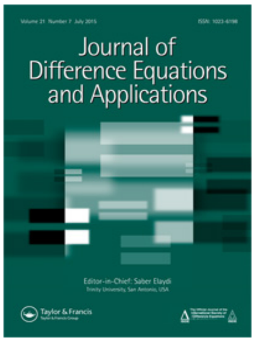

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Two different routes to complex dynamics in an heterogeneous triopoly game

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We study a triopoly game with heterogeneous players. The market is characterized by a nonlinear (isoelastic) demand function and three competitors. The main novelty is the double route to complex dynamics that we find and is quite rare in heterogeneous triopoly models. We show that the two routes have important implications for the economic interpretation of the dynamics emerging when the Cournot–Nash equilibrium becomes locally unstable. Moreover the model displays multistability of different attractors, requiring a global analysis of the dynamical system.

Keywords: game theory; industrial organization; triopoly game; heterogeneous players; global analysis

MSC (2010) Classification: 37G35; 65Q10

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1. Introduction

Oligopoly is the market structure in which the consequences of the bounded rationality of economic agents are more evident. In this kind of markets a higher level of rationality is required in order to make the best choice. In fact, firms do not only have to know the shape of the demand function, but they also have to be able to foresee the output choices of the competitors, because they are in a situation of *strategic interdependence* caused by the influence of each single firm on the market price. In the literature, many papers are devoted to the development and the analysis of the simplest oligopolistic case: duopoly. Both homogeneous and heterogeneous firms cases are considered[1] (see [2,7,9,11,17,20,21,26] for a few papers on homogeneous duopolies, [10] for a recent survey and [4,5,8,24] for heterogeneous duopolies). The authors of these papers underscore the complicated (and complex) dynamics that may emerge whenever firms have some degree of bounded rationality. A more complicated case is the case of triopoly. Differently with respect to duopolies, the case with three firms is not present in the literature as well. One of the main reasons behind this low number of triopoly papers lies on the complexity of the models describing the dynamics of the quantities that must be at least three-dimensional. Only under some particular restriction it is still possible to analytically study this kind of models. For example, it is almost always possible to deal with the equations governing the dynamics of homogeneous triopolies (see [1,3,6,22,23] for some examples), at least for what concerns the local stability of Nash equilibria (NEs). More difficult to study, but more realistic, is the case with three heterogeneous firms. In such a case the study must be necessarily performed by instruments both

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analytic and numeric. The development of quite powerful computers permits now to be able to handle this case (see [14–16]).

To the best of our knowledge, the heterogeneous triopoly game developed by [25] is the only one displaying a route to complex dynamics different from the cascade of period-doubling bifurcations scenario. Such a case only produces realistic disequilibrium dynamics for the combinations of parameters for which a chaotic attractor is reached. In fact, it appears quite unrealistic in the long period the persistence of periodic dynamics. Even if the firms are assumed to be boundedly rational, it seems reasonable that they are able to recognize a periodic path, modifying as a consequence their decisional process. In heterogeneous duopoly games [4,8,24] the possibility that the NE loses stability via flip bifurcation is always accompanied with the possibility to observe a Neimark–Sacker (NS) bifurcation, for some values of the parameters. This is, in our opinion, a quite realistic scenario in which, after the bifurcation, orbits display a quasiperiodic motion, that require a higher degree of rationality for the firms in order to be recognized. In [8,24] the double route to complex dynamics seems to be somehow related to the assumption of isoelastic demand function. In the present model we analyse an heterogeneous triopoly model taking [15] as a benchmark. We adopt the following alternative assumptions:

- We adopt a microfounded isoelastic demand function (see [21]) and linear costs, instead of a linear demand function and quadratic costs.
- We assume a different decisional mechanism for one of the three players: instead of the adaptive player we introduce a firm that approximates the demand function linearly around the last realized couple of quantity and market price (see [12,18–20,27]).

We show that these assumptions permit to obtain the double route to chaos already founded in some heterogeneous duopoly and in [25], where the demand function is again isoelastic. Our work is also related to the triopoly game studied by [25], that shares with our model the demand and cost functions, but differs with respect to the decisional mechanism adopted by oligopolists. This is quite important, because we conjecture that the emergence of the two routes to chaotic dynamics in heterogeneous oligopolies is not related to the number of players or their decisional mechanisms, but to the particular form of nonlinearity of the demand function (i.e. isoelastic). Moreover, the main novelty of our triopoly game with respect to the other heterogeneous triopoly games already existing in the literature consists in the fact that we numerically found multistability of different coexistent attractors. We perform a global analysis through numerical simulations in order to identify the basins of attraction of the attractors, that is the initial conditions leading to one attractor or the other. The paper is organized as follows: in Section 2 we introduce the model whose NE is obtained in Section 3 together with its local stability. Section 4 is devoted to the close examination of the flip bifurcation of the NE. Section 5 concerns the NS bifurcation of the NE, while in Section 6 we show the ambiguous role of the marginal costs. Multistability and some numerical global analysis are given in Section 7. Section 8 concludes.

2. The model

Let us consider a market populated by three firms producing homogeneous goods. The demand function is isoelastic, implying the hypothesis of Cobb–Douglas utility function of the consumers (see [21]):

$$p = f(Q) = \frac{1}{Q} = \frac{1}{q_1 + q_2 + q_3}, \tag{1}$$

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where Q is the total supply and q_i, $i = 1, 2, 3$ represents the level of production of the ith triopolist. The cost function is linear:

$$C_i(q_i) = c_i q_i, \tag{2}$$

where c_i for $i = 1$ to 3 are the constant marginal costs.

The first player does not know the shape of the demand function and at each time period t it builds a *conjectured demand function* through the local knowledge of the real demand function (1). In particular, the firm observes the current market price $p(t)$ and the corresponding total supplied quantity $Q(t)$. By using market experiments, the player is able to linearly approximate the demand function around the point $(Q(t), p(t))$. In other words, it obtains the slope of the demand function in that point and, in the absence of other information, it conjectures the linearity of the demand function that must pass through the point corresponding to the current market price and quantity (see [18,19,25] for recent applications of this mechanism in a duopolistic heterogeneous framework and [12,27] for applications in other contexts). Given this assumption, the first player defines the conjectured demand for the following period $t + 1$:

$$p_1^e(t+1) = p(t) + f^{-1}(Q(t))(Q^e(t+1) - Q(t)), \tag{3}$$

where $f^{-1}(Q(t))$ is the inverse demand function and $Q^e(t+1)$ represents the aggregate conjectured production for time $t + 1$. By using the demand function we obtain

$$p_1^e(t+1) = f(Q(t)) + f^{-1}(Q(t))(q_1(t+1) + q_2^e(t+1) + q_3^e(t+1) - Q(t)). \tag{4}$$

Concerning the expectations about the rivals' outputs, we use the Cournotian hypothesis of static expectations, then the expected quantities for the next period are the same as those supplied in the current one. By using this assumption in (4) we have

$$p_1^e(t+1) = f(Q(t)) + f^{-1}(Q(t))(q_1(t+1) - q_1(t)). \tag{5}$$

The choice of $q_1(t + 1)$ is made in order to maximize the expected profit:

$$q_1(t+1) = \underset{q_1(t+1)}{\operatorname{argmax}}\, \pi_1^e(t+1) = \underset{q_1(t+1)}{\operatorname{argmax}} \left[p_1^e(t+1)q_1(t+1) - c_1 q_1(t+1)\right]. \tag{6}$$

The first-order condition is the following:

$$\frac{\partial \pi_1^e(t+1)}{\partial q_1(t+1)} = f(Q(t)) + 2q_1(t+1)f^{-1}(Q(t)) - q_1(t)f^{-1}(Q(t)) - c_1 = 0. \tag{7}$$

It is easy to verify the second-order condition. So, the evolution of the output of the first player is given the following first-order nonlinear difference equation:

$$q_1(t+1) = \frac{q_1(t)}{2} + \frac{c_1 - f(Q(t))}{2f^{-1}(Q(t))} \tag{8}$$

that is

$$q_1(t+1) = \frac{2q_1(t) + q_2(t) + q_3(t) - c_1(q_1(t) + q_2(t) + q_3(t))^2}{2}. \tag{9}$$

The second player knows the shape of the demand function but it has to conjecture the choices of the other two players. We assume that it also is a *naive* player, that is it uses static expectations like the first player and then it maximizes the expected profit given by

$$q_2(t+1) = \underset{q_2(t+1)}{\operatorname{argmax}}\ \pi_2^e(t+1) = \underset{q_2(t+1)}{\operatorname{argmax}}\ \left[p_2^e(t+1)q_2(t+1) - c_2q_2(t+1)\right]. \tag{10}$$

By using the demand function (1) and static expectations we obtain

$$q_2(t+1) = \underset{q_2(t+1)}{\operatorname{argmax}}\ \pi_2^e(t+1) = \underset{q_2(t+1)}{\operatorname{argmax}}\ \left[\frac{q_2(t+1)}{q_1(t)+q_2(t+1)+q_3(t)} - c_2q_2(t+1)\right] \tag{11}$$

that permits to derive the dynamic equation:

$$q_2(t+1) = \sqrt{\frac{q_1(t)+q_3(t)}{c_2}} - q_1(t) - q_3(t). \tag{12}$$

The third player adopts the so-called *myopic* adjustment mechanism (see [13]), that is

$$q_3(t+1) = q_3(t) + \alpha q_3(t)\phi_3(Q(t)), \tag{13}$$

where $\phi_3(Q(t))$ is the marginal profit of the third triopolist, that is

$$\begin{aligned}\phi_3(Q(t)) = \phi_3(q_1(t)+q_2(t)+q_3(t)) &= \frac{\partial \pi_3(q_1(t)+q_2(t)+q_3(t))}{\partial q_3(t)}\\ &= \frac{q_1(t)+q_2(t)}{(q_1(t)+q_2(t)+q_3(t))^2} - c_3.\end{aligned} \tag{14}$$

In other words, the third firm increases/decreases its output according to the information given by the marginal profit of the last period. The positive parameter α represents the speed of adjustment. By substituting (14) in (13) we finally obtain the dynamic equation:

$$q_3(t+1) = q_3(t) + \alpha q_3(t) + \left[\frac{q_1(t)+q_2(t)}{(q_1(t)+q_2(t)+q_3(t))^2} - c_3\right]. \tag{15}$$

If we use x, y, z instead of q_1, q_2, q_3 in (9), (12) and (15) we have that the dynamics of the firms' outputs are given by the following discrete time dynamical system:

$$(x', y', z') = T(x,y,z): \begin{cases} x' = \frac{2x+y+z-c_1(x+y+z)^2}{2}, \\ y' = \sqrt{\frac{x+z}{c_2}} - x - z, \\ z' = z + \alpha z\left[-c_3 + \frac{x+y}{(x+y+z)^2}\right], \end{cases} \tag{16}$$

where $'$ denotes the unit-time advancement operator.

3. NE stability

In order to analyse the relationship between the stationary state of the dynamical system (16) and the NE, we must seek the equilibrium point as the solution of the following

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algebraic system:

$$\begin{cases} \frac{y^*+z^*-c_1(x^*+y^*+z^*)^2}{2} = 0, \\ \sqrt{\frac{x^*+z^*}{c_2}} - x^* - y^* - z^* = 0, \\ z^*\left[-c_3 + \frac{x^*+y^*}{(x^*+y^*+z^*)^2}\right] = 0, \end{cases} \tag{17}$$

which is obtained by setting $x' = x = x^*$, $y' = y = y^*$ and $z' = z = z^*$ in (16). The algebraic system (17) is solved by the origin O and by the point

$$E : \left(\frac{2(c_2+c_3-c_1)}{(c_1+c_2+c_3)^2}; \frac{2(c_1+c_3-c_2)}{(c_1+c_2+c_3)^2}; \frac{2(c_1+c_2-c_3)}{(c_1+c_2+c_3)^2}\right). \tag{18}$$

We do not consider the origin O because our map is not defined in such a point. It is possible to prove (see [12,20]) that E is the only other stationary state of the system. It is the NE of the static game. We note that such equilibrium is the same equilibrium obtained by [22] in an equivalent triopoly setting. The Jacobian matrix of the map T is the following:

$$J(x,y,z) : \begin{bmatrix} 1 - c_1(x+y+z) & \frac{1}{2}\left[1 - 2c_1(x+y+z)\right] & \frac{1}{2}\left[1 - 2c_1(x+y+z)\right] \\ \left(\frac{x+z}{c_2}\right)^{-1/2}\frac{1}{2c_2} - 1 & 0 & \left(\frac{x+z}{c_2}\right)^{-1/2}\frac{1}{2c_2} - 1 \\ \alpha z\left[\frac{z-x-y}{(x+y+z)^3}\right] & \alpha z\left[\frac{z-x-y}{(x+y+z)^3}\right] & 1 - \alpha c_3 + \alpha(x+y)\left[\frac{x+y-z}{(x+y+z)^3}\right] \end{bmatrix}.$$

In order to analyse the local stability of the NE, you need to evaluate the Jacobian matrix at the NE and calculate the eigenvalues. Unfortunately the expressions defining the eigenvalues are so complicated that nothing can be said analytically. We can still say something through numerical simulations. After several numerical computations, we have found that a generic two-dimensional bifurcation diagram in the (α, c_3) parameters' plane is in almost all the cases qualitatively similar to the one represented in Figure 1.

Figure 1 is representative of the ways by which the NE becomes unstable. We can see a double route to instability: by period doubling or by NS bifurcation.

4. Flip bifurcation

The first one is via period-doubling bifurcation (also called flip bifurcation). This kind of local bifurcation is not new in the literature on both homogeneous and heterogeneous triopolies. It occurs when moving the value of a parameter, one of the eigenvalues of the Jacobian matrix calculated at the NE becomes lower than -1, while the other two are still lower than 1 in absolute value. A 2-cycle appears and it attracts all the orbits previously attracted by the fixed point. The bifurcation diagram Figure 2 shows that this is what happens by increasing the value of the speed of reaction parameter α, keeping fixed the marginal costs at $c_1 = 0.5$, $c_2 = 0.55$ and $c_3 = 0.15$ (the direction A in Figure 1).

The bifurcation diagram also shows that if we keep increasing the value of α a cascade of period doubling bifurcations occurs (see the bifurcation diagram in Figure 2(a)). This is

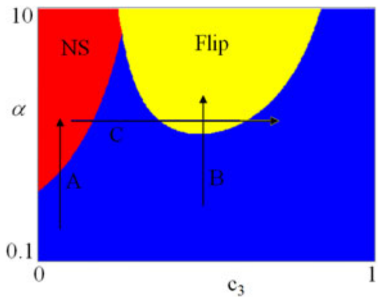

Figure 1. (Colour online) Two-dimensional bifurcation diagram in the (c_3, α) parameters plane. The value of c_1 is fixed to 0.5, while $c_2 = 0.55$. In the blue region the NE is locally stable. Moving the parameter α along the direction A a couple of complex and conjugated eigenvalues becomes higher than 1 in modulus. Along the direction B, the value of one eigenvalue decreases until it becomes lower than -1 entering in the yellow region. Finally, direction C corresponds to a more complicated path altering aperiodic trajectories, convergence to the NE, periodic motion and convergence to the NE again.

a typical route to chaos, as it is confirmed but the maximal Lyapunov exponent, in Figure 2 (b). A chaotic attractor in the three-dimensional phase space is shown in Figure 3(b).

5. NS bifurcation

Another route to complicated dynamics occurs whenever the NE undergoes a NS bifurcation. This happens when increasing the value of α the system enters in the yellow region of the parameters plane as shown in Figure 1. Figure 4(a) shows the locally attractive closed invariant curve that is created after the local bifurcation. Differently from the flip bifurcation case, now the dynamics are quasi-periodic. A further increase in the value of α may lead to chaotic dynamics (as evidenced by the numerical computation of the maximal Lyapunov exponent in Figure 4(b),(c)). The annular chaotic attractor is shown in Figure 5(b). The double route to chaos (via NS and via flip bifurcation) is new with respect to [15] and also with respect to the other triopoly games, with the only exception of [25]. Our model and the [25] model share the isoelasticity of the market demand function, so we can conjecture that it is somehow related to the emergence in a couple of these scenarios. This double route to chaos is an important feature from an economic point of view, because it means that firms may face both periodic and quasi-

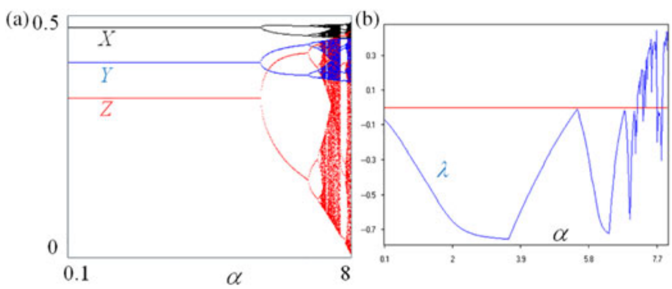

Figure 2. In (a) one-dimensional bifurcation diagram with respect to the parameter α. The fixed parameters are $c_1 = 0.5$, $c_2 = 0.55$ and $c_3 = 0.6$. In (b) the corresponding maximum Lyapunov exponent, displaying a chaotic behaviour for high values of α.

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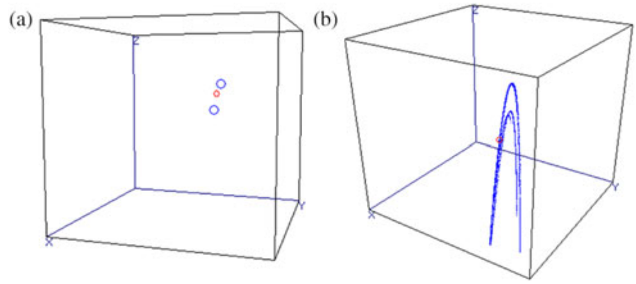

Figure 3. In (a) a 2-cycle obtained with $\alpha = 6.5$. The chaotic attractor in (b) is obtained by using $\alpha = 8$. The values of the variables on the three axes vary between 0 and 0.5.

periodic dynamics. While the former can be recognized by a rational enough firm, the latter are very hard to detect by looking at the time series of the quantities. This means that immediately after the NS bifurcation the system displays complicated dynamics (with the exception of some periodicity windows), differently from what happens after the period doubling bifurcation in which a periodic attractor appears.

6. The role of the marginal costs

The two-dimensional bifurcation diagram shown in Figure 1 permits us to also say something about the role played by the marginal costs. We can see that this role is ambiguous, especially for high values of the speed of adjustment. In fact, it is possible to have a situation in which for low values of the marginal cost the NE is unstable and the orbits converge to a chaotic attractor or a closed invariant curve. For intermediate values

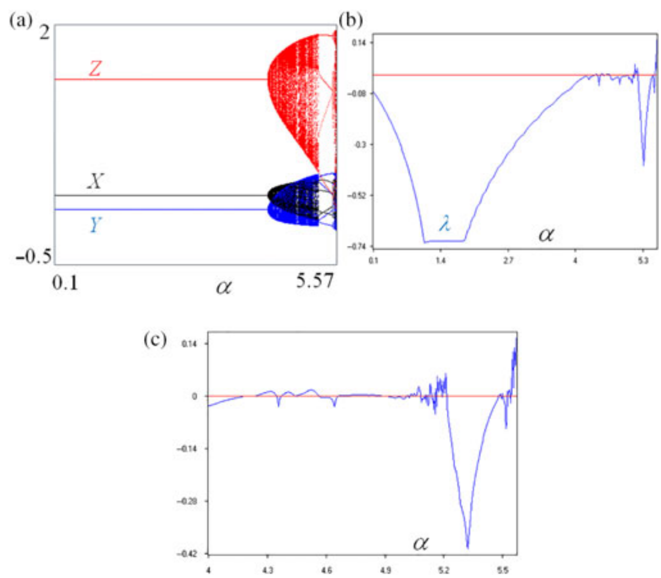

Figure 4. In (a) one-dimensional bifurcation diagram with respect to the parameter α. The fixed parameters are $c_1 = 0.5$, $c_2 = 0.55$ and $c_3 = 0.1$. In (b) the corresponding maximum Lyapunov exponent, displaying a chaotic behaviour for high values of α with a magnification of the relevant region in (c).

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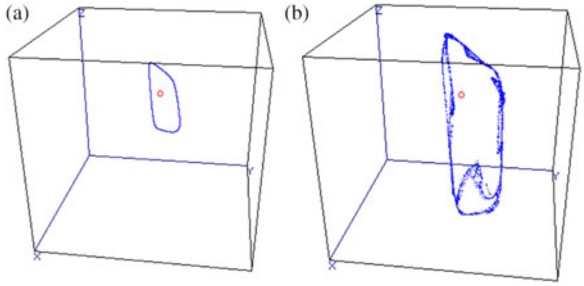

Figure 5. In (a) an attracting closed invariant curve obtained with $\alpha = 4.476$. The annular chaotic attractor in (b) is obtained by using $\alpha = 5.57$. The values of the variables on the three axes vary between 0 and 0.5.

of α the NE is locally stable but increasing again the marginal cost it loses stability via flip bifurcation and then, with higher values of α the NE becomes locally stable again (see the bifurcation diagram in Figure 6 that corresponds to the direction C in Figure 1).

Note that qualitatively nothing would change in Figure 1 by using c_1 or c_2 instead of c_3.

7. Global analysis

In this section we present the main novelty of this model with respect to the other heterogeneous triopolies already studied in the literature. Until now we limited our analysis to the local stability of the NE. We have also shown that the positivity of the NE implies that the other fixed points are locally unstable. This is also what happens in the other triopolies studied so far. The NE, or the attractor originating from its loss of stability, was the unique outcome of feasible trajectories (i.e. excluding divergent trajectories). This is not what happens in our triopoly game. In fact, we can find sets of parameters leading to a bistability of different attractors. Figure 7(b) shows the locally stable NE coexisting with a locally stable 3-cycle whose points are located around it. From the bifurcation diagram of Figure 7(a) we can see that the 3-cycle becomes unstable by increasing the value of α, giving rise to higher periodicity cycles, and even to a chaotic attractor. Nevertheless the NE still remains locally stable.

This is not only a new interesting feature from a mathematical point of view. Coexistence has many economic consequences. The two coexisting attractors are characterized by quite different levels of variance of the quantities produced by the triopolists. In one case what will happen in the future is much more predictable with

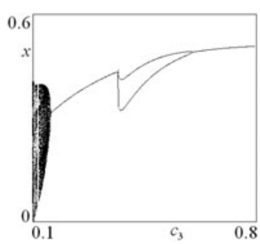

Figure 6. One-dimensional bifurcation diagram obtained by varying the marginal cost c_3 and keeping fixed the other parameters at the values $c_1 = 0.5$, $c_2 = 0.55$ and $\alpha = 5.5$.

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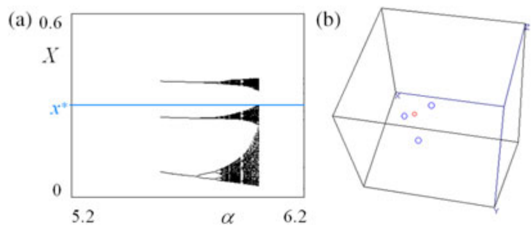

Figure 7. In (a) one-dimensional bifurcation diagram with $c_1 = 0.5$, $c_2 = 0.55$ and $c_3 = 0.2$. α varies between 5.2 and 6.2. The sudden jump from the fixed point to a 3-cycle is caused by the initial conditions that enter the basin of attraction of the 3-cycle. In (b) the NE coexisting with a locally stable 3-cycle at $\alpha = 5.66$. The values of the variables on the three axes vary between 0 and 0.5.

respect to the other case. Initial conditions assume a crucial importance in determining to which attractor the system will asymptotically converge. So, besides the local analysis of the NE, we need to perform some kind of global analysis. In Figure 8 we can see three different sections of the basins of attractions in a situation of coexistence between the locally stable NE (whose basin of attraction is made up of blue points) and a locally stable 3-cycle (whose basin of attraction is in red). The border of the basins is made up of the stable manifolds of the unstable 3-cycle born through fold bifurcation together with the stable 3-cycle, so the points of this unstable cycle are located in the border. The structure of the basins appears quite complicated and this is probably a consequence of the non-invertibility of the map (16).

The shapes of the basins of attractions when the NE coexists with a chaotic attractor (Figure 9) are complicated as well. This permits us to conclude that this heterogeneous triopoly is characterized by a higher degree of unpredictability with respect to other similar models present in the literature.

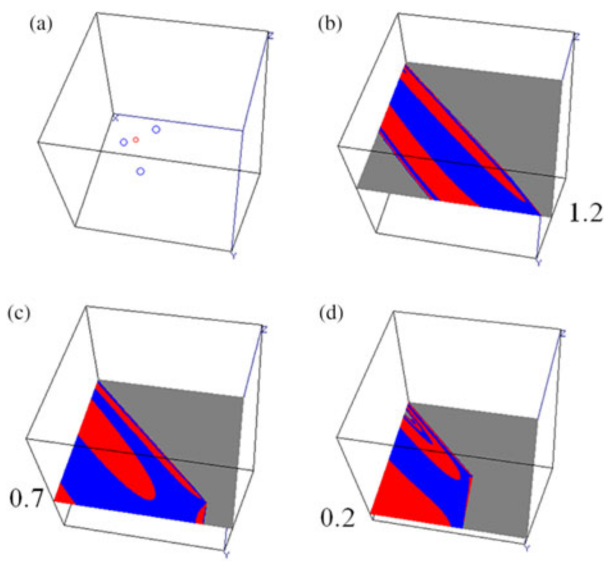

Figure 8. (Colour online) Four different sections of the basins of attraction of the coexisting NE (basin in blue) and 3-cycle (basin in red). In grey is the basin of attraction of diverging trajectories. The values of the variables on the x and y axes vary between 0 and 0.5, while those on the z-axis vary between 0 and 2.

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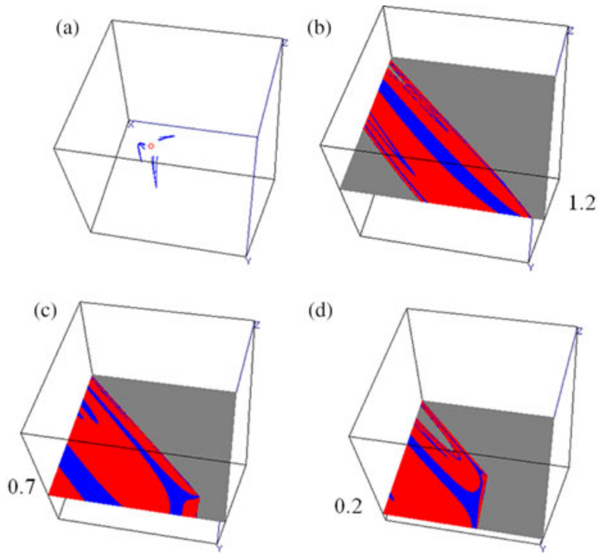

Figure 9. (Colour online) Four different sections of the basins of attraction of the coexisting NE (basin in blue) and 3-pieces chaotic attractor (basin in red). In grey is the basin of attraction of diverging trajectories. The values of the variables on the x and y axes vary between 0 and 0.5, while those on z-axis vary between 0 and 2.

8. Conclusions

A triopoly game with heterogeneous players is analysed in this paper. Nonlinearities are present both in the demand function and in the decisional mechanism adopted by the firms. We have numerically proved the existence of two different routes to complex dynamics: through a flip bifurcation and through NS bifurcation of the NE. Another important feature of this model is the arising, for some parameters' constellation, of multistability between two different attractors. We have numerically performed some global analyses which show how complicated can be the basins of attractions.

Disclosure statement

No potential conflict of interest was reported by the authors.

Note

1. In this branch of the literature the terms *homogeneous* and *heterogeneous* refer to the decisional mechanism adopted by the firms.

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