



Partial Smoothing of the Stochastic Wave Equation and Regularization by Noise Phenomena

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Abstract

We establish partial smoothing properties of the transition semigroup (P_t) associated to the linear stochastic wave equation driven by a cylindrical Wiener noise on a separable Hilbert space. These new results allow the study of related vector-valued infinite-dimensional PDEs in spaces of functions which are Hölder continuous along special directions. As an application we prove strong uniqueness for semilinear stochastic wave equations involving nonlinearities of Hölder type. We stress that we are able to prove well-posedness although the Markov semigroup (P_t) is not strong Feller.

Keywords Partial smoothing · Transition semigroups · Stochastic wave equation · Regularization by noise for semilinear stochastic equations

Mathematics Subject Classification (2020) 60H15 · 60H50 · 35R60

1 Introduction

The aim of this paper is to study regularizing properties of the Ornstein–Uhlenbeck transition semigroup associated to an abstract linear wave equation of the following form

$$\begin{cases} \frac{d^2 y}{d\tau^2}(\tau) = -\Lambda y(\tau) + \dot{W}(\tau), \\ y(0) = x_0, \quad \frac{dy}{d\tau}(0) = x_1, \quad \tau \in (0, T], \end{cases} \quad (1.1)$$

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where $\Lambda : \mathcal{D}(\Lambda) \subset U \rightarrow U$ is a positive self-adjoint operator on a real separable Hilbert space U such that there exists $\Lambda^{-1} : U \rightarrow U$ of trace class (see, for instance, Example 5.8 and Section 5.5.2 in [14], [4] and the references therein) and $\{W(\tau) = W_\tau, \tau \geq 0\}$ is a cylindrical Wiener process on U (see [21] and [11]). Many linear stochastic equations modelling the vibrations of elastic structures can be written in the form (1.1). This includes stochastic wave equations like

$$\begin{cases} \frac{\partial^2}{\partial \tau^2} y(\tau, \xi) = \frac{\partial^2}{\partial \xi^2} y(\tau, \xi) + \dot{W}(\tau, \xi), & \xi \in (0, 1), \\ y(\tau, 0) = y(\tau, 1) = 0, \\ y(0, \xi) = x_0(\xi), \quad \frac{\partial y}{\partial \tau}(0, \xi) = x_1(\xi), & \tau \in (0, T], \quad \xi \in [0, 1], \end{cases} \quad (1.2)$$

where $x_0 \in H_0^1([0, 1])$, $x_1 \in L^2([0, 1]) = U$, and stochastic plate equations in dimension 2, cf. Sect. 3 in [24].

It is known that the Ornstein–Uhlenbeck Markov semigroup associated to (1.1) is not strong Feller (cf. Theorem 9.2.1 in [12]). Here we establish regularizing effects of the Ornstein–Uhlenbeck semigroup along special directions (cf. Sect. 3). These new results allow to study in Section 4.1 a vector-valued Kolmogorov equation also considered in [24] (see (1.8)); this equation is related to the study of well-posedness of singular semilinear stochastic wave equations.

The partial smoothing seems to be an important tool. It replaces the strong Feller property which we have improperly used for the Ornstein–Uhlenbeck semigroup in Sect. 4 of [24].

We also consider interpolation results involving spaces of Hölder continuous functions along special directions (cf. Sect. 3.1).

Related results have been considered in [5] in a different context to investigate infinite dimensional elliptic equations involving the Gross Laplacian; see in particular Lemma 3.2 which is proved in [5]. We partially extend Lemma 3.2 obtaining Lemma 3.4 which deals with interpolation of vector-valued functions. We apply the new regularity results of Section 4.1 to prove strong (or pathwise) uniqueness for semilinear stochastic wave equations involving nonlinearities of Hölder type (cf. (1.7) and see, in particular, the proof of Theorem 4.7).

Since without the noise $\dot{W}(\tau, \xi)$ the corresponding nonlinear deterministic equation is in general not well-posed (cf. Section 3.3 in [24]) our result is a kind of regularization by additive noise for semilinear stochastic wave equations.

Coming into the details, to study Eq. (1.1) we consider two basic Hilbert spaces: K and H . The first one is

$$K = \mathcal{D}(\Lambda^{1/2}) \times U = V \times U. \quad (1.3)$$

This is the usual space for the deterministic wave equation obtained removing $\dot{W}(\tau)$; see for instance [22], [3] and Appendix A.5.4 in [14]. This space is also denoted by $V \oplus U$ in the literature. However solutions to stochastic wave equations (1.1) do not evolve in K but in the larger space

$$H = U \times \mathcal{D}(\Lambda^{-1/2}) = U \times V', \quad (1.4)$$

(see Example 5.8 in [14]). Here V' is the dual space of V . On the other hand, as we mentioned before, the Ornstein–Uhlenbeck semigroup has regularizing effects only along the directions of K .

Notice that when $\Lambda = -\frac{d^2}{dx^2}$ on the interval $[0, 1]$ with Dirichlet boundary conditions, Eq. (1.1) reduces to (1.2) with $U = L^2([0, 1])$, $\mathcal{D}(\Lambda) = H_0^1([0, 1]) \cap H^2([0, 1])$, $\mathcal{D}(\Lambda^{1/2}) = H_0^1([0, 1]) = V$ and $\mathcal{D}(\Lambda^{-1/2}) = H^{-1}([0, 1])$.

Equation (1.1) can be written as an evolution equation in H as

$$dX_\tau^{0,x} = AX_\tau^{0,x}d\tau + GdW_\tau, \tau \in [0, T], X_0^{0,x} = x \in H, \quad (1.5)$$

where A is the generator of a unitary group in H and $GdW_\tau = \begin{pmatrix} 0 \\ dW_\tau \end{pmatrix}$; see Sect. 2 about notations and preliminary results.

At the beginning of Sect. 3 we fix another real and separable Hilbert space J and investigate regularizing properties of the J -valued Ornstein–Uhlenbeck semigroup (R_τ) defined by $R_\tau[\Phi](x) = \mathbb{E}[\Phi(X_\tau^{0,x})]$, $\tau \geq 0$, $\Phi \in B_b(H, J)$, where $X_\tau^{0,x}$ is the Ornstein–Uhlenbeck process solving (1.5).

To prove the differentiability of $R_\tau[\Phi]$, $\tau > 0$, along the directions of K , we use sharp results on the behaviour of the minimal energy for the related deterministic linear control system (A.1), see “Appendix A” (see also the appendix in [25] and the reference therein for more details).

We also study second directional derivatives of $R_\tau[\Phi]$, $\tau > 0$; see in particular our estimate for

$$\sum_{m \geq 1} \sup_{a \in U, |a|_U=1} |\nabla_k \nabla_{Ga} \mathbb{E}[\langle \Phi(X_\tau^{0,\cdot}), f_m \rangle_J](x)|^2, \quad (1.6)$$

$x \in H$, $k \in K$, $\tau > 0$, given in Lemma 3.9 which uses the interpolation result of Lemma 3.4 (here (f_m) denotes any basis of J).

In Sect. 4 we use the regularity results of Sect. 3 to prove well-posedness of the following semilinear stochastic wave equation in H :

$$dX_\tau^{0,x} = AX_\tau^{0,x}d\tau + GB(\tau, X_\tau^{0,x})d\tau + GdW_\tau, \tau \in [0, T], X_0^{0,x} = x \in H. \quad (1.7)$$

Here $B : [0, T] \times H \rightarrow U$ is a measurable and bounded function, Hölder continuous of exponent $\alpha \in (2/3, 1)$ with respect to the variable x (cf. Hypothesis 2 and see also Remark 4.8). The existence of a weak solution to equation (1.7) follows by the Girsanov theorem (cf. [14], [26], [23] and [25, Section 2.1]). We stress that we obtain pathwise uniqueness of equation (1.7) without the strong Feller property for the transition Ornstein–Uhlenbeck semigroup associated to (1.5). This is in contrast with other papers dealing with strong uniqueness (see, for instance, [8], [9], [10]).

To study (1.7) we first prove in Section 4.1 existence of solutions v to the following infinite dimensional PDE of Kolmogorov type for $x \in H, t \in [0, T]$:

$$\begin{aligned} v(t, x) = & \int_t^T R_{s-t} \left[e^{-(s-t)A} G B(s, \cdot) \right] (x) \, ds \\ & + \int_t^T R_{s-t} \left[e^{-(s-t)A} \nabla^G v(s, \cdot) B(s, \cdot) \right] (x) \, ds; \end{aligned} \quad (1.8)$$

here the unknown function v takes values in K and it is regular along the directions of K . Further $\nabla^G v(s, x) B(s, x)$ denotes the directional derivative $\nabla_{GB(s,x)} v(s, x) \in K$ (see (3.4)), $(s, x) \in [0, T] \times H$. To establish Theorem 4.3 about (1.8) we use the results of Sect. 3 when $J = K$. In Remark 4.4 we discuss the use of Lemma 3.4 in the proof of Theorem 4.3.

In Section 4.3 we recall how to get the important identity

$$\begin{aligned} X_\tau^{0,x} = & e^{\tau A} x + e^{\tau A} v(0, x) - v(\tau, X_\tau^x) \\ & + \int_0^\tau e^{(\tau-s)A} \nabla^G v(s, X_s^x) \, dW_s + \int_0^\tau e^{(\tau-s)A} G \, dW_s, \end{aligned} \quad (1.9)$$

which holds for any weak mild solution $(X_\tau^{0,x})$. Note that the irregular coefficient B is not present in (1.9). This identity involves v that solves the Kolmogorov equation (1.8). Identities like (1.9) are established in [17], [8], [9], [32] by the so-called Itô-Tanaka trick which is a variant of the Zvonkin method used in [31] (see also [15]). Here we refer to [24] where (1.9) is proved by using backward stochastic differential equations (BSDEs in the sequel), which, together with the group property of e^{tA} , allow to remove the “bad term” B of (1.7) (cf. Section 4.2).

Note that in contrast with previous papers which use the Itô-Tanaka trick here we have a function v which is regular only along the directions of K (see Theorem 4.3). We can use the previous identity and prove pathwise uniqueness in Theorem 4.7 noting that (see (1.7))

$$X_\tau^{0,x_1} - X_\tau^{0,x_2} \in K, \quad \tau \in [0, T],$$

if $x_1, x_2 \in H$ and $x_1 - x_2 \in K$ (i.e., the difference of two solutions evolves in K when the difference of the initial data is in K , even if the single solution does not evolve on K ; see (4.6) and the related comments).

By Theorem 4.7, using an extension of the Yamada-Watanabe theorem (see [26]) one can obtain that (1.1) has a unique strong mild solution, for any $x \in H$.

2 Notations and Preliminary Results

Given two real separable Hilbert spaces H and J we denote by $L(H, J)$ the space of bounded linear operators from H to J , endowed with the usual operator norm;

$L_2(H, J)$ is the subspace of all Hilbert-Schmidt operators endowed with the Hilbert-Schmidt norm $\|\cdot\|_{L_2(H, J)}$. Let E be a Banach space. $B_b(H, E)$ is the space of all Borel and bounded functions from H into E endowed with the supremum norm $\|\cdot\|_\infty$, $\|f\|_\infty = \sup_{x \in H} \|f(x)\|_E$, $f \in B_b(H, E)$. $C_b(H, E)$ is its subspace consisting of all uniformly continuous and bounded functions from H into E . The space $C_b^1(H, E)$ is the space of all functions in $C_b(H, E)$ which are Fréchet differentiable on H with bounded and uniformly continuous Fréchet derivative $\nabla f : H \rightarrow L(H, E)$; it is a Banach space endowed with the norm $\|\cdot\|_{C_b^1}$, $\|f\|_{C_b^1} = \|f\|_\infty + \|\nabla f\|_\infty$, $f \in C_b^1(H, E)$. We define, for $0 < \alpha < 1$, the space $C_b^\alpha(H, E)$ of all functions f in $C_b(H, E)$ such that

$$[f]_\alpha = \sup_{x', x \in H, x-x' \neq 0} \|f(x) - f(x')\|_E |x - x'|^{-\alpha} < \infty. \quad (2.1)$$

It is a Banach space endowed with the norm $\|\cdot\|_\alpha = \|\cdot\|_\infty + [\cdot]_\alpha$.

By $C([0, T] \times H, E)$ we denote the space of continuous functions from the product space $[0, T] \times H$ into E . Moreover, $B_b([0, T] \times H, E)$ is the Banach space of bounded Borel measurable functions from $[0, T] \times H$ into E endowed with the sup norm.

Let U be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_U$ and norm $|\cdot|_U$. To study (1.1) we assume that

Hypothesis 1 $\Lambda : \mathcal{D}(\Lambda) \subset U \rightarrow U$ is a given positive self-adjoint operator and there exists Λ^{-1} which is a trace class operator from U into U .

We also consider the Hilbert space $V = \mathcal{D}(\Lambda^{1/2}) = \text{Im}(\Lambda^{-1/2})$ endowed with the inner product

$$\langle h, k \rangle_V = \langle \Lambda^{1/2} h, \Lambda^{1/2} k \rangle_U, \quad h, k \in V$$

and its dual space V' which is again a Hilbert space. Note that $|\cdot|_{V'}$ is equivalent to $|\Lambda^{-1/2} \cdot|_U$. Moreover, V' can be identified with the completion of U with respect to the norm $|\Lambda^{-1/2} \cdot|_U$ (see Section 3.4 in [30]). V' is also denoted by $\mathcal{D}(\Lambda^{-1/2})$. We have $V \subset U \simeq U' \subset V'$ with continuous inclusions; Λ can be extended to an unbounded self-adjoint operator on V' with domain V , that we still denote by Λ :

$$\Lambda : V \rightarrow V'. \quad (2.2)$$

In a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_\tau)_{\tau \geq 0}$ satisfying the usual conditions, we consider the linear stochastic wave equation (1.1) where $\{W(\tau) = W_\tau, \tau \geq 0\}$ is a cylindrical Wiener process in U with respect to the filtration $(\mathcal{F}_\tau)_{\tau \geq 0}$. The process W_t is formally given by “ $W_t = \sum_{j \geq 1} \beta_j(t) e_j$ ” where $\beta_j(t)$ are independent real Wiener processes and (e_j) denotes a basis in U (see [14] for more details). We introduce, see (1.4), the reference Hilbert space $H = U \times V'$ for the solutions to (1.1); H is endowed with the inner product $\langle x, y \rangle_H = \langle x_1, y_1 \rangle_U + \langle x_2, y_2 \rangle_{V'} = \langle x_1, y_1 \rangle_U + \langle \Lambda^{-1/2} x_2, \Lambda^{-1/2} y_2 \rangle_U$ and norm $|x|_H = (\langle x, x \rangle_H)^{1/2}$, $x, y \in H$. This space is also denoted by $U \oplus V'$.

In the sequel we will also denote $\langle \cdot, \cdot \rangle_H$ and $|\cdot|_H$ by $\langle \cdot, \cdot \rangle$ and $|\cdot|$.

According to Example 5.8 in [14], the Eq. (1.1) is well-posed in H thanks to Hypothesis 1. On the other hand, (1.1) is not well-posed in the usual space $K = V \times U = \mathcal{D}(\Lambda^{1/2}) \times U$ (see (1.3)) for the deterministic wave equation: (i.e., solutions to (1.1) do not evolve in K even if $x_0 \in V$ and $x_1 \in U$; see Example 5.8 in [14]). Recall the inner product $\langle x, y \rangle_K = \langle x_1, y_1 \rangle_V + \langle x_2, y_2 \rangle_U$, $x, y \in K$. In H one considers the unbounded wave operator A which generates a unitary group e^{tA} :

$$\mathcal{D}(A) = V \times U, A \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \text{ for every } \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(A), \quad (2.3)$$

$$e^{tA} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \cos \sqrt{\Lambda} t & \frac{1}{\sqrt{\Lambda}} \sin \sqrt{\Lambda} t \\ -\sqrt{\Lambda} \sin \sqrt{\Lambda} t & \cos \sqrt{\Lambda} t \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, t \in \mathbb{R}, \quad \begin{pmatrix} y \\ z \end{pmatrix} \in H$$

(see also Appendix A.5.4 in [14]). Let $G : U \rightarrow K \subset H$,

$$Gu = \begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} u, \quad u \in U. \quad (2.4)$$

Notice that $e^{tA} : K \rightarrow K$ and $e^{tA} : H \rightarrow H$, and moreover since $(e^{tA})_t$ is a group of linear operators, then

$$e^{tA}(K) = K, \quad e^{tA}(H) = H, \quad t \in \mathbb{R}. \quad (2.5)$$

We still denote by A the generator A_K of (e^{tA}) in K which has domain $\mathcal{D}(\Lambda) \times V$. Clearly, the operator defined in (2.3) is an extension of A_K .

Equation (1.1) can be rewritten in an abstract form as

$$\begin{cases} dX_\tau = AX_\tau d\tau + GdW_\tau, \tau \in [0, T]. \\ X_0 = x \in H, \end{cases} \quad (2.6)$$

A solution to (2.6) is a particular Ornstein–Uhlenbeck process. We study (1.1) in H since the operators

$$Q_\tau = \int_0^\tau e^{sA} G G^* e^{sA^*} ds, \quad \tau \geq 0, \quad (2.7)$$

are of trace class from H into H thanks to Hypothesis 1 (cf. Example 5.8 in [14]); here G^* denotes the adjoint operator of G in H . Thus the stochastic convolution (i.e., the solution to (2.6) when $x = 0$)

$$S_\tau = \int_0^\tau e^{(\tau-s)A} G dW_s \text{ is well defined in } H. \quad (2.8)$$

Its law at time τ is the Gaussian measure $\mathcal{N}(0, Q_\tau)$ with mean 0 and covariance operator Q_τ (cf. [14]). Moreover, since $\sup_{t \in [0, T]} \|e^{tA} G\|_{L_2(U, H)} < \infty$, $T > 0$, we can apply Theorem 5.11 in [14] and deduce that the process (S_τ) has a continuous version with values in H .

3 The J -Valued Transition Semigroup for the Stochastic Wave Equation

Let J be a real separable Hilbert space. As in Sect. 2 we consider the Hilbert spaces $H = U \times V'$, and $K = V \times U \subset H$. Moreover (e_j) is a basis in U such that $(e_j) \subset \mathcal{D}(\Lambda) \subset U$ and

$$\Lambda e_j = \lambda_j e_j, \quad \lambda_j > 0, \quad j \geq 1; \quad \sum_{j \geq 1} \lambda_j^{-1} < \infty. \quad (3.1)$$

We will prove some regularizing effects for the Ornstein–Uhlenbeck semigroup (R_t) related to stochastic wave equation (1.5) and acting on J -valued functions Φ . Recall that

$$R_\tau [\Phi](x) = R_\tau \Phi(x) = \mathbb{E} \Phi \left(X_\tau^{0,x} \right), \quad \Phi \in B_b(H, J), \quad x \in H, \quad \tau \geq 0, \quad (3.2)$$

where X , defined by (2.6), is the Ornstein–Uhlenbeck process (cf. [8]). Since X is time homogeneous, we have $R_{\tau-t} [\Phi](x) = \mathbb{E} \Phi \left(X_\tau^{t,x} \right)$, $\Phi \in B_b(H, J)$, $\tau \geq t \geq 0$, $x \in H$. Similarly, we consider the usual Ornstein–Uhlenbeck semigroup (P_t) acting on scalar functions $\phi \in B_b(H)$:

$$P_\tau [\phi](x) = P_\tau \phi(x) = \mathbb{E} \phi \left(X_\tau^{0,x} \right), \quad \phi \in B_b(H), \quad \tau \geq 0. \quad (3.3)$$

Using also the results in “Appendix”, for $t > 0$, we show the differentiability of $R_t \Phi$ along the directions of K . Moreover, we prove that, for any $x \in H$, $k \in K$, $t > 0$, the series in (1.6) is finite, and we provide a bound independent of x and k (see Lemma 3.11 and compare with Chapter 6 of [13] and Sect. 3 of [8]).

In order to study differentiability properties of $R_t[\Phi]$ for $t > 0$ we fix some basic definitions. We say that $F : H \rightarrow J$ is differentiable along the subspace $K = V \times U \subset H$ if there exists at any $x \in H$ the directional derivative along any direction $k \in K$ (i.e., $\lim_{s \rightarrow 0} \frac{F(x+sk) - F(x)}{s} \in J$). We denote the directional derivative at x along the direction $k \in K$ as $\nabla_k F(x) \in J$.

If in addition $k \mapsto \nabla_k F(x)$ belongs to $L(K, J)$ we indicate such linear operator with $\nabla^K F(x)$. We say that F is K -differentiable on H if it is differentiable along the subspace K and there exists $\nabla^K F(x) \in L(K, J)$ for any $x \in H$ (if $J = \mathbb{R}$ then $\nabla^K F(x)$ can be identified with an element in K by the Riesz theorem).

Note that the concept of differentiability along subspaces arises naturally in the Malliavin calculus (see also the related concept of Gross differentiability; we refer to [28] and the references therein).

Let $G : U \rightarrow H$, $Ga = \begin{pmatrix} 0 \\ a \end{pmatrix} \in K \subset H$, $a \in U$. If $F : H \rightarrow J$ is differentiable along the subspace $G(U)$ we set

$$\nabla_a^G F(x) = \nabla_{Ga} F(x) \in J, \quad a \in U, x \in H. \quad (3.4)$$

If in addition $a \mapsto \nabla_a^G F(x)$ belongs to $L(U, J)$ we denote such linear operator with $\nabla^G F(x)$. We say that F is G -differentiable on H if it is differentiable along the subspace $G(U)$ and there exists $\nabla^G F(x) \in L(U, J)$ for any $x \in H$. Note that if $F : H \rightarrow J$ is K -differentiable on H then it is also G -differentiable on H and $\nabla^G F(x) = \nabla^K F(x)G \in L(U, J)$.

3.1 Interpolation Results Involving K -Differentiable Functions

We first introduce spaces of functions that are K -differentiable. We say that $f \in C_K^1(H, J)$ if $f \in C_b(H, J)$, f is K -differentiable on H and $\nabla^K f : H \rightarrow L(K, J)$ is uniformly continuous and bounded. It is a Banach space endowed with the norm

$$\|f\|_{C_K^1(H, J)} = \|\nabla^K f\|_\infty + \|f\|_\infty, \quad f \in C_K^1(H, J),$$

setting $\sup_{x \in H} \|\nabla^K f(x)\|_J = \|\nabla^K f\|_\infty$. When $J = \mathbb{R}$ we set $C_K^1(H, \mathbb{R}) = C_K^1(H)$. Recall that for $f \in C_K^1(H)$ one has: $\nabla^K f : H \rightarrow K$ uniformly continuous and bounded.

Let us consider the following operator $Q : H \rightarrow H$,

$$Q = \begin{pmatrix} \Lambda^{-1} & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}, \quad Qh = \begin{pmatrix} \Lambda^{-1}h_1 \\ \Lambda^{-1}h_2 \end{pmatrix}, \quad h = (h_1, h_2) \in H = U \times V'. \quad (3.5)$$

Let (e_j) be the basis in U defined in (3.1). Then $(\sqrt{\lambda_j}e_j)$ is a basis of V' and $\{(e_j, 0)\}_{j \geq 1} \cup \{(0, \sqrt{\lambda_j}e_j)\}_{j \geq 1}$ is a basis of H .

It is not difficult to check that Q is a symmetric positive trace class operator and that

$$C_K^1(H) \text{ coincides with } C_Q^1(H) \text{ introduced in [5] with equivalence of norms.} \quad (3.6)$$

To this purpose we note that $Q^{1/2}H = K$. Then we consider conditions (i), (ii) and (iii) used in the definition of $C_Q^1(H)$ in Section 2.1 of [5]. Let $f \in C_b(H)$. Condition (i) says that there exist all the directional derivatives of f in the directions of $K = Q^{1/2}H$. Let $k = Q^{1/2}h$ with $h \in H$. The directional derivative in x along the direction k is denoted by

$$\nabla_k f(x) = \nabla_{Q^{1/2}h} f(x).$$

Condition (ii) says that for any $x \in H$, there exists $D_Q f(x) \in H$ such that

$$\nabla_{Q^{1/2}h} f(x) = \langle D_Q f(x), h \rangle_H.$$

If $k \in K$ then $k = Q^{1/2}h$ for a unique $h \in H$. We have $\langle D_Q f(x), h \rangle_H = \langle Q^{1/2} D_Q f(x), Q^{1/2} h \rangle_K = \langle Q^{1/2} D_Q f(x), k \rangle_K$. Thus condition (ii) is equivalent to say that $k \mapsto \nabla_k f(x)$ is linear and continuous from K into $\mathbb{R}$. Moreover such linear functional can be identified with $Q^{1/2} D_Q f(x)$. According to our previous notation we can write

$$\nabla^K f(x) = Q^{1/2} D_Q f(x), \quad x \in H.$$

Condition (iii) requires that the mapping: $x \mapsto D_Q f(x)$ is uniformly continuous and bounded from H into H . This is equivalent to say that the mapping: $x \mapsto Q^{1/2} D_Q f(x) = \nabla^K f(x)$ is uniformly continuous and bounded from H into K . This shows (3.6).

Similarly to [5], we define, for $0 < \alpha < 1$, the space $C_K^\alpha(H, J)$ of all functions f in $C_b(H, J)$ such that

$$[f]_{\alpha, K} = \sup_{k', k \in K, k-k' \neq 0} \frac{|f(k) - f(k')|_J}{|k - k'|_K^\alpha} < \infty. \quad (3.7)$$

It is a Banach space endowed with the norm $\|\cdot\|_{\alpha, K} = \|\cdot\|_\infty + [\cdot]_{\alpha, K}$, where $\|f\|_\infty = \sup_{x \in H} |f(x)|_J$.

Note that $C_b^\alpha(H, J) \subset C_K^\alpha(H, J)$ and in general the inclusion is strict (cf. Remark 3.3).

Remark 3.1 Condition (3.7) is equivalent to

$$\sup_{x \in H=U \times V', k \in K, k \neq 0} \frac{|f(x+k) - f(x)|_J}{|k|_K^\alpha} < \infty.$$

Indeed if $x \in H$ and $k \in K$ there exists a sequence $(k_n) \subset K$ such that $k_n \rightarrow x$ in H . Then by (3.7) we find $|f(k_n+k) - f(k_n)|_J \leq C|k|_K^\alpha$. Passing to the limit as $n \rightarrow \infty$ we obtain $|f(x+k) - f(x)|_J \leq C|k|_K^\alpha$.

The space $C_K^\alpha(H) = C_K^\alpha(H, \mathbb{R})$ coincides with the space $C_Q^\alpha(H)$ introduced in Section 2.2 in [5] as the space of all functions $f \in C_b(H)$ such that

$$[f]_{\alpha, Q} = \sup_{h', h \in H, h-h' \neq 0} |f(Q^{1/2}h) - f(Q^{1/2}h')| \cdot |h - h'|_H^{-\alpha} < \infty$$

with equivalence of norms. By [5] we now obtain the following useful result.

Lemma 3.2 *We have, for $\alpha \in (0, 1)$, with equivalence of norms,*

$$(C_b(H), C_K^1(H))_{\alpha, \infty} = C_K^\alpha(H). \quad (3.8)$$

Proof The result is proved in Proposition 2.1 in [5] in the form

$$(C_b(H), C_Q^1(H))_{\alpha, \infty} = C_Q^\alpha(H).$$

We only recall that $f \in X_\alpha = (C_b(H), C_K^1(H))_{\alpha, \infty}$ if $\|f\|_{X_\alpha} = \sup_{t \in (0, 1]} t^{-\alpha} L(t, f) < \infty$ where $L(t, f) = \inf\{\|a\|_{C_b(H)} + t\|b\|_{C_K^1(H)}, f = a + b, a \in C_b(H), b \in C_K^1(H)\}$ (see, for instance, Section 2.3 in [14]). $\square$

Remark 3.3 Theorem 3.1 in [27] implies that $C_b^\alpha(H)$ (the space of real α -Hölder continuous and bounded functions defined on H) is strictly included in $C_Q^\alpha(H)$, $\alpha \in (0, 1)$. Indeed $C_b^\alpha(H)$ is contained in the interpolation space $\mathcal{D}_{\mathcal{A}}(\alpha/2, \infty)$ (see the notation in [27]) which by Theorem 3.1 is strictly included in $C_Q^\alpha(H)$. $\square$

When J is infinite dimensional it is an open problem to characterize both $(C_b(H, J), C_K^1(H, J))_{\alpha, \infty}$ and $(C_b(H, J), C_b^1(H, J))_{\alpha, \infty}$. However we can prove the following inclusion which will be important for the sequel (see in particular the proof of Lemma 3.9).

Lemma 3.4 *For any real separable Hilbert space J we have*

$$C_b^\alpha(H, J) \subset (C_b(H, J), C_K^1(H, J))_{\alpha, \infty}, \quad \alpha \in (0, 1), \quad \text{with a continuous inclusion.} \quad (3.9)$$

Proof Let $f \in C_b^\alpha(H, J)$ and $t \in (0, 1]$. Taking into account Remark 2.3.1 in [13], we prove that there exists $a_t \in C_b(H, J)$ and $b_t \in C_K^1(H, J)$ such that $f = a_t + b_t$ and

$$\|a_t\|_{C_b(H, J)} + t\|b_t\|_{C_K^1(H, J)} \leq c\|f\|_{C_b^\alpha(H, J)} t^\alpha \quad (3.10)$$

with $c > 0$ independent of t and f . This gives (3.9).

Let us consider the trace class operator $Q : H \rightarrow H$ given in (3.5). Recall that Q is injective and $Q^{1/2}(H) = K$. As in Chapter 3 of [13] we consider the heat semigroup (V_t) acting on functions in $C_b(H, J)$:

$$V_r g(x) = \int_H g(x + y) \mathcal{N}(0, rQ)(dy), \quad x \in H, \quad g \in C_b(H, J), \quad r \geq 0.$$

For $t \in (0, T]$ we set

$$a_t = f - V_{t^2} f, \quad b_t = V_{t^2} f$$

and we prove that (3.10) holds. Let us first consider a_t . It is easy to prove that $a_t \in C_b(H, J)$. Moreover,

$$\begin{aligned} |a_t(x)|_J &\leq \int_H |f(x + y) - f(x)|_J \mathcal{N}(0, t^2 Q)(dy) \\ &\leq \|f\|_{C_b^\alpha(H, J)} \int_H |ty|_H^\alpha \mathcal{N}(0, Q)(dy) \leq c_\alpha \|f\|_{C_b^\alpha(H, J)} t^\alpha. \end{aligned}$$

To prove that $b_t \in C_K^1(H, J)$ we consider $k = Q^{1/2}h \in K$ with $h = Q^{-1/2}k \in H$. Arguing as in Theorem 3.3.3 in [14], using the Cameron–Martin theorem, one can prove that, for any $x \in H$, there exists the directional derivative

$$\begin{aligned}\nabla_k b_t(x) &= \lim_{s \rightarrow 0} \frac{V_{t^2} f(x + sQ^{1/2}h) - V_{t^2} f(x)}{s} \\ &= \frac{1}{t} \int_H f(x+y) \langle (t^2 Q)^{-1/2} y, h \rangle \mathcal{N}(0, t^2 Q)(dy) \\ &= \frac{1}{t} \int_H f(x+y) \langle (t^2 Q)^{-1/2} y, Q^{-1/2} k \rangle \mathcal{N}(0, t^2 Q)(dy).\end{aligned}\quad (3.11)$$

It is not difficult to prove that $k \mapsto \nabla_k b_t(x)$ is linear and continuous from K into J . We note that

$$\begin{aligned}|\nabla_k b_t(x)|_J^2 &\leq \frac{1}{t^2} \|f\|_\infty^2 \int_H |\langle (t^2 Q)^{-1/2} y, Q^{-1/2} k \rangle|^2 \mathcal{N}(0, t^2 Q)(dy) \\ &\leq \frac{1}{t^2} \|f\|_\infty^2 |Q^{-1/2} k|^2 = \frac{|k|_K^2}{t^2} \|f\|_\infty^2,\end{aligned}\quad (3.12)$$

where in the last passage we have used that $|Q^{-1/2}k| = |k|_K$. We take into account (3.11) and (3.12), and for $k \in K$, $x, x' \in H$, we consider the difference

$$\begin{aligned}&|\nabla_k b_t(x) - \nabla_k b_t(x')|_J^2 \\ &\leq \frac{1}{t^2} \int_H |f(x+y) - f(x'+y)|^2 \langle (t^2 Q)^{-1/2} y, Q^{-1/2} k \rangle^2 \mathcal{N}(0, t^2 Q)(dy) \\ &\leq \frac{1}{t^2} \int_H |\langle (t^2 Q)^{-1/2} y, Q^{-1/2} k \rangle|^2 \mathcal{N}(0, t^2 Q)(dy) \\ &\quad \cdot \int_H |f(x+y) - f(x'+y)|_J^2 \mathcal{N}(0, t^2 Q)(dy) \leq \frac{|k|_K^2}{t^2} \|f\|_{C_b^\alpha(H, J)}^2 |x - x'|^{2\alpha}.\end{aligned}$$

We check easily that $\nabla^K b_t : H \rightarrow L(K, J)$ is uniformly continuous and bounded. Finally, since

$$\frac{1}{t} \int_H f(x) \langle (t^2 Q)^{-1/2} y, h \rangle \mathcal{N}(0, t^2 Q)(dy) = 0,$$

we obtain by the Cauchy–Schwarz inequality, for any $k \in K$, $x \in H$,

$$\begin{aligned}|\nabla_k b_t(x)|_J^2 &= \left| \frac{1}{t} \int_H \langle (t^2 Q)^{-1/2} y, Q^{-1/2} k \rangle [f(x+y) - f(x)] \mathcal{N}(0, t^2 Q)(dy) \right|_J^2 \\ &\leq \|f\|_{C_b^\alpha(H, J)}^2 \frac{1}{t^2} \int_H |\langle (t^2 Q)^{-1/2} y, Q^{-1/2} k \rangle|^2 \mathcal{N}(0, t^2 Q)(dy) \cdot \\ &\quad \cdot \int_H |ty|_H^{2\alpha} \mathcal{N}(0, Q)(dy) \leq c_\alpha \|f\|_{C_b^\alpha(H, J)}^2 t^{2\alpha-2}.\end{aligned}$$

Collecting the previous estimates we get (3.10). $\square$

3.2 Partial Regularizing Properties of the Transition Semigroup

We first collect some useful properties that we prove in Appendix by classical control theoretic arguments (for more details see Appendix in [25] which is based on similar results proved in [24]).

Remark 3.5 For any $t > 0$, we have $e^{tA}(K) = Q_t^{1/2}(H) = K$ and $Q_t^{-1/2}e^{tA}$ belongs to $L(K, H)$ (we recall that Q_t has been defined in (2.7)). Let $T > 0$. There exists $c = c_T > 0$ such that for any $t \in (0, T]$ we have

$$|Q_t^{-1/2}e^{tA}k|_H \leq \frac{c}{t^{3/2}}|k|_K, \quad k \in K = V \times U; \quad (3.13)$$

$$|Q_t^{-1/2}e^{tA}Ga|_H \leq \frac{c}{t^{1/2}}|a|_U, \quad a \in U. \quad (3.14)$$

Let $\Phi \in B_b(H, J)$ and $x \in H$. Arguing as in the proof of Theorem 6.2.2 of [13] (similar arguments are used in Section 9.4 of [14] and Section 3 of [8]) one can prove the existence of the directional derivative of $R_t[\Phi]$ along the directions of K :

$$\begin{aligned} & \lim_{s \rightarrow 0} \frac{R_t[\Phi](x + sk) - R_t[\Phi](x)}{s} \\ &= \nabla_k R_t[\Phi](x) = \int_H \langle Q_t^{-\frac{1}{2}}e^{tA}k, Q_t^{-\frac{1}{2}}y \rangle \Phi(e^{tA}x + y) \mathcal{N}(0, Q_t)(dy), \\ & k \in K, \quad t > 0. \end{aligned} \quad (3.15)$$

In the sequel we often write $\mu_t = \mathcal{N}(0, Q_t)$ and $|\cdot|_H = |\cdot|$. In the next result we will use (3.15) together with the estimates (3.13) and (3.14).

Lemma 3.6 Assume Hypothesis 1 and let $R = (R_t)$ be the OU semigroup defined in (3.2). If $\Phi \in B_b(H, J)$ and $t > 0$ then $R_t\Phi$ is K -differentiable on H . The directional derivative $\nabla_k R_t[\Phi](x) \in J$ is given by (3.15), for $x \in H$. In particular $R_t[\Phi]$ is G -differentiable on H ; further

$$\nabla_a^G R_t[\Phi](x) = \int_H \langle Q_t^{-\frac{1}{2}} e^{tA} Ga, Q_t^{-\frac{1}{2}} y \rangle \Phi(e^{tA}x + y) \mathcal{N}(0, Q_t)(dy), \quad a \in U. \quad (3.16)$$

Moreover, for $t \in (0, T]$, we have

$$\sup_{x \in H} |\nabla_k R_t[\Phi](x)|_J \leq \frac{c}{t^{\frac{3}{2}}} \|\Phi\|_\infty |k|_K, \quad k \in K; \quad (3.17)$$

$$\sup_{x \in H} |\nabla_a^G R_t[\Phi](x)|_J \leq \frac{c}{t^{\frac{3}{2}}} \|\Phi\|_\infty |Ga|_K = \frac{c}{t^{\frac{3}{2}}} \|\Phi\|_\infty |a|_U, \quad a \in U. \quad (3.18)$$

If in addition $\Phi \in C_b(H, J)$ then $\nabla^K R_t[\Phi] \in C_b(H, L(K, J))$, $\nabla^G R_t[\Phi] \in C_b(H, L(U, J))$ for $t > 0$.

Proof Let us fix $t \in (0, T]$ and $x \in H$. The integral in (3.15) defines a linear operator in $L(K, J)$. Let

$$I_{t,x}k := \int_H \langle Q_t^{-\frac{1}{2}} e^{tA}k, Q_t^{-\frac{1}{2}} y \rangle \Phi(e^{tA}x + y) \mathcal{N}(0, Q_t)(dy), \quad k \in K.$$

We have the well-known estimate (cf. the proof of Theorem 6.2.2 in [13])

$$\begin{aligned} |I_{t,x}k|_J &\leq \int_H |\langle Q_t^{-\frac{1}{2}} e^{tA}k, Q_t^{-\frac{1}{2}} y \rangle \Phi(e^{tA}x + y)|_J \mathcal{N}(0, Q_t)(dy) \\ &\leq \|\Phi\|_\infty \left(\int_H |\langle Q_t^{-\frac{1}{2}} e^{tA}k, Q_t^{-\frac{1}{2}} y \rangle|^2 \mathcal{N}(0, Q_t)(dy) \right)^{1/2} \\ &= \|\Phi\|_\infty |Q_t^{-\frac{1}{2}} e^{tA}k|_H \\ &\leq \frac{c}{t^{\frac{3}{2}}} \|\Phi\|_\infty |k|_K. \end{aligned} \quad (3.19)$$

Similarly, we get (3.18) using (3.14) since

$$\int_H |\langle Q_t^{-\frac{1}{2}} e^{tA}Ga, Q_t^{-\frac{1}{2}} y \rangle|^2 \mathcal{N}(0, Q_t)(dy) \leq |Q_t^{-\frac{1}{2}} e^{tA}Ga|^2 \leq \frac{c|a|_U^2}{t}, \quad a \in U. \quad (3.20)$$

Computing the directional derivative as in (3.15) we obtain the differentiability of $R_t[\Phi]$ at x along the directions of K . We also obtain that $R_t[\Phi]$ is G -differentiable and K -differentiable on H .

If $\Phi \in C_b(H, J)$ we compute, for any $k \in K$, $|k|_K = 1$, $z \in H$,

$$\begin{aligned} &|I_{t,x}k - I_{t,x+z}k|_J^2 \\ &\leq \int_H |\langle Q_t^{-\frac{1}{2}} e^{tA}k, Q_t^{-\frac{1}{2}} y \rangle|^2 \mathcal{N}(0, Q_t)(dy) \end{aligned}$$

$$\begin{aligned} & \cdot \int_H |\Phi(e^{tA}x + y) - \Phi(e^{tA}x + e^{tA}z + y)|_J^2 \mathcal{N}(0, Q_t)(dy) \\ & \leq \frac{c^2 |k|_K^2}{t^3} \int_H \sup_{x \in H} |\Phi(e^{tA}x + y) - \Phi(e^{tA}x + e^{tA}z + y)|_J^2 \mathcal{N}(0, Q_t)(dy) \end{aligned}$$

and so we obtain easily $\lim_{z \rightarrow 0} \sup_{y \in H} \sup_{|k|_K=1} |I_{t,y}k - I_{t,y+z}k|_J = 0$ by the dominated convergence theorem. We deduce $\nabla^K R_t[\Phi] \in C_b(H, L(K, J))$ and $\nabla^G R_t[\Phi] \in C_b(H, L(U, J))$. $\square$

In a similar way we get

Lemma 3.7 *Under the assumptions of Lemma 3.6 let $t > 0$. If $\Phi \in C_b(H, J)$ and $\xi \in U$ the J -valued mapping:*

$$x \mapsto \nabla_\xi^G R_t[\Phi](x)$$

is K -differentiable on H . The second order directional derivative is

$$\begin{aligned} \nabla_k \nabla_\xi^G R_t[\Phi](x) &= \int_H (\langle \Gamma_t k, Q_t^{-\frac{1}{2}} y \rangle \langle \Gamma_t G \xi, Q_t^{-\frac{1}{2}} y \rangle \\ &\quad - \langle \Gamma_t k, \Gamma_t G \xi \rangle) \Phi(e^{tA}x + y) \mu_t(dy), \end{aligned} \quad (3.21)$$

for $x \in H, k \in K, \xi \in U$, and where $\Gamma_t := Q_t^{-\frac{1}{2}} e^{tA}$ is well defined, see Remark 3.5. Moreover, for each $x \in H, k \in K$, the map: $\xi \rightarrow \nabla_k \nabla_\xi^G R_t[\Phi](x)$ belongs to $L(U, J)$ and, for any $t \in (0, T]$,

$$\sup_{x \in H} \|\nabla_k \nabla_\xi^G R_t[\Phi](x)\|_{L(U, J)} \leq \frac{c_T |k|_K}{t^2} \|\Phi\|_\infty, \quad k \in K. \quad (3.22)$$

$$\begin{aligned} & \lim_{x \rightarrow 0} \sup_{y \in H} \|\nabla_k \nabla_\xi^G R_t[\Phi](x + y) - \nabla_k \nabla_\xi^G R_t[\Phi](y)\|_{L(K, J)} \\ &= \lim_{x \rightarrow 0} \sup_{y \in H} \sup_{|k|_K=1} |\nabla_k \nabla_\xi^G R_t[\Phi](x + y) - \nabla_k \nabla_\xi^G R_t[\Phi](y)|_J = 0, \quad \xi \in U. \end{aligned} \quad (3.23)$$

Proof Let us fix $T > 0$ and $t \in (0, T]$, $x \in H$. Let $\xi \in U$. First define $\Gamma_{t,x,k,\xi}$ as the integral in the right hand side of (3.21). Proceeding as in the proof of Lemma 3.6 it is not difficult to show that

$$k \mapsto \Gamma_{t,x,k,\cdot} \text{ is defined from } K \text{ into } L(U, J). \quad (3.24)$$

By (3.13) and using the Cauchy-Schwarz inequality, we get

$$|\Gamma_{t,x,k,\xi}|_J^2 \leq \left| \int_H [\langle \Gamma_t k, Q_t^{-\frac{1}{2}} y \rangle \langle \Gamma_t G \xi, Q_t^{-\frac{1}{2}} y \rangle - \langle \Gamma_t k, \Gamma_t G \xi \rangle] \Phi(e^{tA}x + y) \mu_t(dy) \right|_J^2$$

$$\begin{aligned}
&\leq \int_H |\langle \Gamma_t k, Q_t^{-\frac{1}{2}} y \rangle \langle \Gamma_t G \xi, Q_t^{-\frac{1}{2}} y \rangle - \langle \Gamma_t k, \Gamma_t G \xi \rangle|^2 \mu_t(dy) \\
&\quad \cdot \int_H |\Phi(e^{tA} x + y)|_J^2 \mu_t(dy) \\
&\leq \frac{c|k|_K^2}{t^4} |\xi|_U^2 \|\Phi\|_\infty^2.
\end{aligned} \tag{3.25}$$

Thus we have proved (3.22). Arguing as in Section 9.4 of [14] and as in Sect. 3 of [8] we find that

$$\lim_{s \rightarrow 0} \frac{\nabla_\xi^G R_t[\Phi](x + sk) - \nabla_\xi^G R_t[\Phi](x)}{s} = \Gamma_{t,x,k,\xi}, \quad k \in K.$$

Moreover, arguing as in the proof of Lemma 4.2 in [24], for any $z \in H$, $\xi \in U$,

$$\begin{aligned}
&|\Gamma_{t,x,k,\xi} - \Gamma_{t,x+z,k,\xi}|_J^2 \\
&\leq \frac{c|k|_K^2}{t^4} |\xi|_U^2 \int_H |\Phi(e^{tA} x + y) - \Phi(e^{tA} x + e^{tA} z + y)|_J^2 \mu_t(dy)
\end{aligned}$$

and so

$$\lim_{z \rightarrow 0} \sup_{x \in H} \sup_{|k|_K=1} |\Gamma_{t,x,k,\xi} - \Gamma_{t,x+z,k,\xi}|_J^2 = 0. \tag{3.26}$$

This shows in particular that the mapping $x \mapsto \nabla_\xi^G R_t[\Phi](x)$ verifies (3.23). $\square$

Now we improve the previous estimates in the case when Φ is Hölder continuous along the directions of K using Lemma 3.2.

Lemma 3.8 *Let $T > 0$. Under the assumptions of Lemma 3.6 let $\Phi \in C_K^\alpha(H, J)$, $\alpha \in (0, 1)$, see (3.7) and Remark 3.1. We have all the assertions of Lemmas 3.6 and 3.7 and the following new estimates, for $t \in (0, T]$,*

$$\begin{aligned}
\sup_{x \in H} |\nabla_k R_t[\Phi](x)|_J &\leq \frac{c}{t^{\frac{3}{2}(1-\alpha)}} \|\Phi\|_{\alpha,K} |k|_K, \quad k \in K; \\
\sup_{x \in H} \|\nabla_k \nabla_\xi^G R_t[\Phi](x)\|_{L(U,J)} &\leq \frac{c}{t^{\frac{4-3\alpha}{2}}} \|\Phi\|_{\alpha,K} |k|_K, \quad k \in K.
\end{aligned} \tag{3.27}$$

Proof Let us fix $t \in (0, T]$, $k \in K$ and $\xi \in U$. Using the OU process X defined by (4.5) we can define the Ornstein–Uhlenbeck semigroup (P_t) acting on scalar functions $\phi \in B_b(H)$ (see (3.3)).

For $h \in J$, we introduce the scalar function $\Phi_h(x) = \langle \Phi(x), h \rangle_J$, $x \in H$, which belongs to $C_K^\alpha(H)$ with $\|\Phi_h\|_{\alpha,K} \leq \|\Phi\|_{\alpha,K} |h|_J$. We note arguing as in Sect. 3 of [8] that

$$\langle \nabla_k R_t[\Phi](x), h \rangle_J = \nabla_k P_t[\Phi_h](x), \quad x \in H.$$

To prove the first estimate we consider the linear operators

$$\nabla_k P_t : C_K^1(H) \rightarrow C_b(H), \quad \nabla_k P_t : C_b(H) \rightarrow C_b(H).$$

These operators are well defined by Lemma 3.6. When $\phi \in C_K^1(H)$ we find that (recall that $e^{tA}k \in K$)

$$\begin{aligned} \nabla_k P_t[\phi](x) &= \lim_{s \rightarrow 0} \int_H \frac{\phi(e^{tA}x + se^{tA}k + y) - \phi(e^{tA}x + y)}{s} \mu_t(dy) \\ &= \int_H \nabla_{e^{tA}k} \phi(e^{tA}x + y) \mu_t(dy) = \int_H \langle \nabla^K \phi(e^{tA}x + y), e^{tA}k \rangle_K \mu_t(dy) \end{aligned}$$

(by the Riesz theorem we identify $\nabla^K \phi(e^{tA}x + y)$ with an element in K). We get the estimate

$$\sup_{x \in H} |\nabla_k P_t[\phi](x)| \leq C \|\nabla^K \phi\|_\infty |k|_K, \quad \phi \in C_K^1(H). \quad (3.28)$$

On the other hand we have (cf. (3.17))

$$\sup_{x \in H} |\nabla_k P_t[f](x)| \leq \frac{c}{t^{\frac{3}{2}}} \|f\|_\infty |k|_K, \quad f \in C_b(H). \quad (3.29)$$

Interpolating between (3.28) and (3.29) (see Theorem A.1.1 in [14]) we obtain that, for any $\alpha \in (0, 1)$,

$$\nabla_k P_t : (C_b(H), C_K^1(H))_{\alpha, \infty} \rightarrow C_b(H)$$

is linear and bounded; further

$$\sup_{x \in H} |\nabla_k P_t[\psi](x)| \leq \frac{\tilde{c}}{t^{\frac{3}{2}(1-\alpha)}} \|\psi\|_{(C_b(H), C_K^1(H))_{\alpha, \infty}} |k|_K, \quad \psi \in (C_b(H), C_K^1(H))_{\alpha, \infty}.$$

Now thanks to Lemma 3.2 we deduce

$$\sup_{x \in H} |\nabla_k P_t[\psi](x)| \leq \frac{c}{t^{\frac{3}{2}(1-\alpha)}} \|\psi\|_{\alpha, K} |k|_K, \quad \psi \in C_K^\alpha(H).$$

If we consider now $\psi = \Phi_h$, we have, for each $x \in H, h \in J$,

$$\begin{aligned} |\langle \nabla_k R_t[\Phi](x), h \rangle_J| &= |\nabla_k P_t[\Phi_h](x)| \\ &\leq \frac{c}{t^{\frac{3}{2}(1-\alpha)}} \|\Phi_h\|_{\alpha, K} |k|_K \leq \frac{c}{t^{\frac{3}{2}(1-\alpha)}} \|\Phi\|_{\alpha, K} |k|_K |h|_J. \end{aligned}$$

By taking the supremum over $\{h \in J : |h|_J = 1\}$ we get the first estimate in (3.27).

To prove the second estimate we fix $t > 0$, $k \in K$, with $|k|_K = 1$, $\xi \in U$ with $|\xi|_U = 1$ and argue as before. We first introduce the following linear operators (cf. Lemma 3.7)

$$\nabla_k \nabla_\xi^G P_t : C_K^1(H) \rightarrow C_b(H), \quad \nabla_k \nabla_\xi^G P_t : C_b(H) \rightarrow C_b(H). \quad (3.30)$$

When $\phi \in C_K^1(H)$ we know that

$$\nabla_k \nabla_\xi^G P_t[\phi](x) = \int_H \langle \Gamma_t G \xi, Q_t^{-\frac{1}{2}} y \rangle \nabla_{e^{tA}k} \phi(e^{tA}x + y) \mu_t(dy). \quad (3.31)$$

Moreover, we have, with $\Gamma_t = Q_t^{-1/2} e^{tA}$,

$$\sup_{x \in H} |\nabla_k \nabla_\xi^G P_t[\phi](x)|^2 \leq \|\phi\|_{C_K^1}^2 |Q_t^{-1/2} e^{tA} G \xi|_H^2 \leq \frac{c}{t} |\xi|_U^2 \|\phi\|_{C_K^1}^2 = \frac{c}{t} \|\phi\|_{C_K^1}^2. \quad (3.32)$$

On the other hand if $\phi \in C_b(H)$ then

$$\begin{aligned} \sup_{x \in H} |\nabla_k \nabla_\xi^G P_t[\phi](x)|^2 &\leq C \|\phi\|_\infty^2 |\Gamma_t k|_H^2 |Q_t^{-1/2} e^{tA} G \xi|_H^2 \\ &\leq c \|\phi\|_\infty^2 \frac{1}{t^3} |k|_K^2 \frac{1}{t} |\xi|_U^2 \leq c \|\phi\|_\infty^2 \frac{1}{t^4}. \end{aligned} \quad (3.33)$$

Interpolating between (3.32) and (3.33) as we have done before we get

$$\sup_{x \in H} |\nabla_k \nabla_\xi^G P_t[\psi](x)| \leq \frac{c}{t^{\frac{4-3\alpha}{2}}} \|\psi\|_{\alpha, K}, \quad \psi \in C_K^\alpha(H). \quad (3.34)$$

Now for $x \in H$, $k \in K$, $\Phi \in C_K^\alpha(H, J)$, we compute

$$\begin{aligned} \|\nabla_k(\nabla^G R_t[\Phi])(x)\|_{L(U, J)}^2 &= \sup_{a \in U, |a|_U=1} |\nabla_k(\nabla_a^G R_t[\Phi])(x)|_J^2 \\ &= \sup_{|a|_U=1} \sup_{h \in J, |h|_J=1} |\langle \nabla_k(\nabla_a^G R_t[\Phi])(x), h \rangle_J|^2 = \sup_{|a|_U=1} \sup_{|h|_J=1} |\nabla_k(\nabla_a^G P_t[\Phi_h])(x)|^2 \\ &\leq \frac{c |k|_K^2}{t^{4-3\alpha}} \sup_{|h|_J=1} \|\Phi_h\|_{\alpha, K}^2 \leq \frac{c |k|_K^2}{t^{4-3\alpha}} \|\Phi\|_{\alpha, K}^2. \end{aligned}$$

The second estimate in (3.27) follows easily. $\square$

The next result is crucial for our approach to get pathwise uniqueness (see in particular Theorem 4.3 and the proof of Theorem 4.7). We can only prove the result when $\Phi \in C_b^\alpha(H, J)$ using Lemma 3.4. We do not know if such result holds more generally when $\Phi \in C_K^\alpha(H, J)$ (see the end of the next proof).

We fix a basis (f_m) of J and set $\Phi_m = \langle \Phi, f_m \rangle_J$. We will use the OU semigroup (P_t) given in (3.3).

Lemma 3.9 Let $T > 0$. Under the assumptions of Lemma 3.6 let $\Phi \in C_b^\alpha(H, J)$, $\alpha \in (0, 1)$. We have, for $t \in (0, T]$, $k \in K$,

$$\left(\sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_k \nabla_a^G P_t[\Phi_m](x)|^2 \right)^{1/2} \leq \frac{C}{t^{\frac{4-3\alpha}{2}}} |k|_K \|\Phi\|_{C_b^\alpha(H, J)}. \quad (3.35)$$

where $C > 0$ is independent of x, k, t, Φ and the basis (f_m) of J .

Proof We recall that for $k \in K, a \in U$

$$\begin{aligned} \nabla_k \nabla_a^G P_t[\Phi_m](x) &= \int_H \left(\langle \Gamma_t k, Q_t^{-\frac{1}{2}} y \rangle \langle \Gamma_t G a, Q_t^{-\frac{1}{2}} y \rangle - \right. \\ &\quad \left. - \langle \Gamma_t k, \Gamma_t G a \rangle \right) \Phi_m(e^{tA} x + y) \mu_t(dy). \end{aligned}$$

We fix $k \in K$. We have by the Cauchy-Schwarz inequality (cf. (3.25)) for any $x \in H$, $m \geq 1$,

$$\begin{aligned} &\sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_k \nabla_a^G P_t[\Phi_m](x)|^2 \\ &\leq \sum_{m \geq 1} \sup_{|a|_U=1} \int_H \left| \left(\langle \Gamma_t k, Q_t^{-\frac{1}{2}} y \rangle \langle \Gamma_t G a, Q_t^{-\frac{1}{2}} y \rangle - \langle \Gamma_t k, \Gamma_t G a \rangle \right) \right|^2 \mu_t(dy) \cdot \\ &\quad \cdot \int_H |\Phi_m(e^{tA} x + y)|^2 \mu_t(dy) \\ &\leq \frac{c}{t^4} |k|_K^2 \sum_{m \geq 1} \int_H |\Phi_m(e^{tA} x + y)|^2 \mu_t(dy) = \frac{c}{t^4} |k|_K^2 \int_H |\Phi(e^{tA} x + y)|^2 \mu_t(dy). \end{aligned} \quad (3.36)$$

Now we fix also $x \in H$. For any $l \in J$ we define $\Phi_l : H \rightarrow \mathbb{R}$,

$$\Phi_l(y) = \langle \Phi(y), l \rangle_J, \quad l \in J.$$

We have $\Phi_m = \Phi_{f_m}$.

We can consider the linear operator $T_{x,k} : C_b(H, J) \rightarrow L_2(J, U)$,

$$T_{x,k}(\Phi)(l) = \nabla_k \nabla_a^G P_t[\Phi_l](x) \in U, \quad \Phi \in C_b(H, J), \quad l \in J$$

(we identify U with $L(U, \mathbb{R})$). We have

$$\begin{aligned} &|T_{x,k}(\Phi)(l)|_U^2 \\ &= \sup_{|a|_U=1} \left| \int_H \left(\langle \Gamma_t k, Q_t^{-\frac{1}{2}} y \rangle \langle \Gamma_t G a, Q_t^{-\frac{1}{2}} y \rangle - \langle \Gamma_t k, \Gamma_t G a \rangle \right) \langle \Phi(e^{tA} x + y), l \rangle_J \mu_t(dy) \right|^2. \end{aligned}$$

and

$$\|T_{x,k}(\Phi)\|_{L_2(J,U)}^2 = \sum_{m \geq 1} |T_{x,k}(\Phi)(f_m)|_U^2 = \sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_k \nabla_a^G P_t[\Phi_m](x)|^2.$$

By the bound in (3.36) we deduce that $\|T_{x,k}(\Phi)\|_{L_2(J,U)}^2 \leq \frac{c}{t^4} |k|_K^2 \|\Phi\|_\infty^2$, where c is independent of $t > 0$, $x \in H$, Φ and $k \in K$. The linear operator: $T_{x,k} : C_b(H, J) \rightarrow L_2(J, U)$ is well defined and continuous; we have

$$\|T_{x,k}\|_{L(C_b(H,J), L_2(J,U))} \leq \frac{c}{t^2} |k|_K. \quad (3.37)$$

Now if $\Phi \in C_K^1(H, J)$ we find for $k \in K$, $a \in U$ (cf. (3.31) and (3.20))

$$\begin{aligned} & \sup_{a \in U, |a|_U=1} |\nabla_k \nabla_a^G P_t[\Phi_m](x)|^2 \\ &= \sup_{a \in U, |a|_U=1} \left| \int_H \langle \Gamma_t G a, Q_t^{-\frac{1}{2}} y \rangle \langle \nabla^K \Phi_m(e^{tA} x + y), e^{tA} k \rangle_K \mu_t(dy) \right|^2 \\ &\leq \sup_{|a|_U=1} \int_H |\langle \Gamma_t G a, Q_t^{-\frac{1}{2}} y \rangle|^2 \mu_t(dy) \cdot \int_H |\langle \nabla^K \Phi_m(e^{tA} x + y), e^{tA} k \rangle_K|^2 \mu_t(dy) \\ &\leq \frac{c}{t} \cdot \int_H |\langle \nabla^K \Phi_m(e^{tA} x + y), e^{tA} k \rangle_K|^2 \mu_t(dy) \end{aligned}$$

and so

$$\begin{aligned} \sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_k \nabla_a^G P_t[\Phi_m](x)|^2 &\leq \frac{c}{t} \sum_{m \geq 1} \int_H |\langle \nabla^K \Phi_m(e^{tA} x + y), e^{tA} k \rangle_K|^2 \mu_t(dy) \\ &= \frac{c}{t} \int_H |\nabla^K \Phi(e^{tA} x + y)[e^{tA} k]|_J^2 \mu_t(dy). \end{aligned}$$

Since $|\nabla^K \Phi(e^{tA} x + y)[e^{tA} k]|_J \leq \|\nabla^K \Phi(e^{tA} x + y)\|_{L(K,J)} |k|_K$, $t \geq 0$, it follows that

$$\sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_k \nabla_a^G P_t[\Phi_m](x)|^2 \leq \frac{c}{t} |k|_K^2 \|\Phi\|_{C_K^1(H,J)}^2. \quad (3.38)$$

Hence

$$\|T_{x,k}\|_{L(C_K^1(H,J), L_2(J,U))} \leq \frac{c}{t^{1/2}} |k|_K. \quad (3.39)$$

Interpolating between (3.37) and (3.39) as in the proof of Lemma 3.8, for any $\Phi \in (C_b(H, J), C_K^1(H, J))_{\alpha, \infty}$, we get

$$\left(\sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_k \nabla_a^G P_t[\Phi_m](x)|^2 \right)^{1/2} \leq \frac{c}{t^{\frac{4-3\alpha}{2}}} |k|_K \|\Phi\|_{(C_b(H,J), C_K^1(H,J))_{\alpha, \infty}} \quad (3.40)$$

Now we consider any $\Phi \in C_b^\alpha(H, J) \subset (C_b(H, J), C_K^1(H, J))_{\alpha, \infty}$ (see Lemma 3.4). We finally obtain, for any $t \in (0, T]$,

$$\left(\sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_k \nabla_a^G P_t[\Phi_m](x)|^2 \right)^{1/2} \leq \frac{C}{t^{\frac{4-3\alpha}{2}}} |k|_K \|\Phi\|_{C_b^\alpha(H, J)}.$$

□

Remark 3.10 We provide here an equivalent formulation of Lemma 3.9 when $J = K$ (this will be used in the proof of Theorem 4.3). Let $\Phi \in C_b^\alpha(H, K)$, $\alpha \in (0, 1)$. Let us fix $t \in (0, T]$, $x \in H$ and $k \in K$.

Recall the notation $\Phi_l = \langle \Phi, l \rangle_K$, $l \in K$. We know that the linear operator $\nabla_k \nabla^G R_t[\Phi](x)$ is well defined from K into U by the formula

$$l \mapsto \nabla_k \nabla^G P_t[\Phi_l](x)$$

(cf. Lemma 3.7; note that the operator $a \mapsto \nabla_k \nabla_a^G P_t[\Phi_l](x)$ belongs to $L(U, \mathbb{R})$ and so it can be identified with an element of U). Assertion of Lemma 3.9 is equivalent to say that

$$\begin{aligned} \nabla_k \nabla^G R_t[\Phi](x) &\in L_2(K, U) \text{ and} \\ \|\nabla_k \nabla^G R_t[\Phi](x)\|_{L_2(K, U)} &= \left(\sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_k \nabla_a^G P_t[\langle \Phi, f_m \rangle_K](x)|^2 \right)^{1/2} \\ &\leq \frac{C}{t^{\frac{4-3\alpha}{2}}} |k|_K \|\Phi\|_{C_b^\alpha(H, K)}, \quad t \in (0, T]; \end{aligned} \quad (3.41)$$

here (f_m) is a basis in K . Recall that $C > 0$ is independent of x, k, t and Φ . □

We will also use the following additional regularity result.

Lemma 3.11 *Let $T > 0$. Under the assumptions of Lemma 3.6 let $\Phi \in C_b^\alpha(H, J)$, $\alpha \in (0, 1)$. We have the following estimate, for $t \in (0, T]$, $x, y \in H$*

$$\|\nabla^G R_t[\Phi](x) - \nabla^G R_t[\Phi](x')\|_{L(U, J)} \leq \frac{C}{t^{\frac{1}{2}}} |x - x'|^\alpha \|\Phi\|_\alpha. \quad (3.42)$$

Proof The assertion follows easily by the formula

$$\begin{aligned} &\nabla_a^G R_t[\Phi](x) - \nabla_a^G R_t[\Phi](x') \\ &= \int_H \langle Q_t^{-\frac{1}{2}} e^{tA} G a, Q_t^{-\frac{1}{2}} y \rangle [\Phi(e^{tA} x + y) - \Phi(e^{tA} x' + y)] \mathcal{N}(0, Q_t)(dy), \end{aligned}$$

$a \in U$, using that $|\Phi(e^{tA} x + y) - \Phi(e^{tA} x' + y)|_J \leq \|\Phi\|_\alpha |x - x'|^\alpha$, $t \geq 0$, $y \in H$. □

4 Well-Posedness of Singular Wave Equations

In this section we apply our regularity results to prove strong well-posedness of a semilinear abstract stochastic wave equation. Let us consider the following SPDE

$$\begin{cases} \frac{\partial^2 y}{\partial \tau^2}(\tau) = -\Delta y(\tau) + b(\tau, y(\tau)) + \dot{W}(\tau), \\ y(0) = x_0 \in U, \quad \frac{\partial y}{\partial \tau}(0) = x_1 \in V', \quad \tau \in (0, T]. \end{cases} \quad (4.1)$$

This equation is a semilinear extension of (1.1), with a drift $b : [0, T] \times U \rightarrow U$ which can be a bounded measurable function, Hölder continuous of exponent $\alpha \in (2/3, 1)$ with respect to the y -variable, uniformly in $t \in [0, T]$.

An example of Eq. (4.1) is obtained by adding the term $b(\tau, \xi, y(\tau, \xi))$ to the stochastic wave equation (1.2), i.e.,

$$\begin{cases} \frac{\partial^2 y}{\partial \tau^2}(\tau, \xi) = \frac{\partial^2}{\partial \xi^2} y(\tau, \xi) + b(\tau, \xi, y(\tau, \xi)) + \dot{W}(\tau, \xi), \quad \xi \in (0, 1), \\ y(\tau, 0) = y(\tau, 1) = 0, \\ y(0, \xi) = x_0(\xi), \quad \frac{\partial y}{\partial \tau}(0, \xi) = x_1(\xi), \quad \tau \in (0, T], \quad \xi \in [0, 1]. \end{cases} \quad (4.2)$$

Here $b : [0, T] \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable and, for $\tau \in [0, T]$, a.e. $\xi \in [0, 1]$, the map $b(\tau, \xi, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Moreover, there exists $c_1 \in L^\infty([0, 1])$, $\alpha \in (2/3, 1)$, such that, for $\tau \in [0, T]$ and a.e. $\xi \in [0, 1]$,

$$|b(\tau, \xi, x) - b(\tau, \xi, y)| \leq c_1(\xi) |x - y|^\alpha,$$

$x, y \in \mathbb{R}$. Further, we require that $|b(\tau, \xi, x)| \leq c_2(\xi)$, for $\tau \in [0, T]$, $x \in \mathbb{R}$ and a.e. $\xi \in [0, 1]$, with $c_2 \in L^2([0, 1])$. This example is discussed in Section 3.1 of [24]. In [24] it is also shown that 2-dimensional semilinear stochastic plate equations are of the form (4.1).

Equation (4.1) can be reformulated as an abstract evolution equation in H :

$$dX_\tau^{0,x} = AX_\tau^{0,x}d\tau + GB(\tau, X_\tau^{0,x})d\tau + GdW_\tau, \quad \tau \in [0, T], \quad X_0^{0,x} = x \in H \quad (4.3)$$

where A is the generator of the wave group in H and $GdW_\tau = \begin{pmatrix} 0 \\ dW_\tau \end{pmatrix}$. Concerning the semilinear stochastic Eq. (4.3), we assume that

Hypothesis 2 $B : [0, T] \times H \rightarrow U$ is (Borel) measurable and bounded and moreover there exists $C > 0$ such that

$$|B(t, x + h) - B(t, x)|_U \leq C|h|_H^\alpha, \quad x, h \in H, \quad t \in [0, T], \quad (4.4)$$

for some $\alpha \in (2/3, 1)$. We also write that $B \in B_b([0, T]; C_b^\alpha(H, U))$.

Remark 4.1 We point out that the critical exponent $\alpha = 2/3$ agrees with the one which appears in the study of pathwise uniqueness for degenerate finite dimensional SDEs like (4.3) (with $H = \mathbb{R}^{2d}$); see [6] more details.

Let $x \in H$. Recall that a (weak) mild solution to (4.3) is a tuple

$$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, X)$$

where $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a stochastic basis on which it is defined a cylindrical U -valued $\mathcal{F}_t$ -Wiener process W and a continuous $\mathcal{F}_t$ -adapted H -valued process $X = (X_t)_{t \in [0, T]}$ such that, $\mathbb{P}$ -a.s.,

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}GB(s, X_s)ds + \int_0^t e^{(t-s)A}GdW_s, \quad t \in [0, T]. \quad (4.5)$$

According to Chapter 1 in [26] we say that strong existence holds for equation (4.3) if, for every stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ on which there is defined an U -valued cylindrical $\mathcal{F}_t$ -Wiener process W , for any initial condition $x \in H$, there exists an H -valued continuous $(\mathcal{F}_t)$ -adapted process $X = (X_t)_{t \in [0, T]}$ such that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, X)$ is a weak mild solution. We also write $X_t^{0,x}$ or X_t^x instead of X_t . Similarly, we denote by $(X_\tau^{t,x})_{\tau \geq t}$ the solution to (4.3) starting from $x \in H$ at time $t \in [0, T]$.

We say that for Eq. (4.3) strong or pathwise uniqueness holds (starting from any initial $x \in H$) if given two weak mild solutions $X = (X_t)$ and $Y = (Y_t)$ (defined on the same stochastic basis, solutions with respect to the same cylindrical Wiener process) starting at the same x , we have, $\mathbb{P}$ -a.s., $X_t = Y_t$, $t \in [0, T]$.

Note that if $a \in U$, $Ga = \begin{pmatrix} 0 \\ a \end{pmatrix} \in K$ and $e^{tA} \begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\Lambda}} \sin(\sqrt{\Lambda} t) a \\ \cos(\sqrt{\Lambda} t) a \end{pmatrix} \in K$, $t \in \mathbb{R}$. Hence, since in Hypotheses 2 we assume that the drift B takes its values in U , then $\int_0^t e^{(t-s)A}GB(s, X_s)ds$ evolves in K : it is K -valued and the map $t \mapsto \int_0^t e^{(t-s)A}GB(s, X_s)ds$ is continuous due to the boundedness of B (indeed, let $T > 0$; for any ω , $\mathbb{P}$ -a.s., $s \mapsto GB(s, X_s(\omega))$ is Borel and bounded from $[0, T]$ with values in K and so we can apply Lemma 3.1.5 in [7]).

Therefore even if in general a solution (X_t^x) does not evolve in K (cf. (2.8)) we know that for any initial condition $x_1, x_2 \in H$ such that $x_1 - x_2 \in K$ then any couple of weak mild solutions $X_t^{x_1}$ and $X_t^{x_2}$ (starting at x_1 and x_2 , respectively, and driven by the same noise) verifies the property

$$(X_t^{x_1} - X_t^{x_2})_{t \in [0, T]} \text{ evolves in } K, \quad \text{if } x_1 - x_2 \in K. \quad (4.6)$$

Indeed

$$X_t^{x_1} - X_t^{x_2} = e^{tA}(x_1 - x_2) + \int_0^t e^{(t-s)A}G(B(s, X_s^{x_1}) - B(s, X_s^{x_2}))ds, \quad t \in [0, T];$$

the stochastic integral has disappeared, and since $x_1 - x_2 \in K$, also $e^{tA}(x_1 - x_2) \in K$; the other term we have already discussed that belongs to K . Note that, $\mathbb{P}$ -a.s., the paths of $(X_t^{x_1} - X_t^{x_2})$ are continuous functions from $[0, T]$ with values in K . Property (4.6)

will be important in the proof of our uniqueness result (see Theorem 4.7). Indeed recall that the Ornstein–Uhlenbeck semigroup regularizes only in the directions of K (see Sect. 3).

Note that thanks to the boundedness of B we can apply the Girsanov Theorem, see [24, Remark 2.1] changing H in K , or see [25, Remark 2.1] for more details.

4.1 The Related Infinite Dimensional PDE

In this section we will apply the regularity results proved in Sect. 3 with $J = K = V \times U$. Let $T > 0$. We consider the following integral equation of Kolmogorov type which will be important in the sequel:

$$\begin{aligned} u(t, x) = & \int_t^T R_{s-t} \left[e^{-(s-t)A} G B(s, \cdot) \right] (x) ds \\ & + \int_t^T R_{s-t} \left[e^{-(s-t)A} \nabla^G u(s, \cdot) B(s, \cdot) \right] (x) ds, \end{aligned} \quad (4.7)$$

where $u(t, x)$ takes values in K and $\nabla^G u(s, x) B(s, x) = \nabla_{GB(s, x)} u(s, x) \in K$, $(s, x) \in [0, T] \times H$.

Using results of Sect. 3 we will solve the equation in the Banach space E_0 consisting of all $u \in B_b([0, T] \times H, K)$ such that $u(t, \cdot)$ is K -differentiable on H , with

$\nabla^K u \in B_b([0, T] \times H, L(K, K))$. We also require that there exists $C = C_T > 0$ such that for any $x, y \in H$

$$\sup_{s \in [0, T]} |\nabla^G u(s, x) - \nabla^G u(s, y)|_{L(U, K)} \leq C |x - y|_H^\alpha, \quad (4.8)$$

where $\alpha \in (2/3, 1)$ is given in Hypothesis 2. Finally, to define E_0 we require that, for each $\xi \in U$, $t \in [0, T]$, the mapping:

$$x \mapsto \nabla_\xi^G u(t, x) \text{ is } K\text{-differentiable on } H \quad (4.9)$$

$$\text{with } \sup_{(t, x) \in [0, T] \times H} \sup_{|\xi|_U = 1} \|\nabla^K \nabla_\xi^G u(t, x)\|_{L(K, K)} < \infty.$$

Let $\beta \geq 0$ to be fixed later. It is not difficult to prove that E_0 is a Banach space endowed with the norm

$$\begin{aligned} \|u\|_{E_0, \beta} = & \sup_{(t, x) \in [0, T] \times H} e^{\beta t} |u(t, x)|_K + \sup_{(t, x) \in [0, T] \times H} e^{\beta t} \|\nabla^K u(t, x)\|_{L(K, K)} \\ & + \sup_{t \in [0, T]} e^{\beta t} \|\nabla^G u(t, \cdot)\|_{C_b^g(H, L(U, K))} \\ & + \sup_{(t, x) \in [0, T] \times H} \sup_{|\xi|_U = 1} e^{\beta t} \|\nabla^K \nabla_\xi^G u(t, x)\|_{L(K, K)}. \end{aligned} \quad (4.10)$$

Lemma 4.2 *Let Hypotheses 1 and 2 hold true. There exists a unique solution $u \in E_0$ to (4.7). Moreover, there exists a function $h(r) = h(r, \alpha) > 0$, $r \geq 0$, such that $h(r) \rightarrow 0$ as $r \rightarrow 0^+$ and if $S \in [0, T]$ verifies $h(T - S) \cdot (\sup_{t \in [0, T]} \|B(t, \cdot)\|_\alpha) \leq 1/4$, then*

$$\sup_{t \in [S, T], x \in H} \|\nabla^K u(t, x)\|_{L(K, K)} \leq 1/3. \quad (4.11)$$

Proof We introduce the following operator $\mathcal{T}$ defined on E_0 :

$$\begin{aligned} \mathcal{T}u(t, x) &= \int_t^T R_{s-t} \left[e^{-(s-t)A} GB(s, \cdot) \right] (x) ds \\ &\quad + \int_t^T R_{s-t} \left[e^{-(s-t)A} \nabla^G u(s, \cdot) B(s, \cdot) \right] (x) ds, \end{aligned}$$

$u \in E_0$, $(t, x) \in [0, T] \times H$. In order to apply Lemmas 3.8 and 3.11 with $J = K$ we first check the Hölder regularity of the terms $GB(s, \cdot)$ and $\nabla^G u(s, \cdot) B(s, \cdot)$.

Since, for any $x \in H$, $h \in H$, $s \in [0, T]$,

$$|GB(s, x) - GB(s, x + h)|_K = |B(s, x) - B(s, x + h)|_U \leq C|h|_H^\alpha$$

we get that $GB \in B_b([0, T]; C_b^\alpha(H, K))$, see the definition in Hypothesis (2).

We need to prove that also

$$\nabla^G u(s, \cdot) B(s, \cdot) \text{ belongs to } B_b([0, T]; C_b^\alpha(H, K)). \quad (4.12)$$

We write for $x, y \in H$ with $|x - y|_H \leq 1$

$$[\nabla^G u(s, x) B(s, x) - \nabla^G u(s, y) B(s, x)] + \nabla^G u(s, y) [B(s, x) - B(s, y)].$$

We bound the second term with

$$\begin{aligned} &|\nabla^G u(s, y) [B(s, x) - B(s, y)]|_K \\ &\leq \sup_{(t, x) \in [0, T] \times H} e^{\beta t} \|\nabla^G u(t, x)\|_{L(U, K)} |B(s, x) - B(s, y)|_U \leq C|x - y|_H^\alpha. \end{aligned}$$

Moreover we have

$$\begin{aligned} &|[\nabla^G u(s, x) - \nabla^G u(s, y)] B(s, x)|_K \leq |B(s, x)|_U |\nabla^G u(s, x) - \nabla^G u(s, y)|_{L(K, U)} \\ &\leq \|B\|_\infty \sup_{t \in [0, T]} e^{\beta t} \|\nabla^G u(t, \cdot)\|_{C_b^\alpha(H, L(U, K))} |x - y|_H^\alpha. \end{aligned}$$

By Lemmas 3.8 and 3.11, using that $\alpha > 2/3$, it is not difficult to prove that $\mathcal{T} : E_0 \rightarrow E_0$. Let us check that for a suitable value of β the map $\mathcal{T}$ is a strict contraction

(see (4.10)). We have to consider $\|\mathcal{T}u_1 - \mathcal{T}u_2\|_{E_0, \beta}$, $u_1, u_2 \in E_0$; we only treat the term

$$\sup_{t, x} \sup_{|\xi|_U=1} e^{\beta t} \|\nabla^K \nabla_\xi^G [\mathcal{T}u_1 - \mathcal{T}u_2](t, x)\|_{L(K, K)}.$$

Indeed the other terms of $\|\mathcal{T}u_1 - \mathcal{T}u_2\|_{E_0, \beta}$ can be estimated in a similar way. We have for any $k \in K$ with $|k|_K = 1$

$$\begin{aligned} & e^{\beta t} |\nabla_k \nabla_\xi^G [\mathcal{T}u_1(t, x) - \mathcal{T}u_2(t, x)]|_K \\ & \leq \int_t^T e^{-\beta(s-t)} \left| \nabla_k \nabla_\xi^G R_{s-t} \left[e^{-(s-t)A} e^{\beta s} \{ \nabla^G u_1(s, \cdot) - \nabla^G u_2(s, \cdot) \} B(s, \cdot) \right] (x) \right|_K ds \\ & \leq \int_t^T \frac{c e^{-\beta(s-t)}}{(s-t)^{\frac{4-3\alpha}{2}}} ds \sup_{t \in [0, T]} \|B(t, \cdot)\|_\alpha \|u_1 - u_2\|_{E_0, \beta} \\ & \leq C_{\beta, T} \sup_{t \in [0, T]} \|B(t, \cdot)\|_\alpha \|u_1 - u_2\|_{E_0, \beta}, \end{aligned}$$

where $C_{\beta, T} > 0$ tends to 0 as $\beta \rightarrow +\infty$. Therefore taking the supremum over $k \in K$ with $|k|_K = 1$ we infer

$$e^{\beta t} \|\nabla^K \nabla_\xi^G [\mathcal{T}u_1(t, x) - \mathcal{T}u_2(t, x)]\|_{L(K, K)} \leq C_{\beta, T} \sup_{t \in [0, T]} \|B(t, \cdot)\|_\alpha \|u_1 - u_2\|_{E_0, \beta}.$$

Choosing β large enough, we can apply the fixed point theorem and obtain that there exists a unique solution $u \in E_0$.

In order to prove (4.11), we first introduce $\|u\|_{E_0, 0, S, T}$ which is defined as $\|u\|_{E_0, 0}$ in (4.10) (with $\beta = 0$) but taking all the supremums over $[S, T] \times H$ instead of $[0, T] \times H$. We proceed as before:

$$\begin{aligned} \|u\|_{E_0, 0, S, T} & \leq \sup_{t \in [S, T]} \int_t^T \frac{c}{(s-t)^{\frac{4-3\alpha}{2}}} ds \sup_{t \in [0, T]} \|B(t, \cdot)\|_{\alpha, K} (\|u_0\|_{E_0, 0, S, T} + 1) \\ & \leq h(T - S) \sup_{t \in [0, T]} \|B(t, \cdot)\|_\alpha (\|u_0\|_{E_0, 0, S, T} + 1), \end{aligned}$$

where $h(r) = \int_0^r \frac{c}{s^{\frac{4-3\alpha}{2}}} ds$; now (4.11) follows since we have $\frac{3}{4} \|u\|_{E_0, 0, S, T} \leq 1/4$. $\square$

To prove the next result we will apply Lemma 3.9 (see also Remark 3.10). To this purpose we fix a basis (f_m) of K and set $u_m = \langle u, f_m \rangle_K$, where u is the solution given in Lemma 4.2.

Theorem 4.3 *Let Hypotheses 1 and 2 hold true. Then the unique solution $u \in E_0$ to (4.7) (see Lemma 4.2) verifies in addition*

$$\left(\sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_k \nabla_a^G u_m(t, x)|^2 \right)^{1/2} \leq C |k|_K \sup_{t \in [0, T]} \|B(t, \cdot)\|_{C_b^\alpha(H, U)} \quad k \in K; \quad (4.13)$$

here $C > 0$ is independent of $x \in H$, $k \in K$, $t \in [0, T]$, u and the basis (f_m) in K .

Proof First following the proof of Lemma 4.2 it is not difficult to prove that there exists $C_T > 0$ independent of u such that

$$\sup_{t \in [0, T]} \|\nabla^G u(t, \cdot)\|_{C_b^\alpha(H, L(U, K))} \leq C_T \sup_{t \in [0, T]} \|B(t, \cdot)\|_{C_b^\alpha(H, U)}. \quad (4.14)$$

Then we write $u = v + w$, where

$$\begin{aligned} v(t, x) &= \int_t^T R_{s-t} \left[e^{-(s-t)A} G B(s, \cdot) \right] (x) ds, \\ w(t, x) &= \int_t^T R_{s-t} \left[e^{-(s-t)A} \nabla^G u(s, \cdot) B(s, \cdot) \right] (x) ds. \end{aligned}$$

We need to prove that (4.13) holds when u is replaced by v and w . We concentrate on w (the proof for v is similar). We define, for any $s \in [0, T]$, $x \in H$, $\Phi(s, x) = e^{-(s-t)A} \nabla^G u(s, x) B(s, x)$.

Note that $\Phi \in B_b([0, T]; C_b^\alpha(H, K))$ (see the computations to verify (4.12)). By (4.14) we obtain

$$\begin{aligned} \sup_{s \in [0, T]} \|\Phi(s, \cdot)\|_{C_b^\alpha(H, K)} &\leq C_T \sup_{t \in [0, T]} \|B(t, \cdot)\|_{C_b^\alpha(H, U)} \text{ with} \\ w(t, x) &= \int_t^T R_{s-t} [\Phi(s, \cdot)](x) ds, \quad t \in [0, T], \quad x \in H. \end{aligned}$$

Let us fix $k \in K$ and $t \in [0, T]$; set $w_m = \langle w, f_m \rangle_K$. By Lemma 3.9 (see also Remark 3.10) we know that, for any $s \in (t, T]$, the linear operator $\nabla_k \nabla^G R_{s-t} [\Phi(s, \cdot)](x)$ is well defined from K into U by the formula $l \mapsto \nabla_k \nabla^G P_{s-t} [\langle \Phi(s, \cdot), l \rangle_K](x)$ and it is a Hilbert-Schmidt operator. Moreover, for $s > t$,

$$\begin{aligned} \|\nabla_k \nabla^G R_{s-t} [\Phi(s, \cdot)](x)\|_{L_2(K, U)} &= \left(\sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_k \nabla_a^G P_{s-t} [\langle \Phi(s, \cdot), f_m \rangle_K](x)|^2 \right)^{1/2} \\ &\leq \frac{C}{(s-t)^{\frac{4-3\alpha}{2}}} |k|_K \sup_{r \in [0, T]} \|\Phi(r, \cdot)\|_{C_b^\alpha(H, K)} \leq \frac{c|k|_K}{(s-t)^{\frac{4-3\alpha}{2}}} \sup_{t \in [0, T]} \|B(t, \cdot)\|_{C_b^\alpha(H, U)}. \end{aligned} \quad (4.15)$$

Note that also the linear operator $\nabla_k \nabla^G w(t, x)$ is well defined from K into U ($l \mapsto \nabla_k \nabla^G [\langle w(s, \cdot), l \rangle_K](x)$). Further, we have, for any $x \in H$,

$$\nabla_k \nabla^G w(t, x) = \int_t^T \nabla_k \nabla^G R_{s-t} [\Phi(s, \cdot)](x) ds.$$

Using (4.15) we deduce that $\nabla_k \nabla^G w(t, x) \in L_2(K, U)$ and

$$\|\nabla_k \nabla^G w(t, x)\|_{L_2(K, U)}$$

$$\leq \int_t^T \|\nabla_k \nabla^G R_{s-t}[\Phi(s, \cdot)](x)\|_{L_2(K, U)} ds \leq C_{\alpha, T} |k|_K \sup_{t \in [0, T]} \|B(t, \cdot)\|_{C_b^\alpha(H, U)},$$

where $C_{\alpha, T} > 0$ is independent of $x \in H$, $t \in [0, T]$, B and $k \in K$. $\square$

Remark 4.4 If we try to prove the previous theorem using Lemma 3.2 instead of Lemma 3.4 we could eventually obtain a bound for $\sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_k \nabla_a^G u_m(t, x)|^2$ by requiring that

$$\sum_{j \geq 1} \sup_{t \in [0, T]} \|\langle B(t, \cdot), e_j \rangle_U\|_{C_K^\alpha(H)}^2 < \infty.$$

However this condition is restrictive when it is applied to an example like (4.2) (see also Section 3 in [24]).

4.2 The Related Infinite Dimensional Forward-Backward System

We first collect some results on a related forward-backward system (FBSDE), whose solution will be useful “in removing the bad term B ” in (4.3) (cf. Proposition 4.6). Comparing with [8] this procedure turns out to be an alternative to the use of the Itô’s formula (for recent advances on infinite dimensional Itô formula we refer to [16]). Here, following [24], we use it in the present case when A is the generator of a *group* of operators. In [1] this approach has been extended to the case when A is the generator of a *semigroup* of operators.

In a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let us consider the following FBSDE

$$\begin{cases} d\Xi_\tau^{t,x} = A\Xi_\tau^{t,x} d\tau + GdW_\tau, \tau \in [t, T], \\ \Xi_t^{t,x} = x, \\ -dY_\tau^{t,x} = -AY_\tau^{t,x} + GB(\tau, \Xi_\tau^{t,x}) d\tau + Z_\tau^{t,x} B(\tau, \Xi_\tau^{t,x}) d\tau - Z_\tau^{t,x} dW_\tau, \tau \in [0, T], \\ Y_T^{t,x} = 0, \end{cases} \quad (4.16)$$

$t \in [0, T]$, $x \in H$, and the forward equation is the abstract formulation of the wave equation (1.1) given in (2.6) under Hypothesis 1; here $B : [0, T] \times H \rightarrow U$ satisfies Hypothesis 2, G is defined by (2.4) and W is a cylindrical Wiener process in U .

We extend $\Xi^{t,x}$ to the whole $[0, T]$ by setting $\Xi_\tau^{t,x} = x$ for $0 \leq \tau \leq t$, so $(Y^{t,x}, Z^{t,x})$ are well defined on $[0, T]$. The precise meaning of the BSDE in (4.16) is given by its mild formulation: for $\tau \in [0, T]$

$$\begin{aligned} Y_\tau^{t,x} &= \int_\tau^T e^{-(s-\tau)A} GB(s, \Xi_s^{t,x}) ds \\ &\quad + \int_\tau^T e^{-(s-\tau)A} Z_s^{t,x} B(s, \Xi_s^{t,x}) ds - \int_\tau^T e^{-(s-\tau)A} Z_s^{t,x} dW_s, \end{aligned} \quad (4.17)$$

$\mathbb{P}$ -a.s. (cf. [21], [20] and the references therein).

Notice that the forward equation evolves in H as we have already discussed in Sect. 2, while the backward equation takes values in K . Indeed the term $GB(\cdot, \cdot)$ takes

values in K , and we can use an explicit representation of the solution in terms of $GB(\cdot)$, see [21], Sects. 2 and 3.

We denote by $L^2_{\mathcal{P}}(\Omega, C([0, T], K))$ the space of all predictable (with respect to the completed natural filtration generated by W) K -valued processes Y with continuous paths and such that

$$\mathbb{E}[\sup_{\tau \in [0, T]} |Y_\tau|^2] = \|Y\|_{L^2_{\mathcal{P}}(\Omega, C([0, T], K))}^2 < \infty.$$

We refer to [18], Section 2.2, for the definition of the spaces of Gâteaux differentiable functions $\mathcal{G}^1(H, U)$ and $\mathcal{G}^{0,1}([0, T] \times H, K) \subset C([0, T] \times H, K)$ (such spaces are also considered in [24] and [25]).

Following [21] and Remark 4.5 and estimate (4.19) in [19] (see also Section 5 of [24] where these results are summed up) it is immediate to get existence and pathwise uniqueness of a solution $(Y^{t,x}, Z^{t,x}) \in L^2_{\mathcal{P}}(\Omega, C([0, T], K)) \times L^2_{\mathcal{P}}(\Omega \times [0, T], L_2(U, K))$, such that

$$\mathbb{E}\left[\sup_{\tau \in [0, T]} |Y_\tau^{t,x}|_K^2\right] + \mathbb{E} \int_0^T \|Z_\tau^{t,x}\|_{L_2(U, K)}^2 d\tau \leq C_T \|B\|_\infty. \quad (4.18)$$

Moreover if we further assume that the map $x \mapsto B(\tau, x)$, $H \rightarrow U$, belongs to $\mathcal{G}^1(H, U)$ then, the map: $(t, x) \mapsto Y_t^{t,x}$ belongs to $\mathcal{G}^{0,1}([0, T] \times H, K)$. In addition, $(t, x) \mapsto Y_t^{t,x}$, $[0, T] \times H \rightarrow K$, is deterministic.

So we can define the function

$$v(t, x) = Y_t^{t,x}, \quad (t, x) \in [0, T] \times H. \quad (4.19)$$

and by the Markov property we have:

$$v(\tau, \Xi_\tau^{t,x}) = Y_\tau^{t,x}, \quad \tau \in [t, T], \quad \mathbb{P} - a.s.. \quad (4.20)$$

Theorem 4.5 Assume Hypothesis 1 and let B satisfy Hypothesis 2. Let v be the function defined in (4.19). Then $v \in B_b([0, T] \times H, H)$ and, for any $t \in [0, T]$, $v(t, \cdot) : H \rightarrow K$ is G -differentiable on H (see the definition after (3.4)), for any $(t, x) \in [0, T] \times H$, the map: $\xi \mapsto \nabla_{G\xi} v(t, x) = \nabla_\xi^G v(t, x) \in L(U, K)$ and, for any $\xi \in U$, $\nabla_\xi^G v \in B_b([0, T] \times H, K)$ with $\sup_{(t,x) \in [0, T] \times H} \|\nabla^G v(t, x)\|_{L(U, K)} < \infty$. For any $\tau \in [0, T]$, a.e., we have

$$\nabla^G v(\tau, \Xi_\tau^{t,x}) = Z_\tau^{t,x}, \quad \mathbb{P} - a.s.. \quad (4.21)$$

Finally, by the representation formula given in (4.17) and using the OU semigroup (R_t) , the function v defined in (4.19) satisfies the integral equation

$$v(t, x) = \int_t^T R_{s-t} \left[e^{-(s-t)A} GB(s, \cdot) \right] (x) ds$$

$$+ \int_t^T R_{s-t} \left[e^{-(s-t)A} \nabla^G v(s, \cdot) B(s, \cdot) \right] (x) ds. \quad (4.22)$$

and coincides with the function u , unique solution of (4.7) given in Lemma 4.2 and Theorem 4.3.

Proof For the proof we refer to [24, Lemma 5.3 and Theorem 5.4] for the identification of Z , and to [24, Lemma 5.5] for the identification of v with u . $\square$

4.3 Applications to Strong Uniqueness

We study Eq. (4.3). We show how to remove the “bad” term B and how to apply our regularity results to get pathwise uniqueness. Let $x \in H$. Recall that we consider a (weak) mild solution $(X_\tau^{t,x}) = (X_\tau^{t,x})_{\tau \in [0, T]}$:

$$X_\tau^{t,x} = e^{(\tau-t)A} x + \int_t^\tau e^{(\tau-s)A} G B(s, X_s) ds + \int_t^\tau e^{(\tau-s)A} G dW_s, \tau \in [t, T], \quad (4.23)$$

$X_\tau^{t,x} = x$, $\tau \leq t$; cf. (4.5). This is a continuous H -valued process defined and adapted on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, on which it is defined a cylindrical U -valued $\mathcal{F}_t$ -Wiener process W . The next result shows how to remove the “bad” term B using the function v defined in (4.19), and without applying the Itô formula, but using the BSDE (4.17) in its mild formulation (we refer to [24, Section 6]).

Proposition 4.6 *Let Hypotheses 1 and 2 hold true. Then a (weak) mild solution $X^x = (X_\tau^x)$ of (4.23) starting at $t = 0$ satisfies, for any $\tau \in [0, T]$, $\mathbb{P}$ -a.s.,*

$$\begin{aligned} X_\tau^x &= e^{\tau A} x + e^{\tau A} v(0, x) - v(\tau, X_\tau^x) + \int_0^\tau e^{(\tau-s)A} \tilde{Z}_s^x dW_s + \int_0^\tau e^{(\tau-s)A} G dW_s \\ &= e^{\tau A} x + e^{\tau A} v(0, x) - v(\tau, X_\tau^x) + \int_0^\tau e^{(\tau-s)A} \nabla^G v(s, X_s^x) dW_s \\ &\quad + \int_0^\tau e^{(\tau-s)A} G dW_s. \end{aligned} \quad (4.24)$$

Proof The proof is given in [24, Proposition 6.1] and it is based on BSDEs, see also [24, Remark 6.2] where the novelty of the approach is discussed and [1, Proposition 4.1]. $\square$

We now prove strong (or pathwise) uniqueness of solutions.

Theorem 4.7 *Let Hypotheses 1 and 2 hold true. Then for equation (4.3) pathwise uniqueness holds (starting from any initial condition $x \in H$).*

Moreover, there exists $c_T > 0$ such that if $X_\tau^{x_1}$ and $X_\tau^{x_2}$ are two (weak) mild solutions starting from x_1 and $x_2 \in H$ at $t = 0$ (defined on the same stochastic basis) such that

$x_1 - x_2 \in K$ then $\mathbb{P}$ -a.s. $X_\tau^{x_1} - X_\tau^{x_2} \in K$ for any $\tau \in [0, T]$ and

$$\sup_{\tau \in [0, T]} \mathbb{E} |X_\tau^{x_1} - X_\tau^{x_2}|_K^2 \leq c_T |x_1 - x_2|_K^2. \quad (4.25)$$

Proof We prove (4.25) which implies the pathwise uniqueness starting from any $x \in H$. Indeed if $x_1 = x_2 = x$ then (4.25) implies that $\mathbb{P}$ -a.s., $X_\tau^{x_1} = X_\tau^{x_2}$, $\tau \in [0, T]$.

Let us fix $x_1, x_2 \in H$ with $x_1 - x_2 \in K$ and consider two (weak) mild solutions X^1 and X^2 defined on the same stochastic basis, with respect to the same cylindrical Wiener process and starting respectively from x_1 and x_2 at time $t = 0$. Notice that

$$X_\tau^1 - X_\tau^2 = e^{\tau A}(x_1 - x_2) + \int_0^\tau e^{(\tau-s)A} (GB(s, X_s^1) - GB(s, X_s^2)) ds,$$

and both $e^{\tau A}(x_1 - x_2)$ and the integral take their values in K . Indeed $x_1 - x_2 \in K$, $GB(s, \cdot) \in K$ and $e^{\tau A} : K \rightarrow K$ (cf. (4.6)).

Let $T_0 \in (0, T]$ be such that $h(T_0) \cdot (\sup_{t \in [0, T]} \|B(t, \cdot)\|_\alpha) \leq 1/4$ (see (4.11)).

We consider the FBSDE (4.16) with $T = T_0$ and we denote its solution again $(\tilde{Y}^x, \tilde{Z}^x)$. We find the function $v^{(0)} : [0, T_0] \times H \rightarrow K$ according to (4.19) with $T = T_0$. By (4.24) we know that

$$\begin{aligned} X_\tau^1 - X_\tau^2 &= e^{\tau A}(x_1 - x_2) + e^{\tau A}[v^{(0)}(0, x_1) - v^{(0)}(0, x_2)] \\ &\quad - [v^{(0)}(\tau, X_\tau^1) - v^{(0)}(\tau, X_\tau^2)] + \int_0^\tau e^{(\tau-s)A} [\tilde{Z}_s^{x_1} - \tilde{Z}_s^{x_2}] dW_s, \quad \tau \in [0, T_0], \end{aligned} \quad (4.26)$$

where $\int_0^\tau e^{(\tau-s)A} [\tilde{Z}_s^{x_1} - \tilde{Z}_s^{x_2}] dW_s = \int_0^\tau e^{(\tau-s)A} [\nabla^G v^{(0)}(s, X_s^1) - \nabla^G v^{(0)}(s, X_s^2)] dW_s$.

By the regularity properties of $v^{(0)}$, see Theorem 4.3, Lemma 4.2 and Theorem 4.5, we get

$$\begin{aligned} &|e^{\tau A}(x_1 - x_2)|_K + |e^{\tau A}[v^{(0)}(0, x_1) - v^{(0)}(0, x_2)]|_K + |v^{(0)}(\tau, X_\tau^1) - v^{(0)}(\tau, X_\tau^2)|_K \\ &\leq C|x_1 - x_2|_K + \frac{1}{3}|X_\tau^1 - X_\tau^2|_K, \quad \tau \in [0, T_0]. \end{aligned}$$

Concerning the stochastic integral, we have

(see [12] page 57 or [14], Section 4.3)

$$\begin{aligned} &\mathbb{E} \left| \int_0^\tau e^{(\tau-s)A} [\nabla^G v^{(0)}(s, X_s^1) - \nabla^G v^{(0)}(s, X_s^2)] dW_s \right|_K^2 \\ &\leq \mathbb{E} \int_0^\tau \|\nabla^G v^{(0)}(s, X_s^1) - \nabla^G v^{(0)}(s, X_s^2)\|_{L_2(U, K)}^2 ds. \end{aligned} \quad (4.27)$$

It the sequel we will prove that $\mathbb{E} \int_0^\tau \|\nabla^G v^{(0)}(s, X_s^1) - \nabla^G v^{(0)}(s, X_s^2)\|_{L_2(U, K)}^2 ds$ is finite and we will provide a bound for it.

Let us consider a basis (e_k) in U ; by the regularity properties of $v^{(0)}$ we get

$$\begin{aligned} & \mathbb{E} \int_0^\tau \|\nabla^G v^{(0)}(s, X_s^1) - \nabla^G v^{(0)}(s, X_s^2)\|_{L_2(U, K)}^2 ds \\ &= \sum_{j \geq 1} \mathbb{E} \int_0^\tau |\nabla_{e_j}^G v^{(0)}(s, X_s^1) - \nabla_{e_j}^G v^{(0)}(s, X_s^2)|_K^2 ds. \end{aligned}$$

Now, using a basis (f_m) in K , we write, for any $s \in [0, \tau]$, $\mathbb{P}$ -a.s.,

$$\begin{aligned} & \sum_{j \geq 1} |\nabla_{e_j}^G v^{(0)}(s, X_s^1) - \nabla_{e_j}^G v^{(0)}(s, X_s^2)|_K^2 \\ &= \sum_{m \geq 1} \sum_{j \geq 1} \langle \nabla_{e_j}^G v^{(0)}(s, X_s^1) - \nabla_{e_j}^G v^{(0)}(s, X_s^2), f_m \rangle_K^2 \\ &= \sum_{m \geq 1} \sum_{j \geq 1} [\nabla_{e_j}^G v_m^{(0)}(s, X_s^1) - \nabla_{e_j}^G v_m^{(0)}(s, X_s^2)]^2 \\ &= \sum_{m \geq 1} |\nabla^G v_m^{(0)}(s, X_s^1) - \nabla^G v_m^{(0)}(s, X_s^2)|_U^2, \end{aligned}$$

using $v_m^{(0)} = \langle v^{(0)}, f_m \rangle_K$ and noting that $\nabla^G v_m^{(0)}(s, X_s^1) \in L(U, \mathbb{R})$ can be identified by the Riesz theorem with a unique element in U . We have obtained

$$\begin{aligned} & \sum_{j \geq 1} \int_0^\tau |\nabla_{e_j}^G v^{(0)}(s, X_s^1) - \nabla_{e_j}^G v^{(0)}(s, X_s^2)|_K^2 ds \\ &= \int_0^\tau \sum_{m \geq 1} |\nabla^G v_m^{(0)}(s, X_s^1) - \nabla^G v_m^{(0)}(s, X_s^2)|_U^2 ds \\ &= \int_0^\tau \sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_a^G v_m^{(0)}(s, X_s^1) - \nabla_a^G v_m^{(0)}(s, X_s^2)|^2 ds. \end{aligned}$$

Hence we have, $\mathbb{P}$ -a.s.,

$$\begin{aligned} & \sum_{j \geq 1} \int_0^\tau |\nabla_{e_j}^G v^{(0)}(s, X_s^1) - \nabla_{e_j}^G v^{(0)}(s, X_s^2)|_K^2 ds \\ &= \sum_{m \geq 1} \int_0^\tau \sup_{|a|_U=1} \left| \int_0^1 \nabla^K \nabla_a^G v_m^{(0)}(s, X_s^1 + r(X_s^2 - X_s^1)) [X_s^2 - X_s^1] dr \right|^2 ds \\ &= \int_0^\tau \sum_{m \geq 1} \sup_{|a|_U=1} \left| \int_0^1 \nabla_{[X_s^2 - X_s^1]} \nabla_a^G v_m^{(0)}(s, X_s^1 + r(X_s^2 - X_s^1)) dr \right|^2 ds. \end{aligned}$$

Moreover, we find, $\mathbb{P}$ -a.s., $r \in [0, 1]$, $s \in [0, \tau]$,

$$\begin{aligned} & \sum_{m \geq 1} \sup_{|a|_U=1} \left| \int_0^1 \nabla_{[X_s^2 - X_s^1]} \nabla_a^G v_m^{(0)}(s, X_s^1 + r(X_s^2 - X_s^1)) dr \right|^2 \\ & \leq \int_0^1 \sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_{[X_s^2 - X_s^1]} \nabla_a^G v_m^{(0)}(s, X_s^1 + r(X_s^2 - X_s^1))|^2 dr \\ & \leq C_T |X_s^2 - X_s^1|_K^2 \sup_{t \in [0, T]} \|B(t, \cdot)\|_{C_b^\alpha(H, U)}^2. \end{aligned} \quad (4.28)$$

In the last inequality we have used Theorem 4.3 with $u = v^{(0)}$ and $k = X_s^2 - X_s^1$. Coming back to (4.27) we obtain

$$\begin{aligned} & \mathbb{E} \left| \int_0^\tau e^{(\tau-s)A} [\nabla^G v^{(0)}(s, X_s^1) - \nabla^G v^{(0)}(s, X_s^2)] dW_s \right|_K^2 \\ & \leq \sum_{j \geq 1} \mathbb{E} \int_0^\tau |\nabla_{e_j}^G v^{(0)}(s, X_s^1) - \nabla_{e_j}^G v^{(0)}(s, X_s^2)|_K^2 ds \\ & \leq \mathbb{E} \int_0^\tau ds \int_0^1 \sum_{m \geq 1} \sup_{|a|_U=1} |\nabla_{[X_s^2 - X_s^1]} \nabla_a^G v_m^{(0)}(s, X_s^1 + r(X_s^2 - X_s^1))|^2 dr \\ & \leq C_T \sup_{t \in [0, T]} \|B(t, \cdot)\|_{C_b^\alpha(H, U)}^2 \mathbb{E} \int_0^\tau |X_s^2 - X_s^1|_K^2 ds. \end{aligned}$$

Starting from (4.26) and using the previous estimates, we can apply the Gronwall lemma and obtain

$$\sup_{\tau \in [0, T_0]} \mathbb{E} |X_\tau^{x_1} - X_\tau^{x_2}|_K^2 \leq c_T |x_1 - x_2|_K^2. \quad (4.29)$$

If $T_0 < T$ we consider the FBSDE (4.16) with terminal time $(2T_0) \wedge T$. We find $v^{(1)} : [0, (2T_0) \wedge T] \times H \rightarrow K$ according to (4.22) with T replaced by $(2T_0) \wedge T$. By (4.24) we obtain in particular

$$\begin{aligned} X_\tau^1 - X_\tau^2 &= e^{\tau A} (x_1 - x_2) + e^{\tau A} [v^{(1)}(0, x_1) - v^{(1)}(0, x_2)] \\ &\quad - [v^{(1)}(\tau, X_\tau^1) - v^{(1)}(\tau, X_\tau^2)] + \int_0^\tau e^{(\tau-s)A} [\nabla^G v^{(1)}(s, X_s^1) \\ &\quad - \nabla^G v^{(1)}(s, X_s^2)] dW_s, \end{aligned}$$

$\tau \in [T_0, (2T_0) \wedge T]$. Arguing as before, we get, for $\tau \in [T_0, (2T_0) \wedge T]$,

$$\begin{aligned} & |e^{\tau A} x_1 - x_2|_K + |e^{\tau A} [v^{(1)}(0, x_1) - v^{(1)}(0, x_2)]|_K + |v^{(1)}(\tau, X_\tau^1) - v^{(1)}(\tau, X_\tau^2)|_K \\ & \leq C_T |x_1 - x_2|_K + \frac{1}{3} |X_\tau^1 - X_\tau^2|_K \end{aligned}$$

and

$$\begin{aligned}
 & \mathbb{E} \left| \int_0^\tau e^{(\tau-s)A} [\nabla^G v^{(1)}(s, X_s^1) - \nabla^G v^{(1)}(s, X_s^2)] dW_s \right|_K^2 \\
 & \leq C_T \sup_{t \in [0, T]} \|B(t, \cdot)\|_{C_b^\alpha(H, U)}^2 \int_0^\tau \mathbb{E} |X_s^1 - X_s^2|_K^2 ds \\
 & = C_T \sup_{t \in [0, T]} \|B(t, \cdot)\|_{C_b^\alpha(H, U)}^2 \left(\int_0^{T_0} \mathbb{E} |X_s^1 - X_s^2|_K^2 ds + \int_{T_0}^\tau \mathbb{E} |X_s^1 - X_s^2|_K^2 ds \right) \\
 & \leq C_T \sup_{t \in [0, T]} \|B(t, \cdot)\|_{C_b^\alpha(H, U)}^2 \left(c_T T_0 |x_1 - x_2|_K^2 + \int_{T_0}^\tau \mathbb{E} |X_s^1 - X_s^2|_K^2 ds \right).
 \end{aligned}$$

We have obtained, for $\tau \in [T_0, (2T_0) \wedge T]$,

$$\mathbb{E} |X_\tau^1 - X_\tau^2|_K^2 \leq C'_T |x_1 - x_2|_K^2 + C'_T \int_{T_0}^\tau \mathbb{E} |X_s^1 - X_s^2|_K^2 ds.$$

By the Gronwall lemma we find

$$\sup_{\tau \in [T_0, (2T_0) \wedge T]} \mathbb{E} |X_\tau^{x_1} - X_\tau^{x_2}|_K^2 \leq c_T |x_1 - x_2|_K^2.$$

Proceeding in this way, in finite steps, we get (4.25). $\square$

Remark 4.8 Using a localization argument as in [10], the boundeness of B in (1.7) could be dispensed. In particular, one can prove strong well-posedness of (1.1), for any $x \in H$, under Hypothesis 1 and assuming that $B : [0, T] \times H \rightarrow U$ is continuous on $[0, T] \times H$ and grows at most linearly, uniformly in $t \in [0, T]$; moreover, one requires that for any ball $S \subset H$ the function $B(t, \cdot) : S \rightarrow U$ is α -Hölder continuous, for some $\alpha > 2/3$, uniformly in $t \in [0, T]$ (cf. (4.4)).

A Appendix: The Deterministic Linear Control System in K and in H

In this appendix we prove in particular that (3.13) and (3.14) hold true.

We first consider the linear deterministic control system on $K = \mathcal{D}(\Lambda^{1/2}) \times U = V \times U$ associated to the wave equation

$$\begin{cases} \dot{w}(t) = Aw(t) + Gu(t), \\ w(0) = h \in K, \end{cases} \quad (\text{A.1})$$

where G is defined in (2.4). Here $A = A_K$ is the generator of the wave semigroup in K . As in Sect. 2 we denote by A also the extension of A_K defined in (2.3) (i.e., the generator of the wave semigroup in H). Let $t > 0$ and consider $\tilde{Q}_t : K \rightarrow K$:

$$\tilde{Q}_t = \int_0^t e^{(t-s)A} \begin{pmatrix} 0 & 0 \\ 0 & I_U \end{pmatrix} e^{(t-s)A^*} ds$$

and define the operator $\mathcal{L}_t$, $\mathcal{L}_t u := \int_0^t e^{(t-s)A} G u(s) ds$, acting from $L^2(0, t; U)$ with values in $K \subset H$ (see Appendix in [25] for more details). We know that

$$\tilde{Q}_t^{1/2}(K) = \text{Im} \mathcal{L}_t = \mathcal{L}_t(L^2(0, t; U)) \quad (\text{A.2})$$

(see, for instance, page 253 in [33] and [7]). This operator is different from the operator $Q_t : H \rightarrow H$ considered in (2.7).

It is known that the controlled abstract wave equation (A.1) is exactly controllable in K (see [11] or page 367 in [3]). Hence $\tilde{Q}_t^{1/2}(K) = \text{Im} \mathcal{L}_t = K$. In particular the control system is null controllable in K (i.e., for any $h \in K$, $t > 0$, there exists a control function $u : [0, t] \rightarrow U$ such that the corresponding solution w verifies $w(t) = 0$).

By Theorem 15.3 in [33] it follows that the minimal energy to steer h to 0 in time t is given by

$$\mathcal{E}_C(t, h) = |\tilde{Q}_t^{-1/2} e^{tA} h|_K, \quad h \in K. \quad (\text{A.3})$$

We are interested in the behaviour of $|\tilde{Q}_t^{-1/2} e^{tA} h|_K$ as $t \rightarrow 0^+$ (for related results we refer to [29] and [2]). In [25, Theorem A.1] it is proved that

Theorem A.1 *Let $T_0 > 0$. (i) There exists a positive constant $C_{T_0} > 0$ such that, for any $v \in V$, we have*

$$\left| \tilde{Q}_T^{-1/2} e^{TA} \begin{pmatrix} v \\ 0 \end{pmatrix} \right|_K \leq \frac{C_{T_0} |v|_V}{T^{\frac{3}{2}}}, \quad T \in (0, T_0], \quad (\text{A.4})$$

(ii) There exists $C > 0$ such that, for any $a \in U$, setting $Ga = \begin{pmatrix} 0 \\ a \end{pmatrix}$

$$|\tilde{Q}_T^{-1/2} e^{TA} Ga|_K \leq \frac{C |a|_U}{T^{\frac{1}{2}}}, \quad T > 0. \quad (\text{A.5})$$

The previous estimates imply that there exists $C_{T_0} > 0$ such that

$$|\tilde{Q}_T^{-1/2} e^{TA}|_{L(K, K)} \leq C_{T_0} T^{-3/2}, \quad T \in (0, T_0]. \quad (\text{A.6})$$

Proof We only give a sketch of proof (see [25] for more details). In the proof we will also use the operator A defined in (2.3).

(i) Let $v \in V$ and $T > 0$. We consider $f(t) = t^2(T - t)^2$ and

$$\phi(t) = f(t) \cdot \left(\int_0^T f(s) ds \right)^{-1}, \quad t \in [0, T].$$

Note that $\phi(0) = \phi(T)$, $\int_0^T \phi(s) ds = 1$ and there exists $C > 0$ (independent of $T > 0$) such that $|\phi(t)| \leq \frac{C}{T}$ and $|\phi'(t)| \leq \frac{C}{T^2}$, $t \in [0, T]$. Let $\psi : [0, T] \rightarrow K$,

$$\psi(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = -\phi(t) e^{tA} k = -\begin{pmatrix} \phi(t) \cos(\sqrt{\Lambda} t) v \\ -\phi(t) \sqrt{\Lambda} \sin(\sqrt{\Lambda} t) v \end{pmatrix}, \quad t \in [0, T].$$

Using also the derivative ψ'_1 we introduce the control

$$u(t) = \psi_2(t) + \psi'_1(t) \in U, \quad t \in [0, T]. \quad (\text{A.7})$$

One can show, integrating by parts, that it transfers $k = \begin{pmatrix} v \\ 0 \end{pmatrix}$ to 0 at time T . One compute the energy of the control u , i.e., $\int_0^T |u(s)|_U^2 ds$, finding that $\int_0^T |u(s)|_U^2 dt \leq \frac{c_{T_0} |v|_V^2}{T^3}$, where c_{T_0} is independent of T and v . We finally obtain

$$|\tilde{Q}_T^{-1/2} e^{TA} k|_K \leq \left(\int_0^T |u(s)|_U^2 ds \right)^{1/2} \leq \frac{C_{T_0} |v|_V}{\sqrt{T^3}}, \quad T \in (0, T_0].$$

(ii) Let us fix $T > 0$, $k = \begin{pmatrix} 0 \\ a \end{pmatrix} = Ga$ with $a \in U$. Consider $\phi(t)$ as before. Let

$$\psi(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = -\phi(t) e^{tA} k = -\begin{pmatrix} \phi(t) \frac{1}{\sqrt{\Lambda}} \sin(\sqrt{\Lambda} t) a \\ \phi(t) \cos(\sqrt{\Lambda} t) a \end{pmatrix}, \quad t \in [0, T].$$

Using also the derivative ψ'_1 we introduce as in the first part of the proof the control $u(t) = \psi_2(t) + \psi'_1(t) \in U$, $t \in [0, T]$. It transfers k to 0 at time T . Moreover,

$$|\tilde{Q}_T^{-1/2} e^{TA} Ga|_K \leq \left(\int_0^T |u(s)|_U^2 ds \right)^{1/2} \leq \frac{C |a|_U}{\sqrt{T}}, \quad T > 0.$$

□

In the following we consider the previous control system (A.1) in H , i.e., we take the initial condition $h \in H$. The control function u still belongs to $L^2_{loc}(0, \infty; U)$ with $G : U \rightarrow K \subset H$. Let $t > 0$. We still have

$$Q_t^{1/2}(H) = \text{Im} \mathcal{L}_t = \mathcal{L}_t(L^2(0, t; U)) = K, \quad (\text{A.8})$$

where $\mathcal{L}_t$ is the same operator considered in (A.2) but we use $Q_t : H \rightarrow H$,

$$Q_t = \int_0^t e^{(t-s)A} \begin{pmatrix} 0 & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} e^{(t-s)A^*} ds,$$

instead of $\tilde{Q}_t$ considered before. It follows that $Q_t^{1/2}(H) = K$. Moreover, for any $k \in K$ we define $\mathcal{E}_C(t, k)$ as we have done for the control system in K . By Theorem 15.3 in [33] and (A.8) we infer that

$$\mathcal{E}_C(t, k) = |Q_t^{-1/2} e^{tA} k|_H, \quad k \in K, \quad t > 0. \quad (\text{A.9})$$

By (A.3) we get $|\tilde{Q}_t^{-1/2} e^{tA} k|_K = |Q_t^{-1/2} e^{tA} k|_H$, $k \in K$, $t > 0$. Thus $Q_t^{-1/2} e^{tA} \in L(K, H)$ and, applying (A.6) and (A.5) for any $T > 0$, there exists $c = c_T > 0$ such that for any $t \in (0, T]$, (3.13) and (3.14) hold true.

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Declarations

Conflict of interest There are no conflicts of interest.

Ethical Approval We do not work with any empirical data. For this reason, we are not aware of any ethical issues that could arise within this article.

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