TIME DECAY OF SCALING INVARIANT ELECTROMAGNETIC SCHRÖDINGER EQUATIONS ON THE PLANE

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Abstract. We prove the sharp $L^1 - L^\infty$ time-decay estimate for the $2D$ - Schrödinger equation with a general family of scaling critical electromagnetic potentials.

1. Introduction

Let us consider an electromagnetic Schrödinger equation of the type

$$iu_t = \left(-i\nabla + \frac{\mathbf{A}\big(\frac{x}{|x|}\big)}{|x|}\right)^2 u + \frac{a\big(\frac{x}{|x|}\big)}{|x|^2}\,u, \tag{1.1}$$

where $N \geqslant 2$, $u = u(x,t) : \mathbb{R}^{N+1} \to \mathbb{C}$, $a \in W^{1,\infty}(\mathbb{S}^{N-1},\mathbb{R})$, $\mathbb{S}^{N-1}$ denotes the unit circle, and $\mathbf{A} \in W^{1,\infty}(\mathbb{S}^{N-1},\mathbb{R}^N)$ is a transversal vector field, namely

$$\mathbf{A}(\theta)\cdot\theta = 0 \quad \text{for all } \theta \in \mathbb{S}^{N-1}. \tag{1.2}$$

We always denote by $r := |x|$, $\theta = x/|x|$, so that $x = r\theta$. Notice that the potentials $\mathbf{A}/|x|$ and $a/|x|^2$ preserve the natural scaling $u_\lambda(x,t) := u(x/\lambda, t/\lambda^2)$ of the free Schrödinger equation, and consequently they show a critical behavior with respect to several phenomena.

In [16], we started a program based on the connection between the Schrödinger flow $e^{it\mathcal{L}_{\mathbf{A},a}}$, generated by the hamiltonian

$$\mathcal{L}_{\mathbf{A},a} := \left(-i\nabla + \frac{\mathbf{A}\big(\frac{x}{|x|}\big)}{|x|}\right)^2 + \frac{a\big(\frac{x}{|x|}\big)}{|x|^2}, \tag{1.3}$$

and the spectral properties of the spherical operator $L_{\mathbf{A},a}$, defined by

$$L_{\mathbf{A},a} = \big(-i\,\nabla_{\mathbb{S}^{N-1}} + \mathbf{A}\big)^2 + a(\theta), \tag{1.4}$$

where $\nabla_{\mathbb{S}^{N-1}}$ is the spherical gradient on the unit sphere $\mathbb{S}^{N-1}$. In order to describe the project, let us start by reviewing some well known facts in classical spectral theory.

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The spectrum of the operator $L_{\mathbf{A},a}$ is formed by a diverging sequence of real eigenvalues with finite multiplicity $\mu_1(\mathbf{A},a) \leqslant \mu_2(\mathbf{A},a) \leqslant \cdots \leqslant \mu_k(\mathbf{A},a) \leqslant \cdots$ (see e.g. [19, Lemma A.5]), where each eigenvalue is repeated according to its multiplicity. Moreover we have that $\lim_{k\to\infty}\mu_k(\mathbf{A},a) = +\infty$. To each $k \geqslant 1$, we can associate a $L^2\big(\mathbb{S}^{N-1},\mathbb{C}\big)$-normalized eigenfunction ψ_k of the operator $L_{\mathbf{A},a}$ on $\mathbb{S}^{N-1}$ corresponding to the k-th eigenvalue $\mu_k(\mathbf{A},a)$, i.e. satisfying

$$\begin{cases} L_{\mathbf{A},a}\psi_k = \mu_k(\mathbf{A},a)\,\psi_k(\theta), & \text{in } \mathbb{S}^{N-1}, \\ \int_{\mathbb{S}^{N-1}} |\psi_k(\theta)|^2\,dS(\theta) = 1. \end{cases} \tag{1.5}$$

In particular, if $N = 2$, the functions ψ_k are one-variable 2π periodic functions, i.e. $\psi_k(0) = \psi_k(2\pi)$. Since the eigenvalues $\mu_k(\mathbf{A},a)$ are repeated according to their multiplicity, exactly one eigenfunction ψ_k corresponds to each index $k \geqslant 1$. We can choose the functions ψ_k in such a way that they form an orthonormal basis of $L^2(\mathbb{S}^{N-1},\mathbb{C})$. We also introduce the numbers

$$\alpha_k := \frac{N-2}{2} - \sqrt{\bigg(\frac{N-2}{2}\bigg)^2 + \mu_k(\ \mathbf{A},a)}, \quad \beta_k := \sqrt{\bigg(\frac{N-2}{2}\bigg)^2 + \mu_k(\mathbf{A},a)}, \tag{1.6}$$

so that $\beta_k = \frac{N-2}{2} - \alpha_k$, for $k = 1,2,\dots$, which will come into play in the sequel.

Under the condition

$$\mu_1(\mathbf{A},a) > -\bigg(\frac{N-2}{2}\bigg)^2 \tag{1.7}$$

the quadratic form associated to $\mathcal{L}_{\mathbf{A},a}$ is positive definite (see [16, Section2] and [19]); this implies that the hamiltonian $\mathcal{L}_{\mathbf{A},a}$ is a symmetric semi-bounded operator on $L^2(\mathbb{R}^N;\mathbb{C})$ which then admits a self-adjoint extension (the *Friedrichs extension* which will be still denoted as $\mathcal{L}_{\mathbf{A},a}$) with domain

$$\mathcal{D}(\mathcal{L}_{\mathbf{A},a}) := \left\{ f \in H^1_*(\mathbb{R}^N) :\ \mathcal{L}_{\mathbf{A},a}u \in L^2(\mathbb{R}^N \right\}, \tag{1.8}$$

where $H^1_*(\mathbb{R}^N)$ is the completion of $C^\infty_{\rm c}(\mathbb{R}^N \setminus \{0\},\mathbb{C})$ with respect to the norm

$$\|\phi\|_{H^1_*(\mathbb{R}^N)} = \bigg(\int_{\mathbb{R}^N}\bigg(|\nabla\phi(x)|^2 + \frac{|\phi(x)|^2}{|x|^2} + |\phi(x)|^2\bigg)\,dx\bigg)^{1/2}.$$

From the classical Hardy inequality (see e.g. [26, 29]), $H^1_*(\mathbb{R}^N) = H^1(\mathbb{R}^N)$ with equivalent norms if $N \geqslant 3$, while $H^1_*(\mathbb{R}^N)$ is strictly smaller than $H^1(\mathbb{R}^N)$ if $N = 2$. Furthermore, from condition (1.7) and [19, Lemma 2.2], it follows that $H^1_*(\mathbb{R}^N)$ coincides with the space obtained by completion of $C^\infty_{\rm c}(\mathbb{R}^N \setminus \{0\},\mathbb{C})$ with respect to the norm naturally associated to the operator $\mathcal{L}_{\mathbf{A},a}$, i.e.

$$\bigg(\int_{\mathbb{R}^N}\bigg[\bigg|\bigg(\nabla + i\,\frac{\mathbf{A}\big(x/|x|\big)}{|x|}\bigg)u(x)\bigg|^2 + \frac{a\big(x/|x|\big)}{|x|^2}|u(x)|^2 + |u(x)|^2\bigg]\,dx\bigg)^{1/2}.$$

We notice that $\mathcal{L}_{\mathbf{A},a}$ could be not essentially self-adjoint. For example, in the case $\mathbf{A} \equiv 0$, from a theorem due to Kalf, Schmincke, Walter, and Wüst [31] and Simon [43] (see also [39, Theorems X.11 and X.30], [20], and [21] for non constant a), it is known that $\mathcal{L}_{\mathbf{0},a}$ is essentially self-adjoint if and only if $\mu_1(\mathbf{0},a) \geqslant -\big(\frac{N-2}{2}\big)^2 + 1$ and, consequently, admits a unique self-adjoint extension, which is given by the Friedrichs extension; otherwise, i.e. if $\mu_1(\mathbf{0},a) < -\big(\frac{N-2}{2}\big)^2 + 1$, $\mathcal{L}_{\mathbf{0},a}$ is not essentially self-adjoint and admits many self-adjoint extensions, among which the Friedrichs extension is the only one whose domain is included in the domain of the associated quadratic form (see also [15, Remark 2.5]).

The Friedrichs extension $\mathcal{L}_{\mathbf{A},a}$ naturally extends to a self adjoint operator on the dual of $\mathcal{D}(\mathcal{L}_{\mathbf{A},a})$ and the unitary group of isometries $e^{-it\mathcal{L}_{\mathbf{A},a}}$ generated by $-i\mathcal{L}_{\mathbf{A},a}$ extends to a group of isometries on the dual of $\mathcal{D}(\mathcal{L}_{\mathbf{A},a})$ which will be still denoted as $e^{-it\mathcal{L}_{\mathbf{A},a}}$ (see [8], Section 1.6 for further details). Then for every $u_0 \in L^2(\mathbb{R}^N)$, $u(\cdot,t) = e^{-it\mathcal{L}_{\mathbf{A},a}}u_0(\cdot)$ is the unique solution to the problem

$$\begin{cases} u \in \mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^N)) \cap C^1(\mathbb{R}, (\mathcal{D}(\mathcal{L}_{\mathbf{A},a}))^\star), \\ iu_t = \mathcal{L}_{\mathbf{A},a}u, \\ u(0) = u_0. \end{cases}$$

Now, by means of (1.5) and (1.6) define the following kernel:

$$K(x,y) = \sum_{k=-\infty}^{\infty} i^{-\beta_k} j_{-\alpha_k}(|x||y|)\psi_k\big(\tfrac{x}{|x|}\big)\overline{\psi_k\big(\tfrac{y}{|y|}\big)}, \tag{1.9}$$

where

$$j_\nu(r) := r^{-\frac{N-2}{2}} J_{\nu+\frac{N-2}{2}}(r)$$

and J_ν denotes the usual Bessel function of the first kind

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)}\left(\frac{t}{2}\right)^{2k}.$$

In the main result of [16] we prove that, if $a \in L^\infty(\mathbb{S}^{N-1},\mathbb{R})$ and $\mathbf{A} \in C^1(\mathbb{S}^{N-1},\mathbb{R}^N)$ are such that (1.2) and (1.7) hold, then

$$e^{-it\mathcal{L}_{\mathbf{A},a}}u_0(x) = \frac{e^{\frac{i|x|^2}{4t}}}{i(2t)^{N/2}} \int_{\mathbb{R}^N} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) e^{i\frac{|y|^2}{4t}} u_0(y)\,dy, \tag{1.10}$$

for any $u_0 \in L^2(\mathbb{R}^N)$.

Apart from the interest in itself, formula (1.10) provides a quite solid tool to obtain quantitative informations for the flow $e^{-it\mathcal{L}_{\mathbf{A},a}}u_0(x)$ by the analytical study of the kernel $K(x,y)$. In particular, if

$$\sup_{x,y\in\mathbb{R}^N} |K(x,y)| < \infty \tag{1.11}$$

holds, one automatically obtains by (1.10) the time-decay estimate

$$\left\| e^{-it\mathcal{L}_{\mathbf{A},a}}u_0(\cdot)\right\|_{L^\infty} \lesssim |t|^{-\frac{N}{2}} \|u_0(\cdot)\|_{L^1}. \tag{1.12}$$

In [16], we are able to prove (1.11) (and consequently (1.12)) in two concrete situations:

- the *Aharonov-Bohm* potential: $a \equiv 0$, $\mathbf{A}(x) = \alpha\left(-\frac{x_2}{|x|}, \frac{x_1}{|x|}\right)$, for $\alpha \in \mathbb{R}$, in dimension $N = 2$;
- the positive *inverse square* potential: $\mathbf{A} \equiv 0$, $a \in \mathbb{R}$, $a > 0$.

In both cases, the spectrum of $L_{\mathbf{A},a}$ is explicit, together with a complete set of orthonormal eigenfunctions (spherical harmonics or phase transformations of themselves). These examples give a positive contribution to the recent literature about the topic, which never included before potentials with the critical homogeneity as the ones in (1.1) (see e.g. [3, 6, 7, 9, 10, 11, 12, 13, 14, 22, 23, 24, 37, 38, 40, 41, 42, 44, 45, 48, 49, 51, 52, 53, 54]). Moreover, it is well known that these potentials represent a threshold between the validity and the failure of global (in time) dispersive estimates, as proved in [18, 25]. Recently, Grillo and Kovarik [27] gave a proof of sharp time-decay estimates in the case of the Aharonov-Bohm potential, combined with a compactly supported electric potential, in dimension 2, proving also an interesting remark regarding the connection of diamagnetism with improvement of decay, in suitable weighted spaces.

The aim of this paper is to prove that estimate (1.12) holds, in space dimension $N = 2$, for a general family of potentials of the same kind as in (1.1). Our main result is the following.

Theorem 1.1. *Let* $N = 2$, $a \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R})$, $\mathbf{A} \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R}^2)$ *satisfying* (1.2) *and* $\mu_1(\mathbf{A}, a) > 0$*, and* $\mathcal{L}_{\mathbf{A},a}$ *be given by* (1.3)*. Then, for any* $u_0 \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$*, the following estimate holds:*

$$\left\| e^{-it\mathcal{L}_{\mathbf{A},a}} u_0(\cdot) \right\|_{L^\infty} \leqslant \frac{C}{|t|} \| u_0(\cdot) \|_{L^1}, \tag{1.13}$$

for some $C = C(\mathbf{A}, a) > 0$ *which does not depend on* t *and* u_0*.*

As remarked above, the proof of Theorem 1.1 consists in showing that the kernel $K(x, y)$ in (1.9) is uniformly bounded. The main difficulty is to obtain this information for the queues of the series in (1.9). In order to do this, we need to obtain the precise asymptotic behavior in k of the set of eigenvalues and eigenfunctions of the problem (1.5): this is the topic of Section 2 below. Once this is done, the proof of Theorem 1.1 will be obtained, in Section 3, by suitably comparing the kernel K with the analogous in the case of an Aharonov-Bohm potential with the same average as the potential $\mathbf{A}$ on the sphere $\mathbb{S}^1$.

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2. Spectral properties of spherical laplacians

The fundamental tool which we need in order to prove Theorem 1.1 is the knowledge of the spectral properties of the operator $L_{\mathbf{A},a}$ defined by (1.4). Roughly speaking, we need to obtain informations concerning the asymptotic behavior of the eigenvalues μ_k and the eigenfunctions ψ_k in the eigenvalue problem (1.5), as $k \to \infty$.

An extensive literature has been devoted, in the recent years, to this kind of problems (see e.g. [28, 46, 47, 50] and the references therein). Since we did not find sufficiently explicit results regarding general electromagnetic Laplace operators on the 1D-sphere $\mathbb{S}^1$, we need to show here Lemma 2.1 below, which is possibly of independent interest.

Before starting to settle the eigenvalue problem, we find convenient to briefly sketch the well known consequences which the introduction of lower order terms produces on the spectrum of the spherical Laplacian.

Let us denote by $L_0 := -\Delta_{\mathbb{S}^1}$. Being the inverse of a compact operator on $L^2(\mathbb{S}^1)$, with form domain $H^1(\mathbb{S}^1)$, L_0 has purely discrete spectrum which accumulates at positive infinity. The explicit form is

$$\sigma(L_0) = \{k^2\}_{k=0,1,\dots}.$$

The k^{th}-eigenvalue has multiplicity 2, and the eigenfunctions are combinations of sines and cosines.

The introduction of a 0-order term produces a spectral shift, depending on the average of the potentials, and the formation of clusters of eigenvalues around the free ones (Stark's effect), if the potential is not constant. More precisely, the eigenvalues of the operator $L_a := -\Delta_{\mathbb{S}^1} + a(\theta)$ are of the form

$$\lambda_k = k^2 + \frac{1}{2\pi}\int_0^{2\pi} a(s)\,ds + (\text{rest}),$$

where the rest, depending on k and on the potential a, decays with order $1/k$ as k tends to infinity. For the eigenfunctions ψ_k a similar behavior occurs; for large k, ψ_k looks more and more like a spherical harmonic plus a rest which decays as k tends to $+\infty$ (see e.g. [5, 28, 31, 46, 47] and appendix B of the preprint version [17] of the present paper).

On the other hand, for a purely magnetic potential, a splitting occurs on each eigenvalue. The most famous (and descriptive) example is given by the Aharonov-Bohm potential, namely $a \equiv 0$, $\mathbf{A}(x,y) = \mathbf{A}_{ab}(x,y) = \alpha\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$, with $\alpha \in \mathbb{R}$: in this case, the complete set of eigenvalues and eigenfunctions of problem (1.5) can be computed explicitly, and reads as

$$\lambda_k^{ab} = (k+\alpha)^2, \quad k \in \mathbb{Z} \tag{2.1}$$

$$\phi_k^{ab}(\theta) = \frac{1}{\sqrt{2\pi}} e^{ik\theta}, \quad k \in \mathbb{Z}. \tag{2.2}$$

It is hence quite natural to expect that, in the general case of the operator $L_{\mathbf{A},a}$, the picture is a superposition of the two previously mentioned ones. We did not find in the literature a result written in the generality of Lemma 2.1 below, so that we found convenient to state and prove it in this manuscript.

We recall that, by classical spectral theory, the spectrum of $L_{\mathbf{A},a}$ is formed by a countable family of real eigenvalues with finite multiplicity $\{\mu_k : k \geqslant 1\}$ enumerated in such a way that

$$\mu_1 \leqslant \mu_2 \leqslant \dots,$$

where each eigenvalue is repeated according to its multiplicity. Moreover we have that $\lim_{k\to\infty} \mu_k = +\infty$.

Let $A : [0,2\pi] \to \mathbb{R}$ be defined as $A(\theta) = \mathbf{A}(\cos\theta, \sin\theta)\cdot(-\sin\theta,\cos\theta)$, so that, by assumption (1.2)

$$\mathbf{A}(\cos\theta,\sin\theta) = A(\theta)(-\sin\theta,\cos\theta), \quad \theta \in [0,2\pi]. \tag{2.3}$$

Furthermore, identifying functions defined on $\mathbb{S}^1$ with 2π-periodic functions, the operator $L_{\mathbf{A},a}$ can be identified with the following operator $\mathfrak{L}_{A,a}$ acting on 2π-periodic functions

$$\mathfrak{L}_{A,a}\varphi(\theta) = -\varphi''(\theta) + [a(\theta) + A^2(\theta) - iA'(\theta)]\varphi(\theta) - 2iA(\theta)\varphi'(\theta).$$

The main result of this section is the following asymptotic expansion of eigenvalues and eigenfunctions of the operator $L_{\mathbf{A},a}$ under the non-resonant assumption that the magnetic potential does not have half-integer or integer circulation. The case of half-integer or integer circulation can be reduced through suitable transformations to the magnetic-free problem, for which analogous expansions hold, see Remark 2.2.

Lemma 2.1. *Let* $a \in W^{1,\infty}(\mathbb{S}^1)$, $\widetilde{a} = \frac{1}{2\pi}\int_0^{2\pi} a(s)\,ds$, $A \in W^{1,\infty}(\mathbb{S}^1)$ *such that*

$$\widetilde{A} = \frac{1}{2\pi}\int_0^{2\pi} A(s)\,ds \notin \frac{1}{2}\mathbb{Z}. \tag{2.4}$$

Then there exist $k^*, \ell \in \mathbb{N}$ *such that* $\{\mu_k : \ k > k^*\} = \{\lambda_j : \ j \in \mathbb{Z}, \ |j| \geqslant \ell\}$,

$$\sqrt{\lambda_j - \widetilde{a}} = (\operatorname{sgn} j)\Big(\widetilde{A} - \Big\lfloor \widetilde{A} + \tfrac{1}{2}\Big\rfloor\Big) + |j| + O\big(\tfrac{1}{|j|^3}\big), \quad \text{as } |j| \to +\infty$$

and

$$\lambda_j = \widetilde{a} + \Big(j + \widetilde{A} - \big\lfloor \widetilde{A} + \tfrac{1}{2}\big\rfloor\Big)^2 + O\big(\tfrac{1}{j^2}\big), \quad \text{as } |j| \to +\infty. \tag{2.5}$$

Furthermore, for all $j \in \mathbb{Z}$, $|j| \geqslant \ell$, *there exists a* $L^2\big(\mathbb{S}^1, \mathbb{C}\big)$*-normalized eigenfunction* ϕ_j *of the operator* $L_{\mathbf{A},a}$ *on* $\mathbb{S}^1$ *corresponding to the eigenvalue* λ_j *such that*

$$\phi_j(\theta) = \frac{1}{\sqrt{2\pi}} e^{-i\big([\widetilde{A}+1/2]\theta + \int_0^\theta A(t)\,dt\big)}\Big(e^{i(\widetilde{A}+j)\theta} + R_j(\theta)\Big), \tag{2.6}$$

where $\|R_j\|_{L^\infty(\mathbb{S}^1)} = O\big(\frac{1}{|j|^3}\big)$ *as* $|j| \to \infty$. *In the above formula* $\lfloor \cdot \rfloor$ *denotes the floor function* $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leqslant x\}$.

Lemma 2.1 can be interpreted as follows: asymptotically in k, eigenvalues and eigenfunctions of (1.5) for $L_{\mathbf{A},a}$ are comparable with the ones in the Aharonov-Bohm case (see (2.1), (2.2) above), by means of (2.5), (2.6).

The proof of Lemma 2.1 is based on the idea of reducing the eigenvalue problem (1.5) to another magnetic-free problem, with different boundary conditions, by gauge transformation; this is in fact possible, since $\mathbf{A}(\cos\theta, \sin\theta)$ just depends on the 1D-variable θ. More precisely, we observe that the gauge transformation

$$\psi(\theta) \to e^{-i\int_0^\theta A(s)\,ds}\psi(\theta)$$

transforms the eigenvalue problem (1.5) into the new problem

$$\begin{cases} -\frac{d^2\psi}{d\theta^2} + a(\theta)\psi = \mu_k\psi \\ \psi(0) = e^{-2i\pi\widetilde{A}}\psi(2\pi) \\ \psi'(0) = e^{-2i\pi\widetilde{A}}\psi'(2\pi), \end{cases} \tag{2.7}$$

with non-periodic boundary conditions, where $\widetilde{A}$ is defined in (2.4), which will be analyzed by a usual WKB-strategy.

Remark 2.2. As mentioned above, in the purely electric case $\mathbf{A} \equiv 0$, (2.5) is a well known information about the cluster distribution of the eigenvalues (see e.g. [28] and the references therein). More in general, if $\operatorname{dist}(\widetilde{A}, \mathbb{Z}) = 0$, then the eigenvalue problem (2.7) reduces to

$$\begin{cases} -\frac{d^2\psi_k}{d\theta^2} + a(\theta)\psi_k = \mu_k\psi_k \\ \psi_k(0) = \psi_k(2\pi) \\ \psi_k'(0) = \psi_k'(2\pi), \end{cases} \tag{2.8}$$

i.e. the magnetic-free case. For the proof of Lemma 2.1 in the case $\operatorname{dist}(\widetilde{A}, \mathbb{Z}) = 0$ we mention a classical result by Borg [5] (see also [28]) as a standard reference; in appendix B of the preprint version [17] of the present paper a detailed proof of asymptotics of eigenvalues and eigenfunctions in the purely electric case can be found.

We propose here a proof in the case $\mathrm{dist}(\widetilde{A},\mathbb{Z})\neq 0,\frac{1}{2}$, since we did not find in the literature neither the analogous to [5] for $\mathbf{A}\neq 0$ nor the asymptotic formula for eigenfunctions (2.6), which plays a fundamental role in the proof of our main theorem (see section 3 below). We propose a proof which is based on a usual WKB-strategy.

2.1. **Proof of Lemma 2.1.** Let us denote

$$\bar{A}=\widetilde{A}-\left\lfloor \widetilde{A}+\frac{1}{2}\right\rfloor, \tag{2.9}$$

so that $\bar{A}\in[-1/2,1/2)$; we notice that $\widetilde{A}\in\frac{1}{2}\mathbb{Z}$ if and only if $\bar{A}\in\{-1/2,0\}$. Hence, under assumption (2.4), we have that

$$\bar{A}\in\Big(-\frac{1}{2},\frac{1}{2}\Big)\setminus\{0\}. \tag{2.10}$$

Lemma 2.3. *Let $a,A\in W^{1,\infty}(0,2\pi)$, $\widetilde{A}=\frac{1}{2\pi}\int_0^{2\pi}A(s)\,ds$, and $\bar{A}$ as in (2.9), i.e. $\bar{A}=\widetilde{A}-\lfloor\widetilde{A}+\frac{1}{2}\rfloor$. Then, letting $\mathbf{A}$ as in* (2.3)*, we have that*

$$\sigma(L_{\mathbf{A},a})=\sigma(\mathfrak{L}_{A,a})=\sigma(\mathfrak{L}_{\bar{A},a}).$$

Furthermore, φ is an eigenfunction of $\mathfrak{L}_{A,a}$ associated to the eigenvalue μ if and only if $\widetilde{\varphi}(t)=e^{-i\bar{A}t}e^{i\int_0^t A(s)\,ds}\varphi(t)$ is an eigenfunction of $\mathfrak{L}_{\bar{A},a}$ associated to μ.

Proof. The proof follows by direct calculations. We notice that, since $\widetilde{A}-\bar{A}\in\mathbb{Z}$, function $\widetilde{\varphi}(t)=e^{-i\bar{A}t}e^{i\int_0^t A(s)\,ds}\varphi(t)$ is 2π-periodic if and only if $\varphi(t)$ is 2π-periodic. □

Lemma 2.4. *Let $a\in W^{1,\infty}(\mathbb{S}^1)$, $\widetilde{a}=\frac{1}{2\pi}\int_0^{2\pi}a(s)\,ds$, $\delta>0$, and*

$$I_\delta=\{\lambda\in\mathbb{R}:\ \mathrm{dist}(\sqrt{\lambda-\tilde{a}},\tfrac{1}{2}\mathbb{Z})\geqslant\delta\}.$$

There exist $\overline{\lambda}_\delta>0$ and $C_\delta>0$ such that for every $\lambda\in I_\delta$, $\lambda\geqslant\overline{\lambda}_\delta$, there exists $W_\lambda\in C^0(\mathbb{S}^1)$ such that

$$\|W_\lambda\|_{C^0(\mathbb{S}^1)}\leqslant\frac{C_\delta}{\sqrt{\lambda-\tilde{a}}} \tag{2.11}$$

and

$$T_\lambda(W_\lambda)=W_\lambda,$$

where $T_\lambda : C^0(\mathbb{S}^1) \to C^0(\mathbb{S}^1)$ *is defined as*

$$\begin{aligned}
T_\lambda(W)(\theta) &= e^{-2\sqrt{\lambda-\widetilde{a}}\theta i}\frac{\widetilde{a}-a(0)}{2\sqrt{\lambda-\widetilde{a}}} \\
&\quad + \frac{ie^{-2\sqrt{\lambda-\widetilde{a}}(\theta+2\pi)i}}{1-e^{-4\sqrt{\lambda-\widetilde{a}}\pi i}}\int_0^{2\pi} e^{2\sqrt{\lambda-\widetilde{a}}\theta' i}\bigg(\frac{a'(\theta')}{2\sqrt{\lambda-\widetilde{a}}i} - W^2(\theta')\bigg)d\theta' \\
&\quad + ie^{-2\sqrt{\lambda-\widetilde{a}}\theta i}\int_0^{\theta} e^{2\sqrt{\lambda-\widetilde{a}}\theta' i}\left[\widetilde{a}-a(\theta')-W^2(\theta')\right]d\theta' \\
&= e^{-2\sqrt{\lambda-\widetilde{a}}\theta i}\frac{\widetilde{a}-a(0)}{2\sqrt{\lambda-\widetilde{a}}} \\
&\quad + \frac{ie^{-2\sqrt{\lambda-\widetilde{a}}(\theta+2\pi)i}}{1-e^{-4\sqrt{\lambda-\widetilde{a}}\pi i}}\int_0^{2\pi} e^{2\sqrt{\lambda-\widetilde{a}}\theta' i}\bigg(\frac{a'(\theta')}{2\sqrt{\lambda-\widetilde{a}}i} - W^2(\theta')\bigg)d\theta' \\
&\quad - \frac{a(\theta)-e^{-2\sqrt{\lambda-\widetilde{a}}\theta i}a(0)}{2\sqrt{\lambda-\widetilde{a}}} + \frac{\widetilde{a}-e^{-2\sqrt{\lambda-\widetilde{a}}\theta i}\widetilde{a}}{2\sqrt{\lambda-\widetilde{a}}} \\
&\quad + ie^{-2\sqrt{\lambda-\widetilde{a}}\theta i}\int_0^{\theta} e^{2\sqrt{\lambda-\widetilde{a}}\theta' i}\left[\frac{a'(\theta')}{2\sqrt{\lambda-\widetilde{a}}i}-W^2(\theta')\right]d\theta'.
\end{aligned}$$

Moreover the map $\lambda \mapsto W_\lambda$ *is continuous as a map from* I_δ *to* $C^0(\mathbb{S}^1)$.

PROOF. It is easy to verify that there exist $\bar\lambda_\delta$ and $C_\delta > 0$ such that for every $\lambda \in I_\delta$, $\lambda \geqslant \bar\lambda_\delta$, T_λ maps $\overline{B}_{C_\delta/\sqrt{\lambda-\widetilde{a}}} = \{u \in C^0(\mathbb{S}^1) : \sup_{\mathbb{S}^1}|u| \leqslant C_\delta/\sqrt{\lambda-\widetilde{a}}\}$ into itself and is a contraction there. The conclusion then follows from the Banach contraction mapping theorem. □

For $\lambda \in I_\delta$, $\lambda \geqslant \bar\lambda_\delta$, let W_λ be as in Lemma 2.4. Then it is easy to verify that W_λ satisfies

$$\begin{cases} -iW_\lambda'(\theta) + 2\sqrt{\lambda-\widetilde{a}}\,W_\lambda(\theta) + W_\lambda^2(\theta) = \widetilde{a}-a(\theta), & \text{in } [0,2\pi], \\ W_\lambda(0) = W_\lambda(2\pi). \end{cases} \tag{2.12}$$

Letting

$$S_\lambda(\theta) := \sqrt{\lambda-\widetilde{a}}\,\theta + \int_0^\theta W_\lambda(\theta')\,d\theta', \tag{2.13}$$

we have that S_λ satisfies

$$\begin{cases} -iS_\lambda''(\theta) + (S_\lambda'(\theta))^2 = \lambda - a(\theta), & \text{in } [0,2\pi], \\ S_\lambda'(0) = S_\lambda'(2\pi), \\ S_\lambda(0) = 0. \end{cases} \tag{2.14}$$

Lemma 2.5. *If* λ *is sufficiently large, then* $\int_0^{2\pi} W_\lambda(\theta)\,d\theta \in \mathbb{R}$.

PROOF. Let us define $\eta_\lambda(\theta) = \Re S_\lambda(\theta)$ and $\xi_\lambda(\theta) = \Im S_\lambda(\theta)$. Then (2.14) implies that

$$-\eta_\lambda'' + 2\eta_\lambda'\xi_\lambda' = 0, \quad \text{in } [0,2\pi],$$

so that

$$\eta_\lambda'(\theta) = C_\lambda e^{2\xi_\lambda(\theta)}, \quad \text{in } [0,2\pi], \tag{2.15}$$

where $C_\lambda = \sqrt{\lambda - \widetilde{a}} + \Re W_\lambda(0)$. We notice that (2.11) implies that, if λ is sufficiently large, then $C_\lambda \neq 0$. The condition $S'_\lambda(0) = S'_\lambda(2\pi)$ implies that $\eta'_\lambda(0) = \eta'_\lambda(2\pi)$ and hence from (2.15) it follows that

$$\xi_\lambda(0) = \xi_\lambda(2\pi).$$

Since $S_\lambda(0) = 0$, we have that $\xi_\lambda(0) = 0$ and then

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} W_\lambda(\theta)\,d\theta &= -\sqrt{\lambda - \widetilde{a}} + \frac{1}{2\pi}(\eta_\lambda(2\pi) + i\xi_\lambda(2\pi)) \\ &= -\sqrt{\lambda - \widetilde{a}} + \frac{\eta_\lambda(2\pi)}{2\pi} \in \mathbb{R}.\end{aligned}$$

□

Lemma 2.6. *Let* $\bar{A} \in \mathbb{R}$ *such that* $\bar{A} \notin \frac{1}{2}\mathbb{Z}$ *and let* $0 < \delta < \operatorname{dist}(\bar{A}, \frac{1}{2}\mathbb{Z})$. *Then there exists* $\bar{k} \in \mathbb{N}$ *such that for all* $k \in \mathbb{N}$, $k \geqslant \bar{k}$, *there exist* $\lambda_k^+, \lambda_k^- \in I_\delta$ *such that* $\lambda_k^+ \geqslant \bar{\lambda}_\delta$, $\lambda_k^- \geqslant \bar{\lambda}_\delta$ *and*

$$\sqrt{\lambda_k^+ - \widetilde{a}} = \bar{A} - \frac{1}{2\pi}\int_0^{2\pi} W_{\lambda_k^+}(\theta)\,d\theta + k, \tag{2.16}$$

$$\sqrt{\lambda_k^- - \widetilde{a}} = -\bar{A} - \frac{1}{2\pi}\int_0^{2\pi} W_{\lambda_k^-}(\theta)\,d\theta + k. \tag{2.17}$$

Proof. Let $g : [\bar{\lambda}_\delta, +\infty) \to \mathbb{R}$ be a continuous function such that

$$g(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} W_\lambda(\theta)\,d\theta \quad \text{for all } \lambda \in I_\delta$$

and $|g(\lambda)| \leqslant C_\delta/\sqrt{\lambda - \tilde{a}}$ for all $\lambda \geqslant \bar{\lambda}_\delta$. Then the function

$$f : [\bar{\lambda}_\delta, +\infty) \to \mathbb{R}, \quad f(\lambda) = \sqrt{\lambda - \tilde{a}} - \bar{A} + g(\lambda),$$

is continuous and $\lim_{\lambda\to+\infty} f(\lambda) = +\infty$. Therefore there exists $\bar{k}$ sufficiently large such that, for all $k \geqslant \bar{k}$, there exists $\lambda_k^+ \geqslant \bar{\lambda}_\delta$ such that $f(\lambda_k^+) = k$, i.e.

$$\sqrt{\lambda_k^+ - \tilde{a}} = k + \bar{A} - g(\lambda_k^+). \tag{2.18}$$

If $\bar{k}$ is sufficiently large, (2.18) implies that

$$\operatorname{dist}\left(\sqrt{\lambda_k^+ - \tilde{a}}, \tfrac{1}{2}\mathbb{Z}\right) = \operatorname{dist}(\bar{A}, \tfrac{1}{2}\mathbb{Z}) - g(\lambda_k^+) > \delta,$$

so that $\lambda_k^+ \in I_\delta$ and (2.16) is proved. The proof of (2.17) is analogous. □

Lemma 2.7. *Under the same assumptions as in Lemma 2.6, let, for all* $k \geqslant \bar{k}$, $\lambda_k^+, \lambda_k^- \in I_\delta$ *as in Lemma 2.6. Then*

$$\int_0^{2\pi} W_{\lambda_k^\pm}(\theta)\,d\theta = O\big(\tfrac{1}{k^3}\big), \quad \text{as } k \to +\infty \tag{2.19}$$

and

$$\sqrt{\lambda_k^+ - \widetilde{a}} = \bar{A} + k + O(\tfrac{1}{k^3}), \quad \sqrt{\lambda_k^- - \widetilde{a}} = -\bar{A} + k + O(\tfrac{1}{k^3}), \tag{2.20}$$

$$\lambda_k^+ = \widetilde{a} + (\bar{A} + k)^2 + O(\tfrac{1}{k^2}), \quad \lambda_k^- = \widetilde{a} + (-\bar{A} + k)^2 + O(\tfrac{1}{k^2}), \tag{2.21}$$

as $k \to +\infty$.

Proof. By integrating (2.12) between 0 and 2π and using estimate (2.11) we have that

$$\left|\int_0^{2\pi} W_\lambda(\theta)\,d\theta\right| = \left|-\frac{1}{2\sqrt{\lambda-\widetilde{a}}}\int_0^{2\pi} W_\lambda^2(\theta)\,d\theta\right| \leqslant \frac{\pi C_\delta}{(\lambda-\widetilde{a})^{3/2}}.$$

Since from (2.16) and (2.17) it follows that $\lambda_k^\pm \sim k^2$ as $k \to +\infty$, we derive (2.19), which yields (2.20) (and the (2.21) by squaring) in view of (2.16) and (2.17). ☐

Lemma 2.8. *Let $a \in W^{1,\infty}(\mathbb{S}^1)$ and $\bar{A} \in \mathbb{R} \setminus \frac{1}{2}\mathbb{Z}$. If $k \geqslant \bar{k}$, then*

$$\lambda_k^+, \lambda_k^- \in \sigma(\mathfrak{L}_{\bar{A},a}),$$

where $\mathfrak{L}_{\bar{A},a}\varphi = -(\varphi)'' + [a(\theta) + \bar{A}^2]\varphi - 2i\bar{A}\varphi$. Moreover

$$\varphi_k^+(\theta) = e^{-i\bar{A}\theta} e^{iS_{\lambda_k^+}(\theta)}, \quad \varphi_k^-(\theta) = e^{-i\bar{A}\theta} e^{-i\overline{S_{\lambda_k^-}(\theta)}}, \tag{2.22}$$

are eigenfunctions of $\mathfrak{L}_{\bar{A},a}$ associated to λ_k^+, λ_k^- respectively.

Proof. By direct calculations, we have that $\varphi_k^\pm$ satisfy

$$-(\varphi_k^\pm)''(\theta) + [a(\theta) + \bar{A}^2]\varphi_k^\pm(\theta) - 2i\bar{A}(\varphi_k^\pm)'(\theta) = \lambda_k^\pm \varphi_k^\pm(\theta)$$

in $[0, 2\pi]$, i.e. $\varphi_k^\pm$ are non-trivial solutions to

$$\mathfrak{L}_{\bar{A},a}\varphi_k^\pm = \lambda_k^\pm \varphi_k^\pm \quad \text{in } [0, 2\pi].$$

Furthermore (2.16) and (2.17) imply that

$$\varphi_k^\pm(0) = \varphi_k^\pm(2\pi) \quad \text{and} \quad (\varphi_k^\pm)'(0) = (\varphi_k^\pm)'(2\pi).$$

The lemma is thereby proved. ☐

We recall from [32] the following result, which is based on Kato's Perturbation Theory and which will be the key ingredient in the proof of Lemma 2.10 below.

Lemma 2.9. *Let $L_0, L : \mathcal{H} \to \mathcal{H}$ be two self-adjoint operators on a Hilbert space $\mathcal{H}$. Denote by*

$$R_0(\lambda) := (L_0 - \lambda I)^{-1} \qquad R(\lambda) := (L - \lambda I)^{-1}.$$

Then:

1. *if $R_0(\lambda), R(\lambda) \in \mathcal{L}(\mathcal{H})$, then λ is not an eigenvalue (neither for R_0 nor for $R(\lambda)$);*
2. *if the operator*

$$T := \frac{1}{2\pi i}\int_\Gamma (R(\lambda) - R_0(\lambda))\,d\lambda$$

has operator norm $\|T\|_{\mathcal{L}(\mathcal{H})} < 1$, being Γ a closed curve in the complex plane, then the number of eigenvalues (counted with multiplicity) of L_0 and L contained in the region bounded by Γ is the same.

As a consequence of Lemma 2.9 we can now describe how large eigenvalues distribute aroud the free ones.

Lemma 2.10. *Let* $a \in W^{1,\infty}(\mathbb{S}^1)$ *and* $\bar{A} \in (-\frac{1}{2}, \frac{1}{2}) \backslash \{0\}$*. For every* $\bar{\alpha} \geqslant \|a + \bar{A}^2\|^2_{L^\infty}$ *and* $c \geqslant 0$*, there exist* $\bar{\lambda} > 0$ *and* $k_0 > \bar{k}$ *such that*

$$\sigma(\mathfrak{L}_{\bar{A},a}) \cap [\bar{\lambda}, +\infty) \subset \bigcup_{k=k_0}^{\infty} B\big(k^2, c + \sqrt{\bar{\alpha} + 4k^2\bar{A}^2}\,\big).$$

Furthermore, if $k \geqslant k_0$*, each ball* $B\big(k^2, \sqrt{\bar{\alpha} + 4k^2\bar{A}^2}\big)$ *contains exactly two eigenvalues of* $\mathfrak{L}_{\bar{A},a}$ *(counted with their own multiplicity).*

Proof. We apply lemma 2.9 with

$$L_0 := -\frac{d^2}{d\theta^2} \qquad L := -\frac{d^2}{d\theta^2} - 2i\bar{A}\frac{d}{d\theta} + \alpha(\theta),$$

where $\alpha(\theta) = a(\theta) + \bar{A}^2$. Let

$$R_0(\lambda) = \Big(-\frac{d^2}{d\theta^2} - \lambda I\Big)^{-1}, \quad R(\lambda) = \Big(-\frac{d^2}{d\theta} - 2i\bar{A}\frac{d}{d\theta} + \alpha(\theta) - \lambda I\Big)^{-1}.$$

Notice that, via Fourier we can write, for $f = \sum(\alpha_k \sin(k\theta) + \beta_k \cos(k\theta))$,

$$R_0(\lambda)f = \sum \frac{1}{k^2 - \lambda}(\alpha_k \sin(k\theta) + \beta_k \cos(k\theta)),$$

therefore we have the estimate

$$\|R_0\|_{\mathcal{L}(L^2)} \leqslant \frac{1}{\operatorname{dist}(\lambda, \{k^2 : k \in \mathbb{Z}\})}.$$

Now notice that, formally, we can write

$$R(\lambda) = R_0(\lambda)\left(I + WR_0(\lambda)\right)^{-1}, \tag{2.23}$$

being $W = -2i\bar{A}\frac{d}{d\theta} + \alpha(\theta)$ a first order operator. Since $\frac{d}{d\theta}$ commutes with R_0 (by spectral theorem), we can write as follows:

$$\begin{aligned} WR_0(\lambda)f = -2i\bar{A}\sum_k \frac{k(\alpha_k \cos(k\theta) - \beta_k \sin(k\theta))}{k^2 - \lambda} \\ + \alpha(\theta)\sum_k \frac{1}{k^2 - \lambda}(\alpha_k \sin(k\theta) + \beta_k \cos(k\theta)). \end{aligned}$$

We hence obtain the following: if $\Re\lambda$ is large enough, $\bar{\alpha} \geqslant \|\alpha\|^2_{L^\infty}$, $c \geqslant 0$, and

$$|\lambda - k^2| > c + \big(4k^2\bar{A}^2 + \bar{\alpha}\big)^{\frac{1}{2}},$$

then

$$\|WR_0\|_{\mathcal{L}(L^2)} < 1. \tag{2.24}$$

Then, by (2.23) and (2.24), if k is large enough, outside of any ball with center in k^2 and radius $c + \big(4k^2\bar{A}^2 + \bar{\alpha}\big)^{\frac{1}{2}}$, the operator $R(\lambda)$ is bounded for large λ, hence we do not have large eigenvalues outside that balls. We notice that, if k is large, the balls with center in k^2 and radius $c + \big(4k^2\bar{A}^2 + \bar{\alpha}\big)^{\frac{1}{2}}$ are mutually disjoint, since $|\bar{A}| < \frac{1}{2}$ implies that

$$2c + \big(4k^2\bar{A}^2 + \bar{\alpha}\big)^{\frac{1}{2}} + \big(4(k+1)^2\bar{A}^2 + \bar{\alpha}\big)^{\frac{1}{2}} < (k+1)^2 - k^2$$

provided that k is sufficiently large. On the other hand, if Γ is the circle with center in k^2 and radius $\big(4k^2\bar{A}^2+\bar{\alpha}\big)^{\frac12}$ with k large, we can easily estimate

$$\left\|\frac{1}{2\pi i}\int_\Gamma (R(\lambda)-R_0(\lambda))f\,d\lambda\right\|_{L^2}<\|f\|_{L^2},$$

(use the Born expansion $(I+WR_0)^{-1}=\sum(WR_0)^n$) which together with point (2) of Lemma 2.9 gives the desired result.

Therefore, outside those balls there are no eigenvalues, and inside there are the same number of eigenvalues both for L_0 and L: this number is 2. $\square$

Lemma 2.11. *Let $a\in W^{1,\infty}(\mathbb{S}^1)$ and $\bar{A}\in(-\frac12,\frac12)\setminus\{0\}$. Let $\lambda_k^\pm$ be as in Lemma 2.6. Then there exist $c>0$, $\bar{\alpha}>0$, $\tilde{\lambda}>0$ and $\tilde{k}$ such that*

(i) *for all $k\geqslant\tilde{k}$, $\lambda_k^+,\lambda_k^-\in B\Big(k^2,c+\sqrt{\bar{\alpha}+4k^2\bar{A}^2}\Big)$;*

(ii) $\sigma(\mathfrak{L}_{\bar{A},a})\cap[\tilde{\lambda},+\infty)=\{\lambda_k^+,\lambda_k^-:k\geqslant\tilde{k}\}$.

Proof. From (2.21) we have that, if $c,\bar{\alpha}>0$ are chosen sufficiently large,

$$|\lambda_k^+-k^2|=\Big|\widetilde{a}+\bar{A}^2+2k\bar{A}+O(\tfrac{1}{k^2})\Big|<c+\sqrt{\bar{\alpha}+4k^2\bar{A}^2}$$

if k is large enough, thus proving (i) for λ_k^+. The proof of (i) for λ_k^- is analogous.

The statement (ii) follows by combining (i) and Lemma 2.10. $\square$

Proof of Lemma 2.1. From Lemmas 2.3 and 2.11 it follows that there exist $k^*\in\mathbb{N}$ and $\ell\in\mathbb{Z}$ such that $\{\mu_k:\ k>k^*\}=\{\lambda_j:\ j\in\mathbb{Z},\ |j|\geqslant\ell\}$ where

$$\lambda_j=\begin{cases}\lambda_{|j|}^-, & \text{if } j<0,\\ \lambda_{|j|}^+, & \text{if } j>0.\end{cases}$$

Then, in view of Lemma 2.7

$$\sqrt{\lambda_j-\widetilde{a}}=(\operatorname{sgn} j)\bar{A}+|j|+O\big(\tfrac{1}{|j|^3}\big),\quad \text{as } |j|\to+\infty.$$

From (2.22), (2.13), (2.19), and (2.20), it follows that

$$\varphi_j^+(\theta)=e^{-i\bar{A}\theta}\Big(e^{i(\bar{A}+j)\theta}+O\big(\tfrac{1}{|j|^3}\big)\Big),\quad \text{as } j\to+\infty,$$
$$\varphi_j^-(\theta)=e^{-i\bar{A}\theta}\Big(e^{i(\bar{A}-j)\theta}+O\big(\tfrac{1}{|j|^3}\big)\Big),\quad \text{as } j\to+\infty.$$

Therefore, letting, for $j\in\mathbb{Z}$ such that $|j|\geqslant\ell$,

$$\widetilde{\phi}_j=\begin{cases}\frac{\varphi_{|j|}^-}{\|\varphi_{|j|}^-\|_{L^2(0,2\pi)}}, & \text{if } j<0,\\ \frac{\varphi_{|j|}^+}{\|\varphi_{|j|}^+\|_{L^2(0,2\pi)}}, & \text{if } j>0,\end{cases}$$

we have that, for $|j|\geqslant\ell$, $\widetilde{\phi}_j$ is a $L^2\big((0,2\pi),\mathbb{C}\big)$-normalized eigenfunction of the operator $\mathfrak{L}_{\bar{A},a}$ corresponding to the eigenvalue λ_j and

$$\widetilde{\phi}_j(\theta)=\frac{1}{\sqrt{2\pi}}e^{-i\bar{A}\theta}\Big(e^{i(\bar{A}+j)\theta}+R_j(\theta)\Big),$$

where $\|R_j\|_{L^\infty(0,2\pi)} = O\big(\frac{1}{|j|^3}\big)$ as $j \to \infty$. Hence, in view of Lemma 2.3 we have that $\phi_j(\cos\theta, \sin\theta) = e^{i\bar{A}\theta} e^{-i\int_0^\theta A(s)\,ds} \widetilde{\phi}_j(\theta)$ is a $L^2\big(\mathbb{S}^1, \mathbb{C}\big)$-normalized eigenfunction of the operator $L_{\mathbf{A},a}$ on $\mathbb{S}^1$ corresponding to the eigenvalue λ_j and

$$\phi_j(\cos\theta, \sin\theta) = \frac{1}{\sqrt{2\pi}} e^{-i\left(\lfloor \widetilde{A}+1/2 \rfloor \theta + \int_0^\theta A(t)\,dt\right)} \Big(e^{i(\widetilde{A}+j)\theta} + R_j(\theta)\Big).$$

The proof is thereby complete. ☐

By means of the previous result, we immediately obtain the following Corollary.

Corollary 2.12. *Let k^*, ℓ as in Lemma 2.1 and K be given by (1.9), with ψ_k being any $L^2\big(\mathbb{S}^1, \mathbb{C}\big)$-normalized eigenfunctions of $L_{\mathbf{A},a}$ on $\mathbb{S}^1$ if $k \leqslant k^*$ and $\psi_k = \phi_j$ if $k > k^*$ and $\mu_k = \lambda_j$, with λ_j, ϕ_j being as in Lemma 2.1. Then, we have that*

$$\begin{aligned}(2.25)\quad K(x,y) &= \sum_{k=1}^{k^*} i^{-\beta_k} j_{-\alpha_k}(rr')\psi_k(\theta)\overline{\psi_k(\theta')} \\ &+ \frac{1}{2\pi} e^{-i\int_{\theta'}^{\theta} A(s)\,ds} e^{-i[\widetilde{A}+\frac{1}{2}](\theta-\theta')} \\ &\quad \times \sum_{|j|\geqslant \ell} i^{-\beta(\lambda_j)} j_{-\alpha(\lambda_j)}(rr') \Big(e^{i(\widetilde{A}+j)\theta} + R_j(\theta)\Big) \Big(e^{-i(\widetilde{A}+j)\theta'} + \overline{R_j(\theta')}\Big),\end{aligned}$$

if $x = (r\cos\theta, r\sin\theta)$ and $y = (r'\cos\theta', r'\sin\theta')$, where

$$\alpha(\lambda_j) := -\sqrt{\lambda_j}, \quad \beta(\lambda_j) := \sqrt{\lambda_j}, \tag{2.26}$$

and R_j is as in Lemma 2.1.

3. Proof of the main result

We can now perform the proof of Theorem 1.1. Let us first assume that condition (2.4) holds, so that the asymptotic expansion of eigenvalues and eigenfunctions stated in Lemma 2.1 holds. Let K be defined by (1.9); by formula (1.10), it is sufficient to show that

$$\sup_{x,y\in\mathbb{R}^2} |K(x,y)| < \infty.$$

In particular, the study of the boundedness of K is reduced, thanks to Corollary 2.12, to the study of the boundedness of the two series

$$\Sigma_{k\leqslant k^*} = \sum_{k=1}^{k^*} i^{-\beta_k} j_{-\alpha_k}(rr')\psi_k(\theta)\overline{\psi_k(\theta')}, \tag{3.1}$$

and

$$\Sigma_{|j|\geqslant\ell} = \sum_{|j|\geqslant\ell} i^{-\beta(\lambda_j)} j_{-\alpha(\lambda_j)}(rr') \Big(e^{i(\widetilde{A}+j)\theta} + R_j(\theta)\Big) \Big(e^{-i(\widetilde{A}+j)\theta'} + \overline{R_j(\theta')}\Big) \tag{3.2}$$

uniformly with respect to r, r', θ, θ'. Since $\mu_1(\mathbf{A}, a) > 0$, all the indices α_k in (1.6) are negative. Therefore, the Bessel functions $j_{-\alpha_k}$ are bounded functions, for any k. In addition, the functions ψ_k are obviously bounded, for any k: as a consequence, we obtain that

$$\sup_{\substack{r,r'\geqslant 0\\ \theta,\theta'\in\mathbb{S}^1}} |\Sigma_{k\leqslant k^*}(r, r', \theta, \theta')| < \infty. \tag{3.3}$$

In order to prove that $\Sigma_{|j|\geqslant\ell}$ is uniformly bounded, we compare it with the analogous kernel K_{ab} associated to the Aharonov-Bohm potential $\mathbf{A}_{ab} := \alpha\big(-\frac{x_2}{|x|^2},\frac{x_1}{|x|^2}\big)$, with $\alpha\in\mathbb{R}$, given by

$$K_{ab}(x,y)=\sum_{k\in\mathbb{Z}} i^{-\beta_k^{ab}} j_{-\alpha_k^{ab}}(|x||y|)\psi_k^{ab}\big(\tfrac{x}{|x|}\big)\overline{\psi_k^{ab}\big(\tfrac{y}{|y|}\big)},$$

where ψ_k^{ab} are the eigenfunctions defined in (2.2) of $L_{\mathbf{A}_{ab},0}$ associated to the eigenvalue $\mu_k^{ab}=(k+\alpha)^2$, and $\alpha_k^{ab},\beta_k^{ab}$ are given by (1.6) with μ_k replaced by μ_k^{ab}. We have explicitly

$$\alpha_k^{ab}=-\sqrt{\mu_k^{ab}}=-\left|k+\alpha\right|,\qquad \beta_k^{ab}=\sqrt{\mu_k^{ab}}=\left|k+\alpha\right|.$$

We choose $\alpha=\bar{A}$ with $\bar{A}$ as in (2.9), denote

$$\Sigma^{ab}_{|j|\geqslant\ell}(r,r',\theta,\theta')=\sum_{|j|\geqslant\ell} i^{-|j+\alpha|} j_{|j+\alpha|}(rr')e^{ij\theta}e^{-ij\theta'},$$

and write

$$\Sigma_{|j|\geqslant\ell}=\Big(\Sigma_{|j|\geqslant\ell}-e^{i\bar{A}(\theta-\theta')}\Sigma^{ab}_{|j|\geqslant\ell}\Big)+e^{i\bar{A}(\theta-\theta')}\Sigma^{ab}_{|j|\geqslant\ell}. \tag{3.4}$$

In the paper [16] it has been shown that

$$\sup_{\substack{r,r'\geqslant0\\ \theta,\theta'\in\mathbb{S}^1}}\left|e^{i\bar{A}(\theta-\theta')}\Sigma^{ab}_{|j|\geqslant\ell}(r,r',\theta,\theta')\right|=\sup_{\substack{r,r'\geqslant0\\ \theta,\theta'\in\mathbb{S}^1}}\left|\Sigma^{ab}_{|j|\geqslant\ell}(r,r',\theta,\theta')\right|<\infty. \tag{3.5}$$

To prove the uniform bound of $\Sigma_{|j|\geqslant\ell}$ is hence sufficient to prove the following claim:

$$\sup_{\substack{r,r'\geqslant0\\ \theta,\theta'\in\mathbb{S}^1}}\left|\Sigma_{|j|\geqslant\ell}(r,r',\theta,\theta')-e^{i\bar{A}(\theta-\theta')}\Sigma^{ab}_{|j|\geqslant\ell}(r,r',\theta,\theta')\right|<\infty. \tag{3.6}$$

In view of the above considerations, we now pass to prove that (3.6) holds.

Let us write

$$\Sigma_{|j|\geqslant\ell}-e^{i\bar{A}(\theta-\theta')}\Sigma^{ab}_{|j|\geqslant\ell}=K_1+K_2, \tag{3.7}$$

where

$$\begin{aligned}
K_1&=\sum_{|j|\geqslant\ell}\left[i^{-\beta(\lambda_j)}J_{-\alpha(\lambda_j)}(rr')-i^{-\left|j+\bar{A}\right|}J_{\left|j+\bar{A}\right|}(rr')\right]e^{i(j+\bar{A})\theta}e^{-i(j+\bar{A})\theta'}\\
K_2&=\sum_{|j|\geqslant\ell}i^{-\beta(\lambda_j)}J_{-\alpha(\lambda_j)}(rr')\times\\
&\qquad\times\left[\Big(e^{i(\bar{A}+j)\theta}+R_j(\theta)\Big)\Big(e^{-i(\bar{A}+j)\theta'}+\overline{R_j(\theta')}\Big)-e^{i(j+\bar{A})\theta}e^{-i(j+\bar{A})\theta'}\right].
\end{aligned}$$

Here we used the fact that in dimension $N=2$ we have $j_s\equiv J_s$, for any $s\in\mathbb{R}$.

Let us now recall the estimate

$$|J_\nu(r)|\leqslant\frac{C}{|\nu|^{\frac13}} \tag{3.8}$$

(see e.g. [2, 35]), which holds for some $C>0$ independent of x and ν. Moreover, by (2.5) and (2.26) we have that

$$-\alpha(\lambda_j)\sim|j|\qquad\text{as }|j|\to\infty. \tag{3.9}$$

In addition by Lemma 2.1

$$\left\|\Big(e^{i(\bar{A}+j)\theta}+R_j(\theta)\Big)\Big(e^{-i(\bar{A}+j)\theta'}+\overline{R_j(\theta')}\Big)-e^{i(j+\bar{A})\theta}e^{-i(j+\bar{A})\theta'}\right\|_{L^\infty(\mathbb{S}^1)} = O\Big(\frac{1}{|j|^3}\Big) \tag{3.10}$$

as $|j|\to+\infty$. Hence, by (3.8), (3.9) and (3.10) one easily gets

$$\sup_{\substack{r,r'\geqslant 0\\ \theta,\theta'\in\mathbb{S}^1}} |K_2(r,r',\theta,\theta')| \leqslant C\sum_{|j|\geqslant\ell}|j|^{-\frac{10}{3}}<\infty. \tag{3.11}$$

In order to get the analogous estimate for K_1, we now introduce another well known representation formula for the Bessel functions. Let $\gamma\subset\mathbb{C}$ be the positively oriented contour represented in Figure 1.

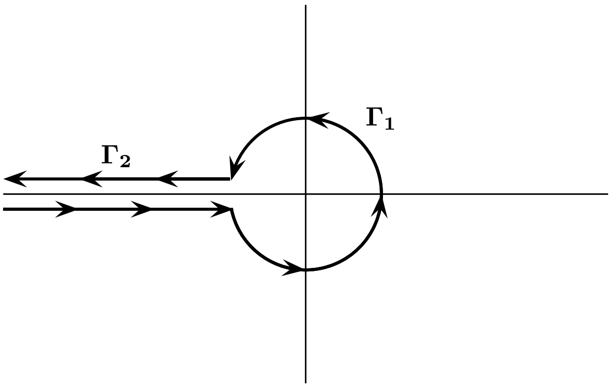

Figure 1. Integration oriented domain γ.

Then we have the representation

$$J_\nu(r)=\frac{1}{2\pi i}\int_\gamma e^{\frac{r}{2}\left(z-\frac{1}{z}\right)}\frac{dz}{z^{\nu+1}}$$

(see [34, 5.10.7]). Consequently, we obtain

$$\begin{aligned}K_1(r,r',\theta,\theta') &= \frac{1}{2\pi i}\sum_{|j|\geqslant\ell}\int_\gamma \frac{1}{z}e^{\frac{rr'}{2}\left(z-\frac{1}{z}\right)}\Big[(iz)^{\alpha(\lambda_j)}-(iz)^{-|j+\bar{A}|}\Big]e^{i(j+\bar{A})(\theta-\theta')}\,dz\\ &= \frac{1}{2\pi i}\sum_{|j|\geqslant\ell}\int_\gamma \frac{1}{z}e^{\frac{rr'}{2}\left(z-\frac{1}{z}\right)}(iz)^{-|j+\bar{A}|}\Big[(iz)^{-\sqrt{\lambda_j}+|j+\bar{A}|}-1\Big]e^{i(j+\bar{A})(\theta-\theta')}\,dz.\end{aligned} \tag{3.12}$$

From (2.5) it follows that

$$-\sqrt{\lambda_j}+|j+\bar{A}|=\sqrt{(j+\bar{A})^2}-\sqrt{\widetilde{a}+(j+\bar{A})^2+O\big(\tfrac{1}{j^2}\big)}=-\frac{\widetilde{a}}{2|j|}+O\left(j^{-2}\right). \tag{3.13}$$

Therefore, a first-order Taylor expansion in the last term of (3.12) gives in turn

$$
\begin{aligned}
&K_1(r,r',\theta,\theta') \\
&\quad = \frac{1}{2\pi i}\sum_{|j|\geqslant\ell}\int_\gamma \frac{1}{z}e^{\frac{rr'}{2}\left(z-\frac{1}{z}\right)}\left[-\frac{\widetilde{a}\log(iz)}{2|j|}\cdot\frac{e^{i(j+\bar{A})(\theta-\theta')}}{(iz)^{|j+\bar{A}|}}+\mathcal{R}_j(z)\right]dz
\end{aligned}
\tag{3.14}
$$

where $\|\mathcal{R}_j(z)\|_{L^\infty(\gamma)} = O(j^{-2})$ as $|j|\to+\infty$.

We observe that it is possible to exchange the order of summation and integration in (3.14), see the proof of Theorem 1.11 in [16] for details. We hence get

$$
\begin{aligned}
&K_1(r,r',\theta,\theta') \\
&\quad = -\frac{1}{2\pi i}\int_\gamma \frac{1}{2z}e^{\frac{rr'}{2}\left(z-\frac{1}{z}\right)}\widetilde{a}\log(iz)\sum_{|j|\geqslant\ell}\left[\frac{1}{|j|}\frac{e^{i(j+\bar{A})(\theta-\theta')}}{(iz)^{|j+\bar{A}|}}+O\left(j^{-2}\right)\right]dz.
\end{aligned}
\tag{3.15}
$$

Finally, we notice that (if ℓ is large enough)

$$
\begin{aligned}
&\sum_{|j|\geqslant\ell}^{\infty}\frac{1}{|j|}\frac{e^{i(j+\bar{A})(\theta-\theta')}}{(iz)^{|j+\bar{A}|}} \\
&= -\frac{e^{i\bar{A}(\theta-\theta')}}{(iz)^{\bar{A}}}\log\left[1-\frac{e^{i(\theta-\theta')}}{iz}\right]-\frac{e^{i\bar{A}(\theta-\theta')}}{(iz)^{-\bar{A}}}\log\left[1-\frac{e^{-i(\theta-\theta')}}{iz}\right] \\
&\quad+\sum_{1\leqslant|j|<\ell}\frac{1}{|j|}\frac{e^{i(j+\bar{A})(\theta-\theta')}}{(iz)^{|j|+\bar{A}\operatorname{sgn}j}},
\end{aligned}
$$

which together with (3.15) leads to

$$
\begin{aligned}
&K_1(r,r',\theta,\theta') \\
&= \frac{e^{i\bar{A}(\theta-\theta')}}{2\pi i}\int_\gamma\frac{1}{2z}e^{\frac{rr'}{2}\left(z-\frac{1}{z}\right)}\widetilde{a}\log(iz)\left(\frac{\log\left(1-\frac{e^{i(\theta-\theta')}}{iz}\right)}{(iz)^{\bar{A}}}+\frac{\log\left(1-\frac{e^{-i(\theta-\theta')}}{iz}\right)}{(iz)^{-\bar{A}}}\right) \\
&\qquad + \text{bounded terms.}
\end{aligned}
$$

In conclusion, since $\left|e^{\frac{r}{2}\left(z-\frac{1}{z}\right)}\right| = 1$ on Γ_1 and $\log\left(1-\frac{e^{\pm i(\theta-\theta')}}{iz}\right)\sim-\frac{e^{\pm i(\theta-\theta')}}{iz}$ as $|z|\to\infty$, we obtain the desired estimate

$$
\sup_{\substack{r,r'\geqslant 0\\ \theta,\theta'\in\mathbb{S}^1}}|K_1(r,r',\theta,\theta')|<\infty,
\tag{3.16}
$$

which together with (3.7) and (3.11) proves claim (3.6). The proof now follows by (3.3), (3.4), (3.5) and (3.6).

In the resonant case $\widetilde{A}\in\frac{1}{2}\mathbb{Z}$, we can repeat exactly the same arguments as above, using the classical estimates by Borg [5] and Gurarie [28] (see Remark 2.2) instead of Lemma 2.1; for more details we refer to the preprint version [17, Lemmas B.9 and B.10] of the present paper where a complete proof of such estimates is given; we observe that, although the control on the remainder terms of the asymptotic expansion is in this case less strong than in the non-resonant case, it is easy to verify that it is enough both for (3.13) and to estimate $\sup|K_2|$ with $C\sum_{|j|\geqslant\ell}|j|^{-\frac{4}{3}}<\infty$ in order to ensure (3.11).

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