The set of equilibria in a class of symmetric conflict models*

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Abstract

This study aims to provide a manageable symmetric two-players conflict model to understand the structural determinants of the properties of its set of equilibria, in particular existence, multiplicity and (a)symmetry. We investigate the effects of introducing spillovers into the marginal productivity of agents' efforts and into the polarization between agents' goals. We show that, without spillovers, the equilibrium efforts' intensity is uniquely connected to the ratio of the marginal productivity of effort to (ex ante) polarization. Then, we connect the existence of multiple symmetric and asymmetric equilibria to the intensity of the spillovers in the outcomes through growing polarization and hostility. We also show that negative spillovers in conflict technology (direct destructiveness) can imply the non-existence of equilibria. Finally, we introduce a measure of the intensity of conflict at equilibrium and we show how, depending on the equilibria configurations, the parameters have different interesting effects on the intensity of conflict.
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1 Introduction

Conflicts are ubiquitous in social contexts, they occur any time agents want to get a goal that is incompatible with other agents' goal (polarization). Situations such as fights, wars, electoral competitions, crimes, strikes and lockouts, sport contests, patent races, projects competitions, litigations, labor market tournaments are all examples of conflicts that can be modelled as games, where players expend costly, non-refundable resources trying to affect the probability of getting a desired outcome. Conflicts share several specific features. Firstly, it is common that an agent cares not only about its goal, but also about the outcome of the defeated party (hostility), whether itself or the opponent. Further, the winner and the loser outcomes may change because of growing polarization and hostility as agents' effort increases, more generally because of possible spillovers. Finally, players' effort may have a direct destructive effect on the conflict technology, reducing its effectivity. This paper contributes to conflict theory providing a manageable symmetric two-players abstract conflict model to understand the structural determinants of the properties of its set of equilibria, in particular existence, multiplicity and (a)symmetry. Often abstract conflict models have a unique interior fully stable equilibrium, as shown in Szidarovszky and Okuguchi (1997). This can be unsatisfactory, as stressed by Hirshleifer (1989) and, more recently, by Blattman and Miguel (2010) and by Baliga and Sjšträöm (2013),

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because it rules out the possibility of boundary equilibria displaying either null or maximum efforts,[1] as well as multiple (a)symmetric Pareto ordered equilibria, while multiplicity or even non-existence may provide insights into real conflict situations, as illustrated by Konrad (2009). We show that, without spillovers, the intensity of the equilibrium efforts is uniquely connected to the ratio of the marginal productivity of effort to (ex ante) polarization. Then, we connect the existence of multiple symmetric and asymmetric equilibria to the intensity of the spillovers in the outcomes through growing polarization and hostility.[2] We also show that spillovers in conflict technology can imply the non-existence of equilibria. Finally, we introduce a measure of the intensity of conflict at the equilibrium and we study the role of parameters on it. A clear and robust connection between on one hand uniqueness or multiplicity, symmetry or asymmetry of the equilibria, and on the other hand the structural properties of the model, is highly desirable to deepen the economic rationale of the emergence of particular conflict scenarios. More generally, understanding the structural origins of heterogeneous behaviors starting from a fully symmetric setting is a central goal of research in different areas of economics and social sciences,[3] and this approach is particularly relevant in conflict settings, where heterogeneity and apparently multiple steady states are both commonly observed, as we argue in Section 2.

Research on conflict theory has developed rapidly and increasingly, as surveyed for example in Anderton and Carter (2019); Sandler and Hartley (2007); Baliga and Sjštröm (2013); Blattman and Miguel (2010); Jackson and Morelli (2011).

It is beyond the scope of this paper to list the enormous number of contributions on conflicts. Roughly, in conflict theory we might distinguish four general classes of models. A first class apply bargaining models to the analysis of conflicts, as reviewed in Baliga and Sjštröm (2013) and Jackson and Morelli (2011). A second class consists of general equilibrium models, which analyze the trade-off between the production of private consumption and of conflict goods.[4] The seminal papers by Esteban and Ray (1999, 2008, 2011) introduce a further class of models, connecting conflicts to various indexes of polarization, fragmentation and inequality. The literature on contests is vast, the interested reader should consult Corchon (2007); Konrad (2009); Nitzan (1994).[5] The model proposed in the present contribution is not within the realm of general equilibrium, and it is related to but different from both contest models and the class of models connecting conflicts and polarization. The present work relies on a class of symmetric two-players games where two symmetric agents submit entries that influence the probability of reaching a desired outcome and bear some costs for the action; an entry might be a bid, an effort, or a commitment of non-refundable resources. Strategic interaction within the model is characterized by four key properties that are easily interpretable in applications: (i) continuous conflict technology (ii) *ex ante* and endogenous polarization (iii) *ex ante* and endogenous hostility and (iv) direct destructiveness. This class of games allows for a clear discussion of all these spillovers, so that a rich variety of equilibria configurations can arise, with multiple symmetric Pareto ordered equilibria or asymmetric equilibria as well as non existence of pure strategy equilibria, even keeping the assumptions on the constitutive elements of the conflict model reasonably simple and transparent. Then, by introducing indexes related to polarization, sabotage and hostility, and considering the possible spillover effects, we connect the characterization of the equilibria set and the intensity of equilibrium conflict to the structural characteristics of the model. Thus, the adopted approach proposes a general methodology

[1] As well-known, zero-effort equilibria are ruled out by assuming the Tullock contest success function, while maximum-effort equilibria are ruled out by assuming unlimited upper bound for agents' effort.

[2] All these concepts will be formally defined in section 3.

[3] See Amir et al. (2010); Matsuyama (2002).

[4] See for example Grossman (1991, 1994, 1995) and Grossman (1999).

[5] Following the seminal contributions of Tullock (1967, 1980) and Krueger (1974), general analysis of rent-seeking games were provided by Pérez-Castrillo and Verdier (1992) and by Riaz et al. (1995), while Corchon (2007) and Konrad (2009) provide formal models for contests, and review their main possible applications and generalizations. Finally, Chowdhury and Gürtler (2015) provide a model of sabotage activities in contests, and review related literature.

for generating inter-agent differences out of a symmetric conflict game with fully rational and completely informed players.

The remainder of the paper is organized as follows. Section 2 quickly discusses some evidence to show the significant heterogeneity in agents' behavior within symmetric conflicts. Section 3 illustrates the proposed approach to model conflicts and discusses the restrictions we use to construct a specific family of conflict models. Section 4 discusses the characteristics of the equilibrium set for three different cases: no spillover, spillovers in outcomes, and spillovers in the effectivity function. Section 5 discusses the relationship between our structural parameters and conflict intensity for the different kind of equilibria. Finally, Section 6 concludes the paper.

2 The Relevance of Multiple Symmetric and Asymmetric Equilibria in Conflict Models

Even if in conflict models players' strategies are costly resources used to affect the probability of getting a desired result, such strategic choices often might be observable only indirectly, as it is the case with agents' efforts. Moreover, the total amount of resources used in equilibrium, called the intensity of conflict by Esteban and Ray (1999, 2011), might not induce a full overt violent conflict, for example as it has been with the Cold War. Actually, abstract conflict models do not explicitly encompass any element of "violence" in them, they just model settings where players have conflicting aims and expend costly, non-refundable resources trying to affect the probability of getting the desired outcome. Hence, the most important observable consequence of the existence of multiple symmetric or asymmetric equilibria is the heterogeneity of observable agents' behavior, possibly with the existence of multiple agglomeration clusters. As argued by Amir et al. (2010) and Matsuyama (2002), the endogenous heterogeneity within a class of symmetric games is important. In conflicts, corner equilibria, multiple symmetric Pareto-ranked equilibria, multiple asymmetric equilibria, and non existence of pure strategy equilibria, are all important cases.

The data we report in what follows do not aim at supporting and grounding a specific model of conflict, they simply suggest that in many conflicts there might be something that it is not easily compatible with the standard prediction of a unique interior equilibrium. The main lesson we draw is that models that explain when and why multiple symmetric and asymmetric equilibria arise can help to analyze real cases because empirical results sometimes are at odd with the prediction of a unique interior equilibrium.

Since empirical analysis of conflicts is somehow problematic because efforts are not directly observable, in order to have a careful interpretation of exogenous and endogenous variables it is crucial to establish a connection between real conflicts and theoretical models. The link between agents' conflict effort levels and observable measures of conflict is empirically complex, and a theoretical view is required to distinguish between different possible interpretations. In particular, in our setting maximum effort does not mean effective conflict, but that the amount of available resource used to try to reach the goal is the maximum possible. For instance, the cold war is an example of high effort in the sense of huge military investment, even if there was no actual overt war. On the other hand, the second world war is an example of maximum intensity conflict with overt war: according to the estimates of Eloranta and Land (2015) during WWII the average military burden for the conflicting countries raised well over 50% of GDP. And we would like to consider both these cases as examples of conflict situations. In general, if we consider some simple examples of, to some extent, similar conflicting countries and their military expenditure as percentage of GDP as a rough indicator of conflict efforts, a relevant degree of heterogeneity emerges. In 2007 North Korea military spending was 22.9% of its GDP against 2.7% of South Korea[6]; Russian military spending was 3.4% of its GDP against 3.8% of USA; Saudi military spending was 8.5% of its GDP against 3.4% of Iran; Pakistan military spending was 3.4% of its GDP against 2.3% of India; Armenia military spending was 3% of its GDP against 2.9% of Azerbaijan.[7] Further, if we consider a minor border dispute such as the one between Bolivia and Brazil on Isla Suárez/Ilha de Guajará-mirim, then in 2007 Brazil and Bolivia were devoting to military spending respectively 1.4% and 1.7 of their GDP.[8]. These simple data have surely many different explanations, however they not only illustrate the heterogeneity of resources devoted to conflicting goals in actual situations, they also indicate that it is possible to have symmetric null, intermediate or maximum intensity, and asymmetric situations too. Another interesting case of conflicts for which evidence is available are litigations. Farber and White in Farber and White (1991) report the disposition of the 252 cases of medical malpractice charges raised against a single large hospital and/or medical personnel on its staff by patients who received treatment there, initiated in 1977 or later and resolved by the end of 1989. Overall, 92 cases (36.5%) were dropped by plaintiffs or dismissed by the judge, 147 cases (58.3%) were settled out of court (with or without mediation), and only 13 (5.2%) were tried to a verdict in court. Again, we get a clue of possible heterogeneous outcomes.

Experimental results on conflicts have been booming in the recent years, in general focused on specific contests, as illustrated by Dechenaux et al. (2014); Sacco and Schmutzler (2008). These papers principally review experiments on standard Tullock contests, on the all-pay auction and on the rank-order tournaments. These models are specific cases of a common strategic model where players exert costly irreversible efforts while competing for a prize and an individual player's probability of winning the prize depends on the players' relative expenditures, however they differ for the specification of such probability. They have obviously different equilibria, when symmetric and with two players, an interior intermediate the first and third case, a mixed one the second. The most important messages that can be drawn from the experimental studies reviewed in Dechenaux et al. (2014) are

1. most studies on Tullock contests and all-pay auctions find significant overbidding relative to the Nash equilibrium prediction, while there is little overbidding in rank-order tournaments;

2. in all three cases there is significant heterogeneity in the behavior of individual subjects;

3. in Tullock contests bids are widely distributed with multiple high frequency bid, while in all-pay auctions the distribution of bids is bimodal, with some subjects submitting very low and others submitting very high bids.

Several explanations have been provided for these results, mostly based on modified utility function (including for instance non-monetary utility from winning the contest, or preferences over payoffs relative to other contestants) or on subject's irrational behaviors. In Sacco and Schmutzler (2008) the authors consider all-pay auctions with endogenous prizes that depend positively on each player own effort and negatively on the effort of competitors, a case somewhat related to our conflict model. They find that, in the two player case (BIG2), the frequency distribution exhibits a global maximum at high bid and a local maximum at 0, and a substantial fraction of the subjects choose bids that are not part of the symmetric mixed strategy equilibrium (MSE). Again, the experimental observations seem to confirm the heterogeneity of observable agents' conflict behavior, consistently with multiple symmetric and asymmetric equilibria.

[6]Data from Anderton and Carter (2019), in turn elaboration of data from CIA and from the International Institute for Strategic Studies.

[7]Data from the World Bank.

[8]Data from the World Bank.

3 Modeling Conflicts

We consider two agents, so that each generic part is indexed by $i \in \{1, 2\}$, and the opponent is identified by $j = 3 - i$. Grounding on the constitutive elements that allow us to introduce a micro-founded *conflict model* (CM), we focus on a sufficiently general class that allows us to describe a suitably wide range of situations, but that, at the same time, is sufficiently specialized to allow us to derive significant results. The main goal is to understand the role and the relation between the structural characteristics of the elements of a CM and the properties of the set of equilibria of the associated game. To this end, we focus on symmetric conflict models, namely those in which the associated strategic-form game is symmetric.[9] We also consider simple functional forms, to guarantee that results on multiplicity and on non-existence are not ascribable to the complexity and on peculiar properties of the constitutive elements.

Each contender uses available resources to try to achieve a particular goal and, while doing so, it influences the probability of obtaining an *outcome* $z_i \in Z \subseteq \mathbb{R}$, exploiting a suitable amount of *effort* $x_i \in X_i \subseteq \mathbb{R}^+$, where X_i represents the *set of agents' possible efforts*. We assume that contenders are exogenously constrained with regard to the amount of effort they can choose, namely we assume a *bounded effort set*, setting w.l.g. $X_i = [0, 1]$. The reasons for this assumption are two. First, it is hardly realistic to assume that players' efforts can increase without limit because of exogenous constraints, such as limited capacity; second, this assumption allows interpreting agents' bids as intensity, i.e. as the percentage of the available resources used for the conflict game.

The agent's effort is the choice variable and has several consequences on the possible outcomes and on the probability of reaching one of them. First, the agents' goals can be affected and can change, depending on the agent's own and the opponent's effort.[10] We represent the possible outcomes of the conflict for each agent using functions of his own/opponent's efforts. In particular, we assume that for each player, the interaction might end in two ways, either reaching its *goal outcome* $g_i(x_i)$, where

$$g_1(x_1) = -\theta - \delta x_1, \quad g_2(x_2) = \theta + \delta x_2,$$

or its *defeat outcome* $d_i(x_j)$, where

$$d_1(x_2) = \gamma x_2, \quad d_2(x_1) = -\gamma x_1,$$

with $\theta > 0$, δ and γ non-negative constants. θ represents the distance between the agents' goals when no effort is exploited, while δ and γ allow taking into account possible *spillover* effects. Thus, the set of i's possible outcomes has just two possibilities: either i reaches its goal g_i or it obtains a defeat outcome d_i, as in Chowdhury and Sheremeta (2011a,b, 2015). Recalling the compactness of efforts, we denote the inferior and the upper bounds of i's outcomes as $\underline{g_i}$ and $\overline{g_i}$, respectively, so that $\underline{g_i} \leq \overline{g_i}$ and $\overline{g_1} \leq 0 \leq \underline{g_2}$.

Each agent's utility function U_i is a linear decreasing function of the distance between $i's$ outcome and its goal, the well known Euclidean preferences:[11]

$$U_1(z) = -(z - g_1(x_1)), \quad U_2(z) = -(g_2(x_2) - z),$$

so that g_i is the bliss point of $i's$ utility function,

$$\arg\max U_i(z_i) = g_i(x_i). \tag{1}$$

[9] We recall that a game $\Gamma = (\{1,2\}, X_i \times X_j, \pi_i(x_i, x_j))$ is symmetric if $X_i = X_j$ and $\pi_1(x_1, x_2) = \pi_2(x_1, x_2), \forall x_i \in X_i$, $i = 1, 2$, and that in a two-player symmetric game, the best reply correspondences are symmetric (i.e., $BR_i(x_j) := BR_j(x_i)$).

[10] When the agents' outcomes do not depend on their efforts, then the model belongs to the class of rent-seeking models. However, in models of production and conflict, the values of the goals are determined endogenously by the agents' choices on how much time is used on conflict and, consequently, on production. See, for example, Garfinkel and Skaperdas (2007); Hausken (2005); Konrad (2009); Neary (1997).

[11] For a recent analysis of the role of this type of utility function, see e.g. Brady and Chambers (2015).

The proposed utility function, while quite common in the political economy literature,[12] is new in conflict theory and has important implications. If there are two agents only, as in our model, the measure of polarization P^* derived in Esteban and Ray (1994) is proportional to the absolute distance between the attributes of individuals.[13] Using this property, in our model it is natural to measure *polarization* as the distance between the agents' bliss points,[14] which is realized from the function $\rho : [0,1]^2 \to \mathbb{R}^+$, given by

$$\rho(x_1, x_2) := g_2(x_2) - g_1(x_1) = 2\theta + \delta(x_1 + x_2).$$

so that θ is a measure of *ex-ante polarization.* Esteban and Ray also prove that the bimodal distribution is more polarized than any other distribution with the same total population, under any measure P^*,[15] a further justification for our measure. The root of any conflict, as discussed by Esteban and Ray (1994, 1999, 2008); Esteban and Schneider (2008); Esteban and Ray (2011); Hirshleifer (1995a,b); Fearon (1995); Jackson and Morelli (2011) is the incompatibility between agents' goals:[16] the attempt to reach the best possible outcome leads the agents to clash and it is one source of the conflicting behavior. In our model, the agents' goal contrast is equivalent to strictly positive polarization, because, by construction, the agents' goals are incompatible:

$$\underline{g_2} \geq \overline{g_1} \Leftrightarrow \rho(x_1, x_2) > 0 \; \forall (x_i, x_j).$$

Polarization is one of the most established concepts in the political economy literature, however the best way of actually measuring polarization has been widely discussed.[17] On one hand, polarization encompasses the distance between agents' goals characterizing political and social conflicts, where polarization refers to the divergence of political attitudes to ideological extremes. On the other hand, more generally, according to the seminal work by Esteban and Ray (1994), polarization results from the interaction of within group identity and across-group alienation. [18] The relation between polarization and conflict has been widely studied, see e.g. Esteban and Ray (2011). In particular Esteban and Ray (1999) show that the level of conflict increases with the magnitude of polarization, and that if there are two groups, the intensity of conflict is most pronounced with a bimodal distribution of the population over opposing goals.

Also the defeat outcome is crucial in conflict models as conflicts may be to the death, as in WWII, or a mere border adjustment, as the Greco-Turkish War of 1897 over the sovereignty of Crete. Similarly, in litigation, a plaintiff can claim the reimbursement only for damages actually incurred or also for potential, moral and indirect losses. Further, in electoral struggles, the defeated party might be protected by constitution or might be seriously limited in its future opportunities. All these possibilities, should be considered for an effective CM.

Efforts have an impact on the likelihood of obtaining a specific outcome through the function $S_i(x_i, x_j) = \beta x_i (1 - \alpha x_j) + k$, where $\beta > 0$ and $\alpha, k \in [0,1]$, so that S_i is linear in x_i and the marginal productivity of i's effort is linearly decreasing in the opponent's effort x_j. These functions represent the *effectivity of agents' efforts.* The term "effectivity function" is used in Corchon and Dahm (2010), while other works (e.g., Rai and Sarin (2009)) label it as the technology that describes the productivity of the investment by contender i. Whatever the terminology, S_i is a formal description of technology and/or institutions that transforms agents' effort into the factors that operate on the likelihood of achieving the desired goal. Note

[12] See, for example, Torsten and Tabellini (2000).
[13] See Esteban and Ray (1994) Theorem 1.
[14] An alternative expression might be *alienation* following Esteban and Ray (1994) formalization.
[15] Esteban and Ray (1994) Theorem 2.
[16] Fearon 1995 call this aspect "issue indivisibilities", and considers it as one of the three mechanisms leadind to conflicts, besides asymmetric information and committment problems.
[17] See e.g. Esteban and Ray (1994) and Schmitt (2016).
[18] There are many recent works on polarization within social and political context, for a recent review see Gentzkow (2016).

that in our formulation, the effectivity of an agent's effort might be affected by the opponent's one, in order to consider the possibility that a part can hamper or disable the opponent's capability directly.[19]

The role of function S_i is to distinguish the agents' effort effectivity, which is part of i's characteristics, from the probability of achieving a specific outcome, represented by the *conflict success function (CSF)*, which is objective and holds for both contenders. Depending on the effectivity of the agent's effort (s_1, s_2), $i's$ probability of achieving an outcome is measured by the ratio conflict success function (Hirshleifer (1989))

$$P_i(z_i|S_i\left(x_i,x_j\right),S_j\left(x_j,x_i\right))=\begin{cases}\frac{S_i(x_i,x_j)}{S_i(x_i,x_j)+S_j(x_j,x_i)} & if \quad z_i=g_i\left(x_i\right),\\ \frac{S_j(x_j,x_i)}{S_i(x_i,x_j)+S_j(x_j,x_i)} & if \quad z_i=d_i\left(x_j\right),\\ 0 & otherwise,\end{cases}$$

where $P_i : \mathbb{R}^+ \times \mathbb{R}^+ \to \Delta(Z)$ [20], determines a random outcome $\widetilde{z}_i \in \widetilde{Z}_i$, distributed according to the probability measure $P_i(S_i\left(x_i,x_j\right),S_j\left(x_j,x_i\right))$. The combination of the ratio CSF with the functional form of the effectivity of agents' efforts, lead to the following reduced form of outcomes probability

$$P_i(z_i|S_i\left(x_i,x_j\right),S_j\left(x_j,x_i\right))=\begin{cases}\frac{\beta x_i(1-\alpha x_j)+k}{\beta x_i(1-\alpha x_j)+\beta x_j(1-\alpha x_i)+2k} & if \quad z_i=g_i\left(x_i\right),\\ \frac{\beta x_j(1-\alpha x_i)+k}{\beta x_i(1-\alpha x_j)+\beta x_j(1-\alpha x_i)+2k} & if \quad z_i=d_i\left(x_j\right),\\ 0 & otherwise.\end{cases}$$

When $k=0$ and $\alpha=0$, we get the standard Tullock CSF. From now on, we assume $k>0$, and w.l.g. $k=1$.[21] The CSF is a key element of any CMs. The proposed ratio CSF is probably the most common functional form in the related literature, where it has been axiomatized by Skaperdas (1996) and generalized by Rai and Sarin (2009). We assume the function is differentiable and satisfies Axioms 1 to 5 of Skaperdas (1996), but it does not satisfy the homogeneity Axiom 6, from which the CSF cannot be continuous. The latter axiom is motivated by the idea that the units in which effort is measured should not count. However, because of the assumption on the compactness of the effort sets, we can interpret bids as the intensity of effort, so that Axiom 6 is not actually relevant to our model. To the best of our knowledge, the proposed form of an agent's effectivity function first appears in Dasgupta and Nti (1998). However, in such study, the function was an implicit specification of the general CSF of Theorem 1 of Skaperdas (1996), which has been discussed subsequently by Amegashie (2006) and Rai and Sarin (2009). Our specification has been considered in Hirshleifer (1995a), in Garfinkel and Skaperdas (2007) reviewing conflict theory, and in Esteban and Ray (2011) discussing the generality of their results.[22] The main point that distinguish this CSF from the usual Tullock type, is that while in the latter zero effort implies zero success probability, our version implies that an agent investing zero effort still retain a strictly positive probability of winning. And this is exactly the reason we don't use the standard Tullock CSF: it implies that if player j were to make no effort, player i could win with certainty by exerting an infinitesimal amount of effort. But then, neither players would leave such an opportunity unexploited, hence a zero effort profile can't be an equilibrium: to exclude such possibility by construction is restrictive since we would like to explore the structural conditions that might imply zero effort as an equilibrium outcome, exactly as we would like to analyze the structural conditions that lead to maximum effort. This point on the restrictiveness of the embedded assumptions behind the

[19] See Chowdhury and Gürtler (2015).

[20] As usual, $\Delta(\cdot)$ denotes the set of all probability measures on the set $\cdot$.

[21] Once $k>0$, its value has no special significance: if we multiply numerator and denominator by any positive number we would get an equivalent functional form.

[22] Actually, in Esteban and Ray (2011) the authors did not emphasize that their generalization may generate non interior equilibria because their proof of proposition 1 does not hold anymore when the conflict success function is continuous in the origin.

standard contest success function, which implies a stable and interior equilibrium with a strictly positive intermediate level of conflict, is also forcefully proposed by Fearon (2008). Moreover, as argued in Hirshleifer (1995b) and Garfinkel and Skaperdas (2007), in conflicts this role of randomness is reasonable, in particular in situations where *frictions* play a role. Indeed, one way of thinking about our CSF, is to consider a third agent, "Nature", which has a constant effectivity function $S_N(x_N) = k$, so that when Nature wins, there is a random draw: in this sense Nature and $S_N(x_N) = k$ describe *frictions*, the role of factors that are independent from players' efforts. Thus, $k > 0$ implies that neither party can ever be fully certain of winning. In this way, we allow for the fact that the outcome of conflicts are typically subject to exogenous uncertainty, as argued by Besley and Persson (2011).[23]

Finally, in exploiting effort x_i, agents face consequent *costs*, described using the linear cost function $C_i(x_i) = x_i$, where, w.l.g., we assume unitary marginal costs[24].

Based on this setting,[25] we can derive a *payoff function* $\pi_i : X_i \times X_j \to \mathbb{R}$, as

$$\pi_i(x_i, x_j) = \int_{\underline{g}_1}^{\overline{g}_2} U_i(z_i)\, dP_i(z_i | S_i(x_i, x_j), S_j(x_i, x_j)) - C_i(x_i),$$

and, thus, obtain the class of *associated strategic-form games* $\mathbf{\Gamma}^{CM} = \left\{ \Gamma = \left(\{1, 2\}, [0, 1]^2, \pi_i(x_i, x_j) \right) \right\}$.

3.1 Endogenous changes in goals, defeat outcomes and effectivity functions

All the functions involved in the proposed CM are assumed to be smooth. This has substantial implications, because of the regularity of the CSF. Assuming continuously differentiable functions, the marginal probability of achieving z_i has symmetric and intuitive behavior in a neighborhood of zero effort and, for a suitable combination of structural parameters, zero-effort equilibria are possible. A further reason to assume smooth function is to be transparent on the reasons that generate non existence of pure strategy equilibria: as we will show, non existence appears with direct sabotage that might generate discontinuity in the best reply correspondence even if all involved function are smooth and the strategy set is compact and convex.

The possible spillovers from the agents' choices are

$$\frac{\partial S_i(x_i, x_j)}{\partial x_j} = -\beta\alpha x_i \leq 0,\ \frac{\partial g_1}{\partial x_1} = -\delta \leq 0,\ \frac{\partial g_2}{\partial x_2} = \delta \geq 0,\ \frac{\partial d_1}{\partial x_2} = \gamma \geq 0,\ \frac{\partial d_2}{\partial x_1} = -\gamma \leq 0.$$

The *spillover effects* can be summarized in the following assumption.

Assumption 1. ***The direct sabotage assumption*** *(*$\alpha > 0$*): the greater each player's effort is, the lower the effectivity of the counterpart's effort becomes;*

The cumulative polarization assumption *(*$\delta > 0$*): the greater each player's effort is, the greater the polarization becomes;*

The hostility assumption *(*$\gamma > 0$*): the greater each player's effort is, the worse the opponent's defeat outcome becomes.*

The direct sabotage assumption states that an agent's effort can directly reduce the effectivity of the counterpart's effort, for physical, economic, or institutional reasons. This is a specific version of what is simply denoted by "sabotage", a deliberate and costly act of damaging a rival's likelihood of winning the

[23] See Assumption 1.

[24] In principle, the cost function may depend on the opponent's effort. In what follows, we ignore this kind of spillover, because we checked that it does not provide any interesting insights into the equilibrium properties.

[25] Actually, a two-agent CM can be defined as the sextuple $(\mathbf{X}, \mathbf{S}, \mathbf{C}, \zeta, \mathbf{P}, \mathbf{U})$, in which each component is a vector of two elements.

conflict.[26] We write of *direct* sabotage because we want to emphasize that in this case the effect on the effectivity function is directly determined by the agent's effort, there is no decision on the portion of effort to use to increase the probability of winning and the portion to use to reduce opponent's effectivity. For example, in innovation conflicts one firm's R&D effort might directly decrease the patent value for the competitor by hastening further innovation, creating a negative spillover. Note that the expected effects on the CSF are reinforced by the direct sabotage assumption, since the probability of an agent achieving a goal is increasing in its effort ($\partial P_i(g_i|S_i, S_j)/\partial x_i > 0$), while that of a defeat outcome is increasing in the opponent's effort ($\partial P_i(g_i|S_i, S_j)/\partial x_j > 0$).

The cumulative polarization assumption is based on the idea that an agent's intensity of effort can push his/her own goal away from that of the opponent. In particular, the assumption has the crucial implication that the polarization increases with both agents' intensity of effort (i.e., $\partial\rho/\partial x_1 = \partial\rho/\partial x_2 = \delta > 0$). This is common to many settings where the intensity of agents' effort increases the distance between players' goals, and it is our synthetic way of modelling polarization as a process. Actually, the literature on polarization has emphasized that polarization is both a state and a process. Polarization as a state refers to the extent to which opinions on an issue are opposed, while polarization as a process refers to the increase in such opposition.[27] We model this latter aspect by assuming that polarization can be affected by the same agents' effort, since as the agents become more involved in the conflict, their goals may further diverge, as argued in Hirshleifer (1991, 1995a,b); Hirshleifer and Osborne (2001).

Hostility is a more disputed concept than polarization. Actually there is no universally accepted definition of hostility. In the scientific literature, the term hostility didn't receive a fixed definition and it has been treated by various authors and disciplines differently.[28] We consider the concept proposed by Barefoot et al. (1994): hostility is the antagonistic attitude towards people including cognitive, affective and behavioral components. Further, Myasishchev (1995) notes that hostility is formed in the course of interaction. This definition covers both steady, so-called personal hostility, and various situational complexes of hostile installations or predisposition in specific conditions. Thus, as a steady, common feature hostility means devaluation of motives and personal qualities of other people, feeling oneself in opposition to people around and wish them evil. Within the present model, the idea behind the hostility assumption is that an agent's effort affects the counterpart's defeat outcome. The point is that the mutual engagement for an outcome might intensify rivalries and anger, i.e. hostility, toward the opposing part.[29] In particular, hostility means that when players are hostile, they are fired up on a mission to defeat the other agent. In this case, a greater intensity of effort worsens the defeat conditions. In particular, the hostility assumption implies that the counterpart's effort intensity pushes the defeat outcome away from a player's bliss point, so that $\partial|g_1(x_1) - d_1(x_2)|/\partial x_1 = \partial|g_2(x_2) - d_2(x_1)|/\partial x_2 = \gamma > 0$. Again, the situation in which an increase in a contender's effort induces a worse outcome for the loser is quite common, and characterizes situations in which the confrontation between counterparts leads to a more radical standing for the loser of the conflict. An example where this kind of spillovers are important is litigation. Depending on the litigation system, losers have to compensate winners for a portion of their legal expenditures or, under one prominent proposal, up to the amount actually spent by the loser.

This CM gives rise to the class of strategic-form games $\mathbf{\Gamma}^L$, in which the payoff function π_i is

$$\pi_i(x_i, x_j) = -\frac{\beta x_j\left(1-\alpha x_i\right)+1}{\beta x_i\left(1-\alpha x_j\right)+\beta x_j\left(1-\alpha x_i\right)+2}\left[\theta+\delta x_i+\gamma x_j\right]-x_i. \tag{2}$$

From (2), it is evident that the constant term "+1" in the expression of the effectivity function is absolutely

[26] See Chowdhury and Gürtler (2015)

[27] See e.g. Baldassarri and Bearman (2007).

[28] E.g., according to Simmel (1904), hostility is the counterpart of the sympathetic impulse.

[29] See Miller and Johnston Conover (2015) for an application to political competition.

general, because the general case can be rephrased by rescaling β, obtaining the same CSF (2). Note that the proposed CSF has several differences with respect to the Tullock CSF (Tullock (1980)), and avoids several disadvantages (e.g., the lack of smoothness). This simplifies the interpretation of the roles of the parameters. The increase in the polarization and in the hostility both negatively affect i's expected outcome in the payoffs of the associated game.

The parameters of this family of CM are crucial to the results described in the next section. Hence, to help the reader with the interpretation, we summarize these parameters in the following table.

Parameters	**Meaning**
$\alpha \in [0,1]$	Direct sabotage
$\beta \in (0,+\infty)$	Productivity of effort
$\gamma \in [0,+\infty)$	Hostility
$\delta \in [0,+\infty)$	Cumulative polarization
$\theta \in (0,+\infty)$	*Ex ante* Polarization

Table 1: The structural parameters of the CM and their meaning

4 Results

In this section, we study the different possible sets of equilibria from our symmetric CM, defined in Section 3. We interpret the intensity of each player's equilibrium effort as the degree of its conflict behavior. We are particularly interested in understanding which conditions foster the emergence of multiple symmetric and/or asymmetric equilibria. To this end, we focus on three scenarios, each obtained by considering some (or no) spillover effects, as illustrated in Assumption 1: (1) no spillovers; (2) spillovers on outcomes, emphasizing the different roles of the polarization and hostility assumptions; (3) spillovers in the conflict success function, the direct destructive assumption.

4.1 No spillovers

The case of no spillovers is obtained by setting $\alpha = \gamma = \delta = 0$. The players' payoff functions reduce to

$$\pi_i(x_i, x_j) = -\frac{\beta x_j + 1}{\beta x_i + \beta x_j + 2}\theta - x_i. \tag{3}$$

In what follows, we identify the corresponding class of strategic-form games by $\mathbf{\Gamma}^{LNS} = \left\{\Gamma = (\{1,2\}, [0,1]^2, \pi)\right\}$, with $\pi(x_i, x_j) = \pi_1(x_1, x_2) \times \pi_2(x_2, x_1)$, where π_i are defined by (7). The expression of the best-response functions $BR_i : [0,1] \to [0,1]$ is given by

$$BR_i(x_j) = \max\left\{\min\left\{-x_j - \frac{2}{\beta} + \frac{1}{\beta}\sqrt{\beta\theta\left(\beta x_j + 1\right)}, 1\right\}, 0\right\}, \tag{4}$$

which are continuous, piecewise smooth functions, with derivatives that vanish at the unique fixed point $BR_i(\bar{x}) = \bar{x}$.

The possible Nash equilibria are investigated in the following Proposition, for which we define

$$x_{IS}^{NE} = \frac{\beta\theta - 4}{4\beta}. \tag{5}$$

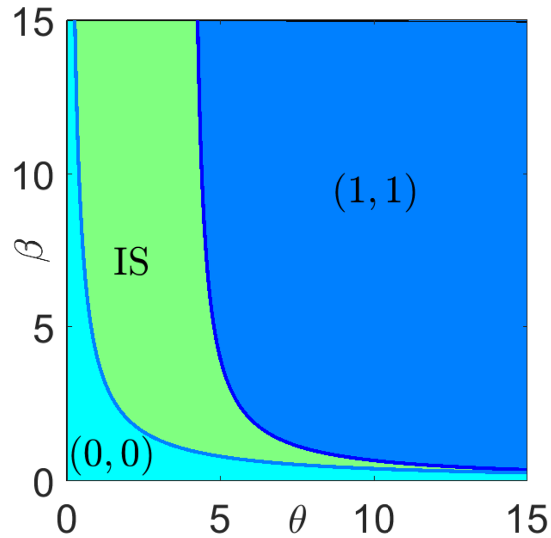

Figure 1: (θ, β) regions where there is the corner equilibrium $(0,0)$ (light blue), the internal symmetric equilibrium $IS = (x_{IS}^{NE}, x_{IS}^{NE})$ (green) or the corner equilibrium $(1,1)$ (blue). Boundaries are represented by darker colors with respect to the region to which they belong.

Proposition 1. *When no spillover effect is present, there always exists exactly one, symmetric, Nash equilibrium (x^{NE}, x^{NE}), which can be either an internal or a corner equilibrium, and is defined by*

$$x^{NE} = \begin{cases} 0 & \text{if } \beta\theta \leq 4, \\ x_{IS}^{NE} & \text{if } 4 < \beta\theta < 4 + 4\beta, \\ 1 & \text{if } \beta\theta \geq 4 + 4\beta. \end{cases} \tag{6}$$

In Figure 1 we graphically represent the result of Proposition 1.

Proposition 1 shows that the occurrence of either a corner or an internal equilibrium is essentially connected to the joint effect of the marginal productivity of effort and to the *ex ante* polarization. The occurrence of maximum equilibrium effort in the conflict is constrained to a suitably large *ex ante* polarization[30]. Further considerations about the effects of β and θ are collected in the following Corollary, the proof of which is straightforward.

Corollary 1. *When no spillover effect is present, then*

1. *the intensity of the effort in equilibrium x_i^{NE} is non- decreasing in the marginal productivity of the effort β and in the agents' polarization;*
2. *an increase in ex ante polarization θ has the effect of reducing the interval $\left[0, \frac{4}{\theta}\right]$ of β for which there is zero effort, while it increases the interval of β for which there is maximum effort. The interval for which there is an intermediate degree of effort increases if $\theta \leq 4$, and decreases if $\theta > 4$.*

From the previous proposition, we have that the intensity of the effort in equilibrium x_i^{NE} is non-decreasing in the marginal productivity of the effort β and in the agents' polarization. The vertical sections of Figure 1 show that an increase in *ex ante* polarization θ increases the interval of β for which there is maximum equilibrium effort (blue vertical sections), and decreases that for which the equilibrium effort is null (light

[30]If $\theta < 4$, the third case of (6) is not possible

blue vertical sections). The interval for which there is an intermediate equilibrium effort increases if a low level ($\theta < 4$) of *ex ante* polarization is increased, and decreases if the *ex ante* polarization is sufficiently large (green vertical sections).

The second result of Corollary 1 shows that when there are no spillovers and polarization is high, then a small increment in the marginal productivity of effort dramatically changes the equilibrium regime from no effort to intermediate, or even to maximum effort. In other words, when *ex ante* polarization is big, small institutional or technological changes may have a significant effect on the equilibrium behavior, which explains why the *ex ante* distance between the counterparts' goals is often the object that we need to monitor to minimize the risk of maximum disruptive antagonism. In terms of example 1, this result implies that an increase in the marginal productivity of effort, for example because of a higher cohesion within the parties, implies a higher equilibrium effort with no effect on the equilibrium probability of victory however with an increase in the effort's costs. This result is consistent with the social studies on conflicts that emphasize that the mounting wave of conflict in the last two centuries can be partially traced to several developments leading to an increase in the marginal productivity of effort, developments such as the growth of technology, of nation-states and its capacity to mobilize resources.[31]

4.2 Spillovers in outcomes: The role of the cumulative polarization and hostility assumptions

The case of a spillover in outcomes is obtained by $\gamma > 0$ and $\delta > 0$, while keeping $\alpha = 0$. The resulting payoff functions are

$$\pi_i(x_i, x_j) = -\frac{\beta x_j + 1}{\beta x_i + \beta x_j + 2}(\theta + \delta x_i + \gamma x_j) - x_i, \tag{7}$$

from which we obtain the class of strategic-form games $\mathbf{\Gamma}^{LSO} = \left\{\Gamma = (\{1,2\}, [0,1]^2, \pi)\right\}$, with $\pi(x_i, x_j) = \pi_1(x_1, x_2) \times \pi_2(x_2, x_1)$, where π_i are defined by (7).

The results we present in this section are simplified if we introduce the following two synthetic parameters:

$$\Delta(\gamma, \delta) \equiv \gamma - \delta \quad \text{and} \quad \Lambda(\beta, \theta, \delta) \equiv \beta\theta - 2\delta,$$

Consider the interpretation of Δ and Λ. By construction

- $\Delta(\gamma, \delta) = \partial_{x_2} d_1 + \partial_{x_1} g_1 = -(\partial_{x_1} d_2 + \partial_{x_2} g_2) = \partial_{x_i}[(g_2 - g_1) + (d_2 - d_1)]$. Hence, Δ is a measure of how both players' efforts affect the divergence between the goal and defeat outcomes; that is, it measures how an agent's effort affects both polarization and hostility. In particular,

  $$\gamma \uparrow \Rightarrow \Delta \uparrow, \quad \delta \downarrow \Rightarrow \Delta \uparrow \quad \text{and} \quad \Delta \uparrow \Rightarrow \gamma \uparrow \vee \delta \downarrow,$$

  which means that an increment in Δ reduces the effects of cumulative polarization ($\delta \downarrow$) and/or worsens the effects of hostility ($\gamma \uparrow$). Moreover

  $$\Delta \geq 0 \Leftrightarrow \gamma \geq \delta$$

  i.e. Δ is positive if and only if hostility prevails over polarization. Hence, we expect that an increment in a positive Δ would increase the equilibrium efforts.

- $\Lambda(\beta, \theta, \delta) = \theta \partial_{x_i} S_i (\alpha = 0) - 2|\partial_{x_i} g_i|$. Hence, Λ is a measure of the combination of *ex ante* polarization and the productivity of i's effort, net of the endogenous polarization effect. In particular,

  $$\beta \uparrow \vee \theta \uparrow \Rightarrow \Lambda \uparrow, \quad \delta \downarrow \Rightarrow \Lambda \uparrow \quad \text{and} \quad \Lambda \uparrow \Rightarrow \beta \uparrow \vee \theta \uparrow \vee \delta \downarrow,$$

[31] See e.g. Bartos and Wehr (2002).

which means that an increment in Λ reduces the effects of i''s effort on cumulative polarization ($\delta \downarrow$), and/or increases the marginal productivity of effort ($\beta \uparrow$), and/or increases the *ex ante* polarization ($\theta \uparrow$). Moreover

$$\Lambda(\beta, \theta, \delta) \geq 0 \Leftrightarrow \beta\theta \geq 2\delta$$

i.e. Λ is positive if and only if the combination of *ex ante* polarization and the productivity of i's effort prevails over the polarization effect. Hence, we expect that an increase in positive Λ would increase the equilibrium efforts.

The best-response functions related to games $\Gamma \in \mathbf{\Gamma}^{LSO}$ are continuous, piecewise smooth functions $BR_i : [0,1] \to [0,1]$, given by

$$BR_i(x_j) = \begin{cases} \min\left\{\max\left\{-x_j - \dfrac{2}{\beta} + \dfrac{1}{\beta}\sqrt{(\beta\Delta x_j + \Lambda)(\beta x_j + 1)}, 0\right\}, 1\right\} & \text{if } \beta\Delta x_j + \Lambda \geq 0, \\ 0 & \text{if } \beta\Delta x_j + \Lambda < 0. \end{cases} \tag{8}$$

Note that the continuity of the best-response function guarantees the existence of the Nash equilibrium. The main difference between this and the case of no spillovers, despite the possibility of having strictly non-increasing best-response functions, lies in its behavior at the fixed points $\bar{x}$. When there are spillovers in polarization and in outcomes, it is no longer true that the best-response functions are flat at $\bar{x}$, and the uniqueness of the equilibrium is no longer guaranteed. Note that when $\Delta = \Lambda = 4$, the best-response function (8) reduces to $BR_i(x_j) = x_j$. Thus, in this peculiar case, any strategy $x_i \in [0,1]$ is a Nash equilibrium. We avoid dealing further with this case. Therefore, in the remainder of this section, we assume that at least one of the two synthetic parameters Δ and Λ is different from 4.

Let us introduce

$$x_{IS}^{NE} = -\frac{1}{\beta}\left[\frac{\Lambda - 4}{\Delta - 4}\right], \tag{9}$$

$$x_{1,AS}^{NE} = \frac{-(\Lambda + \Delta) + (\Lambda - \Delta)\sqrt{\frac{\Delta+4}{\Delta}}}{2\beta\Delta}, \; x_{2,AS}^{NE} = \frac{-(\Lambda + \Delta) - (\Lambda - \Delta)\sqrt{\frac{\Delta+4}{\Delta}}}{2\beta\Delta}, \tag{10}$$

and

$$x_{0,B}^{NE} = -\frac{1}{\beta}\left(2 - \sqrt{\Lambda}\right), \quad x_{1,B}^{NE} = -\frac{1}{\beta}\left(2 + \beta - \sqrt{(\beta + 1)(\beta\Delta + \Lambda)}\right). \tag{11}$$

which, when real and belonging to $[0,1]$, are respectively the coordinates of the internal symmetric, internal asymmetric and boundary asymmetric Nash equilibria. In the following proposition we report the set of possible Nash equilibria.

Proposition 2. *When the polarization and hostility assumptions hold, for any parameter configuration, there exist up to three Nash equilibria. In particular, we can have*

- *a unique symmetric internal* $(x_{IS}^{NE}, x_{IS}^{NE})$ *or corner equilibrium;*
- *three symmetric equilibria, with one internal* $(x_{IS}^{NE}, x_{IS}^{NE})$ *and two corner equilibria;*
- *three internal asymmetric equilibria, corresponding to* $(x_{IS}^{NE}, x_{IS}^{NE}), (x_{1,AS}^{NE}, x_{2,AS}^{NE})$, *and* $(x_{2,AS}^{NE}, x_{1,AS}^{NE})$;
- *the internal symmetric equilibrium* $(x_{IS}^{NE}, x_{IS}^{NE})$ *and two asymmetric boundary/corner equilibria, either* $(x_{0,B}^{NE}, 0), (0, x_{0,B}^{NE})$ *or* $(x_{1,B}^{NE}, 1), (1, x_{1,B}^{NE})$.

In Figure 2, we present the most significant configurations of the equilibria described in Proposition 2. Before commenting on the effects of spillovers on the possible sets of equilibria, we complete the results of Proposition 2 to determine which parameter configurations provide multiplicity of symmetric and asymmetric equilibria.

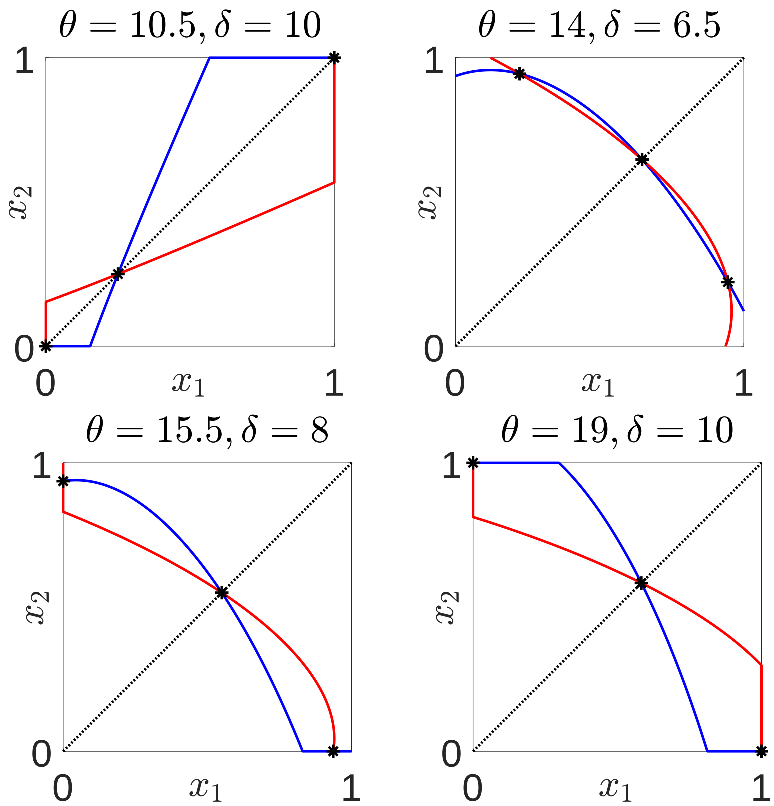

Figure 2: Multiple Nash equilibria for games $\Gamma \in \mathbf{\Gamma}^{LSO}$. Best response functions $BR_i(x_j)$ are respectively sketched using color red ($i = 1$) and blue ($i = 2$), while stars represents the Nash equilibria. From left to right, top to down we respectively have multiple symmetric, multiple asymmetric internal, multiple asymmetric boundary and multiple asymmetric corner equilibria. We set $\beta = 10$ and $\gamma = 2$.

Proposition 3. *For any CM, when the polarization and the hostility assumptions hold, there exist*

- *multiple symmetric Nash equilibria if and only if*

$$\begin{cases} 4 - \beta\,(\Delta - 4) \leq \Lambda \leq 4, \\ \Delta \geq 4; \end{cases} ; \tag{12}$$

- *asymmetric Nash equilibria if and only if*

$$\begin{cases} 4 < \Lambda < \beta(4 - \Delta) + 4, \\ \Delta < -4; \end{cases} ; \tag{13}$$

- *a unique symmetric equilibrium if and only if*

$$\begin{cases} \Lambda \leq 4, \\ \Delta < \frac{4(\beta+1)-\Lambda}{\beta}, \end{cases} \cup \begin{cases} \Lambda > 4, \\ \Delta \geq \frac{4(\beta+1)-\Lambda}{\beta}, \end{cases} \cup \begin{cases} \Delta \geq -4, \\ \Lambda > 4, \\ \Delta < \frac{4(\beta+1)-\Lambda}{\beta}. \end{cases} \tag{14}$$

If condition (12) is fulfilled with a strict inequality, then we have three distinct equilibria. However, when equality occurs, the internal equilibrium coincides with either $(0, 0)$ or $(1, 1)$, and we actually have the two symmetric corner equilibria. In what follows we focus on the case of three distinct equilibria.

The multiplicity of the symmetric equilibria (i.e., condition (12)) necessarily requires that $\Delta > 4$, while the multiplicity of the asymmetric equilibria (i.e., condition (13)) necessarily requires that $\Delta < -4$. If the divergence between the goal and defeat outcomes is suitably small ($|\Delta| < 4$), the possible scenarios are the same as those in the case of no spillovers, and only a unique symmetric equilibrium is possible. This means that the measure of how both players' efforts affect the divergence between the goal and defeat outcomes should be sufficiently large or small to obtain multiple equilibria. However, for intermediate values, we have uniqueness, as if there are no outcome spillovers. Similarly, the multiplicity of symmetric equilibria (i.e., condition (12)) necessarily requires $\beta(4-\Delta)+4 < \Lambda < 4$, while the multiplicity of asymmetric equilibria (i.e., condition (13)) necessarily requires $4 < \Lambda < \beta(4-\Delta)+4$. Thus, the measure of the combination of *ex ante* polarization and the productivity of i's effort, net of the polarization effect, should be high intermediate in order to obtain multiple asymmetric equilibria. Conversely, for low intermediate values, we might obtain multiple symmetric equilibria. More generally, if hostility is sufficiently large with respect to cumulative polarization, multiple symmetric equilibria can arise for intermediate values of Λ. On the other hand, if hostility is sufficiently small with respect to cumulative polarization, multiple asymmetric equilibria can arise, again for intermediate values of Λ. This means that a relatively large level of cumulative polarization can lead to equilibria in which the intensity of the effort of one contender is larger than that of the other, which may become the maximum and minimum values possible, respectively (see the last plot of Figure 2).

Since Δ is a measure of how both players' efforts affect the divergence between the goal and defeat outcomes, that is, a measure of how an agent's effort affects both polarization and hostility, the existence of multiple symmetric equilibria requires a significant endogenous effect of effort on hostility, net of polarization. On the other hand, the existence of multiple asymmetric equilibria requires a significant endogenous effect of effort on polarization, net of hostility. This result shows the different roles played by cumulative polarization and by hostility. In particular, the magnitude of hostility w.r.t. to cumulative polarization is the main factor leading to multiple symmetric equilibria, while multiple asymmetric equilibria requires significant and roughly similar effects of cumulative polarization and of hostility. The point is that the introduction of spillovers on goal and defeat outcomes introduce escalating and de-escalating endogenous mechanisms which might take the form of strategic complementarity, and of multiple symmetric equilibria, or of strategic substitutability, and of multiple asymmetric equilibria, depending whether hostility or cumulative polarization prevail or compensate. This result shows that, as argued by social scientists[32], hostility plays quite a different role in conflict than polarization. The relationship between hostility and conflict behavior is complex. On one hand, hostility adds fuel to and intensifies effort, on the other hand increasing effort increases hostility, pointing to strategic complementarity in players' effort. Cumulative polarization has a different effect because increases the distance between goals, but does not affect the defeat outcomes, meaning that in some cases it might be strategically rational to reply to an opponent effort increment with a reduction in effort, i.e. with strategic substitutability.

Now consider all four parameters: productivity of effort β, endogenous hostility γ, cumulative polarization δ, and *ex ante* polarization θ. Because of these four parameters it is impossible to have a full picture of the joint effects of these parameters. However, it is interesting to consider the necessary and sufficient conditions w.r.t. β and θ, for given γ and δ, and similarly, the necessary and sufficient conditions w.r.t δ and γ, for given β and θ. To help with the discussion, Figures 3 and 4 show two-dimensional sections of the regions of multiple, symmetric, or asymmetric equilibria in the(δ, γ) and (β, θ) planes.

Consider δ and γ, for given β and θ. The necessary and sufficient conditions ((12)) for multiple symmetric

[32]See Bartos and Wehr (2002).

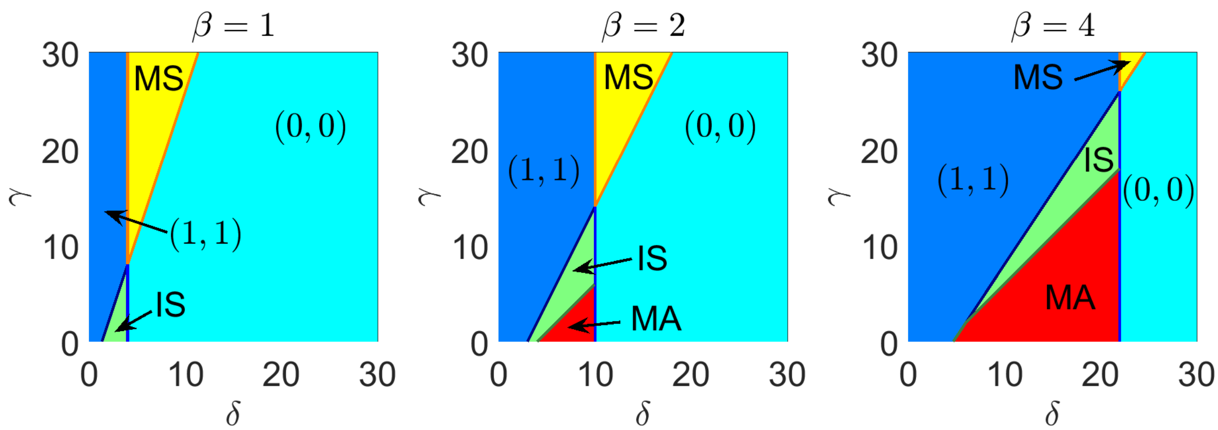

Figure 3: (δ, γ) regions in which there is only the corner equilibrium $(0,0)$ (light blue), only the internal symmetric equilibrium $IS = (x_{IS}^{NE}, x_{IS}^{NE})$ (green), only the corner equilibrium $(1,1)$ (blue) or there are multiple symmetric (MS, yellow) or asymmetric (MA, red) equilibria. Boundaries are represented by darker colors with respect to the region to which they belong. We set $\theta = 12$.

equilibria can be written as

$$\frac{\beta\theta - 4}{2} < \delta < \min\left\{\gamma - 4; \frac{\beta(\gamma + \theta) - 4(\beta + 1)}{\beta + 2}\right\},$$

which means that cumulative polarization should be bounded below by the interaction between the productivity of effort and *ex ante* polarization, and above by a measure of endogenous hostility. Thus, as shown in Figure 3, both cumulative polarization and endogenous hostility should be sufficiently large, even if cumulative polarization cannot grow quickly.

On the other hand, the necessary and sufficient conditions (13) for multiple asymmetric equilibria can be written as

$$\max\left\{\gamma + 4; \frac{\beta(\gamma + \theta) - 4(\beta + 1)}{\beta + 2}\right\} < \delta < \frac{\beta\theta - 4}{2},$$

which means that cumulative polarization should be bounded above by the interaction between the productivity of effort and *ex ante* polarization, and below by a measure of endogenous hostility. Thus, both cumulative polarization and endogenous hostility should be significant, but they are restricted in their values, while a crucial role is played by the interaction between the productivity of effort and *ex ante* polarization, which should be sufficiently large (see Figures 3 and 4).

With regard to the necessary and sufficient conditions w.r.t β and θ, for given δ and γ, we have that the necessary and sufficient conditions (12) for multiple symmetric equilibria can be written as

$$\begin{cases} \dfrac{4 + 2\delta}{\theta + (\Delta - 4)} < \beta < \dfrac{4 + 2\delta}{\theta}, \\ \Delta > 4, \end{cases} .$$

Thus, the productivity of effort should be bounded below and above by two values that are decreasing in *ex ante* polarization, but increasing in cumulative polarization (Figure 4). The necessary and sufficient conditions for multiple asymmetric equilibria (13) can be rewritten as

$$\begin{cases} \dfrac{4 + 2\delta}{\theta} < \beta < \dfrac{4 + 2\delta}{\theta + (\Delta - 4)}, \\ \Delta < -4 \wedge \theta > 4 - \Delta, \end{cases} \vee \begin{cases} \dfrac{4 + 2\delta}{\theta} < \beta, \\ \Delta < -4 \wedge \theta < \Delta - 4, \end{cases} .$$

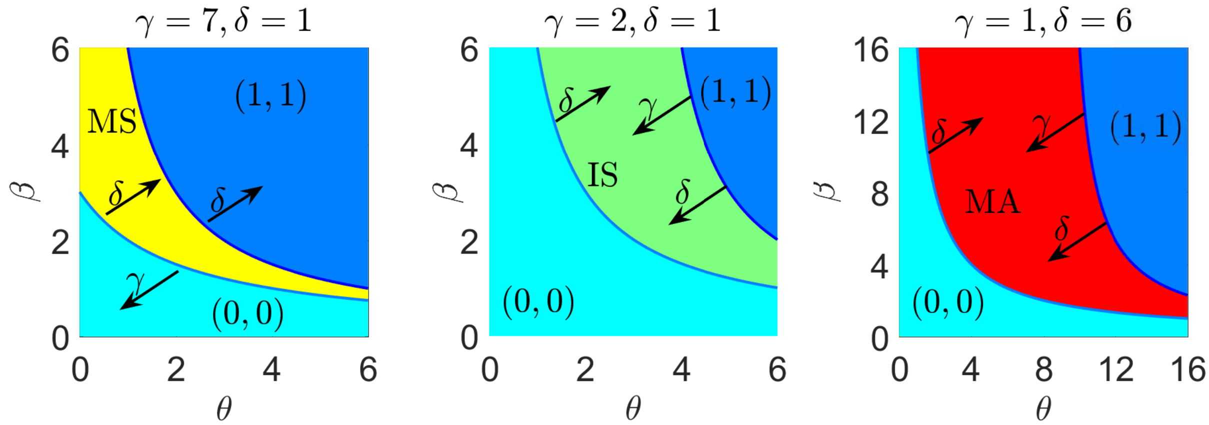

Figure 4: (θ, β) regions in which there is only the corner equilibrium $(0, 0)$ (light blue), only the internal symmetric equilibrium $IS = (x_{IS}^{NE}, x_{IS}^{NE})$ (green), only the corner equilibrium $(1, 1)$ (blue) or there are multiple symmetric (MS, yellow) or asymmetric (MA, red) equilibria. Boundaries are represented by darker colors with respect to the region to which they belong. Arrows show toward which direction boundaries move as the parameter increases.

Thus, the range of possible values for the productivity of effort depends on the value of *ex ante* polarization. These cases are depicted in Figure 4, in which we also show how the thresholds vary depending on δ and γ, when the dependence is uniform with respect to the other parameters. These figures show that the regions of different sets of equilibria in the (θ, β) space have similar behavior w.r.t. the case of no spillovers. Here, a big value of *ex ante* polarization means a small increase in the productivity of effort is sufficient to shift from a region with no effort to an intermediate situation, possibly with multiple symmetric or asymmetric equilibria, to a region with maximum effort only.

In the following Corollary, we show the results concerning the effects of the parameters on the equilibria. We start examining how the intensity of effort at the various Nash equilibria varies depending on β, δ, γ, and θ. We assume that the parameters' perturbation is such that it does not affect the existence of the particular equilibrium. In this sense, the results of Corollary 2 are local. For a graphical representation, refer to Figures 3 and 4.

Corollary 2. *Assume that we have an internal symmetric equilibrium. Then,*

1. *if $\Delta < 4$, then an increase in β, γ, or θ leads to an increase in x_{IS}^{NE}, while an increase in δ leads to a decrease in x_{IS}^{NE};*
2. *if $\Delta > 4$, then an increase in β, γ, or θ leads to a decrease in x_{IS}^{NE}, while an increase in δ leads to an increase in x_{IS}^{NE}.*

In case (1), since we assume that the symmetric equilibrium is internal, recalling Proposition 3, we necessarily have a unique symmetric equilibrium, with the possibility of asymmetric equilibria. In this case, the effort at the symmetric equilibrium is increasing in the marginal productivity of effort, in endogenous hostility, and in the *ex ante* polarization. However, maybe surprisingly, it is decreasing in the cumulative polarization. The point is that with a bigger δ, an increment in effort has two effects: it increases the likelihood of winning the conflict, and it pushes the two players' goals further away. However, an increase in γ makes a default worse, incentivizing more effort.

In case (2), recalling Proposition 3, there are multiple symmetric equilibria. Surprisingly, the effects of the parameters on the symmetric equilibrium are completely reversed. The effort at the internal equilibrium is decreasing in the marginal productivity of effort, in endogenous hostility, and in the *ex ante* polarization, but is increasing in the cumulative polarization. However, it should be emphasized that in this case, an increase in cumulative polarization increases the regions with zero-effort equilibria, while an increase in endogenous hostility increase the regions of maximum-effort equilibria.

The results on asymmetric equilibria are more complex, although are qualitatively similar.

The different roles played by cumulative polarization and hostility in the equilibrium effort, both in terms of the emergence of possible asymmetric and multiple equilibria, and in terms of the effect of the equilibrium intensity, suggest that it is fundamental to distinguish between them in the design, analysis, and interpretation of a conflict model, as it is common in classic social conflict theory.[33]

4.3 Spillovers in effectivity function: Effects of the direct sabotage assumption

Assume $\alpha > 0, \gamma = \delta = 0$; that is, a positive direct destructiveness effect. Then, the resulting payoff functions are

$$\pi_i(x_i, x_j) = -\frac{\beta x_j\,(1-\alpha x_i)+1}{\beta x_i\,(1-\alpha x_j)+\beta x_j\,(1-\alpha x_i)+2}\theta - x_i, \tag{15}$$

from which we obtain the class of strategic-form games $\mathbf{\Gamma}^{LSE} = \left\{\Gamma = (\{1,2\}, [0,1]^2, \pi)\right\}$, with $\pi(x_i, x_j) = \pi_1(x_1, x_2) \times \pi_2(x_2, x_1)$, where π_i are defined by (15).

In this case, the best response is not always a function. In particular, if $\alpha < 1/2$, it is a function expressed by

$$BR_i(x_j) = \max\left\{\min\left\{-\frac{2}{\beta(1-2\alpha x_j)} - \frac{x_j}{(1-2\alpha x_j)} + \frac{\sqrt{\beta\theta(-\alpha\beta x_j^2+\beta x_j+1)}}{\beta(1-2\alpha x_j)}, 1\right\}, 0\right\}. \tag{16}$$

If $\alpha \geq 1/2$, we have that $BR_i(x_j)$ coincides with the right hand side of(16) if $x_j \in [0, 1/2\alpha) \cap [0,1]$, and for $x_j \in (1/(2\alpha), 1] \cap [0,1]$ it is either a constant function (as for example in Figure 5, left plot) or a correspondence (Figure 5, middle and right plots). The main difference between the case of spillovers on the effectivity function and the cases without spillovers or with spillovers in the outcome is that, in the present case, the best response is no longer necessarily convex-valued. Thus, Kakutani's Theorem cannot be applied, and the existence of the Nash equilibrium is no longer guaranteed, as is evident in the right plot of Figure 5. Introducing

$$x_{IS}^{NE} = \frac{1}{2\alpha} - \frac{\sqrt{\beta(4\alpha+\beta-\alpha\beta\theta)}}{2\alpha\beta}, \tag{17}$$

we can state the following result about the set of Nash equilibria under spillovers in the effectivity function.

Proposition 4. *For any CM such that the direct sabotage assumption holds, we can have either a unique symmetric equilibrium, which can be the internal point* $(x_{IS}^{NE}, x_{IS}^{NE})$ *or one of the corner points* $(0,0)$ *and* $(1,1)$, *or no equilibrium.*

The main result on $\mathbf{\Gamma}^{LSE}$ is the possible non-existence of a pure-strategy Nash equilibrium, which is fostered by a sufficiently large value of destructiveness of effort, as shown in the following Corollary. The direct sabotage assumption is a way of modeling a situation where agents use strategies to damage someone else's success rather than improving their own. The discontinuity in the best reply correspondence arises because when the opponent's effort is not too big, the best way to hinder the sabotage effect is to increase his/her own effort, however at some point when the opponent effect is huge, then the best way to avoid the negative effect on the likelihood to get the goal is to suddenly reduce to zero his/her own effort so to cancel the sabotage too.

[33] See Bartos and Wehr (2002).

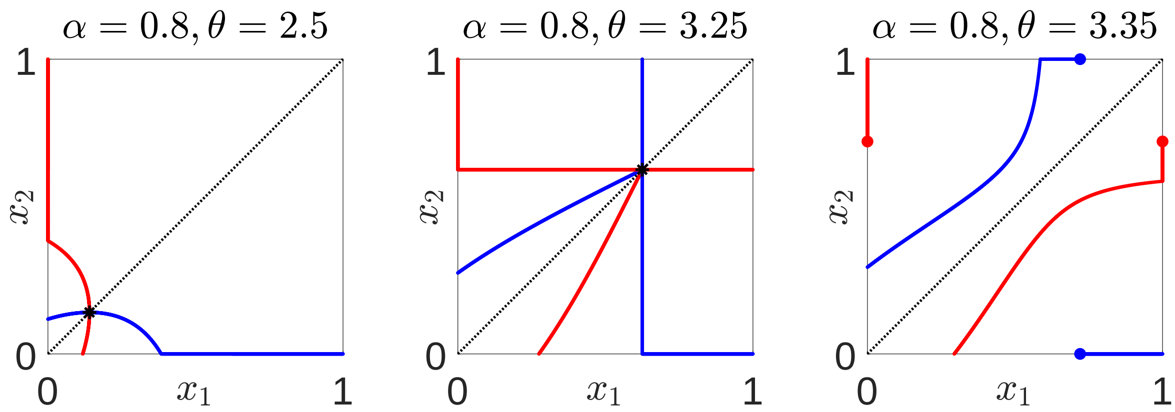

Figure 5: Possible resulting Nash equilibria for games $\Gamma \in \mathbf{\Gamma}^{LSE}$. Best response functions $BR_i(x_j)$ are respectively sketched using color red ($i = 1$) and blue ($i = 2$), while stars represents the Nash equilibria. We set $\beta = 2$.

Corollary 3. *The intensity of effort x_{IS}^{NE} increases as $\beta, \theta,$ and α increase.*
We have no equilibria if and only if

$$\begin{cases} \alpha \geq 1/2, \\ \dfrac{4}{\beta} \leq \theta \leq \dfrac{4\alpha + \beta}{\alpha\beta}. \end{cases}$$

A graphical representation of the possible equilibria, depending on the parameters, is reported in Figure 6. Note that we have the usual effects of the marginal productivity of effort and of *ex ante* polarization on the equilibrium effort, as well as of the direct destructiveness parameter. However, when these effects are sufficiently large, they rule out the existence of a (pure strategy) equilibrium, owing to the discontinuity in the best-response functions.

5 The Changing Patterns of Aggregate Conflict Intensity

In this section, we analyze a measure of the overall conflict intensity, CI, to evaluate the total equilibrium effort of the players. Among the different ways of assessing the aggregate equilibrium effort of the players, we actually follow the practice of Esteban and Ray (1999, 2008, 2011) considering the average players' effort[34]

$$CI(x_1^*, x_2^*) = \frac{x_1^* + x_2^*}{2}.$$

In particular, we define the conflict intensity

- CI_{IS}, at the internal symmetric equilibrium as $CI_{IS} = CI(x_{IS}^{NE}, x_{IS}^{NE})$;
- CI_{AS}, at the internal asymmetric equilibria as $CI_{AS} = CI(x_{AS}^{NE}, x_{AS}^{NE})$;
- $CI_{i,B}$ for $i = 0, 1$ at the boundary equilibria as $CI_{i,B} = CI(x_{i,B}^{NE}, i)$.

Corollary 4. *On increasing the parameters, depending on the equilibria scenario, the intensity of conflict at internal equilibria is*

[34] Esteban and Ray simply adopted the sum of efforts, the linear transformation of it adopted in the present contribution allows considering the aggregate conflict intensity as still a share of the maximum possible effort devoted to conflict.

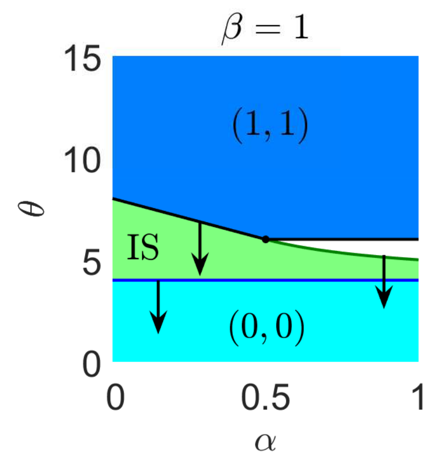

Figure 6: (α, θ) regions in which there is the corner equilibrium $(0,0)$ (light blue), the internal symmetric equilibrium $IS = (x_{IS}^{NE}, x_{IS}^{NE})$ (green), the corner equilibrium $(1,1)$ (blue) or no equilibrium (white). Boundaries are represented by darker colors with respect to the region to which they belong. Arrows show toward which direction boundaries move as β increases.

	C_{IS}			C_{AS}
Scenario / *Parameter*	*Unique int. sym. eq.*	*Multiple int. sym. eq.*	*Multiple asym. eq.*	*Multiple asym. eq.*
marginal productivity of effort β	*increasing concave*	*decreasing convex*	*increasing concave*	*increasing concave*
ex ante polarization θ	*linearly increasing*	*linearly decreasing*	*linearly increasing*	*linearly increasing*
marginal effect on endogenous polarization δ	*decreasing convex*	*increasing convex*	*decreasing convex*	*decreasing convex*
marginal effect on endogenous hostility γ	*increasing convex*	*decreasing convex*	*increasing convex*	*increasing convex*

Monotonicity and concavity properties are always strict.

Corollary 5. *On increasing the parameters, when multiple asymmetric boundary equilibria occurs, the intensity of conflict is*

Scenario / *Parameter*	$C_{1,B}$	$C_{0,B}$
marginal productivity of effort β	*increasing concave*	*not monotonic*
ex ante polarization θ	*increasing concave*	*increasing concave*
marginal effect on endogenous polarization δ	*decreasing concave*	*decreasing concave*
marginal effect on endogenous hostility γ	*increasing concave*	*constant*

Monotonicity and concavity properties are always strict.

Corollary 5 shows that in the case of multiple asymmetric equilibria, the behavior of conflict intensity for border equilibria is first-order similar to the case of interior equilibria apart from the effect of marginal productivity of effort and of hostility on $C_{0,B}$. The most informative and interesting results can be found in Corollary 4. It shows the systematic reverse of the parameters' effects on the aggregate conflict intensity from the case of a unique internal symmetric equilibrium to the situation of multiple internal symmetric equilibria. On the other hand, the intensity of conflict has a similar behavior in both the cases of multiple interior asymmetric equilibria and of a unique interior symmetric equilibrium. This means that for the analysis of the aggregate intensity of conflict, the main crucial distinction is between two polar situations, when there are multiple symmetric equilibria and the other scenarios. In Section 4.2, we showed that the multiplicity of symmetric equilibria necessarily requires $\Delta > 4$, otherwise either we get multiplicity of asymmetric equilibria ($\Delta < -4$) or a unique symmetric equilibrium ($|\Delta| < 4$). This means that the measure of how both players' efforts affect the divergence between the goal and defeat outcomes crucially affects the qualitative behavior of conflict intensity. Moreover, the multiplicity of symmetric equilibria necessarily requires $\beta(4-\Delta)+4 < \Lambda < 4$, while the multiplicity of asymmetric equilibria necessarily requires $4 < \Lambda < \beta(4-\Delta)+4$. Thus, the peculiar scenario of multiple symmetric equilibria requires that the hostility and the cumulative polarization belong to a positively sloped cone that requires higher cumulative polarization and higher hostility as the productivity of effort or the ex ante polarization increase. This is evident looking at the yellow region (labelled with MS) in the diagrams reported in Figure 3.

Thus, according to Corollary 4, cumulative polarization and hostility not only lead to multiple symmetric equilibria as showed in Section 4.2, they also affect the behavior of conflict intensity, confirming that the introduction of spillovers on goal and defeat outcomes introduce new important escalating and de-escalating mechanisms in conflict behavior which might take the form of strategic complementarity when hostility prevail, proving once more that hostility plays quite a different role in conflict than polarization. These results bolster the importance of distinguishing the different scenarios and the role of the different parameters, thus the relevance of our construction of this general class of CMs.

6 Conclusion

This paper focuses on analyzing and understanding the determinants of the properties of the equilibrium set in a class of conflict models. In particular, we focus on five different structural parameters - direct sabotage (α), productivity of effort (β), *ex ante* polarization (θ) , cumulative hostility (γ) and polarization (δ) - and on their effects on the set of equilibria.

To the best of our knowledge, the proposed family of CMs is the first to generate such a multiplicity of possible equilibria scenarios, connecting their characteristics to fundamental micro properties of the model. Based on the proposed approach, it is possible to understand the role of spillovers (or of their absence) on the emergence of multiple and/or asymmetric equilibria and on the behavior of conflict intensity. Within our model, the emergence of these different equilibrium possibilities have a neat strategic interpretation. First, the possibility of having boundary and or/corner equilibria is not ruled out, with a clear interpretation of minimum/maximum equilibrium efforts. Characterizing a conflict only in terms of the constant productivity of effort (β) and polarization (θ) is quite limited, because it allows for the existence of a unique symmetric Nash equilibrium only. Conversely, an essential role is played by the endogenous effects on the outcomes (the polarization and cumulative hostility assumptions), which lead to a complex situation in which equilibria can be multiple and symmetric or asymmetric. In particular, we emphasize the possibility of having multiple Pareto ordered symmetric equilibria. We showed that it is essential to distinguish between different spillover effects, because the effects of cumulative polarization (δ) and of cumulative hostility (γ) lead to significantly different, and possibly opposite results. Moreover, the analysis of the effects of these parameters on the total conflict intensity shows that they are opposite whether the scenario is with multiple symmetric equilibria or not, reinforcing the importance of distinguishing the parameters combinations that justify the emergence of such different cases.

Finally, taking into account the possibility of direct destructive spillovers in a contender's marginal productivity of effort (α), the non-existence of pure-strategy Nash equilibria is possible.

In all these results, a crucial role is played by *ex ante* polarization (θ): an increase in ex ante polarization with constant productivity of effort (β) either increase the likelihood of an equilibrium with high effort (figure 1 and 4), or increase the likelihood of multiple equilibria (figure 4), or lead to non existence of pure strategy equilibria (figure 5).

The results pursued here allow for further investigations in several directions. From a methodological point of view, the properties of conflicts can be analyzed in two different and complementary ways: either using comparative statics, showing how equilibria change as parameters change, or using out-of-equilibrium dynamics to emphasize players' complex and cyclic behavior. This paper belongs to a strand of research in which we pursue both aspects. Here, we analyzed the structural conditions that generate one, zero, or multiple symmetric and asymmetric equilibria, as well as their properties as a function of the structural parameters.

Appendix

In this Section we provide details about how best response relations are obtained and we report the proofs of the Propositions and Corollaries of Sections 4.1, 4.2 and 4.3, respectively concerning the family of games $\mathbf{\Gamma}^{NS}, \mathbf{\Gamma}^{LSO}$ and $\mathbf{\Gamma}^{LSE}$.

6.1 Proofs of Section 4.1

Firstly, we derive the expression for the best response (4). We start noticing that

$$\frac{\partial \pi_i(x_i, x_j)}{\partial x_i} = \frac{\beta\left(\theta + \beta\theta x_j\right)}{\left(\beta x_i + \beta x_j + 2\right)^2} - 1. \tag{18}$$

It is easy to see that $\partial^2_{x_i}\pi_i(x_i, x_j) < 0$, so the payoff function is strictly concave for any parameters' configuration. Let us introduce function $x_+ : [-1/\beta, +\infty) \to \mathbb{R}$ defined by

$$x_+(z) = -z - \frac{2}{\beta} + \sqrt{\left(z + \frac{1}{\beta}\right)\theta}, \tag{19}$$

which will be used for the characterization of the best response function. Function x_+ is strictly concave and has a unique fixed point at $z_m = (\beta\theta - 4)/(4\beta)$, coinciding with the unique maximum point of x_+. Moreover, x_+ is strictly increasing (respectively decreasing) on $[-1/\beta, z_m)$ (respectively in $(z_m, +\infty)$), on which $x_+(z) > z$ (respectively $x_+(z) < z$).

To compute the best response function, we start solving $x_i = \operatorname{argmax}_{z\in[0,1]} \pi_i(z, x_j)$ for a fixed strategy $x_j \in [0, 1]$. From (18), we have that $\partial_{x_i}\pi_i(x_i, x_j) \geq 0$ leads to

$$-\beta^2 x_i^2 + \left(-2x_j\beta^2 - 4\beta\right)x_i - \left(\beta^2 x_j^2 - \theta\beta^2 x_j + 4\beta x_j - \theta\beta + 4\right) \geq 0. \tag{20}$$

Assuming that $x_j > -1/\beta$, inequality (20) is solved for $x_-(x_j) \leq x_i \leq x_+(x_j)$, where

$$x_\pm(x_j) = -x_j - \frac{2}{\beta} \pm \sqrt{\left(x_j + \frac{1}{\beta}\right)\theta}$$

are real values for any positive β and θ. Noticing that $x_-(x_j) < 0$, for each x_j we have three possibilities.

- $x_+(x_j) \leq 0$: the marginal payoff is negative for any $x_i \in [0,1]$, so the payoff function $\pi_i(x_i, x_j)$ is strictly decreasing for any $x_i \in [0,1]$ and attains its maximum at $x_i = 0$. In this case we have $BR_i(x_j) = 0$.

- $x_+(x_j) \in (0,1)$: the marginal payoff is positive for any $x_i \in [0, x_+(x_j))$ and negative for any $x_i \in (x_+(x_j), 1]$, so the payoff function $\pi_i(x_i, x_j)$ is strictly increasing for any $x_i \in [0, x_+(x_j))$ and strictly decreasing for any $x_i \in (x_+(x_j), 1]$, and attains its maximum at $x_i = x_+(x_j)$. We then have $BR_i(x_j) = x_+(x_j)$.

- $x_+(x_j) \geq 1$: the marginal payoff is positive for any $x_i \in [0,1]$, so the payoff function $\pi_i(x_i, x_j)$ is strictly increasing for any $x_i \in [0,1]$ and attains its maximum at $x_i = 1$. We then have $BR_i(x_j) = 1$.

The previous considerations allow concluding that $BR_i(x_j) = \min\{\max\{x_+(x_j), 0\}, 1\}$, namely we find (4).

Proof of Proposition 1. If we solve the system of the two equations $x_1 = BR_1(x_2)$ and $x_2 = BR_2(x_1)$, we easily find that the unique possible solution belongs to $\{(0,0), (1,1), (x_{IS}^{NE}, x_{IS}^{NE})\}$. In particular, recalling the expression of the best response in terms of x_+ and the role of z_m, we have that the Nash equilibrium is $(0,0)$ if and only if $z_m \leq 0$; is $(x_{IS}^{NE}, x_{IS}^{NE})$ if and only if $0 < z_m < 1$; is $(1,1)$ if and only if $z_m \geq 1$. Solving the previous inequalities we obtain (6). $\square$

6.2 Proofs of Section 4.2

Firstly, we derive the expression for the best response (8). We start noticing that, from (7), we have

$$\begin{gathered} \frac{\partial \pi_i(x_i, x_j)}{\partial x_i} = \frac{\beta\left(\beta x_j + 1\right)\left(\theta + \delta x_i + \gamma x_j\right)}{\left(\beta x_i + \beta x_j + 2\right)^2} - \frac{\delta\left(\beta x_j + 1\right)}{\beta x_i + \beta x_j + 2} - 1, \\ \frac{\partial \pi_i(x_i, x_j)}{\partial x_i^2} = -\frac{2\beta\left(\beta x_j + 1\right)\left(\Lambda + \beta \Delta x_j\right)}{\left(\beta x_i + \beta x_j + 2\right)^3}, \end{gathered} \tag{21}$$

and we introduce function $x_+ : D \to \mathbb{R}$, defined on set $D = \{x_j \in [0,1] : \Lambda + \beta\Delta x_j \geq 0\}$, whose expression is given by

$$x_+(x_j) = -\frac{2}{\beta} - x_2 + \frac{\sqrt{(\beta x_j + 1)(\Lambda + \beta \Delta x_j)}}{\beta}. \tag{22}$$

We remark that set D depends on the parameters' configuration. Function (22) is strictly concave for $\Lambda \neq \Delta$, while for $\Lambda = \Delta$ it becomes

$$x_+(x_j) = -x_j\left(1 - \sqrt{\Lambda}\right) + \frac{\sqrt{\Lambda} - 2}{\beta}. \tag{23}$$

To derive the expression for the best response we distinguish three cases, depending on x_j.
a) $x_j \in [0,1]$ is such that $\Lambda + \beta\Delta x_j < 0$. In this case, we have from (21) that the payoff function $\pi_i(x_i, x_j)$ is strictly convex with respect to x_i, so its global maximum is attained at $x_i = 0$ and/or $x_i = 1$. Since

$$\pi_i(0, x_j) - \pi_i(1, x_j) > 0 \Leftrightarrow \omega = \beta^2(1-\Delta)x_j^2 + \beta(4 - \Delta - \Lambda + \beta)x_j + 4 + 2\beta - \Lambda > 0,$$

using $-\Lambda \geq \beta\Delta x_j$ we obtain $\omega > \beta^2 x_j^2 + \beta(4+\beta)x_j + 4 + 2\beta > 0$, which is indeed true. This means that $BR_i(x_j) = 0$.
b) $x_j \in [0,1]$ is such that $\Lambda + \beta\Delta x_j = 0$. In this case we have that

$$\pi_i\left(x_i, -\frac{\Lambda}{\beta\Delta}\right) = -\delta\frac{\Delta - \Lambda}{\beta\Lambda} - x_i$$

is a decreasing function and this again provides $BR_i(x_j) = 0$.
c) $x_j \in [0,1]$ is such that $\Lambda + \beta\Delta x_j > 0$. In this case the payoff function is strictly concave. To find $x_i = \operatorname{argmax}_{z\in[0,1]} \pi_i(z, x_j)$, we start noticing that, from (21), imposing $\partial_{x_i}\pi_i(x_i, x_j) \geq 0$ we obtain

$$- \beta^2 x_i^2 + -2\beta\left(\beta x_j + 2\right) x_i + \beta\theta - 2\delta - 4\beta x_j - \beta^2 {x_j}^2 + \beta^2\theta x_j - \beta^2\delta {x_j}^2 + \beta^2\gamma {x_j}^2 - 3\beta\delta x_j + \beta\gamma x_j - 4 \geq 0. \quad (24)$$

The previous inequality is solved for $x_-(x_j) \leq x_i \leq x_+(x_j)$, where

$$x_\pm(x_j) = -\frac{2}{\beta} - x_j \pm \frac{\sqrt{(\beta x_j + 1)(\Lambda + \beta\Delta x_j)}}{\beta},$$

which, since $\Lambda + \Delta x_j > 0$, are both real and satisfy $x_-(x_j) < -1/\beta < x_+(x_j)$. Since x_+ is concave, the present setting is then very similar to that we obtained in the case of no spillover. Proceeding in the same way, we easily obtain $BR_i(x_j) = \min\{\max\{x_+(x_j), 0\}, 1\}$.
Combining the results of cases (a),(b) and (c) provides (8).

We stress that the best response function is continuous. This is evident if either $\Lambda + \beta\Delta x_j \leq 0$ or $\Lambda + \beta\Delta x_j > 0$ for every $x_j \in [0,1]$. Conversely, if $\bar{x}_j = -\Lambda/(\beta\Delta)$ belongs to $(0,1)$, we need to check the continuity of BR_i at $x_j = \bar{x}_j$. If $\Delta > 0$ (and so $\Lambda < 0$) we have that $BR_i(x_j) = 0$ for $x_j \leq \bar{x}_j$ and since

$$\lim_{x_j \to \bar{x}_j^+} x_+(x_j) = \frac{1}{\beta}\left(\frac{\Lambda}{\Delta} - 2\right) < 0, \quad (25)$$

we have $\lim_{x_j\to\bar{x}_j^+} BR_i(x_j) = 0$, so the best response function is continuous. Similarly, if $\Delta < 0$ (and so $\Lambda > 0$) we have that $BR_i(x_j) = 0$ for $x_j \geq \bar{x}_j$ and, since (25) still holds, we have $\lim_{x_j\to\bar{x}_j^-} BR_i(x_j) = 0$, so the best response function is again continuous.

Proof of Proposition 2. To find symmetric and asymmetric internal equilibria we must solve the system of the two equations $x_+(x_2) = x_1$ and $x_+(x_1) = x_2$, which provides $x = -1/\beta \notin [0,1]$ and (9) and (10). The analytical expressions for (11) are obtained by solving the two equations $x_+(x) = 0$ and $x_+(x) = 1$. We omit to report the simple but very long computations. We just notice that even if $x^{NE}_{i,AS}$ are well-defined even if $\Delta > 0$, they are actually solutions of the aforementioned system only if $\Delta \leq 4$. Moreover, if $\Delta = -4$, we actually have $x^{NE}_{1,AS} = x^{NE}_{2,AS} = x^{NE}_{IS}$. □

Before proving Proposition 3, we need some preliminary results.

Lemma 1. *We have that*

a) $BR_i(0) = 0$ *if and only if* $\Lambda \leq 4$*;*

b) $BR_i(1) = 1$ *if and only if* $\Delta \geq \frac{4(\beta+1)}{\beta} - \frac{\Lambda}{\beta}$*;*

c) $x^{NE}_{IS} \in (0,1)$ *if and only if* $\begin{cases} \Lambda < 4, \\ \Delta > \frac{4(\beta+1)}{\beta} - \frac{\Lambda}{\beta}, \end{cases} \cup \begin{cases} \Lambda > 4, \\ \Delta < \frac{4(\beta+1)}{\beta} - \frac{\Lambda}{\beta}, \end{cases}$

d) asymmetric equilibria exists if and only if $\Delta < -4$ *and there are no symmetric corner equilibria.*

Proof. a,b) The proof is straightforward.

c) It is sufficient to solve $0 < x_{IS} < 1$. From

$$0 < \frac{\Lambda - 4}{\beta(4 - \Delta)} < 1$$

we have

$$\begin{cases} \Lambda - 4 < 0, \\ 4 - \Delta < 0, \\ \Delta > \frac{4(\beta+1)}{\beta} - \frac{\Lambda}{\beta}, \end{cases} \cup \begin{cases} \Lambda - 4 > 0, \\ 4 - \Delta > 0, \\ \Delta < \frac{4(\beta+1)}{\beta} - \frac{\Lambda}{\beta}. \end{cases}$$

Noticing that in each system the first and the last conditions imply the second ones allows concluding.

d) The proof essentially relies on the continuity of the best response function and on the existence of a unique symmetric equilibrium and of a unique couple of asymmetric equilibria.

We start proving that if either symmetric corner equilibria exist or if $\Delta \geq -4$, then no asymmetric equilibria are possible. To this end, let us introduce the two regions

$$T_1 = \{(x_1, x_2) \in [0,1]^2 : x_1 \leq x_2\}, \quad T_2 = \{(x_1, x_2) \in [0,1]^2 : x_1 \geq x_2\},$$

respectively corresponding to the two triangles in which the square $[0,1]^2$ is divided by the line $x_2 = x_1$.

We prove that if $\Delta > -4$ and there are no symmetric corner equilibria, then there are asymmetric equilibria. By contradiction, we actually show that if either there are symmetric corner equilibria or if $\Delta \geq 4$ then no asymmetric equilibria are possible.

Firstly, we assume that a symmetric corner equilibrium is present, for example point $(0,0)$. From $BR_i(0) = 0$ for $i = 1, 2$, we should either have that

d1) the best response functions do not intersect line $x_2 = x_1$ on $(0,1)$, namely $BR_2([0,1]) \subset T_r$ and $BR_1([0,1]) \subset T_s$ with $r, s \in \{1,2\}$ and $r \neq s$

d2) the best response functions intersect line $x_2 = x_1$ at x_{IS}^{NE}, namely $BR_2([0, x_{IS}^{NE}]) \subset T_r \cap [0, x_{IS}^{NE}] \times [0,1], BR_1([0, x_{IS}^{NE}]) \subset T_s \cap [0,1] \times [0, x_{IS}^{NE}]$ and $BR_2([x_{IS}^{NE}, 1]) \subset T_s \cap [x_{IS}^{NE}, 1] \times [0,1], BR_1([x_{IS}^{NE}, 1]) \subset T_r \cap [0,1] \times [x_{IS}^{NE}, 1]$ with $r, s \in \{1,2\}$ and $r \neq s$.

No other configurations are possible since, recalling Proposition 2, the best response functions can have at most one intersection with $x_2 = x_1$ on $(0,1)$. In both cases (d1) and (d2), the graphs of the best response functions are included in sets which have no asymmetric points in common, and so no asymmetric equilibria can exist. As similar argument allows concluding that if point $(1,1)$ is an equilibrium, no asymmetric equilibria can exist.

Now we assume $\Delta \geq -4$. Since we can indeed assume that no corner equilibria exist and noticing that the continuity of the best response function implies that at least a symmetric equilibrium exists, $(x_{IS}^{NE}, x_{IS}^{NE})$ is the unique symmetric equilibrium and we must necessarily have

$$\begin{cases} \Lambda > 4, \\ \Delta < \frac{4(\beta+1)}{\beta} - \frac{\Lambda}{\beta}, \end{cases}$$

as otherwise we would have that either $(0,0)$ or $(1,1)$ are equilibria. Combining the two inequalities, we then have

$$\Delta < 4 + (4 - \Lambda)/\beta < 4 < \Lambda, \tag{26}$$

from which we can conclude that

$$BR_2'(x_{IS}^{NE}) = \frac{(\Delta + 4)\text{sign}(\Delta - \Lambda)}{4\text{sign}(\Delta - 4)} - 1 = \frac{\Delta + 4}{4} - 1. \tag{27}$$

Using arguments similar to those used in case (d2) (from (26) we have $BR_2'(x_{IS}^{NE}) < 1$, so a situation similar to that in case (d1) is excluded), we can conclude that if $BR_2'(x_{IS}^{NE}) \geq 0$, no asymmetric equilibria are possible, which recalling (27), allows concluding that no asymmetric equilibria arise if $\Delta \geq 0$. We then consider $-4 \leq \Delta < 0$. Recalling the proof of Proposition 2, no internal asymmetric equilibria arise. To prove that no boundary/corner equilibria are possible, in what follows, we assume that, as in Figure 2,

functions $x_2 = BR_2(x_1)$ and $x_1 = BR_1(x_2)$ are represented having domains on the horizontal and vertical axis, respectively. Moreover, it is indeed sufficient to focus on what happens, for example, in T_2. We prove that if $\Delta \in [-4, 0)$, function $x_1 = BR_1(x_2)$ is invertible on $D = \{BR_1(x_2) > 0\} \cap [x_{IS}^{NE}, 1]$ and there is left neighborhood of x_{IS}^{NE} on which the graph of BR_2 lies strictly below the graph of BR_1^{-1}. If $-4 < \Delta < 0$, we have $BR_2'(x_{IS}^{NE}) \in (-1, 0)$ so, thanks to the symmetry, the slope of the tangent line of BR_1 (in plane (x_1, x_2)) is $1/BR_2'(x_{IS}^{NE}) < -1$. This provides the neighborhood with the aforementioned property. Moreover, as BR_1 is concave, BR_1 is non-increasing on $(x_{IS}^{NE}, 1]$ and coincides with x_+ where it is strictly positive and strictly decreasing. This allows concluding that BR_1 is invertible on D.
If $\Delta = -4$, the last argument is still valid, and so BR_1 is invertible on D. Comparing the Taylor expansions at $x = x_{IS}^{NE}$ of x_+

$$x_+(x) = x_{IS}^{NE} - (x - x_{IS}^{NE}) - \frac{16\beta}{\Lambda + 4} \frac{(x - x_{IS}^{NE})^2}{2} + o((x - x_{IS}^{NE})^4)$$

and of x_+^{-1}

$$x_+^{-1}(x) = x_{IS}^{NE} - (x - x_{IS}^{NE}) - \frac{16\beta}{\Lambda + 4} \frac{(x - x_{IS}^{NE})^2}{2} - \frac{768\beta^2}{(\Lambda + 4)^2} \frac{(x - x_{IS}^{NE})^3}{6} + o((x - x_{IS}^{NE})^4)$$

we can conclude that there is left neighborhood of x_{IS}^{NE} on which the graph of BR_2 lies strictly below the graph of BR_1^{-1}.
Let $BR_1(1) = \hat{x} < 1$. We first consider $\hat{x} > 0$. Since BR_1 is non-increasing on $[x_{IS}^{NE}, 1]$, we must have $\hat{x} < x_{IS}^{NE}$. If $(\hat{x}, 1)$ were an equilibrium, we would need $BR_2(\hat{x}) = 1$. We can not have $x_+(\hat{x}) = 1$, as otherwise point $(\hat{x}, 1)$ would belong to the intersection of functions $x_2 = x_+(x_1)$ and $x_1 = x_+(x_2)$, which, for $\Delta \geq 4$, have no asymmetric intersections. If $x_+(1) > 1$, the graph of x_+ lies above that of x_+^{-1} in a right neighborhood of $\hat{x}$. This would mean that the reciprocal position of the two best response functions change from a right neighborhood of $\hat{x}$ to a left one of x_{IS}^{NE} and this, since $BR_i x_j$ are continuous, implies that an asymmetric internal equilibrium must exist, which is excluded by $\Delta \geq 4$.
Let us assume $\hat{x} = 0$. Since we can not have $BR_1 \equiv 0$, we must have $\hat{y} > 0$ so that $BR_1 \equiv 0$ on $[\hat{y}, 1]$ and BR_1 is strictly positive and strictly decreasing in a right neighborhood of $\hat{y}$. In such neighborhood it is invertible and its inverse is x_+. In this case we can have a boundary/corner equilibrium only if $BR_2(0) \geq \hat{y}$. Again, this would imply that functions x_+ and x_+^{-1} have an intersection point for $x < x_{IS}^{NE}$, which is not possible. This allows concluding that if $\Delta \geq -4$ no asymmetric equilibria can arise.

To complete the prove we should show that if $\Delta < -4$ and symmetric corner equilibria do not exist, then we have asymmetric equilibria. This can be done proceeding similarly to the last two cases, namely combining geometric and continuity arguments. Since it is simple but very long, we do not report details. □

Proof of Proposition 3. To have just multiple symmetric equilibria it is necessary and sufficient that both $(0, 0)$ and $(1, 1)$ are equilibria. Recalling case (d) of Lemma 1, it is indeed sufficient. It is necessary since if they both were not equilibria we would just have one symmetric equilibrium. Moreover, it is not possible to have just one symmetric corner equilibrium. In this case, to have symmetric multiple equilibria, $(x_{IS}^{NE}, x_{IS}^{NE})$ should be an equilibrium, too. However, the uniqueness of internal symmetric equilibria together with the continuity of the best response would necessarily require that the other corner point is an equilibrium, too. Then (12) is obtained from the union of cases (a) and (b) of Lemma 1. We notice that the second condition reported in (12) can be actually neglected, as it is a necessary consequence of the first one. We left it for the relevance in the interpretation and comparison of the results.

To have a unique equilibrium, we distinguish three situations. We have that $(0, 0)$ is the unique equilibrium if and only if $BR_i(0) = 0$ and $(1, 1)$ is not an equilibrium. It is indeed necessary, it is also sufficient thanks to the continuity of the best response (we recall the considerations about the case of multiple symmetric equilibria). This first situation is then obtained by simultaneously considering case (a) and the negation of case (b) of Lemma 1. Similarly, we have that $(1,1)$ is the unique equilibrium if and only if $BR_i(1) = 1$ and $(0,0)$ is not an equilibrium, and this corresponds to simultaneously considering the negation of case (a) and case (b) of Lemma 1. To have that $(x_{IS}^{NE}, x_{IS}^{NE})$ is the unique equilibrium, we must simply consider case (c) of Lemma 1, excluding the possibility of symmetric corner equilibria (negation of cases (a) and (b)) and of asymmetric equilibria, namely, from case (d), adding condition $\Delta \geq 4$.

The parameters' region (13) in which we have multiple asymmetric equilibria can be then simply obtained as the complement of the union of regions (12) and (14). $\square$

Proof of Corollary 2. Firstly we notice that if x_{IS}^{NE} is an internal equilibrium, then, from $x_{IS}^{NE} > 0$, we necessarily must have either $\Delta > 4$ and $\Lambda - 4 < 0$ or $\Delta < 4$ and $\Lambda - 4 > 0$.

We have

$$\frac{\partial x_{IS}^{NE}}{\partial \beta} = \frac{-2(\delta+2)}{\beta^2(\Delta-4)}, \quad \frac{\partial x_{IS}^{NE}}{\partial \gamma} = \frac{\Lambda-4}{\beta(\Delta-4)^2}, \quad \frac{\partial x_{IS}^{NE}}{\partial \theta} = -\frac{1}{\Delta-4}, \quad \frac{\partial x_{IS}^{NE}}{\partial \delta} = -\frac{\beta\theta-2\gamma+4}{\beta(\Delta+4)^2},$$

so we can immediately conclude that $\partial x_{IS}^{NE}/\partial\beta > 0$ and $\partial x_{IS}^{NE}/\partial\theta > 0$ if and only if $\Delta - 4 < 0$ (or equivalently $\Lambda - 4 > 0$). Similarly, we have $\partial x_{IS}^{NE}/\partial\gamma > 0$ if and only if $\Lambda - 4 > 0$ (or equivalently $\Delta < 4$). Noticing that

$$-\beta\theta + 2\gamma - 4 = (-\Lambda + 4) + 2(\Delta - 4),$$

where the last two addends have the same sign, we can conclude that $\partial x_{IS}^{NE}/\partial\delta > 0$ if and only if $\Delta > 4$ (or equivalently $\Lambda - 4 < 0$). $\square$

7 Proofs of Section 4.3

Firstly, we derive the expression for the best response related to game $\Gamma \in \mathbf{\Gamma}^{LSE}$. As we said in Section 4.3, in this case it can be a correspondence. We start noticing that, from (15), we have

$$\begin{aligned} \partial_{x_i}\pi_i(x_i,x_j) &= \frac{-\alpha\beta x_j^2 + \beta x_j + 1}{(\beta x_i + \beta x_j - 2\alpha\beta x_i x_j + 2)^2}\theta\beta - 1, \\ \partial_{x_i}^2\pi_i(x_i,x_j) &= \frac{2\beta^2\theta(2\alpha x_j - 1)(-\alpha\beta x_j^2 + \beta x_j + 1)}{(\beta x_i + \beta x_j - 2\alpha\beta x_i x_j + 2)^3}. \end{aligned} \tag{28}$$

As for the proofs in the previous sections, we introduce a suitable function $x_+ : [0, 1/(2\alpha)) \to \mathbb{R}$ defined by

$$x_+(x_j) = -\frac{2}{\beta(1-2\alpha x_j)} - \frac{x_j}{(1-2\alpha x_j)} + \frac{\sqrt{\beta\theta(-\alpha\beta x_j^2 + \beta x_j + 1)}}{\beta(1-2\alpha x_j)}, \tag{29}$$

which will be used to obtain the best response depending on strategy x_j. Notice that for $x_j \in [0, 1/(2\alpha))$ we have $-\alpha\beta x_j^2 + \beta x_j + 1 > 0$, so (29) is well-defined. Direct checks show that $x_+(x_j) \leq 0$ for any $x_j \in [0, 1/(2\alpha))$ if and only if $\beta\theta \leq 4$, while for

$$4 < \beta\theta < 4 + \beta/\alpha \tag{30}$$

function x_+ is unimodal and concave on $[0, 1/(2\alpha))$. Moreover, x_+ attains its maximum at $x = x_{IS}^{NE}$ (defined by (17)), which is a fixed point, too.

The next two propositions will respectively deal with the parameters' configurations for which the best response to a player strategy is either a single value or a set consisting of more then one element.

Proposition 5. *Let* $x_j \in [0,1]$. *Then*

$$BR_i(x_j) = \begin{cases} \max\{\min\{x_+(x_j),1\},0\} & \text{if } x_j \in [0,1/(2\alpha)) \cap [0,1], \\ 0 & \text{if } x_j \in [1/(2\alpha),1] \text{ and } \beta\theta < 4+\beta/\alpha, \\ 1 & x_j \in [1/(2\alpha),1] \text{ and } \beta\theta > 2\beta+4. \end{cases}$$

Proof. If $\alpha < 1/2$ and $x_j \in [0,1]$ or $\alpha \geq 1/2$ and $x_j \in [0,1/(2\alpha))$, we indeed have $2\alpha x_j - 1 < 0$, which, as already noticed, guarantees $-\alpha\beta x_j^2 + \beta x_j + 1 < 0$. Moreover, $\beta x_i + \beta x_j - 2\alpha\beta x_i x_j + 2 = \beta x_i(1-2\alpha x_j) + \beta x_j + 2 > 0$, so, recalling 28, we have $\partial^2_{x_i}\pi_i(x_i,x_j) < 0$. For such x_j, the payoff function π_i is then concave with respect to $x_i \in [0,1]$. Solving $\partial_{x_i}\pi_i(x_i,x_j) > 0$ we find $x_- x_j < x_i < x_+ x_j$, where

$$x_\pm(x_j) = \frac{-2-\beta x_j \pm \sqrt{\beta\theta(-\alpha\beta x_j^2 + \beta x_j + 1)}}{\beta(1-2\alpha x_j)},$$

and x_- is strictly negative. Proceeding as in Sections 6.1 and 6.2, we can conclude that $BR_i(x_j) = \max\{\min\{x_+(x_j),1\},0\}$.

Now we consider $\alpha \geq 1/2$ and $x_j = 1/(2\alpha)$. In this case we have

$$\pi_i\left(x_i, \frac{1}{2\alpha}\right) = \frac{-x_i(4\alpha+\beta-\alpha\beta\theta) - 2\alpha\theta - \beta\theta}{4\alpha+\beta}. \tag{31}$$

This means that if $\beta\theta < 4+\beta/\alpha$, the coefficient of x_j is strictly negative and function (31) is strictly decreasing, so $BR_i(1/(2\alpha)) = 0$, while if $\beta\theta > 2\beta+4$ then the coefficient of x_j is strictly positive and function (31) is strictly increasing, so $BR_i(1/(2\alpha)) = 1$.

Now we consider $\alpha > 1/2$ and $x_j \in (1/(2\alpha),1]$. In this case, to have a non-empty interval, we must take $\alpha > 1/2$. We study the sign of

$$f(x_j) = \pi_i(0,x_j) - \pi_i(1,x_j) = \beta^2(1-2\alpha+\alpha\theta)x_j^2 + \beta(4-4\alpha+\beta-\beta\theta)x_j + 4 + 2\beta - \beta\theta. \tag{32}$$

A simple computation shows that

$$\begin{gathered} f\left(\frac{1}{2\alpha}\right) = \frac{(4\alpha+\beta)(4\alpha+\beta-\alpha\beta\theta)}{4\alpha^2},\ f(1) = (2\beta-\beta\theta+4)(\beta-\alpha\beta+1), \\ f'(x_j) = 2\beta^2(1-2\alpha+\alpha\theta)x_j - \beta(4\alpha-\beta+\beta\theta-4), \end{gathered} \tag{33}$$

so if $\beta\theta < 4+\beta/\alpha$, then both $f(1/(2\alpha)) > 0$ and $f(1) > 0$. Moreover, we have that if $(1-2\alpha+\alpha\theta) = 0$, then $f(x_j) = ((4\alpha+\beta)(\beta x_2(1-\alpha)+1))/\alpha > 0$, and then $BR_i(x_j) = 0$ for any $x_j \in (1/(2\alpha),1]$. Conversely, if $(1-2\alpha+\alpha\theta) < 0$, since f is concave and both $f(1/(2\alpha)) > 0$ and $f(1) > 0$, then $BR_i(x_j) = 0$ for any $x_j \in (1/(2\alpha),1]$. Finally, if $(1-2\alpha+\alpha\theta) > 0$, f is convex but $f'(1/(2\alpha)) = \beta(4\alpha+\beta)(1-\alpha)/\alpha > 0$, which, together with $f(1/(2\alpha)) > 0$, guarantees that $BR_i(x_j) = 0$ for any $x_j \in (1/(2\alpha),1]$.

Finally, if $\beta\theta > 2\beta+4$, we have $f(1/(2\alpha)) < 0$, $f(1) < 0$ and $f'(x_j) > 0$, which means that $f(x_j) < 0$ (i.e. $BR_i(x_j) = 1$) for each $x_j \in (1/(2\alpha),1]$. □

We now consider the case of a best response correspondence. To this end we introduce

$$\tilde{x}_j = \frac{4\alpha-\beta+\beta\theta-4+\sqrt{(4\alpha+\beta)(4\alpha+\beta-4\theta-2\beta\theta+\beta\theta^2)}}{2\beta(1-2\alpha+\alpha\theta}. \tag{34}$$

Proposition 6. *Let* $\alpha \geq 1/2$ *and* $x_j \in [1/(2\alpha),1]$. *We have that*

a) if $\beta\theta = 4+\beta/\alpha$ *then* $BR_i(x_j) = \begin{cases} [0,1] & x_j = \frac{1}{2\alpha} \\ 0 & \frac{1}{2\alpha} < x_j = 1 \end{cases}$

b) if $4+\beta/\alpha < \beta\theta \le 2\beta+4$ *then* $BR_i(x_j) = \begin{cases} 1 & \frac{1}{2\alpha} \le x_j < \tilde{x}_j \\ \{0,1\} & x_j = \tilde{x}_j \\ 0 & \tilde{x}_j < x_j \le 1 \end{cases}$

Proof. Firstly, we study $x_j = 1/(2\alpha)$. Recalling (31), if $4+\beta/\alpha < \beta\theta \le 2\beta+4$, then (31) is strictly increasing and $BR_i(1/(2\alpha)) = 1$, while if $4\alpha+\beta-\alpha\beta\theta = 0$, then (31) is constant and $BR_i(1/(2\alpha)) = [0,1]$;

Now we consider $x_j \in (1/(2\alpha), 1]$. In this case, to have a non-empty interval, we must take $\alpha > 1/2$. To study the sign of $\pi_i(0,x_j) - \pi_i(1,x_j)$ we make use of function f defined by (32). Recalling (33) we have that if $\beta\theta = 4+\beta/\alpha$, we have $(1-2\alpha+\alpha\theta) = 2-2\alpha+4\alpha/\beta > 0$, so f is strictly convex and, as just seen, increasing. Since $f(1/(2\alpha)) = 0$ and $f(1) > 0$, we have that $f(x_j) > 0$ (i.e. $BR_i(x_j) = 0$) for each $x_j \in (1/(2\alpha), 1]$.

Conversely if $4+\beta/\alpha < \beta\theta < 2\beta+4$, we have $f(1/(2\alpha)) < 0$, $f(1) > 0$ and $f'(x_j) > 0$. By continuity and increasing monotonicity of f, we have exactly one solution of $f(x_j) = 0$ belonging to $(1/(2\alpha), 1)$, given by (34). Recalling that for $\beta\theta > 4+\beta/\alpha$, x_+ is strictly convex, we have

- $x_j < \tilde{x}_j$ we have $f(x_j) = \pi_i(0,x_j) - \pi_i(1,x_j) < 0$ and then $BR_i(x_j) = 1$;
- $x_j = \tilde{x}_j$ we have $f(x_j) = \pi_i(0,x_j) - \pi_i(1,x_j) = 0$ and then $BR_i(x_j) = \{0,1\}$;
- $x_j > \tilde{x}_j$ we have $f(x_j) = \pi_i(0,x_j) - \pi_i(1,x_j) > 0$ and then $BR_i(x_j) = 0$.

Noticing that for $\beta\theta = 2\beta+4$ we have both $f(1) = 0$, which means that $BR_i(1) = [0,1]$, and $\tilde{x}_j = 1$, we can conclude. □

Combining the conclusions of Propositions 5 and 5 it is easy to obtain the expression of the best response relations for any parameters' configurations.

Proof of Proposition 4. Solving $x_+(x_j) = x_j$ provides

$$\frac{1}{2\alpha} + \frac{\sqrt{\beta(4\alpha+\beta)}}{2\alpha\beta},\ \frac{1}{2\alpha} - \frac{\sqrt{\beta(4\alpha+\beta)}}{2\alpha\beta},\ \frac{1}{2\alpha} + \frac{\sqrt{\beta(4\alpha+\beta-\alpha\beta\theta)}}{2\alpha\beta},\ \frac{1}{2\alpha} - \frac{\sqrt{\beta(4\alpha+\beta-\alpha\beta\theta)}}{2\alpha\beta},$$

which are the potential symmetric equilibria. Thanks to (30), all the four solutions real. However, the first and the third solution are larger than $1/(2\alpha)$ and the second one is negative. Conversely, the last one is indeed smaller than $1/(2\alpha)$ and, using again (30), it is also non negative.

The only possible situation in which we can have asymmetric equilibria, from Proposition 5 and 6, if when $x_j \in [0, \min 1/(2\alpha))$. However, recalling that in this case function x_+ is either negative or unimodal and concave, with a unique fixed point at which the maximum is attained, simple geometrical considerations exclude the possibility to have asymmetric solutions of the system of the two equations $x_2 = x_+(x_1)$ and $x_1 = x_+(x_2)$.

The unique symmetric equilibrium is a simple consequence of the existence of an unique symmetric fixed point for x_+ on $[0, 1/2\alpha]$ and of the possible expression of the best response relations given by Propositions 5 and 5. We limit to notice that the case of no equilibrium corresponds to case (b) of Proposition 6. □

Proof of Proposition 3. The behavior of x_{IS}^{NE} with respect to β, θ and α is straightforwardly obtained computing its partial derivatives with respect to such parameters. The parameters' region on which no equilibrium exists is given by case (b) of Proposition 6. □

8 Proofs of Section 5

Proof of Corollary 4. For simplicity, in the following computations we remove the denominator 2 from the definitions of conflict intensity as it does not affect monotonicity and concavity.

We start recalling the conditions for the occurrence of the following scenarios

a) We have a unique internal symmetric equilibrium if $\Delta < 4$ AND $\Lambda > 4$(as consequence of the third condition in (14)

b) We have multiple symmetric internal equilibria if $\Delta > 4$ and $\Lambda < 4$ (as consequence of condition (12); values $\Delta = 4$ or $\Lambda = 4$ provide boundary equilibria)

c) We have multiple asymmetric equilibria if $\Delta < -4$ and $\Lambda > 4$ (as consequence of condition (13))

We have

$$\frac{\partial CI_{IS}}{\partial \delta} = -\frac{2(\beta\theta - 2\gamma + 4)}{\beta(\delta - \gamma + 4)^2} = -\frac{2(\Lambda - 2\Delta + 4)}{\beta(\Delta - 4)^2}, \quad \frac{\partial^2 CI_{IS}}{\partial \delta^2} = -\frac{4(\Lambda - 2\Delta + 4)}{\beta(\Delta - 4)^3}.$$

When there is a unique internal equilibrium, from conditions in (a), we have $\Lambda + 4 > 2\Delta$ and $\Delta < 4$, from which $\partial CI_{IS}/\partial\delta < 0$ and $\partial CI_{IS}/\partial\delta > 0$.

When we have multiple symmetric equilibria, from conditions in (b), we have $\Lambda + 4 < 2\Delta$ and $\Delta > 4$ (we recall that we do not consider the case of $\Delta = \Lambda = 4$), so $\partial CI_{IS}/\partial\delta > 0$ and $\partial^2 CI_{IS}/\partial\delta^2 > 0$.

When we have multiple asymmetric equilibria, from conditions in (c), we have $\Lambda - 2D + 4 > 0$ and $\Delta < 0$, so $\partial CI_{IS}/\partial\delta < 0$ and $\partial^2 CI_{IS}/\partial\delta^2 > 0$.

We have

$$\frac{\partial CI_{IS}}{\partial \gamma} = -\frac{2(2\delta - \beta\theta + 4)}{\beta(\delta - \gamma + 4)^2} = \frac{2(\Lambda - 4)}{\beta(\Delta - 4)^2}, \quad \frac{\partial^2 CI_{IS}}{\partial \gamma^2} = -\frac{4(2\delta - \beta\theta + 4)}{\beta(\delta - \gamma + 4)^3} = -\frac{4(\Lambda - 4)}{\beta(\Delta - 4)^3}.$$

When there is a unique internal equilibrium, from conditions in (a), we have $\partial CI_{IS}/\partial\gamma > 0$ and $\partial CI_{IS}/\partial\gamma > 0$.

When we have multiple symmetric equilibria, from conditions in (b), we have $\partial CI_{IS}/\partial\gamma < 0$ and $\partial^2 CI_{IS}/\partial\gamma^2 > 0$.

When we have multiple asymmetric equilibria, from conditions in (c), we have $\partial CI_{IS}/\partial\gamma > 0$ and $\partial^2 CI_{IS}/\partial\gamma^2 > 0$.

We have

$$\frac{\partial CI_{IS}}{\partial \theta} = \frac{2}{\delta - \gamma + 4} = -\frac{2}{(\Delta - 4)}, \quad \frac{\partial^2 CI_{IS}}{\partial \theta^2} = 0.$$

When there is a unique internal equilibrium or multiple asymmetric equilibria, from conditions in (a) and (c), we have $\partial CI_{IS}/\partial\theta > 0$, conversely when we have multiple symmetric equilibria, from conditions in (b) we obtain $\partial CI_{IS}/\partial\theta < 0$.

Finally we have

$$\frac{\partial CI_{IS}}{\partial \beta} = \frac{4(\delta + 2)}{\beta^2(\delta - \gamma + 4)} = -\frac{4(\delta + 2)}{\beta^2(\Delta - 4)}, \quad \frac{\partial^2 CI_{IS}}{\partial \beta^2} = -\frac{8(\delta + 2)}{\beta^3(\delta - \gamma + 4)} = \frac{8(\delta + 2)}{\beta^3(\Delta - 4)}.$$

When there is a unique internal equilibrium or multiple asymmetric equilibria, from conditions in (a) and (c), we have $\partial CI_{IS}/\partial\beta > 0$ and $\partial^2 CI_{IS}/\partial\beta^2 < 0$, conversely when we have multiple symmetric equilibria, from conditions in (b) we obtain $\partial CI_{IS}/\partial\beta < 0$ and $\partial^2 CI_{IS}/\partial\beta^2 > 0$.

Now we recall that the existence of internal asymmetric equilibria necessarily requires

$$\begin{cases} 4 < \Lambda < \beta(4 - \Delta) + 4, \\ \Delta < -4. \end{cases}$$

Moreover, from

$$CI_{AS} = -\frac{\Delta + \Lambda}{\Delta\beta} = \frac{\gamma - 3\delta + \beta\theta}{\beta(\delta - \gamma)}$$

We have

$$\frac{\partial CI_{AS}}{\partial \theta} = -\frac{1}{\Delta} > 0,$$

$$\frac{\partial CI_{AS}}{\partial \gamma} = \frac{\Lambda}{\Delta^2\beta} > 0, \qquad \frac{\partial^2 CI_{AS}}{\partial \gamma^2} = -\frac{2\Lambda}{\Delta^3\beta} > 0,$$

$$\frac{\partial CI_{AS}}{\partial \delta} = \frac{2\Delta - \Lambda}{\Delta^2\beta} < 0, \qquad \frac{\partial^2 CI_{AS}}{\partial \delta^2} = \frac{2(2\Delta - \Lambda)}{\Delta^3\beta} > 0,$$

and

$$\frac{\partial CI_{AS}}{\partial \beta} = \frac{\Delta - 2\delta}{\Delta\beta^2} > 0, \qquad \frac{\partial^2 CI_{AS}}{\partial \beta^2} = -\frac{2(\Delta - 2\delta)}{\Delta\beta^3} < 0,$$

which allow concluding. □

Proof of Corollary 5. We start noting that to have $x_{1,B}^{NE} \in (0,1)$ we need

$$\begin{cases} \gamma - \delta < -4, \\ \beta\theta < 2\delta - \beta(\gamma - \delta) + 4(\beta + 1), \\ \beta\theta - 2\delta + \beta(\gamma - \delta) > 0. \end{cases} \tag{35}$$

We have

$$CI_{1,B} = -\frac{\beta - \sqrt{(\beta + 1)(\beta\theta - 2\delta + \beta(\gamma - \delta))} + 2}{\beta} + 1,$$

so the behavior with respect to γ is straightforward and while computing derivatives with respect to θ and δ we have

$$\frac{\partial CI_{1,B}}{\partial \theta} = \frac{\beta + 1}{2\sqrt{(\beta + 1)(\beta\theta - 2\delta + \beta(\gamma - \delta))}} > 0, \quad \frac{\partial^2 CI_{1,B}}{\partial \theta^2} = -\frac{\beta(\beta + 1)^2}{4(-(\beta + 1)(2\delta - \beta\theta + \beta(\delta - \gamma)))^{3/2}} < 0,$$

and

$$\frac{\partial CI_{1,B}}{\partial \delta} = -\frac{(\beta + 1)(\beta + 2)}{2\beta\sqrt{(\beta + 1)(\beta\theta - 2\delta + \beta(\gamma - \delta))}} < 0, \frac{\partial^2 CI_{1,B}}{\partial \delta^2} = -\frac{(\beta + 1)^2(\beta + 2)^2}{4\beta(-(\beta + 1)(2\delta - \beta\theta + \beta(\delta - \gamma)))^{3/2}} < 0.$$

Finally, we have

$$\frac{\partial CI_{1,B}}{\partial \beta} = \frac{4\delta + 3\beta\delta - \beta\gamma - \beta\theta + 4\sqrt{\beta\gamma - 3\beta\delta - 2\delta + \beta\theta - \beta^2\delta + \beta^2\gamma + \beta^2\theta}}{2\beta^2\sqrt{\beta\gamma - 3\beta\delta - 2\delta + \beta\theta - \beta^2\delta + \beta^2\gamma + \beta^2\theta}}$$

and

$$\begin{aligned} \frac{\partial^2 CI_{1,B}}{\partial \beta^2} = &\Big(72\beta\delta^2 - 16(-(\beta + 1)(2\delta - \beta\theta + \beta(\delta - \gamma)))^{3/2} + 32\delta^2 + 51\beta^2\delta^2 + 12\beta^3\delta^2 + 3\beta^2\gamma^2 + 4\beta^3\gamma^2 \\ &+ 3\beta^2\theta^2 + 4\beta^3\theta^2 - 24\beta\delta\gamma - 24\beta\delta\theta - 42\beta^2\delta\gamma - 16\beta^3\delta\gamma - 42\beta^2\delta\theta - 16\beta^3\delta\theta + 6\beta^2\gamma\theta \\ &+ 8\beta^3\gamma\theta\Big) \Big/ \Big(4\beta^3(-(\beta + 1)(2\delta - \beta\theta + \beta(\delta - \gamma)))^{3/2}\Big) \end{aligned}$$

Since $\gamma - \delta < -4$ we have $\delta > 4$, which, together with $\beta\theta < 2\delta - \beta(\gamma - \delta) + 4(\beta + 1)$, leads to

$$\begin{aligned} 4\delta + 3\beta\delta - \beta\gamma - \beta\theta &> 4\delta + 3\beta\delta - \beta\gamma - 2\delta + \beta(\gamma - \delta) - 4(\beta + 1) \\ &= 2\delta + 2\beta\delta - 4\beta - 4 \\ &= 2\delta - 4 + 2\beta(\delta - 2) > 0, \end{aligned}$$

so $CI_{1,B}$ is strictly increasing with respect to β. we have that the denominator of $\frac{\partial^2 CI_{1,B}}{\partial \beta^2}$ is positive, while $-16(-(\beta+1)(2\delta-\beta\theta+\beta(\delta-\gamma)))^{3/2}<0$. The remaining terms at the numerator can be written as

$$\begin{aligned} f(\gamma) =& (4\beta^3+3\beta^2)\gamma^2+(6\beta^2\theta-42\beta^2\delta-16\beta^3\delta-24\beta\delta+8\beta^3\theta)\gamma \\ &+12\beta^3\delta^2-16\beta^3\delta\theta+4\beta^3\theta^2+51\beta^2\delta^2-42\beta^2\delta\theta+3\beta^2\theta^2+72\beta\delta^2-24\beta\delta\theta+32\delta^2, \end{aligned}$$

which is a convex parabola in γ. From the second and third condition in (35) we must have

$$\gamma_0=\frac{2\delta+\beta\delta-\beta\theta}{\beta}<\gamma<\frac{4\beta+2\delta+\beta\delta-\beta\theta+4}{\beta}=\gamma_1,$$

so

$$f(\gamma_0)=-4\delta^2(\beta+1)^2<0, \quad f(\gamma_1)=-4(\beta+1)^2(\beta(8\delta-16)+\delta^2+12\delta-12)<0,$$

since $\delta>4$ provides $8\delta-16>0$ e $\delta^2+12\delta-12>0$. This guarantees $\frac{\partial^2 CI_{1,B}}{\partial \beta^2}<0$, so $CI_{1,\beta}$ is strictly concave.

The proof of the comparative static of $CI_{0,B}$ is straightforward. □

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