A tâtonnement process with fading memory, stabilization and optimal speed of convergence

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Abstract

The purpose of this work is to provide a way to improve stability and convergence rate of a price adjustment mechanism that converges to a Walrasian equilibrium. We focus on a discrete tâtonnement based on a two-agent, two-good exchange economy, and we introduce memory, assuming that the auctioneer adjusts prices not only using the current excess demand, but also making use of the past excess demand functions. In particular, we study the effect of computing a weighted average of the current and the previous excess demands (finite two level memory) and of all the previous excess demands (infinite memory). We show that suitable weights' distributions have a stabilizing effect, so that the resulting price adjustment process converge toward the competitive equilibrium in a wider range of situations than the process without memory. Finally, we investigate the convergence speed toward the equilibrium of the proposed mechanisms. In particular, we show that using infinite memory with fading weights approaches the competitive equilibrium faster than with a distribution of quasi-uniform weights.

Keywords: Price adjustment, tâtonnement, memory, stabilization, convergence speed, period doubling, bifurcation

JEL classification: C62,C61

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1. Introduction

Over the past decades, a considerable number of studies concerned tâtonnement processes. The classical papers by Arrow and Hurwicz [1], Arrow and Hahn [2], and Negishi [3] proved that the tâtonnement process in continuous time converges to the unique equilibrium price under global gross substitutability. However, if such assumption is removed, the continuous tâtonnement process does not necessarily converge. Smale in [4], Saari and Simon in [5] and Saari in [6] proposed alternative price adjustment mechanisms in order to recover convergence, whose main drawbacks are that they "require a considerable informational requirement" ([5, p. 1097]) and guarantee the convergence of continuous tâtonnement processes only. In fact, as noticed by Weddephol in [7, p. 551], the discrete price adjustment is more troublesome than the continuous one, as "the discrete time tâtonnement process need not converge to an equilibrium, even if the economy satisfies conditions (gross substitutability for example) that guarantee a continuous time tâtonnement to converge." As a result, in the last twenty years several contributions have focused on understanding and studying the instabilities arising in discrete tâtonnement processes. The papers by Saari [6], Day and Pianigiani [8], Bala and Majumdar [9], Weddepohl [10], Tuinstra [11], Mukherji [12] and Kaizoji [13, 14] provide several examples of discrete tâtonnement processes that become unstable and exhibit chaos. In particular, if the trajectories are unstable (either periodic or chaotic) prices do not converge toward equilibrium values and, consequently, transactions cannot take place. It is worth noticing that, even if parameters are such that stability conditions are satisfied and the trajectories converge, the time required to approach equilibrium can become very long when we are close to the instability threshold.

Since, "without an argument establishing the existence of a price adjustment process that converges to Walrasian equilibrium, Walrasian states, even if they exist and are optimal, lose both descriptive and normative relevance" [15, p. 209-210] the above portrayed scenarios started several research efforts in the direction both of finding conditions under which the tâtonnement process converges and of proposing modified price adjustment mechanisms that recover convergence. As an example, we can mention the contributions of Bala, Majumdar and Mitra [16] and of Saari [17], who studied convergence to equilibrium in discrete tâtonnement process. Among the techniques trying to recover equilibrium stability, Weddepohl [18, 10] and Goeree et al. [19] proposed to introduce bounds to price variation, and this allowed the price dynamic to converge or, at least, to be restricted to a neighborhood of the equilibrium price. Conversely, Fujimoto [20] proposed a way to stabilize nonlinear difference equations by introducing a sufficiently long time-lag and showed its effectiveness for the stabilization of Scarf's Examples proposed by Scarf in [21]. Bala and Kiefer [22] proposed a generalization of the families of mechanisms studied by Saari in [6] (an improved version of Newton iterations), Kumar and Shubik [23] tested the effectiveness of two alternative price mechanisms, respectively based on a proportional-integral-derivative (PID) controller and on a Cournot-Shubik mechanism. Several other contributions can be found in the book by Bryant [15].

However, most of the above mentioned mechanisms have unsatisfactory aspects. The most mathematically refined mechanisms, as for example those proposed in [4, 6, 5], as remarked by Saari himself in [5], necessarily require a very large informational endowment (in practice all the gradients of almost all the excess demands of each iterations of the adjustment process), which is unrealistic from the economic point of view. Other stabilization techniques (for example those studied in [18, 10, 19]) are definitely more realistic, but allow stabilizing of smaller class of economies and may be only able to limit price dynamics in a neighborhood of the equilibrium.

Our contribution places in the research strand which investigates alternative discrete price adjustment mechanisms that be able to recover the convergence toward the equilibrium. In particular, we look for a mechanism that be

- *effective*, to allow recovering, at least locally, convergence toward equilibrium for a sufficiently large class of excess demand functions;

- *informationally undemanding*, namely, in being effective it, should not require a too excessive amount of informational endowment of the auctioneer;
- *efficient*, to allow approaching the equilibrium, up to a desired precision, as fast as possible.

We notice that the effectiveness requirement is implicit with respect to the goal of obtaining a price adjustment that converges. The requirement about the auctioneer endowment is *economically* essential, as the mathematical tools exploited for example in [4, 5, 6, 23] can indeed lead to effective mechanisms, which, however, are hardly justifiable from the economic point of view. Finally, efficiency, even if seldom investigated in similar works, is essential to satisfy performance criteria, as noticed by Kumar and Shubik in [23]. In fact, even in a stable context, a too long adjustment process can prevent transactions from taking place in practice.

We pursue the previous aims by studying the effect of memory introduction in the tâtonnement process. To this end, we focus on a setting similar to that considered by Mukherkji [24], who focused on a particular two-person, two-good exchange economy, and we employ a price adjustment mechanism in which a central auctioneer, who aggregates demand and supply and announces a new price so as, to reduce the demand-supply differential, does not only consider the current excess demand but he makes use of a weighted cumulative excess demand, namely a weighted average of the present and past excess demand functions. This means that prices vary with respect to a weighted cumulative excess demand instead of the current excess demand.

We remark that the introduction of memory, which allow taking into account the strategies previously expressed by the agents, is a stabilizing technique used in several economic contexts, as for example in cobweb models by Hommes [25, 26, 27], Bischi and Naimzada [28], Hommes et al. [29], or in evolutionary selection in asset model with heterogeneous beliefs [30].

In our contribution, we focus on two cases:

- the price adjustment mechanism is based on two (the current and the last) excess demands (finite memory);
- the current and all the past excess demands are used (infinite memory).

The weights' distribution is regulated by a memory ratio parameter, which weighs the present and past excess demands. In particular, we will mainly focus on the case of fading weights, i.e. the weights given to the latest excess demand functions are more relevant than those given to the previous ones. The use of weighted averages instead of a simple average is motivated by several considerations. First of all, it seems reasonable that the auctioneer give more credit to the most recent excess demands. Moreover, we show that improved stability properties and/or convergence speed are obtained using weighted averages instead of quasi-uniform averages.

We remark that the possibility to use *finite length* memory in order to obtain a convergent price adjustment is explicitly contemplated in the theoretical price mechanism studied [6]. Memory indeed represents just one of the possible technique to obtain improved price mechanism. We investigate it because it fulfills the three requirements we mentioned above. Firstly, we notice that the approach we propose does not require an elevated informational requirement, which is actually the same of the classical Walrasian tâtonnement. In fact, past excess demands are known and available to the auctioneer, who is simply endowed with a modest amount of supplementary computational capabilities, in order to compute a weighted average of (some or all) the past excess demands. Moreover, we provide a natural way to define weights so that, even if the number of used past excess demands indefinitely grow, the price adjustment can be studied by means of a fixed, low dimensional system.

Moreover, we prove several results to show the effectiveness and efficiency of the proposed mechanism, which concern the stability and convergence speed improvement of the tâtonnement process obtained introducing the weighted memory. After noticing that the steady states of the price mechanism we propose coincide with the Walrasian equilibria, we study their local asymptotic stability when the process without and with memory are considered, showing that local asymptotic stability of equilibria in processes without memory implies the stability of those with memory, if fading weights are used. Then, considering an equilibrium at which the excess demand function is decreasing, we prove that

(i) if the equilibrium is unstable with respect to the price mechanism without memory, the two level memory with suitable memory ratio parameter can allow for recovering equilibrium stability for a limited set of situations, which must be not too far from the stability threshold;

(ii) for suitable memory ratio parameters, the equilibrium is stable in the price adjustment mechanism with infinite memory, even if it is unstable in both processes with and without memory;

(iii) for both finite and infinite memory, when the equilibrium is stable, we find the optimal memory ratio which provides the fastest convergence;

(iv) considering the optimal memory ratio, the tâtonnement processes with (both finite and infinite) memory converge faster than those without memory and than those with infinite memory and quasi-uniform weights.

Finally, we compare the convergence speed of the processes with finite and infinite memory.

The plan of the paper is the following: in Section 2, starting from the example analyzed by Day and Pianigiani [8] and Mukherkji [24], we present the general price mechanism without memory. In Section 3, we show how introducing finite memory, under suitable hypothesis, can improve the stability of equilibria. In Section 4, we propose and study the introduction of infinite length memory. In Section 5 we study and compare the convergence speed of the introduced models.

2. Model without memory

The model studied by Day and Pianigiani in [8] and by Mukherji in [24] takes into account a standard pure exchange economy consisting of two individuals (A, B) and in two goods (x, y). In particular, Mukherji focuses on Cobb-Douglas utility functions

$$u_A(x, y) = x^\alpha y^{1-\alpha}, \quad u_B(x, y) = x^\beta y^{1-\beta},$$

tuned by two possibly different parameters $\alpha, \beta \in (0, 1)$, and he supposes that agents A and B respectively have initial endowment allocations $(x^0, 0)$ and $(0, y^0)$, where x^0, y^0 are both positive quantities. Considering good x, the resulting excess demand function is

$$z(p) = \frac{\beta y^0}{p} - (1 - \alpha)x^0, \tag{1}$$

in which p represents the price of x relative to y. Imposing $z(p) = 0$, we find the unique equilibrium price p^*

$$p^* = \frac{\beta y^0}{(1 - \alpha)x^0}. \tag{2}$$

The goal of Mukherji was to investigate the evolution of prices when a *discrete* adjustment process is considered rather than a continuous one, and he considers the price adjustment process

$$p(t + 1) = p(t) + \gamma z(p(t)) = f(p(t)), \tag{3}$$

where $\gamma > 0$ represent the constant and exogenous speed of adjustment and $z(p(t))$ is the excess demand function. The steady state of iteration (3) coincides with equilibrium (2), which turns out to be locally stable provided that

$$K < 2, \tag{4}$$

where

$$K = \frac{\gamma(x^0(1 - \alpha))^2}{\beta y^0} \tag{5}$$

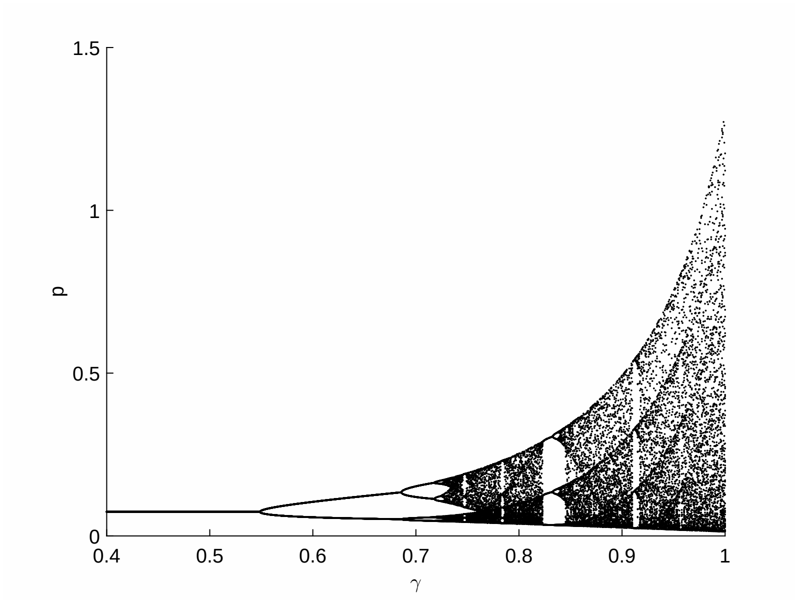

Figure 1: Bifurcation diagram for the model without memory.

is a synthetic parameter which depends on $\alpha, \beta, x^0, y^0, \gamma$ and characterizes the price mechanism. When condition (4) is violated, a flip bifurcation occurs, as shown in Figure 1 for $\alpha = 0.1, \beta = 0.1, x^0 = 0.3, y^0 = 0.2$ on varying γ.

In Sections 3 and 4 we will investigate the effect of replacing in (3) the current excess demand with a suitably weighted average of current and past excess demands. To this end, we consider a general excess demand which, at any equilibrium price p^*, is locally downward sloping

$$z'(p^*) < 0. \tag{6}$$

Also in this more general setting, local stability can be studied in terms of the synthetic positive parameter

$$K = -\gamma z'(p^*) > 0, \tag{7}$$

which is actually a straightforward generalization of (5). We remark that, when (6) is satisfied, the steady states of (3) are stable under condition (4). Conversely, when $z'(p^*) > 0$, equilibrium p^* is always unstable for (3). However, in this case, the approaches we will describe in the following Sections are ineffective and are not able to provide equilibrium stability. Finally, all the results we are going to focus on concern local asymptotic stability.

3. Short memory model

In this section we assume that, in order to adjust the price, the market-maker tries to learn from the last excess demand, using *two* excess demands $z(p(t)), z(p(t-1))$ for the determination of $p(t+1)$. The goal is to investigate if the excess demand at time $t-1$, suitably weighted with respect to that at time t, can be used to provide a modified price mechanism in which the equilibrium price has improved stability properties with respect to (3). We remark that the introduction of finite length memory was studied by Fujimoto in [20], in which, however memory is introduced using a weighted average of the past prices and not of excess demands, as in our case.

3.1. Model construction

Assuming that the auctioneer use a linear convex combination of the excess demands $z(p(t))$ and $z(p(t-1))$, we obtain

$$p(t+1) = p(t) + \gamma\left((1-\rho)z(p(t)) + \rho z(p(t-1))\right) = f_\rho(p(t), p(t-1)), \qquad (8)$$

where $\rho \in [0,1]$ is a *short memory ratio* that "weighs" the amount of memory we use for the determination of $p(t+1)$. When $\rho = 0$, no memory is involved in (8) and so $f_0(p) = f(p)$ and we recover (3). Conversely, if $\rho = 1$ we only use the excess demand at time $t-1$ to compute $p(t+1)$. We are particularly interested in studying (8) for $\rho \in (0, 1/2]$, when the current excess demand has more influence than the previous one on $p(t+1)$, even if (8) makes sense for $\rho > 1/2$ too. We remark that, when $\rho > 0$, two initial values of p are needed to start the process described in (8).

To study the price adjustment mechanism (8), we notice that it can be recast into a two-dimensional system introducing an auxiliary variable $q(t)$ for price

$p(t-1)$, obtaining

$$T_2(p,q) = \begin{cases} p(t+1) = p(t) + \gamma\left((1-\rho)z(p(t)) + \rho z(q(t))\right), \\ q(t+1) = p(t). \end{cases} \tag{9}$$

3.2. Dynamical analysis

Firstly, we notice that the steady states of (9) are exactly the prices for which the excess demand vanishes. In fact, if $z(p^*) = 0$, then, setting $p(t-1) = p^*$ in (9), we have $p(t+1) = p(t) = p^*$. Conversely, if p^* is a steady state for (9), from $p(t+1) = p(t) = q(t) = p^*$ we have $z(p^*) = 0$.

Now we investigate the local stability of the steady states of (9).

Proposition 1. *Let p^* be a steady state of* (9) *at which* (6) *is valid. Then p^* is locally asymptotically stable provided that*

$$\begin{aligned} &0 < K < \frac{2}{1-2\rho} && \textit{when } \rho \le 1/4, \\ &0 < K < \frac{1}{\rho} && \textit{when } \rho > 1/4. \end{aligned} \tag{10}$$

where K is defined by (7).

Proof. To study the stability of (9), we compute the Jacobian matrix of $T_2(p,q)$

$$J(p,q) = \begin{pmatrix} 1 + \gamma(1-\rho)z'(p) & \gamma\rho z'(q) \\ 1 & 0 \end{pmatrix}.$$

We recall that a steady state (p^*, p^*) is stable provided that

$$\begin{cases} 1 - \mathrm{Tr}(J(p^*,p^*)) + \det(J(p^*,p^*)) > 0, \\ 1 + \mathrm{Tr}(J(p^*,p^*)) + \det(J(p^*,p^*)) > 0, \\ 1 - \det(J(p^*,p^*)) > 0. \end{cases} \tag{11}$$

Since

$$J(p^*,p^*) = \begin{pmatrix} 1 - K(1-\rho) & -K\rho \\ 1 & 0 \end{pmatrix}, \tag{12}$$

the first condition of (11) reduces to $K > 0$, which, recalling (7), is automatically satisfied. The second condition reduces to $(2\rho - 1)K + 2 > 0$, which, for $\rho \in [0, 1/2)$, requires

$$K < \frac{2}{1-2\rho}, \tag{13}$$

while the third condition of (11) requires $1 - \rho K > 0$, which is indeed true for $\rho = 0$ and requires

$$K < \frac{1}{\rho} \tag{14}$$

for $\rho > 0$. Noticing that for $\rho \in (0, 1/4)$

$$\frac{2}{1-2\rho} < \frac{1}{\rho},$$

we can conclude. $\square$

We notice that (10) provides the stability region with respect to a given value of ρ. Now we reformulate (10) in order to understand for which values of $\gamma z'(p^*)$ we can find a set of stabilizing memory ratios.

Corollary 1. *Under the hypotheses of Proposition 1, we have that a steady state p^* is stable*

$$\begin{array}{ll} \textit{for all } \rho \in [0,1] & \textit{if } 0 < K < 1 \\ \textit{for } \rho \in [0, 1/K) & \textit{if } 1 \leq K < 2 \\ \textit{for } \rho \in (1/2 - 1/K, 1/K) & \textit{if } 2 \leq K < 4. \end{array}$$

If $K \geq 4$, no memory ratio ρ can be found in order to have a stable equilibrium.

The previous result, which can be easily proved using (10), says that, if $-\gamma z'(p^*) < 4$, we can always find a non-empty interval of values of ρ for which the equilibrium is stable. In what follows, we will use S_2 to indicate the set of $(\rho, -\gamma z'(p^*))$ which satisfies the conditions of Proposition 1. Stability region S_2 is reported in Figure 2. As we can see, for $\rho = 0$ we retrieve the stability region provided by (3), while for $\rho \in (0, 1/2)$, we have that model (9) is stable for a larger interval of values $-\gamma z'(p^*)$. If $\rho \in (0, 1/2)$, the current excess demand $z(p(t))$ has a greater relevance with respect to the previous one $z(p(t-1))$. Finally, we notice that the stability region is maximal when $\rho = 1/4$, for which the curves described by (14) and (13) intersect in $-\gamma z'(p^*) = 4$.

Conversely, increasing memory ratio above $\rho = 1/2$, the stability region reduces, becoming $0 < K < 1$ when $\rho = 1$. The previous results seem to point out that the introduction of memory has a positive stabilizing effect, provided that the coefficients favor the nearest excess demands in time. However, the equilibrium is still unstable if $-\gamma z'(p^*) \geq 4$.

We want to draw the attention on the fact that the stability region provided by Proposition 1 is not symmetric with respect to $\rho = 1/2$, as it is maximized

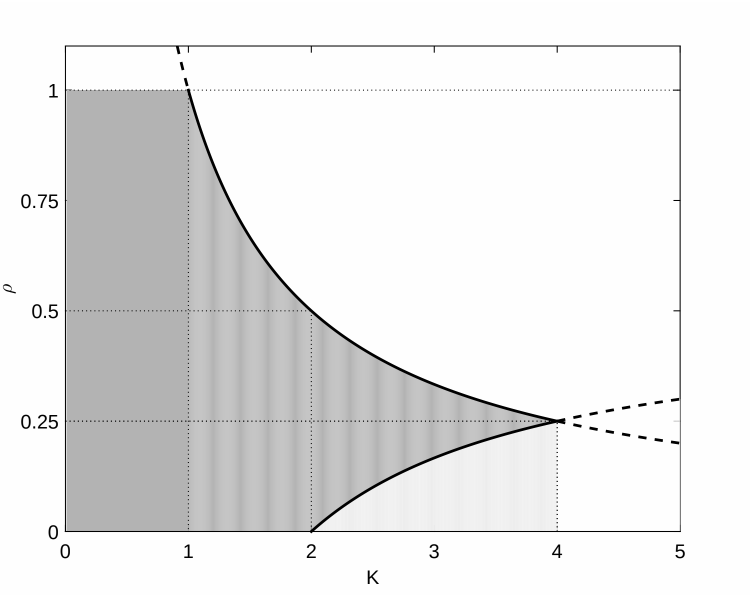

Figure 2: The gray area represents the region S_2 in which system (9) is stable.

for $\rho = 1/4$ and not for the uniform weight distribution. We remark that, as a result of this, giving more relevance to $z(p(t-1))$ with respect to $z(p(t))$ actually reduces the stability region. In the existing literature about iteration processes, (for a detailed survey we refer to the book by Berinde [31]), the most effective distributions of weights are the uniform ones (e.g. for the Mann iterations). However, we remark that, differently than in short memory model (9), in most of the stabilization techniques proposed in [31], a succession of infinite weights is involved and that weights directly affect the state variable, and not a function of it. Conversely, an example of short memory is proposed by Fujimoto in [20], who studies

$$p(t+1) = ap(t) + bp(t-1) + cp(t-2) + \gamma z(p(t)).$$

We remark that, as shown in [20], the previous approach is only effective for a system of dimension greater than 2. The author shows that the previous iterative method converges if a particular non-uniform combination of weights a, b, c is used. We do not further investigate the connection between finite memory and uniform weights' distribution, which we aim to address in future

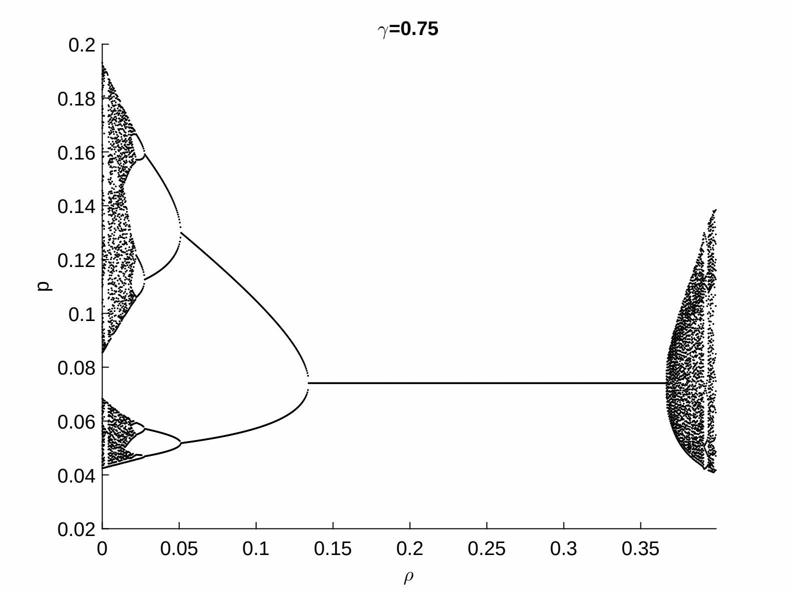

Figure 3: Bifurcation diagram on varying the memory ratio ρ for the model with short memory.

researches.

3.3. Numerical investigations

We present some computational results focusing on the example studied by Mukherji in [24], so we consider the excess demand (1).

Firstly, we investigate what happens when conditions of Proposition 1 are violated. We use the same parameters of the simulation reported in Section 2 and we focus on $\gamma = 0.75$, for which, as reported in Figure 1, the equilibrium is unstable, as we have $K = 2.6973$. For more details about chaotic behaviors of discrete dynamical systems, we refer to the book by Elaydi [32] .

If $K > 2$, when $\rho = 1/2 - 1/K$ the third condition of (11) is violated and a flip bifurcation occurs, as in the original model of Mukherji when $K = 2$. Conversely, if $K > 2$, when $\rho = 1/K$ the second condition of (11) is violated and a Neimark-Sacker bifurcation occurs. The bifurcation diagram obtained on varying ρ is reported in Figure 3. Therefore, we have different chaotic behaviors depending on ρ, K. For decreasingly smaller values of $\rho < 1/2 - 1/K$, the prices predicted by (9) first start oscillating between two or more values, then, after a

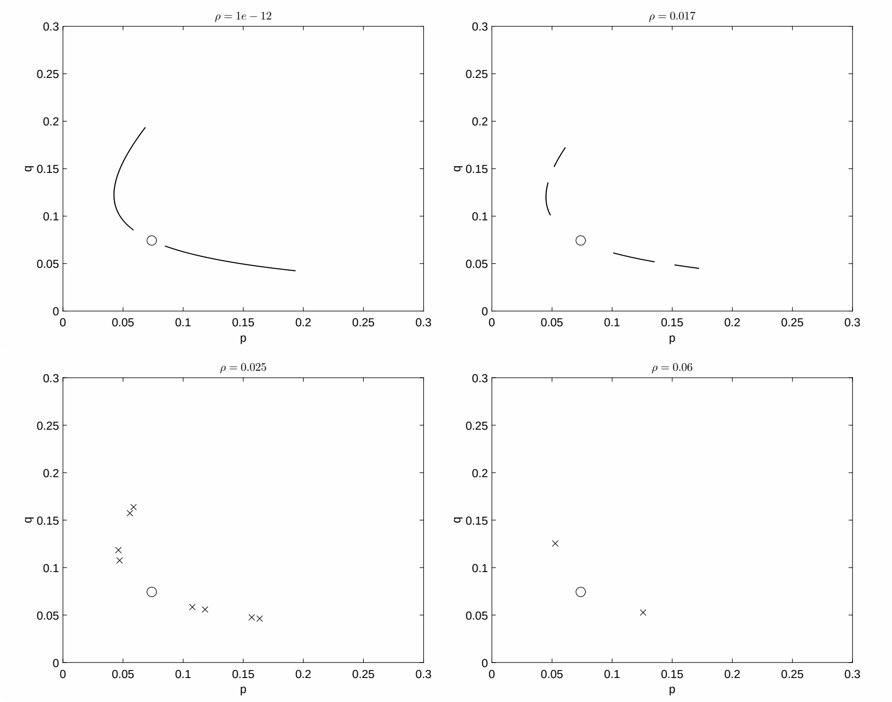

Figure 4: Attractors for model (9) when $\rho < 1/4$ for increasingly large values of ρ. Circle represents the steady state (2)

cascade of flip bifurcations, converge to a chaotic attractor, as shown in Figure 4. Conversely, increasing ρ above $1/K$, prices follow a quasi-periodic evolution and converge toward a closed attracting curve, as shown in Figure 5.

All the previous proofs and simulations show that two level memory allows stabilizing the process only for a limited amount of economies. Moreover, decreasing weights behave better than uniform weights. This suggests to investigate the case in which all the previous excess demands are taken into account, but with fading weights.

4. Infinite memory model

In Section 3 we proved that introducing memory can help to improve equilibrium stability, provided that the latest excess demand $z(p(t))$ has a greater influence than the previous one $z(p(t-1))$. It is predictable that, increasing the

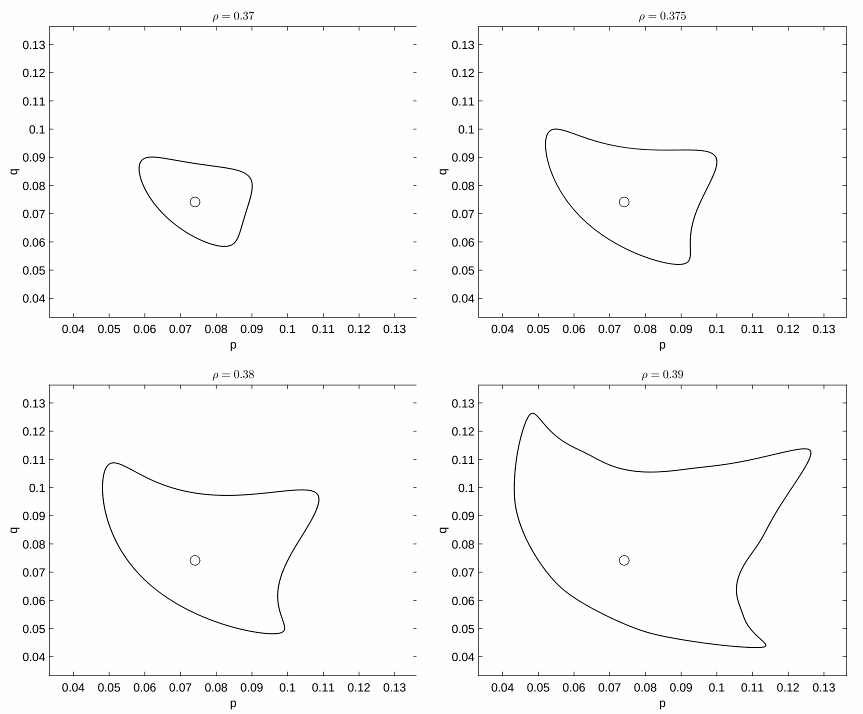

Figure 5: Attractors for model (9) when $\rho > 1/4$ for increasingly large values of ρ. Circle represents the steady state (2)

number of considered past excess demands, stability can be further improved. In particular, may we always obtain stable price adjustments if we take into account, in a suitable way, *all* the previous excess demands?

If, at each time $t > 0$, the auctioneer takes into account all the known excess demands to decide $p(t+1)$, the resulting process can be represented by means of the recurrence relation

$$\begin{aligned} p(t+1) &= p(t) + \gamma \sum_{k=0}^{t} a_{t-k}^{(t)} z(p(t-k)) \\ &= p(t) + \gamma \left(a_t^{(t)} z(p(t)) + a_{t-1}^{(t)} z(p(t-1)) + \cdots + a_1^{(t)} z(p(1)) + a_0^{(t)} z(p(0)) \right), \end{aligned} \tag{15}$$

in which $\{a_k^{(t)}\}_{k=0,\dots,t}$ are nonnegative weights, normalized so that

$$\sum_{k=0}^{t} a_{t-k}^{(t)} = 1.$$

Equation (15) says that, at each time t, the auctioneer adjusts the price on the base of a weighted average of the excess demands $z(p(0)), \dots, z(p(t-1)), z(p(t))$, to which we will refer as *weighted cumulative excess demand*. In particular, when the process starts ($t = 0$), the auctioneer only knows the initial excess demand $z(p(0))$, whose weight is indeed $a_0^{(0)} = 1$. Then, when $t = 1$, he uses both $z(p(1))$ and $z(p(0))$ to decide the new price $p(2)$, using two weights satisfying $a_0^{(1)} + a_1^{(1)} = 1$. We notice that, at each time t, the weights can be collected in a vector $(a_k^{(t)})_{k=0,\dots,t}$ of length $t+1$ and that $a_n^{(t)}$ is the weight given to the excess demand $z(p(n))$ at time t.

We notice that the model is more and more complicated as memory becomes increasingly long, as recurrence (15) is actually a dynamical system of increasing dimension, and that, at least in principle, any distribution of weights could be used.

Among the possible sequence of weights' vectors, we focus on a particular family, regulated by a parameter $\sigma > 0$ and consisting of *exponentially decreasing weights*, similar to that used in [28] for a nonlinear quadratic model with learning. Firstly, we introduce the sequence of unnormalized weights $(\omega_k^t)_{k=0,\dots,t}$. At any fixed time step $t \geq 0$, we set the weight of the current excess demand $z(p(t))$ to 1 ($\omega_t^{(t)} = 1$). Then, we set the remaining t weights so that the ratios between any consecutive weights $\omega_k^{(t)}$ and $\omega_{k+1}^{(t)}$ are identical to the same *infinite memory ratio* $\sigma \in [0, 1)$, namely

$$\frac{\omega_k^{(t)}}{\omega_{k+1}^{(t)}} = \sigma, \quad \forall k = 0, \dots, t, \forall t.$$

This means that, at each time step t, the weights can be collected in the unnormalized vector $\omega^{(t)} = (\omega_k^{(t)})_{k=0,1,\dots,t} \in \mathbb{R}^{t+1}$ which results

$$\omega^{(t)} = (\sigma^{t-k})_{k=0,1,\dots,t} = (\sigma^t, \sigma^{t-1}, \dots, \sigma^2, \sigma, 1). \tag{16}$$

We remark that the weights, ordered from that related to the current excess demand to that related to $z(p(0))$, form an exponentially decreasing sequence, meaning that an exponentially decreasing relevance is given by the auctioneer to the past excess demands. We notice that the unnormalized vectors can be organized in an infinite lower triangular matrix,

$$\begin{array}{cccccc}
1 & & & & & \\
\sigma & 1 & & & & \\
\sigma^2 & \sigma & 1 & & & \\
\vdots & & & \ddots & & \\
\sigma^t & \sigma^{t-1} & \cdots & \cdots & 1 & \\
\vdots & & & & & \ddots
\end{array}$$

in which the n-th row collects the weights of time step n, the j-th column the weights related to excess demand $z(p(j))$ (provided that $j \geq t$) and the diagonal elements collect the weights of the current excess demand. In particular, focusing on the columns of the previous matrix, the relevance given to a particular demand excess decreases exponentially and vanishes as t increases.

To normalize $\omega^{(t)}$ when $t > 0$, we simply need to divide its elements by

$$W(t) = \sum_{k=0}^{t} \sigma^{t-k} = \frac{1 - \sigma^{k+1}}{1 - \sigma} \tag{17}$$

which represents the unnormalized cumulative weight at time t.

Going back to equation (15), an exponentially fading weights' distribution is realized by choosing

$$a_k^{(t)} = \frac{\omega_k^{(t)}}{W(t)}. \tag{18}$$

In the following proposition, we show that exponentially fading weights allow rewriting (15) into an equivalent three dimensional system. To do this, we use the unnormalized cumulative weights defined by (17) and the weighted cumulative excess demand variable

$$Z(t) = \frac{1}{W(t)} \sum_{k=0}^{t} \omega_k^{(t)} z(p(k)). \tag{19}$$

Proposition 2. *Consider a price adjustment mechanism in which, at time t, each excess demand $z(p(s))$ for $s = 0, \ldots, t$ is taken into account using normalized weights*

$$\frac{\omega_k^{(t)}}{\sum_{k=0}^{t} \omega_k^{(t)}}, \tag{20}$$

where $\omega_k^{(t)}$ *are defined by* (16). *Then, equation* (15) *is equivalent to the following three-dimensional dynamical system*

$$T_\infty(p, Z, W) = \begin{cases} p(t+1) = p(t) + \gamma Z(t), & \text{(21a)} \\ Z(t+1) = \dfrac{\sigma W(t) Z(t) + z(p(t) + \gamma Z(t))}{1 + \sigma W(t)}, & \text{(21b)} \\ W(t+1) = 1 + \sigma W(t). & \text{(21c)} \end{cases}$$

provided that we set $W(0) = 1$ *and* $Z(0) = z(p(0))$, *where* $z(p(0))$ *is the given initial excess demand.*

Proof. Equation (21c) can be obtained by evaluating (17) at $t+1$ and noticing that

$$W(t+1) = \sum_{k=0}^{t+1} \sigma^{t+1-k} = 1 + \sum_{k=0}^{t} \sigma^{t+1-k} = 1 + \sigma W(t). \tag{22}$$

The price adjustment mechanism can be achieved from (15) and (18), obtaining

$$\begin{aligned} p(t+1) &= p(t) + \gamma \sum_{k=0}^{t} \frac{\omega_k^{(t)}}{W(t)} z(p(k)) \\ &= p(t) + \gamma \sum_{k=0}^{t} \frac{\sigma^{t-k}}{W(t)} z(p(k)). \end{aligned} \tag{23}$$

Recalling (16), the weighted cumulative excess demand function (19) can be rewritten as

$$Z(t) = \frac{1}{W(t)} \sum_{k=0}^{t} \sigma^{t-k} z(p(k)), \tag{24}$$

which allows rewriting (23) in terms of the new variable $Z(t)$, obtaining

$$p(t+1) = p(t) + \gamma Z(t). \tag{25}$$

Writing (24) at time $t+1$ we have

$$\begin{aligned} Z(t+1) &= \frac{1}{W(t+1)} \sum_{k=0}^{t+1} \sigma^{t+1-k} z(p(k)) \\ &= \frac{1}{W(t+1)} \left[z(p(t+1)) + \sum_{k=0}^{t} \sigma^{t+1-k} z(p(k)) \right] \\ &= \frac{1}{W(t+1)} \left[z(p(t+1)) + \sigma \sum_{k=0}^{t} \sigma^{t-k} z(p(k)) \right]. \end{aligned} \tag{26}$$

Finally, substituting (22), (25) and (24) into (26), we have the evolution equation for the cumulative excess demand function

$$Z(t+1) = \frac{\sigma W(t) Z(t) + z(p(t) + \gamma Z(t))}{1 + \sigma W(t)}. \tag{27}$$

The infinite memory model is then the three-dimensional system made by (22), (25) and (27). $\square$

Thanks to the choice of exponentially fading weights, Proposition 2 allows us to state that we only need one more variable than in the short memory model to describe and study the evolution of p. Moreover, process (21) needs only one price value to start.

First of all we notice that System (21) is formally defined also for $\sigma = 0$, but in this case, since $\omega_t^{(t)} = 1$ and $\omega_s^{(t)} = 0$ for $s < t$, we find again (3). For small values of σ, we have that σ^{t-s} decreases rapidly as s is far from t, and in this case only a few excess demands are significantly taken into account for the determination of $p(t+1)$. In this case, the infinite memory model can be seen as an approximation of a particular short memory model with suitable finite memory. On the contrary, when $\sigma \approx 1$, several $z(p(t-s))$ have substantial influence, as they have similar weights. When $\sigma \to 1$, we have that the weights tend to the uniform distribution $\omega_k^t = 1/(t+1)$. System (21) can not be considered for $\sigma = 1$, which would give rise to a uniform weights' distribution. However, if σ is sufficiently close to 1, a suitable finite set of weights related to the most recent excess demands are very close to an uniform average. In this sense, system (21) can provide a suitable approximation of a uniform weights' distribution, which we will call quasi-uniform.

4.1. Dynamical analysis

Steady states of system (21) are characterized by the following proposition.

Proposition 3. *Steady states of system* (21) *are characterized by*

$$p^*, \quad Z^* = 0, \quad W^* = \frac{1}{1-\sigma}. \tag{28}$$

where p^* are such that $z(p^*) = 0$.

Proof. If in (21) we set $(p(s), Z(s), W(s)) = (p^*, Z^*, W^*)$ for $s = t, t+1$, by the first equation we have $Z^* = 0$ and the third equation gives $W^* = 1/(1-\sigma)$. Substituting Z^* into the second equation, we obtain $z(p^*) = 0$. $\square$

We underline that the price component of a steady state of (21) is a Walrasian equilibrium ($z(p^*) = 0$) and that the cumulative excess demand Z^* is null too.

To study the stability of steady states, we make use of the conditions for a three-dimensional discrete dynamical system obtained in [33]

$$\begin{cases} 1 - |\det(J)| > 0, \\ 1 - \mathrm{M}(J) + \mathrm{tr}(J)(\det(J)) - (\det(J))^2 > 0, \\ 1 + \mathrm{M}(J) - \mathrm{tr}(J) - \det(J) > 0, \\ 1 + \mathrm{M}(J) + \mathrm{tr}(J) + \det(J) > 0, \end{cases} \tag{29}$$

where M is the sum of the three principal minors of the Jacobian matrix J. We remark that also in this case we need to assume hypothesis (6) and that stability can be expressed in terms of (7).

Proposition 4. *Let p^* be a steady state of* (9) *in which* (6) *is valid. Then p^* is locally asymptotically stable provided that*

$$K < \frac{2(\sigma+1)}{1-\sigma}. \tag{30}$$

Proof. The Jacobian matrix of System (21) is

$$J(p, Z, W) = \begin{pmatrix} 1 & \gamma & 0 \\ \dfrac{z'(p+\gamma Z)}{W\sigma+1} & \dfrac{\gamma z'(p+Z\gamma)+W\sigma}{W\sigma+1} & \dfrac{\sigma Z - \sigma z(p+Z\gamma))}{(W\sigma+1)^2} \\ 0 & 0 & \sigma \end{pmatrix}$$

which, evaluated at (28), gives

$$J_f = J(p^*, 0, 1/(1-\sigma)) = \begin{pmatrix} 1 & \gamma & 0 \\ \dfrac{K(\sigma-1)}{\gamma} & (K+1)\sigma - K & 0 \\ 0 & 0 & \sigma \end{pmatrix}. \tag{31}$$

Since $\det(J_f) = \sigma^2$, the first condition of (29) is satisfied, as $\sigma \in [0, 1)$. The second condition of (29) is equivalent to $(\sigma-1)^2(1-\sigma^2+K\sigma) > 0$ and it is verified since $\sigma \in [0, 1)$. The third condition is equivalent to $K(\sigma-1)^2 > 0$, which is true thanks the hypothesis on the parameters and $\sigma < 1$. The last

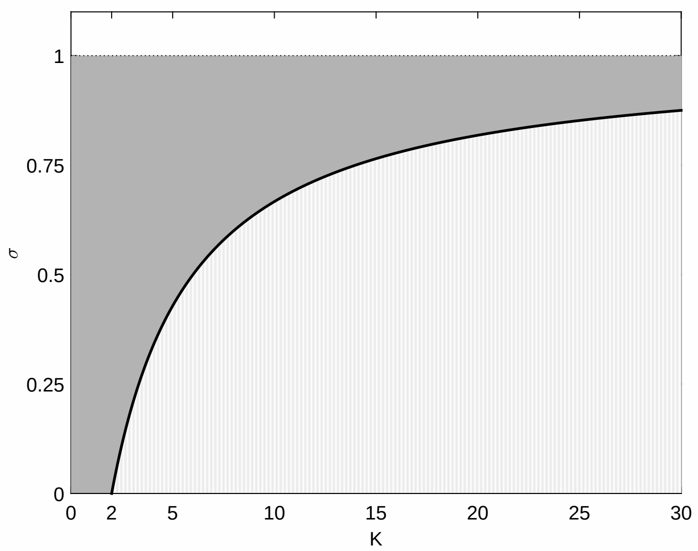

Figure 6: The gray area represents the region S_∞ in which system (21) is stable.

condition of (29) is equivalent to $(K+2)\sigma^2 + 4\sigma - K + 2 > 0$ and can be simplified as

$$2(\sigma + 1) + K(\sigma - 1) > 0,$$

which gives (30). □

The stability region given by the previous Proposition, which will be indicated in the following with S_∞, is shown in Figure 6. The main consequence of (30) is that for *any* economy and reaction parameter γ, we can *always* find an interval of values for σ so that the price adjustment mechanism is stable. In fact, we can rewrite Proposition 4 as

Corollary 2. *For any given $K > 0$, we have that an equilibrium p^* is locally asymptotically stable*

$$\begin{array}{ll} \textit{for all } \sigma \in [0, 1) & \textit{if } K < 2, \\ \textit{for } \sigma \in ((K-2)/(K+2), 1) & \textit{if } K \geq 2. \end{array}$$

Hence, the mechanism we introduced in this section, in which we take into account all the available weighed excess demand, is able to produce a stable price adjustment mechanism for any excess demand function that satisfies (6).

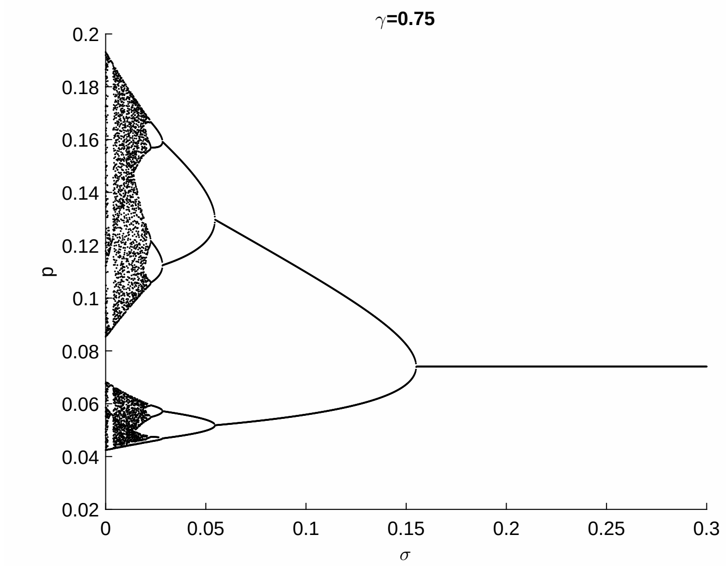

Figure 7: Bifurcation diagram for the model with infinite memory.

4.2. Numerical investigations

In this Section, we investigate the behavior of (21) when stability condition is violated. Again, we consider the example studied in Section 2. As we can see in Figure 7, when $\sigma = (K-2)/(K+2)$ a cascade of period doubling bifurcations starts, leading to chaos. Only the leftmost part of the bifurcation diagram is similar to that reported in Figure 3, since in the present case, as σ increases, we always have convergence. For small memory ratios, the chaotic behavior of (21) is similar to that of (8) and as shown by the chaotic/periodic attractors reported in Figure 8. We remark that since W converges to $1/(1-\sigma)$, the curves of Figure 8 lie on a plane $W = const$.

5. Convergence speed

In the previous sections we showed that the introduction of memory allows stabilizing the model and increasing the stability region of K up to 4, using short memory, or to $+\infty$ with infinite memory. However, as well as stability, it is important that a price mechanism, when convergent, be able to approach the equilibrium prices as fast as possible, involving the minimum number of

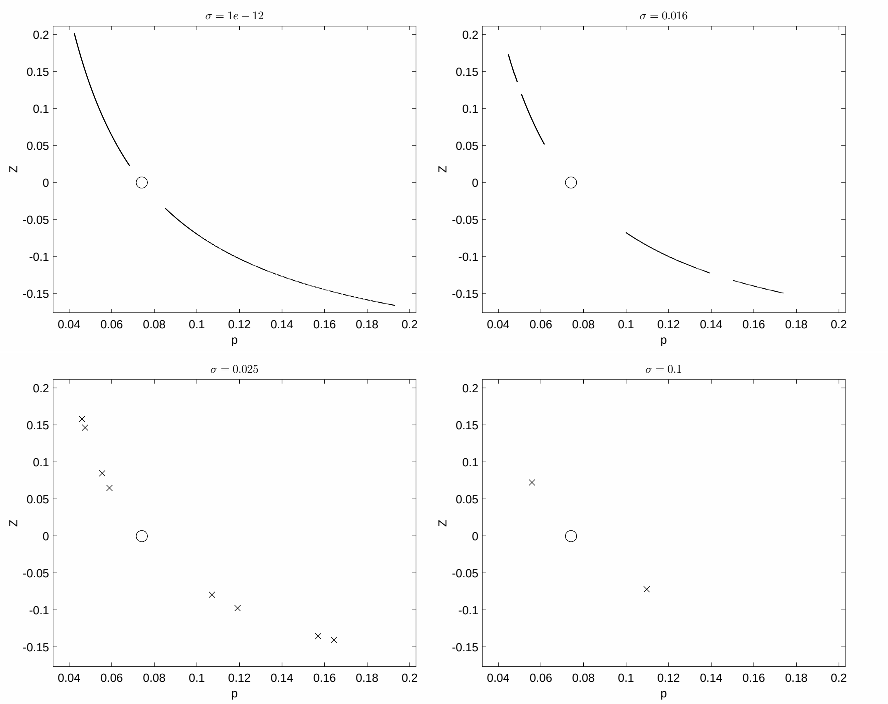

Figure 8: Attractors for model (21) when $\sigma < (K-2)/(K+2)$. Circle represents steady state (28)

iterations, in order to allow transactions. How can we evaluate the convergence speed of an iterative map? According to the literature about fixed point iterations (see for example [34]), for a one-dimensional map $x(t+1) = f(x(t))$ we can define the asymptotic convergence factor

$$C = |f'(x_f)|, \tag{32}$$

where x_f is the (stable) fixed point of the iteration map, while for multidimensional maps $\vec{x}(t+1) = g(\vec{x}(t))$, it is defined by

$$C = \rho(J(\vec{x}_f)), \tag{33}$$

where $\rho(J) = \max_i\{|\lambda_i|\}$ is the spectral radius of the Jacobian matrix J of map g, being λ_i the eigenvalues of J.

We notice that, thanks to the stability of x_f, we have $C < 1$. Moreover, $C \neq 0$ represents the (asymptotic) reduction factor of the absolute error at each iteration. For example, in the one dimensional case, we have

$$|e(t+1)| < C|e(t)|,$$

where $e(t) = x(t) - x_f$. We remark that when $C \neq 0$, the convergence is linear, while for $C = 0$ we have a superlinear convergence. Moreover, the convergence to the steady state is faster as C gets smaller.

In the following sections we want to compute and compare C for (3), (9) and (21). In particular, since for both (9) and (21) we have that an equilibrium is stable for a range of memory ratios (ρ or σ), we will investigate for which memory ratio we obtain the smallest asymptotic convergence factor. We will refer to this value as the *optimal memory ratio*, which gives the corresponding *optimal convergence factor*. The proofs of the following Propositions can be found in Appendix. Again, we can express the results in terms of K, under condition (6) on the excess demand function at the equilibrium.

5.1. Asymptotic convergence rates
Model without memory

For the original model of Mukherji (3), we have the following result.

Proposition 5. *When an equilibrium point is stable for* (3) *($K < 2$), the asymptotic convergence factor C_0 is*

$$C_0 = |1 - K|. \tag{34}$$

Short memory model

For the short memory model (9) we start computing the asymptotic convergence factor on varying ρ. Let us introduce

$$\Delta_2 = (1 - K + K\rho)^2 - 4K\rho,$$

and

$$\rho^* = 1 + \frac{1}{K} - \frac{2}{\sqrt{K}},$$

for which, when $K \in (0, 4)$, we have

$$\frac{1}{2} - \frac{1}{K} < \rho^* < \frac{1}{K}. \tag{35}$$

We have

Proposition 6. *The asymptotic convergence factor* C_2 *for* (9) *is*

$$C_2(\rho,K) = \begin{cases} \dfrac{1-K+K\rho+\sqrt{\Delta_2}}{2} & \textit{if } K\in(0,1], 0\le\rho\le\rho^* \quad (36a)\\ \dfrac{K-1-K\rho+\sqrt{\Delta_2}}{2} & \textit{if } K\in(1,4), \frac{1}{2}-\frac{1}{K}<\rho\le\rho^* \quad (36b)\\ \sqrt{K\rho} & \textit{if } K\in(0,4), \rho^*<\rho<\frac{1}{K}, \quad (36c)\end{cases}$$

provided that $\rho\in[0,1]$.

We remark that function (36) is continuous, in particular for $\rho=\rho^*$. Now we are able to find, for any given $K\in(0,4)$, the value of ρ which gives the optimal convergence factor.

Proposition 7. *For* $K\in[0,4)$, *the optimal memory ratio is given by*

$$\rho^{opt}(K) = 1+\frac{1}{K}-\frac{2}{\sqrt{K}}, \tag{37}$$

to which corresponds the optimal asymptotic convergence factor

$$C_2^{opt}(K) = |1-\sqrt{K}|. \tag{38}$$

Infinite memory model

For the infinite memory model (21), we start computing the asymptotic convergence factor on varying σ. Let us introduce

$$\Delta_\infty = (1-\sigma)[(K-1)^2-\sigma(K+1)^2],$$

and

$$\sigma^* = \left(\frac{K-1}{K+1}\right)^2.$$

We notice that $\sigma^*\in S^\infty$ since

$$\frac{K-2}{K+2}<\sigma^*.$$

We have

Proposition 8. *The asymptotic convergence factor* C_∞ *for* (21) *is*

$$C_\infty(\sigma,K) = \begin{cases} \dfrac{(\sigma+1)-K(1-\sigma)+\sqrt{\Delta_\infty}}{2} & K\in(0,1], 0\le\sigma\le\sigma^* \quad (39a)\\ \dfrac{K(1-\sigma)-(\sigma+1)+\sqrt{\Delta_\infty}}{2} & K>1, \frac{K-2}{K+2}<\sigma\le\sigma^* \quad (39b)\\ \sqrt{\sigma} & K>0, \sigma>\sigma^* \quad (39c)\end{cases}$$

provided that $\sigma\in[0,1)$.

We remark that also function (39) is continuous, in particular for $\sigma = \sigma^*$. As in the case of short memory, we can identify, for any given $K > 0$, the value of σ to which the optimal convergence factor corresponds.

Proposition 9. *For $K > 0$, the optimal memory ratio is given by*

$$\sigma^{opt}(K) = \left(\frac{K-1}{K+1}\right)^2 \tag{40}$$

to which corresponds the optimal asymptotic convergence factor

$$C_\infty^{opt}(K) = \left|1 - \frac{2}{1+K}\right|. \tag{41}$$

In both (34), (38) and (41) the asymptotic convergence factor vanishes for $K = 1$ and the convergence of the fixed point iteration is superlinear. In this case (32) gives no information about the convergence speed and one should investigate the convergence order of the fixed point iteration. But for $K = 1$, both (9) and (21) achieve their optimal convergence factors (38) and (41) for $\rho = \sigma = 0$, so they actually reduces to the model without memory. On the basis of this, we do not investigate further the case $K = 1$.

5.2. Convergence speed comparison

The comparison of the asymptotic convergence factor of the model without memory (34) and of the optimal convergence factors for the model with short memory (38) and infinite memory (41) is made in the following Proposition. We recall that convergence is faster when (32) is smaller.

Proposition 10. *For all $K \in (0, 2), K \neq 1$, convergence is slower for the model with no memory with respect to those with short and infinite memory. When $0 < K < 1$, the model with short memory converges faster than that with infinite memory, while for $1 < K < 4$ the model with infinite memory is faster.*

To prove the previous Proposition it is sufficient to compare C_0, C_2^{opt} and C_∞^{opt}. We omit the details and we report in Figure the plots of 9 (34), (38) and (41). Proposition 10 says that the introduction of a certain level of memory always allows increasing the convergence speed with respect to the model without memory. Moreover,if K is small, the short memory model is more efficient, while the adjustment process based on infinite memory should be preferred for

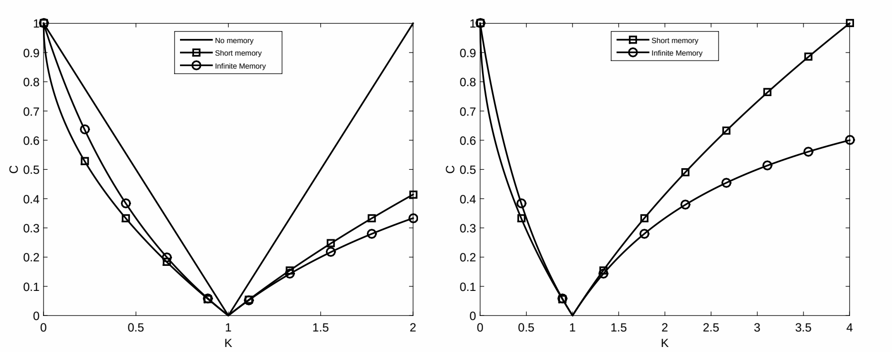

Figure 9: Comparison of the asymptotic convergence factors for all the models when $0 < K < 2$ (left plot) and for the models with memory when $0 < K < 4$ (right plot).

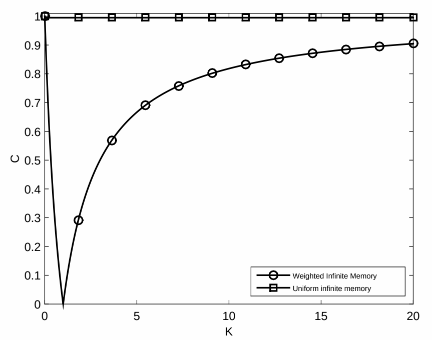

Figure 10: Comparison of the asymptotic convergence factors for infinite memory process with quasi-uniform and fading weights, which give faster convergence.

$K > 1$. Finally, if we consider quasi-uniform weights, we can see from Figure 10 that the price adjustment obtained with optimal memory ratio is always much faster.

6. Concluding remarks

We proposed two alternative tâtonnement processes based on the introduction of memory to improve the stability and the convergence speed to the equilibrium price of a two-agent, two-good exchange economy. Introducing short memory, we proved that the stability of the price mechanism can be guaranteed for a larger bounded set of conditions, for suitable choices of the memory ratio. With infinite memory we was able to find a set of memory ratio values for which the equilibrium is stable for all economies. Moreover, both strategies allowed to accelerate the convergence to the equilibrium with respect to the model with no memory. We aim to extend this approach to more general economies and to investigate and compare different stabilization techniques. In particular, we want to study price adjustment mechanisms obtained by replacing the true unknown excess demand with a suitable approximation (for example by interpolating some past excess demands). Moreover, we aim to deepen the understanding of the results about the improved stability properties of the fading short memory distribution with respect to the uniform memory distribution.

Acknowledgments

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Appendix A. Proofs

Proof of Proposition 5. We just need to notice that, from (3), we have $f'(p^*) = 1 - K$. □

Proof of Proposition 6. To estimate the spectral radius of (12), we need to compute its eigenvalues. In what follows, we assume that (ρ, K) belongs to stability region S_2. The characteristic polynomial of (12) is

$$\lambda^2 + (K(1-\rho) - 1)\lambda + K\rho, \tag{A.1}$$

and, since for $\rho \in [0, 1]$ we have $\Delta_2 \geq 0$ provided that

$$\rho \leq \rho^* = 1 + \frac{1}{K} - \frac{2}{\sqrt{K}}, \tag{A.2}$$

the zeros of (A.1) are the real values

$$\lambda_{+,-} = \frac{1 - K + K\rho \pm \sqrt{\Delta_2}}{2},$$

when (A.2) is valid, while are the couple of complex conjugate values

$$\lambda_{1,2} = \frac{1 - K + K\rho \pm i\sqrt{-\Delta_2}}{2},$$

when (A.2) is violated.

We need to find the largest in modulus eigenvalue. When $\Delta_2 < 0$ we have that

$$|\lambda_1| = |\lambda_2| = \frac{1}{2}\sqrt{(1 - K(1-\rho))^2 - \Delta_2} = \sqrt{K\rho},$$

which, recalling (35), gives (36c).

When $\Delta_2 \geq 0$, we need to distinguish between two cases. First we consider $K \in (0, 1]$. Thanks to 1, we have $(\rho, K) \in S_2$ if $\rho \in [0, 1]$, so $\lambda_+ \geq \lambda_- \geq 0$. In fact $\lambda_- \geq 0$, since we have $1 - K + K\rho \geq 0$ and $1 - K + K\rho \geq \sqrt{\Delta_2}$ is true since it is equivalent to

$$K\rho \geq 0. \tag{A.3}$$

This allows obtaining (36a).

Conversely, when $1 < K < 4$ we have $\lambda_- < \lambda_+ < 0$. In fact $\lambda_+ < 0$, since we have that $1 - K + K\rho < 0$ is true as we are considering ρ such that

$$\rho \leq \rho^\star = 1 + 1/K - 2/\sqrt{K} < (K - 1)/K$$

Moreover, $-(1 - K(1 - \rho)) \geq \sqrt{\Delta_2}$ reduces again to (A.3). Hence $\rho(J_2) = |\lambda_-|$ and this gives (36b). □

Proof of Proposition 7. We notice that for a fixed K, $C_2(\rho, K)$ is decreasing for $\rho < \rho^*$. In fact, if $K \in (0, 1)$, we consider (36a) and

$$\partial_\rho C_2(\rho, K) = \frac{K\sqrt{\Delta_2} - K(K(1 - \rho) + 1)}{2\sqrt{\Delta_2}},$$

which is negative since

$$K\sqrt{\Delta_2} - K(K(1 - \rho) + 1) < 0 \Leftrightarrow -4K^3 < 0.$$

If $K \in (1, 4)$, we have to consider (36a), so

$$\partial_\rho C_2(\rho, K) = \frac{-K\sqrt{\Delta_2} - K(K(1 - \rho) + 1)}{2\sqrt{\Delta_2}}$$

which is negative.

Conversely, when $\rho > \rho^*$, $C_2(\rho, K) = \sqrt{K\sigma}$ is indeed increasing. This allows concluding that, for each K, the minimum is attained for $\rho = \rho^*$. Expression (38) is a straightforward consequence of (37). □

Proposition 8. We compute the eigenvalues of (31), considering (ρ, K) belonging to stability region S_∞. The characteristic polynomial of (31) is

$$-(\lambda - \sigma)\left(-\lambda^2 + \lambda(1 - K + \sigma + K\sigma) - \sigma\right) \tag{A.4}$$

which always has solution $\lambda_1 = \sigma$. Since for $\sigma \in [0, 1]$ we have $\Delta_\infty \geq 0$ provided that

$$\sigma \leq \sigma^* = \left(\frac{K - 1}{K + 1}\right)^2, \tag{A.5}$$

the zeros of (A.4) are the real values

$$\lambda_{+,-} = \frac{1 - K + \sigma(1 + K) \pm \sqrt{\Delta_\infty}}{2},$$

when (A.5) is valid, while are the couple of complex conjugate values

$$\lambda_{2,3} = \frac{1 - K + \sigma(1+K) \pm \sqrt{\Delta_\infty}}{2},$$

when (A.5) is violated.

We need to find the largest in modulus eigenvalue. When $\Delta_\infty < 0$ we have that

$$\sigma < |\lambda_1| = |\lambda_2| = \frac{1}{2}\sqrt{(1 - K + \sigma(1+K))^2 - \Delta^2} = \sqrt{\sigma},$$

which gives (39c).

When $\Delta_\infty \geq 0$, we have to distinguish between two cases. First we consider $K \in (0, 1]$. Thanks to Corollary 2, we have $(\sigma, K) \in S_\infty$ if $\sigma \in [0, 1]$, so $\lambda_+ \geq \lambda_- \geq 0$. In fact, $\lambda_- \geq 0$, since $1 - K + \sigma(1 + K) \geq 0$, and

$$1 - K + \sigma(1-\rho) \geq \sqrt{\Delta_\infty}$$

is true since it is equivalent to

$$\sigma \geq 0.$$

Since

$$\lambda_+ - \sigma = \frac{(1-K)(1-\sigma) + \sqrt{\Delta_\infty}}{2} > 0,$$

we can conclude that $\rho(J_\infty) = \lambda_+$ and we obtain (39a).

Conversely, when $K > 1$ we have $\lambda_- < \lambda_+ < 0$. In fact $\lambda_+ < 0$ since we have that $1 - K + \sigma(1 + K) < 0$ is true as we are considering σ such that

$$0 < \sigma \leq \left(\frac{K-1}{K+1}\right)^2 < \frac{K-1}{K}.$$

Moreover, $-(1 - K + \sigma(1 + K)) < \sqrt{\Delta}$ is again equivalent to $\sigma > 0$.

Finally, since

$$|\lambda_-| - \sigma = \frac{-(K+3)\sigma + K - 1 + \sqrt{\Delta_\infty}}{2},$$

and since $-(K + 3)\sigma + K - 1 > 0$ for

$$\sigma \leq \left(\frac{K-1}{K+1}\right)^2 < \frac{K-1}{K+3},$$

we have that $|\lambda_-| > \sigma$ and (39b) is valid. □

Proposition 9. We notice that for a fixed K, $C_\infty(\sigma, K)$ is decreasing for $\sigma < \sigma^*$. In fact, if $K \in (0, 1)$, we consider (39a) and

$$\partial_\sigma C_\infty(\sigma, K) = \frac{(K+1)\sqrt{\Delta_\infty} + \sigma(1+K)^2 - K^2 - 1}{2\sqrt{\Delta_\infty}},$$

is negative since $\sigma(1+K)^2 - K^2 - 1 < 0$ as

$$\sigma \leq \sigma^* < \frac{K^2+1}{(K+1)^2},$$

and

$$(K+1)\sqrt{\Delta_2} + \sigma(1+K)^2 - K^2 - 1 < 0 \Leftrightarrow -4K^2 < 0.$$

If $K > 1$, we have to consider (39b), so

$$\partial_\sigma C_\infty(\sigma, K) = \frac{-(K+1)\sqrt{\Delta_\infty} + \sigma(1+K)^2 - K^2 - 1}{2\sqrt{\Delta_\infty}}$$

which is negative.

Conversely, when $\sigma > \sigma^*$, $C_\infty(\sigma, K) = \sqrt{\sigma}$ is indeed increasing. This allows concluding that, for each K, the minimum is attained for $\sigma = \sigma^*$. Expression (41) is a straightforward consequence of (40). $\square$

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