$, whence, by abstract nonsense, $\|h\|_p \le \sup_k \|G(\cdot,\varepsilon_k)\|_p$.

Next we prove (ii). By elliptic regularity, $\mathscr{G}_\Sigma F$ is continuous on Σ, for F is continuous therein by assumption. Consequently so is G. For the sake of completeness we give a proof of the continuity of $\mathscr{G}_\Sigma F$, which is, we believe, quite standard. Suppose that $X \in \Sigma$. Recall that $\mathscr{G}_\Sigma F$ is a distributional solution of $\mathbb{\Delta} V = F$ on Σ. Choose a harmonic coordinate system (U,ϕ_U) with $U \subset \Sigma$ open set containing X and $\phi_U : U \to \mathbb{R}^{n+1}$. For any function V on U, set $\widetilde{V} := V \circ \phi_U^{-1}$. Clearly, V is continuous if and only if $\widetilde{V}$ is. Then

$$L\widetilde{\mathscr{G}_\Sigma F} = \widetilde{F}$$

in the sense of distributions on U, where L is the elliptic operator defined in $\phi_U(U)$ by $\sum_{i,j}(g\circ\phi_U^{-1})^{ij}\partial^2_{ij}$. Since $\widetilde{F}$ is continuous, it is in $L^p_{loc}(U)$ for every $p \in [1,\infty)$. By elliptic theory (see, for instance, [27, Theorem 6.33]), $\widetilde{\mathscr{G}_\Sigma F} \in W^{2,2}_{loc}(U) \subset W^{1,\frac{2n'}{n'-2}}(U)$. The latter inclusion, with $n' = n+1$, is a consequence of local Sobolev embeddings. Moreover the Euclidean Laplacian $\mathbb{\Delta}_0$ of $\widetilde{\mathscr{G}_\Sigma F}$ satisfies

$$|\mathbb{\Delta}_0\widetilde{\mathscr{G}_\Sigma F}| \le C\,|L\widetilde{\mathscr{G}_\Sigma F}| = C\,\widetilde{F} \in L^{\frac{2n'}{n'-2}}_{loc}(U).$$

By a local Euclidean Calderón–Zygmund inequality, we obtain that $\widetilde{\mathscr{G}_\Sigma F} \in W^{2,\frac{2n'}{n'-2}}_{loc}(U)$. Indeed, from [30, Theorem 9.9] there exists a function $w \in W^{2,\frac{2n'}{n'-2}}_{loc}$ solving $Lw = \widetilde{F}$ in a neighbourhood of X. Since $w - \widetilde{\mathscr{G}_\Sigma F}$ solves $L(w - \widetilde{\mathscr{G}_\Sigma F}) = 0$ and is thus smooth, we get that also $\widetilde{\mathscr{G}_\Sigma F} \in W^{2,\frac{2n'}{n'-2}}_{loc}$. In particular, $\widetilde{\mathscr{G}_\Sigma F} \in W^{1,\frac{2n'}{n'-4}}_{loc}(U)$ by a local Sobolev embedding. Since $\mathbb{\Delta}_0 \widetilde{\mathscr{G}_\Sigma F} \in L^{\frac{2n'}{n'-4}}_{loc}(U)$, we can iterate the argument, thus obtaining that $\widetilde{\mathscr{G}_\Sigma F} \in W^{2,\frac{2n'}{n'-2k}}_{loc}(U)$ for every positive integer k such that $2k < n'$. As soon as $2k > n' - 4$, by a local Sobolev embedding $W^{2,\frac{2n'}{n'-2k}}_{loc} \subset C^0$, thereby concluding the proof of the continuity of $\mathscr{G}_\Sigma F$.

To prove (ii), first notice that both sides of the desired inequality are continuous on $N \times [\eta, 2\sigma - \eta]$ (the continuity of the right hand side follows from Proposition 3.1), the left hand side and the right hand side are subharmonic and harmonic in $N \times (\eta, 2\sigma - \eta)$, respectively.

We claim that $G(\cdot, \eta)$ is in $C_0(N)$. By assumption, $F(\cdot, \eta)$ is in $C_0(N)$. Thus, it remains to prove that $\mathscr{G}_\Sigma F(\cdot, \eta)$ is in $C_0(N)$. Suppose that $\varepsilon > 0$. Choose γ in $(0, \eta)$ so that $\varrho(\gamma) < \varepsilon$ (recall that $\varrho(\gamma) = \sin \lambda_1 \gamma$, see the beginning of Section 4; clearly it suffices to choose $\gamma < \varepsilon/\lambda_1$). The estimate (4.6) implies that there exists a constant C such that $k_{\mathscr{G}_\Sigma}(X, Y) \leq C \min\big(\varrho(v), \mathrm{e}^{-2\lambda_1 d(x,y)\sqrt{c}}\big)$ whenever $X = (x, \eta)$, $Y := (y, v)$ and $D(X, Y) \geq \eta - \gamma$. Thus, in particular, $k_{\mathscr{G}_\Sigma}(X, Y) < C\,\varepsilon$ if $Y := (y, v)$ belongs either to $N \times (0, \gamma]$ or to $N \times [2\sigma - \gamma, 2\sigma)$ (this just because $D(X, Y) \geq \eta - \gamma > 0$). Therefore

$$k_{\mathscr{G}_\Sigma}(X, Y) = k_{\mathscr{G}_\Sigma}(X, Y)^\delta\, k_{\mathscr{G}_\Sigma}(X, Y)^{1-\delta} \leq C\, \varepsilon^\delta\, \mathrm{e}^{-2(1-\delta)\lambda_1 d(x,y)\sqrt{c}}$$

for any δ in $(0, 1)$. Therefore, if δ is small enough, then $\tau := 2(1-\delta)\lambda_1\sqrt{c} > 2\beta$ and

$$\begin{aligned} \int_{\Sigma\setminus\Sigma_\gamma} k_{\mathscr{G}_\Sigma}(X, Y)\, F(Y)\, \mathrm{d}\mathscr{Y}(Y) &\leq C\, \varepsilon^\delta \int_{\Sigma\setminus\Sigma_\gamma} \mathrm{e}^{-\tau d(x,y)}\, F(Y)\, \mathrm{d}\mathscr{Y}(Y) \\ &\leq C\, \varepsilon^\delta \Big[\int_\Sigma \mathrm{e}^{-\tau p' d(x,y)}\, \mathrm{d}\mathscr{Y}(Y)\Big]^{1/p'} \big\|F\big\|_{L^p(\Sigma)} \\ &\leq C\, \varepsilon^\delta \big\|F\big\|_{\mathscr{B}^p}\,. \end{aligned} \tag{6.1}$$

Furthermore, by assumption, there exists $R > 0$ such that $F(y, u) < \varepsilon$ when (y, u) belongs to $B_R(o)^c \times [\gamma, 2\sigma - \gamma]$. Hence

$$\int_{B_R(o)^c \times [\gamma, 2\sigma-\gamma]} k_{\mathscr{G}_\Sigma}(X, Y)\, F(Y)\, \mathrm{d}\mathscr{Y}(Y) \leq C\, \varepsilon; \tag{6.2}$$

we have used Proposition 4.4 (ii) in the last inequality. Finally, if Y belongs to $B_R(o) \times [\gamma, 2\sigma - \gamma] =: Q_{\gamma,R}$ and $d(x, o)$ is big enough, then there exists a constant C such that $k_{\mathscr{G}_\Sigma}(X, Y) \leq C\, \mathrm{e}^{-2\lambda_1 d(x,y)\sqrt{c}}$ (see Proposition 4.2 (i)). This and the fact that $d(x, y) \geq d(x, o) - R$ whenever $d(x, o)$ is large and y belongs to $B_R(o)$ imply that

$$\int_{Q_{\gamma,R}} k_{\mathscr{G}_\Sigma}(X, Y)\, F(Y)\, \mathrm{d}\mathscr{Y}(Y) \leq C\, \mathrm{e}^{CR}\, \mathrm{e}^{-2\lambda_1 d(x,o)\sqrt{c}} \int_{Q_{\gamma,R}} F(Y)\, \mathrm{d}\mathscr{Y}(Y).$$

By Hölder's inequality

$$\int_{Q_{\gamma,R}} F(Y)\,\mathrm{d}\mathscr{Y}(Y) \leq C\,\nu\big(B_R(o)\big)^{1/p'} \Big[\int_{Q_{\gamma,R}} F(Y)^p\,\mathrm{d}\mathscr{Y}(Y)\Big]^{1/p} \leq C\,\mathrm{e}^{CR}\,\,\big\|F\big\|_{\mathscr{B}^p}\,.$$

Thus, we may conclude that

$$\int_{Q_{\gamma,R}} k_{\mathscr{G}_\Sigma}(X,Y)\,F(Y)\,\mathrm{d}\mathscr{Y}(Y) \leq C\,\mathrm{e}^{CR}\,\mathrm{e}^{-2\lambda_1 d(x,o)\sqrt{c}}\,\,\big\|F\big\|_{\mathscr{B}^p}\,, \tag{6.3}$$

By combining (6.1), (6.2) and (6.3), we see that

$$\mathscr{G}_\Sigma F(x,\eta) \leq C\,\varepsilon\,\big(\,\big\|F\big\|_{\mathscr{B}^p} + 1\big) + C\,\mathrm{e}^{CR}\,\mathrm{e}^{-2\lambda_1 d(x,o)\sqrt{c}}\,\,\big\|F\big\|_{\mathscr{B}^p}\,.$$

Now we take the limit of both sides as $d(x,o)$ tends to infinity, and obtain that

$$\lim_{d(x,o)\to\infty} \mathscr{G}_\Sigma F(x,\eta) \leq C\,\varepsilon\,\big(\,\big\|F\big\|_{\mathscr{B}^p} + 1\big),$$

from which the claim follows directly.

Note that

$$\lim_{d(x,o)\to\infty}\,\sup_{t\in[\eta,2\sigma-\eta]} \mathscr{P}^\eta\big[G(\cdot,\eta)\big](x,t) = 0. \tag{6.4}$$

Indeed, since $G(\cdot,\eta)$ is in $C_0(N)$, for every $\varepsilon>0$ there exists R such that $G(x,\eta)<\varepsilon$ for every x such that $d(x,o)>R$. Then

$$\big|\big[\mathscr{P}^\eta G(\cdot,\eta)\big](x,t)\big| \leq \Big|\int_{B_R(o)} k_{M_t^\eta(\mathscr{D})}(x,y)\,G(y,\eta)\,\mathrm{d}\nu(y)\Big| + \varepsilon \int_{B_R(o)^c} \big|k_{M_t^\eta(\mathscr{D})}(x,y)\big|\,\mathrm{d}\nu(y).$$

The operator $M_t^\eta(\mathscr{D})$ satisfies on the slice Σ_η estimates similar to those of $M_t(\mathscr{D})$ on Σ. The proofs of such estimates for $M_t^\eta(\mathscr{D})$ are almost *verbatim* the same as the corresponding proofs for $M_t(\mathscr{D})$. In particular, for each $\varepsilon>0$, there exists a positive constant C such that for every function f in $L^1(N)$ with support contained in $B_R(o)$

$$\sup_{t\in(\eta,2\sigma-\eta)} \big|M_t^\eta(\mathscr{D})f(x)\big| \leq C\,\mathrm{e}^{(\varepsilon-\lambda_1^\eta)d(x,o)}\,\,\big\|f\big\|_1$$

for every x in $B_{2R}(o)^c$, where $\lambda_1^\eta = \dfrac{\pi}{2(\sigma-\eta)}$ (see the proof of Lemma 3.3 (iii)). Therefore for every x in $B_{2R}(o)^c$ we have the estimate

$$\sup_{t\in(\eta,2\sigma-\eta)} \Big|\int_{B_R(o)} k_{M_t^\eta(\mathscr{D})}(x,y)\,G(y,\eta)\,\mathrm{d}\nu(y)\Big| \leq C\,\mathrm{e}^{(\varepsilon-\lambda_1^\eta)d(x,o)}\,\,\big\|\mathbf{1}_{B_R(o)}\,G(\cdot,\eta)\big\|_1\,,$$

which tends to 0 as $d(x,o)$ tends to infinity. Furthermore,

$$\int_{B_R(o)^c} \big|k_{M_t^\eta(\mathscr{D})}(x,y)\big|\,\mathrm{d}\nu(y) \leq \big\|k_{M_t^\eta(\mathscr{D})}(x,\cdot)\big\|_1 \leq C\,\big[1+\big\|M_\sigma^\eta\big\|_{\mathscr{E}(\mathbf{S}_\varphi)}\,\big];$$

the last inequality follows from Lemma 3.3 (iv) (with $p=\infty$). The right hand side is independent of t in $(\eta,2\sigma-\eta)$. By combining the estimates above, we get that

$$\lim_{|x|\to\infty}\,\sup_{t\in[\eta,2\sigma-\eta]} \big|\big[\mathscr{P}^\eta G(\cdot,\eta)\big](x,t)\big| \leq \varepsilon,$$

which, of course, implies the required estimate (6.4).

Now, consider the function $\Xi(x,t) := G(x,t) - \big[\mathscr{P}^\eta G(\cdot,\eta)\big](x,t)$, which is continuous on $\overline{\Sigma}_\eta$ (by Proposition 3.1). Notice that $\Xi(x,\eta) = 0 = \Xi(x, 2\sigma-\eta)$ for every x in N. Since G is subharmonic on Σ and $\mathscr{P}^\eta\big[G(\cdot,\eta)\big]$ is harmonic on Σ_η, Ξ is subharmonic on Σ_η. For $R>0$ consider the compact set $K_R := \overline{B}_R(o)\times[\eta, 2\sigma-\eta]$. Fix $\varepsilon>0$. Our assumptions and (6.4) yield $\sup\limits_{(x,t)\in\partial K_R} \Xi(x,t) < \varepsilon$ for R large enough. By the maximum principle for subharmonic functions (see, for instance, [29, Corollary 1, p. 479]) applied to Ξ and K_R, we have the estimate $\Xi\leq\varepsilon$ on K_R. By letting ε tend to 0 (and R to infinity), we may conclude that $\Xi\leq 0$ on Σ_η, as required.

Finally, we prove (iii). It suffices to show that

$$\lim_{k\to\infty} \big[\mathscr{P}^{\varepsilon_k} G(\cdot,\varepsilon_k)\big](x,t) = \mathscr{P}^0 h(x,t) \tag{6.5}$$

for almost every (x,t) in Σ, where $\{\varepsilon_k\}$ denotes (possibly a subsequence of) the sequence whose existence is established in (i). Indeed, this and (ii) would imply that $G(x,t)\leq \mathscr{P}^0 h(x,t)$ for almost every (x,t) in Σ. Since both $G(x,t)$ and $\mathscr{P}^0 h(x,t)$ are continuous functions on Σ, the latter equality would hold everywhere, as required.

In order to prove (6.5), we consider preliminarily the function $m_t^\varepsilon(z) := M_t^\varepsilon(z) - M_t(z)$ for $\varepsilon>0$ and t in $(0,\sigma]$. We *claim* that for each φ in $(0,\pi/2)$

$$\lim_{\varepsilon\downarrow 0}\sup_{z\in\overline{\mathbf{S}}_\varphi} \big|m_t^\varepsilon(z)\big| = 0. \tag{6.6}$$

Here $\mathbf{S}_\varphi$ and $\overline{\mathbf{S}}_\varphi$ denote the sector $\{z\in\mathbb{C} : |\arg z|<\varphi\}$ and its closure, respectively.

A straightforward computation shows that given φ in $(0,\pi/2)$ and t in $(0,\sigma]$ there exists a constant C such that $\sup\limits_{z\in\overline{\mathbf{S}}_\varphi}\big|m_t^\varepsilon(z)\big|\leq C$ for every $\varepsilon\leq t/2$. By the Phragmen–Lindelöf principle $\sup\limits_{z\in\overline{\mathbf{S}}_\varphi}\big|m_t^\varepsilon(z)\big| \leq \sup\limits_{z\in\partial\overline{\mathbf{S}}_\varphi}\big|m_t^\varepsilon(z)\big|$. Observe that

$$\begin{aligned}
\big|m_t^\varepsilon(r\mathrm{e}^{i\varphi})\big| &= \big|\cosh[(\sigma-t)r\mathrm{e}^{i\varphi}]\big|\,\Big|\frac{1}{\cosh[\sigma r\mathrm{e}^{i\varphi}]} - \frac{1}{\cosh[(\sigma-\varepsilon)r\mathrm{e}^{i\varphi}]}\Big| \\
&\leq \big|\cosh[(\sigma-t)r\mathrm{e}^{i\varphi}]\big|\; r\varepsilon \sup_{u\in(\sigma-\varepsilon,\sigma)} \frac{\big|\sinh(ur\mathrm{e}^{i\varphi})\big|}{\big|\cosh^2[ur\mathrm{e}^{i\varphi}]\big|} \\
&\leq C\varepsilon r\,\mathrm{e}^{(\varepsilon-t)r\cos\varphi};
\end{aligned} \tag{6.7}$$

the first inequality follows from the mean value theorem, applied to the function $u\mapsto 1/\cosh[ur\mathrm{e}^{i\varphi}]$. Now we take the supremum of both sides with respect to r in $(0,\infty)$, and obtain

$$\sup_{r>0}\big|m_t^\varepsilon(r\mathrm{e}^{i\varphi})\big| \leq \frac{C\varepsilon}{(t-\varepsilon)\cos\varphi} \to 0$$

as ε tends to 0. Since $m_t^\varepsilon(\overline{z}) = \overline{m_t^\varepsilon(z)}$, we can conclude that $\lim\limits_{\varepsilon\downarrow 0}\sup\limits_{z\in\partial\overline{\mathbf{S}}_\varphi}\big|m_t^\varepsilon(z)\big| = 0$, thereby concluding the proof of the claim.

By using (6.7), it is easy to check that the function m_t^ε belongs to the algebra $H_0^\infty(\mathbf{S}_\varphi)$, which is included in the Dunford class $\mathscr{E}(\mathbf{S}_\varphi)$ (see page 15 for the definitions). Thus, if

$\pi/4 < \varphi < \pi/2$, then the natural functional calculus [35, Theorem 2.3.3] implies that

$$\left\| m_t^{\varepsilon_k}(\mathscr{D}) \right\|_p \leq C \left\| m_t^{\varepsilon} \right\|_{\mathscr{E}(\mathbf{S}_\varphi)} = C \left\| m_t^{\varepsilon} \right\|_{H_0^\infty(\mathbf{S}_\varphi)}, \tag{6.8}$$

because $\mathscr{D}$ is sectorial of angle $\pi/4$. Write

$$M_t^{\varepsilon_k}(\mathscr{D})G(\cdot,\varepsilon_k) - M_t(\mathscr{D})h = m_t^{\varepsilon_k}(\mathscr{D})G(\cdot,\varepsilon_k) + M_t(\mathscr{D})\big[G(\cdot,\varepsilon_k) - h\big]. \tag{6.9}$$

In order to prove (6.5), it suffices to show that both summands on the right hand side of (6.9) tend to 0 pointwise a.e. Since $\{G(\cdot,\varepsilon_k)\}$ is uniformly bounded in $L^p(N)$,

$$\left\| m_t^{\varepsilon_k}(\mathscr{D})G(\cdot,\varepsilon_k) \right\|_p \leq C \left\| m_t^{\varepsilon_k}(\mathscr{D}) \right\|_p,$$

which, by (6.8) and (6.6), tends to 0 as k tends to infinity. Hence, by abstract nonsense, a suitable subsequence of $m_t^{\varepsilon_k}(\mathscr{D})G(\cdot,\varepsilon_k)$ is pointwise convergent a.e. to 0. Next,

$$M_t(\mathscr{D})\big[G(\cdot,\varepsilon_k) - h\big](x) = \int_N k_{M_t(\mathscr{D})}(x,y)\,\big[G(\cdot,\varepsilon_k) - h\big](y)\,\mathrm{d}\nu(y),$$

which tends to 0 as k tends to infinity, because $G(\cdot,\varepsilon_k) - h$ is weakly convergent to 0 in $L^p(N)$, and $\left\| k_{M_t(\mathscr{D})}(x,\cdot) \right\|_{p'}$ is uniformly bounded with respect to x in N by Lemma 3.3 (v) (and the symmetry of $M_t(\mathscr{D})$).

This concludes the proof of (iii), and of the lemma. □

We shall apply Lemma 6.1 to the case where $F = |\nabla u|^q$ and u is a harmonic function on Σ. Suppose that β is a positive number. We say that a function S is *β-subharmonic* provided that $\Delta S \geq -\beta\, S$ in the sense of distributions. The following result, which generalises old ideas of Stein and Weiss (see for instance [55, pp. 217–220]), is due to Dindoš [19, Section 6.3].

Proposition 6.2. *Suppose that* $\mathrm{Ric}_{N\times\mathbb{R}} \geq -\kappa^2$. *If* $(n-1)/n \leq q \leq 1$ *and* u *is a harmonic function on an open subset* Ω *of* $N\times\mathbb{R}$*, then* $|\nabla u|^q$ *is* $q\kappa^2$*-subharmonic in* Ω.

Theorem 6.3. *Suppose that* u *is a harmonic function on* Σ. *Then the following are equivalent:*

(i) $|\nabla u|^*$ *is in* $L^1(N)$;
(ii) $|\nabla u|$ *is in* $\mathscr{B}^1$, *and* $\displaystyle\lim_{d(x,o)\to\infty} \sup_{t\in[\eta,2\sigma-\eta]} |\nabla u(x,t)| = 0$ *for every* η *in* $(0,2\sigma)$.

Furthermore, there exists a constant C*, independent of* u*, such that*

$$\big\| |\nabla u| \big\|_{\mathscr{B}^1} \leq \big\| |\nabla u|^* \big\|_1 \leq C\, \big\| |\nabla u| \big\|_{\mathscr{B}^1}. \tag{6.10}$$

Proof. We prove that (i) implies (ii), and that the left hand inequality in (6.10) holds. Observe that $\big|\nabla u\big|^*(x) \geq \big|\nabla u(x,t)\big|$ for every t in $(0,2\sigma)$, so that

$$\big\| |\nabla u| \big\|_{\mathscr{B}^1} \leq \int_N \big|\nabla u\big|^*(x)\,\mathrm{d}\nu(x).$$

It remains to prove that $\sup_{t\in[\eta,2\sigma-\eta]}\big|\nabla u(\cdot,t)\big|$ vanishes at infinity for each η in $(0,2\sigma)$. We argue by contradiction. Suppose that $\limsup_{d(x,o)\to\infty}\sup_{t\in[\eta,2\sigma-\eta]}\big|\nabla u(x,t)\big| =: \beta > 0$ for some η in $(0,2\sigma)$. Then there exists a sequence $\{x_k\}$ such that $d(x_k,o)$ tends to infinity as k does and $\limsup_{k\to\infty}\sup_{t\in[\eta,2\sigma-\eta]}\big|\nabla u(x_k,t)\big| = \beta$. Clearly $\big|\nabla u\big|^*(x)\geq \beta/2$ for all x in $B_{\alpha\eta}(x_k)$ and k large enough (here α denotes the aperture of the cone Γ_α in (5.1)). By possibly passing to a subsequence, we may assume that the balls $B_{\alpha\eta}(x_k)$ are mutually disjoint. Therefore

$$\int_N \big|\nabla u\big|^*(x)\,\mathrm{d}\nu(x) \geq \sum_k \int_{B_{\alpha\eta}(x_k)}\big|\nabla u\big|^*(x)\,\mathrm{d}\nu(x) = \infty,$$

which clearly contradicts our assumption.

Next we prove that (ii) implies (i) and the right hand inequality in (6.10) holds. Choose q in $\big((n-1)/n,1\big)$. Since $\big|\nabla u\big|$ is in $\mathscr{B}^1$, $\big|\nabla u\big|^q$ is in $\mathscr{B}^{1/q}$. Furthermore $\big|\nabla u\big|^q$ is $q\kappa^2$-subharmonic in Σ by Proposition 6.2, whence $G := \big|\nabla u\big|^q - q\kappa^2\mathscr{G}_\Sigma\big|\nabla u\big|^q$ is subharmonic therein. We may apply Lemma 6.1 (iii) (with $\big|\nabla u\big|^q$, $q\kappa^2$ and $1/q$ in place of F, α and p, respectively), and conclude that

$$|\nabla u|^q \leq \mathscr{P}^0 h + q\kappa^2\mathscr{G}_\Sigma\big(|\nabla u|^q\big). \tag{6.11}$$

Fix an integer $J > (n+1)/2$. The inequality (6.11) can be iterated J times, to wit

$$|\nabla u|^q \leq \mathscr{P}^0 h + C\Big(\sum_{j=1}^{J-1}\mathscr{G}_\Sigma^j\mathscr{P}^0 h + \mathscr{G}_\Sigma^J\big(|\nabla u|^q\big)\Big).$$

We raise both sides of the last inequality to the power $1/q$, and obtain that

$$|\nabla u| \leq C\Big\{(\mathscr{P}^0 h)^{1/q} + \sum_{j=1}^{J-1}\big(\mathscr{G}_\Sigma^j\mathscr{P}^0 h\big)^{1/q} + \big(\mathscr{G}_\Sigma^J|\nabla u|^q\big)^{1/q}\Big\}.$$

This implies the following inequality for the associated nontangential maximal functions

$$|\nabla u|^* \leq C\Big\{\big[\big(\mathscr{P}^0 h\big)^*\big]^{1/q} + \sum_{j=1}^{J-1}\big[\big(\mathscr{G}_\Sigma^j\mathscr{P}^0 h\big)^*\big]^{1/q} + \big[\big(\mathscr{G}_\Sigma^J|\nabla u|^q\big)^*\big]^{1/q}\Big\}. \tag{6.12}$$

Suppose that p is in $(1,\infty)$. Observe that there exists a constant C such that

$$\big\|\big(\mathscr{P}^0 h\big)^*\big\|_p \leq C\,\big\|h\big\|_p \qquad \forall h\in L^p(N). \tag{6.13}$$

Indeed, a straightforward argument, using the subordination formula (2.12), shows that

$$\sup_{t\in(0,\sigma]}\big|\mathscr{P}_t^N f\big| \leq \sup_{0<t}\big|\mathscr{H}_t^N f\big|.$$

By the Littlewood–Paley–Stein theory [57, p. 73] (see also [14, Theorem 7]), for each p in $(1,\infty]$, there exists a constant A_p such that

$$\big\|\sup_{0<t}\big|\mathscr{H}_t^N f\big|\big\|_p \leq A_p\,\big\|f\big\|_p \qquad \forall f\in L^p(N).$$

The subordination formula (2.12) implies that a similar estimate holds for the Poisson maximal operator. Since $\mathscr{P}^0 h(\cdot,t) = \mathscr{P}^0 h(\cdot, 2\sigma - t)$,

$$\sup_{X\in\Gamma_\alpha(z)} \big|\mathscr{P}^0 h(X)\big| = \sup_{X\in\Gamma_\alpha(z)'} \big|\mathscr{P}^0 h(X)\big|,$$

where $\Gamma_\alpha(z)' := \big\{(x,t)\in\Gamma_\alpha(z) : 0<t<\sigma\big\}$. In the case where $X=(x,t)$ is in $\Gamma_\alpha(z)'$, formula (3.6) and the Markovianity of the Poisson semigroup imply that

$$\big|\mathscr{P}^0 h(x,t)\big| \leq \mathscr{P}_t^N h(x) + \frac{1}{2}\Big[\mathscr{P}_{\sigma-t}^N |M_\sigma\big(\mathscr{D}\big)h|(x) + \mathscr{P}_t^N |\mathrm{e}^{-\sigma\mathscr{D}} M_\sigma\big(\mathscr{D}\big)h|(x)\Big]. \tag{6.14}$$

Clearly, the right hand side of the above inequality is a positive harmonic function on $N\times(0,\infty)$. If the aperture α is small enough and (x,t) is in $\Gamma_\alpha(z)'$, then the ball in $N\times\mathbb{R}$ with centre (z,t) and radius $2d(x,z)$ is contained in $N\times(0,\infty)$. Therefore, by Harnack's principle (see, for instance, [54, Theorem 5.4.3]), there exists a constant C such that

$$\mathscr{P}_t^N h(x) \leq C\,\mathscr{P}_t^N h(z) \qquad \forall (x,t)\in\Gamma_\alpha(z)' \quad \forall z\in N,$$

and a similar estimate holds for the other summands on the right hand side of (6.14). Set $h_0 := h$, $h_1 := \big|M_\sigma\big(\mathscr{D}\big)h\big|$ and $h_2 := \big|\mathrm{e}^{-\sigma\mathscr{D}} M_\sigma\big(\mathscr{D}\big)h\big|$. Then

$$\big\|\big(\mathscr{P}^0 h\big)^*\big\|_p \leq C \sum_{j=0}^2 \big\|\sup_{0<t} |\mathscr{P}_t^N h_j|\big\|_p \leq C \sum_{j=0}^2 \big\|h_j\big\|_p \leq C\,\big\|h\big\|_p\,;$$

the last inequality follows from the boundedness of the operators $M_\sigma\big(\mathscr{D}\big)$ and $\mathrm{e}^{-\sigma\mathscr{D}}M_\sigma\big(\mathscr{D}\big)$ on $L^p(N)$, which follows from Lemma 3.3 (iv) and the contractivity of the Poisson semigroup. This proves (6.13).

Similarly, by Theorem 5.1 (i), for every positive integer j there exists a constant C such that

$$\big\|\big(\mathscr{G}_\Sigma^j \mathscr{P}^0 h\big)^*\big\|_p \leq C\,\big\|\big(\mathscr{P}^0 h\big)^*\big\|_p\,. \tag{6.15}$$

By combining (6.13) and (6.15) we obtain that $\big\|\big(\mathscr{G}_\Sigma^j \mathscr{P}^0 h\big)^*\big\|_p \leq C\,\big\|h\big\|_p$ In particular, the last estimate holds for $p=1/q$. This and (6.12) imply that

$$\begin{aligned}\big\||\nabla u|^*\big\|_1 &\leq C\Big\{\,\big\|\big(\mathscr{P}^0 h\big)^*\big\|_{1/q}^{1/q} + \sum_{j=1}^{J-1}\big\|\big(\mathscr{G}_\Sigma^j \mathscr{P}^0 h\big)^*\big\|_{1/q}^{1/q} + \big\|\big(\mathscr{G}_\Sigma^J |\nabla u|^q\big)^*\big\|_{1/q}^{1/q}\,\Big\}\\ &\leq C\Big\{\,\big\|h\big\|_{1/q}^{1/q} + \big\|\big(\mathscr{G}_\Sigma^J |\nabla u|^q\big)^*\big\|_{1/q}^{1/q}\,\Big\}.\end{aligned}$$

Since $J>(n+1)/2$, Theorem 5.1 (ii) yields

$$\big\|\big(\mathscr{G}_\Sigma^J |\nabla u|^q\big)^*\big\|_{1/q}^{1/q} \leq C\,\big\||\nabla u|^q\big\|_{L^{1/q}(\Sigma)}^{1/q} \leq C\,\big\||\nabla u|\big\|_{\mathscr{B}^1}\,.$$

By combining the estimates above and using the estimate for $\big\|h\big\|_{1/q}$ in Lemma 6.1 (i), we get that $\big\||\nabla u|^*\big\|_1 \leq C\,\big\||\nabla u|\big\|_{\mathscr{B}^1}$, as required. ∎

7. Analysis of the local Riesz transform

7.1. **Goldberg-type spaces.** We introduce the *Goldberg-type* space $\mathfrak{h}^1(N)$ (also referred to as *local Hardy space*), which generalises the Goldberg space $\mathfrak{h}^1(\mathbb{R}^n)$ and plays a fundamental role in our analysis.

Definition 7.1. Fix a positive number s. Suppose that p is in $(1,\infty]$. A *standard p-atom* at scale s is a function a in $L^1(N)$ supported in a ball B of radius at most s satisfying the following conditions:

(i) *size condition*: $\|a\|_p \leq \nu(B)^{-1/p'}$;

(ii) *cancellation condition*: $\displaystyle\int_B a \,\mathrm{d}\nu = 0.$

A *global p-atom* at scale s is a function a in $L^1(N)$ supported in a ball B of radius *exactly equal to* s satisfying the size condition above (but possibly not the cancellation condition). Standard and global p-atoms will be referred to simply as *p-atoms*.

Definition 7.2. Suppose that s is a positive number. The *local atomic Hardy space* $\mathfrak{h}^{1,p}_s(N)$ is the space of all functions f in $L^1(N)$ that admit a decomposition of the form $f = \sum_{j=1}^\infty \lambda_j\, a_j$, where $\lambda_j \in \mathbb{C}$, the a_j's are p-atoms at scale s and $\sum_{j=1}^\infty |\lambda_j| < \infty$. The norm $\|f\|_{\mathfrak{h}^{1,p}_s}$ is the infimum of $\sum_{j=1}^\infty |\lambda_j|$ over all decompositions of f as above.

It is well known that $\mathfrak{h}^{1,p}_s(N)$ is independent of p and of s and the corresponding norms $\|\cdot\|_{\mathfrak{h}^{1,p}_s}$ are pairwise equivalent (see [51, Proposition 1]); henceforth, the space $\mathfrak{h}^{1,2}_s(N)$ will be denoted simply by $\mathfrak{h}^1(N)$. The fact that $\mathfrak{h}^{1,2}_s(N)$ is independent of s and p will be used without further comment in the sequel. Hereafter, atomic decompositions of functions in $\mathfrak{h}^1(N)$ will consist of atoms at scale 1.

The definition of the space $\mathfrak{h}^1(N)$ is similar to that of the atomic Hardy space $H^1(N)$, introduced by A. Carbonaro, G. Mauceri and Meda [6, 7], the only difference being that atoms in $H^1(N)$ are just standard atoms in $\mathfrak{h}^1(N)$, and there are no global atoms. As a consequence, the integral of functions in $H^1(N)$ vanishes, a property not enjoyed by all the functions in $\mathfrak{h}^1(N)$. Thus, trivially, $H^1(N)$ is properly and continuously contained in $\mathfrak{h}^1(N)$.

We need the following result, which is one of the main contributions of [45].

Theorem 7.3. *Suppose that $\sigma > 0$. Under our geometric assumptions, $\mathfrak{h}^1(N)$ agrees with the space $\mathfrak{h}^1_{\mathscr{P}}(N)$ of all functions f in $L^1(N)$ such that $\mathscr{P}^N_* f := \sup_{s\in(0,\sigma)} |\mathscr{P}^N_s f|$ is in $L^1(N)$. Furthermore, there exist positive constants C_1 and C_2 such that*

$$C_1 \, \|f\|_{\mathfrak{h}^1(N)} \leq \left\|\mathscr{P}^N_* f\right\|_1 \leq C_2 \, \|f\|_{\mathfrak{h}^1(N)} \,.$$

We need also the following simple result.

Proposition 7.4. *Suppose that $\varepsilon > 0$. Then there exists a constant C such that if h is a measurable function on N satisfying $|h(x)| \leq A\,\mathrm{e}^{-(2\beta+\varepsilon)d(x,o)}$ for some o in N, then $\|h\|_{\mathfrak{h}^1(N)} \leq C\,A$.*

Proof. A corollary of [51, proof of Lemma 2] is that for any p in $(1,\infty]$ there exists a constant C such that every function f in $L^p(N)$ supported in a ball B is in $\mathfrak{h}^1(N)$ and

$$\|f\|_{\mathfrak{h}^1(N)} \leq C\,\nu(B)^{1/p'}\,\|f\|_p\,. \tag{7.1}$$

Consider an exhaustion of N with $B_1(o)$ and annuli $A_j := B_{j+1}(o) \setminus B_j(o)$, where $j = 1,2,\ldots$. Correspondingly, write

$$h = \mathbf{1}_{B_1(o)}\,h + \sum_{j=1}^{\infty} \mathbf{1}_{A_j}\,h.$$

Clearly $\mathbf{1}_{B_1(o)}\,h$ is in $\mathfrak{h}^1(N)$, for it is a multiple of a global $\mathfrak{h}^1(N)$ atom. Next, by (7.1) (with $p = 2$),

$$\|\mathbf{1}_{A_j}\,h\|_{\mathfrak{h}^1(N)} \leq C\,\nu\big(B_{j+1}(o)\big)^{1/2}\,\|h\|_{L^2(A_j)}$$

The pointwise estimate of h implies that $\|h\|_{L^2(A_j)} \leq CA\,\mathrm{e}^{-(2\beta+\varepsilon)j}\,\nu(A_j)^{1/2}$, whence

$$\|\mathbf{1}_{A_j}\,h\|_{\mathfrak{h}^1(N)} \leq CA\,\nu\big(B_{j+1}(o)\big)\,\mathrm{e}^{-(2\beta+\varepsilon)j} \leq CA\,\mathrm{e}^{-\varepsilon j/2}.$$

The required estimate follows by summing the estimates above with respect to j. ☐

7.2. **Analysis of the Riesz transform.** For any $\tau > 0$ consider the operator $\mathscr{D}_\tau := \sqrt{\mathscr{L}_\tau}$, obtained by analytic continuation from the analytic family of operators $\{\mathscr{L}_\tau^{-\alpha/2} : \operatorname{Re}\alpha > 0\}$. We write $\mathscr{D}_\tau^{-1} = \mathscr{J}_\tau^0 + \mathscr{J}_\tau^\infty$, where $\mathscr{J}_\tau^0$ and $\mathscr{J}_\tau^\infty$ are the operators associated to the kernels

$$k_{\mathscr{J}_\tau^0} = \varphi\,k_{\mathscr{D}_\tau^{-1}} \qquad \text{and} \qquad k_{\mathscr{J}_\tau^\infty} = (1-\varphi)\,k_{\mathscr{D}_\tau^{-1}}, \tag{7.2}$$

where $\varphi : N \times N \to [0,1]$ is the smoooth function introduced in Lemma 2.6 (ii) (with $R = 1$). We further decompose $\mathscr{J}_\tau^0$ as $\mathscr{J}_\tau^{0,0} + \mathscr{J}_\tau^{0,\infty}$, where

$$k_{\mathscr{J}_\tau^{0,0}} = \frac{\varphi}{\sqrt{\pi}} \int_0^1 t^{-1/2}\,\mathrm{e}^{-\tau t}\,h_t^N\,\mathrm{d}t \qquad \text{and} \qquad k_{\mathscr{J}_\tau^{0,\infty}} = \frac{\varphi}{\sqrt{\pi}} \int_1^\infty t^{-1/2}\,\mathrm{e}^{-\tau t}\,h_t^N\,\mathrm{d}t.$$

Recall the definition of Υ_R (see (2.16)).

Lemma 7.5. *There exists a positive constant C such that the following hold:*

(i) $k_{\mathscr{D}_\tau^{-1}} \leq C\left[d^{1-n}\,\mathbf{1}_{\Upsilon_1} + \mathrm{e}^{-2d\sqrt{\tau c}}\,\mathbf{1}_{\Upsilon_1^c}\right]$, *where c is as in (2.3);*
(ii) $\mathscr{D}_\tau^{-1}$ *is bounded from $L^1(N)$ to $\mathfrak{h}^1(N)$ provided that $\tau > \beta^2/c$.*

Proof. First we prove (i). From the estimates (2.3) for h_t^N, we deduce that

$$k_{\mathscr{D}_\tau^{-1}} \leq C \int_0^1 t^{(1-n)/2}\,\mathrm{e}^{-\tau t - cd^2/t}\,\frac{\mathrm{d}t}{t} + C \int_1^\infty \mathrm{e}^{-\tau t - cd^2/t}\,\frac{\mathrm{d}t}{t}.$$

Changing variables ($d^2/t = u$) we see that the first integral above is $\asymp d^{1-n}$ as d tends to 0 and decays superexponentially as d tends to infinity.

Clearly, the second integral above is bounded as d tends to 0. For d large we write $\tau t + cd^2/t = 2d\sqrt{\tau c} + \big(\sqrt{\tau t} - d\sqrt{c/t}\big)^2$, and the second integral above may be written as $\mathrm{e}^{-2d\sqrt{\tau c}}\, J(d) \asymp d^{-1/2}\,\mathrm{e}^{-2d\sqrt{\tau c}}$, where $J(d)$ is as in Lemma 4.1 (with $j = 1/2$ and $\sqrt{\tau}$ instead of λ_1). The required estimate follows by combining the estimates above.

To prove (ii), write $g = \sum_j g_j$, where $g_j := g\,\mathbf{1}_{B_j}$ and $\{B_j\}$ is a covering of N by balls of radius 1 with the finite overlapping property. Choose p in $\big(1, n/(n-1)\big)$. We *claim* that there exist a constant C, independent of j, and a function h_j, with support contained in $2B_j$, such that

$$\big|\mathscr{D}_\tau^{-1} g_j(x)\big| \le h_j + C\,\mathrm{e}^{-2\sqrt{\tau c}\, d(x,c_j)}\, \big\|g_j\big\|_1 \tag{7.3}$$

and $\big\|h_j\big\|_p \le C\,\big\|g_j\big\|_1$. Here c_j denotes the centre of B_j.

Note that $h_j/\big(\nu(2B_j)^{1/p'}\,\big\|h_j\big\|_p\big)$ is a global $\mathfrak{h}_2^{1,p}(N)$ atom. Hence $\big\|h_j\big\|_{\mathfrak{h}^1(N)} \le \nu(2B_j)^{1/p'}\,\big\|h_j\big\|_p \le C\,\big\|g_j\big\|_1$. Notice also that, by Proposition 7.4, each of the functions $x \mapsto \mathrm{e}^{-2\sqrt{\tau c}\, d(x,c_j)}$ is in $\mathfrak{h}^1(N)$ with norm uniformly bounded with respect to j. Thus, given (7.3), we may conclude that

$$\big\|\mathscr{D}_\tau^{-1} g\big\|_{\mathfrak{h}^1(N)} \le \sum_j \big\|\mathscr{D}_\tau^{-1} g_j\big\|_{\mathfrak{h}^1(N)} \le C \sum_j \big\|g_j\big\|_1 \le C\,\big\|g\big\|_1\,,$$

as required.

Thus, it remains to prove (7.3). Note that, by (i),

$$\big|\mathscr{D}_\tau^{-1} g_j(x)\big| \le C\,h_j(x) + C\,\mathbf{1}_{(2B_j)^c}(x) \int_{B_j} \mathrm{e}^{-2\sqrt{\tau c}\, d(x,y)}\, g_j(y)\,\mathrm{d}\nu(y),$$

where $h_j(x) := \mathbf{1}_{2B_j}(x) \int_{B_j} d(x,y)^{1-n}\, g_j(y)\,\mathrm{d}\nu(y)$. Since the integral operator with kernel $(x,y) \mapsto \mathbf{1}_{2B_j}(x)\, d(x,y)^{1-n}\,\mathbf{1}_{B_j}(y)$ is bounded from $L^1(N)$ to $L^p(N)$, the required estimate for $\big\|h_j\big\|_p$ follows. By the triangle inequality $d(x,y) \ge d(x,c_j) - d(y,c_j) \ge d(x,c_j) - 1$, so that

$$\Big|\int_{B_j} \mathrm{e}^{-2\sqrt{\tau c}\, d(x,y)}\, g_j(y)\,\mathrm{d}\nu(y)\Big| \le C\,\mathrm{e}^{-2\sqrt{\tau c}\, d(x,c_j)}\, \big\|g_j\big\|_1\,.$$

This concludes the proof of the claim and of (ii). ☐

Lemma 7.6. *Suppose that $\tau > \beta^2/c$, where c is as in* (2.3). *Then there exists a constant C such that*

$$\big\|\mathscr{D}_\tau \mathscr{J}_\tau^{0,\infty} f\big\|_{\mathfrak{h}^1(N)} \le C\,\big\|f\big\|_1 \quad \textit{and} \quad \big\|\mathscr{D}_\tau \mathscr{J}_\tau^{\infty} f\big\|_{\mathfrak{h}^1(N)} \le C\,\big\|f\big\|_1$$

for every function f in $L^1(N)$ with compact support in a ball of radius ≤ 1.

Proof. We assume that the support of f is contained in $B_R(o)$, with $R \le 1$. Since $\mathscr{D}_\tau \mathscr{J}_\tau^{0,\infty} f = \mathscr{D}_\tau^{-1} \mathscr{L}_\tau \mathscr{J}_\tau^{0,\infty} f$ and $\mathscr{D}_\tau \mathscr{J}_\tau^{\infty} f = \mathscr{D}_\tau^{-1} \mathscr{L}_\tau \mathscr{J}_\tau^{\infty} f$, Lemma 7.5 (ii) implies that it suffices to show that there exists a constant C, independent of f, such that

$$\big\|\mathscr{L}_\tau \mathscr{J}_\tau^{0,\infty} f\big\|_1 \le C\,\big\|f\big\|_1 \quad \text{and} \quad \big\|\mathscr{L}_\tau \mathscr{J}_\tau^{\infty} f\big\|_1 \le C\,\big\|f\big\|_1\,. \tag{7.4}$$

The first inequality above will follow from the fact that $\mathscr{L}_\tau \mathscr{J}_\tau^{0,\infty} f$ is a bounded function with support contained in $B_2(o)$ and the estimate $\big\|\mathscr{L}_\tau \mathscr{J}_\tau^{0,\infty} f\big\|_\infty \leq C \, \|f\|_1$. Since $\mathscr{L}_\tau \mathscr{J}_\tau^{0,\infty} f = \mathscr{L} \mathscr{J}_\tau^{0,\infty} f + \tau \mathscr{J}_\tau^{0,\infty} f$, it suffices to show that both $\mathscr{J}_\tau^{0,\infty} f$ and $\mathscr{L} \mathscr{J}_\tau^{0,\infty} f$ are bounded functions with compact support contained in $B_2(o)$ and the estimates $\big\|\mathscr{J}_\tau^{0,\infty} f\big\|_\infty \leq C \, \|f\|_1$ and $\big\|\mathscr{L} \mathscr{J}_\tau^{0,\infty} f\big\|_\infty \leq C \, \|f\|_1$ hold. Observe that

$$\mathscr{J}_\tau^{0,\infty} f(x) = \frac{1}{\sqrt{\pi}} \int_{B_R(o)} \mathrm{d}\nu(y) \, \varphi(x,y) \, f(y) \int_1^\infty t^{-1/2} \, \mathrm{e}^{-\tau t} \, h_t^N(x,y) \, \mathrm{d}t.$$

By our choice of φ and the fact that the support of f is contained in $B_1(o)$, the support of $\mathscr{J}_\tau^{0,\infty} f$ is contained in $B_2(o)$. The upper estimate (2.3) for $h_t^N(x,y)$ implies that the inner integral above is dominated by a constant independent of x and y. Therefore $\big\|\mathscr{J}_\tau^{0,\infty} f\big\|_1 \leq C \, \|f\|_1$, as required. We now prove that the same is true of $\mathscr{L} \mathscr{J}_\tau^{0,\infty} f$. Notice that for each y in N

$$\mathscr{L}\big[\varphi h_t^N\big](\cdot,y) = \mathscr{L}\varphi(\cdot,y) \, h_t^N(\cdot,y) - 2\big\langle \nabla\varphi(\cdot,y), \nabla h_t^N(\cdot,y)\big\rangle + \varphi(\cdot,y) \, \mathscr{L} h_t^N(\cdot,y);$$

note also that, by our choice of φ, the first and the second summand on the right hand side vanish when $d(\cdot,y) \leq 1/4$. Correspondingly, $\mathscr{L} \mathscr{J}_\tau^{0,\infty} f$ may be written as $\mathscr{B}_1 f - \mathscr{B}_2 f + \mathscr{B}_3 f$, where

$$\mathscr{B}_1 f = \frac{1}{\sqrt{\pi}} \int_{B_R(o)} \mathrm{d}\nu(y) \, \mathscr{L}\varphi(\cdot,y) \, f(y) \int_1^\infty t^{-1/2} \, \mathrm{e}^{-\tau t} \, h_t^N(\cdot,y) \, \mathrm{d}t,$$

$$\mathscr{B}_2 f = \frac{2}{\sqrt{\pi}} \int_{B_R(o)} \mathrm{d}\nu(y) \, \Big\langle \nabla\varphi(\cdot,y) f(y), \int_1^\infty t^{-1/2} \, \mathrm{e}^{-\tau t} \, \nabla h_t^N(\cdot,y) \, \mathrm{d}t \Big\rangle$$

and

$$\mathscr{B}_3 f = \frac{1}{\sqrt{\pi}} \int_{B_R(o)} \mathrm{d}\nu(y) \, \varphi(\cdot,y) \, f(y) \int_1^\infty t^{-1/2} \, \mathrm{e}^{-\tau t} \, \mathscr{L} h_t^N(\cdot,y) \, \mathrm{d}t.$$

We estimate $\mathscr{B}_3 f$. The estimates of $\mathscr{B}_1 f$ and $\mathscr{B}_2 f$ are easier, for the kernels of these operators are supported in $\Upsilon_1 \setminus \Upsilon_{1/4}$ (recall also that $|\nabla\varphi(\cdot,y)|$ and $|\mathscr{L}\varphi(\cdot,y)|$ are uniformly bounded; see Lemma 2.6 (ii)), and are left to the interested reader. Notice that $\mathscr{L} h_t^N(\cdot,y) = -\partial_t h_t^N(\cdot,y)$. Then, by integrating by parts in the inner integral, we find that

$$\mathscr{B}_3 f = \frac{1}{2\sqrt{\pi}} \int_{B_R(o)} \mathrm{d}\nu(y) \, \varphi(\cdot,y) \, f(y) \Big[2\mathrm{e}^{-\tau} \, h_1^N - \int_1^\infty t^{-3/2} \, \mathrm{e}^{-\tau t} \, (1 + 2\tau t) \, h_t^N(\cdot,y) \, \mathrm{d}t\Big].$$

The upper estimate in (2.3) and simple considerations show that $\big|\mathscr{B}_3 f\big|$ is dominated by $\displaystyle\int_{B_R(o)} k(\cdot,y) \, \big|f(y)\big| \, \mathrm{d}\nu(y)$, where k is bounded, nonnegative and supported in Υ_1 (see (2.16) for the notation). Consequently, $\big|\mathscr{B}_3 f\big|$ is a bounded function with support contained in $B_2(o)$ and $\big\|\mathscr{B}_3 f\big\|_\infty \leq C \, \|f\|_1$. This concludes the proof of the first inequality in (7.4).

Next we prove the second inequality in (7.4). Recall that

$$\begin{aligned}\mathscr{J}_\tau^\infty f &= \frac{1}{\sqrt{\pi}} \int_{B_R(o)} \mathrm{d}\nu(y) \big[1-\varphi(\cdot,y)\big]\, f(y) \int_0^\infty t^{-1/2}\, \mathrm{e}^{-\tau t}\, h_t^N(\cdot,y)\,\mathrm{d}t \\ &= \frac{1}{\sqrt{\pi}} \int_{B_R(o)} \big[1-\varphi(\cdot,y)\big]\, k_{\mathscr{D}_\tau^{-1}}(x,y)\, f(y)\,\mathrm{d}\nu(y).\end{aligned}$$

This and the estimates for $k_{\mathscr{D}_\tau^{-1}}$ in Lemma 7.5 (i) imply that there exists a constant C such that $\big|\mathscr{J}_\tau^\infty f(x)\big| \leq C\,\mathrm{e}^{-2d(x,o)\sqrt{\tau c}}\,\big\|f\big\|_1$ for every x in N. Thus, $\big\|\mathscr{J}_\tau^\infty f\big\|_1 \leq C\,\big\|f\big\|_1$, because, by assumption, $\tau > \beta^2/c$. Since $\mathscr{L}_\tau \mathscr{J}_\tau^\infty f = \mathscr{L}\mathscr{J}_\tau^\infty f + \tau\,\mathscr{J}_\tau^\infty f$, it remains to show that $\mathscr{L}\mathscr{J}_\tau^\infty f$ satisfies a similar estimate. Now,

$$\mathscr{L}\big[(1-\varphi)h_t^N\big](\cdot,y) = -\mathscr{L}\varphi(\cdot,y)\,h_t^N(\cdot,y) + 2\,\big\langle \nabla\varphi(\cdot,y), \nabla h_t^N(\cdot,y)\big\rangle + \big[1-\varphi(\cdot,y)\big]\,\mathscr{L}h_t^N(\cdot,y);$$

note that each of the summands on the right hand side vanishes in $\Upsilon_{1/4}$. Correspondingly, $\mathscr{L}\mathscr{J}_\tau^\infty f$ may be written as $\mathscr{A}_1 f + \mathscr{A}_2 f + \mathscr{A}_3 f$, where

$$\mathscr{A}_1 f = -\frac{1}{\sqrt{\pi}} \int_{B_R(o)} \mathrm{d}\nu(y)\,\mathscr{L}\varphi(\cdot,y)\,f(y) \int_0^\infty t^{-1/2}\,\mathrm{e}^{-\tau t}\,h_t^N(\cdot,y)\,\mathrm{d}t,$$

$$\mathscr{A}_2 f = -\frac{2}{\sqrt{\pi}} \int_{B_R(o)} \mathrm{d}\nu(y)\,\Big\langle \nabla\varphi(\cdot,y) f(y), \int_0^\infty t^{-1/2}\,\mathrm{e}^{-\tau t}\,\nabla h_t^N(\cdot,y)\,\mathrm{d}t\Big\rangle$$

and

$$\mathscr{A}_3 f = \frac{1}{\sqrt{\pi}} \int_{B_R(o)} \mathrm{d}\nu(y)\,\big[1-\varphi(\cdot,y)\big]\,f(y) \int_0^\infty t^{-1/2}\,\mathrm{e}^{-\tau t}\,\mathscr{L}h_t^N(\cdot,y)\,\mathrm{d}t.$$

We estimate $\mathscr{A}_3 f$. The estimates of $\mathscr{A}_1 f$ and $\mathscr{A}_2 f$ are easier, for the kernel of these operators are supported in $\Upsilon_1 \setminus \Upsilon_{1/4}$ (recall also that $|\nabla\varphi(\cdot,y)|$ and $|\mathscr{L}\varphi(\cdot,y)|$ are uniformly bounded; see Lemma 2.6 (ii)), and are left to the interested reader.

Notice that $\mathscr{L}h_t(\cdot,y) = -\partial_t h_t(\cdot,y)$. Then, by integrating by parts in the inner integral, we find that

$$\mathscr{A}_3 f = -\frac{1}{2\sqrt{\pi}} \int_{B_R(o)} \mathrm{d}\nu(y)\,\big[1-\varphi(\cdot,y)\big]\,f(y) \int_0^\infty t^{-3/2}\,\mathrm{e}^{-\tau t}\,(1+2\tau t)\,h_t(\cdot,y)\,\mathrm{d}t.$$

The inner integral is dominated by

$$C\int_0^1 t^{-(n+3)/2}\,\mathrm{e}^{-cd^2/t}\,\mathrm{d}t + C\int_1^\infty t^{-1}\,\mathrm{e}^{-(\tau t + cd^2/t)}\,\mathrm{d}t.$$

We need to estimate these integrals in the case where d is large (because of the cutoff $1-\varphi$). The first is bounded above by

$$C\,\mathrm{e}^{-cd^2/2}\int_0^1 t^{-(n+3)/2}\,\mathrm{e}^{-cd^2/(2t)}\,\mathrm{d}t = C\,\frac{\mathrm{e}^{-cd^2/2}}{d^{n+1}} \int_{d^2}^\infty u^{(n-1)/2}\,\mathrm{e}^{-cu/2}\,\mathrm{d}u \leq C\,\frac{\mathrm{e}^{-cd^2/2}}{d^{n+1}},$$

and the second by $C\,\mathrm{e}^{-2d\sqrt{\tau c}}$ (see the proof of Lemma 7.5 (i)), which is integrable at infinity because $\tau > \beta^2/c$. These estimates imply that $\big\|\mathscr{A}_3 f\big\|_1 \leq C\,\big\|f\big\|_1$. A similar conclusion applies also to $\mathscr{A}_1 f$ and $\mathscr{A}_2 f$. Thus, $\big\|\mathscr{L}\mathscr{J}_\tau^\infty f\big\|_1 \leq C\,\big\|f\big\|_1$, as required to conclude the proof of the second inequality in (7.4). ∎

Denote by $k_{\mathscr{R}_\tau}$ the distributional kernel of $\mathscr{R}_\tau$, and write $k_{\mathscr{R}_\tau}$ as the sum of $\varphi\, k_{\mathscr{R}_\tau}$ and $(1-\varphi)\, k_{\mathscr{R}_\tau}$, where φ is the smooth function on $N \times N$ given by Lemma 2.6 (with $R=1$). Denote by $\mathscr{R}_\tau^0$ and by $\mathscr{R}_\tau^\infty$ the operators associated to the kernels $\varphi\, k_{\mathscr{R}_\tau}$ and $(1-\varphi)\, k_{\mathscr{R}_\tau}$, respectively. Obviously,

$$\mathscr{R}_\tau = \mathscr{R}_\tau^0 + \mathscr{R}_\tau^\infty. \tag{7.5}$$

Observe that

$$k_{\mathscr{R}_\tau^0}(x,y) = \frac{\varphi(x,y)}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\, \mathrm{e}^{-\tau t}\, \nabla_x h_t^N(x,y)\, \mathrm{d}t\,.$$

It is convenient to further decompose the operator $\mathscr{R}_\tau^0$ as the sum of the operators $\mathscr{R}_\tau^{0,0}$ and $\mathscr{R}_\tau^{0,\infty}$, which are associated to the kernels $k_{\mathscr{R}_\tau^{0,0}}$ and $k_{\mathscr{R}_\tau^{0,\infty}}$, defined by

$$k_{\mathscr{R}_\tau^{0,0}}(x,y) = \frac{\varphi(x,y)}{\sqrt{\pi}} \int_0^1 t^{-1/2}\, \mathrm{e}^{-\tau t}\, \nabla_x h_t^N(x,y)\, \mathrm{d}t$$

and

$$k_{\mathscr{R}_\tau^{0,\infty}}(x,y) = \frac{\varphi(x,y)}{\sqrt{\pi}} \int_1^\infty t^{-1/2}\, \mathrm{e}^{-\tau t}\, \nabla_x h_t^N(x,y)\, \mathrm{d}t.$$

Notice that $k_{\mathscr{R}_\tau^{0,0}}(x,y) = \nabla_x k_{\mathscr{J}_\tau^{0,0}}(x,y) - k_{\mathscr{V}}(x,y)$, where $k_{\mathscr{V}} := \dfrac{\nabla_x\varphi}{\sqrt{\pi}} \displaystyle\int_0^1 t^{-1/2}\, \mathrm{e}^{-\tau t} h_t^N\, \mathrm{d}t$.

Lemma 7.7. *For each $\varepsilon > 0$ there exists a constant C such that for every p in N and every function $f \in L^1(N)$ with support contained in a ball with centre p and radius ≤ 1 the following hold:*

(i) $\big|M_\sigma(\mathscr{D})\, \mathscr{J}_\tau^{0,0} f(x)\big| \leq C\, \mathrm{e}^{(\varepsilon-\lambda_1) d(x,p)}\, \big\|f\big\|_1$;

(ii) $\big|\mathscr{L} M_\sigma(\mathscr{D})\, \mathscr{J}_\tau^{0,0} f(x)\big| \leq C\, \mathrm{e}^{(\varepsilon-\lambda_1) d(x,p)}\, \big\|f\big\|_1$.

Consequently, $M_\sigma(\mathscr{D})\, \mathscr{J}_\tau^{0,0} f$ and $\mathscr{L} M_\sigma(\mathscr{D})\, \mathscr{J}_\tau^{0,0} f$ are in $\mathfrak{h}^1(N)$ and their norms in $\mathfrak{h}^1(N)$ are controlled by $C\, \big\|f\big\|_1$.

Proof. By Proposition 2.2 (ii), $M_\sigma(\mathscr{D})$ and $\mathscr{L} M_\sigma(\mathscr{D})$ are bounded from $L^1(N)$ to $L^\infty(N)$. Moreover, clearly $\mathscr{J}_\tau^{0,0}$ is bounded in $L^1(N)$. Therefore,

$$\big\|M_\sigma(\mathscr{D}) \mathscr{J}_\tau^{0,0} f\big\|_\infty \leq C\, \big\|\mathscr{J}_\tau^{0,0} f\big\|_1 \leq C\, \big\|f\big\|_1 \tag{7.6}$$

and a similar estimate holds for $\mathscr{L} M_\sigma(\mathscr{D})\, \mathscr{J}_\tau^{0,0} f$. Furthermore, since $\mathscr{J}_\tau^{0,0} f$ is supported in $B_2(p)$ for some $p \in N$, Lemma 3.3 (iii) gives that

$$\begin{aligned} \big|M_\sigma(\mathscr{D})\, \mathscr{J}_\tau^{0,0} f(x)\big| &\leq C\, \mathrm{e}^{(\varepsilon-\lambda_1) d(x,p)}\, \big\|\mathscr{J}_\tau^{0,0} f\big\|_1 \\ &\leq C\, \mathrm{e}^{(\varepsilon-\lambda_1) d(x,p)}\, \big\|f\big\|_1 \qquad \forall x \in B_4(p)^c. \end{aligned} \tag{7.7}$$

Now (i) follows by combining (7.6) and (7.7).

The assertion in (ii) follows in a similar way. The last statement of the lemma is a direct consequence of (i), (ii) and Proposition 7.4. $\square$

Theorem 7.8. *There exists a constant C such that*

$$\|f\|_{\mathfrak{h}^1(N)} \leq C\,\big[\,\||\mathscr{R}_\tau^{0,0} f|\|_1 + \|f\|_1\,\big].$$

for every function f with support contained in a ball of radius ≤ 1 for which the right hand side is finite.

Proof. Let o be the centre of the ball of radius $R \leq 1$ which contains the support of f. Then the support of $\mathscr{J}_\tau^{0,0} f$ is contained in a ball of radius $R+1 \leq 2$. Define $H := \mathscr{P}^0(\mathscr{J}_\tau^{0,0} f)$. By Theorem 3.4, there exists a constant C such that

$$\||\nabla\!\!\!\!\nabla H|\|_{\mathscr{B}^1} \leq C\,\mathscr{N}\big(\mathscr{J}_\tau^{0,0} f\big).$$

Furthermore, by Lemma 3.3 (iii), for every $\varepsilon > 0$ there exists a constant C such that

$$\sup_{t\in(0,2\sigma)} |\nabla\!\!\!\!\nabla H(x,t)| \leq C\,\mathrm{e}^{(\varepsilon-\lambda_1)d(x,o)}\,\|\mathscr{J}_\tau^{0,0} f\|_1 \leq C\,\mathrm{e}^{(\varepsilon-\lambda_1)d(x,o)}\,\|f\|_1$$

for every x in $B_4(o)^c$. Hence $|\nabla\!\!\!\!\nabla H|$ satisfies the assumptions of Theorem 6.3 (ii), whence there exists a constant such that $\||\nabla\!\!\!\!\nabla H|^*\|_1 \leq C\,\||\nabla\!\!\!\!\nabla H|\|_{\mathscr{B}^1}$. By combining the estimates above we see that

$$\||\nabla\!\!\!\!\nabla H|^*\|_1 \leq C\,\mathscr{N}(\mathscr{J}_\tau^{0,0} f),$$

provided that the right hand side is finite (as we shall prove below). The required norm estimate will follow from the following two facts:

(a) there exists a constant C such that $\mathscr{N}(\mathscr{J}_\tau^{0,0} f) \leq C\,\big[\,\||\mathscr{R}_\tau^{0,0} f|\|_1 + \|f\|_1\,\big]$;
(b) there exists a constant C such that $\|f\|_{\mathfrak{h}^1(N)} \leq C\,\big[\,\||\nabla\!\!\!\!\nabla H|^*\|_1 + \|f\|_1\,\big]$.

First we prove (a). Recall that $\mathscr{N}(\mathscr{J}_\tau^{0,0} f) := \|\mathscr{J}_\tau^{0,0} f\|_1 + \||\nabla\mathscr{J}_\tau^{0,0} f|\|_1 + \|\mathscr{D}\mathscr{J}_\tau^{0,0} f\|_1$. Clearly $\|\mathscr{J}_\tau^{0,0} f\|_1 \leq C\,\|f\|_1$. Notice that

$$\nabla(\mathscr{J}_\tau^{0,0} f) = \mathscr{R}_\tau^{0,0} f + \mathscr{V}(\mathscr{J}_\tau^{0,0} f),$$

where $\mathscr{V}$ is the operator with kernel $k_{\mathscr{V}} := \dfrac{\nabla_x\varphi}{\sqrt{\pi}}\displaystyle\int_0^1 t^{-1/2}\,\mathrm{e}^{-\tau t} h_t^N\,\mathrm{d}t$. It is straightforward to check that there exists a constant C such that

$$\||\mathscr{V}(\mathscr{J}_\tau^{0,0} f)|\|_1 \leq C\,\|\mathscr{J}_\tau^{0,0} f\|_1 \leq C\,\|f\|_1\,.$$

We leave the verification of this fact to the interested reader. Therefore

$$\||\nabla(\mathscr{J}_\tau^{0,0} f)|\|_1 \leq C\,\big[\,\||\mathscr{R}_\tau^{0,0} f|\|_1 + \|f\|_1\,\big].$$

The proof of (a) will be complete, once the following *claim* will be proved. There exists a constant C such that

$$\|\mathscr{D}\mathscr{J}_\tau^{0,0} f\|_1 \leq C\,\|f\|_1\,. \tag{7.8}$$

Write

$$\mathscr{D}\mathscr{J}_\tau^{0,0} f = \big[\mathscr{D} - \mathscr{D}_\tau\big]\,\mathscr{J}_\tau^{0,0} f + \mathscr{D}_\tau\mathscr{J}_\tau^{0,0} f. \tag{7.9}$$

The operator $\mathscr{D}-\mathscr{D}_\tau$ corresponds to the spectral multiplier $\sqrt{\lambda}-\sqrt{\tau+\lambda}$ of $\mathscr{L}$. The latter function is in $\mathscr{E}(\mathbf{S}_\varphi)$ for every φ in $(0,\pi)$. Indeed,

$$\sqrt{\lambda}-\sqrt{\tau+\lambda}=-\frac{\tau}{\sqrt{\lambda}+\sqrt{\tau+\lambda}}=\tau\Big[\frac{1}{\sqrt{\tau+\lambda}}-\frac{1}{\sqrt{\lambda}+\sqrt{\tau+\lambda}}\Big]-\frac{\tau}{\sqrt{\tau+\lambda}}.$$

It is straightforward to check that the function within square brackets is in $H^\infty_0(\mathbf{S}_\varphi)$ and $(\tau+\lambda)^{-1/2}$ is in $\mathscr{E}(\mathbf{S}_\varphi)$ by [35, Lemma 2.2.3]. The operator $\mathscr{L}$ is sectorial of angle $\pi/2$ on $\mathfrak{h}^1(N)$ [44, Theorem 3.1]. Therefore the natural functional calculus [35, Theorem 2.3.3] implies that $\mathscr{D}-\mathscr{D}_\tau$ is bounded on $\mathfrak{h}^1(N)$. It is straightforward to check that $\mathscr{J}^{0,0}_\tau f$ is a function in $L^p(N)$ for each $p\in[1,n/(n-1))$, and that its support is contained in $B_2(o)$. Furthermore,

$$\big\|\mathscr{J}^{0,0}_\tau f\big\|_{\mathfrak{h}^1(N)}\leq C\,\nu\big(B_2(o)\big)^{1/p'}\,\big\|f\big\|_1\,.$$

Thus,

$$\big\|[\mathscr{D}-\mathscr{D}_\tau]\,\mathscr{J}^{0,0}_\tau f\big\|_{\mathfrak{h}^1(N)}\leq C\,\big\|\mathscr{J}^{0,0}_\tau f\big\|_{\mathfrak{h}^1(N)}\leq C\,\big\|f\big\|_1\,.$$

In particular, $\mathscr{D}\mathscr{J}^{0,0}_\tau f-\mathscr{D}_\tau\mathscr{J}^{0,0}_\tau f$ is in $L^1(N)$. By (7.9), $\mathscr{D}\mathscr{J}^{0,0}_\tau f$ is in $L^1(N)$ if and only if $\mathscr{D}_\tau\mathscr{J}^{0,0}_\tau f$ is. Recall that $\mathscr{J}^{0,0}_\tau f+\mathscr{J}^{0,\infty}_\tau f+\mathscr{J}^{\infty}_\tau f=\mathscr{D}_\tau^{-1}f$. Thus,

$$\mathscr{D}_\tau\mathscr{J}^{0,0}_\tau f=\mathscr{D}_\tau\mathscr{D}_\tau^{-1}f-\mathscr{D}_\tau\mathscr{J}^{0,\infty}_\tau f-\mathscr{D}_\tau\mathscr{J}^{\infty}_\tau f. \tag{7.10}$$

By Lemma 7.6, the second and the third summands on the right hand side are in $\mathfrak{h}^1(N)$, hence in $L^1(N)$. Furthermore, f belongs to $L^1(N)$ by assumption, whence so does $\mathscr{D}_\tau\mathscr{J}^{0,0}_\tau f$, equivalently so does $\mathscr{D}\mathscr{J}^{0,0}_\tau f$. Thus, the L^1 norm of each of the summands is dominated by $C\,\big\|f\big\|_1$. This implies the claim (7.8), and concludes the proof of (a).

Next we prove (b). Notice that

$$\partial_t H=-\mathscr{P}^N_t(\mathscr{D}\mathscr{J}^{0,0}_\tau f)+\frac{1}{2}\big[\mathscr{P}^N_{\sigma-t}+\mathscr{P}^N_{\sigma+t}\big]\,\mathscr{D}M_\sigma(\mathscr{D})(\mathscr{J}^{0,0}_\tau f). \tag{7.11}$$

We *claim* that there exists a constant C, independent of f, such that

$$\big\|\mathscr{D}M_\sigma(\mathscr{D})(\mathscr{J}^{0,0}_\tau f)\big\|_{\mathfrak{h}^1(N)}\leq C\,\big\|f\big\|_1\,. \tag{7.12}$$

Given the claim, Theorem 7.3 implies that

$$\Big\|\sup_{s\in(0,2\sigma)}\big|\mathscr{P}^N_s\mathscr{D}M_\sigma(\mathscr{D})(\mathscr{J}^{0,0}_\tau f)\big|\Big\|_1\leq C\;\big\|\mathscr{D}M_\sigma(\mathscr{D})(\mathscr{J}^{0,0}_\tau f)\big\|_{\mathfrak{h}^1(N)}\leq C\;\big\|f\big\|_1\,.$$

This and (7.11) imply that

$$\Big\|\sup_{t\in(0,\sigma)}\big|\mathscr{P}^N_t(\mathscr{D}\mathscr{J}^{0,0}_\tau f)\big|\Big\|_1\leq C\,\big[\,\big\||\partial_t H|^*\big\|_1+\big\|f\big\|_1\,\big].$$

A further application of Theorem 7.3 and the trivial inequality $\big\||\partial_t H|^*\big\|_1\leq\big\||\nabla H|^*\big\|_1$ yield

$$\big\|\mathscr{D}\mathscr{J}^{0,0}_\tau f\big\|_{\mathfrak{h}^1(N)}\leq C\,\big\|\mathscr{P}^N_*\mathscr{D}\mathscr{J}^{0,0}_\tau f\big\|_1\leq C\,\big[\,\big\||\nabla H|^*\big\|_1+\big\|f\big\|_1\,\big].$$

as required.

Thus it remains to prove (7.12). In the proof of fact (a) above we have shown that $\mathscr{D} - \mathscr{D}_\tau$ is bounded on $\mathfrak{h}^1(N)$. Then Lemma 7.7 implies that

$$\big\|(\mathscr{D} - \mathscr{D}_\tau)M_\sigma(\mathscr{D})(\mathscr{J}_\tau^{0,0} f)\big\|_{\mathfrak{h}^1(N)} \leq C \,\big\|f\big\|_1 \,.$$

Thus, in order to prove the claim it suffices to prove that $\big\|\mathscr{D}_\tau M_\sigma(\mathscr{D})(\mathscr{J}_\tau^{0,0} f)\big\|_{\mathfrak{h}^1(N)} \leq C\,\big\|f\big\|_1$. Write

$$\mathscr{D}_\tau M_\sigma(\mathscr{D})(\mathscr{J}_\tau^{0,0} f) = \mathscr{D}_\tau^{-1}[(\mathscr{L}+\tau)M_\sigma(\mathscr{D})(\mathscr{J}_\tau^{0,0} f)].$$

By Lemma 7.7 (i)-(ii), for each $\varepsilon > 0$ there exists a constant C such that

$$\Big|(\mathscr{L}+\tau)M_\sigma(\mathscr{D})\,\mathscr{J}_\tau^{0,0} f(x)\Big| \leq C\,\mathrm{e}^{(\varepsilon-\lambda_1)d(x,o)}\,\big\|f\big\|_1 \qquad \forall x \in N.$$

We use the estimate for the kernel of $\mathscr{D}_\tau^{-1}$ contained in Lemma 7.5, and obtain that

$$\begin{aligned}\big|\mathscr{D}_\tau[M_\sigma(\mathscr{D})(\mathscr{J}_\tau^{0,0} f)](x)\big| \leq C\ \big\|f\big\|_1\,\Big[\int_{B_1(x)} & d(x,y)^{1-n}\mathrm{e}^{(\varepsilon-\lambda_1)d(y,o)}\,\mathrm{d}\nu(y)\\ &+\int_{B_1(x)^c}\mathrm{e}^{-2d(x,y)\sqrt{\tau c}+(\varepsilon-\lambda_1)d(y,o)}\,\mathrm{d}\nu(y)\Big]\end{aligned}$$

By the triangle inequality, the sum of the last two integrals is dominated by

$$\mathrm{e}^{(\varepsilon-\lambda_1)d(x,o)}\Big[\int_{B_1(x)} d(x,y)^{1-n}\mathrm{e}^{(\lambda_1-\varepsilon)d(y,x)}\,\mathrm{d}\nu(y)+\int_{B_1(x)^c}\mathrm{e}^{(\lambda_1-\varepsilon-2\sqrt{\tau c})d(x,y)}\,\mathrm{d}\nu(y)\Big]$$

By integrating in polar coordinates centred at x, it is straightforward to see that the integral above are convergent, provided that $\tau > \lambda_1^2/(4c)$ and ε is small enough. Thus, we may conclude that

$$\big|\mathscr{D}_\tau[M_\sigma(\mathscr{D})(\mathscr{J}_\tau^{0,0} f)](x)\big| \leq C\,\mathrm{e}^{(\varepsilon-\lambda_1)d(x,o)}\,\big\|f\big\|_1 \qquad \forall x \in N.$$

Then Proposition 7.4 yields $\big\|\mathscr{D}_\tau M_\sigma(\mathscr{D})(\mathscr{J}_\tau^{0,0} f)\big\|_{\mathfrak{h}^1(N)} \leq C\,\big\|f\big\|_1$. This concludes the proof of (7.12) and of the theorem. □

Recall that the *local Riesz–Hardy space* $\mathfrak{h}^1_{\mathscr{R}_\tau}(N)$ is defined in (1.1). The main result of the paper is the following.

Theorem 7.9. *Suppose that N is an n-dimensional complete, connected noncompact Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius. Assume that τ is a large positive number. Then $\mathfrak{h}^1_{\mathscr{R}_\tau}(N) = \mathfrak{h}^1(N)$ and their norms are equivalent.*

In particular, the conclusion of Theorem 7.9 holds provided that $\tau > \lambda_1^2/4c$ (this implies that $\tau > \beta^2/c$, where β and c are as in (2.2) and (2.3)), and is so large that Proposition 7.12 holds.

Remark 7.10. We observe that the claim of Theorem 7.9 is invariant under rescaling of the Riemannian metric by a constant conformal factor, since the spaces $\mathfrak{h}^1_{\mathscr{R}_\tau}(N)$ and $\mathfrak{h}^1(N)$ are invariant, and their norms rescale by the same factor. Accordingly, instead of choosing σ small enough depending on β (see (3.2)), one could have fixed σ and rescaled the Riemannian metric of N in order to make β small enough.

The proof of Theorem 7.9 occupies the rest of this section. First we analyse the kernel of $\mathscr{R}_\tau^{0,0}$.

Lemma 7.11. *Under the same assumption as in Theorem 7.9, there exists a constant C such that*

$$\big|k_{\mathscr{R}_\tau^{0,0}}(x,y)\big| \leq C\,\varphi(x,y)\,d(x,y)^{-n}$$

off the diagonal.

Proof. The proof is a straightforward consequence of the definition of $\mathscr{R}_\tau^{0,0}$ and the pointwise estimate [18, Theorem 6, Case II] for the gradient of the heat kernel on N. We leave the details to the interested reader. □

Denote by $\{\psi_j\}$ a *locally uniformly finite partition of unity* on N such that the following holds: the support of ψ_j is contained in the ball B_j with radius 1, $0 \leq \psi_j \leq 1$, $\psi_j = 1$ on $(1/4)B_j$, and there exists a constant C, independent of j, such that

$$\big|\psi_j(x) - \psi_j(y)\big| \leq C\,d(x,y) \qquad \forall x,y \in N. \tag{7.13}$$

For the construction of such a partition of unity see, for instance, [37, Lemma 1.1 and pp. 59–60]. We recall the following norm estimate for the local Riesz transform on N, due to E. Russ [53, proof of Theorem 14]; see also [51, Theorem 8].

Proposition 7.12. *For every $\tau > 0$ large enough there exists a constant C such that $\big\||\mathscr{R}_\tau f|\big\|_1 \leq C\,\|f\|_{\mathfrak{h}^1(N)}$ for every f in $\mathfrak{h}^1(N)$.*

Lemma 7.13. *Under the same assumption as in Theorem 7.9, the following hold:*

- (i) *the operator $\mathscr{R}_\tau^\infty$ is bounded on $L^1(N)$;*
- (ii) *the operator $\mathscr{R}_\tau^{0,\infty}$ is bounded on $L^1(N)$.*
- (iii) *if f is in $\mathfrak{h}^1_{\mathscr{R}_\tau}(N)$, then $\big|\mathscr{R}_\tau^{0,0} f\big|$ is in $L^1(N)$;*
- (iv) *for each p such that $1 \leq p < n/(n-1)$ there exists a constant C, independent of j, such that if f is in $\mathfrak{h}^1_{\mathscr{R}_\tau}(N)$, then*

$$\big\||\mathscr{R}_\tau^{0,0}(\psi_j f) - \psi_j\,\mathscr{R}_\tau^{0,0} f|\big\|_p \leq C\,\|f\|_{L^1(2B_j)}\,;$$

- (v) *there exists a constant C, independent of j, such that*

$$\big\||\mathscr{R}_\tau^{0,0}(\psi_j f)|\big\|_{L^1(2B_j)} \leq C\,\big[\,\|f\|_{L^1(2B_j)} + \big\||\mathscr{R}_\tau^{0,0} f|\big\|_{L^1(2B_j)}\,\big].$$

Proof. First we prove (i). By [53, Theorem 14], $\sup_{y\in N}\int_N \big|k_{\mathscr{R}_\tau^\infty}(x,y)\big|\,\mathrm{d}\nu(x) < \infty$. Consequently the operator $\mathscr{R}_\tau^\infty$ is bounded on $L^1(N)$, as required.

To prove (ii) observe that, at least formally,

$$\mathscr{R}_\tau^{0,\infty} f(x) = \frac{1}{\sqrt{\pi}} \int_1^\infty \frac{\mathrm{d}t}{t^{1/2}}\, \mathrm{e}^{-\tau t} \int_N \varphi(x,y)\, \nabla_x h_t^N(x,y)\, f(y)\, \mathrm{d}\nu(y).$$

Therefore

$$\begin{aligned} \big\| |\mathscr{R}_\tau^{0,\infty} f| \big\|_1 &\leq \frac{1}{\sqrt{\pi}} \int_1^\infty \frac{\mathrm{d}t}{t^{1/2}}\, \mathrm{e}^{-\tau t} \int_N \mathrm{d}\nu(y)\, \big|f(y)\big| \int_N \varphi(x,y)\, \big|\nabla_x h_t^N(x,y)\big|\, \mathrm{d}\nu(x) \\ &\leq \frac{1}{\sqrt{\pi}} \int_1^\infty \frac{\mathrm{d}t}{t^{1/2}}\, \mathrm{e}^{-\tau t} \int_N \mathrm{d}\nu(y)\, \big|f(y)\big| \int_{B_1(y)} \big|\nabla_x h_t^N(x,y)\big|\, \mathrm{d}\nu(x) \\ &\leq C \int_1^\infty \frac{\mathrm{d}t}{t^{1/2}}\, \mathrm{e}^{-\tau t} \int_N \big|f(y)\big|\, \big\| |\nabla_x h_t^N(\cdot,y)| \big\|_2 \, \mathrm{d}\nu(y), \end{aligned}$$

where the last inequality follows from Schwarz's inequality and the uniform ball size condition of N. Observe that

$$\begin{aligned} \big\| |\nabla_x h_t^N(\cdot,y)| \big\|_2^2 &= \big(\nabla_x h_t^N(\cdot,y), \nabla_x h_t^N(\cdot,y)\big) \\ &= \big(\mathscr{L}_x h_t^N(\cdot,y), h_t^N(\cdot,y)\big) \\ &\leq \big\|\mathscr{L}_x h_t^N(\cdot,y)\big\|_2 \, \big\|h_t^N(\cdot,y)\big\|_2 \,. \end{aligned}$$

Now, the ultracontractivity of the heat semigroup and [47, Proposition 2.2] imply that the supremum with respect to y in N of the right hand side is dominated by a constant multiple of $t^{-3/2}$. Therefore we may conclude that

$$\big\| |\mathscr{R}_\tau^{0,\infty} f| \big\|_1 \leq C \int_1^\infty \frac{\mathrm{d}t}{t^{5/4}}\, \mathrm{e}^{-\tau t} \int_N \big|f(y)\big|\, \mathrm{d}\nu(y) \leq C\, \big\|f\big\|_1 \,,$$

i.e., the operator $\mathscr{R}_\tau^{0,\infty}$ is bounded on $L^1(N)$, as required.

Next we prove (iii). The assumption that f is in $\mathfrak{h}^1_{\mathscr{R}_\tau}(N)$ together with the decomposition (7.5) and (i) above yields that $\big|\mathscr{R}_\tau^0 f\big|$ in $L^1(N)$. Since $\mathscr{R}_\tau^{0,0} f = \mathscr{R}_\tau^0 f - \mathscr{R}_\tau^{0,\infty} f$ and both $\big|\mathscr{R}_\tau^0 f\big|$ and $\big|\mathscr{R}_\tau^{0,\infty} f\big|$ are in $L^1(N)$ (by (i) and (ii)), the same is true of $\big|\mathscr{R}_\tau^{0,0} f\big|$, as required.

To prove (iv) observe that, at least formally,

$$\mathscr{R}_\tau^{0,0}(\psi_j f)(x) - \psi_j(x)\, \mathscr{R}_\tau^{0,0}(f)(x) = \int_N k_{\mathscr{R}_\tau^{0,0}}(x,y)\, \big[\psi_j(y) - \psi_j(x)\big]\, f(y)\, \mathrm{d}\nu(y).$$

If x is not in $2B_j$, then $\psi_j(x)$ vanishes, and so does $k_{\mathscr{R}_\tau^{0,0}}(x,y)$ as long as y belongs to B_j. Hence $\mathscr{R}_\tau^{0,0}(\psi_j f) - \psi_j\, \mathscr{R}_\tau^{0,0} f$ vanishes at x. In particular

$$\big\|\mathscr{R}_\tau^{0,0}(\psi_j f) - \psi_j\, \mathscr{R}_\tau^{0,0} f\big\|_p = \big\|\mathscr{R}_\tau^{0,0}(\psi_j f) - \psi_j\, \mathscr{R}_\tau^{0,0} f\big\|_{L^p(2B_j)}$$

If, instead, x is in $2B_j$, then we use the estimates for $k_{\mathscr{R}_\tau^{0,0}}$ in Lemma 7.11, the uniform Lipschitz property of ψ_j, and conclude that there exists a constant C, independent of j, such that

$$\big|\mathscr{R}_\tau^{0,0}(\psi_j f)(x) - \psi_j(x)\, \mathscr{R}_\tau^{0,0} f(x)\big| \leq C \int_{2B_j} \frac{\varphi(x,y)}{d(x,y)^{n-1}}\, \big|f(y)\big|\, \mathrm{d}\nu(y).$$

It is not hard to check that if $1/p > 1 - 1/n$, then the integral operator with kernel $\mathbf{1}_{2B_j}(x)\,\frac{\varphi(x,y)}{d(x,y)^{n-1}}\,\mathbf{1}_{2B_j}(y)$ is bounded from $L^1(2B_j)$ to $L^p(2B_j)$ uniformly in j. Therefore there exists a constant C such that

$$\big\|\mathscr{R}_\tau^{0,0}(\psi_j f) - \psi_j\, \mathscr{R}_\tau^{0,0} f\big\|_{L^p(2B_j)} \leq C\, \big\|f\big\|_{L^1(2B_j)}\,,$$

as required to conclude the proof of (iv).

Now we prove (v). Clearly $\mathscr{R}_\tau^{0,0}(\psi_j f) = \mathscr{R}_\tau^{0,0}(\psi_j f) - \psi_j \mathscr{R}_\tau^{0,0} f + \psi_j \mathscr{R}_\tau^{0,0} f$. By (iv), the function $\mathscr{R}_\tau^{0,0}(\psi_j f) - \psi_j \mathscr{R}_\tau^{0,0}(f)$ is in $L^p(2B_j)$ with norm $\leq C\,\big\|f\big\|_{L^1(2B_j)}$. Hölder's inequality, together with local Ahlfors regularity, imply that

$$\big\|\mathscr{R}_\tau^{0,0}(\psi_j f) - \psi_j \mathscr{R}_\tau^{0,0} f\big\|_{L^1(2B_j)} \leq C\, \big\|f\big\|_{L^1(2B_j)}\,.$$

Therefore

$$\begin{aligned}\big\|\mathscr{R}_\tau^{0,0}(\psi_j f)\big\|_{L^1(2B_j)} &\leq C\,\big[\ \big\|\mathscr{R}_\tau^{0,0}(\psi_j f) - \psi_j \mathscr{R}_\tau^{0,0} f\big\|_{L^1(2B_j)} + \big\|\mathscr{R}_\tau^{0,0}(f)\big\|_{L^1(2B_j)}\ \big]\\ &\leq C\,\big[\ \big\|f\big\|_{L^1(2B_j)} + \big\|\mathscr{R}_\tau^{0,0}(f)\big\|_{L^1(2B_j)}\ \big],\end{aligned}$$

as required to conclude the proof of (v), and of the lemma. □

Proof of Theorem 7.9. The containment $\mathfrak{h}^1(N) \subseteq \mathfrak{h}^1_{\mathscr{R}_\tau}(N)$ is a direct consequence of Proposition 7.12.

It remains to prove that $\mathfrak{h}^1_{\mathscr{R}_\tau}(N) \subseteq \mathfrak{h}^1(N)$. Suppose that f is in $\mathfrak{h}^1_{\mathscr{R}_\tau}(N)$. By Lemma 7.13 (iii), $\big|\mathscr{R}_\tau^{0,0} f\big|$ is in $L^1(N)$. Then, by Lemma 7.13 (v),

$$\big\||\mathscr{R}_\tau^{0,0}(\psi_j f)|\big\|_{L^1(2B_j)} \leq C\,\big[\ \big\|f\big\|_{L^1(2B_j)} + \big\||\mathscr{R}_\tau^{0,0} f|\big\|_{L^1(2B_j)}\ \big].$$

By Theorem 7.8, there exists a constant C, independent of j, such that

$$\big\|\psi_j f\big\|_{\mathfrak{h}^1(N)} \leq C\,\big\||\mathscr{R}_\tau^{0,0}(\psi_j f)|\big\|_1 + C\,\big\|\psi_j f\big\|_1\,.$$

Then, using also Lemma 7.13 (i), (ii) and (iv),

$$\begin{aligned}\big\|f\big\|_{\mathfrak{h}^1(N)} &\leq \sum_j \big\|\psi_j f\big\|_{\mathfrak{h}^1(N)}\\ &\leq C\,\sum_j \big\||\mathscr{R}_\tau^{0,0}(\psi_j f)|\big\|_1 + C\,\sum_j \big\|\psi_j f\big\|_1\\ &\leq C\,\sum_j \big\|\psi_j |\mathscr{R}_\tau^{0,0} f|\big\|_1 + C\,\sum_j \Big[\ \big\||\mathscr{R}_\tau^{0,0}(\psi_j f) - \psi_j \mathscr{R}_\tau^{0,0} f|\big\|_1 + \big\|\psi_j f\big\|_1\ \Big]\\ &\leq C\,\big\||\mathscr{R}_\tau^{0,0} f|\big\|_1 + C\,\big\|f\big\|_1\\ &\leq C\,\big[\ \big\||\mathscr{R}_\tau f|\big\|_1 + \big\||\mathscr{R}_\tau^{0,\infty} f|\big\|_1 + \big\||\mathscr{R}_\tau^{\infty} f|\big\|_1 + \big\|f\big\|_1\ \big]\\ &\leq C\,\big[\||\mathscr{R}_\tau f|\|_1 + \|f\|_1\big],\end{aligned}$$

as required. □

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