ASIDE:

A GROWTH ESTIMATE

Lemma 1

IF $B \in \mathbb{B}_{ad}$; $\dfrac{\phi_{xx}}{\phi_x} \in L^1(D)$;

AND $\left(\dfrac{\rho}{\phi_x}\right)^{[-1]} \subset L^{\infty}(D)$; $\exists \lim_{x \to x_0^+} \left(\dfrac{\rho}{\phi_x}\right)^{[-1]}$

AND $B_x\phi_x + B\phi_{xx} - \rho =_{\text{d.w.}} 0$

THEN

$$||B||_{0,\infty} \leq ||\left(\frac{\rho}{\phi_x}\right)_0^{[-1]}||_{0,\infty} \exp[\, ||\frac{\phi_{xx}}{\phi_x}||_{0,1}]$$

Proof:

relies on extended form of Gronwall – Bellman inequality

$D = (x_0, x_1)$; $a \in \mathbb{A}_{ad}$;

$g \in \mathbb{G}_{ad} := \{g \mid g \in L^1(D),\ g \geq_{\text{a.e.}} 0\}$

$c_+ \geq 0$ (constant)

IF $\quad a \leq c_+ + \int_{x_0}^{x} ag\, d\xi$, a.e. in D

THEN

$$||a||_{0,\infty} \leq c_+ \exp ||g||_{0,1}$$

STABILITY ESTIMATES:
2 – APPLICATION TO LAYERED MEDIA

Rem.: (Implications of Hp.)

$$a' = \sum_{i=1}^{\infty} c_i \, \delta(x - \xi_i) + \sum_{i=1}^{\infty} \psi_i \, \chi_{(\xi_i, \xi_{i+1})} \, ,$$

$$\sum_{i=1}^{\infty} |c_i| < \infty \text{ , etc. } \Rightarrow$$

i) $|\hat{a}_x|_0^{[-1]} \subset L^{\infty}(D)$;

rules out e.g., $a = 2 + \sin\frac{1}{x}$ in $[-1, +1]$

ii) $\exists \lim_{x \to x_0^+} \hat{a}$; $\exists \lim_{x \to x_1^-} \hat{a}$

STABILITY ESTIMATES:
3 – *SINGULAR* CAUCHY – UNIQUE SOLN.

Let $\bar{I} \subset D$

Thm.

IF $\quad u, v \in \mathbb{X}$

$$\exists\, \tau \in \bar{T} \cdot\!\ni\!\cdot \left(\tfrac{1}{u_x}\right)(\tau) \in L^1(D)\,;$$

(∗) $\quad E_u(\tau) \neq \emptyset \wedge \{\text{meas}[E_u(\tau)] = 0\}$

$$E_v(\tau) \neq \emptyset\,,\ \text{meas}[E_v(\tau)] \geq 0$$

$$E_u(\tau) \subset \bar{I},\ \ E_v(\tau) \subset \bar{I}$$

$$\left(\tfrac{1}{v_x}\right)(\tau) \in L^1(D \setminus E_v(\tau)) := \mathbb{Y}(\tau)$$

$$\left\|\tfrac{1}{v_x}\right\|_{\mathbb{Y}_\tau} \leq c_v$$

$$\exists\, \hat{a}(u, f) \in \mathbb{A}_{ad} \cap C^0(\bar{I})$$

$$\exists\, b(v, f) \in \mathbb{A}_{ad}$$

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THEN

$$\|B\|_{0,1} \leq c_v\,[1 + 2\|\hat{a}\|_{0,\infty}]\,\|V\|_{\mathbb{X}_\tau}$$

Rem.

i) Second conductivity, *b*, need not be unique.

ii) Same estimate from uniqueness condition (∗∗); much more complicated proof.

STABILITY ESTIMATES:
4 – SYNOPSIS OF *PROOFS*

Regular Cauchy | **Singular Cauchy (*)**

$$B := b - a \;;\; V := v - u \;;\; r := -V_t + (aV_x)_x$$

starting point: the *defect* equation

$$(Bv_x)_x + r =_{\text{d.w.}} 0 \;@\; t = \tau$$

Regular Cauchy:

$B(x_0) = 0$

$B_x = -\frac{r}{v_x} - B\frac{v_{xx}}{v_x}$

apply Gronwall – Bellman inequality and Lemma 1

$R := \frac{r}{v_x}$

$\|B\|_{0,\infty} \leq$

$\leq [\|R_0^{[-1]}\|_{0,\infty} \exp\|\frac{v_{xx}}{v_x}\|](\tau)$

Singular Cauchy (*):

$(Bv_x)(\xi^+_v(\tau), \tau) = 0$

$r^{[-1]} = -V_t^{[-1]} + \hat{a}\, V_x + \text{const.}$

$Bv_x =_{\text{a.e.}} -(r_0^{[-1]})(\tau)$

need $\exists\; \hat{a}\,(\xi_v(\tau))$ now

$B =_{\text{a.e.}} -(\frac{r_0^{[-1]}}{v_x})(\tau)$

$\|B\|_{0,1} \leq$

$\leq c_v \;\|r_0^{[-1]}\|_{0,\infty}(\tau)$

bring in V instead of R, resp. r ;

all remaining Hp. on data used up here

CONCLUSION:

L^∞–estimates | L^1–estimates

STABILITY ESTIMATES:
5 – INCOMPATIBLE CONDITIONS

$$\left(\frac{v_{xx}}{v_x}\right)(\tau) \in L^1(D) \qquad \nparallel \qquad (Bv_x)(\xi^+{}_v(\tau), \tau) = 0$$

$$\Updownarrow$$

$$\frac{1}{v_x} \in AC(\bar{D}) \qquad\qquad \frac{1}{v_x} \in L^1(\bar{D})$$

$$\Downarrow$$

necessary for the non–
uniqueness of $b(v,f)$,
given existence

L^1– estimate

Related results:

Baumeister & Kunisch, 1991, *Applicable Analysis*

Marcellini, 1982, *Ric. di Matem. dell' Univ. di Napoli*

CONCLUSION

i) Defect equation as main device.

ii) Cauchy problem thereof.

iii) Unified view over

uniqueness conditions and stability estimates.

iv) Supplementary (regularization) conditions needed to attain stability estimates.

v) Admissible a in stability estimates affected by type of Cauchy problem:

a bounded, measurable when Cauchy pbm. regular;

continuity of a @ critical points cannot be relaxed.

vi) Application to time – independent inverse problems:

technically straightforward;

distinction between regular and singular Cauchy problems left unaltered;

same stability estimates.