Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition *

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Abstract

For the positive solutions of the Gross–Pitaevskii system

$$\begin{cases} -\Delta u_\beta + \lambda_\beta u_\beta = \omega_1 u_\beta^3 - \beta u_\beta v_\beta^2 \\ -\Delta v_\beta + \mu_\beta v_\beta = \omega_2 v_\beta^3 - \beta u_\beta^2 v_\beta \end{cases}$$

we prove that L^∞–boundedness implies $C^{0,\alpha}$–boundedness, uniformly as $\beta \to +\infty$, for every $\alpha \in (0,1)$. Moreover we prove that the limiting profile, as $\beta \to +\infty$, is Lipschitz continuous. The proof relies upon the blow–up technique and the monotonicity formulae by Almgren and Alt–Caffarelli–Friedman. This system arises in the Hartree-Fock approximation theory for binary mixtures of Bose–Einstein condensates in different hyperfine states. Extensions to systems with $k > 2$ densities are given.

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1 Introduction

The purpose of this paper is to prove uniform bounds in Hölder norm for families of positive solutions to nonlinear Schrödinger equations of the form

$$\begin{cases} -\Delta u_\beta + \lambda_\beta u_\beta = \omega_1 u_\beta^3 - \beta u_\beta v_\beta^2 \\ -\Delta v_\beta + \mu_\beta v_\beta = \omega_2 v_\beta^3 - \beta u_\beta^2 v_\beta \\ u_\beta, v_\beta \in H_0^1(\Omega), \end{cases} \tag{1}$$

for the competition parameter $\beta \in (0, +\infty)$. Such systems arise in different physical applications, such as the determination of standing waves in a binary mixture of Bose-Einstein condensates in two different hyperfine states. While the sign of the parameter ω_i discriminates between the focusing and defocusing behavior of a single component, the sign of β determines the type of interplay between the two states. When positive, the two states are in competition and repel each other. In this paper we deal with diverging interspecific competition rates (both in the focusing and defocusing case). The limiting behavior is known for the ground state solutions: as $\beta \to +\infty$ the wave amplitudes segregate, that is, their supports tend to be disjoint. This phenomenon, called phase separation, has been studied, in the case of $\omega_i > 0$ (focusing), starting from [7, 8], and, when $\omega_i < 0$ (defocusing), in [6]. As far as the excited states are concerned, the recent literature

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shows that other families of solutions exist for large β's ([18, 12, 17]). The asymptotic behavior of such families of solutions has been investigated in [19], where, in the case of planar systems, it is proved uniform convergence to a segregated limiting profile (u, v), where each component satisfies the equation

$$\begin{cases} -\Delta u + \lambda u &= \omega_1 u^3 \quad \text{in } \{u > 0\}, \\ -\Delta v + \mu v &= \omega_2 v^3 \quad \text{in } \{v > 0\}. \end{cases} \tag{2}$$

In this paper we improve the result of [19], proving bounds in Hölder norms whenever $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, in dimension $N = 2, 3$ (and also in higher dimension, provided the cubic nonlinearities are replaced with subcritical ones). Besides the validity of the equations above, we prove Lipschitz regularity of the limiting profile. Our result relies upon the blow–up technique (section 3) and suitable Liouville–type theorems (section 2). Such a strategy has been already adopted by some of the authors in [9] in proving uniform Hölder estimates for competition–diffusion systems with Lotka–Volterra type of interactions. The arguments there, however, though helpful in the present situation, need to be complemented with some new ideas, including a proper use of the Almgren's frequency formula [1]. This requires the systems to have a gradient form. Let us mention that Hölder estimates for (non gradient) coupling arising in combustion theory have been obtained in [5]. Regularity of the limiting profile and its nodal set, for ground states and other minimizing vector solutions has been established in [11, 4]. Our main results write as follows.

(1.1) Theorem. *Let u_β, v_β be positive solutions of* (1) *uniformly bounded in $L^\infty(\Omega)$, where λ_β, μ_β are bounded in $\mathbb{R}$ and ω_1, ω_2 are fixed constants. Then for every $\alpha \in (0, 1)$ there exists $C > 0$, independent of β, such that*

$$\|(u_\beta, v_\beta)\|_{C^{0,\alpha}(\overline{\Omega})} \le C \quad \textit{for every } \beta > 0.$$

(1.2) Theorem. *Under the assumptions of the previous theorem, there exists a pair (u, v) of Lipschitz continuous functions such that, up to a subsequence, there holds*

(i) $u_\beta \to u, v_\beta \to v$ *in* $C^{0,\alpha}(\overline{\Omega}) \cap H^1(\Omega)$, $\forall\ \alpha \in (0, 1)$;

(ii) $u \cdot v \equiv 0$ *in* Ω *and* $\displaystyle\int_\Omega \beta u_\beta^2 v_\beta^2 \to 0$ *as* $\beta \to +\infty$;

(iii) the limiting functions u, v satisfy system (2) *with* $\lambda := \lim_{\beta\to+\infty} \lambda_\beta$, $\mu := \lim_{\beta\to+\infty} \mu_\beta$.

For the sake of simplicity we consider here systems of two components, but all the results extend to the case of systems of k equations

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \omega_i u_i^3 - \beta u_i \displaystyle\sum_{i\neq j} \beta_{ij} u_j^2, \quad i = 1, \dots, k \\ u_i \in H_0^1(\Omega), \end{cases} \tag{3}$$

provided that it possesses a gradient structure, i.e. $\beta_{ij} = \beta_{ji}$ (see Remark 3.11).

The study of system (1) will be carried out as a particular case of a more general one, where L^2–perturbations are allowed. The reason for this approach is that, in a forthcoming paper, the authors intend to present a variational construction to obtain, for every fixed β, several solutions of (1); the present estimates, in their more general version, will then be used to study how, and in which sense, such a variational structure passes to the limit as $\beta \to +\infty$. To be more precise, let us consider the system

$$\begin{cases} -\Delta u_\beta + \lambda_\beta u_\beta = \omega_1 u_\beta^3 - \beta u_\beta v_\beta^2 + h_\beta & \text{in } \Omega \\ -\Delta v_\beta + \mu_\beta v_\beta = \omega_2 v_\beta^3 - \beta u_\beta^2 v_\beta + k_\beta & \text{in } \Omega \\ u_\beta,\ v_\beta \in H_0^1(\Omega), \quad u_\beta, v_\beta \ge 0 & \text{in } \Omega \end{cases} \tag{4}$$

under the assumptions: h_β, k_β are uniformly bounded in $L^2(\Omega)$, $\lambda_\beta, \mu_\beta \in \mathbb{R}$ are bounded in $\mathbb{R}$, $\omega_1, \omega_2 \in \mathbb{R}$ are fixed constants. Defining

$$\alpha^* = \begin{cases} 1 & \text{if } N = 2 \\ 1/2 & \text{if } N = 3, \end{cases} \tag{5}$$

then, by Sobolev embedding, we have that any solution of (4) belongs to $C^{0,\alpha}$, for every $\alpha \in (0, \alpha^*)$ (and even $\alpha = \alpha^*$ if $N = 3$). As a consequence, in the general case, we can not expect boundedness for every Hölder exponent. In fact we have the following.

(1.3) Theorem. *Let u_β, v_β be solutions of (4) uniformly bounded in $L^\infty(\Omega)$. Then for every $\alpha \in (0, \alpha^*)$ there exists $C > 0$, independent of β, such that*

$$\|(u_\beta, v_\beta)\|_{C^{0,\alpha}(\overline{\Omega})} \leq C \quad \textit{for every } \beta > 0.$$

(1.4) Theorem. *Let u_β, v_β be solutions of (4) uniformly bounded in $L^\infty(\Omega)$. Then there exists $(u, v) \in C^{0,\alpha}$, $\forall\ \alpha \in (0, \alpha^*)$, such that (up to a subsequence) there holds, as $\beta \to +\infty$,*

(i) $u_\beta \to u, v_\beta \to v$ *in* $C^{0,\alpha}(\overline{\Omega}) \cap H^1(\Omega)$, $\forall\ \alpha \in (0, \alpha^*)$;

(ii) $u \cdot v \equiv 0$ *in* Ω *and* $\displaystyle\int_\Omega \beta u_\beta^2 v_\beta^2 \to 0$;

(iii) the limiting functions u, v satisfy the system

$$\begin{cases} -\Delta u + \lambda u &= \omega_1 u^3 + h \quad \textit{in } \{u > 0\}, \\ -\Delta v + \mu v &= \omega_2 v^3 + k \quad \textit{in } \{v > 0\}, \end{cases} \tag{6}$$

where $\lambda := \lim \lambda_\beta$, $\mu := \lim \mu_\beta$, and h, k denote the L^2–weak limits of h_β, k_β as $\beta \to +\infty$.

Even though the actual result is stronger (no limitation on α), the proof of Theorem 1.1 is in fact a particular case of the one of Theorem 1.3, once one observes that, if $h_\beta \equiv k_\beta \equiv 0$, then u_β and v_β, at any fixed β, belong to $C^{1,\alpha}$ for every $\alpha \in (0, 1)$. For this reason, we will prove in the details all the results in the case of system (4), except the Lipschitz continuity of the limiting state (section 4), that requires $h_\beta \equiv k_\beta \equiv 0$.

We wish to mention that system (1) is of great interest also in the complementary case we do not face, namely when β is negative, for instance because of its application to the study of incoherent solitons in nonlinear optics. For results in this direction we refer the reader to [3, 12, 15, 16] and references therein.

2 Liouville–type results

In this section we prove some nonexistence results in $\mathbb{R}^N$. The main tools will be the monotonicity formula by Alt, Caffarelli, Friedman originally stated in [2], as well as some generalizations made by Conti, Terracini, Verzini ([10, 9]).

(2.1) Lemma (Monotonicity formula). *Let $u, v \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ be nonnegative functions such that $u \cdot v \equiv 0$. Assume moreover that $-\Delta u \leq 0, -\Delta v \leq 0$ in $\mathbb{R}^N$ and $u(x_0) = v(x_0) = 0$. Then the function*

$$J(r) := \frac{1}{r^2} \int_{B_r(x_0)} \frac{|\nabla u|^2}{|x - x_0|^{N-2}} \cdot \frac{1}{r^2} \int_{B_r(x_0)} \frac{|\nabla v|^2}{|x - x_0|^{N-2}}$$

is non decreasing for $r \in (0, +\infty)$.

(2.2) Proposition. *Under the same assumptions of Lemma 2.1, assume moreover that for some* $\alpha \in (0,1)$ *there holds*

$$\sup_{x,y\in\mathbb{R}^N} \frac{|u(x)-u(y)|}{|x-y|^\alpha}, \sup_{x,y\in\mathbb{R}^N} \frac{|v(x)-v(y)|}{|x-y|^\alpha} < \infty. \tag{7}$$

Then either $u \equiv 0$ *or* $v \equiv 0$.

Proof. In the following we will denote $B_r := B_r(x_0)$. Assume by contradiction that neither u nor v is zero, then none of them is constant since $u(x_0) = v(x_0) = 0$. Hence Lemma 2.1 ensures the existence of a constant $C > 0$ such that

$$\int_{B_r} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} \cdot \int_{B_r} \frac{|\nabla v|^2}{|x-x_0|^{N-2}} \geq Cr^4 \tag{8}$$

for every r sufficiently large. Let $\eta_{a,b}$ $(0 < a < b)$ be any smooth, radial, cut-off function with the following properties: $0 \leq \eta_{a,b} \leq 1$, $\eta_{a,b} = 0$ in $\mathbb{R}^N \setminus B_b$, $\eta_{a,b} = 1$ in B_a and $|\nabla \eta_{a,b}| \leq C/(b-a)$. Given $0 < \varepsilon << r$, let $A_\varepsilon := B_{2r} \setminus B_\varepsilon$ and $\eta := \eta_{r,2r}(1 - \eta_{\varepsilon,2\varepsilon})$. Testing the inequality $-\Delta u \leq 0$ with the function $\eta^2 u/|x-x_0|^{N-2}$ in the annulus A_ε, we obtain

$$\begin{aligned}
\int_{A_\varepsilon} \frac{\eta^2|\nabla u|^2}{|x-x_0|^{N-2}} &\leq -\int_{A_\varepsilon} \left[\frac{2\eta u}{|x-x_0|^{N-2}} \nabla u \cdot \nabla \eta + \eta^2 u \nabla u \cdot \nabla \left(\frac{1}{|x-x_0|^{N-2}}\right)\right] \\
&\leq \int_{A_\varepsilon} \left[\frac{1}{2}\frac{\eta^2|\nabla u|^2}{|x-x_0|^{N-2}} + 2\frac{u^2|\nabla \eta|^2}{|x-x_0|^{N-2}} - \eta^2 u \nabla u \cdot \nabla \left(\frac{1}{|x-x_0|^{N-2}}\right)\right].
\end{aligned}$$

We can rewrite the last term using the fact that $1/|x-x_0|^{N-2}$ is harmonic in A_ε:

$$\begin{aligned}
0 &= \int_{A_\varepsilon} \nabla\left(\frac{\eta^2 u^2}{2}\right) \cdot \nabla\left(\frac{1}{|x-x_0|^{N-2}}\right) = \\
&= \int_{A_\varepsilon} \left[\eta u^2 \nabla \eta \cdot \nabla\left(\frac{1}{|x-x_0|^{N-2}}\right) + \eta^2 u \nabla u \cdot \nabla\left(\frac{1}{|x-x_0|^{N-2}}\right)\right],
\end{aligned}$$

obtaining

$$\frac{1}{2}\int_{A_\varepsilon} \frac{\eta^2|\nabla u|^2}{|x-x_0|^{N-2}} \leq \int_{A_\varepsilon} \left[2\frac{u^2|\nabla \eta|^2}{|x-x_0|^{N-2}} + \eta u^2 \nabla \eta \cdot \nabla\left(\frac{1}{|x-x_0|^{N-2}}\right)\right].$$

By the definition of η, the last expression becomes

$$\frac{1}{2}\int_{B_r\setminus B_{2\varepsilon}} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} \leq \frac{C}{\varepsilon^N}\int_{B_{2\varepsilon}} u^2 + \frac{C'}{r^2}\int_{B_{2r}\setminus B_r} \frac{u^2}{|x-x_0|^{N-2}} + \frac{C''}{r}\int_{B_{2r}\setminus B_r} \frac{u^2}{|x-x_0|^{N-1}}.$$

Keeping in mind that $u(x_0) = 0$, we let now $\varepsilon \to 0$, obtaining

$$\int_{B_r} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} \leq \frac{C'}{r^2}\int_{B_{2r}\setminus B_r} \frac{u^2}{|x-x_0|^{N-2}} + \frac{C''}{r}\int_{B_{2r}\setminus B_r} \frac{u^2}{|x-x_0|^{N-1}}.$$

Using the assumptions on u, this implies

$$\int_{B_r} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} \leq \frac{C}{r^2}\int_0^{2r} \frac{\rho^{2\alpha}}{\rho^{N-2}}\rho^{N-1}d\rho + \frac{C'}{r}\int_0^{2r} \frac{\rho^{2\alpha}}{\rho^{N-1}}\rho^{N-1}d\rho \leq C''r^{2\alpha}.$$

Since the same result holds for v, we finally obtain

$$\int_{B_r} \frac{|\nabla u|^2}{|x-x_0|^{N-2}} \int_{B_r} \frac{|\nabla v|^2}{|x-x_0|^{N-2}} \leq Cr^{4\alpha},$$

which contradicts (8) for r large and $\alpha < 1$. □

(2.3) Corollary. *Let u be an harmonic function in $\mathbb{R}^N$ such that for some $\alpha \in (0,1)$ there holds*

$$\sup_{x,y\in\mathbb{R}^N} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty.$$

Then u is constant.

Proof. If $u \geq 0$ or $u \leq 0$, then since u is harmonic it holds that it is a constant (this is the usual nonexistence Liouville result). Otherwise if u changes sign, then we can apply the previous result to its positive and negative parts. □

(2.4) Remark. The previous result does not hold for $\alpha = 1$: consider for instance the function $u(x) = x_1$ (analogously, it is possible to see that also system (9) below admits non trivial solutions which are globally bounded in Lipschitz norm; these are the main reasons for which our strategy, as it is, can not apply to prove uniform Lipschitz estimates).

We shall need a result similar to Proposition 2.2, for functions u, v which do not have disjoint supports, but are positive solutions in $H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ of the system

$$\begin{cases} -\Delta u &= -uv^2 \quad \text{in } \mathbb{R}^N \\ -\Delta v &= -u^2 v \quad \text{in } \mathbb{R}^N. \end{cases} \tag{9}$$

Again, to obtain a Liouville–type result for the previous system, we will use a suitable generalization of the monotonicity formula (a similar idea, even though with slightly different equations, can be found in [10, 9]). To this aim we introduce a C^1 auxiliary function

$$f(r) = \begin{cases} \dfrac{2-N}{2} r^2 + \dfrac{N}{2} & r \leq 1 \\ \dfrac{1}{r^{N-2}} & r > 1 \end{cases}$$

and denote $m(|x|) := -\Delta f(|x|)/2$. Notice that $m(|x|)$ is bounded on $\mathbb{R}^N$, vanishes in $\mathbb{R}^N \setminus B_1$ and $m(|x|) \geq 0$ for a.e. x.

(2.5) Lemma. *Let u, v be positive solutions of (9) and let $\varepsilon > 0$ be fixed. Then there exists $\bar{r} > 1$ such that the function*

$$J(r) := \frac{1}{r^{4-\varepsilon}} \int_{B_r(0)} \left[f(|x|) \left(|\nabla u|^2 + u^2 v^2 \right) + m(|x|) u^2 \right] \cdot \int_{B_r(0)} \left[f(|x|) \left(|\nabla v|^2 + u^2 v^2 \right) + m(|x|) v^2 \right]$$

is increasing for $r \in (\bar{r}, +\infty)$.

Proof. Let us first evaluate the derivative of $J(r)$ for $r > 1$. In order to simplify notations we shall denote $J(r) = J_1(r) J_2(r)/r^{4-\varepsilon}$. Then we have

$$\frac{J'(r)}{J(r)} = -\frac{4-\varepsilon}{r} + \frac{\int_{\partial B_r} f(|x|)(|\nabla u|^2 + u^2 v^2)}{J_1(r)} + \frac{\int_{\partial B_r} f(|x|)(|\nabla v|^2 + u^2 v^2)}{J_2(r)} \tag{10}$$

(recall that $m(r) = 0$ for $r > 1$). We can rewrite the term J_1 in a different way: by testing the equation for u with $f(|x|)u$ on B_r, we obtain

$$\begin{aligned} \int_{B_r} f(|x|)(|\nabla u|^2 + u^2 v^2) &= -\int_{B_r} \nabla(\frac{u^2}{2}) \cdot (\nabla f(|x|)) + \int_{\partial B_r} f(|x|) u \partial_\nu u = \\ &= -\int_{B_r} m(|x|) u^2 + \int_{\partial B_r} \{ f(|x|) u \partial_\nu u - \frac{u^2}{2} \partial_\nu (f(|x|)) \}, \end{aligned}$$

which gives

$$J_1(r) = \frac{1}{r^{N-2}} \int_{\partial B_r} u\partial_\nu u + \frac{N-2}{r^{N-1}} \int_{\partial B_r} \frac{u^2}{2}. \tag{11}$$

In order to estimate this quantity we define

$$\Lambda_1(r) := \frac{r^2 \int_{\partial B_r} (|\nabla_\theta u|^2 + u^2 v^2)}{\int_{\partial B_r} u^2}, \qquad \Lambda_2(r) := \frac{r^2 \int_{\partial B_r} (|\nabla_\theta v|^2 + u^2 v^2)}{\int_{\partial B_r} v^2},$$

where $|\nabla_\theta u|^2 = |\nabla u|^2 - |\partial_\nu u|^2$. Then for every $\delta \in \mathbb{R}$, by Young's inequality, there holds

$$\begin{aligned}
\left| \int_{\partial B_r} u\partial_\nu u \right| &\leq \left(\int_{\partial B_r} u^2 \right)^{1/2} \left(\int_{\partial B_r} (\partial_\nu u)^2 \right)^{1/2} \\
&\leq \frac{\sqrt{\Lambda_1(r)}}{2\delta^2 r} \int_{\partial B_r} u^2 + \frac{\delta^2 r}{2\sqrt{\Lambda_1(r)}} \int_{\partial B_r} (\partial_\nu u)^2 \\
&\leq \frac{1}{2} \left[\frac{1}{\delta^2} \int_{\partial B_r} \left(|\nabla_\theta u|^2 + u^2 v^2 \right) + \delta^2 \int_{\partial B_r} (\partial_\nu u)^2 \right] \frac{r}{\sqrt{\Lambda_1(r)}}.
\end{aligned}$$

Substituting in (11) we obtain

$$J_1(r) \leq \frac{1}{2r^{N-3}} \left[\left(\frac{1}{\delta^2 \sqrt{\Lambda_1(r)}} + \frac{N-2}{\Lambda_1(r)} \right) \int_{\partial B_r} (|\nabla_\theta u|^2 + u^2 v^2) + \frac{\delta^2}{\sqrt{\Lambda_1(r)}} \int_{\partial B_r} (\partial_\nu u)^2 \right].$$

Now we choose δ in such a way that $\dfrac{1}{\delta^2 \sqrt{\Lambda_1(r)}} + \dfrac{N-2}{\Lambda_1(r)} = \dfrac{\delta^2}{\sqrt{\Lambda_1(r)}}$, or equivalently, after some calculation,

$$\frac{\sqrt{\Lambda_1(r)}}{\delta^2} = \gamma(\Lambda_1(r)),$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$\gamma(x) = \sqrt{\left(\frac{N-2}{2} \right)^2 + x} - \frac{N-2}{2}.$$

We remark that this function plays a crucial role in the proof of the Alt-Caffarelli-Friedman Monotonicity Formula (see [2]). Of particular importance is the following property: let E_1, E_2 be any couple of disjoint subsets of the sphere S^{N-1} and denote with $\lambda(E_i)$ the first eigenvalue of the Dirichlet Laplacian on S^{N-1}, then

$$\gamma(\lambda(E_1)) + \gamma(\lambda(E_2)) \geq 2. \tag{12}$$

With this choice of δ we have

$$J_1(r) \leq \frac{r}{2\gamma(\Lambda_1(r))} \int_{\partial B_r} f(|x|)(|\nabla u|^2 + u^2 v^2),$$

(recall that $r > 1$ and consequently $f(r) = 1/r^{N-2}$) and a similar expression holds also for J_2. Substituting in (10) we obtain

$$\frac{J'(r)}{J(r)} \geq -\frac{4-\varepsilon}{r} + \frac{2\gamma(\Lambda_1(r))}{r} + \frac{2\gamma(\Lambda_2(r))}{r},$$

therefore it only remains to prove that there exists a $\bar{r} > 1$ such that for every $r \geq \bar{r}$ there holds

$$\gamma(\Lambda_1(r)) + \gamma(\Lambda_2(r)) > \frac{4-\varepsilon}{2}. \tag{13}$$

To this aim we define the functions $u_{(r)}(\theta), v_{(r)}(\theta) : \partial B_1(0) \to \mathbb{R}$ as $u_{(r)}(\theta) := u(r\theta)$, $v_{(r)}(\theta) := v(r\theta)$. Then a change of variables gives

$$\Lambda_1(r) = \frac{\int_{\partial B_1}(|\nabla u_{(r)}|^2 + r^2 u_{(r)}^2 v_{(r)}^2)}{\int_{\partial B_1} u_{(r)}^2}, \qquad \Lambda_2(r) = \frac{\int_{\partial B_1}(|\nabla v_{(r)}|^2 + r^2 u_{(r)}^2 v_{(r)}^2)}{\int_{\partial B_1} v_{(r)}^2}.$$

The idea now is to show that the functions $u_{(r)}, v_{(r)}$ (normalized in $L^2(\partial B_1)$) converge as $r \to +\infty$ to some functions having disjoint supports, and then to take advantage of (12). Notice first of all that there exists a constant $C > 0$ such that $\int_{\partial B_1} u_{(r)}^2 \geq C$ for r sufficiently large. Indeed assume by contradiction this is not true, then $\frac{1}{|\partial B_r|}\int_{\partial B_r} u \to 0$ as $r \to +\infty$, which implies $u(0) = 0$ since u is subharmonic, and this contradicts the assumption $u > 0$. The same result clearly holds also for $v_{(r)}$.

Assume (13) does not hold, then there exists $r_n \to +\infty$ such that

$$\gamma(\Lambda_1(r_n)) + \gamma(\Lambda_2(r_n)) \leq \frac{4-\varepsilon}{2} < 2.$$

In particular, $\Lambda_1(r_n)$ and $\Lambda_2(r_n)$ are bounded. As a consequence the function

$$\tilde{u}_{(r_n)} := \frac{u_{(r)}}{\|u_{(r)}\|_{L^2(\partial B_1)}} \qquad \text{satisfies} \qquad C \geq \Lambda_1(r_n) \geq \int_{\partial B_1} |\nabla \tilde{u}_{(r_n)}|^2$$

(and an analogous property holds for $\tilde{v}_{(r_n)} := v_{(r)}/\|v_{(r)}\|_{L^2(\partial B_1)}$). This ensures the existence of $\bar{u}, \bar{v} \neq 0$ such that $\tilde{u}_{(r_n)} \rightharpoonup \bar{u}$, $\tilde{v}_{(r_n)} \rightharpoonup \bar{v}$ in $H^1(\partial B_1(0))$. Moreover, since

$$C \geq \Lambda_1(r_n) \geq r_n^2 \int_{\partial B_1} \tilde{u}_{(r_n)}^2 \tilde{v}_{(r_n)}^2$$

we infer that $\bar{u} \cdot \bar{v} \equiv 0$. This immediately provides

$$\liminf_{n\to+\infty}[\gamma(\Lambda_1(r_n)) + \gamma(\Lambda_2(r_n))] \geq \gamma(\lambda(\{\mathrm{supp}(\bar{u})\})) + \gamma(\lambda(\{\mathrm{supp}(\bar{v})\})),$$

that is in contradiction with (12). □

Now that we have a suitable monotonicity formula we are ready to prove a Liouville–type result for the considered system.

(2.6) Proposition. *Let u, v be non negative solutions of (9). Assume moreover that (7) holds for some $\alpha \in (0,1)$. Then one of the functions is identically zero and the other is a constant.*

Proof. We start by noticing that, due to the form of system (9), if one of the functions is 0 or a positive constant, then the other must be a constant or 0 respectively. Hence we assume by contradiction that neither u nor v is constant. Then by the maximum principle u and v are positive, and Lemma 2.5 ensures the existence of a constant $C > 0$ such that

$$\int_{B_r} \left[f(|x|)\left(|\nabla u|^2 + u^2v^2\right) + m(|x|)u^2\right] \int_{B_r} \left[f(|x|)\left(|\nabla v|^2 + u^2v^2\right) + m(|x|)v^2\right] \geq Cr^{4-\varepsilon} \tag{14}$$

for r sufficiently large. Let $\eta = \eta_{r,2r}$ be the cut-off function defined in the proof of Proposition 2.2. By testing the equation for u with $\eta^2 f u$ on B_{2r} we obtain

$$\begin{aligned}\int_{B_{2r}} \eta^2 f \cdot (|\nabla u|^2 + u^2v^2) &= -\int_{B_{2r}} \left[2f\eta u \nabla u \cdot \nabla\eta + \eta^2 \nabla\left(\frac{u^2}{2}\right)\cdot \nabla f\right] \leq \\ &\leq \int_{B_{2r}} \left[\frac{1}{2} f\eta^2|\nabla u|^2 + 2fu^2|\nabla\eta|^2 - \nabla\left(\frac{\eta^2u^2}{2}\right)\cdot\nabla f + u^2\eta\nabla\eta\cdot\nabla f\right].\end{aligned}$$

Recalling that $\Delta f = -2m$ and testing it with $\eta^2 u^2/2$ in B_{2r} we have

$$\int_{B_{2r}} \nabla\left(\frac{\eta^2 u^2}{2}\right)\cdot\nabla f = \int_{B_{2r}} \eta^2 u^2 m,$$

which substituted in the previous inequality, together with $m \geq 0$, gives

$$\int_{B_{2r}} \eta^2 \left[f\cdot(|\nabla u|^2 + u^2 v^2) + m u^2\right] \leq 2\int_{B_{2r}} \left[2 f u^2 |\nabla\eta|^2 + u^2 \eta \nabla\eta\cdot\nabla f\right].$$

Now, recalling the definition of η and f and using assumption (7), we finally obtain

$$\int_{B_r} \left[f\cdot(|\nabla u|^2 + u^2 v^2) + m u^2\right] \leq C\int_{B_{2r}\setminus B_r} \frac{u^2}{|x|^N} \leq C\int_0^{2r} \frac{\rho^{2\alpha}}{\rho^N}\rho^{N-1} d\rho \leq C r^{2\alpha},$$

which contradicts (14) for r large enough. □

Arguing as above, one can prove the following Liouville–type theorem for systems with an arbitrary number of densities.

(2.7) Proposition. *Let $k \geq 3$ and $u_1, \ldots, u_k$ be non negative solutions of*

$$-\Delta u_i = -u_i \sum_{j\neq i} u_j^2 \quad \text{in } \mathbb{R}^N, \tag{15}$$

with the property that, for some $\alpha \in (0,1)$,

$$\sup_{x,y\in\mathbb{R}^N} \frac{|u_i(x) - u_i(y)|}{|x-y|^\alpha} < \infty \quad \text{for every } i.$$

Then $k-1$ functions are identically zero and the remaining one is constant.

Sketch of the proof. We want to see that, for any $i \neq j$, (at least) one between u_i and u_j is identically zero (this, exploiting every possible choice of i and j, will readily complete the proof). Assume not, then, by the maximum principle, $u = u_i$ and $v = u_j$ are positive subsolutions of system (9). It is easy to see that Lemma 2.5 also holds for positive subsolutions of that system; as a consequence, (14) holds for $u = u_i$ and $v = u_j$. But this, reasoning as in the proof of the previous proposition, is in contradiction with the global bound of the Hölder quotients. □

3 Uniform Hölder continuity

This section is mainly devoted to the proof of Theorem 1.3, that will provide, as a byproduct, also Theorem 1.4. As we said the strategy we follow is reasoning by contradiction, in order to perform a blow–up analysis, and then to use the results of the previous section to obtain an absurd. To start with, we need the following technical lemma, which refines the estimate in [9], Lemma 4.4 (to which we refer for more details).

(3.1) Lemma. *Let $B_R \subset \mathbb{R}^N$ be any ball of radius R. Let M, A be positive constants, $h \in L^2(B_R)$, and let $u \in H^1(B_R)$ be a solution of*

$$\begin{cases} -\Delta u \leq -Mu + h & \text{in } B_R \\ u \geq 0 & \text{in } B_R \\ u \leq A & \text{on } \partial B_R. \end{cases}$$

Then for every $\varepsilon, \theta > 0$ such that $0 < \theta < \varepsilon < R$ there holds

$$\|u\|_{L^2(B_{R-\varepsilon})} \leq \frac{2AR}{\varepsilon - \theta} e^{-\theta\sqrt{M}} + \frac{1}{M}\|h\|_{L^2(B_R)},$$

where $B_{R-\varepsilon}$ is the ball of radius $R - \varepsilon$ which shares its center with B_R.

Proof. We can estimate u as $|u| \leq |u_1| + |u_2|$, where u_1, u_2 are defined by

$$\begin{cases} -\Delta u_1 &= -Mu_1 & \text{in } B_R \\ u_1 &\geq 0 & \text{in } B_R \\ u_1 &= u & \text{on } \partial B_R \end{cases} \qquad \begin{cases} -\Delta u_2 &= -Mu_2 + h & \text{in } B_R \\ u_2 &= 0 & \text{on } \partial B_R. \end{cases}$$

In order to estimate u_1 we shall make use of Lemma 4.4 in [9], where it is proved that

$$\begin{cases} \psi_1''(r) + \frac{N-1}{r}\psi_1'(r) = M\psi_1(r) \\ \psi_1(R) = A > 0 \\ \psi_1'(0) = 0 \end{cases} \Rightarrow \begin{cases} \psi_1(0) > 0, \quad \psi_1'(r) > 0 & r \in [0, +\infty) \\ \psi_1(r) \leq \psi_1(0)e^{r\sqrt{M}} & r \in [0, +\infty) \\ \psi_1(r) \geq \frac{\psi_1(0)\bar{r}}{2e^{\bar{r}\sqrt{M}}} \frac{e^{r\sqrt{M}}}{r} & r \in [\bar{r}, +\infty). \end{cases}$$

Choosing $\bar{r} = \varepsilon - \theta$, with $\theta \in (0, \varepsilon)$, we obtain the following inequalities

$$\begin{aligned} &\psi_1(R - \varepsilon) \leq \psi_1(0)e^{(R-\varepsilon)\sqrt{M}} \\ &A = \psi_1(R) \geq \frac{\psi_1(0)(\varepsilon-\theta)}{2e^{(\varepsilon-\theta)\sqrt{M}}} \frac{e^{R\sqrt{M}}}{R}, \end{aligned}$$

which imply

$$\psi_1(r) \leq \psi_1(R - \varepsilon) \leq \frac{2AR}{\varepsilon - \theta} e^{-\theta\sqrt{M}}, \qquad \text{for all } r \in [0, R - \varepsilon].$$

Defining $v_1(x) = \psi_1(|x - x_0|)$ (where x_0 is the center of B_R) and using the maximum principle we infer $0 \leq u_1(x) \leq v_1(x)$. To obtain an upper estimate for u_2, let us now multiply the equation for u_2 by u_2 itself and integrate; having zero boundary conditions we have

$$\int_{B_R} Mu_2^2 \leq \int_{B_R} |\nabla u_2|^2 + Mu_2^2 = \int_{B_R} hu_2 \leq \left(\int_{B_R} \frac{h^2}{M}\right)^{1/2} \left(\int_{B_R} Mu_2^2\right)^{1/2},$$

and therefore $\|u_2\|_{L^2(B_R)} \leq \frac{1}{M}\|h\|_{L^2(B_R)}$. In conclusion we have $\|u\|_{L^2(B_{R-\varepsilon})} \leq \|u_1\|_{L^2(B_{R-\varepsilon})} + \|u_2\|_{L^2(B_R)}$ which gives the desired estimates. □

3.1 Normalization and blow–up

To start with, we recall the standard non–uniform regularity properties for solutions to system (4).

(3.2) Remark. Let u_β, v_β be solutions of (4). Then, since h_β, k_β belong to $L^2(\Omega)$, and Ω is bounded and regular, by elliptic regularity theory it holds

$$u_\beta, v_\beta \in H^2(\Omega) \quad \text{that implies} \quad u_\beta, v_\beta \in C^{0,\alpha}(\overline{\Omega})$$

for every $\alpha \in (0, \alpha^*)$, where α^* is defined as in (5). Let us mention that, if $h_\beta \equiv k_\beta \equiv 0$, then, by a bootstrap argument, we can choose $\alpha^* = 1$ also in dimension $N = 3$.

Coming to the proof of Theorem 1.3, let us assume by contradiction that, for some $\alpha \in (0, \alpha^*)$, up to a subsequence it holds

$$L_\beta := \max\left\{\max_{x,y\in\overline{\Omega}} \frac{|u_\beta(x) - u_\beta(y)|}{|x - y|^\alpha}, \max_{x,y\in\overline{\Omega}} \frac{|v_\beta(x) - v_\beta(y)|}{|x - y|^\alpha}\right\} \longrightarrow +\infty$$

as $\beta \to +\infty$. We can assume that L_β is achieved, say, by u_β at the pair (x_β, y_β). We observe that

$$|x_\beta - y_\beta| \to 0 \quad \text{as} \quad \beta \to +\infty,$$

since we have $|x_\beta - y_\beta|^\alpha = |u_\beta(x_\beta) - u_\beta(y_\beta)|/L_\beta \le 2\|u_\beta\|_\infty/L_\beta \le 2C/L_\beta \to 0$.

The idea now is to consider an uniformly α-Hölder continuous blow-up with center at x_β. Keeping this in mind, let us define the rescaled functions

$$\bar{u}_\beta(x) = \frac{1}{L_\beta r_\beta^\alpha} u_\beta(x_\beta + r_\beta x), \quad \bar{v}_\beta(x) = \frac{1}{L_\beta r_\beta^\alpha} v_\beta(x_\beta + r_\beta x), \qquad \text{for } x \in \Omega_\beta := \frac{\Omega - x_\beta}{r_\beta},$$

where $r_\beta \to 0$ will be chosen later. Depending on the asymptotic behavior of the distance $d(x_\beta, \partial\Omega)$ and on r_β, we have $\Omega_\beta \to \Omega_\infty$, where Ω_∞ is either $\mathbb{R}^N$ or an half-space (when $d(x_\beta, \partial\Omega)/r_\beta \to \infty$ or the limit is finite, respectively).

First of all we observe that the $\bar{u}_\beta, \bar{v}_\beta$'s are uniformly α-Hölder continuous for every choice of r_β, with Hölder constant equal to one:

$$\max\left\{ \max_{x,y\in\overline{\Omega}_\beta} \frac{|\bar{u}_\beta(x) - \bar{u}_\beta(y)|}{|x-y|^\alpha}, \max_{x,y\in\overline{\Omega}_\beta} \frac{|\bar{v}_\beta(x) - \bar{v}_\beta(y)|}{|x-y|^\alpha} \right\} = \frac{\left|\bar{u}_\beta(0) - \bar{u}_\beta\left(\frac{y_\beta - x_\beta}{r_\beta}\right)\right|}{\left|\frac{y_\beta - x_\beta}{r_\beta}\right|^\alpha} = 1. \tag{16}$$

Moreover the rescaled functions satisfy the following system in Ω_β:

$$\begin{cases} -\Delta\bar{u}_\beta + \lambda_\beta r_\beta^2 \bar{u}_\beta &= \omega_1 M_\beta \bar{u}_\beta^3 - \beta M_\beta \bar{u}_\beta \bar{v}_\beta^2 + \bar{h}_\beta(x) \\ -\Delta\bar{v}_\beta + \mu_\beta r_\beta^2 \bar{v}_\beta &= \omega_2 M_\beta \bar{v}_\beta^3 - \beta M_\beta \bar{u}_\beta^2 \bar{v}_\beta + \bar{k}_\beta(x) \\ \bar{u}_\beta,\ \bar{v}_\beta \in H_0^1(\Omega_\beta), \end{cases} \tag{17}$$

where

$$M_\beta := L_\beta^2 r_\beta^{2\alpha+2},$$

and

$$\bar{h}_\beta(x) := \frac{r_\beta^{2-\alpha}}{L_\beta} h_\beta(x_\beta + r_\beta x), \quad \bar{k}_\beta(x) := \frac{r_\beta^{2-\alpha}}{L_\beta} k_\beta(x_\beta + r_\beta x).$$

(3.3) Remark. Since u_β, v_β are $L^\infty(\Omega)$–bounded, h_β, k_β are $L^2(\Omega)$–bounded, λ_β, μ_β are bounded in $\mathbb{R}$, and $r_\beta \to 0$, $L_\beta \to +\infty$, by direct calculations it is easy to see that

$$\begin{aligned} &\lambda_\beta r_\beta^2 \bar{u}_\beta,\ \mu_\beta r_\beta^2 \bar{v}_\beta \to 0 \quad \text{in } L^\infty(\Omega_\beta) \\ &\omega_1 M_\beta \bar{u}_\beta^3,\ \omega_2 M_\beta \bar{v}_\beta^3 \to 0 \quad \text{in } L^\infty(\Omega_\beta) \\ &\bar{h}_\beta,\ \bar{k}_\beta \to 0 \quad \text{in } L^2(\Omega_\beta). \end{aligned}$$

In order to manage the different parts of the proof, we will need to make different choices of the sequence r_β. Once r_β is chosen, we wish to pass to the limit (on compact sets), and to this aim we will use Ascoli–Arzelà's Theorem. Now, since the $\bar{u}_\beta, \bar{v}_\beta$'s are uniformly α-Hölder continuous, it suffices to show that $\{\bar{u}_\beta(0)\}$, $\{\bar{v}_\beta(0)\}$ are bounded in β. The following lemma provides a sufficient condition on r_β for such a bound to hold.

(3.4) Lemma. *Under the previous notations, let $r_\beta \to 0$ as $\beta \to +\infty$ be such that*

(i) $\dfrac{|y_\beta - x_\beta|}{r_\beta} \le R'$ *for some* $R' > 0$,

(ii) $\beta M_\beta \nrightarrow 0$.

Then $\{\bar{u}_\beta(0)\}, \{\bar{v}_\beta(0)\}$ *are uniformly bounded in* β.

Proof. Assume by contradiction that $\{\bar{u}_\beta(0)\}$ is unbounded, and let $R \geq R'$. Since the $\bar{u}_\beta$'s are uniformly Hölder continuous and vanish on $\partial\Omega_\beta$, we can consider β sufficiently large such that $B_{2R}(0) \subset \Omega_\beta$. Moreover since $\beta M_\beta \nrightarrow 0$, we have that

$$I_\beta := \inf_{B_{2R}(0)} \beta M_\beta \bar{u}_\beta \longrightarrow +\infty.$$

Now since $\beta M_\beta \bar{u}_\beta \leq \beta M_\beta \bar{u}_\beta^2$ in $B_{2R}(0)$ and (similarly to Remark 3.3) $\|\omega_2 M_\beta \bar{v}_\beta^2\|_{L^\infty(B_{2R})} \to 0$ as $\beta \to +\infty$, we have

$$\begin{aligned} -\Delta \bar{v}_\beta &= -\mu_\beta r_\beta^2 \bar{v}_\beta + \omega_2 M_\beta \bar{v}_\beta^3 - \beta M_\beta \bar{u}_\beta^2 \bar{v}_\beta + \bar{k}_\beta \\ &\leq -\frac{I_\beta}{2} \bar{v}_\beta + \bar{k}_\beta. \end{aligned}$$

In order to use Lemma 3.1, we need to show that $\bar{v}_\beta$ is bounded on $\partial B_{2R}(0)$. With this in mind, let us choose a cut-off function η that vanishes outside $B_{2R}(0)$. Then by testing the second equation in (17) with $\eta^2 \bar{v}_\beta$ in $B_{2R}(0)$, we obtain

$$\int_{B_{2R}(0)} \{\eta^2 |\nabla \bar{v}_\beta|^2 + 2\eta \bar{v}_\beta \nabla \eta \cdot \nabla \bar{v}_\beta + \mu_\beta r_\beta^2 \bar{v}_\beta^2 \eta^2\} \leq \int_{B_{2R}(0)} \{\omega_2 M_\beta \bar{v}_\beta^4 \eta^2 - I_\beta \eta^2 \bar{v}_\beta^2 + \bar{k}_\beta \eta^2 \bar{v}_\beta\},$$

and thus

$$\begin{aligned} \int_{B_{2R}(0)} \{\frac{1}{2}\eta^2 |\nabla \bar{v}_\beta|^2 + I_\beta \eta^2 \bar{v}_\beta^2\} &\leq 2 \int_{B_{2R}(0)} \{|\nabla \eta|^2 \bar{v}_\beta^2 + |\mu_\beta| r_\beta^2 v_\beta^2 \eta^2 + |\omega_2| M_\beta \bar{v}_\beta^4 \eta^2 + \bar{k}_\beta \eta^2 \bar{v}_\beta\} \\ &\leq C(R)(\sup_{B_{2R}(0)} \bar{v}_\beta^2 + 1). \end{aligned}$$

On the other hand, since $\bar{v}_\beta$ is uniformly Hölder continuous,

$$I_\beta \int_{B_{2R}(0)} \eta^2 \bar{v}_\beta^2 \geq I_\beta C'(R) \inf_{B_{2R}(0)} \bar{v}_\beta^2 \geq I_\beta C'(R) \sup_{B_{2R}(0)} \bar{v}_\beta^2 - I_\beta C''(R).$$

Therefore, putting together the two previous inequalities, we obtain

$$I_\beta C(R) \sup_{B_{2R}(0)} \bar{v}_\beta^2 \leq C'(R)(\sup_{B_{2R}(0)} \bar{v}_\beta^2 + 1) + I_\beta C''(R)$$

which implies the boundedness of $\bar{v}_\beta$ in $B_{2R}(0)$ (in particular on $\partial B_{2R}(0)$).

Thus we can apply Lemma 3.1, which gives

$$\|\bar{v}_\beta\|_{L^2(B_R)} \leq C e^{-C'\sqrt{I_\beta}} + \frac{2}{I_\beta} \|\bar{k}_\beta\|_{L^2(B_{2R})}.$$

Hence

$$\|\beta M_\beta \bar{u}_\beta \bar{v}_\beta\|_{L^2(B_R)} \leq (I_\beta + \beta M_\beta (4R)^\alpha) \|\bar{v}_\beta\|_{L^2(B_r)} \leq 2 I_\beta (C e^{-C'\sqrt{I_\beta}} + \frac{2}{I_\beta} \|\bar{k}_\beta\|_{L^2(B_{2R})}) \to 0$$

when $\beta \to +\infty$. This, together with Remark 3.3 and the boundedness of $\bar{v}_\beta$, gives

$$\|\Delta \bar{u}_\beta\|_{L^2(B_R)} \to 0 \tag{18}$$

for every $R \geq R'$.

Consider now $\tilde{u}_\beta(x) := \bar{u}_\beta(x) - \bar{u}_\beta(0)$. By the uniform Hölder continuity and Ascoli–Arzelà's Theorem we know that $\tilde{u}_\beta \to \tilde{u}_\infty$ on compact sets. Moreover by (18) we have that $\bar{u}_\beta$ is bounded in $C^{0,\gamma}_{\mathrm{loc}}$, with $\gamma \in (0, \alpha^*)$ (in fact, Theorem 8.12 of [14] gives us boundedness in $W^{2,2}$, and the result follows by Sobolev imbbedings). As a consequence we obtain:

$$\max_{x,y\in\overline{\Omega}_\infty} \frac{|\tilde{u}_\infty(x) - \tilde{u}_\infty(y)|}{|x-y|^\alpha} = 1. \tag{19}$$

Indeed notice that by assumption (i), $(y_\beta - x_\beta)/r_\beta$ must converge up to a subsequence. But it can not be $(y_\beta - x_\beta)/r_\beta \to 0$, otherwise we would have (considering an $\varepsilon > 0$ sufficiently small)

$$\frac{\left|\bar{u}_\beta(0) - \bar{u}_\beta\left(\frac{y_\beta - x_\beta}{r_\beta}\right)\right|}{\left|\frac{y_\beta - x_\beta}{r_\beta}\right|^\alpha} = \left|\frac{y_\beta - x_\beta}{r_\beta}\right|^\varepsilon \frac{\left|\tilde{u}_\beta(0) - \tilde{u}_\beta\left(\frac{y_\beta - x_\beta}{r_\beta}\right)\right|}{\left|\frac{y_\beta - x_\beta}{r_\beta}\right|^{\alpha+\varepsilon}} \le C\left|\frac{y_\beta - x_\beta}{r_\beta}\right|^\varepsilon \to 0 \tag{20}$$

with β, which contradicts (16). Therefore, there is an $a \in \mathbb{R}^N \setminus \{0\}$ such that $(y_\beta - x_\beta)/r_\beta \to a$, and hence the left hand side of (16) also passes to the limit in β, providing (19).

Finally, we have that $\Delta\tilde{u}_\infty = 0$ in Ω_∞. Now if $\Omega_\infty = \mathbb{R}^N$, by Corollary 2.3 $\tilde{u}_\infty$ is a constant, in contradiction with (19). On the other hand, if Ω_∞ is an half space, we have that $\tilde{u}_\infty = 0$ on $\partial\Omega_\infty$ and thus we can extend it by symmetry as an harmonic function in the whole $\mathbb{R}^N$, obtaining the same contradiction.

We have shown that $\{\bar{u}_\beta(0)\}$ is bounded. Let us now check that the same happens with $\{\bar{v}_\beta(0)\}$. In order to do so, we have to make some small changes to the previous argument. Assume then that $\{\bar{v}_\beta(0)\}$ is unbounded, and consider the quantity (for $R \ge R'$ fixed)

$$\tilde{I}_\beta := \inf_{B_{2R}(0)} \beta M_\beta \bar{v}_\beta^2 \to +\infty.$$

We have

$$-\Delta\bar{u}_\beta \le -\frac{\tilde{I}_\beta}{2}\bar{u}_\beta + \bar{h}_\beta$$

and $\bar{u}_\beta$ is bounded on $\partial B_{2R}(0)$. Therefore by Lemma 3.1

$$\|\bar{u}_\beta\|_{L^2(B_r)} \le Ce^{-C'\sqrt{\tilde{I}_\beta}} + \frac{2}{\tilde{I}_\beta}\|\bar{h}_\beta\|_{L^2(B_{2R})}$$

and hence

$$\|\beta M_\beta \bar{u}_\beta \bar{v}_\beta^2\|_{L^2(B_R)} \to 0$$

as $\beta \to +\infty$. Once again this gives $\|\Delta\bar{u}_\beta\|_{L^2(B_R)} \to 0$ and the proof follows as before. □

Using the previous lemma we can now quantify the asymptotic relation between β, L_β and $|x_\beta - y_\beta|$.

(3.5) Lemma. *Under the previous notation, we have (up to a subsequence)*

$$\beta L_\beta^2 |x_\beta - y_\beta|^{2\alpha+2} \to +\infty.$$

Proof. By contradiction assume that $\beta L_\beta^2|x_\beta - y_\beta|^{2\alpha+2}$ is bounded. Then we can choose

$$r_\beta = (\beta L_\beta^2)^{-\frac{1}{2\alpha+2}} \qquad \text{(and thus } \beta M_\beta = 1\text{)},$$

in such a way that the assumptions of Lemma 3.4 are satisfied and thus $\{\bar{u}_\beta(0)\}, \{\bar{v}_\beta(0)\}$ are bounded. By uniform Hölder continuity and Ascoli–Arzelà's theorem we have that, up to a subsequence, there exist u_∞, v_∞ such that $\bar{u}_\beta \to u_\infty$, $\bar{v}_\beta \to v_\infty$ uniformly in the compact subsets of $\overline{\Omega}_\infty$.

Since $\beta M_\beta = 1$ and by Remark 3.3, we have that $\Delta\bar{u}_\beta$, $\Delta\bar{v}_\beta$ are bounded in $L^2_{\rm loc}$ and therefore the same happens to $\bar{u}_\beta, \bar{v}_\beta$ in $C^{0,\gamma}_{\rm loc}(\overline{\Omega}_\infty)$, for all $\gamma \in (0, \alpha^*)$. We are now going to show that, as a consequence, u_∞, v_∞ are α-Hölder continuous and that the maximum of the Hölder quotients is given by:

$$\max_{x,y\in\overline{\Omega}_\infty} \frac{|u_\infty(x) - u_\infty(y)|}{|x-y|^\alpha} = 1. \tag{21}$$

Indeed notice that we cannot have $(y_\beta - x_\beta)/r_\beta \to 0$, otherwise we would obtain the same contradiction as in (20). Therefore, there is an $a \in \mathbb{R}^2\backslash\{0\}$ such that $(y_\beta - x_\beta)/r_\beta \to a$, and hence the left hand side of (16) also passes to the limit in β, providing (21). Moreover, at the limit we have

$$\begin{cases} -\Delta u_\infty &= -u_\infty v_\infty^2 & \text{in } \Omega_\infty \\ -\Delta v_\infty &= -u_\infty^2 v_\infty & \text{in } \Omega_\infty. \end{cases}$$

If $\Omega_\infty = \mathbb{R}^N$, then by Proposition 2.6 u_∞, v_∞ are constants, which contradicts (21).

On the other hand, let Ω_∞ be equal to an half-space. Since $u_\infty = 0$ on $\partial\Omega_\infty$, we can extend it to the whole space by even symmetry and obtain a function satisfying the hypotheses of Proposition 2.2 (apply it to the pair $(u_{\infty|_{\Omega_\infty}}, u_{\infty|_{\mathbb{R}^N\setminus\overline{\Omega}_\infty}})$ – were we consider both functions extended by 0 – and choose for x_0 any point of $\partial\Omega_\infty$). Therefore $u_\infty \equiv 0$, which contradicts (21). □

Now we are in a position to define our choice of r_β and to deduce the convergence of the blow–up sequences.

(3.6) Lemma. *Let*

$$r_\beta = |x_\beta - y_\beta|.$$

Then there exist $u_\infty, v_\infty \in C^{0,\alpha}(\mathbb{R}^N)$ such that, as $\beta \to +\infty$ (up to subsequences), there holds

(i) $\bar{u}_\beta \to u_\infty$, $\bar{v}_\beta \to v_\infty$, uniformly in compact subsets of $\Omega_\infty = \mathbb{R}^N$; moreover

(ii) for any fixed $r > 0$ and $x_0 \in \mathbb{R}^N$ there holds $\displaystyle\int_{B_r(x_0)} \beta M_\beta \bar{u}_\beta^2 \bar{v}_\beta^2 \to 0$; as a consequence

(iii) $\|\bar{u}_\beta - u_\infty\|_{H^1(B_r(x_0))} \to 0$, $\|\bar{v}_\beta - v_\infty\|_{H^1(B_r(x_0))} \to 0$.

Proof. With this choice of r_β, we obtain $\beta M_\beta = \beta L_\beta^2 |x_\beta - y_\beta|^{2\alpha+2} \to +\infty$ by Lemma 3.5. Once again the assumptions of Lemma 3.4 are satisfied and hence, reasoning as in the initial part of the proof of Lemma 3.5, we deduce that the rescaled functions $\bar{u}_\beta, \bar{v}_\beta$ converge uniformly to some u_∞, v_∞, in every compact set of $\overline{\Omega}_\infty$. In this situation (16) writes

$$1 = \max\left\{ \max_{x,y\in\overline{\Omega}_\beta} \frac{|\bar{u}_\beta(x) - \bar{u}_\beta(y)|}{|x-y|^\alpha}, \max_{x,y\in\overline{\Omega}_\beta} \frac{|\bar{v}_\beta(x) - \bar{v}_\beta(y)|}{|x-y|^\alpha} \right\} = \left| \bar{u}_\beta(0) - \bar{u}_\beta\left(\frac{y_\beta - x_\beta}{r_\beta}\right) \right|$$

and hence by $L^\infty_{\rm loc}(\overline{\Omega}_\beta)$ convergence, u_∞, v_∞ are globally α–Hölder continuous and in particular

$$\max_{x\in\partial B_1(0)\cap\overline{\Omega}_\infty} |u_\infty(0) - u_\infty(x)| = 1. \tag{22}$$

Now if Ω_∞ is an half-space we can proceed exactly as in the last part of the proof of Lemma 3.5, obtaining a contradiction. Therefore $\Omega_\infty = \mathbb{R}^N$, and (i) is proved.

In order to prove the second part of the lemma, let us fix any ball $B_r(x_0)$ of $\mathbb{R}^N$, and let β be large so that $B_r(x_0) \subset \Omega_\beta$. Let us consider a smooth cut-off function $0 \le \eta \le 1$ such that $\eta = 1$ in

B_r, $\eta = 0$ in $\mathbb{R}^N \setminus B_{2r}$. Testing the equation for $\bar{u}_\beta$ with η, we obtain (since the $\bar{u}_\beta$'s are uniformly bounded in B_{2r})

$$\int_{B_r} \beta M_\beta \bar{u}_\beta \bar{v}_\beta^2 \leq \int_{B_{2r}} |\bar{u}_\beta \Delta\eta - \lambda_\beta r_\beta^2 \eta \bar{u}_\beta + \omega_1 M_\beta \eta \bar{u}_\beta^3 + \eta \bar{h}_\beta| \leq C \tag{23}$$

and analogously $\displaystyle\int_{B_r} \beta M_\beta \bar{u}_\beta^2 \bar{v}_\beta \leq C$. This immediately implies that

$$u_\infty \cdot v_\infty \equiv 0 \quad \text{ in } \mathbb{R}^N, \tag{24}$$

providing

$$\begin{aligned} \int_{B_r} \beta M_\beta \bar{u}_\beta^2 \bar{v}_\beta^2 &\leq \|\bar{u}_\beta\|_{L^\infty(B_r \cap \{u_\infty = 0\})} \int_{B_r} \beta M_\beta \bar{u}_\beta \bar{v}_\beta^2 + \|\bar{v}_\beta\|_{L^\infty(B_r) \cap \{v_\infty = 0\}} \int_{B_r} \beta M_\beta \bar{u}_\beta^2 \bar{v} \\ &\leq C \left(\|\bar{u}_\beta\|_{L^\infty(B_r \cap \{u_\infty = 0\})} + \|\bar{v}_\beta\|_{L^\infty(B_r \cap \{v_\infty = 0\})} \right) \to 0, \end{aligned} \tag{25}$$

which is (ii).

Finally, integrating the equation for $\bar{u}_\beta$ in B_r, we have

$$\left| \int_{\partial B_r} \partial_\nu \bar{u}_\beta \, d\sigma \right| \leq \int_{B_r} \beta M_\beta \bar{u}_\beta \bar{v}_\beta^2 + \int_{B_r} |\lambda_\beta r_\beta^2 \bar{u}_\beta - \omega_1 M_\beta \bar{u}_\beta^3 - \bar{h}_\beta| \leq C, \tag{26}$$

which also gives, testing the equation for $\bar{u}_\beta$ with $\bar{u}_\beta$ itself, $\int_{B_r} |\nabla \bar{u}_\beta|^2 \leq C$. Doing the same with $\bar{v}_\beta$, we obtain the weak H^1–convergence $\bar{u}_\beta \rightharpoonup u_\infty$, $\bar{v}_\beta \rightharpoonup v_\infty$. Finally by testing the equation for $\bar{u}_\beta$ with $\bar{u}_\beta - u_\infty$ we obtain

$$\begin{aligned} \int_{B_r} \nabla \bar{u}_\beta \cdot \nabla(\bar{u}_\beta - u_\infty) \leq \|\bar{u}_\beta - u_\infty\|_{L^\infty(B_r)} \cdot \Bigg(&\int_{\partial B_r} |\partial_\nu \bar{u}_\beta| + \\ &+ \int_{B_r} | - \lambda_\beta r_\beta^2 \bar{u}_\beta + \omega_1 M_\beta \bar{u}_\beta^3 - \beta M_\beta \bar{u}_\beta \bar{v}_\beta^2 + \bar{h}_\beta| \Bigg) \end{aligned}$$

and therefore we proved (iii) by uniform convergence and estimates (26), (23) (the convergence of $\bar{v}_\beta$ is analogous). □

In the following lemma we collect the properties enjoyed by the limiting states u_∞, v_∞.

(3.7) Lemma. *Let u_∞, v_∞ be defined as in Lemma 3.6. Then the following holds.*

(i) $u_\infty \cdot v_\infty \equiv 0$ *in* $\mathbb{R}^N$*;*

(ii) $\displaystyle\max_{x \in \partial B_1(0)} |u_\infty(0) - u_\infty(x)| = 1$ *(in particular, u_∞ is not constant);*

(iii) $\begin{cases} -\Delta u_\infty = 0 & \text{in } \{u_\infty > 0\}, \\ -\Delta v_\infty = 0 & \text{in } \{v_\infty > 0\}. \end{cases}$

Proof. Properties (i) and (ii) are simply (24) and (22), respectively. Let us check that u_∞ is harmonic in the (open) set $\{x \in \mathbb{R}^N : \ u_\infty(x) > 0\}$ (the same is true for v_∞ in the set $\{x \in \mathbb{R}^N : v_\infty(x) > 0\}$). Given any point x_0 such that $u_\infty(x_0) > 0$, we have to find a neighborhood of it where u_∞ is harmonic. By continuity we can consider a ball $B_\delta(x_0)$ where $u_\infty \geq 2\gamma > 0$, and hence by locally L^∞ convergence $\bar{u}_\beta \geq \gamma > 0$ in $B_\delta(x_0)$ for large β. Therefore we have

$$-\Delta \bar{v}_\beta \leq -\beta M_\beta \frac{\gamma^2}{2} \bar{v}_\beta + \bar{k}_\beta$$

and thus, using Lemma 3.1, we obtain

$$\|\bar{v}_\beta\|_{L^2(B_{\delta/2})} \le Ce^{-C'\sqrt{\beta M_\beta}} + \frac{1}{\beta M_\beta}\|\bar{k}_\beta\|_{L^2(B_{\delta/2})}.$$

Hence

$$\|\beta M_\beta \bar{u}_\beta \bar{v}_\beta^2\|_{L^2(B_{\delta/2})} \to 0$$

and, using also Remark 3.3, we conclude that $\|\Delta \bar{u}_\beta\|_{L^2(B_{\delta/2})} \to 0$, which implies the harmonicity of u_∞ in $B_{\delta/2}(x_0)$. □

(3.8) Remark. By the previous lemmas we obtain that u_∞ must vanish somewhere in $\mathbb{R}^N$ (indeed if not u_∞ would be a positive non–constant harmonic function in $\mathbb{R}^N$, a contradiction), and also v_∞ must vanish somewhere (otherwise we would have $u_\infty \equiv 0$ in $\mathbb{R}^N$, again a contradiction). This, by continuity, implies that u_∞ and v_∞ must have a common zero, thus they satisfy all the assumptions of Proposition 2.2. Since u_∞ is not constant, we deduce that

$$v_\infty \equiv 0 \qquad \text{in } \mathbb{R}^N.$$

Moreover, we have

$$\{x : u_\infty(x) = 0\} \neq \emptyset, \qquad \text{and} \qquad \{x : u_\infty(x) > 0\} \text{ is connected.}$$

This last claim is due to the fact that, was $\{u_\infty > 0\}$ non trivially decomposed into $\Omega_1 \cup \Omega_2$, then again $u = u_\infty|_{\Omega_1}$ and $v = u_\infty|_{\Omega_2}$ would be non–zero and satisfy the assumptions of Proposition 2.2, a contradiction.

3.2 Almgren's Formula

In order to conclude the proof of Theorem 1.3 we will show that u_∞ is radially homogeneous; this crucial information will come from a generalization of the Almgren's Monotonicity Formula. This formula was first introduced in [1] and used for instance in [4, 13] to prove some regularity issues related to free boundary problems. The aim is to study the monotonicity properties of the functions

$$E(r) = \frac{1}{r^{N-2}} \int_{B_r} \left\{ |\nabla u_\infty|^2 + |\nabla v_\infty|^2 \right\}, \qquad H(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} \left\{ u_\infty^2 + v_\infty^2 \right\},$$

and of the Almgren's quotient (where it is defined)

$$N(r) = \frac{E(r)}{H(r)},$$

where u_∞, v_∞ are defined in Lemma 3.6 and B_r is centered at a fixed x_0 (with respect to the literature, our definition of H involves the averages of the densities, not of their oscillations). It is worthwhile noticing that the result we prove for u_∞, v_∞ in fact holds for any non trivial, strong H^1_{loc}–limits of variational systems; indeed, we will perform the proof without using all the other properties we collected about u_∞, v_∞. The reason for this is that we will need a similar result, for different functions, in Section 4.

(3.9) Proposition. *Under the above notations, for every $x_0 \in \mathbb{R}^N$ there exists $r_0 \ge 0$ such that, for every $r > r_0$, $H(r) \neq 0$, and*

$N(r)$ is an absolutely continuous, non decreasing function

such that

$$\frac{d}{dr}\log(H(r)) = \frac{2N(r)}{r}. \tag{27}$$

Moreover if $N(r) \equiv \gamma$ for all $r > r_0$, then $r_0 = 0$ and $u_\infty(x) = r^\gamma g_1(\theta)$, $v_\infty(x) = r^\gamma g_2(\theta)$ in $\mathbb{R}^N$, for some functions g_1, g_2 (where (r, θ) denote the polar coordinates).

Proof. Up to a translation, we can suppose $B_r = B_r(0)$. We divide the proof into steps.

Approximated quotients. Let $0 < r_1 < r_2$ be such that $H(r) \neq 0$ in $[r_1, r_2]$ (they exist for sure, since $u_\infty \not\equiv 0$ and it is continuous). Let us check that the conclusions of the proposition follow in this interval (the existence of r_0 as claimed will be obtained only later). To evaluate derivatives of $E(r)$, $H(r)$ and $N(r)$ we have to face two main problems: first, it is not clear how regular these functions are; second, we have no global equation for u_∞, v_∞. To overcome these difficulties, the idea is to consider analogous functions, that will result to be C^1, for the approximated problem (17), and then to pass to the limit as $\beta \to +\infty$. In order to simplify notations we will denote for the moment $u := \bar{u}_\beta$ and the same for $\bar{v}_\beta, \bar{h}_\beta, \bar{k}_\beta$. We then define the approximated Almgren's quotient

$$N_\beta(r) = \frac{E_\beta(r)}{H_\beta(r)},$$

where

$$E_\beta(r) = \frac{1}{r^{N-2}} \int_{B_r} \left\{ |\nabla u|^2 + |\nabla v|^2 + r_\beta^2(\lambda_\beta u^2 + \mu_\beta v^2) - M_\beta(\omega_1 u^4 + \omega_2 v^4) + 2\beta M_\beta u^2 v^2 \right\},$$

$$H_\beta(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} \left\{ u^2 + v^2 \right\}.$$

We also observe that, by multiplying system (17) by (u, v) and integrating in B_r, we obtain

$$E_\beta(r) = \frac{1}{r^{N-2}} \int_{\partial B_r} \{ u\, \partial_\nu u + v\, \partial_\nu v \} + \frac{1}{r^{N-2}} \int_{B_r} \{ h(x) u + k(x) v \} \tag{28}$$

(the boundary integrals above, and all the following ones, are well defined, for β fixed and for *every* r, by Remark 3.2 and by the continuous immersion of $H^2(B_r)$ into $H^1(\partial B_r)$).

Derivatives of E_β, H_β. In order to compute the derivatives of these expressions, we consider the rescaled function $u_r(x) := u(rx)$ and similar expressions for v, h, k. System (17) now becomes

$$\begin{cases} -\Delta u_r + r^2 \lambda_\beta r_\beta^2 u_r &= r^2 \omega_1 M_\beta u_r^3 - r^2 \beta M_\beta u_r v_r^2 + r^2 h_r \\ -\Delta v_r + r^2 \mu_\beta r_\beta^2 v_r &= r^2 \omega_2 M_\beta v_r^3 - r^2 \beta M_\beta u_r^2 v_r + r^2 k_r. \end{cases} \tag{29}$$

Performing a change of variables $x = ry$ in $E_\beta(r)$ we obtain

$$E_\beta(r) = \int_{B_1} \left\{ |\nabla u_r|^2 + |\nabla v_r|^2 + r^2 r_\beta^2 (\lambda_\beta u_r^2 + \mu_\beta v_r^2) - r^2 M_\beta(\omega_1 u_r^4 + \omega_2 v_r^4) + 2r^2 \beta M_\beta u_r^2 v_r^2 \right\},$$

and hence (Remark 3.2 implies that E_β is in fact C^1 in r)

$$\begin{aligned} E_\beta'(r) \;=\; & 2 \int_{B_1} \{ \nabla u_r \cdot \nabla(\nabla u(rx) \cdot x)) + \nabla v_r \cdot \nabla(\nabla v(rx) \cdot x) \} + \\ & + 2r^2 \int_{B_1} (r_\beta^2 \lambda_\beta u_r - 2M_\beta \omega_1 u_r^3 + 2\beta M_\beta u_r v_r^2)(\nabla u(rx) \cdot x) + \\ & + 2r^2 \int_{B_1} (r_\beta^2 \mu_\beta v_r - 2M_\beta \omega_2 v_r^3 + 2\beta M_\beta u_r^2 v_r)(\nabla v(rx) \cdot x) + \\ & + 2r \int_{B_1} \left\{ r_\beta^2 (\lambda_\beta u_r^2 + \mu_\beta v_r^2) - M_\beta(\omega_1 u_r^4 + \omega_2 v_r^4) + 2\beta M_\beta u_r^2 v_r^2 \right\}. \end{aligned}$$

Multiplying the first equation in (29) by $(\nabla u(rx) \cdot x)$, the second one by $(\nabla v(rx) \cdot x)$, integrating

by parts in B_1 and substituting the result in the previous expression, it follows:

$$\begin{aligned} E'_\beta(r) &= \frac{2}{r^{N-2}}\int_{\partial B_r}\left\{(\partial_\nu u)^2+(\partial_\nu v)^2\right\}+\frac{2}{r^{N-1}}\int_{B_r}\{h(x)(\nabla u\cdot x)+k(x)(\nabla v\cdot x)\}+\\ &+\frac{2}{r^{N-1}}\int_{B_r}\left\{r_\beta^2(\lambda_\beta u^2+\mu_\beta v^2)-M_\beta(\omega_1 u^4+\omega_2 v^4)+2\beta M_\beta u^2v^2\right\}-\\ &-\frac{2}{r^{N-1}}\int_{B_r}\left\{\omega_1 M_\beta u^3(\nabla u\cdot x)+\omega_2 M_\beta v^3(\nabla v\cdot x)\right\}+\\ &+\frac{2}{r^{N-1}}\int_{B_r}\left\{\beta M_\beta u(\nabla u\cdot x)v^2+\beta M_\beta u^2 v(\nabla v\cdot x)\right\}. \end{aligned}$$

Using the divergence theorem, we can rewrite some terms:

$$\begin{aligned} \frac{2}{r^{N-1}}\int_{B_r}\left\{\beta M_\beta u(\nabla u\cdot x)v^2+\beta M_\beta u^2 v(\nabla v\cdot x)\right\} &= \frac{\beta M_\beta}{r^{N-1}}\int_{B_r}\nabla(u^2v^2)\cdot x=\\ &= -\frac{N}{r^{N-1}}\int_{B_r}\beta M_\beta u^2v^2+\frac{1}{r^{N-2}}\int_{\partial B_r}\beta M_\beta u^2v^2; \end{aligned}$$

$$\begin{aligned} -\frac{2}{r^{N-1}}\int_{B_r}&\left\{\omega_1 M_\beta u^3(\nabla u\cdot x)+\omega_2 M_\beta(\nabla(v)\cdot x)\right\}=\\ &= -\frac{2}{r^{N-1}}\int_{B_r}\left\{\omega_1 M_\beta(\nabla(u^4)\cdot x)+\omega_2 M_\beta(\nabla(v^4)\cdot x)\right\}=\\ &= \frac{N}{2r^{N-1}}\int_{B_r}\left\{\omega_1 M_\beta u^4+\omega_2 M_\beta v^4\right\}-\frac{1}{2r^{N-2}}\int_{\partial B_r}\left\{\omega_1 M_\beta u^4+\omega_2 M_\beta v^4\right\}; \end{aligned}$$

obtaining at the end

$$\begin{aligned} E'_\beta(r) &= \frac{2}{r^{N-2}}\int_{\partial B_r}\left\{(\partial_\nu u)^2+(\partial_\nu v)^2\right\}+\frac{2}{r^{N-1}}\int_{B_r}\{h(x)(\nabla u\cdot x)+k(x)(\nabla v\cdot x)\}+\\ &+\frac{1}{r^{N-1}}\int_{B_r}\left\{2r_\beta^2(\lambda_\beta u^2+\mu_\beta v^2)+\left(\frac{N}{2}-2\right)M_\beta(\omega_1 u^4+\omega_2 v^4)+(4-N)\beta M_\beta u^2v^2\right\}+\\ &+\frac{1}{2r^{N-2}}\int_{\partial B_r}\left\{2\beta M_\beta u^2v^2-\omega_1 M_\beta u^4-\omega_2 M_\beta v^4\right\}. \end{aligned}$$

Using the same ideas, we also obtain, for the C^1 function H_β,

$$H'_\beta(r)=\frac{2}{r^{N-1}}\int_{\partial B_r}\{u\,\partial_\nu u+v\,\partial_\nu v\}.$$

Estimate of $N_\beta(r+\delta)-N_\beta(r)$. At this point, let us recover the original notations $\bar u_\beta,\bar v_\beta,\bar h_\beta,\bar k_\beta$. Recalling equation (28) we can compute N'_β in (r_1,r_2) as

$$\begin{aligned} N'_\beta(r)=\frac{2}{r^{2N-3}H_\beta^2(r)}\Bigg\{&\int_{\partial B_r}\left[(\partial_\nu\bar u_\beta)^2+(\partial_\nu\bar v_\beta)^2\right]\cdot\int_{\partial B_r}(\bar u_\beta^2+\bar v_\beta^2)-\\ &-\left[\int_{\partial B_r}(\bar u_\beta\partial_\nu\bar u_\beta+\bar v_\beta\partial_\nu\bar v_\beta)\right]^2\Bigg\}+R_\beta(r), \end{aligned}$$

where

$$
\begin{aligned}
R_\beta(r) = & \frac{2}{r^{N-1}H_\beta(r)}\int_{B_r}\left\{\bar h_\beta(x)(\nabla\bar u_\beta\cdot x)+\bar k_\beta(x)(\nabla\bar v_\beta\cdot x)\right\}+\\
& +\frac{1}{r^{N-1}H_\beta(r)}\int_{B_r}\left\{2r_\beta^2(\lambda_\beta\bar u_\beta^2+\mu_\beta\bar v_\beta^2)+\frac{N-4}{2}M_\beta(\omega_1\bar u_\beta^4+\omega_2\bar v_\beta^4)+(4-N)\beta M_\beta\bar u_\beta^2\bar v_\beta^2\right\}+\\
& +\frac{1}{2r^{N-2}H_\beta(r)}\int_{\partial B_r}\left\{2\beta M_\beta\bar u_\beta^2\bar v_\beta^2-\omega_1 M_\beta\bar u_\beta^4-\omega_2 M_\beta^4\right\}-\\
& -\frac{2}{r^{2N-3}H_\beta^2(r)}\int_{\partial B_r}\left\{\bar u_\beta\,\partial_\nu\bar u_\beta+\bar v_\beta\,\partial_\nu\bar v_\beta\right\}\int_{B_r}\left\{\bar h_\beta(x)\bar u_\beta+\bar k_\beta(x)\bar v_\beta\right\}.
\end{aligned}
$$

Notice that, since $H_\beta(r)\neq 0$, for every $\delta>0$ such that $r, r+\delta\in(r_1,r_2)$, there exists a constant $C>0$ depending only on r_1, r_2 and δ such that

$$
\begin{aligned}
\int_r^{r+\delta}|R_\beta(s)|ds \quad \le \quad & C\int_{B_{r_2}}\{|\bar h_\beta||\nabla\bar u_\beta|+|\bar k_\beta||\nabla\bar v_\beta|+r_\beta^2(\bar u_\beta^2+\bar v_\beta^2)+M_\beta(\bar u_\beta^4+\bar v_\beta^4)+\\
& +\beta M_\beta\bar u_\beta^2\bar v_\beta^2+|\omega_1|M_\beta\bar u_\beta^3|\nabla\bar u_\beta|+|\omega_2|M_\beta\bar v_\beta^3|\nabla\bar v_\beta|\}+\\
& +\|M_\beta\bar u_\beta^4+M_\beta\bar v_\beta^4\|_{L^\infty(B_r)}+C\int_{B_{r_2}}\left\{|\bar h_\beta||\bar u_\beta|+|\bar k_\beta||\bar v_\beta|\right\}\longrightarrow 0
\end{aligned}
$$

as $\beta\to+\infty$, where we used Remark 3.3, Lemma 3.6, (iii) and (26). Therefore,

$$
\begin{aligned}
N_\beta(r+\delta)-N_\beta(r) \quad = \quad & \int_r^{r+\delta}\frac{2}{s^{2N-3}H_\beta^2(s)}\left\{\int_{\partial B_s}\left[(\partial_\nu\bar u_\beta)^2+(\partial_\nu\bar v_\beta)^2\right]\cdot\int_{\partial B_s}(\bar u_\beta^2+\bar v_\beta^2)-\right.\\
& \left.-\left[\int_{\partial B_s}(\bar u_\beta\partial_\nu\bar u_\beta+\bar v_\beta\partial_\nu\bar v_\beta)\right]^2\right\}+o_\beta(1).
\end{aligned}
$$

Derivatives of N, H, E, $\log H$. Now we are in a position to pass to the limit in β. Indeed, Lemma 3.6, (iii) (that is, strong convergence) ensures that $N_\beta(r)\to N(r)$ for every r. Moreover it implies the existence of a function $f(\rho)\in L^1(r_1,r_2)$ such that, up to a subsequence, $\int_{\partial B_\rho}|\nabla\bar u_\beta|^2\le f(\rho)$ and $\int_{\partial B_\rho}|\nabla\bar u_\beta|^2\to\int_{\partial B_\rho}|\nabla u_\infty|^2$ a.e. for $\rho\in(r_1,r_2)$ (and analogously for $\bar v_\beta$). Hence, letting $\beta\to+\infty$ in the previous equation we readily obtain that N is absolutely continuous and that (for almost every r)

$$
\begin{aligned}
N'(r) \quad = \quad & \frac{2}{r^{2N-3}H^2(r)}\left\{\int_{\partial B_r}\left[(\partial_\nu u_\infty)^2+(\partial_\nu v_\infty)^2\right]\cdot\int_{\partial B_r}(u_\infty^2+v_\infty^2)-\right.\\
& \left.-\left[\int_{\partial B_r}(u_\infty\partial_\nu u_\infty+v_\infty\partial_\nu v_\infty)\right]^2\right\}\ge 0, \qquad (30)
\end{aligned}
$$

by Hölder inequality. This implies that $N(r)$ is increasing in $[r_1,r_2]$ and in addition gives an explicit expression for the derivative. Reasoning as above, we can conclude that

$$
\begin{aligned}
H'(r) \quad &= \quad \frac{2}{r^{N-1}}\int_{\partial B_r}\{u_\infty\,\partial_\nu u_\infty+v_\infty\partial_\nu v_\infty\},\\
H(r) \quad &= \quad \lim_{\beta\to+\infty}H_\beta(r)=\frac{1}{r^{N-1}}\int_{\partial B_r}\left\{u_\infty^2+v_\infty^2\right\},\\
E(r) \quad &= \quad \lim_{\beta\to+\infty}E_\beta(r)=\frac{1}{r^{N-2}}\int_{\partial B_r}\{u_\infty\,\partial_\nu u_\infty+v_\infty\partial_\nu v_\infty\},
\end{aligned}
$$

(where we used (28) to obtain the last limit) and therefore a direct calculation gives (27) for $r \in (r_1, r_2)$. Incidentally, we observe that equation (27) implies that $\log H$, and hence H, are C^1–functions.

Existence of r_0. Equality (27) also implies that when $H(r) > 0$, then $H'(r) \geq 0$ and therefore there exists $r_0 := \inf\{r > 0 : \ H(r) \neq 0\}$ such that $H(r) \neq 0$ for every $r > r_0$. Hence everything we have done so far is true in $(r_0, +\infty)$.

Case $N(r)$ constant. Let us now analyze what happens when $N(r) \equiv \gamma$ for all $r > r_0$. By (27) we have

$$\frac{d}{dr}\log(H(r)) = \frac{2\gamma}{r} = \frac{d}{dr}\log(r^{2\gamma})$$

for all $r > r_0$. By considering $\bar{r} > r > r_0$ and integrating the previous equality between r and $\bar{r}$ we have

$$H(r) = H(\bar{r})\left(\frac{r}{\bar{r}}\right)^{2\gamma}.$$

Now, if $H(r_0) = 0$, since H is continuous, the previous equation implies that $r_0 = 0$; on the other hand, if $H(r_0) \neq 0$, then $r_0 = 0$ by definition. Moreover, by (30), if $N(r)$ is constant then there exists $C(r)$ such that $(\partial_\nu u_\infty, \partial_\nu v_\infty) = C(r)(u_\infty, v_\infty)$, which gives $u_\infty = f(r)g_1(\theta)$, $v_\infty = f(r)g_2(\theta)$, with $f(r) > 0$ for $r > 0$. Now since u_∞ is harmonic in $\{u_\infty > 0\}$ and v_∞ is harmonic in $\{v_\infty > 0\}$, we finally infer $u_\infty(x) = r^\gamma g_1(\theta)$, $v_\infty(x) = r^\gamma g_2(\theta)$ in $\mathbb{R}^N$. □

(3.10) Remark. Starting from system (3), one can perform the same blow–up argument than above, obtaining in particular that, for the limiting states $(u_{1,\infty}, \dots, u_{k,\infty})$, a result analogous to Proposition 3.9 holds, with the choice

$$E(r) = \frac{1}{r^{N-2}}\int_{B_r}\sum_{i=1}^{k}|\nabla u_{i,\infty}|^2, \qquad H(r) = \frac{1}{r^{N-1}}\int_{\partial B_r}\sum_{i=1}^{k}u_{i,\infty}^2.$$

3.3 Proof of the main results.

End of the proof of Theorem 1.3. By Lemma 3.6 we know that the blow–up limiting profiles u_∞ and v_∞ are globally α–Hölder continuous. Moreover, by Remark 3.8, $v_\infty \equiv 0$ and we can choose

$$x_0 \text{ such that } u(x_0) = 0.$$

If $N(r)$ is defined as in Proposition 3.9, then we claim that $N(r) \equiv \alpha$ for all $r > r_0$. Indeed, according to that proposition, assume that there exists $\bar{r} > r_0$ such that $N(\bar{r}) \leq \alpha - \varepsilon$. Then by monotonicity, for all $r_0 < r < \bar{r}$ we have $N(r) \leq \alpha - \varepsilon$ and

$$\frac{d}{dr}\log(H(r)) \leq \frac{2}{r}(\alpha - \varepsilon),$$

hence (integrating between r and $\bar{r}$) we have $Cr^{2\alpha-2\varepsilon} \leq H(r)$ for all $r_0 < r < \bar{r}$. On the other hand, by the α–Hölder continuity and the fact that $u_\infty(x_0) = v_\infty(x_0) = 0$, we also have $H(r) \leq C'(r - r_0)^{2\alpha}$, a contradiction. On the other hand, if $N(\bar{r}) \geq \alpha + \varepsilon$, then by monotonicity $N(r) \geq \alpha + \varepsilon$ for all $r > \bar{r}$, and thus

$$\frac{d}{dr}\log H(r) \geq \frac{2}{r}(\alpha + \varepsilon),$$

which implies (integrating between $\bar{r}$ and r and again by the α–Hölder continuity) that $Cr^{2\alpha+2\varepsilon} \leq H(r) \leq C'r^{2\alpha}$ for large r, a contradiction.

Therefore $N(r) \equiv \alpha$ for all $r > r_0$, and by the previous proposition we know that $r_0 = 0$ and $u_\infty(x) = r^\alpha g_1(\theta)$. This implies that the null set $\Gamma = \{u_\infty = 0\}$ is a cone with respect to x_0. Since

this can be done for any $x_0 \in \Gamma$, we obtain that Γ is in fact a cone with respect to each of its points, and thus it is a linear subspace of $\mathbb{R}^N$. Moreover, again by Remark 3.8, Γ has dimension strictly smaller than $N-1$, otherwise $\{u_\infty > 0\}$ would be disconnected. But then u_∞ turns out to be a non–negative, non–constant function in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$, which is harmonic on the complement of a set of zero (local) capacity. That is, it is harmonic on the whole $\mathbb{R}^N$, a contradiction. □

Proof of Theorem 1.4. By Theorem 1.3, for every $\alpha < \alpha' < \alpha^*$ there exists a constant $C > 0$ such that $\|(u_\beta, v_\beta)\|_{C^{0,\alpha'}} \le C$, for every $\beta > 0$. By compact embedding, we obtain, up to a subsequence, the existence of $(u, v) \in C^{0,\alpha'}$ that are strong $C^{0,\alpha}$–limits of (u_β, v_β). By uniqueness of the limit, this proves that

$$(u_\beta, v_\beta) \to (u, v) \quad \text{in } C^{0,\alpha}(\overline{\Omega}) \text{ for every } \alpha < \alpha^*.$$

To obtain the other claims of the theorem, we reason as in the proof of Lemma 3.6. Testing system (4) with (u_β, v_β) we obtain

$$\int_\Omega |\nabla u_\beta|^2 \le \int_\Omega \left(-\lambda_\beta u_\beta + \omega_1 u_\beta^3 + h_\beta\right) u_\beta,$$

and an analogous inequality for v_β. By uniform convergence, the right hand side is bounded and then (u_β, v_β) is bounded in H^1_0. Thus, again up to a subsequence, we have

$$(u_\beta, v_\beta) \rightharpoonup (u, v) \quad \text{weakly in } H^1_0(\Omega).$$

On the other hand, integrating system (4) we have

$$-\int_{\partial\Omega} \partial_\nu u_\beta + \beta \int_\Omega u_\beta v_\beta^2 = \int_\Omega \left(-\lambda_\beta u_\beta + \omega_1 u_\beta^3 + h_\beta\right).$$

Again, the right hand side is bounded and, by Hopf lemma, $\partial_\nu u_\beta < 0$ on $\partial\Omega$. We infer

$$\beta \int_\Omega u_\beta v_\beta^2 \le C, \qquad \beta \int_\Omega u_\beta^2 v_\beta \le C$$

not depending on β. This immediately provides $u \cdot v \equiv 0$ almost everywhere in Ω, and, in turn, reasoning as (25),

$$\beta \int_\Omega u_\beta^2 v_\beta^2 \to 0 \quad \text{as } \beta \to +\infty,$$

that completes the proof of (ii). Now we can test system (4) with $(u_\beta - u, v_\beta - v)$, obtaining

$$\int_\Omega \nabla u_\beta \cdot \nabla(u_\beta - u) \le \|u_\beta - u\|_{L^\infty} \int_\Omega \left(-\lambda_\beta u_\beta + \omega_1 u_\beta^3 - u_\beta v_\beta^2 + h_\beta\right),$$

and the same for v. By uniform convergence we infer convergence in norm, and hence strong H^1_0–convergence of (u_β, v_β) to (u, v), and also (i) is proved. Finally, to prove (iii), we observe that, by continuity of the limiting profile, we know that $\{u > 0\}$ is an open set. Therefore, given $x_0 \in \{u > 0\}$, there exists $B_\delta(x_0)$ such that $u \ge 2\gamma > 0$ in $B_\delta(x_0)$, for some positive constant γ. Let us show that the equation is satisfied in this open neighborhood. By (i) there holds $u_\beta \ge \gamma$ in $B_\delta(x_0)$ for large β, therefore

$$\int_{B_\delta(x_0)} \beta u_\beta v_\beta^2 \le \frac{1}{\gamma} \int_{B_\delta(x_0)} \beta u_\beta^2 v_\beta^2 \to 0,$$

because of (ii). By testing the equation with a test function $\phi \in C^1_0(B_\delta(x_0))$ we obtain

$$\int_{B_\delta(x_0)} (\nabla u_\beta \cdot \nabla \phi + \lambda_\beta u_\beta \phi) = \int_{B_\delta(x_0)} (\omega_1 u_\beta^3 - \beta u_\beta v_\beta^2 + h_\beta)\phi,$$

and the previous estimate together with the H^1–convergence conclude the proof. □

Proof of Theorems 1.1 and 1.2. As we just noticed, with one small change in the previous arguments one can prove also these two theorems, except for the Lipschitz continuity of the limiting profile (u, v), which will be the object of the following section. In dimension $N = 2$, since $\alpha^* = 1$, then the theorems follow directly from Theorems 1.3 and 1.4. In dimension $N = 3$, according to Remark 3.2, if $h_\beta \equiv k_\beta \equiv 0$ then we can choose $\alpha^* = 1$ and repeat, as they are, all the arguments in this section. Then Theorem 1.1 straightly follows, while the proof of Theorem 1.2 will be completed by Proposition 4.1 and Remark 4.9 below. □

(3.11) Remark. With exactly the same strategy it is also possible to prove analogous results for L^∞–bounded, positive solutions of system (3). The only differences are pointed out in Proposition 2.7 and in Remark 3.10.

4 Lipschitz continuity of the limiting profile

Throughout all this section, let $(u, v) \in C^{0,\alpha} \cap H_0^1$ denote the limiting profile introduced in Theorem 1.4, and $h_\beta, k_\beta \equiv 0$ (that is, we are dealing with system (1)). As we noticed, in this case the uniform Hölder continuity result holds for every $\alpha \in (0, 1)$ also if $N = 3$. In such a situation, although we are not able to prove uniform Lipschitz continuity of the solutions with respect to β (see Remark 2.4), one can prove that the limiting profile is in fact Lipschitz continuous. To be more precise, we will first give the details of the proof of the local Lipschitz continuity of (u, v), and then we will advise (in Remarks 4.8 and 4.9) how this proof can be modified in order to obtain the Lipschitz regularity up to the boundary of Ω. Also, after Remarks 3.10 and 3.11, the reader will easily see how this result holds true for k–tuples of densities that are solutions of system (3).

Let us fix a (regular) domain $\tilde{\Omega} \subset\subset \Omega$, and let us define the null set

$$\Gamma = \{x \in \tilde{\Omega} : u(x) = v(x) = 0\} \neq \tilde{\Omega}$$

(from now on, we will exclude the trivial case $(u, v) \equiv (0, 0)$ in $\tilde{\Omega}$, which obviously enjoys Lipschitz continuity).

(4.1) Proposition. *Let (u, v) be the limiting profile introduced in Theorem 1.4, $h_\beta \equiv k_\beta \equiv 0$ and $\tilde{\Omega} \subset\subset \Omega$. Then $(u, v) \in W^{1,\infty}(\tilde{\Omega})$.*

Again, in order to prove the proposition, the main tool will be the Almgren's Monotonicity Formula introduced in Section 3.2, with some small change in its definition. In fact, due to the fact that the limiting profiles satisfy system (2), the natural definitions for $E(r)$ and $H(r)$ are

$$\begin{aligned} E(r) = E_{x_0}(r) &= \frac{1}{r^{N-2}} \int_{B_r(x_0)} \left(|\nabla u|^2 + |\nabla v|^2 + \lambda u^2 + \mu v^2 - \omega_1 u^4 - \omega_2 v^4\right), \\ H(r) = H_{x_0}(r) &= \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} (u^2 + v^2), \end{aligned}$$

where, here and in the following,

$$x_0 \in \overline{\Gamma} \quad \text{and} \quad r < \bar{r}_1 := \operatorname{dist}(\tilde{\Omega}, \partial\Omega).$$

In this setting, we have that $E(r)/H(r)$ is no longer necessarily positive. To overcome this fact, we define a modified Almgren's quotient as

$$N(r) = \frac{E(r)}{H(r)} + 1 = \frac{E(r) + H(r)}{H(r)}.$$

With this choice, N turns out to be non negative (where it is defined).

(4.2) Lemma. *There exists $\bar r_2 < \bar r_1$ such that for every $0 < r \le \bar r_2$ and for every x_0 we have*

$$E(r) + H(r) \ge \frac{1}{2}\left[\frac{1}{r^{N-2}}\int_{B_r(x_0)}(|\nabla u|^2 + |\nabla v|^2) + \frac{1}{r^{N-1}}\int_{\partial B_r(x_0)}(u^2+v^2)\right] \ge 0.$$

Proof. We shall make use of the following formulation of Poincaré's inequality: for every $w \in H^1_{\rm loc}(\mathbb{R}^N)$, every x_0 and every $r > 0$ there holds

$$\frac{1}{r^N}\int_{B_r} w^2 \le \frac{1}{N-1}\left[\frac{1}{r^{N-2}}\int_{B_r}|\nabla w|^2 + \frac{1}{r^{N-1}}\int_{\partial B_r} w^2\right].$$

Recalling that $u, v \in L^\infty(\Omega)$, let now $C > 0$ be such that

$$\left|\frac{1}{r^N}\int_{B_r}\lambda u^2 + \mu v^2 - \omega_1 u^4 - \omega_2 v^4\right| \le \frac{C}{r^N}\int_{B_r}(u^2+v^2)$$

(hence C depends on u, v, ω_i, λ, μ, but not on x_0 and r). Then

$$E(r) + H(r) \ge \frac{1}{r^{N-2}}\int_{B_r}(|\nabla u|^2 + |\nabla v|^2) + \frac{1}{r^{N-1}}\int_{\partial B_r}(u^2+v^2) - r^2\frac{C}{r^N}\int_{B_r}(u^2+v^2),$$

and Poincaré's inequality immediately implies that, for $r \le \bar r_2$ sufficiently small (independent of the choice of x_0), the lemma holds. □

Now, with the new notations of this section, let us present a result which corresponds to Proposition 3.9 in this context.

(4.3) Proposition. *There exist $\bar r \le \bar r_2$ and $C > 0$ such that, for every $x_0 \in \overline{\Gamma}$ and $0 < r \le \bar r$, we have $H(r) \ne 0$,*

$$N'(r) \ge -2CrN(r), \qquad \text{and thus } \tilde N(r) := e^{Cr^2}N(r) \text{ is non decreasing.}$$

Moreover

$$\frac{d}{dr}\log(H(r)) = \frac{2}{r}(N(r) - 1). \tag{31}$$

(4.4) Remark. During the proof of this proposition we will also see that Γ has empty interior.

Proof. We will follow closely the proof of Proposition 3.9.

Proof when $H(r) \ne 0$. Let us first suppose that there is an interval $[r_1, r_2]$, with $r_2 < \bar r_2$ (defined in the previous lemma), such that $H(r) > 0$ in $[r_1, r_2]$. Again, we first consider the approximated problem

$$\begin{aligned}
E_\beta(r) &= \frac{1}{r^{N-2}}\int_{B_r}\left(|\nabla u_\beta|^2 + |\nabla v_\beta|^2 + \lambda_\beta u_\beta^2 + \mu_\beta v_\beta^2 - \omega_1 u_\beta^4 - \omega_2 v_\beta^4 + 2\beta u_\beta^2 v_\beta^2\right),\\
H_\beta(r) &= \frac{1}{r^{N-1}}\int_{\partial B_r}(u_\beta^2 + v_\beta^2),\\
N_\beta(r) &= \frac{E_\beta(r) + H_\beta(r)}{H_\beta(r)}.
\end{aligned}$$

Proceeding exactly as in Proposition 3.9 we obtain

$$\begin{aligned}
H'_\beta(r) &= \frac{2}{r^{N-1}}\int_{\partial B_r}\{u_\beta\partial_\nu u_\beta + v_\beta\partial_\nu v_\beta\} = \frac{2}{r}E_\beta(r),\\
E'_\beta(r) &= \frac{2}{r^{N-2}}\int_{\partial B_r}\{(\partial_\nu u_\beta)^2 + (\partial_\nu v_\beta)^2\} + R_\beta(r),
\end{aligned}$$

where

$$R_\beta(r) = \frac{1}{r^{N-1}}\int_{B_r}\left\{2\lambda_\beta u_\beta^2+\mu_\beta v_\beta^2+\frac{N-4}{2}(\omega_1 u_\beta^4+\omega_2 v_\beta^4)+(4-N)\beta u_\beta^2 v_\beta^2\right\} - \\ -\frac{1}{2r^{N-2}}\int_{\partial B_r}\{\omega_1 u_\beta^4+\omega_2 v_\beta^4-2\beta u_\beta^2 v_\beta^2\}.$$

Hence in (r_1, r_2) there holds, by Hölder inequality,

$$\frac{N'_\beta(r)}{N_\beta(r)} = \frac{E'_\beta(r)H_\beta(r)-E_\beta(r)H'_\beta(r)}{(E_\beta(r)+H_\beta(r))H_\beta(r)} \geq \frac{R_\beta(r)}{E_\beta(r)+H_\beta(r)},$$

(recall that, by Lemma 4.2, $H(r) > 0$ implies $E(r) + H(r) > 0$, and $N(r) > 0$). Now we can let $\beta \to +\infty$, obtaining

$$\frac{N'(r)}{N(r)} \geq \frac{R(r)}{E(r)+H(r)},$$

with

$$R(r) = \frac{1}{r^{N-1}}\int_{B_r}\left\{2\lambda u^2+2\mu v^2+\frac{N-4}{2}(\omega_1 u^4+\omega_2 v^4)\right\} - \frac{1}{2r^{N-2}}\int_{\partial B_r}\{\omega_1 u^4+\omega_2 v^4\}.$$

Finally, by using the same arguments as in the proof of Lemma 4.2, we can prove the existence of a constant $C > 0$ (depending only on r_1, r_2, independent of x_0) such that

$$|R(r)| \leq 2C(E(r)+H(r)), \qquad \text{and thus } \frac{N'(r)}{N(r)} \geq -2C.$$

Finally, (31) comes from a direct calculation as in Proposition 3.9.

Therefore, at this point, we have proved the lemma for every interval $[r_1, r_2]$ with $r_2 < \bar{r}_1$ and for every x_0 where $H(r) > 0$. Now we need only to check that in fact $H(r) \neq 0$ for r small, and the proof will be complete. This will be done in two more steps.

Γ has empty interior. Assume not, and let $x_1 \in \Gamma$ be such that $d_1 := \mathrm{dist}(x_1, \partial\Gamma) < \bar{r}_1$ (recall that we are assuming that $u^2 + v^2$ is not identically zero in $\tilde{\Omega}$). We have $H(r) > 0$ for $r \in (d_1, d_1 + \varepsilon)$ for some small $\varepsilon > 0$. By what we have done so far $H(r)$ verifies, in $(d_1, d_1 + \varepsilon)$, the initial value problem

$$\begin{cases} H'(r) = a(r)H(r) & r \in (d_1, d_1+\varepsilon) \\ H(d_1) = 0, \end{cases}$$

with $a(r) = 2(N(r) - 1)/r$, which is continuous also at d_1 by the monotonicity of $\tilde{N}$. Then by uniqueness $H(r) \equiv 0$ for $r > d_1$, a contradiction with the definition of d_1.

Definition of $\bar{r}$. Finally we observe that, by (2), we have

$$-\Delta u \leq (\omega_1 u^2 - \lambda)u < \lambda_1(B_r(x_0))u \qquad \text{in } \Omega$$

for small r, let us say for $0 < r < \bar{r}_3$, independent of x_0 (indeed $\lambda_1(B_r(x_0)) \to +\infty$ when $r \to 0$); an analogous inequality holds for v. Fixing now $\bar{r} < \min\{\bar{r}_2, \bar{r}_3\}$, so that all we have done so far holds, for $0 < r \leq \bar{r}$ we must have $H_{x_0}(r) \neq 0$ for every x_0. Otherwise, for some $x_1 \in \Gamma$, we would have $u, v = 0$ on $\partial B_r(x_1)$. This, together with the previous inequality, would give $u, v \equiv 0$ in $B_r(x_1)$, a contradiction since Γ has empty interior. □

The previous lemma immediately provides some estimates of N for small r.

(4.5) Lemma. *Under the previous notations, for every $x_0 \in \Gamma$, $0 < r \leq \bar{r}$,*

$$N(0^+) \geq 2 \qquad \textit{and thus} \qquad N(r) \geq 2e^{-Cr^2}.$$

Proof. First of all, the limit exists finite because of the monotonicity of $\tilde{N}(r) = e^{Cr^2}N(r)$, and $N(0^+) = \tilde{N}(0^+)$. Let us assume by contradiction that, for some x_0, $N(0^+) < 2$. As a consequence there exist $r^* < \bar{r}$ and $\varepsilon > 0$ such that, for $0 < r < r^*$, we have $N(r) \le 2 - \varepsilon$. Integrating (31) between r and r^*, we obtain

$$\frac{H(r^*)}{H(r)} \le \left(\frac{r^*}{r}\right)^{2(1-\varepsilon)}.$$

This and the fact that u, v are α–Hölder continuous for every $\alpha \in (0,1)$ implies $C(r^*)r^{2(1-\varepsilon)} \le H(r) \le C' r^{2\alpha}$ for every $\alpha \in (0,1)$, a contradiction. □

(4.6) Remark. We recall that, for any fixed $0 < r < \bar{r}$, the maps

$$x_0 \mapsto E_{x_0}(r), \quad x_0 \mapsto H_{x_0}(r) \qquad \text{are continuous in } \overline{\Gamma}.$$

As a consequence, for $\bar{r}$ as in the previous lemma, we deduce the existence of a constants C_1, C_2, not depending on x_0, such that

$$0 < C_1 \le H_{x_0}(\bar{r}) \le C_2 \qquad \text{for every } x_0 \in \overline{\Gamma},$$

indeed H is lower–bounded as a result of the fact that Γ has an empty interior (Remark 4.4). Thus also

$$\tilde{N}_{x_0}(r) \le \tilde{N}_{x_0}(\bar{r}) = \frac{e^{C\bar{r}^2}\left(E_{x_0}(r) + H_{x_0}(r)\right)}{H_{x_0}(r)} \le C_3 \qquad \text{for every } x_0 \in \overline{\Gamma},\ 0 < r \le \bar{r},$$

whit C_3 not depending on x_0.

(4.7) Lemma. *Under the previous notations there exists a constant $C > 0$, not depending on x_0 and r, such that*

$$\frac{1}{r^N}\int_{B_r(x_0)} \left\{|\nabla u|^2 + |\nabla v|^2\right\} \le C \qquad \textit{for every } x_0 \in \overline{\Gamma},\ 0 < r \le \bar{r}.$$

Proof. By Lemma 4.2 and the definitions of N, $\tilde{N}$, we know that

$$\frac{1}{r^{N-2}}\int_{B_r(x_0)} \left\{|\nabla u|^2 + |\nabla v|^2\right\} \le 2(E(r) + H(r)) = 2e^{-Cr^2}\tilde{N}(r)H(r) \le 2\tilde{N}(\bar{r})H(r),$$

and thus

$$\frac{1}{r^N}\int_{B_r(x_0)} \left\{|\nabla u|^2 + |\nabla v|^2\right\} \le 2C_3\,\frac{H(r)}{r^2}, \tag{32}$$

where C_3, not depending on r and x_0, is as in Remark 4.6. In order to estimate the right hand side above, we can use (31) to write

$$\frac{H(\bar{r})}{\bar{r}^2} - \frac{H(r)}{r^2} = \int_r^{\bar{r}} \frac{d}{d\rho}\frac{H(\rho)}{\rho^2}\,d\rho = \int_r^{\bar{r}} \frac{2}{\rho}\left(N(\rho) - 2\right)\,d\rho,$$

that, taking into account Lemma 4.5 and Remark 4.6, implies

$$\frac{H(r)}{r^2} \le \frac{C_2}{\bar{r}^2} + \int_0^{\bar{r}} \frac{4}{\rho}\left(1 - e^{-C\rho^2}\right)\,d\rho \le C',$$

not depending on x_0 and r. Substituting into (32), the lemma is proved. □

Finally, we are ready to prove the local Lipschitz regularity of the limiting profile.

Proof of Proposition 4.1. Let us assume by contradiction that (u, v) is not Lipschitz continuous in $\tilde{\Omega}$ (we follow some of the ideas of the proof of Theorem 5.1 in [8], to which we refer for more details). Then there exists $\{x_n\} \subset \tilde{\Omega}$, $r_n \to 0$ such that

$$\lim_{n\to+\infty} \frac{1}{r_n^N} \int_{B_{r_n}(x_n)} (|\nabla u|^2 + |\nabla v|^2) = +\infty. \tag{33}$$

We claim that (33) holds also for a different choice of the centers $y_n \in \overline{\Gamma}$ instead of x_n. This will contradict Lemma 4.7 and prove the proposition.

Clearly $d(x_n, \Gamma) \to 0$ (u and v solve (2) where they are positive), hence, up to a subsequence, we can assume the existence of $x_0 \in \overline{\Gamma}$ such that $x_n \to x_0$. Let us start by showing that (33) holds for a choice of $\{x'_n\}$ such that $d(x'_n, \Gamma) \le K r_n$, with K independent of n. If, up to a subsequence, $x_n \in \overline{\Gamma}$, then there is nothing to prove. Otherwise, in every set $A_n = \{x \in \tilde{\Omega} : d(x, \Gamma) \ge r_n\}$ there holds

$$\begin{cases} -\Delta u + \lambda u = \omega_1 u^3 \\ -\Delta v + \mu v = \omega_2 v^3, \end{cases}$$

and hence

$$-\Delta(|\nabla u|^2) \le 2\nabla u \cdot \nabla(-\Delta u) = 2(3\omega_1 u^2 - \lambda)|\nabla u|^2$$

in every A_n and similarly for v. If we set

$$\Phi(x) = \frac{1}{r^N} \int_{B_r(x)} (|\nabla u(y)|^2 + |\nabla v(y)|^2) dy = \frac{1}{r^N} \int_{B_r(0)} (|\nabla u(x+y)|^2 + |\nabla v(x+y)|^2) dy,$$

then we just have proved the existence of a constant $C \ge 0$ (independent on n) such that $-\Delta\Phi \le C\Phi$ in A_n, for every n. Let now ρ be so small that $-\Delta\varphi \le C\varphi$ admits a strictly positive solution φ in $B_\rho(x_0)$ and let $n \ge \bar{n}$ such that $x_n \in B_\rho(x_0)$ $\forall n$. Then on $A_{n,\rho} = A_n \cup B_\rho(x_0)$ there holds $-\mathrm{div}(\varphi^2 \nabla \frac{\Phi}{\varphi}) \le 0$, and hence by the maximum principle

$$\max_{A_{n,\rho}} \Phi \le C' \max_{\partial A_n} \Phi.$$

This immediately implies that (33) holds for a choice $\{x'_n\}$ such that $d(x'_n, \Gamma) \le K r_n$. Let now $y_n \in \overline{\Gamma}$ be such that $|y_n - x'_n| = d(x'_n, \Gamma)$ and define $s_n = r_n + d(x'_n, \Gamma) \le (K+1) r_n$, then there holds

$$\frac{1}{s_n^N} \int_{B_{s_n}(y_n)} (|\nabla u|^2 + |\nabla v|^2) \ge \frac{1}{(K+1) r_n} \int_{B_{r_n}(x'_n)} (|\nabla u|^2 + |\nabla v|^2) \to +\infty,$$

but this, as we just noticed, is in contradiction with Lemma 4.7, and hence (u, v) is Lipschitz in $\tilde{\Omega}$. □

(4.8) Remark. Following [13], one can see that all the Almgren–type formulae can in fact be proved in a more general setting, that is when the Laplace operator is replaced with uniformly elliptic operators of the type

$$-Lu = -\mathrm{div}\,(A(x)\nabla u)\,,$$

where A is smooth (at least C^1). The key ingredient is to replace the usual polar coordinates with coordinates which are polar with respect to the geodesic distance associated to A. Of course the energies in the Almgren's quotient must be defined in a suitable way. We refer to [13] for further details.

(4.9) Remark. Once suitable Almgren's formulae are settled as in the previous remark, one can treat the Lipschitz continuity of u and v up to $\partial\Omega$ in the following way: with a local change of coordinates, and hence changing the differential operator, it is possible to assume that $\partial\Omega$ is locally a hyperplane, and reflect u and v with respect to this hyperplane. It turns out that we find new functions $\tilde{u}$, $\tilde{v}$ which satisfy a new system of equations, with different differential operators, in a larger domain $\Omega' \supset\supset \Omega$. We can then prove Lipschitz regularity of $(\tilde{u}, \tilde{v})$, locally in Ω', and deduce Lipschitz regularity of (u, v) in $\overline{\Omega}$.

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References

[1] Almgren, Jr., F. J. Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents. In *Minimal submanifolds and geodesics (Proc. Japan-United States Sem., Tokyo, 1977)*. North-Holland, Amsterdam, 1979, pp. 1–6.

[2] Alt, H. W., Caffarelli, L. A., and Friedman, A. Variational problems with two phases and their free boundaries. *Trans. Amer. Math. Soc. 282*, 2 (1984), 431–461.

[3] Ambrosetti, A., and Colorado, E. Standing waves of some coupled nonlinear Schrödinger equations. *J. Lond. Math. Soc. (2) 75*, 1 (2007), 67–82.

[4] Caffarelli, L. A., and Lin, F.-H. Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries. *J. Amer. Math. Soc. 21*, 3 (2008), 847–862.

[5] Caffarelli, L. A., and Roquejoffre, J.-M. Uniform Hölder estimates in a class of elliptic systems and applications to singular limits in models for diffusion flames. *Arch. Ration. Mech. Anal. 183*, 3 (2007), 457–487.

[6] Chang, S., Lin, C., Lin, T., and Lin, W. Segregated nodal domains of two-dimensional multispecies Bose-Einstein condensates. *Phys. D 196*, 3-4 (2004), 341–361.

[7] Conti, M., Terracini, S., and Verzini, G. Nehari's problem and competing species systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire 19*, 6 (2002), 871–888.

[8] Conti, M., Terracini, S., and Verzini, G. An optimal partition problem related to nonlinear eigenvalues. *J. Funct. Anal. 198*, 1 (2003), 160–196.

[9] Conti, M., Terracini, S., and Verzini, G. Asymptotic estimates for the spatial segregation of competitive systems. *Adv. Math. 195*, 2 (2005), 524–560.

[10] Conti, M., Terracini, S., and Verzini, G. On a class of optimal partition problems related to the Fučík spectrum and to the monotonicity formulae. *Calc. Var. Partial Differential Equations 22*, 1 (2005), 45–72.

[11] Conti, M., Terracini, S., and Verzini, G. A variational problem for the spatial segregation of reaction-diffusion systems. *Indiana Univ. Math. J. 54*, 3 (2005), 779–815.

[12] Dancer, E., Wei, J., and Weth, T. A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system. *preprint* (2007).

[13] Garofalo, N., and Lin, F.-H. Monotonicity properties of variational integrals, A_p weights and unique continuation. *Indiana Univ. Math. J. 35*, 2 (1986), 245–268.

[14] Gilbarg, D., and Trudinger, N. S. *Elliptic partial differential equations of second order*, second ed., vol. 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1983.

[15] Maia, L., Montefusco, E., and Pellacci, B. Infinitely many nodal solutions for a weakly coupled nonlinear Schrödinger system. *Comm. Cont. Math. 10* (2008), 651–669.

[16] Sirakov, B. Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$. *Comm. Math. Phys. 271*, 1 (2007), 199–221.

[17] Terracini, S., and Verzini, G. Multipulse phases in k–mixtures of bose–einstein condensates. *Arch. Rational Mech. Anal. to appear*.

[18] Wei, J., and Weth, T. Radial solutions and phase separation in a system of two coupled Schrödinger equations. *Arch. Rational Mech. Anal. to appear*.

[19] Wei, J., and Weth, T. Asymptotic behaviour of solutions of planar elliptic systems with strong competition. *Nonlinearity 21*, 2 (2008), 305–317.

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