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SOME DEVELOPMENTS ON EXISTENCE AND UNIQUENESS OF DG-ENHANCEMENTS

PHD THESIS

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Introduction

Triangulated categories have attracted great interest since their introduction in Verdier's PhD thesis [76]. In the beginning, the focus was mainly on derived categories of coherent sheaves (the curious reader may refer to the brief historic dissertation [39, §1.1]). However, derived categories are far more rigid than triangulated categories, as they can be described using a higher categorical approach. In order to broaden our framework, the triangulated categories considered should allow such description as well. For this reason, we consider *DG-categories*, i.e. categories where the hom-sets are cochain complexes and the composition behaves accordingly. It turns out that the categories given by the 0-th cohomology of the hom-sets of special DG-categories, called *pretriangulated DG-categories*, come with a natural triangulated structure. A triangulated category admitting such a construction is called *algebraic*. From a different viewpoint, we say that a *(DG-)enhancement* of a triangulated category is a pretriangulated DG-category corresponding to it. As one may expect, derived categories and homotopy categories of complexes are algebraic. It should be noted that other enhancement theories exist. For instance, another well-studied concept is the one of topological triangulated categories, obtained by (stable) model categories.

In this thesis, we present the results of [49] and [50]. The former deals with the existence of enhancements for triangulated categories with a full strong exceptional sequence. Roughly speaking, such a sequence is a collection of simple objects generating the triangulated category, useful in studies of both Representation Theory and Algebraic Geometry. A famous theorem on the topic is the following.

Theorem – Keller-Orlov. [63, Corollary 1.9] (cf. [6, Theorem 6.2]). *Let $\mathscr{T}$ be an algebraic $\mathbb{K}$-linear triangulated category with $\mathbb{K}$ a field. Assume that $\mathscr{T}$ has a full strong exceptional sequence $\langle E_1, \dots, E_n\rangle$. Then $\mathscr{T}$ is triangulated equivalent to the bounded derived category $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}(A))$, where $A = \mathrm{End}(\bigoplus_{i=1}^{n} E_i)$ and $\mathrm{mod}(A)$ is the category of finitely generated (right) modules over A.*

From the statement, the connection with Representation Theory becomes more evident. Indeed, we are able to describe the bounded derived categories associated to a large class of finite-dimensional algebras (see §4.2 for more details). Concerning Algebraic Geometry, the bounded derived category of (coherent sheaves on) the projective space has a full strong exceptional sequence. This example motivates the study of rationality using derived categories; the interested reader may refer to [46].

We remark that very few examples of non-algebraic triangulated categories are known. For instance, the stable homotopy category is a topological triangulated category that is not algebraic, as proved in [44, §7.6]. The first example without any enhancement is discussed in [56]. For the case of linearity over a field, the reader may refer to [68], where the triangulated category is obtained by a semiorthogonal decomposition with algebraic components. More precisely, we can describe the triangulated category using two triangulated subcategories which are both algebraic. This result may suggest that non-algebraic settings can arise from exceptional sequences, since they give rise to semiorthogonal decompositions.

The aim of [49] is to drop the algebraic requirement in the theorem above. In order to do so, we describe a construction to obtain a heart of a t-structure in the triangulated category from a semiorthogonal decomposition (see Theorem 4.7). Roughly speaking, a heart of a t-structure is an abelian subcategory that defines a cohomology study inside the triangulated category. It turns out that a full strong exceptional sequence of length 2 gives a hereditary heart, i.e. a heart for which $\mathrm{Ext}^i = 0$ for $i > 1$. This special heart forces the triangulated category to be uniquely determined up to triangulated equivalence; this is the content of Hubery's Theorem 2.56. Therefore, in the case at hand, the statement holds in the whole generality of triangulated categories (see Corollary 4.13). For a full strong exceptional sequence with length greater than 2, we require the triangulated category $\mathscr{T}$ to be realized, i.e. for every admissible abelian subcategory $\mathscr{A} \subset \mathscr{T}$ there exists a triangulated functor real : $\mathcal{D}^{\mathrm{b}}(\mathscr{A}) \to \mathscr{T}$ extending the inclusion of $\mathscr{A}$ in $\mathscr{T}$.

4.28. Theorem. *Let $\mathbb{K}$ be a field and let $\mathscr{T}$ be a realized $\mathbb{K}$-linear triangulated category with a full strong exceptional sequence $\langle E_1, \dots, E_n \rangle$ such that $\bigoplus_i \mathrm{Hom}(X, Y[i])$ is a finite-dimensional vector space for any $X, Y \in \mathscr{T}$. Then $\mathscr{T} \cong \mathcal{D}^{\mathrm{b}}(\mathrm{mod}(A))$, where $A = \mathrm{End}(\bigoplus_{i=1}^n E_i)$. In particular, $\mathscr{T}$ is algebraic.*

The fact that a triangulated category is algebraic, however, does not prevent bizarre behaviours. For instance, its structure may come from non-quasi-equivalent DG-categories, i.e. "homologically different" DG-categories. This possibility motivates the following definition: whenever the enhancements are all quasi-equivalent, we say that an algebraic triangulated category has a *unique enhancement*.

As the reader may expect, not all algebraic triangulated categories have a unique enhancement. An example is $\mathrm{mod}(\mathbb{K})$, the category of finite dimensional vector spaces over the field $\mathbb{K}$. This category becomes triangulated with shift the identity and distinguished triangles generated by short exact sequences. In [71], Schlichting proved that $\mathrm{mod}(\mathbb{K})$ does not have a unique enhancement when $\mathbb{K} = \mathbb{F}_p$ (with p prime), giving two explicit enhancements that are not quasi-equivalent (for the transposition of the result in the DG-world, one may refer to [14, Corollary

3.10]). It is important to notice that one of the enhancements is not $\mathbb{K}$-linear. As a matter of fact, the only known example of a triangulated category with a non-unique enhancement linear over a field is discussed in [67].

The motivating question that led to the birth of [50] is the following: does $\mathrm{mod}(\mathbb{K})$ have a unique $\mathbb{K}$-linear enhancement? The answer is yes (see Corollary 5.41), and it follows by studying the associated graded algebra $\mathbb{K}[t, t^{-1}]$, where t has degree 1. Indeed, if such graded algebra is intrinsically formal, we obtain the uniqueness of enhancements for the associated triangulated category, as proved in Proposition 5.1. The conclusion that $\mathbb{K}[t, t^{-1}]$ is intrinsically formal follows from Proposition 1.46 (see also Proposition 1.47).

After this discovery, we wanted to understand how intrinsic formality relates to stricter requirements of the uniqueness of enhancements, namely the *strong uniqueness of enhancements*. This property tells us that the triangulated autoequivalences of the triangulated category come from the DG-world (cf. Proposition 3.64). Very few examples of triangulated categories with a strongly unique enhancement are known, the most important one being the bounded derived categories of projective varieties, investigated by Lunts and Orlov in the celebrated article [51]. The procedure used to obtain such a result has been generalized to other cases: Canonaco and Stellari worked on coherent sheaves of a quasi-projective scheme supported in a projective subscheme [14]; Olander studied the case of a proper algebraic space over an Artinian ring [60]; Li, Pertusi, and Zhao considered the case of Kuznetsov components [48].

Currently, there is only one explicit example of a triangulated category with a unique but not strongly unique enhancement (see [33, Corollary 5.4.12]). However, we do not know whether the uniqueness of enhancements implies the strong uniqueness of enhancements for derived categories or homotopy categories of complexes. It is worth noting that the examples of triangulated categories with a unique enhancement are by far more general: the most recent paper in this direction is [12], where it is proved that all the derived categories and all the homotopy categories of complexes over an abelian category have a unique enhancement.

For this reason, we are interested in characterizing the strong uniqueness of enhancements. With this aim, we define the notions of triangulated formal DG-categories, extending the concept of intrinsic formality, and formally standard DG-categories, inspired by D-standard and K-standard categories introduced by Chen and Ye in [19]. When we restrict to graded categories, the combination of triangulated formality and formal standardness is equivalent to the strong uniqueness of enhancements (see Theorem 5.33). Furthermore, since D-standardness and K-standardness are instances of formal standardness in a proper sense, they are proven to be equivalent to the strong uniqueness of enhancements.

5.47. Proposition. *An additive category $\mathscr{A}$ is K-standard if and only if $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$ has a strongly unique enhancement.*

5.51. Theorem. *An exact category $\mathscr{E}$ is D-standard if and only if $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ has a strongly unique enhancement.*

These results follow from the fact that $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$ and $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ have a (semi-strongly) unique enhancement for every choice of $\mathscr{A}$ additive and $\mathscr{E}$ exact (see Proposition 5.7 and Proposition

5.10). We emphasize that the corollary above extends [19, Theorem 5.10], which is valid for the category of finitely generated modules over a finite-dimensional $\mathbb{K}$-algebra, with $\mathbb{K}$ a field.

Overview Chapter 1 introduces some basic concepts on DG-rings, as well as the notion of intrinsic formality and a discussion on its interpretation via A_∞-algebras. In Chapter 2, we deal with full exact subcategories of triangulated categories and show Hubery's Theorem for hereditary hearts. The crucial concept of (pretriangulated) DG-categories is defined in Chapter 3, where we also show some preliminary results on the different notions of the uniqueness of enhancements. Chapter 4 and Chapter 5 present the content of [49] and [50] respectively.

The symbol ★ in the first three chapters indicates results that the author proved independently; we will quote a reference if known. All the results in the last two chapters are new unless otherwise specified.

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CHAPTER 1.

Intrinsic formality

In this chapter, we present some basic concepts on DG-rings and A_∞-algebras, as well as conventions that will be respected throughout the thesis. Moreover, we prove some expected facts on formality (see Proposition 1.40 and Corollary 1.41) and an interesting result linking Hochschild cohomology with group cohomology for a specific example (see Proposition 1.46).

§1.1. DG-rings

1.1. Definition. Let $\Bbbk$ be a (unital associative) commutative ring. A (unital associative) ring R, together with a ring homomorphism $i_R : \Bbbk \to R$ factoring through the center of R, is called a *central* $\Bbbk$*-ring*. A morphism of central $\Bbbk$-rings is a (unital) ring homomorphism $f : R \to S$ satisfying a commutative diagram

$$\begin{tikzcd} \Bbbk \arrow[d, "i_R"] \arrow[dr, bend left, "i_S"] & \\ R \arrow[r, "f"] & S. \end{tikzcd}$$

Central $\Bbbk$-rings are better known as (unital associative) $\Bbbk$-algebras. However, as the standard naming may suggest to the reader we are assuming that $\Bbbk$ is a field, we prefer to follow the choice of [77]. Accordingly, we reserve the term algebra when linearity over a field is required. We give a crucial example to motivate our decision.

1.2. Example. When $\Bbbk = \mathbb{Z}$, notice that central $\Bbbk$-rings are simply rings.

1.3. Convention. We will work under the following conventions:
- $\Bbbk$ is a fixed (unital associative) commutative ring;
- A ring is a central $\Bbbk$-ring;

- A ring homomorphism is a morphism of central $\Bbbk$-rings.
- The tensor product $\otimes$ is used to denote the tensor product over $\Bbbk$.
- In fact, everything will be considered to be $\Bbbk$-linear.
- When not explicitly said, modules are to be considered right modules.

1.4. Remark. Let R be a (central $\Bbbk$-)ring. Notice that any (right) R-module is in fact also a $\Bbbk$-module by i_R. Moreover, as a morphism of modules is R-linear, it is also $\Bbbk$-linear.

1.5. Definition. A *graded module* is a $\Bbbk$-module M with a direct sum decomposition $M = \bigoplus_{i\in\mathbb{Z}} M^i$ into $\Bbbk$-modules.

An element $m \in M^i$ is called a *homogeneous element of degree i*. Sometimes, to denote the degree we will use $|m| = i$.

1.6. Definition. Given two graded modules M,N, we define the *tensor product*

$$M \otimes N := \bigoplus_i (M \otimes N)^i, \qquad \text{where } (M \otimes N)^i := \bigoplus_j (M^j \otimes N^{i-j}).$$

1.7. Definition. A morphism $f : M \to N$ between graded modules is *homogeneous of degree i*, in symbols $|f| = i$, if $f(M^j) \subset N^{j+i}$ for any $j \in \mathbb{Z}$. The set of homogeneous morphisms of degree i is denoted by $\operatorname{Hom}(M,N)^i$. Alternatively,

$$\operatorname{Hom}(M,N)^i := \prod_j \operatorname{Hom}(M^j, N^{j+i}).$$

The *module of graded morphisms* from M to N is given by

$$\operatorname{Hom}(M,N) := \bigoplus_i \operatorname{Hom}(M,N)^i.$$

1.8. Definition. A *graded ring* is a ring A with a direct sum decomposition $A = \bigoplus_{i\in\mathbb{Z}} A^i$ into $\Bbbk$-modules such that $A^iA^j \subset A^{i+j}$. In particular, $1 \in A^0$ and A^0 is a subring of A.

1.9. Definition. A ring homomorphism $f : A \to B$ between graded rings is a *graded ring homomorphism* if $f(A^i) \subset B^i$ for all $i \in \mathbb{Z}$. In other words, $f \in \operatorname{Hom}(A,B)^0$.

1.10. Definition. A *DG-module* (differential graded module), or *complex*, is a couple (M, d_M) where M is a graded module and $d_M : M \to M$ is a graded morphism of degree 1, called *differential*, such that $d_M^2 = 0$. Habitually, we will omit d_M and say that M is a DG-module. With a slight abuse of notation, we will also use d instead of d_M.

A *DG-morphism* (or *chain map*) $f : M \to N$ between DG-modules is a graded morphism of degree 0 commuting with the differentials, i.e. $f d_M = d_N f$.

1.11. Definition. Given a DG-module $(M = \bigoplus_i M^i, d_M)$, its *cohomology* is the graded module $H^*M := \ker d_M / \operatorname{im} d_M$. More explicitly, denoted with $d_M^i : M^i \to M^{i+1}$ the morphism obtained by restricting the differential d_M to M^i, we have

$$H^*M := \bigoplus_i H^iM, \qquad \text{where } H^iM := \ker d_M^i / \operatorname{im} d_M^{i-1}.$$

1.12. Definition. The *tensor product* $M \otimes N$ of two DG-modules M, N is the graded tensor product introduced in Definition 1.6 equipped with the differential

$$d_{M \otimes N}(m \otimes n) := d_M(m) \otimes n + (-1)^{|m|} m \otimes d_N(n)$$

for $m \in M$ and $n \in N$ homogeneous. In particular, in this way $M \otimes N$ is a DG-module.

1.13. Definition. Let M, N be two DG-modules. We denote with $\mathrm{Hom}(M,N)$ the *DG-module of graded morphisms* given by the module of graded morphisms of Definition 1.7 equipped with the differential

$$d_{\mathrm{Hom}(M,N)}(f) := d_N f - (-1)^{|f|} f d_M$$

for $f \in \mathrm{Hom}(M,N)$ homogeneous.

1.14. Definition. A *DG-ring* (differential graded ring) is a graded ring A equipped with a differential, i.e. a morphism $d_A = (d_A^i : A^i \to A^{i+1})_{i \in \mathbb{Z}}$ of degree 1 such that $d_A^2 = 0$, satisfying the graded Leibniz rule:

$$d_A^{i+j}(a \cdot b) = d_A^i(a) \cdot b + (-1)^i a \cdot d_A^j(b),$$

for every $a \in A^i$ and $b \in A^j$. Sometimes, we will write d instead of d_A.

1.15. Definition. Let A and B be DG-rings. A graded ring homomorphism $f : A \to B$ is a *DG-ring homomorphism* if it commutes with the differentials, i.e. $f d_A = d_B f$.

1.16. Example. Any graded ring is a DG-ring with trivial differential, i.e. $d = 0$. Moreover, any ring can be considered a graded ring concentrated in degree 0, so it is also a DG-ring.

1.17. Remark. Given a DG-ring A, its *cohomology* $H^*A = \bigoplus_i H^i A$ is a graded ring. Moreover, it can be proven that any DG-ring homomorphism $f : A \to B$ induces a graded ring homomorphism $H^*(f) : H^*(A) \to H^*(B)$.

1.18. Definition. A DG-ring homomorphism $f : A \to B$ is a *quasi-isomorphism* if $H^*(f)$ is an isomorphism of graded rings.

1.19. Definition. A DG-ring A is *formal* if there exists a zig-zag of quasi-isomorphisms

$$\begin{tikzcd} & A_1 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & \\ A & & A_2 & & H^*A. \end{tikzcd}$$

A graded ring B is *intrinsically formal* if any DG-ring A with cohomology isomorphic to B is formal.

1.20. Proposition. [22, Lemma 6.6]. *A ring R is always intrinsically formal.*

PROOF. Let A be a DG-ring with $H^*(A) \cong R$. Set $\tau_{\leq 0} A$ the DG-ring with $\tau_{\leq 0} A^i = 0$ for $i > 0$, $\tau_{\leq 0} A^0 = \ker d^0$ and $\tau_{\leq 0} A^i = A^i$ for $i < 0$. The inclusion $\tau_{\leq 0} A \to A$ and the quotient $\tau_{\leq 0} A \to R$ are quasi-isomorphisms. We conclude that A is formal. $\square$

§1.2. A_∞-algebras

1.21. Convention. We reserve the capital letter $\mathbb{K}$ for fields. A central $\mathbb{K}$-ring is called *algebra*. Accordingly, we will talk about graded algebras and DG-algebras.

1.22. Convention. Given $f : A \to C$ and $g : B \to D$ two graded homogeneous morphisms, we adopt the Koszul sign rule:

$$f \otimes g(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b)$$

for $a \in A$ and $b \in B$ homogeneous elements.

1.23. Definition. An *A_∞-algebra over (a field)* $\mathbb{K}$ is a graded $\mathbb{K}$-module $A = \bigoplus_{i\in\mathbb{Z}} A^i$ equipped with a family of graded morphisms $(m_n : A^{\otimes n} \to A)_{n\geq 1}$ such that $|m_n| = 2-n$ and the following holds for every $n \geq 1$:

$$\sum_{n=r+s+t} (-1)^{r+st} m_{r+t+1}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0, \tag{1.24}$$

where $\mathrm{id} : A \to A$ is the identity.

1.25. Remark. To better understand the notion of A_∞-algebra, we analyze (1.24) for $n = 1,2,3$. Let A be an A_∞-algebra.

For $n = 1$, the equation gives $m_1 m_1 = 0$. In particular, (A,m_1) is a DG-module. Setting $n = 2$, we obtain $m_1 m_2 = m_2(m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1)$. Roughly speaking, this means that m_1 satisfies the graded Leibniz rule with respect to m_2. With $n = 3$, we get the identity

$$m_2(\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) = m_1 m_3 + m_3(m_1 \otimes \mathrm{id}^{\otimes 2} + \mathrm{id} \otimes m_1 \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes m_1)$$

as morphisms $A^{\otimes 3} \to A$. The left-hand side is equal to zero if m_2 is associative. In a sense we will not explore, this formula tells us that m_2 is associative up to homotopy. In particular, if $m_3 = 0$, then m_2 is associative and can be used as a multiplication. The reader interested in the topic may refer to [40].

1.26. Remark. One may wonder whether a DG-algebra can be defined as an A_∞-algebra with $m_n = 0$ for $n \geq 3$. However, A_∞-algebras do not require a unit to exist; instead, such characterization defines *non-unital* DG-algebras. Similarly, non-unital graded algebras are exactly A_∞-algebras with $m_n = 0$ for $n = 1$ and $n \geq 3$.

As seen in Remark 1.25, an A_∞-algebra is not necessarily associative. However, its cohomology is.

1.27. Definition. The *cohomology* H^*A of an A_∞-algebra A is the cohomology of the DG-module (A,m_1). It can be proven that H^*A is a non-unital graded algebra with product induced by m_2.

1.28. Definition. A *morphism of A_∞-algebras* $f : A \to B$ is a family of graded morphisms

$$f := (f_n : A^{\otimes n} \to B)_{n \geq 1}, \qquad |f_n| = 1 - n$$

satisfying, for every $\ell \geq 1$,

$$\sum_{\ell = r+s+t} (-1)^{r+st} f_{r+1+t}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = \sum_{\substack{1 \leq r \leq \ell \\ \ell = i_1 + \cdots + i_r}} (-1)^u m_r (f_{i_1} \otimes \cdots \otimes f_{i_r}),$$

where

$$u = \sum_{j=1}^{r} (r-j)(i_j - 1) = (r-1)(i_1 - 1) + (r-2)(i_2 - 1) + \cdots + (i_{r-1} - 1).$$

The composition of two morphisms $f = (f_n) : A \to B$ and $g = (g_n) : B \to C$ is given by

$$(g \circ f)_\ell := \sum_{\substack{1 \leq r \leq \ell \\ \ell = i_1 + \cdots + i_r}} (-1)^u f_r \circ (g_{i_1} \otimes \cdots \otimes g_{i_r}),$$

while the identity id is defined by $\mathrm{id}_1 := \mathrm{id}$ and $\mathrm{id}_n := 0$ for $n > 1$.

A morphism of A_∞-algebras f is a *quasi-isomorphism* if f_1 is a quasi-isomorphism of DG-modules.

1.29. Remark. Given a morphism of A_∞-algebras $f : A \to B$, one can notice that f_1 is a morphism of DG-modules, i.e. $f_1 m_1 = m_1 f_1$, and moreover f_1 commutes with m_2 "up to homotopy", i.e.

$$f_1 m_2 = m_2(f_1 \otimes f_1) + m_1 f_2 + f_2(m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1).$$

In fact, the homology can be equipped with an A_∞-structure, so that it encodes everything of the A_∞-algebra we started with.

1.30. Theorem – Kadeishvili. [34] (see, for instance, [41, Theorem 2.3] for a modern formulation). *Given an A_∞-algebra A, its cohomology H^*A has an A_∞-algebra structure $H^*_\infty A$ such that:*

- *$m_1 = 0$ and m_2 is induced from m_2^A;*
- *There is a quasi-isomorphism of A_∞-algebras $H^*_\infty A \to A$ lifting the identity of H^*A.*

In addition, this structure is unique up to (non-unique) isomorphism.

1.31. Remark. The notation $H^*_\infty A$ is not standard, but it helps to distinguish between the natural A_∞-algebra structure of H^*A, given by setting $m_n = 0$ for $n > 2$, and the A_∞-algebra structure adopted for the statement of Theorem 1.30.

1.32. Remark. By [41, Proposition 2.1], every A_∞-algebra is quasi-isomorphic to a non-unital DG-algebra.

1.33. Definition. An A_∞-algebra A is called *minimal* if $m_1 = 0$. In Theorem 1.30, $H^*_\infty A$ is a *minimal model* for A.

An A_∞-algebra is *non-unital formal* if its minimal model $H^*_\infty A$ can be chosen to be a graded algebra, meaning that $m_n = 0$ for $n > 2$. More briefly, if $H^*_\infty A \cong H^* A$.

A graded algebra B is *non-unital intrinsically formal* if all minimal A_∞-structures on B are quasi-isomorphic. In other words, by Theorem 1.30, it is non-unital intrinsically formal if any A_∞-algebra A with $H^*A \cong B$ is non-unital formal.

Notice that, until now, A_∞-algebras do not have a unit. There are in fact two standard definitions.

1.34. Definition. An A_∞-algebra A is *cohomologically unital* if H^*A is a unital graded algebra. It is *strictly unital* if there exists an element $1 \in A$ of degree 0 such that $m_1(1) = 0$, $m_2(a \otimes 1) = a = m_2(1 \otimes a)$ and for any $i \geq 3$,

$$m_i(a_1, \dots, a_i) = 0$$

whenever $a_j = 1$ for some $j = 1, \dots, i$.

A *cohomologically unital morphism* is a morphism of cohomologically unital A_∞-algebras $f : A \to B$ such that f_1 induces a unital morphism $H^*A \to H^*B$.

A morphism of strictly unital A_∞-algebras $f : A \to B$ is *strictly unital* if $f_1(1) = 1$ and for every $i > 1$, $f_i(a_1, \dots, a_i) = 0$ if $a_j = 1$ for some $j = 1, \dots, i$.

1.35. Definition. We say that a strictly unital A_∞-algebra A is *formal* if there exists a strictly unital quasi-isomorphism $H^*A \to A$ (where H^*A has $m_n = 0$ for $n > 2$).

A graded algebra B is *(A_∞) intrinsically formal* if all strictly unital A_∞-algebras with $H^*A \cong B$ are formal.

1.36. Remark. A DG-algebra is formal if and only if it is formal as an A_∞-algebra. Indeed, any quasi-isomorphism between A_∞-algebras gives rise to a zig-zag of quasi-isomorphisms between DG-algebras (see [40, §3.3]). In particular, a graded algebra B is intrinsically formal if and only if it is (A_∞) intrinsically formal from Remark 1.32.

Before stating the next proposition, we need to define the notion of homotopy between morphisms.

1.37. Definition. Let $f, g : A \to B$ be two morphisms of A_∞-algebras. A *homotopy* between f and g is a family of morphisms

$$h := (h_i : A^{\otimes i} \to B)_{i \geq 1}, \qquad |h_i| = -i$$

such that, for all $n \geq 1$,

$$\begin{aligned} f_n - g_n = & \sum_{\substack{0 \leq r,t \leq n \\ i_1 + \cdots + i_r + k + j_1 + \cdots + j_t = n}} (-1)^s m_{r+1+t}(f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes \cdots \otimes g_{j_t}) \\ & + \sum_{j+k+l=n} (-1)^{jk+l} h_n(\mathrm{id}^{\otimes j} \otimes m_k \otimes \mathrm{id}^{\otimes l}), \end{aligned} \tag{1.38}$$

where

$$s = t + \sum_{1\le v\le t}\left((1-j_v)\left(n-\sum_{u\le v} j_u\right)\right) + k\sum_{1\le u\le r} i_u + \sum_{2\le v\le r}\left((1-i_v)\sum_{u<v} i_u\right).$$

Whenever there exists a homotopy between f and g, we say that f and g are *homotopic*.

1.39. Remark. For $n=1$, the equation (1.38) reads

$$f_1 - g_1 = m_1 h_1 + h_1 m_1,$$

which is exactly the standard idea of homotopy (cf. Definition 2.6).

1.40. Proposition. ★. *A strictly unital A_∞-algebra A is formal if it is non-unital formal.*

PROOF. Assume A is non-unital formal. By [47, Proposition 3.2.4.1], we have a minimal model A' with a strictly unital quasi-isomorphism $A' \to A$. Being $H^*_\infty A \cong H^*A$ and A' strictly unital minimal models, by Theorem 1.30 we have an isomorphism $f : H^*A \to A'$. Moreover, $f_1 : H^*A^{\otimes 1} \to A'$ satisfies $f_1(a) = f_1(a1) = f_1(a)f_1(1)$ since the differentials of H^*A and A' are zero, and we can find $a \in H^*(A)$ such that $f_1(a) = 1$ because f is an isomorphism. Therefore, $1 = f_1(a) = f_1(a)f_1(1) = 1f_1(1) = f_1(1)$. In other words, f is cohomologically unital. By [47, Theorem 3.2.2.1], f is homotopic to a strictly unital morphism g. Since H^*A and A' are minimal models, f_1 agrees with g_1, so g is a quasi-isomorphism. The composition $H^*A \xrightarrow{g} A' \to A$ concludes the proof. □

1.41. Corollary. ★. *A (unital) graded algebra B (over a field $\mathbb{K}$) is intrinsically formal if it is non-unital intrinsically formal.*

PROOF. Let A be a strictly unital A_∞-algebra such that $H^*A \cong B$. Since B is non-unital intrinsically formal, A is non-unital formal. By Proposition 1.40, we conclude that A is formal. □

§1.3. Hochschild cohomology

In this section, we introduce Hochschild cohomology to give new examples of intrinsically formal graded algebras. We work over a field $\Bbbk = \mathbb{K}$.

1.42. Definition. *[1].* Let A be an algebra and M be an A-bimodule. The *Hochschild complex* is the DG-module (morphisms Hom are meant as morphisms of vector spaces)

$$M \xrightarrow{d^0} \mathrm{Hom}(A,M) \xrightarrow{d^1} \mathrm{Hom}(A^{\otimes 2},M) \xrightarrow{d^2} \mathrm{Hom}(A^{\otimes 3},M) \xrightarrow{d^3} \cdots$$

where

$$\begin{aligned} d^n f(a_0,\dots,a_n) = {} & a_0 f(a_1,\dots,a_n) \\ & - \sum_{i=0}^{n-1} (-1)^i f(a_0,\dots,a_{i-1},a_i a_{i+1},a_{i+2},\dots,a_n) \\ & + (-1)^{n+1} f(a_0,\dots,a_{n-1})a_n. \end{aligned}$$

(We use commas instead of tensor products for the sake of a simpler notation). The cohomology of such DG-module is called *Hochschild cohomology* and denoted by $\mathrm{HH}^n(A,M)$. If $M = A$, we will use $\mathrm{HH}^n(A)$ instead of $\mathrm{HH}^n(A,A)$.

If A and M are graded, $\mathrm{HH}^n(A,M)$ becomes graded by considering the complexes given by morphisms of homogeneous degree: $\mathrm{HH}^n(A,M) = \bigoplus_{k\in\mathbb{Z}} \mathrm{HH}^{n,k}(A,M)$. If $M = A$, we will write $\mathrm{HH}^{n,k}(A) = \mathrm{HH}^{n,k}(A,A)$.

We can now state an important result linking Hochschild cohomology and intrinsic formality.

1.43. Theorem – Kadeishvili. [35]. *A graded algebra A such that*

$$\mathrm{HH}^{n,2-n}(A) = 0 \quad \textit{for } n \geq 3$$

is non-unital intrinsically formal.

By Corollary 1.41, we immediately obtain the following.

1.44. Corollary. ★. *Let A be a unital graded algebra such that* $\mathrm{HH}^{n,2-n}(A) = 0$ *for* $n \geq 3$. *Then A is intrinsically formal.*

Interestingly, this corollary has not been proven in the literature. In [73], it was shown for augmented A_∞-algebras, while in [69] the result is stated without proof under the assumption that all A_∞-algebras are strictly unital.

We conclude this section by giving some examples of intrinsically formal graded algebras obtained by Theorem 1.43.

1.45. Example. Let $\mathbb{K}$ be algebraically closed. By [28, Lemma 2.8], for any choice of positive integers n,k, the graded algebra $\mathbb{K}[t]/(t^{n+1})$ with $\deg t = k$ is intrinsically formal.

We work out the details of another example: $\mathbb{K}[t,t^{-1}]$, which is a shorthand for the quotient $\mathbb{K}[t,s]/(ts = 1)$ where the grading is required to satisfy $\deg(s) = -\deg(t)$. Interestingly, intrinsic formality is obtained by comparing Hochschild cohomology with group cohomology. For an introduction to group cohomology, we refer to [10] (in [20, §5.1] there is an explicit description that may be useful to understand the following proof).

1.46. Proposition. ★. *For any* $n,k \in \mathbb{Z}$*, with n non-negative, we have that*

$$\mathrm{HH}^{n,k}(\mathbb{K}[t,t^{-1}]) \cong H^n(\mathbb{Z},\mathbb{K})$$

where t is homogeneous of positive degree, H^n *is the group cohomology,* $\mathbb{Z}$ *and* $\mathbb{K}$ *are considered only as additive groups and* $\mathbb{K}$ *is equipped with the structure of a* $\mathbb{Z}$*-group module by the trivial action.*[*]

In particular, $\mathbb{K}[t,t^{-1}]$ *is intrinsically formal.*

[*]For this result, I would like to thank Giorgio Leoni, whose knowledge on Group Theory has been crucial in finding the link between the two notions of cohomology.

PROOF. For the sake of brevity, we set $A := \mathbb{K}[t,t^{-1}]$. We denote with $\mathrm{Hom}^k(A^{\otimes n},A)$ the $\mathbb{K}$-linear morphisms of homogeneous degree k.

Notice that a morphism $f \in \mathrm{Hom}^k(A^{\otimes n},A)$ is described by a function $\lambda : \mathbb{Z}^n \to \mathbb{K}$ such that $f(t^{p_1},\dots,t^{p_n}) = \lambda(p_1,\dots,p_n)t^{\sum_i p_i + k/\deg(t)}$, where $\deg(t)$ is the homogeneous degree of t (and for f to be nonzero, k is always divisible by $\deg(t)$). Conversely, every such function λ give rise to a unique homogeneous morphism f of degree k.

With this new notation, we can reinterpret the differentials d^n:

$$\begin{aligned} d^n\lambda(q_0,\dots,q_n) = \lambda(q_1,\dots,q_n) & \\ & - \sum_{i=0}^{n-1}(-1)^i\lambda(q_0,\dots,q_{i-1},q_i+q_{i+1},q_{i+2},\dots,q_n) \\ & + (-1)^{n+1}\lambda(q_0,\dots,q_{n-1}). \end{aligned}$$

As the differential of Hochschild cohomology does not change the degree of the maps, whenever λ is associated to a morphism of degree k, $d^n\lambda$ is still associated to a morphism of degree k.

Then the Hochschild complex (for homogeneous morphisms of degree k) can be reinterpreted as the following:

$$\mathbb{K} \xrightarrow{d^0} \mathrm{Hom}_{\mathrm{Set}}(\mathbb{Z},\mathbb{K}) \xrightarrow{d^1} \mathrm{Hom}_{\mathrm{Set}}(\mathbb{Z}^2,\mathbb{K}) \xrightarrow{d^2} \mathrm{Hom}_{\mathrm{Set}}(\mathbb{Z}^3,\mathbb{K}) \xrightarrow{d^3} \dots$$

where Set is the category of sets. This is exactly the group cohomology for the $\mathbb{Z}$-group module $\mathbb{K}$, where both $\mathbb{Z}$ and $\mathbb{K}$ are considered as additive groups and the action of $\mathbb{Z}$ on $\mathbb{K}$ is the trivial one.

From [10, Example III.1], we have that $H^n(\mathbb{Z},M) = 0$ for any $\mathbb{Z}$-group module M and any $n \geq 2$. Therefore, $\mathrm{HH}^{n,k}(\mathbb{K}[t,t^{-1}]) = 0$ for $n \geq 2$ and all k. We conclude by Corollary 1.44. □

In fact, this result holds in a more general setting.

1.47. Proposition. [70, Corollary 4.2]. *Let Λ be an algebra with finite projective dimension d as a Λ-bimodule. Consider the graded algebra $\Lambda[t,t^{-1}]$ with $\deg t = m$. Then $\Lambda[t,t^{-1}]$ is intrinsically formal if $d \leq m$.*

1.48. Remark. It is very important to notice that intrinsic formality depends on the linearity: in Remark 5.42, we give an example of a graded algebra which is intrinsically formal with linearity over a field, but not intrinsically formal with linearity over a ring. This follows from non-uniqueness of a DG-enhancement.

CHAPTER 2.

Deriving exact categories

The purpose of this chapter is to study full exact subcategories of triangulated categories, having in mind the key example of the derived categories of exact categories. The discussion in §2.3 deals with the relation of Yoneda extensions and hom-sets in triangulated categories. In the last sections, we focus on hearts (of bounded t-structures) and show Hubery's Theorem 2.56, proved independently by the author in the first draft of [49].

§2.1. Triangulated categories

For the sake of fixing notation, we give a brief introduction of triangulated categories. This is not to be intended for readers who are unfamiliar with the basic concepts. A minimal knowledge on the topic is covered in [31, §1.2], or [59, §1.1].

2.1. Convention. Categories will always be locally small, meaning that all hom-sets are sets. Moreover, according to Convention 1.3, we suppose that all categories are $\Bbbk$-linear. This means that the hom-sets are $\Bbbk$-modules and the composition is $\Bbbk$-bilinear. Similarly, all functors are $\Bbbk$-linear, i.e. the induced morphisms between hom-sets are $\Bbbk$-linear.

2.2. Definition. A *triangulated category* $\mathscr{T}$ is an additive category together with a *shift functor* (also called suspension functor) $\Sigma : \mathscr{T} \to \mathscr{T}$, which is an automorphism*, and a class of *distinguished triangles*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma(A)$$

satisfying the following axioms.

*Some authors require Σ to be an autoequivalence. Here we want Σ to have a proper inverse to avoid technicality on the choice of the inverse Σ^{-1} (cf. [77, Remark 5.1.2]).

TR1 1. Each sequence of the form

$$A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow \Sigma(A)$$

is a distinguished triangle.

2. If $A \to B \to C \to \Sigma(A)$ is a distinguished triangle and $D \to E \to F \to \Sigma(D)$ is *isomorphic* to it, i.e. there exist vertical isomorphisms a, b, c making the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & \Sigma(A) \arrow[d, "\Sigma(a)"] \\ D \arrow[r] & E \arrow[r] & F \arrow[r] & \Sigma(D) \end{tikzcd}$$

commutative, then also $D \to E \to F \to \Sigma(D)$ is a distinguished triangle.

3. Any morphism $f : A \to B$ gives rise to a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow \Sigma(A),$$

which can be proved to be unique up to isomorphism. The object C, sometimes denoted by $\mathrm{Cone}(f)$, is called *cone of* f, and it is determined up to non-unique isomorphism (see TR3). When we do not need to make f explicit, we will also use the notation $\mathrm{Cone}(A \to B)$.

TR2 A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$$

is a distinguished triangle if and only if

$$B \xrightarrow{g} C \xrightarrow{h} \Sigma(A) \xrightarrow{-\Sigma(f)} \Sigma(B)$$

is a distinguished triangle.

TR3 For any two distinguished triangles, given a and b vertical arrows making the left square commutative as in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, dashed, "c"] & \Sigma(A) \arrow[d, "\Sigma(a)"] \\ D \arrow[r] & E \arrow[r] & F \arrow[r, "h"] & \Sigma(D), \end{tikzcd}$$

there exists c making the other two squares commutative. In general, such c is not unique.

TR4 Let $f : A \to B$ and $g : B \to C$ be two morphisms. From TR1, we get three distinguished triangles:

$$A \xrightarrow{f} B \xrightarrow{f'} C' \xrightarrow{s} \Sigma(A),$$

$$A \xrightarrow{gf} C \xrightarrow{h} B' \xrightarrow{r} \Sigma(A),$$

$$B \xrightarrow{g} C \xrightarrow{g'} A' \xrightarrow{t} \Sigma(B).$$

In this situation, the *octahedral axiom* requires that there exists a distinguished triangle

$$C' \xrightarrow{u} B' \xrightarrow{v} A' \xrightarrow{w} \Sigma(C')$$

making the following diagram commutative:

$$\begin{tikzcd}
A \arrow[rr, bend left, "gf"] \arrow[dr, "f"] & & C \arrow[rr, bend left, "g'"] \arrow[dr, "h"] & & A' \arrow[rr, bend left, "w"] \arrow[dr, "t"] & & \Sigma(C') \\
& B \arrow[ur, "g"] \arrow[dr, "f'"] & & B' \arrow[ur, "v"] \arrow[dr, "r"] & & \Sigma(B) \arrow[ur, "\Sigma(f')"] & \\
& & C' \arrow[ur, "u"] \arrow[rr, bend right, "s"] & & \Sigma(A) \arrow[ur, "\Sigma(f)"] & &
\end{tikzcd}$$

2.3. Definition. A *triangulated functor* between two triangulated categories $\mathscr{T}$ and $\mathscr{T}'$ is a couple (F,η) such that

- $F : \mathscr{T} \to \mathscr{T}'$ is a functor;
- $\eta : F\Sigma_{\mathscr{T}} \to \Sigma_{\mathscr{T}'}F$ is a natural isomorphism;
- This couple sends distinguished triangles to distinguished triangles:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$$

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \xrightarrow{\eta_A Fh} \Sigma(FA).$$

For the sake of simplicity, we will say that F is a triangulated functor whenever η is not explicitly needed. The identity functor id is the couple $(\mathrm{id}, \mathrm{id}_\Sigma)$.

For a triangulated functor $(F,\eta) : \mathscr{T} \to \mathscr{T}'$, we define $\eta^i : F\Sigma^i \to \Sigma^i F$ inductively as follows:

$$\begin{aligned}
&\eta^0 = \mathrm{id}, \quad \eta^1 = \eta, \quad \eta^{-1} = (\Sigma_{\mathscr{T}'}^{-1}\eta\Sigma_{\mathscr{T}}^{-1})^{-1},\\
&\text{for } i > 1, \quad \eta^i = \Sigma_{\mathscr{T}'}^{i-1}(\eta)\eta^{i-1}\Sigma_{\mathscr{T}}, \quad \eta^{-i} = \Sigma_{\mathscr{T}'}^{1-i}(\eta^{-1})\eta^{1-i}\Sigma_{\mathscr{T}}^{-1}.
\end{aligned}$$

Notice that, since Σ is an automorphism, η^{-i} is properly defined.

A *natural transformation of triangulated functors* $(F,\eta),(F',\eta') : \mathscr{T} \to \mathscr{T}'$ is a natural transformation $f : F \to F'$ such that

$$\begin{tikzcd}
F\Sigma_{\mathscr{T}} \arrow[r, "\eta"] \arrow[d, "f\Sigma_{\mathscr{T}}"] & \Sigma_{\mathscr{T}'}F \arrow[d, "\Sigma_{\mathscr{T}'}f"] \\
F'\Sigma_{\mathscr{T}} \arrow[r, "\eta'"] & \Sigma_{\mathscr{T}'}F'
\end{tikzcd}$$

commutes.

Given two composable triangulated functors (F,η) and (G,μ), we define the composition as $(GF,(\mu F)(G\eta))$.

2.4. Notation. It is common to indicate the shift functor Σ with $[1]$; more precisely, $A[1] := \Sigma(A)$ and $f[1] := \Sigma(f)$. In general, we may write $[n]$ instead of Σ^n for any $n \in \mathbb{Z}$.

We will almost always use the $[n]$-notation for shifts, but sometimes Σ allows us to state more clearly some properties.

2.5. Definition. A *triangulated subcategory* $\mathscr{S}$ of $\mathscr{T}$ is a full subcategory of $\mathscr{T}$ such that the inclusion $\mathscr{S} \subset \mathscr{T}$ is a triangulated functor.

An *extension closed subcategory* $\mathscr{E}$ of $\mathscr{T}$ is a full subcategory of $\mathscr{T}$ such that, whenever there exists a distinguished triangle $X \to Y \to Z \to X[1]$ with $X, Z \in \mathscr{E}$, also $Y \in \mathscr{E}$.

2.6. Definition. Let $\mathscr{A}$ be an additive category. A *complex* $M = (M^i, d_M^i)_{i\in\mathbb{Z}}$ is a sequence

$$\dots \longrightarrow M^{i-1} \xrightarrow{d_M^{i-1}} M^i \xrightarrow{d_M^i} M^{i+1} \longrightarrow \dots$$

of objects and morphisms of $\mathscr{A}$ such that $d_M^{i+1} d_M^i = 0$ for all integers i. A *chain map* between complexes $f : M \to N$ is a collection of morphisms $\{f^i : M^i \to N^i\}_{i\in\mathbb{Z}}$ such that $d_N^i f^i = f^{i+1} d_M^i$.

We say that two chain maps $f, g : M \to N$ are *homotopy equivalent* if there exists a collection of morphisms $\{h^i : M^i \to N^{i-1}\}_{i\in\mathbb{Z}}$ such that

$$f^i - g^i = d_N^{i-1} h^i + h^{i+1} d_M^i$$

for all i. The *homotopy category of complexes* $\mathcal{K}(\mathscr{A})$ is the category whose objects are complexes and morphisms are chain maps up to homotopy equivalence. By an abuse of notation, we will denote chain maps and homotopy classes of chain maps in the same way.

	bounded above		for $i \gg 0$
We say that a complex M is	*bounded below*	if $M^i = 0$	for $i \ll 0$
	bounded		for $\lvert i\rvert \gg 0$

The full subcategory of $\mathcal{K}(\mathscr{A})$ given by bounded above (resp. bounded below, bounded) complexes is denoted with $\mathcal{K}^-(\mathscr{A})$ (resp. $\mathcal{K}^+(\mathscr{A})$, $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$). We will use the notation $\mathcal{K}^*(\mathscr{A})$ as a short hand to refer to homotopy categories without restricting to a particular boundedness requirement. In particular, $\mathcal{K}(\mathscr{A})$ is also considered by setting $* = \varnothing$.

2.7. Remark. Notice that DG-modules and DG-morphisms of a ring R are respectively complexes and chain maps associated to the category $\mathrm{Mod}(R)$ of all R-modules.

2.8. Definition/Proposition. Let $\mathscr{A}$ be an additive category and consider the homotopy category $\mathcal{K}^*(\mathscr{A})$.

- For a complex M, the *shifted complex* $M[1]$ is described by $M[1]^i := M^{i+1}$ and $d_{M[1]}^i := -d_M^{i+1}$. Given a chain map f, the *shifted morphism* $f[1] : M[1] \to N[1]$ is defined by $f[1]^i := f^{i+1}$. This association is compatible with homotopy, hence it gives a shift functor on $\mathcal{K}^*(\mathscr{A})$.

- Let $f : M \to N$ a chain map. The *cone* of f is the complex

$$\mathrm{Cone}(f) := N \oplus M[1], \qquad d^i_{\mathrm{Cone}(f)} := \begin{pmatrix} d^i_N & f^i \\ 0 & d^i_{M[1]} \end{pmatrix}$$

Then $\mathcal{K}^*(\mathscr{A})$ is a triangulated category whose distinguished triangles are isomorphic to

$$M \xrightarrow{f} N \xrightarrow{\binom{\mathrm{id}}{0}} \mathrm{Cone}(f) \xrightarrow{(0\ \mathrm{id})} M[1]$$

(cf. [32, Lemma I.4.16, I.4.19 and Theorem XI.5.5]).

2.9. Definition/Proposition. [76, II.2.2.10]. Given a triangulated category $\mathscr{T}$ and a triangulated full subcategory $\mathscr{S}$, the *Verdier quotient* $\mathscr{T}/\mathscr{S}$ is a triangulated category, together with a triangulated functor $Q : \mathscr{T} \to \mathscr{T}/\mathscr{S}$, satisfying the following universal property: if $F : \mathscr{T} \to \mathscr{T}'$ is a triangulated functor such that $F(X) = 0$ for all $X \in \mathscr{S}$, then F factors uniquely through Q.

IDEA OF PROOF. In order to define $\mathscr{T}/\mathscr{S}$, we recall a construction of localization via multiplicative systems. A multiplicative system S is a selection of morphisms in $\mathscr{T}$ satisfying some properties (see [76, II.2.1.1] or [32, §XI.1]) such that we can create morphisms from X to Y by *(left) fractions*

$$X \xrightarrow{f} Z \xleftarrow{s} Y,$$

where Z is some object of $\mathscr{T}$ and $s \in S$. The category whose objects are the ones of $\mathscr{T}$ and whose morphisms are fractions is denoted with $S^{-1}\mathscr{T}$.[†]

Since the collection

$$S := \{f : X \to Y \mid \text{there exists a distinguished triangle } X \xrightarrow{f} Y \to Z \to X[1] \text{ with } Z \in \mathscr{S}\}$$

is a multiplicative system by [76, Proposition II.2.1.8], we simply define $\mathscr{T}/\mathscr{S} := S^{-1}\mathscr{T}$. □

2.10. Definition. A triangulated subcategory $\mathscr{S}$ of $\mathscr{T}$ is *thick* if it is closed under isomorphism and whenever $X \oplus Y \in \mathscr{S}$, then $X, Y \in \mathscr{S}$. For any subcategory $\mathscr{U} \subset \mathscr{T}$, we denote with $\mathrm{Thick}(\mathscr{U})$ the smallest thick subcategory of $\mathscr{T}$ containing $\mathscr{U}$.

2.11. Remark. Notice that the kernel $\mathscr{K}$ of a triangulated functor $F : \mathscr{T} \to \mathscr{T}'$, i.e. the full subcategory of $\mathscr{T}$ whose objects satisfy $F(X) = 0$, is a thick subcategory. In particular, in the case of a Verdier quotient $Q : \mathscr{T} \to \mathscr{T}/\mathscr{S}$, we have that $\mathscr{S} \subset \mathscr{K}$; by the universal property, $\mathscr{T}/\mathscr{K} \cong \mathscr{T}/\mathscr{S}$. Also, one can show that $\mathscr{K} = \mathrm{Thick}(\mathscr{S})$.

2.12. Remark. Let $\mathscr{S}$ be a thick subcategory of $\mathscr{T}$. Then a morphism $f : X \to Y$ in $\mathscr{T}$ is an isomorphism in $\mathscr{T}/\mathscr{S}$ if and only if $\mathrm{Cone}(f) \in \mathscr{S}$ (cf. [32, Proposition XI.1.7] and [76, Proposition II.2.1.8]).

[†] Analogously, one may define the category of right fractions. The resulting category is isomorphic to $S^{-1}\mathscr{T}$ because it is characterized by the same universal property [32, Proposition XI.1.5].

We conclude by recalling the concept of triangulated envelope.

2.13. Definition. Let $\mathscr{T}$ be a triangulated category and $\mathscr{S}$ a full subcategory. The *triangulated envelope* of $\mathscr{S}$, denoted with $\langle\mathscr{S}\rangle$, is the smallest full triangulated subcategory of $\mathscr{T}$ containing $\mathscr{S}$. For the sake of simplicity, we will say that $\mathscr{T}$ *is the triangulated envelope of* $\mathscr{S}$, and write $\mathscr{T} = \langle\mathscr{S}\rangle$, whenever $\langle\mathscr{S}\rangle$ is triangulated equivalent to $\mathscr{T}$ via inclusion.

2.14. Remark. In order to prove that some property holds for every object of $\langle\mathscr{S}\rangle$, we can use the following induction principle.
Base case The property holds for every finite direct sum of shifts of objects of $\mathscr{S}$.
Induction step If the property holds for X and Y, then it holds also for $\mathrm{Cone}(X \to Y)$.
This follows from the fact that all objects in $\langle\mathscr{S}\rangle$ are obtained by iterating cones.[‡]

§2.2. Derived categories of exact categories

The main reference for derived categories of exact categories is [11]. Here we present some results that are of interest for our studies, especially for §2.3.

2.15. Definition. Let $\mathscr{A}$ be an additive category. A *kernel-cokernel pair* (in $\mathscr{A}$) is a pair (i, p) of composable morphisms such that i is a kernel of p and p is a cokernel of i.

For a class of kernel-cokernel pairs E, we say that a morphism i is an *admissible monic* if there exists p such that $(i, p) \in E$. Dually, we define *admissible epic*. We say that a class of kernel-cokernel pairs E is an *exact structure* if it is closed under isomorphisms and the following axioms are satisfied:

1. All identities are admissible monics and admissible epics.
2. The composition of two admissible monics (resp. epics) is an admissible monic (resp. epic).
3. The push-out of an admissible monic along an arbitrary morphism exists and gives an admissible monic. The same can be said for pull-backs of admissible epics.

An *exact category* $\mathscr{E}$ is given by a couple $(\mathscr{A}, E)$, where $\mathscr{A}$ is an additive category and E is an exact structure on $\mathscr{A}$. A kernel-cokernel pair in $\mathscr{E}$ is called *conflation* and it is represented as a short exact sequence in the context of abelian categories, i.e. $0 \to A \to B \to C \to 0$. For the sake of simplicity, $\mathscr{E}$ will also denote the underlying additive category $\mathscr{A}$.

[‡]The interested reader may check this as an exercise. *Hint.* Consider the function $d_{\mathscr{S}} : \mathscr{T} \to \mathbb{N} \cup \{\infty\}$ defined by

$$d_{\mathscr{S}}(X) = \begin{cases} 0 & \text{if } X \text{ is a finite direct sum of shifts of objects in } \mathscr{S} \\ n & \text{if there exists a distinguished triangle } Y_1 \to Y_2 \to X \to Y_1[1] \\ & \quad \text{with } d_{\mathscr{S}}(Y_i) < n \text{ for } i = 1,2 \text{ and } d_{\mathscr{S}}(X) \not< n \\ \infty & \text{if } d_{\mathscr{S}}(X) \neq n \text{ for all } n \in \mathbb{N}. \end{cases}$$

2.16. Definition. A complex

$$\dots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^i \longrightarrow \dots$$

is *acyclic* if, for all $i \in \mathbb{Z}$, d^i factors through an object $C_i \in \mathscr{E}$ and $0 \to C_{i-1} \to X_i \to C_i \to 0$ is a conflation.

We choose the term "acyclic complex" instead of "long exact sequence" because the latter may recall the reader too much of the abelian situation, and we want to avoid confusion as exact categories have a more peculiar behaviour. Indeed, it is not always possible to define cohomology objects associated to complexes (see Definition 2.24).

2.17. Definition. The *derived category* $\mathcal{D}^*(\mathscr{E})$ of an exact category $\mathscr{E}$ is defined as the quotient $\mathcal{K}^*(\mathscr{E})/\operatorname{Ac}^*(\mathscr{E})$, where $\operatorname{Ac}^*(\mathscr{E})$ is the triangulated subcategory of $\mathcal{K}^*(\mathscr{E})$ given by acyclic complexes (see [57, Lemma 1.1]). A morphism in $\mathcal{K}^*(\mathscr{E})$ is a *quasi-isomorphism* if its cone belongs to $\operatorname{Thick}(\operatorname{Ac}^*(\mathscr{E}))$.

2.18. Remark.

1. An additive category with split exact sequences is exact, and the derived category associated to it is simply the homotopy category of complexes. This follows from the fact that split exact sequences are always homotopy equivalent to 0.
2. An abelian category with all its short exact sequences is exact, and the derived category associated is the expected one.
3. A morphism of $\mathcal{K}^*(\mathscr{E})$ becomes an isomorphism in $\mathcal{D}^*(\mathscr{E})$ if and only if it is a quasi-isomorphism; this follows from Remark 2.12.

Let us state a first preliminary result on distinguished triangles of derived categories.

2.19. Proposition. *Let $\mathscr{E}$ be an exact category. A conflation in $\mathscr{E}$ gives rise to a distinguished triangle in $\mathcal{D}^*(\mathscr{E})$.*

PROOF. Let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a conflation. Then g can be used to define a morphism $\operatorname{Cone}(f) \to Z$ in $\mathcal{K}^*(\mathscr{E})$, which is an isomorphism in $\mathcal{D}^*(\mathscr{E})$ because its cone is the conflation we started with. $\square$

The following generalizes the last part of the statement of [45, Lemma 3.1].

2.20. Lemma. ★. *Let $\mathscr{E}$ be an exact category. Then $\operatorname{Ac}^{\mathrm{b}}(\mathscr{E}) \subset \mathcal{K}^{\mathrm{b}}(\mathscr{E})$, the category of bounded acyclic complexes, is the triangulated envelope of the full subcategory given by conflations (intended as complexes).*

PROOF. Let $X = (X^i, d^i)$ be a bounded acyclic complex. Up to shift, we can assume $X^i = 0$ for $i < 1$ and $i > n$ for some $n > 3$. Let us write

$$\begin{aligned} X &:= \dots \to 0 \to X^1 \to X^2 \to \dots \qquad\qquad\qquad\qquad \to X^n \to 0 \to \dots \\ X^{\leq n-2} &:= \dots \to 0 \to X^1 \to X^2 \to \dots \to X^{n-2} \xrightarrow{p} \operatorname{coker}(d^{n-3}) \to 0 \to \dots \\ X^{\geq n-1}[-1] &:= \qquad\qquad\qquad\qquad \dots \to 0 \to \operatorname{coker}(d^{n-3}) \xrightarrow{j} X^{n-1} \xrightarrow{-d^{n-1}} X^n \to 0 \to \dots \end{aligned}$$

where the composition of $p : X^{n-2} \to \operatorname{coker}(d^{n-3})$ and $j : \operatorname{coker}(d^{n-3}) \to X^{n-1}$ gives $-d^{n-2}$.

We claim that $X \cong \operatorname{Cone}(f) =: Y$, where $f : X^{\geq n-1}[-1] \to X^{\leq n-2}$ is defined to be the identity on $\operatorname{coker}(d^{n-3})$ and 0 elsewhere. This will suffice to conclude the proof of the statement. Let us describe Y explicitly: it is the chain complex

$$\dots \to 0 \to X^1 \to X^2 \to \dots \to X^{n-3} \to X^{n-2} \oplus \operatorname{coker}(d^{n-3}) \to \operatorname{coker}(d^{n-3}) \oplus X^{n-1} \to X^n \to 0 \dots$$

with differential

$$d_Y^i := \begin{cases} \begin{pmatrix} d^{n-3} \\ 0 \end{pmatrix} & \text{if } i = n-3 \\ \begin{pmatrix} p & \mathrm{id} \\ 0 & -j \end{pmatrix} & \text{if } i = n-2 \\ \begin{pmatrix} 0 & d^{n-1} \end{pmatrix} & \text{if } i = n-1 \\ d^i & \text{otherwise.} \end{cases}$$

We now define chain maps $h : X \to Y$ and $g : Y \to X$ as follows:

$$h^i := \begin{cases} \begin{pmatrix} \mathrm{id} \\ -p \end{pmatrix} & \text{if } i = n-2 \\ \begin{pmatrix} 0 \\ -\mathrm{id} \end{pmatrix} & \text{if } i = n-1 \\ -\mathrm{id} & \text{if } i = n \\ \mathrm{id} & \text{otherwise} \end{cases}, \qquad g^i := \begin{cases} \begin{pmatrix} \mathrm{id} & 0 \end{pmatrix} & \text{if } i = n-2 \\ \begin{pmatrix} -j & -\mathrm{id} \end{pmatrix} & \text{if } i = n-1 \\ -\mathrm{id} & \text{if } i = n \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

Notice that $gh = \mathrm{id}$, so it remains to prove that $hg \cong \mathrm{id}$ up to homotopy; this is true using the maps $\varphi^i : Y^i \to Y^{i-1}$ such that $\varphi^i = 0$ for all $i \neq n-1$ and $\varphi^{n-1} := \begin{pmatrix} 0 & 0 \\ -\mathrm{id} & 0 \end{pmatrix}$. □

We now wonder when quasi-isomorphisms are completely described by acyclic complexes, i.e. when acyclic complexes give thick subcategories.

2.21. Definition. Let $\mathscr{A}$ be an additive category. An *idempotent* is a morphism $e : X \to X$, such that $e^2 = e$. We say that $\mathscr{A}$ is *idempotent complete* if for every idempotent $e : X \to X$ there exist $s : Y \to X$ and $r : X \to Y$ such that $e = sr$ and $rs = \mathrm{id}_Y$.

An additive category is *weakly idempotent complete* if one of the following equivalent requirements hold:

1. Every retraction has a kernel;
2. Every section has a cokernel.

2.22. Remark. Given any exact category $\mathscr{E}$, we can consider its idempotent closure $\mathscr{E}^{ic}$ and its weak idempotent completion $\mathscr{E}^{wic}$. Any object in $\mathscr{E}^{ic}$ is a direct summand of an object in $\mathscr{E}$; this is clear from the description given in [36, §1.2]. The same is true also for $\mathscr{E}^{wic}$ (see [11, Remark 7.8]), since it is contained in $\mathscr{E}^{ic}$.

It is important to remember that the inclusions $\mathscr{E} \to \mathscr{E}^{wic}$ and $\mathscr{E} \to \mathscr{E}^{ic}$ reflect exactness, meaning that a conflation in the target with objects in $\mathscr{E}$ is still a conflation in $\mathscr{E}$. In addition, their essential image is closed under extensions, i.e. if $0 \to A \to B \to C \to 0$ is a conflation such that A, C are in the essential image of the inclusion, then also B is.

2.23. Proposition. [11, Corollary 10.11 and Proposition 10.14]. *Let $\mathscr{E}$ be an exact category. Then:*

- *$\mathscr{E}$ is weakly idempotent complete if and only if $\mathrm{Ac}^{\mathrm{b}}(\mathscr{E})$ is thick in $\mathcal{K}^{\mathrm{b}}(\mathscr{E})$, or if and only if $\mathrm{Ac}^*(\mathscr{E})$ is thick in $\mathcal{K}^*(\mathscr{E})$ for both $* = +$ and $* = -$.*
- *$\mathscr{E}$ is idempotent complete if and only if $\mathrm{Ac}(\mathscr{E})$ is thick in $\mathcal{K}(\mathscr{E})$.*

2.24. Definition. An *admissible morphism* $f : X \to Y$ in an exact category is a morphism admitting a decomposition $f = ip$, with i admissible monic and p admissible epic. Accordingly, a complex $(X^i, d^i)_i$ is *admissible* if d^i is admissible for each $i \in \mathbb{Z}$.

Notice that any acyclic complex is admissible, and any admissible morphism admits kernel, cokernel and image.

For an admissible complex X, if $\mathrm{im}(d^{i-1}) \to \ker(d^i)$ is admissible we define the *i-th cohomology object*

$$H^i(X) := \frac{\ker(d^i)}{\mathrm{im}(d^{i-1})} = \mathrm{coker}(\mathrm{im}(d^{i-1}) \to \ker(d^i)) \in \mathscr{E}.$$

2.25. Remark. ★. Cohomology objects of admissible complexes are always defined (according to the definition above) if and only if we are working on a weakly idempotent complete exact category.

This follows from the following characterization: an exact category $\mathscr{E}$ is weakly idempotent complete if and only if any morphism j for which we can find a commutative diagram

$$\begin{tikzcd} A \arrow[rr, hook, "ij"] \arrow[dr, "j"'] & & C, \\ & B \arrow[ur, hook, "i"'] & \end{tikzcd}$$

where $\hookrightarrow$ indicates admissible monics, is an admissible monic. This is the dual statement of [11, Corollary 7.7]

Indeed, given any admissible complex $X = (X^i, d^i)_i$ in a weakly idempotent complete exact category $\mathscr{E}$, we have the diagram above with $A = \mathrm{im}(d^{i-1})$, $B = \ker(d^i)$ and $C = X^i$. Since $j : \mathrm{im}(d^{i-1}) \to \ker(d^i)$ is admissible, $H^i(X)$ exists.

Conversely, if $\mathscr{E}$ is an exact category which is not weakly idempotent complete, we can find a monic $j : A \to B$ which is not admissible and satisfies the diagram above for some $i : B \to C$. Then the complex $A \to C \to \mathrm{coker}(i)$ is clearly admissible, but by construction we cannot compute its cohomology.

The following remark is of utter importance, as it allows to restrict our studies to (weakly) idempotent complete categories without loss of generality.

2.26. Remark. By inclusions, we obtain that $\mathcal{D}^{\mathrm{b}}(\mathscr{E}) \cong \mathcal{D}^{\mathrm{b}}(\mathscr{E}^{wic})$ and $\mathcal{D}^{*}(\mathscr{E}) \cong \mathcal{D}^{*}(\mathscr{E}^{ic})$ for $* = +, -, \varnothing$ (see [57, Remark 1.12]). The notation is the one used in Remark 2.22.

2.27. Lemma. ★. *Let $\mathscr{E}$ be a weakly idempotent complete exact category and $* = \mathrm{b}, +, -$. Any complex $X \in \mathcal{D}^{*}(\mathscr{E})$ admitting a quasi-isomorphism $B[n] \to X$, with $B \in \mathscr{E}$, is admissible. More precisely, its cohomology objects are well-defined and*

$$H^{i}(X) = \begin{cases} B & \text{if } i = -n \\ 0 & \text{otherwise}. \end{cases}$$

The same is true in $\mathcal{D}(\mathscr{E})$ if $\mathscr{E}$ is idempotent complete.

PROOF. We work under the assumption that $\mathrm{Ac}^{*}(\mathscr{E})$ is thick in $\mathcal{K}^{*}(\mathscr{E})$; Proposition 2.23 shows us the reason why we have two different assumptions in the case of $* = \mathrm{b}, +, -$ and $* = \varnothing$.

For the sake of simplicity, assume $n = -1$. Since $s : B[-1] \to X$ is a quasi-isomorphism, $\mathrm{Cone}(s)$ is acyclic. Moreover, $\mathrm{Cone}(s)^{i} = X^{i}$ for $i \neq 0$ and $d^{i}_{\mathrm{Cone}(s)} = d^{i}$ for $i \neq -1, 0$ (d denotes the differential of X). We also notice that $d^{-1}_{\mathrm{Cone}(s)} = \left(\begin{smallmatrix} d^{-1} \\ 0 \end{smallmatrix}\right)$, so $\operatorname{im} d^{-2} = \operatorname{im} d^{-2}_{\mathrm{Cone}(s)} = \ker d^{-1}_{\mathrm{Cone}(s)} = \ker d^{-1}$. We therefore have the statement for all degrees but $i = 0, 1$, since acyclicity implies admissibility. Considering $\mathrm{Cone}(s)$, we have

$$\begin{tikzcd} X^{-1} \arrow[rr, "\left(\begin{smallmatrix} d^{-1} \\ 0 \end{smallmatrix}\right)"] \arrow[dr, two heads] & & X^{0} \oplus B \arrow[rr, "(d^{0}\ s)"] \arrow[dr, two heads] & & X^{1}. \\ & K^{0} \arrow[ur, hook] & & K^{1} \arrow[ur, hook] & \end{tikzcd}$$

Notice that the natural projection $X^{0} \oplus B \twoheadrightarrow B$ factors through $K^{1} = \operatorname{coker} \left(\begin{smallmatrix} d^{-1} \\ 0 \end{smallmatrix}\right)$, from which we obtain a decomposition of the identity of B through K^{1}. Since $\mathscr{E}$ is weakly idempotent complete, we immediately get that $K^{1} = B \oplus K^{1}_{X}$ for some object $K^{1}_{X} \in \mathscr{E}$. Therefore, the conflation $K^{0} \to X^{0} \oplus B \to K^{1}$ can be divided in two direct summands: one is given by the identity of B, while the other is $K^{0} \to X^{0} \to K^{1}_{X}$. The latter is a conflation by [11, Corollary 2.18], so that $H^{0}(X) = 0$. Since $K^{1}_{X} \to K^{1}$ is a split monomorphism, X is admissible and the cohomology is exactly as expressed in the statement (in particular, $H^{1}(X) = \operatorname{coker}(K^{1}_{X} \to K^{1}) = B$). □

2.28. Remark. Under the same conditions, Lemma 2.27 holds true also for quasi-isomorphisms $X \to B[n]$.

§2.3. Yoneda extensions in triangulated categories

In the theory of abelian categories, it is well-known that Yoneda extensions are associated to hom-sets in the derived categories. As a matter of fact, the whole idea can be extended to exact categories, and the expected result holds also in this wider generality (see Corollary 2.42). We will conclude by showing a generalization of [16, Corollary 2.8].

2.29. Definition – Yoneda extensions. Let $\mathscr{E}$ be an exact category. The elements of $\mathrm{Ext}^n(A,B)$ are *n-extensions* for $n > 0$, i.e. classes of acyclic complexes

$$\mathbf{X} : 0 \to B \to X_1 \to \cdots \to X_n \to A \to 0$$

under the equivalence relation generated by identifying two acyclic complexes $\mathbf{X}$, $\mathbf{Y}$ if there is a family of morphisms $\psi = \{\psi_1, \dots, \psi_n\}$ satisfying the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{id}"] & X_1 \arrow[r] \arrow[d, "\psi_1"] & \dots \arrow[r] & X_n \arrow[r] \arrow[d, "\psi_n"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & B \arrow[r] & Y_1 \arrow[r] & \dots \arrow[r] & Y_n \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

(cf. [24, Theorem III.5.5]). For $n = 0$, $\mathrm{Ext}^0(A,B) = \mathrm{Hom}(A,B)$.

The *Yoneda product* is given by maps $Y^{n,m}_{A,B,C} : \mathrm{Ext}^n(A,B) \times \mathrm{Ext}^m(B,C) \to \mathrm{Ext}^{n+m}(A,C)$ for any $n,m \geq 0$ and any $A,B,C \in \mathscr{A}$. For $n,m \geq 1$, the Yoneda product is the map

$$\begin{gathered} \big(\mathbf{X} : 0 \to B \to X_1 \to \cdots \to X_n \to A \to 0\ ,\ \mathbf{Y} : 0 \to C \to Y_1 \to \cdots \to Y_m \to B \to 0\big) \\ \big\downarrow \\ \mathbf{Y} \cdot \mathbf{X} : 0 \to C \to Y_1 \to \cdots \to Y_m \to X_1 \to \cdots \to X_n \to A \to 0. \end{gathered}$$

If $n = m = 0$, the product is simply the composition of maps. The case $n > 0$ and $m = 0$ requires a more sophisticated definition. If $n = 1$, let $X_1 \in \mathrm{Ext}^1(A,B)$ and $g : B \to C$. Then $g \cdot X_1$ is described by the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "g"] & X_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & C \arrow[r] & g \cdot X_1 \arrow[r] & A \arrow[r] & 0, \end{tikzcd} \tag{2.30}$$

where $g \cdot X_1$ is the pushout of g and $B \to X_1$. For $n > 1$, considered an n-extension

$$\mathbf{X} : 0 \to B \to X_1 \to X_2 \to \cdots \to X_n \to A \to 0$$

and $g : B \to C$, the Yoneda product is given by substituting $0 \to B \to X_1$ with $0 \to C \to g \cdot X_1$:

$$g \cdot \mathbf{X} : 0 \to C \to g \cdot X_1 \to X_2 \to \cdots \to X_n \to A \to 0.$$

Dually, one can describe the case $n = 0$ and $m > 0$.

We can equip $\mathrm{Ext}^n(A,B)$ with a structure of abelian group via the *Baer sum*, described as follows. Let $\mathbf{X}, \mathbf{Y} \in \mathrm{Ext}^n(A,B)$. Consider the direct sum of the n-extensions

$$\mathbf{X} \oplus \mathbf{Y} : 0 \to B \oplus B \to X_1 \oplus Y_1 \to \cdots \to X_n \oplus Y_n \to A \oplus A \to 0,$$

the diagonal map $\Delta_A = \binom{\mathrm{id}}{\mathrm{id}} : A \to A \oplus A$ and the codiagonal map $\nabla_B = (\mathrm{id}\ \mathrm{id}) : B \oplus B \to B$. Then the Baer sum is given by $\mathbf{X} + \mathbf{Y} := \nabla_B \cdot (\mathbf{X} \oplus \mathbf{Y}) \cdot \Delta_A$.

The neutral element of the Baer sum is the n-extension represented by

$$\mathbf{0} : 0 \to B \overset{\mathrm{id}}{\to} B \to 0 \to \cdots \to 0 \to A \overset{\mathrm{id}}{\to} A \to 0.$$

In the case of $\mathrm{Ext}^1(A,B)$, this reduces to the split exact sequence $0 \to B \to B \oplus A \to A \to 0$. The opposite of $\mathbf{X}$ is simply given by choosing the opposite of the first morphism $B \to X_1$.[§]

The *(absolute) homological dimension of* $\mathscr{E}$, denoted by $\dim \mathscr{E}$, is the greatest integer n such that $\mathrm{Ext}^n(A,B) \neq 0$ for some $A, B \in \mathscr{E}$. If such an integer does not exist, we will set $\dim \mathscr{E} := \infty$.

2.31. Proposition. ★. [65, §A.7]. *Let $\mathscr{E}$ be an exact category, $* = \mathrm{b}, +, -, \varnothing$. Given $A, B \in \mathscr{E}$,*

- *if $n < 0$, $\mathrm{Hom}_{\mathcal{D}^*(\mathscr{E})}(A, B[n]) = 0$;*
- *if $n > 0$, every morphism in $\mathrm{Hom}_{\mathcal{D}^*(\mathscr{E})}(A, B[n])$ is associated to an n-extension in a natural way. More explicitly, we have a family of isomorphisms $\mathrm{Ext}^n(A,B) \to \mathrm{Hom}_{\mathcal{D}^*(\mathscr{E})}(A, B[n])$ which, a posteriori, will satisfy item 4 of Proposition 2.35.*

PROOF. First, let us restrict to the case of weakly idempotent complete exact categories if $* = \mathrm{b}, +, -$ or idempotent complete if $* = \varnothing$. We recall that a morphism $f : A \to B[n]$ is represented by a fraction

$$A \xrightarrow{\ g\ } X \xleftarrow{\ s\ } B[n]$$

where g is a morphism in $\mathcal{K}^*(\mathscr{E})$ and s is a quasi-isomorphism. By Lemma 2.27, even if truncation functors are not defined in general, such X can be truncated as in the case of abelian categories. Indeed, in $\mathcal{D}^*(\mathscr{E})$ the definition of truncation is well-defined for admissible complexes. This suffices to conclude that the reasonings of [76, Proposition III.1.2.10] and [32, Proposition XI.4.5-8] can be applied in this case.

For the general case, notice that from Remark 2.26, we are reduced to show that an n-extension of two objects of $\mathscr{E}$ in $\mathscr{E}^{ic}$ (which denotes the idempotent closure of $\mathscr{E}$) always admits a representation by an acyclic complex of $\mathscr{E}$. Let $A, B \in \mathscr{E}$ and consider the acyclic complex $\mathbf{X}$

$$0 \longrightarrow B \longrightarrow X_1 \xrightarrow{\ \xi_1\ } X_2 \xrightarrow{\ \xi_2\ } \dots \xrightarrow{\ \xi_{n-1}\ } X_n \longrightarrow A \longrightarrow 0.$$

with $X_i \in \mathscr{E}^{ic}$ for all i. As observed in Remark 2.22, $\mathrm{im}\,\xi_1$ is a direct summand of an object of $\mathscr{E}$. Let Y_1 such that $\mathrm{im}\,\xi_1 \oplus Y_1 \in \mathscr{E}$. Since the essential image of $\mathscr{E}$ is closed under extensions in $\mathscr{E}^{ic}$ and the inclusion reflects exactness, $X_1 \oplus Y_1 \in \mathscr{E}$. Analogously, if we consider the conflation $0 \to \mathrm{im}\,\xi_1 \oplus Y_1 \to X_2 \oplus Y_1 \to \mathrm{im}\,\xi_2 \to 0$, we can find Y_2 such that $\mathrm{im}\,\xi_2 \oplus Y_2 \in \mathscr{E}$, so that $X_2 \oplus Y_1 \oplus Y_2 \in \mathscr{E}$. By induction, we define $Z_i := X_i \oplus Y_{i-1} \oplus Y_i$, where Y_i is obtained as above for

[§]This fact becomes easy to check after Corollary 2.42, in view of the morphisms defined in Proposition 2.35. We also remark that the opposite of $\mathbf{X}$ can also be represented by changing the sign of any morphism occurring in $\mathbf{X}$, or, even more generally, by changing the sign for an odd number of morphisms.

$i = 1, \dots n-1$, while $Y_0 = Y_n = 0$. We also define

$$\mu_i = \begin{pmatrix} \xi_i & 0 & 0 \\ 0 & 0 & \mathrm{id} \\ 0 & 0 & 0 \end{pmatrix} : Z^i \to Z^{i+1}.$$

for $i = 1, \dots, n-1$. By these choices, we obtain an acyclic complex $\mathbf{Z}$ of $\mathscr{E}$ which represents the class of $\mathbf{X}$ by setting $\begin{pmatrix} \mathrm{id} & 0 & 0 \end{pmatrix} : Z_i \to X_i$ for all $i = 1, \dots, n$. □

2.32. Definition/Proposition. *[23].* Let $\mathscr{E}$ be a full extension closed additive subcategory of a triangulated category $\mathscr{T}$ such that $\mathrm{Hom}(A, B[-1]) = 0$ for any $A, B \in \mathscr{E}$.

Then $\mathscr{E}$ has a natural exact structure, given by defining $0 \to A \to B \to C \to 0$ a conflation if $A \to B \to C \to A[1]$ is a distinguished triangle in $\mathscr{T}$ for some $C \to A[1]$. This association gives rise to a natural isomorphism $\mathrm{Ext}^1_{\mathscr{E}}(A, B) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{T}}(A, B[1])$ for all $A, B \in \mathscr{E}$.

When $\mathscr{E}$ is considered with this natural exact structure, we say that $\mathscr{E}$ is *Dyer* (in $\mathscr{T}$).

In particular, whenever $\mathscr{E}$ is a Dyer exact subcategory of $\mathscr{T}$, the *homological dimension of $\mathscr{E}$ in $\mathscr{T}$*, denoted by $\dim_{\mathscr{T}}(\mathscr{E})$, is the greatest integer n such that $\mathrm{Hom}(A, B[n]) \neq 0$ for some $A, B \in \mathscr{E}$. If such an integer does not exist, we set $\dim_{\mathscr{T}}(\mathscr{E}) := \infty$.

2.33. Definition. A full subcategory $\mathscr{E}$ of a triangulated category $\mathscr{T}$ is called *admissible* if $\mathrm{Hom}(A, B[n]) = 0$ for all $n < 0$. Whenever it is extension closed, by Definition/Proposition 2.32 we will say that $\mathscr{E}$ is *admissible exact*. In particular, $\mathscr{E}$ is *admissible abelian* if the exact structure gives rise to an abelian category (cf. [30, §. 2]).

2.34. Example. By Proposition 2.19 and Proposition 2.31, any exact category $\mathscr{E}$ is admissible exact in $\mathcal{D}^*(\mathscr{E})$.

2.35. Proposition. ★. [65, §A.8]. *Let $\mathscr{E}$ be a Dyer exact subcategory of a triangulated category $\mathscr{T}$. Then there is a well-defined map $f_{n,A,B} : \mathrm{Ext}^n(A, B) \to \mathrm{Hom}_{\mathscr{T}}(A, B[n])$ for any $A, B \in \mathscr{E}$ and $n \geq 0$. The following facts are true.*

1. *The image of $f_{n,A,B}$ is given by all the maps $A \to B[n]$ factoring as*

$$A \to C_{n-1}[1] \to \cdots \to C_1[n-1] \to B[n]$$

   for some $C_i \in \mathscr{E}$, $i \in \{1, \dots, n-1\}$.
2. *The Yoneda product is sent to composition as expected: therefore, $f_{n,-,-}$ is a natural transformation and $f_{n,A,B}$ is a group homomorphism with respect to the Baer sum on* $\mathrm{Ext}^n(A, B)$.
3. *If $f_{n-1,A,B}$ is an isomorphism for any $B \in \mathscr{E}$, then $f_{n,A,B}$ is injective.*
4. *Let $g_{n,A,B} : \mathrm{Ext}^n(A, B) \to \mathrm{Hom}_{\mathscr{T}}(A, B[n])$ be a map for any $n \geq 0$ and $A, B \in \mathscr{E}$. If $g_{0,A,B}$ is the isomorphism induced by the inclusion $\mathscr{E} \subset \mathscr{T}$, $g_{1,A,B}$ is the natural isomorphism of Definition/Proposition 2.32, and the Yoneda product is sent to the composition, then $g_{n,A,B} = f_{n,A,B}$.*

PROOF. For $n = 0$, $f_{0,A,B} : \mathrm{Hom}_{\mathscr{E}}(A,B) \to \mathrm{Hom}_{\mathscr{T}}(A,B)$ is an isomorphism since $\mathscr{E}$ is a full subcategory of $\mathscr{T}$. Let $n > 0$ and consider $\mathbf{X}$ an acyclic complex

$$0 \longrightarrow B \xrightarrow{\xi_0} X_1 \xrightarrow{\xi_1} X_2 \xrightarrow{\xi_2} \dots \xrightarrow{\xi_{n-1}} X_n \xrightarrow{\xi_n} A \longrightarrow 0.$$

To $\mathbf{X}$ we can associate conflations

$$\begin{array}{c} 0 \longrightarrow B = \operatorname{im}\xi_0 \longrightarrow X_1 \longrightarrow \operatorname{im}\xi_1 \longrightarrow 0 \\ 0 \longrightarrow \operatorname{im}\xi_1 \longrightarrow X_2 \longrightarrow \operatorname{im}\xi_2 \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow \operatorname{im}\xi_{n-1} \longrightarrow X_n \longrightarrow \operatorname{im}\xi_n = A \longrightarrow 0 \end{array}$$

which are associated to distinguished triangles. Therefore, we define $f_{n,A,B}(\mathbf{X})$ to be the map

$$A \to \operatorname{im}\xi_{n-1}[1] \to \cdots \to \operatorname{im}\xi_2[n-2] \to \operatorname{im}\xi_1[n-1] \to B[n]$$

obtained by the composition of the morphisms appearing in the distinguished triangles. We need to show that if $(\mathbf{X},\xi)$ and $(\mathbf{Y},\eta)$ give the same n-extension, then the associated map $A \to B[n]$ obtained is the same. Without loss of generality, assume there is a family of morphisms ψ as in Definition 2.29. Then for each $i \in \{0,\dots,n-1\}$ we have

$$\begin{tikzcd} \operatorname{im}\xi_i \arrow[r] \arrow[d, "\varphi_i"] & X_{i+1} \arrow[r] \arrow[d, "\psi_{i+1}"] & \operatorname{im}\xi_{i+1} \arrow[r] \arrow[d, "\varphi_{i+1}"] & \operatorname{im}\xi_i[1] \arrow[d, "\varphi_i[1]"] \\ \operatorname{im}\eta_i \arrow[r] & Y_{i+1} \arrow[r] & \operatorname{im}\eta_{i+1} \arrow[r] & \operatorname{im}\eta_i[1], \end{tikzcd}$$

where φ_i is obtained by the universal property of the kernel. In order to prove that the middle square is commutative, we notice that

$$\begin{aligned} X_{i+1} \to \operatorname{im}\xi_{i+1} \to \operatorname{im}\eta_{i+1} \hookrightarrow Y_{i+2} &= X_{i+1} \to \operatorname{im}\xi_{i+1} \to X_{i+2} \to Y_{i+2} \\ &= X_{i+1} \to Y_{i+1} \to Y_{i+2} \\ &= X_{i+1} \to Y_{i+1} \to \operatorname{im}\eta_{i+1} \hookrightarrow Y_{i+2}, \end{aligned}$$

so $X_{i+1} \to \operatorname{im}\xi_{i+1} \to \operatorname{im}\eta_{i+1} = X_{i+1} \to Y_{i+1} \to \operatorname{im}\eta_{i+1}$. Since φ_{i+1} is the only one making the middle square commutative by the universal property of the cokernel, TR3 entails that also the right-hand square is commutative.

We obtain a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi_n"] & \operatorname{im}\xi_{n-1}[1] \arrow[r] \arrow[d, "\varphi_{n-1}[1]"] & \dots \arrow[r] & \operatorname{im}\xi_2[n-2] \arrow[r] \arrow[d, "\varphi_2[n-2]"] & \operatorname{im}\xi_1[n-1] \arrow[r] \arrow[d, "\varphi_1[n-1]"] & B[n] \arrow[d, "\varphi_0[n]"] \\ A \arrow[r] & \operatorname{im}\eta_{n-1}[1] \arrow[r] & \dots \arrow[r] & \operatorname{im}\eta_2[n-2] \arrow[r] & \operatorname{im}\eta_1[n-1] \arrow[r] & B[n], \end{tikzcd}$$

where $\varphi_n = \mathrm{id}$ and $\varphi_0 = \mathrm{id}$, so that the rows are in fact the same map. This gives the well-definition of every $f_{n,A,B}$.

1. Let us consider a map $\alpha : A \to B[n]$ factoring through $A = C_n \to C_{n-1}[1] \to \cdots \to C_1[n-1] \to C_0[n] = B[n]$. To any $C_i[-1] \to C_{i-1}$, we can associate a cone, which is in $\mathscr{E}$ because it is Dyer. Let us call such cone X_i. We have the following conflations: $0 \to C_{i-1} \to X_i \to C_i \to 0$. Since C_i is also the kernel of $X_{i+1} \to C_{i+1}$, we manage to create an acyclic complex
$$0 \to B \to X_1 \to X_2 \to \cdots \to X_n \to A \to 0.$$
It is easy to notice that such acyclic complex is associated to the map $\alpha : A \to B[n]$ via $f_{n,A,B}$.
2. In the case of Ext^n and Ext^m with $n, m > 0$, the Yoneda product is sent to composition with a reasoning similar to item 1. Therefore, it suffices to show it is true when either m or n is zero. First, we recall that $f_{1,A,B}$ is exactly the map considered in Definition/Proposition 2.32, which is a natural transformation for both entries. So (2.30) can be translated to
$$\begin{tikzcd} B \arrow[r] \arrow[d, "g"] & X_1 \arrow[r] \arrow[d] & A \arrow[r, "h"] \arrow[d, "\mathrm{id}"] & B[1] \arrow[d, "g[1]"] \\ C \arrow[r] & g \cdot X_1 \arrow[r] & A \arrow[r, "g[1]h"] & C[1] \end{tikzcd} \tag{2.36}$$
in $\mathscr{T}$. Let us prove that $f_{n,A,-}$ is a natural transformation, the proof of $f_{n,-,B}$ being dual. For a general n-extension
$$\mathbf{X} : 0 \to B \to X_1 \to X_2 \to \cdots \to X_n \to A \to 0$$
and $g : B \to C$, the map $A \to C[n]$ associated to $g \cdot \mathbf{X}$ factors through $K[n-1] \to C[n]$, where $K = \mathrm{im}(g \cdot X_1 \to X_2) = \mathrm{im}(X_1 \to X_2)$, according to (2.36). Furthermore, the same diagram shows that $K \to C[1]$ is obtained as a composition $K \to B[1] \to C[1]$, where the latter morphism is $g[1]$. Therefore, $A \to C[n]$ can be written as the composition of $A \to B[n]$, obtained by $\mathbf{X}$, and $g[n] : B[n] \to C[n]$, as wanted.
Finally, $f_{n,-,-}$ is a natural transformation for both entries A and B. Moreover, $f_{n,A,B}$ is a group homomorphism since the Baer sum of two extensions is given by Yoneda products as explained in Definition 2.29.
3. We want to show that the zero map $A \to B[n]$ is associated to only one equivalence class of extensions, i.e. the neutral element of the Baer sum, whenever $f_{n-1,A,X}$ is an isomorphism for any $X \in \mathscr{E}$. Let us consider
$$\mathbf{X} : 0 \to B \to X_1 \to X_2 \to \cdots \to X_n \to A \to 0$$
such that $f_{n,A,B}(\mathbf{X}) = 0$ and the associated factorization
$$A \to C_{n-1}[1] \to \cdots \to C_2[n-2] \to C_1[n-1] \to B[n].$$

We have the following diagram, where the rows are distinguished triangles:

$$\begin{tikzcd} A \arrow[r, "0"] \arrow[d] & B[n] \arrow[r] \arrow[d, "\mathrm{id}"] & B[n]\oplus A[1] \arrow[r] \arrow[d] & A[1] \arrow[d] \\ C_1[n-1] \arrow[r] & B[n] \arrow[r] & X_1[n] \arrow[r, "g[n]"] & C_1[n] \end{tikzcd} \tag{2.37}$$

Now we pick the map $A[1] \to B[n] \oplus A[1] \to X_1[n]$. Since f_{n-1,A,X_1} is surjective, we get that $A \to X_1[n-1]$ is associated to an acyclic complex

$$\mathbf{Y} : 0 \to X_1 \to Y_1 \to \cdots \to Y_{n-1} \to A \to 0.$$

Composing $\mathbf{Y}$ with $0 \to B \to X_1 \oplus B \to X_1 \to 0$, we have the following:
(2.38)

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "\binom{0}{\mathrm{id}}"] \arrow[d, "\mathrm{id}"] & X_1\oplus B \arrow[r] \arrow[d, "(\mathrm{id},\iota)"] & Y_1 \arrow[r] & \dots \arrow[r] & Y_{n-1} \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & B \arrow[r, "\iota"] & X_1 \arrow[r] & X_2 \arrow[r] & \dots \arrow[r] & X_n \arrow[r] & A \arrow[r] & 0. \end{tikzcd}$$

We want to prove there are maps $Y_i \to X_{i+1}$ making every square of the diagram above commutative, so that the two rows represent the same n-extension. We consider the sequences starting at X_1 and C_1 respectively (remember that C_1 is the image of $X_1 \to X_2$). The Yoneda product of $\mathbf{Y}$ and $g : X_1 \to C_1$ gives us $g \cdot \mathbf{Y}$, whose associated map $A \to X_1[n-1] \to C_1[n-1]$ factors as $A \to C_{n-1}[1] \to \cdots \to C_1[n-1]$ because of the right-hand commutative square in (2.37). Since f_{n-1,A,C_1} is injective by assumption, we know that $g \cdot \mathbf{Y}$ is in the same equivalence class of

$$\mathbf{X}' : 0 \to C_1 \to X_2 \to \cdots \to X_n \to A \to 0.$$

Therefore, we can assume, up to equivalence, that $\mathbf{X}$ is in fact

$$0 \to B \to X_1 \to g \cdot Y_1 \to Y_2 \to \cdots \to Y_{n-1} \to A \to 0,$$

since it is obtained by the Yoneda product of $\mathbf{X}'$ and $0 \to B \to X_1 \to C_1 \to 0$. With this assumption, (2.38) can be completed with maps $Y_i \to X_{i+1}$ as wanted: the first morphism is given according to (2.30), while all the others are the identity. It remains to show that the equivalence class of

$$0 \to B \to X_1 \oplus B \to Y_1 \to \cdots \to Y_{n-1} \to A \to 0$$

is the one associated to the neutral element of the Baer sum, which is obvious because the diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "\binom{0}{\mathrm{id}}"] \arrow[d, "\mathrm{id}"] & X_1\oplus B \arrow[r] \arrow[d, "(0\ \mathrm{id})"] & Y_1 \arrow[r] \arrow[d] & \dots \arrow[r] & Y_{n-2} \arrow[r] \arrow[d] & Y_{n-1} \arrow[r, "\pi"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & B \arrow[r, "\mathrm{id}"] & B \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & A \arrow[r, "\mathrm{id}"] & A \arrow[r] & 0 \end{tikzcd}$$

commutes.

4. Let $g_{n,A,B}$ as in the statement and assume by induction that $g_{m,C,D} = f_{m,C,D}$ for any $m < n$ and $C, D \in \mathscr{E}$. We consider $\mathbf{X} \in \mathrm{Ext}^n(A,B)$ given by

$$0 \to B \to X_1 \xrightarrow{\xi_1} X_2 \to \cdots \to X_n \to A \to 0.$$

Such an extension can be split into two shorter extensions:

$$\begin{aligned}\mathbf{X}_1 : &\quad 0 \to B \to X_1 \to \mathrm{im}(\xi_1) \to 0\\ \mathbf{X}_2 : &\quad 0 \to \mathrm{im}(\xi_1) \to X_2 \to \cdots \to X_n \to A \to 0.\end{aligned}$$

Moreover, $\mathbf{X}_1 \cdot \mathbf{X}_2 = \mathbf{X}$. As $g_{n,A,B}$ sends Yoneda product to composition, we have

$$\begin{aligned}g_{n,A,B}(\mathbf{X}) &= g_{n,A,B}(\mathbf{X}_1 \cdot \mathbf{X}_2)\\ &= g_{1,\mathrm{im}(\xi_1),B}(\mathbf{X}_1) \circ g_{n-1,A,\mathrm{im}(\xi_1)}(\mathbf{X}_2)\\ &= f_{1,\mathrm{im}(\xi_1),B}(\mathbf{X}_1) \circ f_{n-1,A,\mathrm{im}(\xi_1)}(\mathbf{X}_2)\\ &= f_{n,A,B}(\mathbf{X}_1 \cdot \mathbf{X}_2) = f_{n,A,B}(\mathbf{X}).\end{aligned}$$

□

2.39. Remark. By Proposition 2.35, for any Dyer exact subcategory $\mathscr{E} \subset \mathscr{T}$, it holds that $\mathrm{Ext}^2(A,B) \subset \mathrm{Hom}(A,B[2])$ for any $A, B \in \mathscr{E}$. In particular, $\dim_{\mathscr{T}} \mathscr{E} \leq 1$ implies that $\dim \mathscr{E} \leq 1$.

2.40. Definition. Let $\mathscr{T}$ be a triangulated category and $\mathscr{E}$ an admissible exact subcategory. We say that *$\mathscr{T}$ has all the Ext groups of $\mathscr{E}$* if the morphism $f_{n,A,B}$ defined in Proposition 2.35 is an isomorphism for any $A, B \in \mathscr{E}$ and all n.

2.41. Corollary. ★. *A triangulated category $\mathscr{T}$ has all the Ext groups of an admissible exact subcategory $\mathscr{E}$ if and only if for every morphism $A \to B[n]$ there exists a factorization*

$$A \to C_{n-1}[1] \to \cdots \to C_1[n-1] \to B[n]$$

with $C_i \in \mathscr{E}$ for $i \in \{1, \ldots, n-1\}$. In particular, if $\dim_{\mathscr{T}} \mathscr{E} \leq 1$, then $\mathscr{T}$ has all the Ext groups of $\mathscr{E}$ and $\dim \mathscr{E} = \dim_{\mathscr{T}} \mathscr{E}$.

PROOF. The only if part is obvious: if $f_{n,A,B}$ is an isomorphism, then the image of such map contains all morphisms $A \to B[n]$: item 1 of Proposition 2.35 concludes.

Conversely, item 1 of Proposition 2.35 shows that $f_{n,A,B}$ is surjective for all $n \in \mathbb{N}$. By assumption, $f_{1,A,B}$ is an isomorphism: we obtain that $f_{2,A,B}$ is injective according to item 3 of Proposition 2.35, so it is an isomorphism. An induction proves that this holds for every n. □

2.42. Corollary. ★. [65, Corollary A.1 and Proposition A.7]. *If $\mathscr{E}$ is an exact category, then $\mathcal{D}^*(\mathscr{E})$ has all the Ext-groups of $\mathscr{E}$.*

PROOF. It follows from Proposition 2.31 and Proposition 2.35. □

2.43. Definition. Let $\mathscr{T}$ be a triangulated category. Given an admissible exact subcategory $\mathscr{E} \subset \mathscr{T}$, we call *realization functor* (of $\mathscr{E}$ in $\mathscr{T}$) a triangulated functor real : $\mathcal{D}^{\mathrm{b}}(\mathscr{E}) \to \mathscr{T}$ such that $\mathrm{real}_{|\mathscr{E}} = \mathrm{id}_{\mathscr{E}}$.

Before stating Proposition 2.45, we begin with a lemma that will make some reasonings easier.

2.44. Lemma. ★. *Let $\mathscr{T}$ be a triangulated category, $\mathscr{E} \subset \mathscr{T}$ an admissible exact subcategory and* real : $\mathcal{D}^{\mathrm{b}}(\mathscr{E}) \to \mathscr{T}$ *a realization functor. Then the following are equivalent:*

- *$\mathscr{T}$ has all the Ext groups of $\mathscr{E}$.*
- *The realization functor is full on shifts of objects of $\mathscr{E}$.*
- *The realization functor is fully faithful on shifts of objects of $\mathscr{E}$.*

PROOF. For $A, B \in \mathscr{E}$ and $n \geq 0$, we consider the morphisms

$$g_{n,A,B} : \mathrm{Ext}^n(A,B) \to \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathscr{E})}(A, B[n]) \overset{\mathrm{real}}{\to} \mathrm{Hom}_{\mathscr{T}}(A, B[n]).$$

Since to each element of $\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathscr{E})}(A, B[1])$ we can associate a distinguished triangle $B \to C \to A \to B[1]$ with $C \in \mathscr{E}$, and this is sent by real to a distinguished triangle $B \to C \to A \to B[1]$ in $\mathscr{T}$, we have that $g_{1,A,B}$ is in fact the natural isomorphism defined in Definition/Proposition 2.32. Moreover, $g_{n,A,B}$ sends the Yoneda product to the composition: from item 4 of Proposition 2.35, we have that $g_{n,A,B} = f_{n,A,B}$.

By Corollary 2.42, $\mathrm{Ext}^n(A,B) \to \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathscr{E})}(A, B[n])$ is an isomorphism. Furthermore, $g_{n,A,B}$ is surjective if and only if it is an isomorphism by item 1 of Proposition 2.35 and Corollary 2.41. We conclude by noticing that $\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathscr{E})}(A, B[n]) \to \mathrm{Hom}_{\mathscr{T}}(A, B[n])$ is surjective (resp. an isomorphism) if and only if $g_{n,A,B}$ is surjective (resp. an isomorphism). □

2.45. Proposition. ★. *Let $\mathscr{T}$ be a triangulated category, $\mathscr{E} \subset \mathscr{T}$ an admissible exact subcategory and* real : $\mathcal{D}^{\mathrm{b}}(\mathscr{E}) \to \mathscr{T}$ *a realization functor. The following assertions are equivalent:*

1. *$\mathscr{T}$ has all the Ext groups of $\mathscr{E}$.*
2. *The realization functor is fully faithful.*
3. *The realization functor is full.*

If, in addition, $\mathscr{T}$ is the triangulated envelope of $\mathscr{E}$, real *is a triangulated equivalence (cf. [16, Corollary 2.8]).*

PROOF. $1 \Rightarrow 2$. As explained in Remark 2.14, we can proceed by induction. Notice that the base case is equivalent to item 1, as discussed by Lemma 2.44. Assume the thesis is known for $E_i, F_i \in \mathcal{D}^{\mathrm{b}}(\mathscr{E})$ with $i = 1, 2$, and let $E = \mathrm{Cone}(E_1[-1] \to E_2)$ and $F = \mathrm{Cone}(F_1[-1] \to F_2)$. Since real is a triangulated functor, we have that $\mathrm{real}(E) = \mathrm{Cone}(\mathrm{real}(E_1)[-1] \to \mathrm{real}(E_2))$ and $\mathrm{real}(F) = \mathrm{Cone}(\mathrm{real}(F_1)[-1] \to \mathrm{real}(F_2))$. By the exact hom-sequences

$$\dots \longrightarrow \mathrm{Hom}(E, F_2) \longrightarrow \mathrm{Hom}(E, F) \longrightarrow \mathrm{Hom}(E, F_1) \longrightarrow \dots$$

$$\dots \longrightarrow \mathrm{Hom}(E_1, F_i) \longrightarrow \mathrm{Hom}(E, F_i) \longrightarrow \mathrm{Hom}(E_2, F_i) \longrightarrow \dots$$

we conclude that $\mathrm{Hom}(E,F) \cong \mathrm{Hom}(\mathrm{real}(E),\mathrm{real}(F))$ by induction hypothesis and the five lemma.

$2 \Rightarrow 3$ is obvious, while $3 \Rightarrow 1$ follows from Lemma 2.44.

Whenever real is fully faithful, $\mathrm{real}(\mathcal{D}^{\mathrm{b}}(\mathscr{E}))$ is a full subcategory of $\mathscr{T}$ equivalent to the triangulated envelope of $\mathscr{E}$ via inclusion. Therefore, if $\mathscr{T}$ is the triangulated envelope of $\mathscr{E}$, real is an equivalence. □

§2.4. t-structures

In this section, we briefly discuss hearts, which are admissible abelian categories obtained by a truncation of the triangulated category.

2.46. Definition. A *t-structure* on a triangulated category $\mathscr{T}$ is a full subcategory $\mathscr{T}^{\leq 0}$ closed by left shifts, i.e. $\mathscr{T}^{\leq 0}[1] \subset \mathscr{T}^{\leq 0}$, and such that for any object $E \in \mathscr{T}$ there is a distinguished triangle $A \to E \to B \to A[1]$, where $A \in \mathscr{T}^{\leq 0}$ and $B \in \mathscr{T}^{\geq 1} := (\mathscr{T}^{\leq 0})^{\perp}$.

We remember that for any full subcategory $\mathscr{C} \subset \mathscr{T}$, we write $\mathscr{C}^{\perp}$ to mean the full subcategory whose objects are Y such that $\mathrm{Hom}(X,Y) = 0$ for any $X \in \mathscr{C}$.

We will write $\mathscr{T}^{\leq i} := \mathscr{T}^{\leq 0}[-i]$ and $\mathscr{T}^{\geq j} := \mathscr{T}^{\geq 1}[-j+1]$ for any i, j integers. A t-structure is said to be *bounded* if

$$\mathscr{T} = \bigcup_{i,j\in\mathbb{Z}} (\mathscr{T}^{\leq i} \cap \mathscr{T}^{\geq j}).$$

Moreover, the t-structure is *non-degenerate* if $\bigcap_i \mathscr{T}^{\leq i} = \bigcap_j \mathscr{T}^{\geq j} = 0$.

The *heart of a t-structure* $\mathscr{T}^{\leq 0}$ is the additive category $\mathscr{A} := \mathscr{T}^{\geq 0} \cap \mathscr{T}^{\leq 0}$, and it is always admissible abelian (cf. [4, Théorème 1.3.6]).

2.47. Convention. We will use the short-hand *heart* only for hearts of bounded t-structures. In fact, we are not interested in studying unbounded cases.

2.48. Definition/Proposition. [8, Lemma 3.2]. Let $\mathscr{T}$ be a triangulated category. A *heart (of a bounded t-structure)* on $\mathscr{T}$ is a full additive subcategory $\mathscr{A}$ satisfying the following properties:

1. It is admissible, i.e. for any two objects $A, B \in \mathscr{A}$, $\mathrm{Hom}(A, B[n]) = 0$ for every $n < 0$.
2. Given an object $E \in \mathscr{T}$, we can find integers $k_1 > \cdots > k_m$ and a filtration

$$0 = E_0 \to E_1 \to \cdots \to E_{m-1} \to E_m = E$$

such that $\mathrm{Cone}(E_{i-1} \to E_i) = A_i[k_i]$ for some $A_i \in \mathscr{A}$. The *cohomology objects* (with respect to $\mathscr{A}$) are defined as $H^{-k_i}(E) := A_i$. As one may expect, any distinguished triangle gives rise to a long exact sequence of cohomology objects.

2.49. Lemma. ★. *For an additive subcategory $\mathscr{A}$ of $\mathscr{T}$, the following are equivalent:*

1. *$\mathscr{A}$ is a heart on $\mathscr{T}$;*
2. *$\mathscr{A}$ is admissible abelian and $\mathscr{T}$ is its triangulated envelope.*

PROOF. The implication $1 \Rightarrow 2$ follows from Definition/Proposition 2.48 because a heart is always admissible abelian. Let us deal with the converse. Since admissibility is already assumed, we just need to show that the filtration exists for all objects of $\mathscr{T}$. Notice all objects in $\mathscr{T}$ are obtained by iterating cones by assumption (see Remark 2.14), therefore we just need to show that whenever X, Y admit a filtration, then $\mathrm{Cone}(f)$ has a filtration as well, for all $f : X \to Y$.

Set n and m to be the length of the filtration of X and Y respectively. We proceed by double induction.

If $n = m = 1$, assume $X \in \mathscr{A}$ and $Y = Z[j]$ for some $Z \in \mathscr{A}$. If $j > 1$, then $Z[j] \to \mathrm{Cone}(f) \to X[1]$ yields the wanted filtration. If $j < 0$, by assumption $f = 0$, so that the cone is just a direct sum and we can consider $X[1] \to X[1] \oplus Z[j] \to Z[j]$. If $j = 1$, then $\mathrm{Cone}(f)[-1] \in \mathscr{A}$ and there is nothing to prove. We are left with the case of $j = 0$. By considering the octahedron associated to $X \to \mathrm{im}\, f \to Y$, we obtain a distinguished triangle $\ker f[1] \to \mathrm{Cone}(f) \to \mathrm{coker}\, f$, which yields the wanted filtration.¶

If $n = 1$ and $m > 1$, assume as before that $X \in \mathscr{A}$. Since Y has a filtration, we have a distinguished triangle $Y^{\leq 0} \to Y \to Y^{>0}$, where $Y^{\leq 0}$ is the last object in the filtration whose associated cohomology object $Y_h \in \mathscr{A}$ appears with a non-negative shift k_h. In other words, $H^i(Y^{\leq 0}) = 0$ for all $i > 0$. By definition, $X \to Y^{>0}$ is zero, so that $f : X \to Y$ factors through $f^{\leq 0} : X \to Y^{\leq 0}$. In particular, if the cone of $f^{\leq 0}$ has a filtration, we can construct a filtration of Y by induction using the following step: assume we have a filtration for $\mathrm{Cone}(f^{\leq \ell})$, $\ell \geq 0$. Then the octahedron associated to $X \to Y^{\leq \ell} \to Y^{\leq \ell+1}$ shows that the cone of $\mathrm{Cone}(f^{\leq \ell}) \to \mathrm{Cone}(f^{\leq \ell+1})$ is $H^{\ell+1}(Y)[-\ell-1]$. We get the filtration

$$\cdots \to \mathrm{Cone}(f^{\leq 0}) \to \mathrm{Cone}(f^{\leq 1}) \to \mathrm{Cone}(f^{\leq 2}) \to \dots$$

Therefore, we can restrict to $X \to Y^{\leq 0}$. We consider the octahedron associated to $X \to Y^{\leq 0} \to Y_h[k_h]$ and call $f_h : X \to Y_h[k_h]$ and $Y^h = \mathrm{Cone}(Y^{\leq 0}[-1] \to Y_h[k_h - 1])$. Such octahedron shows that $Y^h[1] = \mathrm{Cone}(\mathrm{Cone}(f^{\leq 0}) \to \mathrm{Cone}(f_h))$. Moreover, notice that the length of the filtration of Y^h is one less than the one of $Y^{\leq 0}$. Our aim is now to study what happens to $\mathrm{Cone}(f_h)$ and apply the induction hypothesis.

If $k_h = 1$, then $Y^h \to \mathrm{Cone}(f^{\leq 0}) \to \mathrm{Cone}(f_h)$ gives already the right filtration, while if $k_h > 1$ then the filtration is obtained by $Y^{\leq 0} \to \mathrm{Cone}(f^{\leq 0}) \to X[1]$.

If $k_h = 0$, we have a distinguished triangle $\ker f_h[1] \to \mathrm{Cone}(f_h) \to \mathrm{coker}\, f_h \to \ker f_h[2]$. Let us consider the octahedron

$$\begin{tikzcd}
\mathrm{Cone}(f^{\leq 0}) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{Cone}(f_h) \arrow[r] \arrow[d] & Y^h[1] \arrow[d] \\
\mathrm{Cone}(f^{\leq 0}) \arrow[r] \arrow[d] & \mathrm{coker}\, f_h \arrow[r] \arrow[d, "\mathrm{id}"] & \tilde{Y}^h[1] \arrow[d] \\
\mathrm{Cone}(f_h) \arrow[r] & \mathrm{coker}\, f_h \arrow[r] & \ker f_h[2]
\end{tikzcd}$$

¶Please note that $\ker f$, $\mathrm{coker}\, f$ and $\mathrm{im}\, f$ mean respectively the kernel, the cokernel and the image of f in $\mathscr{A}$, and not in the triangulated category, where in general kernels, cokernels and images do not exist.

associated to $\mathrm{Cone}(f^{\leq 0}) \to \mathrm{Cone}(f_h) \to \operatorname{coker} f_h$, where $\tilde{Y}^h = \mathrm{Cone}(\ker f_h)[1] \to Y^h[1])$. By induction hypothesis, we notice that $\tilde{Y}^h$ has a filtration whose length is at most m. By substituting Y with $\tilde{Y}^h$ and X with $\operatorname{coker} f_h$, we notice that we have reduced the case of $k_h = 0$ to the one of $k_h > 0$, which has already been discussed above.

For $n > 1$, set X^h to be the last object in the filtration of X before X itself, so that $\mathrm{Cone}(X^h \to X)$ is of the form $X_h[k_h]$ with $X_h \in \mathscr{A}$. Take the octahedron associated to $f^h : X^h \to X \to Y$, which writes the cone of $f : X \to Y$ as the cone of $X_h[k_h] \to \mathrm{Cone}(f^h)$, where the latter has a filtration by induction hypothesis. The fact that such a cone has a filtration is exactly what we showed in the step with $n = 1$ and $m > 1$. □

2.50. Lemma. *Every bounded t-structure $\mathscr{T}^{\leq 0}$ is non-degenerate. In particular, the collection of functors H^i is conservative and $H^i(E) = 0$ for all $i > 0$ (respectively $i < 0$) if and only if $E \in \mathscr{T}^{\leq 0}$ (respectively $\mathscr{T}^{\geq 0}$); this is [4, Proposition 1.3.7].*

PROOF. Let E be in the intersection of all $\mathscr{T}^{\leq i}$. Since $\mathscr{T}^{\leq 0}$ is bounded, E must be in $\mathscr{T}^{\leq j} \cap \mathscr{T}^{\geq h}$ for some j, h. Then E is in $\mathscr{T}^{\geq h}$, but also in $\mathscr{T}^{\leq h-1}$. By definition,

$$\mathscr{T}^{\geq h} = \mathscr{T}^{\geq 1}[-h+1] = (\mathscr{T}^{\leq 0})^{\perp}[-h+1] = (\mathscr{T}^{\leq 0}[-h+1])^{\perp} = (\mathscr{T}^{\leq h-1})^{\perp}.$$

So $\mathrm{Hom}(E,E) = 0$, therefore E is a zero object. In the same way one proves that also $\bigcap_i \mathscr{T}^{\geq i} = 0$. □

§2.5. Hubery's Theorem

This section is devoted to the proof of an incredibly strong result: a triangulated category $\mathscr{T}$ with a hereditary heart is uniquely determined up to triangulated equivalence (see Theorem 2.56). This result has been proved by Hubery [30]; in the first draft of [49], we showed it independently. Let us start by recalling the definition of hereditary category.

2.51. Definition. An abelian category $\mathscr{A}$ is *hereditary* if $\dim \mathscr{A} \leq 1$. Similarly, an *admissible hereditary subcategory* $\mathscr{A} \subset \mathscr{T}$ is an admissible abelian subcategory satisfying $\dim_{\mathscr{T}} \mathscr{A} \leq 1$. A *hereditary heart* is a heart which is admissible hereditary.

2.52. Remark. We recall Remark 2.39 with this new notions: if $\mathscr{A}$ is admissible hereditary, then it is hereditary, but the converse is not true in general (cf. Remark 4.25).

2.53. Proposition. *Let $\mathscr{T}$ be a triangulated category with an admissible hereditary subcategory $\mathscr{A}$, and consider $\mathscr{S} := \langle \mathscr{A} \rangle$ the triangulated envelope of $\mathscr{A}$ in $\mathscr{T}$. Then any object of $\mathscr{S}$ can be written as a (finite) direct sum $\bigoplus_i E_i[i]$, with $E_i \in \mathscr{A}$.*

PROOF. By Lemma 2.49, $\mathscr{A}$ is a heart on $\mathscr{S}$, therefore for every $E \in \mathscr{S}$ we have an associated filtration. Let us proceed by induction on $n := \#\{i \in \mathbb{Z} : H^i(E) \neq 0\}$. If $n = 0$, then $E = 0$ and we have nothing to prove (this follows from the last sentence of Lemma 2.50). If $n > 0$, there exists a distinguished triangle

$$F \longrightarrow E \xrightarrow{f} H^m(E)[-m] \longrightarrow F[1],$$

where $m := \max\{i \in \mathbb{Z} : H^i(E) \neq 0\}$, and $H^m(f)$ is an isomorphism. By induction hypothesis, $F = \bigoplus_{i<m} H^i(F)[-i]$. In order to conclude, we need to prove that $E = F \oplus H^m(E)[-m]$. But this is true since

$$\begin{aligned}\mathrm{Hom}(H^m(E)[-m-1]\ ,\ F) &= \mathrm{Hom}\left(H^m(E)[-m-1]\ ,\ \bigoplus_{i<m} H^i(F)[-i]\right)\\ &= \bigoplus_{i<m} \mathrm{Hom}(H^m(E)[-m-1]\ ,\ H^i(F)[-i])\\ &= \bigoplus_{i<m} \mathrm{Ext}^{m-i+1}(H^m(E)\ ,\ H^i(F)) = 0.\end{aligned}$$

The last equality holds because $m - i + 1 > 1$ for any $i < m$. $\square$

The condition of the last proposition is also necessary. In order to prove this, let us first recall a well-known result.

2.54. Proposition. [4, Proposition I.1.11]. *Let $\mathscr{T}$ be a triangulated category. Any commutative square*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

gives rise to a diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & A''[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A[1] & \longrightarrow & B[1] & \longrightarrow & C[1] & \longrightarrow & A[2] \end{array}$$

where all the squares are commutative, except the lower right-hand one, which is anticommutative. Further, each of the rows and each of the columns are distinguished triangles; in particular, $\mathrm{Cone}(C \to C') = \mathrm{Cone}(A'' \to B'')$.

2.55. Proposition. ★. *Let $\mathscr{A} \subset \mathscr{T}$ be an admissible exact subcategory of a triangulated category. Assume that for any $E \in \mathscr{S} := \langle \mathscr{A} \rangle$, we have a decomposition $E = \bigoplus_i E_i[i]$ with $E_i \in \mathscr{A}$. Then $\mathscr{A}$ is an admissible hereditary subcategory.*

PROOF. The decomposition immediately implies that the filtration of Definition/Proposition 2.48 exists, so $\mathscr{A}$ is a heart in $\mathscr{S} = \langle \mathscr{A} \rangle$. Therefore, $\mathscr{A}$ is admissible abelian.

We are now reduced to prove that $\mathrm{Hom}_{\mathscr{T}}(A, B[j]) = 0$ for any $j \geq 2$ and any $A, B \in \mathscr{A}$. Let $f : A \to B[j]$ be any map and consider the cone C. Such cone will have $H^{-1}(C) = A$, $H^{-j}(C) = B$ and $H^i(C) = 0$ for $i \neq -1, -j$. By assumption, we have that $C = \bigoplus_i C_i[i]$, with $C_i \in \mathscr{A}$, so we obtain that $H^k(C) = H^k(\bigoplus_i C_i[i]) = H^k(C_{-k}[-k]) = H^0(C_{-k}) = C_{-k}$. This entails that $C = A[1] \oplus B[j]$. We consider the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, dashed, "\alpha"] & B[j] \arrow[r] \arrow[d, dashed, "\beta"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & A[1] \arrow[d, dashed] \\
A \arrow[r, "0"] & B[j] \arrow[r] & A[1] \oplus B[j] \arrow[r] & A[1]
\end{tikzcd}$$

where the dashed maps are obtained since $B[j] \to A[1]$ is zero; indeed, $\mathrm{Hom}(B[j], A[1]) \cong \mathrm{Hom}(B, A[1-j]) = 0$ for $j \geq 2$ by admissibility of $\mathscr{A}$. By Proposition 2.54, we also have that $\mathrm{Cone}(\alpha) \cong \mathrm{Cone}(\beta)$. For the sake of simplicity, we write K to mean both $\mathrm{Cone}(\alpha)$ and $\mathrm{Cone}(\beta)$, since our reasoning will be up to isomorphism. Being $K = \mathrm{Cone}(\alpha)$, the long exact cohomology sequence associated to α proves that $H^i(K) = 0$ for any $i \neq -1, 0$. Further, since $K = \mathrm{Cone}(\beta)$, we also have that $H^i(K) = 0$ for any $i \neq -j-1, -j$. Since $-j \leq -2$, K must be zero. In particular, α and β are isomorphisms, so $f = \beta^{-1} 0 \alpha = 0$. □

2.56. Theorem – Hubery. ★. [30, Theorem 3.2]. *Let $\mathscr{T}$ be a triangulated category with a hereditary heart $\mathscr{A}$. Then $\mathscr{T}$ is uniquely determined by $\mathscr{A}$ up to triangulated equivalence. More precisely, there is a realization functor $\mathcal{D}^{\mathrm{b}}(\mathscr{A}) \to \mathscr{T}$ which is an equivalence.*

PROOF. For any couple of triangulated categories $\mathscr{T}, \mathscr{T}'$ satisfying the conditions of the statement with the same $\mathscr{A}$, we want to construct a triangulated equivalence. By Proposition 2.53, we know that every object in both $\mathscr{T}$ and $\mathscr{T}'$ is a finite direct sum $\bigoplus_i E_i[i]$, with $E_i \in \mathscr{A}$. We define a functor $F : \mathscr{T} \to \mathscr{T}'$ as follows:

1. Objects. $F(\bigoplus_i E_i[i]) := \bigoplus_i \mathrm{id}_{\mathscr{A}}(E_i)[i]$.
2. Morphisms. Let $E = \bigoplus_i E_i[i]$ and $G = \bigoplus_j G_j[j]$ two objects of $\mathscr{T}$, with $E_i, G_j \in \mathscr{A}$. Given a map $f : E \to G$, we consider its matrix decomposition f_{ij}. Then

$$F((f_{ij})_{i,j}) = (f'_{ij})_{i,j}, \qquad f'_{ij} := \begin{cases} 0 (= f_{ij}) & \text{if } i < j-1 \text{ or } i > j \\ \mathrm{id}_{\mathscr{A}}(f_{i,i}) & \text{if } i = j \\ f'_{i,i+1} & \text{if } i = j-1 \end{cases}$$

where $f'_{i,i+1}$ is obtained by the identifications $\mathrm{Hom}_{\mathscr{T}}(E_i, G_{i+1}[1]) \cong \mathrm{Ext}^1(E_i, G_{i+1}) \cong \mathrm{Hom}_{\mathscr{T}'}(E_i, G_{i+1}[1])$ according to Definition/Proposition 2.32.

It is clear that $F(\mathrm{id}) = \mathrm{id}$. The fact that F preserves composition follows by the fact that the isomorphism in Definition/Proposition 2.32 is natural. Therefore, F is a functor; by definition, it is fully faithful, essentially surjective and it preserves shifts and direct sums. In order to conclude, it remains to show that F also preserves distinguished triangles.

We recall that $\mathscr{T}$ is in fact given by two data: an additive category with a shift $\mathscr{T}_\Sigma$ and a class of distinguished triangles $\mathscr{D}$, in brief $\mathscr{T} = (\mathscr{T}_\Sigma, \mathscr{D})$. With the same meaning, we write $\mathscr{T}' = (\mathscr{T}'_\Sigma, \mathscr{D}')$. Since the functor F above gives an equivalence $\mathscr{T}_\Sigma \cong \mathscr{T}'_\Sigma$, we may assume without loss of generality that $\mathscr{T}_\Sigma = \mathscr{T}'_\Sigma$ and $F = \mathrm{id}$. We denote with $\mathrm{Cone}_{\mathscr{T}}(-)$ the cone of $\mathscr{T}$ and with $\mathrm{Cone}_{\mathscr{T}'}(-)$ the cone of $\mathscr{T}'$.

Our aim is to prove that $\mathscr{D} \subseteq \mathscr{D}'$, which will directly imply $\mathscr{D} = \mathscr{D}'$ since a class of distinguished triangles is closed under isomorphisms. Let $A = \bigoplus_{i=0}^{a} A_i[n_i]$ and $B = \bigoplus_{j=0}^{b} B_j[m_j]$, with $A_i, B_j \in \mathscr{A}$, and consider $f : A \to B$. We want to show that $A \to B \to \mathrm{Cone}_{\mathscr{T}}(f) \to A[1] \in \mathscr{D}$ belongs to $\mathscr{D}'$ by induction on a and b.

Step 1. The case $a = 0$ and $b = 0$.

Up to shifting, we can assume $A = A_0$ and $B = B_0[n]$. If $n \neq 0, 1$, then the cone is simply a direct sum $B_0[n] \oplus A[1]$, and the statement holds true. If $n = 1$, the cone $\mathrm{Cone}_{\mathscr{T}}(f)$ is an object C such that $C[-1] \in \mathscr{A}$ is an extension of B_0 and A. Then, since $\mathscr{A}$ is a heart for both $\mathscr{T}$ and $\mathscr{T}'$, $\mathrm{Cone}_{\mathscr{T}}(f) = \mathrm{Cone}_{\mathscr{T}'}(f)$ with the same distinguished triangle.

If $n = 0$, the cone of $f : A \to B$ is in fact isomorphic to $\operatorname{coker} f \oplus \ker f[1]$,[‖] because of the octahedron obtained from $A \to \operatorname{im} f \to B$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\mathrm{id}"] & \operatorname{im} f \arrow[r] \arrow[d] & \ker f[1] \arrow[r, "h_1"] \arrow[d] & A[1] \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d, "\mathrm{id}"] & \operatorname{coker} f \oplus \ker f[1] \arrow[r, "h"] \arrow[d] & A[1] \arrow[d] \\
\operatorname{im} f \arrow[r] & B \arrow[r, "g_0"] & \operatorname{coker} f \arrow[r] \arrow[dr, "0"'] & \operatorname{im} f[1] \arrow[d] \\
 & & & \ker f[2]
\end{tikzcd}$$

(first and last row are distinguished triangles since they are short exact sequences in a heart). In particular, $\mathrm{Cone}_{\mathscr{T}}(f) = \mathrm{Cone}_{\mathscr{T}'}(f)$. We are left to study the maps of the distinguished triangle in $\mathscr{T}$ and in $\mathscr{T}'$. Let us consider the notation in the octahedron for the distinguished triangle in $\mathscr{T}$ and

$$A \xrightarrow{f} B \xrightarrow{\alpha} \operatorname{coker} f \oplus \ker f[1] \xrightarrow{\beta} A[1]$$

for the distinguished triangle in $\mathscr{T}'$. We would like to show that, up to some isomorphism of $\operatorname{coker} f \oplus \ker f[1]$, $\alpha = g$ and $\beta = h$. We use the matrix notation: $g = \binom{g_0}{g_1}$, where g_0 :

[‖] Please note that $\ker f$ and $\operatorname{coker} f$ mean the kernel and the cokernel of f in $\mathscr{A}$, and not in the triangulated category, where in general kernels and cokernels do not exist.

$B \to \operatorname{coker} f \oplus \ker f[1] \to \operatorname{coker} f$ and $g_1 : B \to \operatorname{coker} f \oplus \ker f[1] \to \ker f[1]$; $h = (h_0\ h_1)$, with $h_0 : \operatorname{coker} f \to A[1]$ and $h_1 : \ker f[1] \to A[1]$. Up to isomorphism, the octahedron above can be rewritten in $\mathscr{T}'$ simply by changing g and h since the other maps can be fixed by the case $n = 1$ and by taking the same morphisms considered in $\mathscr{T}$ for the vertical distinguished triangle associated to $\operatorname{coker} f \oplus \ker f[1]$. Then $\alpha = \left(\begin{smallmatrix} g_0 \\ \alpha_1 \end{smallmatrix}\right)$ and $\beta = (\beta_0\ h_1)$. We are reduced to prove $\beta_0 = h_0$ and $\alpha_1 = g_1$. In order to do that, we consider the extensions C_{h_0} and C_{g_1} associated to $h_0 : \operatorname{coker} f \to A[1]$ and $g_1 : B \to \ker f[1]$. First of all, we need to prove that these extensions are in fact isomorphic. Let us consider, in $\mathscr{T}$, the commutative diagram

(2.57)
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\mathrm{id}"] & C_{h_0} \arrow[r] \arrow[d, dashed] & \operatorname{coker} f \arrow[r, "h_0"] \arrow[d, "\left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right)"] & A[1] \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & \operatorname{coker} f \oplus \ker f[1] \arrow[r, "h"] & A[1],
\end{tikzcd}$$

where rows are distinguished triangles. Proposition 2.54 entails that $\operatorname{Cone}_{\mathscr{T}}(C_{h_0} \to B) \cong \ker f[1]$, as $\operatorname{Cone}(\mathrm{id}) = 0$. Up to isomorphisms of $\ker f[1]$, we may assume that $\operatorname{Cone}_{\mathscr{T}}(C_{h_0} \to B) \to \ker f[1]$ is in fact the identity; the map $B \to \ker f[1]$ so obtained has to be g_1 by the commutative diagram

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d] & \operatorname{coker} f \oplus \ker f[1] \arrow[d, "(0\ \mathrm{id})"] \\
\ker f[1] \arrow[r, "\mathrm{id}"] & \ker f[1].
\end{tikzcd}$$

Thus, $C_{h_0} \cong C_{g_1}$, and this holds also in $\mathscr{T}'$ since $C_{h_0}, C_{g_1} \in \mathscr{A}$. We now consider, in $\mathscr{T}'$, the octahedron

(2.58)
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\mathrm{id}"] & C_{h_0} \arrow[r] \arrow[d] & \operatorname{coker} f \arrow[r, "h_0"] \arrow[d] & A[1] \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "f"] \arrow[d] & B \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"] & \operatorname{coker} f \oplus \ker f[1] \arrow[r, "\beta"] \arrow[d] & A[1] \arrow[d] \\
C_{h_0} \arrow[r] & B \arrow[r, "g_1"] & \ker f[1] \arrow[r] \arrow[dr] & C_{h_0}[1] \arrow[d] \\
 & & & \operatorname{coker} f[1]
\end{tikzcd}$$

associated to the left hand square of (2.57). The first and the last row have the same maps as in $\mathscr{T}$ by the case $n = 1$. We want to study the maps of $\operatorname{coker} f \to \operatorname{coker} f \oplus \ker f[1] \to \ker f[1]$. By using the cohomology with respect to $\mathscr{A}$, we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\mathrm{id}"] & C_{h_0} \arrow[r] \arrow[d] & \operatorname{coker} f \arrow[r] \arrow[d, dashed] & 0 \\
A \arrow[r, "f"] & B \arrow[r, "g_0"] & \operatorname{coker} f \arrow[r] & 0
\end{tikzcd}$$

where the dashed map is uniquely determined by the cokernel property in $\mathscr{A}$, and the identity does make this diagram commute by (2.57); therefore, the map $\operatorname{coker} f \to \operatorname{coker} f \oplus \ker f[1]$ in the octahedron (2.58) must be of the form $\binom{\mathrm{id}}{\mu}$. Analogously, the map $\operatorname{coker} \oplus \ker f[1] \to \ker f[1]$ must be of the form $(\nu \;\; \mathrm{id})$. The fact that the composition $(\nu \;\; \mathrm{id})\binom{\mathrm{id}}{\mu}$ must be zero shows that $\nu = -\mu$. Now we modify α and β up to the isomorphism $\begin{pmatrix} \mathrm{id} & 0 \\ \mu & \mathrm{id} \end{pmatrix}$: the commutative diagrams of (2.58) prove the last part of these equalities.

$$\begin{pmatrix} \mathrm{id} & 0 \\ -\mu & \mathrm{id} \end{pmatrix} \alpha = \begin{pmatrix} \mathrm{id} & 0 \\ -\mu & \mathrm{id} \end{pmatrix} \begin{pmatrix} g_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} g_0 \\ -\mu g_0 + \alpha_1 \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}$$

$$\beta \begin{pmatrix} \mathrm{id} & 0 \\ \mu & \mathrm{id} \end{pmatrix} = \begin{pmatrix} \beta_0 & h_1 \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ \mu & \mathrm{id} \end{pmatrix} = \begin{pmatrix} \beta_0 + h_1 \mu \\ h_1 \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \end{pmatrix}.$$

This concludes the case $a = 0$ and $b = 0$.
Step 2. The case $a = 0$ and $b > 0$.

As before, we can assume without loss of generality that $A = A_0$. Since direct sums of distinguished triangles are distinguished (see, for instance, [59, Proposition 1.2.1]), we can restrict to the case of a nontrivial map $f = \binom{f_0}{f_1} : A \to B_0 \oplus B_1[1] =: B$, since by assumption $f_n : A \to B_n[n]$ is trivial for any $n \neq 0, 1$. We consider the following octahedron in $\mathscr{T}$ and labelling of morphisms

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & B_0 \oplus B_1[1] \arrow[r, "g"] \arrow[d, "(0\ \mathrm{id})"] & \mathrm{Cone}_{\mathscr{T}}(f) \arrow[r, "h"] \arrow[d, "x"] & A[1] \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "f_1"] \arrow[d, "f"] & B_1[1] \arrow[r, "s[1]"] \arrow[d, "\mathrm{id}"] & C_{f_1}[1] \arrow[r, "t[1]"] \arrow[d, "y[1]"] & A[1] \arrow[d, "f[1]"] \\
B_0 \oplus B_1[1] \arrow[r, "(0\ \mathrm{id})"] & B_1[1] \arrow[r, "0"] & B_0[1] \arrow[r, "\binom{\mathrm{id}}{0}"] \arrow[dr, "g_0[1]"'] & B_0[1] \oplus B_1[2] \arrow[d, "g[1]"] \\
 & & & \mathrm{Cone}_{\mathscr{T}}(f)[1].
\end{tikzcd}$$

We also name $g = (g_0 \; g_1)$. First of all, notice the commutative diagrams show that $h = t[1]x$, $\binom{\mathrm{id}}{0} y = ft$, which implies $y = f_0 t$, and $s[1] = x g_1$.

We claim that the octahedron holds with the same maps also in $\mathscr{T}'$, aside from g and h. By induction hypothesis, we can choose $B_1[1] \to C_{f_1}[1]$ and $C_{f_1} \to A[1]$ to be exactly $s[1]$ and $t[1]$ also in $\mathscr{T}'$ up to an isomorphism of C_{f_1}. Since $y = f_0 t$, this choice forces y to appear also in the analogous of the octahedron above in $\mathscr{T}'$. Therefore, the associated distinguished triangle $C_{f_1} \to B_0 \to \mathrm{Cone}_{\mathscr{T}'}(f) \to C_{f_1}[1]$, appearing (up to shift) in the analogous of the octahedron above in $\mathscr{T}'$, is given by $-y$. In particular, from induction hypothesis, $\mathrm{Cone}_{\mathscr{T}}(f) = \mathrm{Cone}_{\mathscr{T}'}(f)$, and the same triangle with $-g_0$ and x is still distinguished. It follows that, up to an isomorphism of $\mathrm{Cone}_{\mathscr{T}'}(f)$, we can choose g_0 and x as in the octahedron above. Finally, the claim holds.

Now let $g' : B_0 \oplus B_1[1] \to \mathrm{Cone}_{\mathscr{T}'}(f)$ and $h' : \mathrm{Cone}_{\mathscr{T}'}(f) \to A[1]$ appearing in the octahedron in $\mathscr{T}'$ with all the other maps determined as stated in the claim. The commutative triangle proves that $g' \left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right) = g_0$, so that $g' = (g_0 \; g_1')$ for some $g_1' : B_1[1] \to \mathrm{Cone}_{\mathscr{T}'}(f)$. Moreover, $h' = t[1]x = h$. It remains to prove that $g_1' = g_1$. From the octahedron, $xg_1' = s[1] = xg_1$. By the fact that g_1 must have target $\ker y[1]$, i.e. the direct summand of $\mathrm{Cone}_{\mathscr{T}'}(f)$ in $\mathscr{A}[1]$, the equality can be considered to be $x_1 g_1' = x_1 g_1$, where $x_1 : \ker y[1] \to C_{f_1}[1]$. We know, by the long exact sequence of cohomology with respect to $\mathscr{A}$, that x_1 is a monomorphism of $\mathscr{A}[1]$. Therefore, $x_1 g_1' = x_1 g_1$ implies that $g_1' = g_1$.

Step 3. The case $a > 0$.

Let us consider the following octahedron in $\mathscr{T}$ and its notation:

$$\begin{tikzcd}
A_0[n_0] \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"] & A \arrow[r, "\pi"] \arrow[d, "f"] & \bigoplus_{i=1}^{a} A_i[n_i] \arrow[r, "0"] \arrow[d, "x"] & A_0[n_0+1] \arrow[d, "\mathrm{id}"] \\
A_0[n_0] \arrow[r, "f_0"] \arrow[d, "\iota"] & B \arrow[r, "g_0'"] \arrow[d, "\mathrm{id}"] & \mathrm{Cone}_{\mathscr{T}}(f_0) \arrow[r, "h_0'"] \arrow[d, "y"] & A_0[n_0+1] \arrow[d, "\iota[1]"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & \mathrm{Cone}_{\mathscr{T}}(f) \arrow[r, "h"] \arrow[dr, "z"] & A[1] \arrow[d, "\pi[1]"] \\
 & & & \bigoplus_{i=1}^{a} A_i[n_i+1],
\end{tikzcd}$$

where

$$\iota = \begin{pmatrix} \mathrm{id} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \pi = \begin{pmatrix} 0 & \mathrm{id} & & \\ \vdots & & \ddots & \\ 0 & & & \mathrm{id} \end{pmatrix}.$$

The entries left blank are to be intended as 0. Up to isomorphism, we can consider the first two rows to have the same maps also in $\mathscr{T}'$ (first one being obvious, while for the second we use the case of $a = 0$ and $b > 0$). The map

$$x = x\pi \begin{pmatrix} 0 & \dots & 0 \\ \mathrm{id} & & \\ & \ddots & \\ & & \mathrm{id} \end{pmatrix} = g_0' f \begin{pmatrix} 0 & \dots & 0 \\ \mathrm{id} & & \\ & \ddots & \\ & & \mathrm{id} \end{pmatrix}$$

is uniquely determined from g_0' and f, which are fixed. In particular, $\mathrm{Cone}_{\mathscr{T}}(f) = \mathrm{Cone}_{\mathscr{T}'}(f)$. By induction hypothesis, up to isomorphism we can also choose y and z as in the octahedron above. We immediately get that $g = yg_0'$, $\pi[1]h = z$ and $hy = \iota[1]h_0'$. Let us consider the notation

$h = (h_0 \ \dots \ h_a)^T$. Then

$$z = \pi[1]h = \begin{pmatrix} 0 & \mathrm{id} & & \\ \vdots & & \ddots & \\ 0 & & & \mathrm{id} \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_a \end{pmatrix} = \begin{pmatrix} h_1 \\ \vdots \\ h_a \end{pmatrix}.$$

It remains to prove that h_0 is determined, as g and the other components of h are fixed. First, the fact that $hy = \iota[1]h_0'$ shows that $h_0 y = h_0'$ by applying $(\mathrm{id}\ 0\dots 0)$ on the left. As h_0 and h_0' have target $A_0[n_0+1]$, we may consider only the part of y that composed with h_0 is not necessarily zero. For a better understanding, let us denote with $\mathrm{Cone}(f_0)_m$ the direct summand of $\mathrm{Cone}_{\mathscr{T}'}(f_0) = \mathrm{Cone}_{\mathscr{T}}(f_0)$ belonging to $\mathscr{A}[m]$ (recall Proposition 2.53). Similarly, we define $\mathrm{Cone}(f)_m$. Then the part of y we are interested in is given by

$$y' : \mathrm{Cone}(f_0)_{n_0} \oplus \mathrm{Cone}(f_0)_{n_0+1} \to \mathrm{Cone}(f)_{n_0} \oplus \mathrm{Cone}(f)_{n_0+1}, \qquad y' = \begin{pmatrix} \phi & 0 \\ y'' & p \end{pmatrix}.$$

We now assume that n_0 is the least integer for which there exists a nonzero direct summand of A. This does not change what we have done so far, as it holds for any nonzero direct summand. By this minimality, from the long exact cohomology sequence of the distinguished triangle described by x, y, z

$$\dots \longrightarrow \mathrm{Cone}(f_0)_{n_0+1} \xrightarrow{p} \mathrm{Cone}(f)_{n_0+1} \longrightarrow 0 \longrightarrow \mathrm{Cone}(f_0)_{n_0} \xrightarrow{\phi} \mathrm{Cone}(f)_{n_0} \longrightarrow 0,$$

we conclude that p is an epimorphism and ϕ is an isomorphism. Now, let us consider some map $k : \mathrm{Cone}(f)_{n_0} \oplus \mathrm{Cone}(f)_{n_0+1} \to A_0[n_0+1]$ such that $ky' = h_0'$. We get

$$\begin{pmatrix} h_{0,0}' & h_{0,1}' \end{pmatrix} = h_0' = ky' = \begin{pmatrix} k_0 & k_1 \end{pmatrix} \begin{pmatrix} \phi & 0 \\ y'' & p \end{pmatrix} = \begin{pmatrix} k_0\phi + k_1 y'' & k_1 p \end{pmatrix}$$

This holds also replacing k with $h_0 = (h_{0,0}\ h_{0,1})$. Then $k_1 p = h_{0,1}' = h_{0,1}p$, and by epimorphism definition $k_1 = h_{0,1}$ (these maps are in $\mathscr{A}[n_0+1]$). Furthermore, $k_0\phi = h_{0,0}' - k_1 y'' = h_{0,0}' - h_{0,1}y'' = h_{0,0}\phi$. Being ϕ an isomorphism, $k_0 = h_{0,0}$, and so h_0 is uniquely determined. □

2.59. Corollary – Realization functors. ★. *Let $\mathscr{A}$ be a hereditary heart on a triangulated category $\mathscr{T}$. We consider a triangulated category $\mathscr{T}'$ and a fully faithful additive functor $F : \mathscr{A} \to \mathscr{T}'$ whose image is an admissible hereditary subcategory. Then there exists a fully faithful triangulated functor $G : \mathscr{T} \to \mathscr{T}'$ extending F.*

PROOF. Let us consider the full subcategory $\mathscr{S}' := \langle F(\mathscr{A}) \rangle \subset \mathscr{T}'$. Since $F(\mathscr{A})$ is an admissible hereditary subcategory equivalent to $\mathscr{A}$, by Theorem 2.56 we can extend F to a triangulated equivalence with target $\mathscr{S}'$. Finally, we obtain $G : \mathscr{T} \xrightarrow{\cong} \mathscr{S}' \subset \mathscr{T}'$. □

2.60. Proposition. ★. *Let $\mathscr{A}$ be a hereditary category such that* $\operatorname{Ext}^1(A,B)$ *is a set for any* $A,B \in \mathscr{A}$*. Then* $\mathcal{D}^{\mathrm{b}}(\mathscr{A})$ *is a category, i.e. the classes of morphisms are sets. In particular, in this case* $\mathcal{D}^{\mathrm{b}}(\mathscr{A})$ *is the only triangulated category with hereditary heart* $\mathscr{A}$ *up to triangulated equivalence.*

PROOF. Since $\mathscr{A}$ is hereditary, each element in $\mathcal{D}^{\mathrm{b}}(\mathscr{A})$ is a direct sum $E = \bigoplus_i E_i[i]$ with $E_i \in \mathscr{A}$. Then

$$\operatorname{Hom}(E,F) = \operatorname{Hom}\left(\bigoplus_i E_i[i], \bigoplus_j F_j[j]\right) = \bigoplus_{i,j} \operatorname{Hom}(E_i[i], F_j[j])$$

As $\operatorname{Hom}(E_i[i], F_j[j])$ is always a set, we conclude that $\operatorname{Hom}(E,F)$ is also a set. $\square$

CHAPTER 3.

DG-categories

In general, triangulated categories do not offer enough tools to provide a meaningful framework. For this reason, it is common to equip such categories with additional structure. Concerning our studies, we mainly focus on algebraic triangulated categories, which are associated to a (pretriangulated) DG-category.

§3.1. Basics

To begin with, we introduce some basic notions and examples of the theory. We work under Convention 2.1, requiring categories and functors to be $\Bbbk$-linear with $\Bbbk$ a commutative ring, and the following convention, which offers a safe framework for some crucial definitions in §3.2.

3.1. Convention. All categories are $\mathbb{U}$-small for an appropriate Grothendieck universe $\mathbb{U}$.

3.2. Definition. A *graded category* is a ($\Bbbk$-linear) category C such that $\operatorname{Hom}(X,Y)$ is a graded module for any objects $X,Y \in \mathsf{C}$ and the composition

$$\operatorname{Hom}(X,Y) \otimes \operatorname{Hom}(Y,Z) \to \operatorname{Hom}(X,Z)$$

is a graded morphism of degree 0.

Recalling Definition 1.7, a morphism f is *homogeneous of degree i* if $f \in \operatorname{Hom}(X,Y)^i$, and we may indicate the degree with $|f|$. A *(graded) isomorphism* in C is a morphism f of degree 0 with an inverse (which is automatically of degree 0) g, i.e. $gf = \mathrm{id}$ and $fg = \mathrm{id}$.

We emphasize that a morphism with inverse is just an isomorphism in the classical categorical sense, but to avoid confusion between graded isomorphism and isomorphism we will avoid the latter term.

3.3. Remark. Given a graded category C, for any object X the endomorphism ring $\mathrm{Hom}(X,X)$ is in fact a graded ring. This comes from a direct calculation. Conversely, any graded ring A can be seen as a graded category with one object $\diamond$ and $\mathrm{Hom}(\diamond,\diamond) := A$.

3.4. Definition. A functor $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ between graded categories is *graded* if

$$\mathsf{f}_{X,Y} : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(\mathsf{f}X,\mathsf{f}Y)$$

is a graded morphism of degree 0 for every $X,Y \in \mathsf{C}$.

3.5. Remark. Let A,B be graded rings. Then A,B can be also seen as graded categories. A map $f : A \to B$ is a graded ring homomorphism if and only if it gives rise to a graded functor.

3.6. Definition. A *DG-category* is a ($\Bbbk$-linear) category C such that $\mathrm{Hom}(X,Y)$ is a DG-module for any $X,Y \in \mathsf{C}$ and the composition $\mathrm{Hom}(X,Y)\otimes\mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$ is a DG-morphism.

The *differential* of C, sometimes denoted with d_{C}, is the family of all differentials, i.e. $d_{\mathsf{C}} := (d_{X,Y})_{(X,Y)\in\mathsf{C}\times\mathsf{C}}$ where $d_{X,Y}$ is the differential of $\mathrm{Hom}(X,Y)$. The differential d_{C} is said to be trivial if $d_{X,Y} = 0$ for any $(X,Y) \in \mathsf{C}\times\mathsf{C}$. By an abuse of notation, d_{C} will be used also to denote each $d_{X,Y}$. A *closed morphism* is a morphism f such that $d_{\mathsf{C}}(f) = 0$. A *(DG-)isomorphism* in C is a closed morphism of degree 0 with an inverse.

3.7. Remark. Any DG-category can be considered a graded category by discarding the differential. Moreover, a graded category is a DG-category with trivial differential.

DG-rings are exactly all the DG-categories with one object, using the same reasoning of Remark 3.3.

3.8. Definition. Let C,D be two DG-categories. A functor $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ is a *DG-functor* if

$$\mathsf{f}_{X,Y} : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(\mathsf{f}X,\mathsf{f}Y)$$

is a DG-morphism for every $X,Y \in \mathsf{C}$. In other words, it is a graded functor commuting with the differentials.

We say that a DG-functor f is a *DG-equivalence* if it is fully faithful and every object of D is DG-isomorphic to $\mathsf{f}(X)$ for some $X \in \mathsf{C}$.

3.9. Remark. Analogously to Remark 3.5, given two DG-rings, DG-functors and DG-ring homomorphisms are linked in a natural way.

3.10. Notation. For the sake of simplicity, DG-rings and DG-categories with one object are identified, and the only object of a DG-ring A will be denoted by O_A.

We recall the constructions of the tensor product and the functor category.

3.11. Definition. Given C,D two DG-categories, we define $\mathsf{C}\otimes\mathsf{D}$ the DG-category whose objects are couples $X\otimes Y := (X,Y)$ with $X \in \mathsf{C}$ and $Y \in \mathsf{D}$ and whose morphisms are given by the graded module

$$\mathrm{Hom}(X\otimes Y, X'\otimes Y') = \mathrm{Hom}(X\otimes X')\otimes\mathrm{Hom}(Y,Y')$$

whose differential satisfies

$$d(f \otimes g) = d(f) \otimes g + (-1)^{|f|} f \otimes d(g)$$

for any homogeneous elements f, g. The composition respects the Koszul sign rule:

$$(f \otimes g)(f' \otimes g') = (-1)^{|g||f'|} ff' \otimes gg'$$

for all homogeneous elements f, g, f', g'.

3.12. Definition. Let C, D be two DG-categories. We define the *functor DG-category*, denoted by $\mathscr{H}om(\mathsf{C}, \mathsf{D})$, as follows:

- The objects are the DG-functors $\mathsf{f} : \mathsf{C} \to \mathsf{D}$;
- A morphism $\eta : \mathsf{f} \to \mathsf{g}$ of degree i is a family of morphisms $(\eta_X : \mathsf{f}X \to \mathsf{g}X)_{X \in \mathsf{C}}$ of degree i in D such that

  $$\mathsf{g}(h)\eta_X = (-1)^{|\eta||h|} \eta_Y \mathsf{f}(h) \tag{3.13}$$

  for any homogeneous $h : X \to Y$. With this definition, we can create $\mathrm{Hom}(\mathsf{f}, \mathsf{g})$ as a graded module whose i-th component is the set of all morphisms of degree i. The differential

  $$(d\eta)_X := d(\eta_X)$$

  gives to $\mathscr{H}om(\mathsf{C}, \mathsf{D})$ the structure of a DG-category.

In fact, morphisms of $\mathscr{H}om(\mathsf{C}, \mathsf{D})$ are also called *DG-natural transformations*. Similarly, DG-isomorphisms, i.e. closed DG-natural transformations of degree 0 with an inverse, are also called *DG-natural isomorphisms*.

3.14. Remark. One may show that a DG-functor $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ is a DG-equivalence if and only if there exists an inverse up to DG-natural isomorphisms, i.e. if and only if there exists $\mathsf{g} : \mathsf{D} \to \mathsf{C}$ such that $\mathsf{gf} \cong \mathrm{id}_{\mathsf{C}}$ and $\mathsf{fg} \cong \mathrm{id}_{\mathsf{D}}$, in the sense of DG-natural isomorphisms. This result is analogous to the classical one for the equivalence of categories.

An important class of DG-categories is given by complexes.

3.15. Definition. Let $\mathscr{A}$ be a ($\Bbbk$-linear) category. The *DG-category of complexes*, denoted with $\mathsf{C}_{\mathrm{DG}}(\mathscr{A})$, is described as follows:

- Its objects are complexes $M = (M^i, d^i)_{i \in \mathbb{Z}}$, as in Definition 2.6.
- Given M, N complexes, the ℓ-th homogeneous morphisms are

  $$\mathrm{Hom}_{\mathsf{C}_{\mathrm{DG}}(\mathscr{A})}(M, N)^{\ell} := \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(M^i, N^{\ell+i}).$$

  The differential is defined by

  $$d(f) := d_N f - (-1)^{|f|} f d_M$$

  for any homogeneous element f.

According to Definition 2.6, we can consider complexes that are bounded above, bounded below or bounded, and write $\mathsf{C}^*_{\mathrm{DG}}(\mathscr{A})$, with $* = -, +$ or b respectively, for the full DG-subcategory obtained. The notation $\mathsf{C}^*_{\mathrm{DG}}(\mathscr{A})$ will refer to all four categories of complexes defined (analogously to the case of the homotopy category of complexes, $* = \varnothing$ indicates $\mathsf{C}_{\mathrm{DG}}(\mathscr{A})$).

An important example is the *category of DG-modules* $\mathsf{C}_{\mathrm{DG}}(\mathrm{Mod}(\Bbbk))$, where $\mathrm{Mod}(\Bbbk)$ is the category of all (right) $\Bbbk$-modules.

3.16. Definition. The *homotopy category* $H^0(\mathsf{C})$ of a DG-category C has the same objects of C and the morphisms are defined by

$$\mathrm{Hom}_{H^0(\mathsf{C})}(X,Y) := H^0(\mathrm{Hom}_{\mathsf{C}}(X,Y))$$

for any $X,Y \in \mathsf{C}$; compositions and identities of $H^0(\mathsf{C})$ are naturally induced by the ones of C. Two objects X,Y in C are *homotopy equivalent* if they are isomorphic in $H^0(\mathsf{C})$. The *graded homotopy category* $H^*(\mathsf{C})$ has the same objects of C and $\mathrm{Hom}_{H^*(\mathsf{C})}(X,Y) := H^*(\mathrm{Hom}_{\mathsf{C}}(X,Y))$.

Any DG-functor $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ gives rise to a functor $H^0(\mathsf{f}) : H^0(\mathsf{C}) \to H^0(\mathsf{D})$ and a graded functor $H^*(\mathsf{f}) : H^*(\mathsf{C}) \to H^*(\mathsf{D})$ in a natural way.

3.17. Example. Notice that, for any additive category $\mathscr{A}$, $H^0(\mathsf{C}^*_{\mathrm{DG}}(\mathscr{A})) = \mathcal{K}^*(\mathscr{A})$. This motivates the naming choice.

3.18. Definition. Let C be a DG-category. Its *truncation* is the couple $(\tau_{\leq 0}\mathsf{C}, \mathsf{p}_{\leq 0})$ where $\tau_{\leq 0}\mathsf{C}$ is the DG-category whose objects are the same of C and

$$\mathrm{Hom}_{\tau_{\leq 0}\mathsf{C}}(X,Y)^n := \begin{cases} \mathrm{Hom}(X,Y)^n & \text{if } n<0 \\ \ker d^0_{\mathrm{Hom}(X,Y)} & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases}$$

for every $X,Y \in \mathsf{C}$, while $\mathsf{p}_{\leq 0} : \tau_{\leq 0}\mathsf{C} \to \mathsf{C}$ is the natural DG-functor which is the identity on objects and the inclusion on morphisms.

3.19. Remark. Dually, one may expect to define the truncation $(\tau_{\geq 0}\mathsf{C}, \mathsf{p}_{\geq 0} : \mathsf{C} \to \tau_{\geq 0}\mathsf{C})$. However, $\mathsf{p}_{\geq 0}$ is not always well-defined: pick two composable homogeneous morphisms f and g, respectively of degree $-i$ and $n+i$ with $n,i > 0$, such that $gf \neq 0$. Then $gf = \mathsf{p}_{\geq 0}(gf) \neq \mathsf{p}_{\geq 0}(g)\mathsf{p}_{\geq 0}(f) = \mathsf{p}_{\geq 0}(g)0 = 0$.*

Despite such situation, notice that in the case of $\tau_{\leq 0}\mathsf{C}$, the left truncation exists because the DG-functor $\tau_{\leq 0}\mathsf{C} \to H^0(\mathsf{C})$ is actually well-defined, as the obstruction above cannot happen.

3.20. Remark. Given a DG-functor $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, we have a natural DG-functor $\tau_{\leq 0}\mathsf{f} : \tau_{\leq 0}\mathsf{C} \to \tau_{\leq 0}\mathsf{D}$ satisfying $\mathsf{f}\mathsf{p}_{\leq 0} = \mathsf{p}_{\leq 0}\tau_{\leq 0}\mathsf{f}$.

3.21. Definition. Let C be a DG-category. The *opposite DG-category* C^o has the same objects of C, while morphisms have opposite direction

$$\mathrm{Hom}_{\mathsf{C}^o}(X,Y) := \mathrm{Hom}_{\mathsf{C}}(Y,X)$$

*The author would like to thank Amnon Neeman for pointing out the impossibility of this dual truncation.

and composition is reversed, with a change of sign:

$$f \circ_{\mathsf{C}^o} g := (-1)^{|f||g|} g \circ_{\mathsf{C}} f,$$

where f, g are homogeneous.

We recall that, by Convention 1.3, all modules are considered to be right. Accordingly, we give the following key definition.

3.22. Definition. Let C be a DG-category. A *(right) DG* C*-module* is a DG-functor

$$M : \mathsf{C}^o \to \mathsf{C}_{\mathrm{DG}}(\mathrm{Mod}(\Bbbk)).$$

The DG-category of (right) DG C-modules is $\mathrm{DGMod}(\mathsf{C}) := \mathscr{H}om(\mathsf{C}^o, \mathsf{C}_{\mathrm{DG}}(\mathrm{Mod}(\Bbbk)))$.

3.23. Lemma – Yoneda DG-embedding. *The Yoneda embedding*

$$\mathsf{y} : \mathsf{C} \to \mathrm{DGMod}(\mathsf{C}) : X \mapsto \mathrm{Hom}_{\mathsf{C}}(-, X)$$

is a fully faithful DG-functor.

3.24. Remark. Given a DG-functor $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, we have an induced DG-functor

$$\mathrm{Ind}(\mathsf{f}) : \mathrm{DGMod}(\mathsf{C}) \to \mathrm{DGMod}(\mathsf{D})$$

such that $\mathrm{Ind}(\mathsf{f})\mathsf{y} \cong \mathsf{y}\mathsf{f}$. We refer to [21, C.9] for its definition.

§3.2. Pretriangulated DG-categories

In this section, we discuss the notion of pretriangulated DG-categories, given by the DG-versions of shifts and cones, and provide some expected results.

3.25. Definition. Let C be a DG-category. Given $X \in \mathsf{C}$, its *suspension* is an object $Y \in \mathsf{C}$ equipped with two closed morphisms $\sigma_X : Y \to X$ of degree 1 and $\tau_X : X \to Y$ of degree -1 such that $\tau_X \sigma_X = \mathrm{id}_Y$ and $\sigma_X \tau_X = \mathrm{id}_X$. In this case, we will write $\Sigma(X) := Y$, since Y is determined up to unique DG-isomorphism. Conversely, we will also write $\Sigma^{-1}(Y) := X$.

A *suspension functor* $\Sigma : \mathsf{C} \to \mathsf{C}$ is a DG-functor such that there exist two closed DG-natural transformations $\sigma : \Sigma \to \mathrm{id}$, of degree 1, and $\tau : \mathrm{id} \to \Sigma$, of degree -1, such that $\tau\sigma = \sigma\tau = \mathrm{id}$.

3.26. Remark. By definition, a DG-category with a suspension functor has suspensions for all objects. The converse is also true in view of Lemma 3.23 and Corollary 3.31 below.

3.27. Definition. Let $f : X \to Y$ be a closed morphism of degree 0 in a DG-category C. A *cone* of f is an object $\mathrm{Cone}_{\mathrm{DG}}(f)$ together with 4 morphisms of degree 0

$$\begin{tikzcd} Y \arrow[r, "j"] & \mathrm{Cone}_{\mathrm{DG}}(f) \arrow[r, "p"] \arrow[l, bend left, dashed, "q"] & \Sigma(X) \arrow[l, bend left, dashed, "i"] \end{tikzcd}$$

satisfying

$$pi = \mathrm{id}, \quad qj = \mathrm{id}, \quad qi = 0, \quad pj = 0, \quad ip + jq = \mathrm{id}$$

and

$$d(j) = d(p) = 0, \quad d(i) = jf\sigma_X, \quad d(q) = -f\sigma_X p.$$

We emphasize that j and p are required to be closed, while q and i are generally not. Notice that cones are uniquely determined up to DG-isomorphism (see [7, Lemma 4.8]).

3.28. Definition. A DG-category C is *strongly pretriangulated* if it is closed under cones and suspensions, meaning that:

- Any object $X \in \mathsf{C}$ has a suspension $Y \in \mathsf{C}$, and there exists $Z \in \mathsf{C}$ such that X is a suspension for Z;
- Any closed morphism of degree 0 have a cone.

The previous definitions are inspired by Definition/Proposition 2.8. In particular, we will see a description of strongly pretriangulated DG-categories using DG-modules (see Lemma 3.36).

3.29. Definition. Let M be a DG-module. The associated *suspended DG-module* $\Sigma(M)$ is defined by

$$\Sigma(M)^p := M^{p+1} \qquad \text{and} \qquad d_{\Sigma(M)} := -d_M.$$

For a homogeneous morphism $f : M \to N$, we set $\Sigma(f) : \Sigma(M) \to \Sigma(N)$ by $\Sigma(f)^i := (-1)^{|f|} f^{i+1}$.

3.30. Proposition. [77, Theorem 4.1.8]. *The association*

$$\Sigma : \mathsf{C}_{\mathrm{DG}}(\mathrm{Mod}(\Bbbk)) \to \mathsf{C}_{\mathrm{DG}}(\mathrm{Mod}(\Bbbk))$$

is a suspension functor.

3.31. Corollary. ★. *For any DG-category* C*,* $\mathrm{DGMod}(\mathsf{C})$ *has a suspension functor.*

PROOF. Given M a DG C-module, its suspension is given by the composition

$$\Sigma M : \ \mathsf{C}^o \xrightarrow{M} \mathsf{C}_{\mathrm{DG}}(\mathrm{Mod}(\Bbbk)) \xrightarrow{\Sigma} \mathsf{C}_{\mathrm{DG}}(\mathrm{Mod}(\Bbbk)).$$

This clearly extends to a DG-functor $\mathrm{DGMod}(\mathsf{C}) \to \mathrm{DGMod}(\mathsf{C})$. Accordingly, the associated natural DG-transformations are induced by the ones of Σ on $\mathsf{C}_{\mathrm{DG}}(\mathrm{Mod}(\Bbbk))$. □

3.32. Proposition. *For any DG-category* C*,* $\mathrm{DGMod}(\mathsf{C})$ *is strongly pretriangulated.*

PROOF. Corollary 3.31 deals with suspensions, so it remains to prove the closure under cones. Given a closed morphism $f : M \to N$ of degree 0, we define $\mathrm{Cone}_{\mathrm{DG}}(f)$ to be the graded module $N \oplus \Sigma M$ with differential

$$d_{\mathrm{Cone}_{\mathrm{DG}}(f)} := \begin{pmatrix} d_N & f\sigma_M \\ 0 & d_{\Sigma M} \end{pmatrix}.$$

We obtain a diagram

$$\begin{tikzcd} N \arrow[r, "\binom{\mathrm{id}}{0}"] & \mathrm{Cone}_{\mathrm{DG}}(f) \arrow[r, "(0\ \mathrm{id})"] \arrow[l, dashed, bend left, "(\mathrm{id}\ 0)"] & \Sigma M \arrow[l, dashed, bend left, "\binom{0}{\mathrm{id}}"] \end{tikzcd}$$

satisfying the definition of cone (cf. Definition/Proposition 2.8). □

Thanks to Lemma 3.23 and Proposition 3.32, any DG-category admits an embedding into a strongly pretriangulated DG-category.

3.33. Definition. For any DG-category C, its *pretriangulated closure* is the smallest full DG-subcategory $\mathsf{C}^{\mathrm{pretr}}$ of DGMod(C) that contains $\mathsf{y}(\mathsf{C})$ and is strongly pretriangulated. In particular, notice that the Yoneda embedding factors through $\mathsf{C}^{\mathrm{pretr}}$. By an abuse of notation, we call $\mathsf{y} : \mathsf{C} \hookrightarrow \mathsf{C}^{\mathrm{pretr}}$ the natural embedding so obtained.

3.34. Remark. From Corollary 3.31, notice that $\mathsf{C}^{\mathrm{pretr}}$ has a suspension functor associated, as already argued in Remark 3.26.

3.35. Remark. Given a DG-functor $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, we have an induced $\mathsf{f}^{\mathrm{pretr}} : \mathsf{C}^{\mathrm{pretr}} \to \mathsf{D}^{\mathrm{pretr}}$ given by restricting Ind(f) introduced in Remark 3.24.

3.36. Lemma. [15, Lemma 6.2.2]. *A DG-category C is strongly pretriangulated if and only if $\mathsf{y} : \mathsf{C} \to \mathsf{C}^{\mathrm{pretr}}$ is a DG-equivalence.*

3.37. Remark. We emphasize that any strongly pretriangulated DG-category C has a suspension functor, denoted by Σ_{C}.

3.38. Example. A direct check shows that $\mathsf{C}^*_{\mathrm{DG}}(\mathscr{A})$ is a strongly pretriangulated DG-category for any additive category $\mathscr{A}$ (cf. Definition/Proposition 2.8 and Proposition 3.32).

Here we strengthen [7, Proposition 4.11].

3.39. Proposition. ★. *Let C, D be DG-categories, with D strongly pretriangulated. Then the embedding $\mathsf{y} : \mathsf{C} \hookrightarrow \mathsf{C}^{\mathrm{pretr}}$ induces a DG-equivalence $\mathscr{H}om(\mathsf{C}^{\mathrm{pretr}}, \mathsf{D}) \to \mathscr{H}om(\mathsf{C}, \mathsf{D})$.*

More explicitly,

(Essential surjectivity). *Any DG-functor $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ admits an extension $\mathsf{f}' : \mathsf{C}^{\mathrm{pretr}} \to \mathsf{D}$.*

(Fully faithfulness). *Let us consider another DG-functor $\mathsf{g} : \mathsf{C} \to \mathsf{D}$ and an extension $\mathsf{g}' : \mathsf{C}^{\mathrm{pretr}} \to \mathsf{D}$. Then any DG-natural transformation $\eta : \mathsf{f} \to \mathsf{g}$ extends to a unique DG-natural transformation $\eta' : \mathsf{f}' \to \mathsf{g}'$. In particular, if η is a DG-natural isomorphism, then η' is a DG-natural isomorphism.*

Proof. An extension $\mathsf{C}^{\mathrm{pretr}} \to \mathsf{D}$ is given by $\mathsf{y}^{-1}\mathsf{f}^{\mathrm{pretr}}$, since y is a DG-equivalence because of Lemma 3.36.

For the sake of simplicity, assume $\mathsf{D} = \mathsf{D}^{\mathrm{pretr}}$, $\mathsf{C} \subset \mathsf{C}^{\mathrm{pretr}}$ and $\mathsf{f}'_{|\mathsf{C}} = \mathsf{f}$. We aim to prove that η can be extended to η'. For suspensions, we consider $\mathsf{f}'(\Sigma M)$ for $M \in \mathsf{C}$. Then $\mathsf{f}'(\sigma_M)$ and $\mathsf{f}'(\tau_M)$

shows that $\mathsf{f}'(\Sigma M) \cong \Sigma(\mathsf{f}M)$ (via unique DG-isomorphism). As the same can be said about g', we define

$$\eta'_{\Sigma M} : \mathsf{f}'(\Sigma M) \xrightarrow{\cong} \Sigma(\mathsf{f}M) \xrightarrow{\Sigma(\eta_M)} \Sigma(\mathsf{g}M) \xrightarrow{\cong} \mathsf{g}'(\Sigma M).$$

By construction, $\eta'_{\Sigma M} = \eta_{\Sigma M}$ if $\Sigma M \in \mathsf{C}$. Therefore, from now on we may assume that C is closed under suspensions. Let $\varphi_h : M_h \to N_h$ be a closed morphism of C, with $h = 1, 2$. We consider $\mathrm{Cone}_{\mathrm{DG}}(\varphi_h)$ with the following notation

$$\begin{tikzcd} N_h \arrow[r, "j_h"] & \mathrm{Cone}_{\mathrm{DG}}(\varphi_h) \arrow[r, "p_h"] \arrow[l, bend left, dashed, "q_h"] & \Sigma M_h \arrow[l, bend left, dashed, "i_h"] \end{tikzcd}$$

and define $\eta'_{\mathrm{Cone}_{\mathrm{DG}}(\varphi_h)} := \mathsf{g}'(j_h)\eta_{N_h}\mathsf{f}'(q_h) + \mathsf{g}'(i_h)\eta_{\Sigma M_h}\mathsf{f}'(p_h)$. In order to conclude the proof, we aim to prove that for any homogeneous morphism $\nu : \mathrm{Cone}_{\mathrm{DG}}(\varphi_1) \to \mathrm{Cone}_{\mathrm{DG}}(\varphi_2)$, the requirement (3.13) of DG-natural transformation holds. This will be done in two steps.

First of all, we restrict the claim to $\mu : P \to \mathrm{Cone}_{\mathrm{DG}}(\varphi_2)$ with $P \in \mathsf{C}$. For the sake of simplicity, in the following computations we avoid the subscript 2.

$$\begin{aligned} \eta'_{\mathrm{Cone}_{\mathrm{DG}}(\varphi)}\mathsf{f}'(\mu) &= \mathsf{g}'(j)\eta_N\mathsf{f}'(q)\mathsf{f}'(\mu) + \mathsf{g}'(i)\eta_{\Sigma M}\mathsf{f}'(p)\mathsf{f}'(\mu) \\ &= \mathsf{g}'(j)\eta_N\mathsf{f}(q\mu) + \mathsf{g}'(i)\eta_{\Sigma M}\mathsf{f}(p\mu) \\ &= (-1)^{|\eta||q\mu|}\mathsf{g}'(j)\mathsf{g}(q\mu)\eta_P + (-1)^{|\eta||p\mu|}\mathsf{g}'(i)\mathsf{g}(p\mu)\eta_P \\ &= (-1)^{|\eta||\mu|}(\mathsf{g}'(j)\mathsf{g}(q\mu) + \mathsf{g}'(i)\mathsf{g}(p\mu))\eta_P \\ &= (-1)^{|\eta||\mu|}\mathsf{g}'(jq+ip)\mathsf{g}'(\mu)\eta_P = (-1)^{|\eta||\mu|}\mathsf{g}'(\mu)\eta_P \end{aligned}$$

We are now able to address the general situation of $\nu : \mathrm{Cone}_{\mathrm{DG}}(\varphi_1) \to \mathrm{Cone}_{\mathrm{DG}}(\varphi_2)$:

$$\begin{aligned} \mathsf{g}'(\nu)\eta'_{\mathrm{Cone}_{\mathrm{DG}}(\varphi_1)} &= \mathsf{g}'(\nu j_1)\eta_{N_1}\mathsf{f}'(q_1) + \mathsf{g}'(\nu i_1)\eta_{\Sigma M_1}\mathsf{f}'(p_1) \\ &= (-1)^{|\nu j_1||\eta|}\eta'_{\mathrm{Cone}_{\mathrm{DG}}(\varphi_2)}\mathsf{f}'(\nu j_1)\mathsf{f}'(q_1) + (-1)^{|\nu i_1||\eta|}\eta'_{\mathrm{Cone}_{\mathrm{DG}}(\varphi_2)}\mathsf{f}'(\nu i_1)\mathsf{f}'(p_1) \\ &= (-1)^{|\nu||\eta|}\eta'_{\mathrm{Cone}_{\mathrm{DG}}(\varphi_2)}\mathsf{f}'(\nu(j_1q_1 + i_1p_1)) = (-1)^{|\nu||\eta|}\eta'_{\mathrm{Cone}_{\mathrm{DG}}(\varphi_2)}\mathsf{f}'(\nu) \end{aligned}$$

It is important to notice that the same passages also entail $\eta_{\mathrm{Cone}_{\mathrm{DG}}(\varphi_h)} = \eta'_{\mathrm{Cone}_{\mathrm{DG}}(\varphi_h)}$ whenever $\mathrm{Cone}_{\mathrm{DG}}(\varphi_h) \in \mathsf{C}$. The last sentence of the statement follows by definition of η'. □

3.40. Definition. A DG-functor $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ is

- *quasi-fully faithful* if $H^*(\mathsf{f}) : H^*(\mathsf{C}) \to H^*(\mathsf{D})$ is fully faithful;
- *quasi-essentially surjective* if $H^0(\mathsf{f}) : H^0(\mathsf{C}) \to H^0(\mathsf{D})$ is essentially surjective;
- a *quasi-equivalence* if it is quasi-fully faithful and quasi-essentially surjective.

From now on, $\simeq$ is used to indicate that a DG-functor is a quasi-equivalence.

3.41. Remark. If a DG-functor $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ is a quasi-equivalence, then also $\mathsf{f}^{\mathrm{pretr}}$ is a quasi-equivalence (see [21, Proposition 2.5]). Similarly, if f is fully faithful, then $\mathsf{f}^{\mathrm{pretr}}$ is fully faithful as well.

Moreover, by the suspension functor, one can show that a DG-functor f between strongly pretriangulated DG-categories is a quasi-equivalence if and only if $H^0(\mathsf{f})$ is an equivalence.

3.42. Definition. A *quasi-functor* $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ is a (suitable equivalence class of) zig-zag of DG-functors

$$\begin{tikzcd} & \mathsf{C}_1 \arrow[dl, "\approx"'] \arrow[dr] & & \cdots \arrow[dl, "\approx"'] \arrow[dr] & \\ \mathsf{C} & & \mathsf{C}_2 & & \mathsf{D}. \end{tikzcd}$$

A *quasi-equivalence quasi-functor* is a quasi-functor given by a zig-zag of quasi-equivalences. Two DG-categories are *quasi-equivalent* if there exists a quasi-equivalence quasi-functor. In particular, a quasi-functor is a morphism of the homotopy category with respect to the model structure on the category of DG-categories with weak equivalences the quasi-equivalences (cf. [29, Definition 1.2.1] and [74]).† We will not investigate this aspect further.

3.43. Remark. According to Remark 3.35 and Remark 3.41, for every quasi-functor $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ we have an induced quasi-functor $\mathsf{f}^{\mathrm{pretr}} : \mathsf{C}^{\mathrm{pretr}} \to \mathsf{D}^{\mathrm{pretr}}$.

3.44. Remark. The only quasi-functor $\mathsf{f} : \mathsf{A} \to \mathsf{B}$ such that $H^0(\mathsf{f}) = 0$ is the trivial one.

Indeed, if $H^0(\mathsf{f}) = 0$, then the image of f is quasi-equivalent to 0. We conclude that $\mathsf{f} = 0$ by the following commutative diagram:

$$\begin{tikzcd} \mathsf{A} \arrow[r] \arrow[rr, bend left, "\mathsf{f}"] \arrow[dr] & \mathrm{im}(\mathsf{f}) \arrow[r, hook] \arrow[d, "\simeq"] & \mathsf{B}. \\ & 0 & \end{tikzcd}$$

3.45. Definition. A DG-category C is *pretriangulated* if $H^0(\mathsf{y}) : H^0(\mathsf{C}) \to H^0(\mathsf{C}^{\mathrm{pretr}})$ is an equivalence.

3.46. Remark. The second part of Remark 3.41 can be generalized to quasi-functors between pretriangulated DG-categories using $H^0(\mathsf{y})$. However, a pretriangulated DG-category does not have a suspension functor in general.

We now introduce some standard examples of strongly pretriangulated DG-categories.

3.47. Definition. Let C be a DG-category. The *DG-category of semi-free DG-modules* $\mathrm{SF}(\mathsf{C})$ is the full DG-subcategory of $\mathrm{DGMod}(\mathsf{C})$ whose objects have a filtration of free DG C-modules, i.e.

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{i-1} \subset M_i \subset \cdots = M$$

such that M_i/M_{i-1} is isomorphic to a direct sum of $\Sigma^n \mathsf{y}(X)$ for some $n \in \mathbb{Z}$ and $X \in \mathsf{C}$.

† In the literature, quasi-functors are generally a zig-zag of DG-functors as above, and our definition of quasi-functor is often called an isomorphism class of quasi-functors. For the purposes of this thesis, our choice is preferable for the sake of brevity.

3.48. Definition. Let C be a DG-category. The *DG-category of perfect complexes* $\mathrm{Perf}(\mathsf{C})$ is the full DG-subcategory of $\mathrm{SF}(\mathsf{C})$ whose objects are homotopy equivalent to a direct summand of $\mathsf{C}^{\mathrm{pretr}}$.

3.49. Remark. In the same fashion of Remark 3.35, any DG-functor $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ admits extensions $\mathrm{Perf}(\mathsf{f}) : \mathrm{Perf}(\mathsf{C}) \to \mathrm{Perf}(\mathsf{D})$ and $\mathrm{SF}(\mathsf{f}) : \mathrm{SF}(\mathsf{C}) \to \mathrm{SF}(\mathsf{D})$, given by restricting the DG-functor $\mathrm{Ind}(\mathsf{f}) : \mathrm{DGMod}(\mathsf{C}) \to \mathrm{DGMod}(\mathsf{D})$, defined in [21, C.9].

3.50. Remark. The following chain of inclusions of strongly pretriangulated DG-categories holds:

$$\mathsf{C}^{\mathrm{pretr}} \subset \mathrm{Perf}(\mathsf{C}) \subset \mathrm{SF}(\mathsf{C}) \subset \mathrm{DGMod}(\mathsf{C}).$$

3.51. Notation. Let C be a DG-category. We will use the following notation:

$$\mathrm{tr}(\mathsf{C}) := H^0(\mathsf{C}^{\mathrm{pretr}}), \quad \mathrm{D}(\mathsf{C})^{\mathrm{c}} := H^0(\mathrm{Perf}(\mathsf{C})), \quad \mathrm{D}(\mathsf{C}) := H^0(\mathrm{SF}(\mathsf{C})).$$

3.52. Remark. The notation $\mathrm{D}(\mathsf{C})^{\mathrm{c}}$ comes from the fact that $H^0(\mathrm{Perf}(\mathsf{C}))$ is given by all the *compact objects* of $\mathrm{D}(\mathsf{C})$ ([38]). We recall that an object X in a triangulated category with all coproducts is compact if $\mathrm{Hom}(X, -)$ commutes with coproducts.

Another important fact to keep in mind is that $\mathrm{D}(\mathsf{C})^{\mathrm{c}}$ is the idempotent closure of $\mathrm{tr}(\mathsf{C})$.

We notice that these homotopy categories are always triangulated.

3.53. Proposition. *Let* C *be a DG-category. Then* $\mathrm{tr}(\mathsf{C})$, $\mathrm{D}(\mathsf{C})^{\mathrm{c}}$, $\mathrm{D}(\mathsf{C})$ *and* $H^0(\mathrm{DGMod}(\mathsf{C}))$ *are triangulated categories (cf [77, Corollary 5.4.14]).*

In particular, the shift functor Σ *is the homotopy functor* $H^0(\Sigma_{\mathrm{DG}})$, *where* Σ_{DG} *is a suspension functor as in Definition 3.25, and the distinguished triangles are generated by*

$$X \xrightarrow{f} Y \xrightarrow{j} \mathrm{Cone}_{\mathrm{DG}}(f) \xrightarrow{p} \Sigma_{\mathrm{DG}}(X)$$

with notation of Definition 3.27, meaning that a distinguished triangle is a triangle $X' \to Y' \to Z' \to \Sigma(X')$ *admitting an isomorphism of triangles*

$$\begin{tikzcd}
X \arrow[r, "H^0(f)"] \arrow[d, "x"] & Y \arrow[r] \arrow[d, "y"] & \mathrm{Cone}_{\mathrm{DG}}(f) \arrow[r] \arrow[d, "z"] & \Sigma(X) \arrow[d, "\Sigma(x)"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma(X')
\end{tikzcd}$$

(cf. [77, Theorem 5.4.3]).

3.54. Definition. Let C be a pretriangulated category. Then the triangulated structure considered on $H^0(\mathsf{C})$ is the one obtained from the equivalence $H^0(\mathsf{y}) : H^0(\mathsf{C}) \to \mathrm{tr}(\mathsf{C})$.

3.55. Proposition. ★. *Consider a DG-functor* $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ *between strongly pretriangulated DG-categories. Then* $H^0(\mathsf{f})$ *is a triangulated functor.*

PROOF. Since f is additive, the same holds for $H^0(\mathsf{f})$.

Let us consider Σ_{C}, Σ_{D} the suspension functors according to Remark 3.37. Set $\sigma_{\mathsf{C}} : \Sigma_{\mathsf{C}} \to \mathrm{id}$ the DG-natural transformation of degree 1 and $\tau_{\mathsf{D}} : \mathrm{id} \to \Sigma_{\mathsf{D}}$ the DG-natural transformation of degree -1 as in Definition 3.25. We construct a DG-natural isomorphism η by composition:

$$\mathsf{f}\Sigma_{\mathsf{C}} \xrightarrow{\mathsf{f}(\sigma_{\mathsf{C}})} \mathsf{f} \xrightarrow{\tau_{\mathsf{D}}\mathsf{f}} \Sigma_{\mathsf{D}}\mathsf{f}.$$

We emphasize that η is closed because it is composition of closed DG-natural transformations. It remains to show that $(H^0(\mathsf{f}), H^0(\eta))$ sends distinguished triangles to distinguished triangles. Take a closed morphism $g : X \to Y$ of degree 0 in C and consider its cone

$$\begin{tikzcd} Y \arrow[r, "j"] & \mathrm{Cone}_{\mathrm{DG}}(g) \arrow[r, "p"] \arrow[l, bend left, dashed, "q"] & \Sigma(X) \arrow[l, bend left, dashed, "i"] \end{tikzcd}$$

with the notation of Definition 3.27 (we use g instead of f to avoid misunderstandings between the morphism in C and the functor). Notice that the image of this diagram via f gives a cone in D (the requirements of Definition 3.27 are immediately verified because f commutes with the differential and respects the composition). To conclude the proof, we show that

$$\begin{tikzcd} \mathsf{f}(Y) \arrow[r, "\mathsf{f}(j)"] & \mathsf{f}(\mathrm{Cone}_{\mathrm{DG}}(g)) \arrow[r, "\eta\mathsf{f}(p)"] \arrow[l, bend left, dashed, "\mathsf{f}(q)"] & \Sigma_{\mathsf{D}}(\mathsf{f}(X)) \arrow[l, bend left, dashed, "\mathsf{f}(i)\eta^{-1}"] \end{tikzcd}$$

gives a cone for $\mathsf{f}(g)$. Indeed, every property of Definition 3.27 is satisfied easily from what we observed above, except the last two equalities, which follow from these computations:

$$\begin{aligned} d(\mathsf{f}(i)\eta^{-1}) &= \mathsf{f}(d(i))\eta^{-1} = \mathsf{f}(jg\sigma_{\mathsf{C},X})\eta^{-1} = \mathsf{f}(jg\sigma_{\mathsf{C},X})\mathsf{f}(\tau_{\mathsf{C},X})\sigma_{\mathsf{D},\mathsf{f}(X)} \\ &= \mathsf{f}(j)\mathsf{f}(g)\sigma_{\mathsf{D},\mathsf{f}(X)} \\ d(\mathsf{f}(q)) &= -\mathsf{f}(g\sigma_{\mathsf{C},X}p) = -\mathsf{f}(g)\mathsf{f}(\sigma_{\mathsf{C},X})\mathsf{f}(p) = -\mathsf{f}(g)\sigma_{D,\mathsf{f}(X)}\tau_{D,\mathsf{f}(X)}\mathsf{f}(\sigma_{\mathsf{C},X})\mathsf{f}(p) \\ &= -\mathsf{f}(g)\sigma_{D,\mathsf{f}(X)}\eta\mathsf{f}(p), \end{aligned}$$

where τ_{C} is the inverse of σ_{C} and σ_{D} is the inverse of τ_{D}. □

3.56. Corollary. ★. *Let* $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ *be a DG-functor between pretriangulated DG-categories. Then* $H^0(\mathsf{f})$ *is a triangulated functor.*

PROOF. We extend f to a DG-functor $\mathsf{f}^{\mathrm{pretr}} : \mathsf{C}^{\mathrm{pretr}} \to \mathsf{D}^{\mathrm{pretr}}$ by Proposition 3.39. Then this extension gives a triangulated functor by Proposition 3.55. Let $\mathsf{y}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}^{\mathrm{pretr}}$ and $\mathsf{y}_{\mathsf{D}} : \mathsf{D} \to \mathsf{D}^{\mathrm{pretr}}$ be the Yoneda embeddings. Then $H^0(\mathsf{f}) \cong H^0(\mathsf{y}_{\mathsf{D}})^{-1}H^0(\mathsf{f}^{\mathrm{pretr}})H^0(\mathsf{y}_{\mathsf{C}})$ is triangulated by Definition 3.54. □

3.57. Definition/Proposition. [21, Theorem 1.6.2] Let C be a DG-category and $\mathsf{B} \subset \mathsf{C}$ a full DG-subcategory. A *DG-quotient*, often denoted by C/B, is a DG-category D together with a quasi-functor $\mathsf{q} : \mathsf{C} \to \mathsf{D}$ satisfying the following equivalent properties:

1. The functor $H^0(\mathsf{q})$ is essentially surjective and $H^0(\mathsf{q}^{\mathrm{pretr}})$ induces a triangulated equivalence $\mathrm{tr}(\mathsf{C})/\,\mathrm{tr}(\mathsf{B}) \to \mathrm{tr}(\mathsf{D})$.
2. For every DG-category K, the category of quasi-functors $\mathsf{D} \to \mathsf{K}$ is equivalent by composition to the category of quasi-functors $\mathsf{C} \to \mathsf{K}$ such that $\mathsf{B} \to \mathsf{C} \to \mathsf{K}$ is zero (see [21, Appendix E] for a discussion on these categories).

The DG-quotient is determined up to quasi-equivalence, i.e. given another DG-quotient D' with $\mathsf{q}' : \mathsf{C} \to \mathsf{D}'$, we can find a quasi-equivalence quasi-functor $\mathsf{f} : \mathsf{D} \to \mathsf{D}'$ such that $\mathsf{q}' \cong \mathsf{f}\mathsf{q}$.

3.58. Remark. With the same notation above, we can choose D so that q becomes a DG-functor. Additionally, if C is pretriangulated, then so is D. The reader may refer to [51, Remark 1.4 and Lemma 1.5].

3.59. Example – Main examples. Here we give two very important examples of triangulated categories.

- Let $\mathscr{A}$ be any additive category. Let us consider $\mathsf{C}_{\mathrm{DG}}(\mathscr{A})$, whose homotopy category is the homotopy category of complexes $\mathcal{K}(\mathscr{A})$. By Proposition 3.39 and Remark 3.41, the inclusion $\mathscr{A} \to \mathsf{C}_{\mathrm{DG}}(\mathscr{A})$ extends to a fully faithful DG-functor $\mathscr{A}^{\mathrm{pretr}} \to \mathsf{C}_{\mathrm{DG}}(\mathscr{A})$ because $\mathsf{C}_{\mathrm{DG}}(\mathscr{A})$ is strongly pretriangulated (as noted in Example 3.38). Therefore, $\mathrm{tr}(\mathscr{A})$ is equivalent to a triangulated subcategory of $\mathcal{K}(\mathscr{A})$. Moreover, it is the triangulated envelope of $\mathscr{A}$. This suffices to conclude that $\mathrm{tr}(\mathscr{A}) \cong \mathcal{K}^{\mathrm{b}}(\mathscr{A})$, since $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$ is the triangulated envelope of $\mathscr{A}$ in $\mathcal{K}(\mathscr{A})$. In addition, $\mathsf{C}^{\mathrm{b}}_{\mathrm{DG}}(\mathscr{A})$ is quasi-equivalent to $\mathscr{A}^{\mathrm{pretr}}$.
  For a general ($\Bbbk$-linear) category $\mathscr{A}$, the same argument proves that $\mathrm{tr}(\mathscr{A})$ is equivalent to the bounded homotopy category of complexes in its additive closure.
- Let $\mathscr{E}$ be an exact category. Let $\mathrm{Ac}^*_{\mathrm{DG}}\,\mathscr{E}$ be the full DG-subcategory of $\mathsf{C}^*_{\mathrm{DG}}(\mathscr{E})$ whose objects are acyclic complexes. Consider the DG-quotient $\mathcal{D}^*_{\mathrm{DG}}(\mathscr{E}) := \mathsf{C}^*_{\mathrm{DG}}(\mathscr{E})/\,\mathrm{Ac}^*_{\mathrm{DG}}\,\mathscr{E}$. Its homotopy category is equivalent to $\mathcal{D}^*(\mathscr{E})$ since, by definition, we have

$$H^0(\mathcal{D}^*_{\mathrm{DG}}(\mathscr{E})) \cong H^0(\mathsf{C}^*_{\mathrm{DG}}(\mathscr{E}))/H^0(\mathrm{Ac}^*_{\mathrm{DG}}\,\mathscr{E}) = \mathcal{K}^*(\mathscr{E})/\,\mathrm{Ac}^*(\mathscr{E}) = \mathcal{D}^*(\mathscr{E}).$$

§3.3. Algebraic triangulated categories

For a fixed triangulated category, a (DG-)enhancement is a pretriangulated DG-category associated to it. If such an enhancement exists, the triangulated category is called algebraic and we can wonder whether the enhancement is unique. In this section, we present some preliminary results on uniqueness and (semi-)strong uniqueness of enhancements.

3.60. Definition. Let $\mathscr{T}$ be a triangulated category. A *(DG-)enhancement* of $\mathscr{T}$ is a couple (C, E) where C is a pretriangulated DG-category and $E : H^0(\mathsf{C}) \to \mathscr{T}$ is a triangulated equivalence. If a triangulated category admits a (DG-)enhancement, it is called *algebraic*.

3.61. Definition. An algebraic triangulated category $\mathscr{T}$ has a *unique (DG-)enhancement* if, given two enhancements (C,E) and (C',E'), there exists a quasi-equivalence quasi-functor $\mathsf{f}:\mathsf{C}\to\mathsf{C}'$. In other words, there exists a zig-zag of quasi-equivalences

$$\begin{tikzcd} & \mathsf{D}_1 \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & & \cdots \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ \mathsf{C} & & \mathsf{D}_2 & & \mathsf{C}'. \end{tikzcd}$$

3.62. Definition. An algebraic triangulated category $\mathscr{T}$ has a *strongly unique enhancement* (respectively, *semi-strongly unique*) if, given two enhancements (C,E) and (C',E'), there exists a quasi-equivalence quasi-functor $\mathsf{f}:\mathsf{C}\to\mathsf{C}'$ such that $E\cong E'H^0(\mathsf{f})$ (respectively, $E(X)\cong E'H^0(\mathsf{f})(X)$ for all $X\in\mathsf{C}$).

The following is a non conventional definition.

3.63. Definition. Let $F:\mathscr{T}\to\mathscr{T}'$ be a triangulated functor between algebraic triangulated categories. Given enhancements (C,E) and (C',E') of $\mathscr{T}$ and $\mathscr{T}'$ respectively, a $(\mathsf{C},E)-(\mathsf{C}',E')$*-lift* of F is a quasi-functor $\mathsf{f}:\mathsf{C}\to\mathsf{C}'$ such that $F\cong E'H^0(\mathsf{f})E^{-1}$. If $\mathscr{T}=\mathscr{T}'$ and $(\mathsf{C}',E')=(\mathsf{C},E)$, we will say that f is a (C,E)*-lift* of F. If for every enhancement (C,E) there exists a (C,E)-lift for F, we say that F has a *good DG-lift*.

Similarly, a $(\mathsf{C},E)-(\mathsf{C}',E')$*-semilift* of F is a quasi-functor $\mathsf{f}:\mathsf{C}\to\mathsf{C}'$ such that $F(X)\cong E'H^0(\mathsf{f})E^{-1}(X)$ for any $X\in\mathscr{T}$. In the same fashion as above, we define (C,E)*-semilift* and *good DG-semilift*.

3.64. Proposition. ★. *Let $\mathscr{T}$ be an algebraic triangulated category with a unique enhancement. The following are equivalent.*

1. *$\mathscr{T}$ has a strongly unique enhancement.*
2. *Every triangulated autoequivalence $F:\mathscr{T}\to\mathscr{T}$ has a good DG-lift.*
3. *There exists an enhancement (C,E) such that any triangulated autoequivalence $F:\mathscr{T}\to\mathscr{T}$ has a (C,E)-lift.*
4. *There exist two enhancements (C,E) and (C',E') such that any triangulated autoequivalence $F:\mathscr{T}\to\mathscr{T}$ has a $(\mathsf{C},E)-(\mathsf{C}',E')$-lift.*

PROOF. $1\Rightarrow 4$. Given two arbitrary enhancements (C,E) and (C',E'), we can find a quasi-functor f such that $FE\cong E'H^0(\mathsf{f})$ by considering the enhancements (C,FE) and (C',E').

$4\Rightarrow 3$. Let us consider a quasi-equivalence $\mathsf{g}:\mathsf{C}'\to\mathsf{C}$. This gives an equivalence $G=EH^0(\mathsf{g})(E')^{-1}:\mathscr{T}\to\mathscr{T}$. We now consider h a $(\mathsf{C},E)-(\mathsf{C}',E')$-lift for $G^{-1}F$. The quasi-functor $\mathsf{gh}:\mathsf{C}\to\mathsf{C}$ is then a (C,E)-lift for F:

$$\begin{aligned} EH^0(\mathsf{gh})E^{-1} &\cong EH^0(\mathsf{g})H^0(\mathsf{h})E^{-1} \\ &\cong EH^0(\mathsf{g})(E')^{-1}E'H^0(\mathsf{h})E^{-1} \\ &\cong GG^{-1}F\cong F \end{aligned}$$

$3 \Rightarrow 2$. Let (C',E') be another enhancement and let $\mathsf{g} : \mathsf{C}' \to \mathsf{C}$ be a quasi-equivalence. We denote with L the autoequivalence $\mathscr{T} \to \mathscr{T}$ given by $EH^0(\mathsf{g})(E')^{-1}$. From assumption, LFL^{-1} has a (C,E)-lift h. The quasi-functor $\mathsf{g}^{-1}\mathsf{h}\mathsf{g} : \mathsf{C}' \to \mathsf{C}'$ is the wanted (C',E')-lift of F:

$$\begin{aligned} E'H^0(\mathsf{g}^{-1}\mathsf{h}\mathsf{g})(E')^{-1} &\cong E'H^0(\mathsf{g}^{-1})H^0(\mathsf{h})H^0(\mathsf{g})(E')^{-1} \\ &\cong L^{-1}EH^0(\mathsf{h})E^{-1}L \\ &\cong L^{-1}LFL^{-1}L \cong F. \end{aligned}$$

$2 \Rightarrow 1$. Let (C,E) and (C',E') be two enhancements. By assumption, we have a quasi-equivalence $\mathsf{g} : \mathsf{C} \to \mathsf{C}'$. Now we consider the equivalence $EH^0(\mathsf{g})^{-1}(E')^{-1} : \mathscr{T} \to \mathscr{T}$. Since it has a good DG-lift, we can consider its (C,E)-lift f, satisfying $EH^0(\mathsf{f})E^{-1} \cong EH^0(\mathsf{g})^{-1}(E')^{-1}$. From this natural isomorphism, we obtain

$$H^0(\mathsf{g}\mathsf{f}) = H^0(\mathsf{g})H^0(\mathsf{f}) \cong (E')^{-1}E.$$

In particular, $E'H^0(\mathsf{g}\mathsf{f}) \cong E$. □

Analogously to the previous proposition, one can show the following.

3.65. Proposition. ★. *Let $\mathscr{T}$ be an algebraic triangulated category with a unique enhancement. The following are equivalent.*

1. *$\mathscr{T}$ has a semi-strongly unique enhancement.*
2. *Every triangulated autoequivalence $F : \mathscr{T} \to \mathscr{T}$ has a good DG-semilift.*
3. *There exists an enhancement (C,E) such that any triangulated autoequivalence $F : \mathscr{T} \to \mathscr{T}$ has a (C,E)-semilift.*
4. *There exist two enhancements (C,E) and (C',E') such that any triangulated autoequivalence $F : \mathscr{T} \to \mathscr{T}$ has a $(\mathsf{C},E)-(\mathsf{C}',E')$-semilift.*

3.66. Notation. Let (C,E) be an enhancement of a triangulated category $\mathscr{T}$, and let $\mathscr{S} \subset \mathscr{T}$ be a full subcategory (not necessarily triangulated). With the notation $\mathsf{C}^E_{|\mathscr{S}}$ we mean the full DG-subcategory of C whose objects X are such that $E(X) \cong Y \in \mathscr{S}$. In particular, $EH^0(\mathsf{C}_{|\mathscr{S}})$ is equivalent to $\mathscr{S}$. Usually, we will simply write $\mathsf{C}_{|\mathscr{S}}$ instead of $\mathsf{C}^E_{|\mathscr{S}}$ if there is no misunderstanding.

3.67. Lemma. *Let (C,E) be an enhancement of a triangulated category $\mathscr{T}$ and consider $\mathscr{S} \subset \mathscr{T}$ a full subcategory. Then $\mathsf{C}_{|\mathscr{S}}$ is closed under homotopy equivalence in C.*

PROOF. Let $Y \in \mathsf{C}$ homotopy equivalent to $X \in \mathsf{C}_{|\mathscr{S}}$. In other words, X and Y are isomorphic in $H^0(\mathsf{C})$. Since an equivalence sends isomorphisms to isomorphisms, $E(Y) \cong E(X)$. By definition, there exists $Z \in \mathscr{S}$ such that $E(X) \cong Z$. We obtain $E(Y) \cong Z$, which implies that $Y \in \mathsf{C}_{|\mathscr{S}}$. □

We now investigate some relations between uniqueness of enhancements of the triangulated categories associated to a DG-category. We start with a technical lemma.

3.68. Lemma. ★. *Let $\mathscr{T}$ be a triangulated category and* A *be a DG-category. If*

$$F : \mathscr{T} \to H^0(\mathrm{DGMod}(\mathsf{A}))$$

is a full triangulated functor and $H^0(\mathsf{A}) \subset \mathrm{EssIm}(F)$, then $\mathrm{tr}(\mathsf{A}) \subset \mathrm{EssIm}(F)$.

Moreover, if $\mathscr{T}$ is idempotent complete and $G : \mathscr{T} \to H^0(\mathrm{DGMod}(\mathsf{A}))$ is a fully faithful triangulated functor such that $H^0(\mathsf{A}) \subset \mathrm{EssIm}(G)$, then $\mathrm{D}(\mathsf{A})^{\mathrm{c}} \subset \mathrm{EssIm}(G)$.

PROOF. Let $X \in \mathrm{tr}(\mathsf{A})$. We aim to show that $X \in \mathrm{EssIm}(F)$. If $X \in H^0(\mathsf{A})$, this is true by hypothesis. Therefore, it suffices to show that $\mathrm{EssIm}(F)$ is closed under shifts and cones (cf. Definition 3.33). A trivial reasoning shows that $X \in \mathrm{EssIm}(F)$ if it is the shift of an object in $\mathrm{EssIm}(F)$. It remains to study the case $X = \mathrm{Cone}(f)$ for a morphism $f : Y_1 \to Y_2$ where $Y_i \in \mathrm{EssIm}(F)$ for $i = 1,2$. Notice there exist an object $Z_i \in \mathscr{T}$ and an isomorphism $\varphi_i : FZ_i \to Y_i$. Since F is full, we can find $g : Z_1 \to Z_2$ such that $Fg = \varphi_2^{-1} f \varphi_1$. Then $X \cong \mathrm{Cone}(Fg) \cong F(\mathrm{Cone}(g)) \in \mathrm{EssIm}(F)$.

When $\mathscr{T}$ is idempotent complete and $G : \mathscr{T} \to H^0(\mathrm{DGMod}(\mathsf{A}))$ is a fully faithful triangulated functor, we have that $\mathrm{EssIm}(G)$ is idempotent complete. Indeed, if $f : GX \to GX$ is idempotent, by fully faithfulness there exists an idempotent $e : X \to X$ such that $f = G(e)$. Hence, $e = sr$ and $rs = \mathrm{id}$ for some $s : Y \to X$ and $r : X \to Y$, so that $f = G(e) = G(s)G(r)$ and $G(r)G(s) = \mathrm{id}$. Since $\mathrm{D}(\mathsf{A})^{\mathrm{c}}$ is the idempotent completion of $\mathrm{tr}(\mathsf{A})$ and the latter is contained in $\mathrm{EssIm}(G)$, we conclude. □

3.69. Remark. It directly follows from Lemma 3.68 that every fully faithful triangulated functor $F : \mathrm{tr}(\mathsf{A}) \to \mathrm{tr}(\mathsf{A})$ with $H^0(\mathsf{A}) \subset \mathrm{EssIm}(F)$ is an equivalence. Analogously, any fully faithful triangulated functor $G : \mathrm{D}(\mathsf{A})^{\mathrm{c}} \to \mathrm{D}(\mathsf{A})^{\mathrm{c}}$ with $H^0(\mathsf{A}) \subset \mathrm{EssIm}(G)$ is an equivalence. In particular, in this situation $G_{|\mathrm{tr}(\mathsf{A})} : \mathrm{tr}(\mathsf{A}) \to G(\mathrm{tr}(\mathsf{A}))$ is an equivalence and G is its idempotent extension up to natural isomorphism by [2, Theorem 1.5].

If $H^0(\mathsf{A})$ is a full and essentially wide‡ subcategory of $\mathrm{EssIm}(G_{|H^0(\mathsf{A})})$, the inclusion $\mathrm{tr}(\mathsf{A}) \subset \mathrm{EssIm}(G_{|\mathrm{tr}(\mathsf{A})})$ obtained by Lemma 3.68 is in fact an equivalence. This follows by considering the fully faithful triangulated functor

$$L : \mathrm{tr}(\mathsf{A}) \xhookrightarrow{\mathrm{incl}} \mathrm{EssIm}(G_{|\mathrm{tr}(\mathsf{A})}) \xrightarrow{(G_{|\mathrm{tr}(\mathsf{A})})^{-1}} \mathrm{tr}(\mathsf{A}),$$

which is an equivalence since $H^0(\mathsf{A}) \subset \mathrm{EssIm}(L)$ by assumption. Therefore, since the second functor is already an equivalence, incl is an equivalence as well. In particular, we obtain an induced functor $G' : \mathrm{tr}(\mathsf{A}) \to \mathrm{tr}(\mathsf{A})$ whose idempotent extension is G, since G' is defined as $L^{-1} = (\mathrm{incl})^{-1} G_{|\mathrm{tr}(\mathsf{A})}$.

3.70. Proposition. ★. *Let* A *be a DG-category. If* $\mathrm{D}(\mathsf{A})^{\mathrm{c}}$ *has a strongly unique enhancement, then* $\mathrm{tr}(\mathsf{A})$ *has a strongly unique enhancement.*

‡i.e. it contains at least one object for each isomorphism class.

PROOF. Take a triangulated equivalence $F : \mathrm{tr}(\mathsf{A}) \to \mathrm{tr}(\mathsf{A})$ and consider its idempotent extension $F' : \mathsf{D}(\mathsf{A})^{\mathsf{c}} \to \mathsf{D}(\mathsf{A})^{\mathsf{c}}$, which is unique (up to natural isomorphism) by [2, Theorem 1.5]. Given an enhancement (C, E) of $\mathsf{D}(\mathsf{A})^{\mathsf{c}}$, we have a quasi-functor $\mathsf{f}' : \mathsf{C} \to \mathsf{C}$ which is a (C, E)-lift of F'. We restrict f' to $\mathsf{C}_{|\mathrm{tr}(\mathsf{A})}$ (see Notation 3.66). Since $F'_{|\mathrm{tr}(\mathsf{A})}$ is F, the restiction of f' gives a quasi-functor $\mathsf{f} : \mathsf{C}_{|\mathrm{tr}(\mathsf{A})} \to \mathsf{C}_{|\mathrm{tr}(\mathsf{A})}$ by Lemma 3.67. We conclude that f is a $(\mathsf{C}_{|\mathrm{tr}(\mathsf{A})}, E_{|\mathrm{tr}(\mathsf{A})})$-lift of F.

It remains to show that $\mathrm{tr}(\mathsf{A})$ has a unique enhancement, so that we can apply Proposition 3.64 to conclude. Let (D, E) and (D', E') be two enhancements of $\mathrm{tr}(\mathsf{A})$. Then $\mathrm{Perf}(\mathsf{D})$ and $\mathrm{Perf}(\mathsf{D}')$ are enhancements of $\mathsf{D}(\mathsf{A})^{\mathsf{c}}$. Indeed, $\mathrm{tr}(\mathsf{D}) \cong H^0(\mathsf{D}) \cong \mathrm{tr}(\mathsf{A})$, which implies $\mathsf{D}(\mathsf{D})^{\mathsf{c}} \cong \mathsf{D}(\mathsf{A})^{\mathsf{c}}$ by [2, Theorem 1.5]. By hypothesis, we have a quasi-equivalence quasi-functor $\mathsf{g} : \mathrm{Perf}(\mathsf{D}) \to \mathrm{Perf}(\mathsf{D}')$ lifting the identity of $\mathsf{D}(\mathsf{A})^{\mathsf{c}}$. We now consider the restrictions $\mathrm{Perf}(\mathsf{D})_{|\mathrm{tr}(\mathsf{A})}$ and $\mathrm{Perf}(\mathsf{D}')_{|\mathrm{tr}(\mathsf{A})}$. Since g is a lift of the identity, it induces a quasi-equivalence quasi-functor $\mathsf{g}' : \mathrm{Perf}(\mathsf{D})_{|\mathrm{tr}(\mathsf{A})} \to \mathrm{Perf}(\mathsf{D}')_{|\mathrm{tr}(\mathsf{A})}$. We conclude by the following diagram:

$$\mathsf{D} \xhookrightarrow{\simeq} \mathrm{Perf}(\mathsf{D})_{|\mathrm{tr}(\mathsf{A})} \xrightarrow{\mathsf{g}'} \mathrm{Perf}(\mathsf{D}')_{|\mathrm{tr}(\mathsf{A})} \xhookleftarrow{\simeq} \mathsf{D}'.$$

□

3.71. Remark. The previous result holds also if we replace strongly unique enhancement with semi-strongly unique enhancement. The only difference in the proof is that we need to consider semilifts.

3.72. Proposition. ★. *Let* A *be a DG-category.*

1. *If* $\mathrm{tr}(\mathsf{A})$ *has a unique enhancement, then* $\mathsf{D}(\mathsf{A})^{\mathsf{c}}$ *has a unique enhancement.*
2. *If* $\mathsf{D}(\mathsf{A})^{\mathsf{c}}$ *has a unique enhancement, then* $\mathsf{D}(\mathsf{A})$ *has a unique enhancement.*

PROOF.

1. Let C and C' be two enhancements of $\mathsf{D}(\mathsf{A})^{\mathsf{c}}$. Then $\mathsf{C}_{|\mathrm{tr}(\mathsf{A})}$ and $\mathsf{C}'_{|\mathrm{tr}(\mathsf{A})}$ are enhancements of $\mathrm{tr}(\mathsf{A})$. In particular, there exists a quasi-equivalence quasi-functor $\mathsf{f} : \mathsf{C}_{|\mathrm{tr}(\mathsf{A})} \to \mathsf{C}'_{|\mathrm{tr}(\mathsf{A})}$. Consider g as the composition

$$\mathsf{C} \xhookrightarrow{\mathsf{y}} \mathrm{Perf}(\mathsf{C}) \xhookleftarrow[\mathrm{Perf(incl)}]{} \mathrm{Perf}(\mathsf{C}_{|\mathrm{tr}\mathsf{A}}) \xrightarrow{\mathrm{Perf}(\mathsf{f})} \mathrm{Perf}(\mathsf{C}'_{|\mathrm{tr}(\mathsf{A})}) \xhookrightarrow{\mathrm{Perf(incl)}} \mathrm{Perf}(\mathsf{C}') \xhookleftarrow[\mathsf{y}]{} \mathsf{C}'$$

   Lemma 3.68 shows that y and Perf(incl) are quasi-equivalences for both C and C'. Since $\mathrm{Perf}(\mathsf{f})$ is a quasi-equivalence as well, g becomes a quasi-equivalence.
2. The proof is very similar to item 1: consider C and C' two enhancements of $\mathsf{D}(\mathsf{A})$. By hypothesis, there exists a quasi-equivalence quasi-functor $\mathsf{f} : \mathsf{C}_{|\mathsf{D}(\mathsf{A})^{\mathsf{c}}} \to \mathsf{C}'_{|\mathsf{D}(\mathsf{A})^{\mathsf{c}}}$. Let h be the composition

$$\mathsf{C} \xrightarrow{\phi} \mathrm{SF}(\mathsf{C}_{|\mathsf{D}(\mathsf{A})^{\mathsf{c}}}) \xrightarrow{\mathrm{SF}(\mathsf{f})} \mathrm{SF}(\mathsf{C}'_{|\mathsf{D}(\mathsf{A})^{\mathsf{c}}}) \xleftarrow{\phi'} \mathsf{C}',$$

where ϕ, ϕ' are defined as in [51, §1]. By [51, Proposition 1.17], they both are quasi-equivalences. Since f is a quasi-equivalence, the same holds for $\mathrm{SF}(\mathsf{f})$ (see [5, Theorem 10.12.5.1], or [38, Example 7.2]). Finally, h is a quasi-equivalence.

□

CHAPTER 4.

Exceptional sequences and existence of enhancements

Let $\mathbb{K}$ be a field. An exceptional object E in a $\mathbb{K}$-linear triangulated category is such that $\mathrm{Hom}(E,E) = \mathbb{K}$ and $\mathrm{Hom}(E,E[n]) = 0$ for $n \neq 0$. Roughly speaking, an exceptional object is a very simple object in the theory of triangulated categories. Indeed, its triangulated envelope is triangulated equivalent to the bounded derived category of finite-dimensional vector spaces or, more geometrically, coherent sheaves over a point. We focus on full strong exceptional sequences, which are examples of ordered sets of exceptional objects in a $\mathbb{K}$-linear triangulated category. Let us recall the following well-known theorem by Bondal, expressing the rigidity of full strong exceptional sequences.

Theorem – Bondal. [6, Theorem 6.2]. *Assume that the bounded derived category $\mathcal{D}^{\mathrm{b}}(X)$ of coherent sheaves on a smooth manifold X has a full strong exceptional sequence $\langle E_1,\dots,E_n\rangle$. Then $\mathcal{D}^{\mathrm{b}}(X) \cong \mathcal{D}^{\mathrm{b}}(\mathrm{mod}(A))$, where $\mathrm{mod}(A)$ is the category of finitely generated (right) modules over the algebra of endomorphisms $A = \mathrm{End}(\bigoplus_{i=1}^n E_i)$.*

This result has been generalized by Keller (see [42, Theorem 8.7]). In particular, for the situation at hand, we have the following.

Theorem – Keller-Orlov. [63, Corollary 1.9]. *Let $\mathscr{T}$ be an algebraic $\mathbb{K}$-linear triangulated category with $\mathbb{K}$ a field. Assume that $\mathscr{T}$ has a full strong exceptional sequence $\langle E_1,\dots,E_n\rangle$. Then $\mathscr{T}$ is triangulated equivalent to the bounded derived category $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}(A))$, where $A = \mathrm{End}(\bigoplus_{i=1}^n E_i)$.*

In this chapter, we present the content of [49], where we wonder whether we can drop the algebraic requirement in the theorem above. With this purpose in mind, we give a construction of a global t-structure starting with compatible t-structures on semiorthogonal components (see Theorem 4.7). As a corollary, a full strong exceptional sequence of length 2 gives a hereditary heart. By Hubery's Theorem 2.56, we obtain the following.

4.13. Corollary. *Let $\mathbb{K}$ be a field. Any $\mathbb{K}$-linear triangulated category $\mathscr{T}$ with a full strong exceptional sequence $\langle E_1, E_2\rangle$ such that $\dim_{\mathbb{K}} \operatorname{Hom}(E_1, E_2) < \infty$ is triangulated equivalent to $\mathcal{D}^{\mathrm{b}}(\operatorname{mod}(A))$, where $A = \operatorname{End}(\bigoplus_{i=1}^{2} E_i)$.*

In the case of a full strong exceptional sequence of length greater than 2, we deal with realized triangulated categories, meaning that they admit a realization functor for every admissible abelian subcategory. In particular, all algebraic triangulated categories are realized (see Example 4.24 for examples of realized triangulated categories). An induction on the length of the exceptional sequence proves the main result below.

4.28. Theorem. *Let $\mathbb{K}$ be a field and let $\mathscr{T}$ be a realized $\mathbb{K}$-linear triangulated category with a full strong exceptional sequence $\langle E_1, \dots, E_n\rangle$ such that $\bigoplus_i \operatorname{Hom}(X, Y[i])$ is a finite-dimensional vector space for any $X, Y \in \mathscr{T}$. Then $\mathscr{T} \cong \mathcal{D}^{\mathrm{b}}(\operatorname{mod}(A))$, where $A = \operatorname{End}(\bigoplus_{i=1}^{n} E_i)$.*

§4.1. Semiorthogonal decompositions and compatibility of t-structures

After recalling the notion of semiorthogonal decomposition, we define compatibility between t-structures with respect to such decomposition. In Theorem 4.7 we show how this situation gives rise to a global t-structure. As an application of the result, we study exceptional sequences and state Corollary 4.13, which generalizes Bondal's theorem [6, Theorem 6.2] for exceptional sequences of length 2. Throughout this chapter, we will use Convention 2.1 (Convention 3.1 is not needed).

4.1. Definition. Let $\mathscr{T}$ be a triangulated category. A *semiorthogonal decomposition* is a sequence of triangulated subcategories $\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_n$ such that

1. $\operatorname{Hom}(\mathscr{T}_i, \mathscr{T}_j) = 0$ when $i > j$;
2. For any $E \in \mathscr{T}$, there is a filtration
$$0 = E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 = E$$
such that $\operatorname{Cone}(E_i \to E_{i-1}) \in \mathscr{T}_i$ for any $i \in \{1, \dots, n\}$.

In this situation, we will write $\mathscr{T} = \langle \mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_n\rangle$.

4.2. Remark. Item 1 entails that both the filtration and its cones are unique up to isomorphism and functorial, i.e. we can define functors $\sigma_i : \mathscr{T} \to \mathscr{T}_i$ described by $\sigma_i(E) = \operatorname{Cone}(E_i \to E_{i-1})$. Moreover, if $\mathscr{T} = \langle \mathscr{T}_1, \mathscr{T}_2\rangle$, then σ_1 is the left adjoint to the inclusion $\mathscr{T}_1 \hookrightarrow \mathscr{T}$ and σ_2 is the right adjoint to $\mathscr{T}_2 \hookrightarrow \mathscr{T}$.

4.3. Definition. Let $\mathscr{T}$ be a triangulated category. Given two full subcategories $\mathscr{X}$ and $\mathscr{Y}$ of $\mathscr{T}$, we define $\mathscr{X} * \mathscr{Y}$ to be the full subcategory of $\mathscr{T}$ whose objects are

$$\{Z \in \mathscr{T} \mid \text{there exists a distinguished triangle } X \to Z \to Y \to X[1], \text{ with } X \in \mathscr{X}, Y \in \mathscr{Y}\}.$$

This construction gives rise to an operation $*$ between full subcategories of $\mathscr{T}$.

4.4. Proposition. [4, Lemma 1.3.10]. *The operation $*$ is associative.*

4.5. Example. Let $\mathscr{T}$ be a triangulated category. Given a semiorthogonal decomposition $\mathscr{T} = \langle \mathscr{T}_1, \dots, \mathscr{T}_n \rangle$, we can write $\mathscr{T} = \mathscr{T}_n * \cdots * \mathscr{T}_2 * \mathscr{T}_1$. If we consider a t-structure $\mathscr{T}^{\leq 0}$ on $\mathscr{T}$, we have $\mathscr{T} = \mathscr{T}^{\leq 0} * \mathscr{T}^{\geq 1}$.

4.6. Definition. Let $\mathscr{T} = \langle \mathscr{T}_1, \mathscr{T}_2 \rangle$ be a semiorthogonal decomposition, $\mathscr{T}$ any triangulated category. Assume that $\mathscr{T}_i$ has a t-structure $\mathscr{T}_i^{\leq 0}$ for $i = 1, 2$. Then $\mathscr{T}_1^{\leq 0}$ and $\mathscr{T}_2^{\leq 0}$ are *compatible in $\mathscr{T}$* if $\operatorname{Hom}(\mathscr{T}_1^{\leq 0}, \mathscr{T}_2^{\geq 1}) = 0$.

Denoted by $\mathscr{A}_1$ and $\mathscr{A}_2$ the hearts of $\mathscr{T}_1^{\leq 0}$ and $\mathscr{T}_2^{\leq 0}$ respectively, the *relative dimension of $\mathscr{A}_1$ and $\mathscr{A}_2$ in $\mathscr{T}$* is the number

$$\operatorname{rdim}_{\mathscr{T}}(\mathscr{A}_1, \mathscr{A}_2) := \begin{cases} \sup\{m \in \mathbb{Z} \mid \operatorname{Hom}(\mathscr{A}_1, \mathscr{A}_2[m]) \neq 0\} & \text{if the set is nonempty} \\ -1 & \text{otherwise.} \end{cases}$$

Notice that, whenever the set above is nonempty, $\operatorname{rdim}_{\mathscr{T}}(\mathscr{A}_1, \mathscr{A}_2) \geq 0$ by compatibility. The reason why we have chosen the value -1 in case the set is empty will become clear reading the statement of Theorem 4.7.

4.7. Theorem. *Let $\mathscr{T}$ be a triangulated category with a semiorthogonal decomposition $\mathscr{T} = \langle \mathscr{T}_1, \mathscr{T}_2 \rangle$. Given two compatible t-structures $\mathscr{T}_1^{\leq 0}$ and $\mathscr{T}_2^{\leq 0}$ on $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively, the full subcategory of $\mathscr{T}$ defined by*

$$\mathscr{T}^{\leq 0} := \mathscr{T}_2^{\leq 0} * (\mathscr{T}_1^{\leq 0}[1])$$

is a t-structure on $\mathscr{T}$. Furthermore,

1. *If $\mathscr{T}_1^{\leq 0}$ and $\mathscr{T}_2^{\leq 0}$ are bounded (respectively non-degenerate), then $\mathscr{T}^{\leq 0}$ is bounded (respectively non-degenerate).*
2. *Let $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}$ be the hearts of t-structures associated to $\mathscr{T}_1^{\leq 0}$, $\mathscr{T}_2^{\leq 0}$ and $\mathscr{T}^{\leq 0}$ respectively. Then*
$$\mathscr{A} = \mathscr{A}_2 * (\mathscr{A}_1[1]).$$
3. *The following equality holds true whenever at least one of the two hearts of t-structures $\mathscr{A}_1, \mathscr{A}_2$ is nonzero:*
$$\dim_{\mathscr{T}} \mathscr{A} = \max\{\dim_{\mathscr{T}_1} \mathscr{A}_1, \dim_{\mathscr{T}_2} \mathscr{A}_2, \operatorname{rdim}_{\mathscr{T}}(\mathscr{A}_1, \mathscr{A}_2) + 1\}.$$

PROOF. Since $\mathscr{T}_i^{\leq 0}[1] \subset \mathscr{T}_i^{\leq 0}$ for $i = 1, 2$, it is clear that also $\mathscr{T}^{\leq 0}$ is closed by left shifts. We aim to show that $\mathscr{T} = \mathscr{T}^{\leq 0} * \mathscr{T}^{\geq 1}$, where $\mathscr{T}^{\geq 1} := (\mathscr{T}^{\leq 0})^{\perp}$. Notice that

$$\mathscr{T} = \mathscr{T}_2 * \mathscr{T}_1 = \mathscr{T}_2^{\leq 0} * \mathscr{T}_2^{\geq 1} * (\mathscr{T}_1^{\leq 0}[1]) * (\mathscr{T}_1^{\geq 1}[1]).$$

Since $\langle \mathscr{T}_1, \mathscr{T}_2 \rangle$ is a semiorthogonal decomposition and compatibility holds, we have

$$\mathscr{T}_2^{\geq 1} * (\mathscr{T}_1^{\leq 0}[1]) = \{X \oplus Y \mid X \in \mathscr{T}_2^{\geq 1}, Y \in \mathscr{T}_1^{\leq 0}[1]\} = (\mathscr{T}_1^{\leq 0}[1]) * \mathscr{T}_2^{\geq 1}.$$

Therefore, $\mathscr{T} = \mathscr{T}_2^{\leq 0} * (\mathscr{T}_1^{\leq 0}[1]) * \mathscr{T}_2^{\geq 1} * (\mathscr{T}_1^{\geq 1}[1])$. We claim that $\mathscr{T}_2^{\geq 1} * (\mathscr{T}_1^{\geq 1}[1]) = \mathscr{T}^{\geq 1}$.

Let $A \in \mathscr{T}_2^{\geq 1} * (\mathscr{T}_1^{\geq 1}[1])$. There exists a distinguished triangle $A_2^{\geq 1} \to A \to A_1^{\geq 1}[1] \to A_2^{\geq 1}[1]$ with $A_2^{\geq 1} \in \mathscr{T}_2^{\geq 1}$ and $A_1^{\geq 1}[1] \in \mathscr{T}_1^{\geq 1}[1]$. Now let $B \in \mathscr{T}^{\leq 0}$ and consider a distinguished triangle $B_2^{\leq 0} \to B \to B_1^{\leq 0}[1] \to B_2^{\leq 0}[1]$, where $B_2^{\leq 0} \in \mathscr{T}_2^{\leq 0}$ and $B_1^{\leq 0}[1] \in \mathscr{T}_1^{\leq 0}[1]$. These distinguished triangles give rise to the following hom-exact sequences:

$$\dots \longrightarrow \mathrm{Hom}(B, A_2^{\geq 1}) \longrightarrow \mathrm{Hom}(B, A) \longrightarrow \mathrm{Hom}(B, A_1^{\geq 1}[1]) \longrightarrow \dots$$

$$\dots \longrightarrow \mathrm{Hom}(B_1^{\leq 0}[1], A_1^{\geq 1}[1]) \longrightarrow \mathrm{Hom}(B, A_1^{\geq 1}[1]) \longrightarrow \mathrm{Hom}(B_2^{\leq 0}, A_1^{\geq 1}[1]) \longrightarrow \dots$$

$$\dots \longrightarrow \mathrm{Hom}(B_1^{\leq 0}[1], A_2^{\geq 1}) \longrightarrow \mathrm{Hom}(B, A_2^{\geq 1}) \longrightarrow \mathrm{Hom}(B_2^{\leq 0}, A_2^{\geq 1}) \longrightarrow \dots$$

Since $\langle \mathscr{T}_1, \mathscr{T}_2 \rangle$ is a semiorthogonal decomposition, $\mathrm{Hom}(B_2^{\leq 0}, A_1^{\geq 1}[1]) = 0$, and the properties of t-structures tell us that $\mathrm{Hom}(B_1^{\leq 0}[1], A_1^{\geq 1}[1]) = 0 = \mathrm{Hom}(B_2^{\leq 0}, A_2^{\geq 1})$. By compatibility, we also have $\mathrm{Hom}(B_1^{\leq 0}[1], A_2^{\geq 1}) = 0$. Then the last two exact sequences prove that $\mathrm{Hom}(B, A_1^{\geq 1}[1]) = \mathrm{Hom}(B, A_2^{\geq 1}) = 0$. The first exact sequence concludes that $\mathrm{Hom}(B, A) = 0$. Finally, $A \in \mathscr{T}^{\geq 1}$.

Conversely, if $A \in \mathscr{T}^{\geq 1}$, then there exists a distinguished triangle

$$A^{\leq 0} \to A \to A^{\geq 1} \to A^{\leq 0}[1]$$

with $A^{\leq 0} \in \mathscr{T}^{\leq 0}$ and $A^{\geq 1} \in \mathscr{T}_2^{\geq 1} * (\mathscr{T}_1^{\geq 1}[1])$. Notice $A^{\leq 0} \to A$ must be zero because $A \in \mathscr{T}^{\geq 1}$. Since $A^{\geq 1}$ cannot have a direct summand in $\mathscr{T}^{\leq 0}$, we get that $A^{\leq 0} = 0$. In particular, $A = A^{\geq 1}$; as wanted, $\mathscr{T}_2^{\geq 1} * (\mathscr{T}_1^{\geq 1}[1]) = \mathscr{T}^{\geq 1}$.

1. First, we deal with boundedness. Let $A \in \mathscr{T}$. From the semiorthogonal decomposition, we get a distinguished triangle $A_2 \to A \to A_1[1] \to A_2[1]$ where $A_i \in \mathscr{T}_i$ for $i = 1, 2$. Since $\mathscr{T}_i^{\leq 0}$ is bounded, $A_i \in \mathscr{T}_i^{\leq k_i} \cap \mathscr{T}_i^{\geq h_i}$ for some integers k_i, h_i. Let $k := \max\{k_1, k_2\}$ and $h := \min\{h_1, h_2\}$.
   By assumption, $A_i \in \mathscr{T}_i^{\leq 0}[-k_i]$, so $A_i[k] \in \mathscr{T}_i^{\leq 0}[k - k_i] \subseteq \mathscr{T}_i^{\leq 0}$ because t-structures are closed by left shifts. Therefore, $A[k] \in \mathscr{T}^{\leq 0}$; in other words $A \in \mathscr{T}^{\leq 0}[-k] = \mathscr{T}^{\leq k}$.
   Similarly, $A_i \in \mathscr{T}_i^{\geq 1}[1 - h_i]$ implies $A_i[h-1] \in \mathscr{T}_i^{\geq 1}[1 - h_i + h - 1] \subseteq \mathscr{T}_i^{\geq 1}$ (here we use the closure by right shifts). We conclude that $A[h-1] \in \mathscr{T}^{\geq 1}$, which means that $A \in \mathscr{T}^{\geq 1}[1-h] = \mathscr{T}^{\geq h}$. As wanted, $A \in \mathscr{T}^{\leq k} \cap \mathscr{T}^{\geq h}$.
   To prove non-degeneracy when $\mathscr{T}_1^{\leq 0}$ and $\mathscr{T}_2^{\leq 0}$ are non-degenerate, let $C \in \bigcap_j \mathscr{T}^{\leq j}$. By Remark 4.2, we have $C = \mathrm{Cone}(E \to F)$ for $E \in \bigcap_j \mathscr{T}_1^{\leq j}$ and $F \in \bigcap_j \mathscr{T}_2^{\leq j}$. By hypothesis, both intersections are zero, so $C \cong 0$ as wanted. The proof of $\bigcap_j \mathscr{T}^{\geq j} = 0$ is analogous since $\mathscr{T}^{\geq 1} = \mathscr{T}_2^{\geq 1} * (\mathscr{T}_1^{\geq 1}[1])$.
2. For any $A \in \mathscr{A}$ we can find two distinguished triangles, according to the fact that $A \in \mathscr{T}^{\leq 0}$ and $A[-1] \in \mathscr{T}^{\geq 1} = \mathscr{T}_2^{\geq 1} * (\mathscr{T}_1^{\geq 1}[1])$. Then Remark 4.2 proves that $\mathscr{A}$ is exactly as described in the statement.
3. Let $A = \mathrm{Cone}(A_1 \to A_2)$ and $B = \mathrm{Cone}(B_1 \to B_2)$ be two objects of $\mathscr{A}$, with $A_i, B_i \in \mathscr{A}_i$, $i = 1, 2$. For any m, we consider the long exact sequence

$$\dots \to \mathrm{Hom}(A_1[1], B[m]) \to \mathrm{Hom}(A, B[m]) \to \mathrm{Hom}(A_2, B[m]) \to \dots$$

associated to the distinguished triangle $A_1 \to A_2 \to A \to A_1[1]$. By considering the first and the last term, we can create two exact sequences associated to the distinguished triangle $B_1 \to B_2 \to B \to B_1[1]$:

$$\cdots \to \mathrm{Hom}(A_1[1], B_2[m]) \to \mathrm{Hom}(A_1[1], B[m]) \to \mathrm{Hom}(A_1[1], B_1[m+1]) \to \dots$$
$$\cdots \to \mathrm{Hom}(A_2, B_2[m]) \to \mathrm{Hom}(A_2, B[m]) \to \mathrm{Hom}(A_2, B_1[m+1]) \to \dots$$

Notice that $\mathrm{Hom}(A_2, B_1[m+1]) = 0$ since $A_2 \in \mathscr{T}_2$ and $B_1 \in \mathscr{T}_1$. Defined

$$\ell := \max\{\dim_{\mathscr{T}_1} \mathscr{A}_1, \dim_{\mathscr{T}_2} \mathscr{A}_2, \mathrm{rdim}_{\mathscr{T}}(\mathscr{A}_1, \mathscr{A}_2) + 1\},$$

the exact sequences above prove that $\mathrm{Hom}(A, B[m]) = 0$ for any $m > \ell$, so $\dim_{\mathscr{T}} \mathscr{A} \leq \ell$. To conclude, it suffices to show that $\dim_{\mathscr{T}} \mathscr{A} \geq \ell$.

We have two cases. If ℓ is realized by the homological dimension of $\mathscr{A}_1$ or $\mathscr{A}_2$, we notice that $\mathscr{A}_1[1], \mathscr{A}_2 \subset \mathscr{A}$ by item 2, so $\dim_{\mathscr{T}} \mathscr{A} \geq \ell$.

Assume $\ell = \mathrm{rdim}_{\mathscr{T}}(\mathscr{A}_1, \mathscr{A}_2) + 1$. If $0 < \ell < +\infty$, for some choices of $A_1[1]$ and B_2 in $\mathscr{A}$ we have $\mathrm{Hom}(A_1[1], B_2[\ell]) \neq 0$. Similarly, if $\ell = +\infty$, there is a sequence $(a_n) \subset \mathbb{Z}$ such that $a_n \to +\infty$ and $\mathrm{Hom}(A_1^n[1], B_2^n[a_n]) \neq 0$ for any a_n and some $A_1^n[1], B_2^n \in \mathscr{A}$. In both cases, $\dim_{\mathscr{T}} \mathscr{A}$ cannot be less than ℓ. If $\ell = 0$, then ℓ is also equal to the homological dimensions of $\mathscr{A}_1$ or $\mathscr{A}_2$, and this possibility has already been addressed. □

4.8. Remark. The t-structure constructed in Theorem 4.7 may not behave as wanted. For instance, using the notation of the statement, $\mathscr{A}_1$ is not contained in $\mathscr{A}$: we need to consider its shift $\mathscr{A}_1[1]$.

One may think this shifting could be easily adjusted, but the assumption needed is incredibly strong. The first idea it comes to mind is to consider the t-structure $\mathscr{T}_1^{\leq 1} = \mathscr{T}_1^{\leq 0}[-1]$ instead of $\mathscr{T}_1^{\leq 0}$. Indeed, if we ask $\mathscr{T}_1^{\leq 1}$ and $\mathscr{T}_2^{\leq 0}$ to be compatible, no shift will be involved, and in particular $\mathscr{A}_1, \mathscr{A}_2 \subset \mathscr{A}$. However, such requirement implies that $\mathrm{Hom}(\mathscr{A}_1, \mathscr{A}_2) = 0$, which is generally too restrictive.

4.9. Remark. Theorem 4.7 is incredibly linked to torsion pairs (for an introduction of the concept, we refer to [27, §I.2]). Let $\mathscr{T}$ be a triangulated category with a semiorthogonal decomposition $\langle \mathscr{T}_1, \mathscr{T}_2 \rangle$ and a t-structure $\mathscr{T}^{\leq 0}$ such that $\mathscr{T}_i^{\leq 0} = \mathscr{T}^{\leq 0} \cap \mathscr{T}_i$ is a t-structure on $\mathscr{T}_i$ for $i = 1, 2$. If these t-structures are compatible in $\mathscr{T}$, Theorem 4.7 gives rise to a t-structure $\mathscr{T}_{\#}^{\leq 0}$, which is different from $\mathscr{T}^{\leq 0}$. Indeed, $E \in \mathscr{T}_1^{\leq 0} \cap \mathscr{T}_1^{\geq 0}$ is an object in $(\mathscr{T}_{\#}^{\leq 0} \cap \mathscr{T}_{\#}^{\geq 0})[-1]$.

As a matter of fact, $\mathscr{T}_{\#}^{\leq 0}$ gives rise to a heart which is a tilted version of the heart $\mathscr{A}$ of $\mathscr{T}^{\leq 0}$. This is simply true by picking the couple $\mathscr{F} = \mathscr{A} \cap \mathscr{T}_1$ and $\mathscr{T} = \mathscr{A} \cap \mathscr{T}_2$, which is a torsion pair by [53, Exercise 6.5].

4.10. Remark. Theorem 4.7 is very similar to [4, Theorem 1.4.10], which constructs global t-structures via recollements instead of semiorthogonal decompositions. Let us explain this relation in detail.

First of all, we recall that any recollement gives rise to a semiorthogonal decomposition. We consider $\mathscr{T} = \langle \mathscr{T}_1, \mathscr{T}_2 \rangle$ a semiorthogonal decomposition and let $\mathscr{T}_i^{\leq 0}$ a t-structure on $\mathscr{T}_i$ for

$i = 1,2$. Then from Theorem 4.7 we get the global t-structure $\mathscr{T}_2^{\leq 0} * (\mathscr{T}_1^{\leq 0}[1])$, while, under the assumption that the semiorthogonal decomposition comes from a recollement, [4, Theorem 1.4.10] gives the t-structure $\mathscr{T}_2^{\leq 0} * \mathscr{T}_1^{\leq 0}$. In other words, the new result gives a tilted version of the old statement (see Remark 4.9).

Moreover, the two theorems deal with different situations. Indeed, although it is possible to construct a left adjoint i^* to the inclusion $i_* : \mathscr{T}_1 \to \mathscr{T}$ (i.e. $\mathscr{T}_1$ is *left admissible*) and a right adjoint j^* to the inclusion $j_! : \mathscr{T}_2 \to \mathscr{T}$ (i.e. $\mathscr{T}_2$ is *right admissible*) by [6, Lemma 3.1], in general a left (respectively right) admissible subcategory does not need to be right (respectively left) admissible. Conversely, a recollement does not ensure that the compatibility requirement is satisfied, since $\mathscr{T}_2^{\geq 1}$ is not necessarily equal to $\mathscr{T}^{\geq 1} \cap \mathscr{T}_2$.

Concerning our studies, Theorem 4.7 is to be preferred because it computes the homological dimension of the obtained heart; this is crucial, especially for Corollary 4.13.

The definition of compatible t-structures can be generalized so that Theorem 4.7 holds for semiorthogonal decompositions of any length, but the requirement may result unnatural since we need to consider some shifts.

4.11. Definition. Let $\mathscr{T} = \langle \mathscr{T}_1, \dots, \mathscr{T}_m \rangle$ and assume $\mathscr{T}_i$ has a t-structure $\mathscr{T}_i^{\leq 0}$ for $i = 1, \dots, m$. Then all the t-structures are *compatible* if $\mathrm{Hom}(\mathscr{T}_i^{\leq 0}[k-i-1], \mathscr{T}_k^{\geq 1}) = 0$ for any $k > i$.

With this notion of compatibility, we can apply Theorem 4.7 by recursion. Considering the same notation of the definition above, if $\mathscr{A}_i$ is the heart of $\mathscr{T}_i^{\leq 0}$, the obtained heart $\mathscr{A}$ of $\mathscr{T}$ is described as

$$\mathscr{A} = \mathscr{A}_m * \mathscr{A}_{m-1}[1] * \cdots * \mathscr{A}_2[m-2] * \mathscr{A}_1[m-1].$$

4.12. Example – Exceptional sequences. Let $\mathbb{K}$ be a field and consider a $\mathbb{K}$-linear triangulated category $\mathscr{T}$. We recall that an *exceptional object* is an object $E \in \mathscr{T}$ such that

$$\mathrm{Hom}(E, E[n]) = \begin{cases} \mathbb{K} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

A sequence of exceptional objects $E_1, \dots, E_m \in \mathscr{T}$, such that $\mathrm{Hom}(E_i, E_j[n]) = 0$ for any $i > j$ and all n, is called an *exceptional sequence*. It is *full* if $\mathscr{T}$ is the triangulated envelope of $\{E_1, \dots, E_m\}$.

Consider a $\mathbb{K}$-linear triangulated category with a full exceptional sequence $E_1, \dots, E_m$ such that $\bigoplus_i \mathrm{Hom}(A, B[i])$ is a finite-dimensional vector space for any $A, B \in \mathscr{T}$*. By [31, §1.4], it is known that such a full exceptional sequence gives rise to a semiorthogonal decomposition given by $\mathscr{T}_i = \{\bigoplus_\ell E_i^{\oplus b_\ell}[\ell] : b_\ell \in \mathbb{N}\}$. Accordingly, we will use the notation $\langle E_1, \dots, E_m \rangle$ to indicate exceptional sequences. Notice that on each $\mathscr{T}_i$ we can consider a bounded t-structure with heart $\mathscr{A}_i = \{E_i^{\oplus b} : b \in \mathbb{N}\}$.

*In fact, it suffices to require this property for $A, B \in \{E_1, \dots, E_m\}$.

If the full exceptional sequence is also *strong*†, i.e. $\mathrm{Hom}(E_i, E_j[n]) = 0$ for any i, j and $n \neq 0$, the above t-structures are compatible: indeed, taking $k > i$,

$$\mathrm{Hom}\left(\bigoplus_{\ell \geq 0} E_i^{\oplus b_\ell}[\ell][k-i-1], \bigoplus_{j<0} E_k^{\oplus c_j}[j]\right) = 0.$$

Moreover, the t-structure induced on $\mathscr{T}$ is bounded.

4.13. Corollary. *Let $\mathbb{K}$ be a field. Any $\mathbb{K}$-linear triangulated category $\mathscr{T}$ with a full strong exceptional sequence $\langle E_1, E_2 \rangle$ such that $\dim_{\mathbb{K}} \mathrm{Hom}(E_1, E_2) < \infty$ is algebraic. In particular, $\mathscr{T} \cong \mathcal{D}^{\mathrm{b}}(\mathrm{mod}(A))$, where $A = \mathrm{End}(\bigoplus_{i=1}^{2} E_i)$ and $\mathrm{mod}(A)$ is the category of finitely generated (right) A-modules.*

PROOF. Theorem 4.7 and Example 4.12 prove that $\mathscr{T}$ has a hereditary heart $\mathscr{A}$. By Theorem 2.56, $\mathscr{T} \cong \mathcal{D}^{\mathrm{b}}(\mathscr{A})$ is algebraic. We conclude by [63, Corollary 1.9]. $\square$

4.14. Example. Let $\mathbb{K}$ be a field. By [25, §1], $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^1) := \mathcal{D}^{\mathrm{b}}(\mathrm{Coh}(\mathbb{P}^1_{\mathbb{K}})) = \langle \mathscr{O}, \mathscr{O}(1) \rangle$ is a strong full exceptional sequence, where $\mathrm{Coh}(\mathbb{P}^1_{\mathbb{K}})$ is the category of coherent sheaves on $\mathbb{P}^1_{\mathbb{K}}$, $\mathscr{O}$ is the structure sheaf and $\mathscr{O}(1)$ is its twist. From Corollary 4.13, we conclude that $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^1)$ is the unique $\mathbb{K}$-linear triangulated category with a full strong exceptional sequence $\langle E_1, E_2 \rangle$ such that $\dim_{\mathbb{K}} \mathrm{Hom}(E_1, E_2) = 2$.

§4.2. Quivers

In order to study exceptional sequences of length greater than 2, we need some basic knowledge on quivers. Here we give a brief introduction, mostly following [6, §5].

4.15. Definition. A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is a set of vertices, Q_1 a set of arrows between vertices and $s, t : Q_1 \to Q_0$ are the maps indicating source and target respectively. A quiver is *finite* if Q_0 and Q_1 are finite. It is *ordered* if the vertices are ordered and for every arrow a, $s(a) < t(a)$.

A *path p of length n* is a sequence of arrows $a_1, \ldots, a_n \in Q_1$ such that $t(a_i) = s(a_{i+1})$. Moreover, with the same notation, $t(p) := t(a_n)$ and $s(p) := s(a_1)$. We also allow paths of length 0: such paths are in correspondence with the vertices. Let p, q be two paths. Then the composition of paths $q \circ p$ is defined to be the concatenated path whenever $s(q) = t(p)$.

Let $\mathbb{K}$ be a field. The *path algebra* $\mathbb{K}Q$ is the $\mathbb{K}$-vector space with basis the paths. The product is described as follows:

$$\lambda q \cdot \mu p = \begin{cases} (\lambda\mu)\, q \circ p & \text{if } s(q) = t(p) \\ 0 & \text{otherwise,} \end{cases}$$

†This condition can be weakened.

where $\lambda, \mu \in \mathbb{K}$ and p, q are paths. In particular, paths of length 0 are idempotents in $\mathbb{K}Q$.

If $S \subset \mathbb{K}Q$ is any subset, (Q, S) is called *quiver with relations* and its associated path algebra is given by $\mathbb{K}Q/\langle S\rangle$, where $\langle S\rangle$ is the ideal generated by S.

Now, let us consider $A = \mathbb{K}Q/\langle S\rangle$ the path algebra associated to the quiver with relations (Q, S). A left A-module is a vector space V over $\mathbb{K}$ with the left action of the algebra A. This is also called *representation of a quiver*. When dealing with right A-modules, one can consider the *opposite quiver* Q^{op} where s, t are swapped with respect to Q. In other words, arrows go in the other direction, analogously to what happens with the notion of the opposite category. As one expects, left modules associated to $(Q^{\mathrm{op}}, S^{\mathrm{op}})$ are right modules of A.

In case the quiver Q is finite and ordered, let $X_1, \dots, X_n$ be the vertices and p_i the idempotent in A associated to X_i. Every right A-module V has a decomposition $V = \bigoplus_{i \in Q_0} G_i V$, where $G_i V := V p_i$. Let us denote with S_i the representation for which $G_j S_i = \delta_{ij}\mathbb{K}$, where δ_{ij} is the Kronecker delta, and all arrows are represented by the zero morphisms. Notice that for each right A-module V we can create a filtration

$$0 = F^0V \hookrightarrow F^1V = G^1V \hookrightarrow F^2V = \bigoplus_{j=1}^{2} G_jV \hookrightarrow \cdots \hookrightarrow F^{n-1}V = \bigoplus_{j=1}^{n-1} G_jV \hookrightarrow F^nV = V \tag{4.16}$$

such that each quotient $F^iV/F^{i-1}V$ is a direct sum of copies of S_i. The modules $P_i = p_iA$ are projective and the decomposition $A = \bigoplus_{i=1}^n P_i$ holds. As a matter of fact,

$$A \cong \mathrm{Hom}_A(A, A) \cong \mathrm{Hom}_A\left(\bigoplus_{i=1}^{n} P_i, \bigoplus_{i=1}^{n} P_i\right) \cong \bigoplus_{i,j} \mathrm{Hom}(P_i, P_j).$$

These isomorphisms allow to interpret the arrows of a quiver as morphisms between projective modules. In particular, being A the path algebra of an ordered quiver, $\mathrm{Hom}(P_i, P_j) = 0$ for $i > j$. Furthermore, it is possible to consider the exact sequence

$$0 \to F^{i-1}P_i \to P_i \to S_i \to 0 \tag{4.17}$$

for every $i = 1, \dots, n$. Notice that $P_1 = S_1$.

Let $\mathscr{T}$ be a $\mathbb{K}$-linear algebraic triangulated category with a full strong exceptional sequence $\langle E_1, \dots, E_n\rangle$. Then $A = \mathrm{End}(\bigoplus_{i=1}^n E_i)$ is the path algebra of an ordered and finite quiver with relations. In particular, the equivalence $F : \mathscr{T} \to \mathcal{D}^{\mathrm{b}}(\mathrm{mod}(A))$ obtained in [63, Corollary 1.9] is such that $F(E_i) = P_i$, the projective modules of the path algebra A.

§4.3. Filtered enhancements

In this section, we explore the definition of filtered triangulated categories and give a fairly simple result that we could not find in the literature, namely if a triangulated category admits a filtered enhancement, then every triangulated subcategory admits a filtered enhancement in a natural way (see Proposition 4.21). Main reference is [3, Appendix A]. In Remark 4.22, we discuss the relation of filtered enhancements with realization functors.

4.18. Definition. Let us consider a quintuple $(\mathscr{F}, \mathscr{F}(\leq 0), \mathscr{F}(\geq 0), s, \alpha)$, where $\mathscr{F}$ is a triangulated category, $\mathscr{F}(\leq 0)$ and $\mathscr{F}(\geq 0)$ are strictly full triangulated subcategories, $s : \mathscr{F} \to \mathscr{F}$ is a triangulated isomorphism and $\alpha : \mathrm{id}_{\mathscr{F}} \to s$ is a natural transformation. We set $\mathscr{F}(\leq n) = s^n\mathscr{F}(\leq 0)$ and $\mathscr{F}(\geq n) = s^n\mathscr{F}(\geq 0)$. In this picture, $\mathscr{F}$ is called a *filtered triangulated category* if it satisfies the following axioms:

fcat1 $\mathscr{F}(\leq 0) \subset \mathscr{F}(\leq 1)$ and $\mathscr{F}(\geq 1) \subset \mathscr{F}(\geq 0)$.
fcat2 $\mathscr{F} = \bigcup_n \mathscr{F}(\leq n) = \bigcup_n \mathscr{F}(\geq n)$.
fcat3 $\mathrm{Hom}(\mathscr{F}(\geq 1), \mathscr{F}(\leq 0)) = 0$.
fcat4 For any $X \in \mathscr{F}$ there exists a distinguished triangle $A \to X \to B \to A[1]$ where $A \in \mathscr{F}(\geq 1)$ and $B \in \mathscr{F}(\leq 0)$; in other words, $\mathscr{F} = \mathscr{F}(\geq 1) * \mathscr{F}(\leq 0)$.
fcat5 For any $X \in \mathscr{F}$, it holds that $\alpha_{s(X)} = s(\alpha_X)$.
fcat6 For any $X \in \mathscr{F}(\geq 1)$ and $Y \in \mathscr{F}(\leq 0)$, α induces isomorphisms

$$\mathrm{Hom}(Y, X) \cong \mathrm{Hom}(Y, s^{-1}X) \cong \mathrm{Hom}(sY, X).$$

A triangulated category $\mathscr{T}$ admits a *filtered enhancement* if there exists a filtered triangulated category $\mathscr{F}$ such that $\mathscr{T} \cong \mathscr{F}(\leq 0) \cap \mathscr{F}(\geq 0)$ in the sense of triangulated categories. For the sake of simplicity, we assume that $\mathscr{T} = \mathscr{F}(\leq 0) \cap \mathscr{F}(\geq 0)$.

4.19. Proposition. [3, Proposition A.3]. *Let $\mathscr{F}$ be a filtered triangulated category. Then the following assertions hold true:*

1. *The inclusion $i_{\leq n} : \mathscr{F}(\leq n) \to \mathscr{F}$ has a left adjoint $\sigma_{\leq n}$, and the inclusion $i_{\geq n} : \mathscr{F}(\geq n) \to \mathscr{F}$ has a right adjoint $\sigma_{\geq n}$. In particular, these adjoints are exact (see, for instance, [31, Proposition 1.41]).*
2. *There is a unique natural transformation $\delta : \sigma_{\leq n} \to \sigma_{\geq n+1}[1]$ such that, for any $X \in \mathscr{F}$,*

$$\sigma_{\geq n+1}(X) \to X \to \sigma_{\leq n}(X) \xrightarrow{\delta(X)} \sigma_{\geq n+1}(X)[1]$$

   is a distinguished triangle. Up to unique isomorphism, this is the only distinguished triangle $A \to X \to B \to A[1]$ with $A \in \mathscr{F}(\geq n+1)$ and $B \in \mathscr{F}(\leq n)$.
3. *For any two integers m, n, we have the following natural isomorphisms:*

$$\sigma_{\leq m}\sigma_{\leq n} \cong \sigma_{\leq \min\{m,n\}}, \qquad \sigma_{\geq m}\sigma_{\geq n} \cong \sigma_{\geq \max\{m,n\}}, \qquad \sigma_{\geq m}\sigma_{\leq n} \cong \sigma_{\leq n}\sigma_{\geq m}.$$

PART OF THE PROOF. We want to prove the first two isomorphisms of item 3, since it is the only part of the statement not considered in [3]. Being the reasoning analogous, let us focus just on the first isomorphism. Let $X \in \mathscr{F}$. If $m \geq n$, then $\mathscr{F}(\leq m) \supset \mathscr{F}(\leq n)$. We recall that $\sigma_{\leq m} i_{\leq m} \cong \mathrm{id}$ because the inclusion $i_{\leq m}$ is fully faithful. Since $\sigma_{\leq n}(X) \in \mathscr{F}(\leq m)$, we simply have that $\sigma_{\leq m}\sigma_{\leq n}(X) \cong \sigma_{\leq n}(X)$ by the natural isomorphism mentioned before. We conclude that $\sigma_{\leq m}\sigma_{\leq n} \cong \sigma_{\leq n}$.

Let $m \leq n$, so that $\mathscr{F}(\leq m) \subset \mathscr{F}(\leq n)$. Then, by adjunction, we have the following isomorphisms for any $X \in \mathscr{F}$ and $Y \in \mathscr{F}(\leq m)$:

$$\begin{aligned}\mathrm{Hom}_{\mathscr{F}(\leq m)}(\sigma_{\leq m}\sigma_{\leq n}(X), Y) &\cong \mathrm{Hom}_{\mathscr{F}}(\sigma_{\leq n}(X), Y)\\ &\cong \mathrm{Hom}_{\mathscr{F}(\leq n)}(\sigma_{\leq n}(X), Y)\\ &\cong \mathrm{Hom}_{\mathscr{F}}(X, Y).\end{aligned}$$

In particular, $\sigma_{\leq m}\sigma_{\leq n}$ is left adjoint to $i_{\leq m}$. Since adjoints are determined up to a natural isomorphism, $\sigma_{\leq m}\sigma_{\leq n} \cong \sigma_{\leq m}$ as wanted. □

4.20. Remark. By item 2 of Proposition 4.19, we also have the following isomorphisms:

$$s\sigma_{\leq n} \cong \sigma_{\leq n+1}s, \qquad s\sigma_{\geq n} \cong \sigma_{\geq n+1}s.$$

Let us set $\mathrm{gr}^n := \sigma_{\leq n}\sigma_{\geq n}$. This is not the definition used in [3], but it will come in handy in the proof of the following statement.

4.21. Proposition. *Let $\mathscr{T}$ be a triangulated category admitting a filtered enhancement $\mathscr{F}$. Then any triangulated subcategory $\mathscr{S}$ of $\mathscr{T}$ has a filtered enhancement given by the full subcategory $\mathscr{G}$ of $\mathscr{F}$ with objects*

$$\{X \in \mathscr{F} \mid s^{-n}\,\mathrm{gr}^n(X) \in \mathscr{S}\ \forall n\}.$$

PROOF. First of all, we would like to show that $\mathscr{G}$ is a triangulated subcategory of $\mathscr{F}$. Notice that the shift functor of $\mathscr{F}$ obviously restricts to $\mathscr{G}$ since $s^{-n}\,\mathrm{gr}^n$ is exact, being composition of triangulated functors. Let us consider $X \to Y$ with $X, Y \in \mathscr{G}$. This gives a distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathscr{F}$. We get that

$$s^{-n}\,\mathrm{gr}^n(X) \to s^{-n}\,\mathrm{gr}^n(Y) \to s^{-n}\,\mathrm{gr}^n(Z) \to s^{-n}\,\mathrm{gr}^n(X[1])$$

is a distinguished triangle in $\mathscr{T}$, with $s^{-n}\,\mathrm{gr}^n(X)$ and $s^{-n}\,\mathrm{gr}^n(Y)$ objects of $\mathscr{S}$. This suffices to conclude that $s^{-n}\,\mathrm{gr}^n(Z) \in \mathscr{S}$, so that $Z \in \mathscr{G}$. Next, we set $\mathscr{G}(\leq 0) := \mathscr{G} \cap \mathscr{F}(\leq 0)$ and $\mathscr{G}(\geq 0) := \mathscr{G} \cap \mathscr{F}(\geq 0)$. We would like to prove that the autoequivalence $s : \mathscr{F} \to \mathscr{F}$ can be restricted to $\mathscr{G}$. Let $X \in \mathscr{G}$. Then, by Remark 4.20, we have

$$\begin{aligned}s^{-n}\,\mathrm{gr}^n(sX) &= s^{-n}\sigma_{\leq n}\sigma_{\geq n}s(X)\\ &\cong s^{-n}\sigma_{\leq n}s\sigma_{\geq n-1}(X)\\ &\cong s^{-n}s\sigma_{\leq n-1}\sigma_{\geq n-1}(X)\\ &= s^{-n+1}\,\mathrm{gr}^{n-1}(X) \in \mathscr{S}.\end{aligned}$$

So we can restrict s and create an exact autoequivalence $s : \mathscr{G} \to \mathscr{G}$, called s as well by an abuse of notation. Of course, the restriction of $\alpha : \mathrm{id}_{\mathscr{F}} \to s$ gives us the required natural transformation and fcat5 is ensured. We set $\mathscr{G}(\leq n)$ and $\mathscr{G}(\geq n)$ via s as described in Definition 4.18. Being s an equivalence, we have the following

$$\mathscr{G}(\geq n) = s^n(\mathscr{G}(\geq 0)) = s^n(\mathscr{G} \cap \mathscr{F}(\geq 0)) = s^n(\mathscr{G}) \cap s^n(\mathscr{F}(\geq 0)) = \mathscr{G} \cap \mathscr{F}(\geq n),$$

and analogously $\mathscr{G}(\leq n) = \mathscr{G} \cap \mathscr{F}(\leq n)$. This immediately shows that fcat1,2,3,6 hold. As fcat5 has already been dealt with, it remains to show fcat4. In order to do that, we recall the distinguished triangle in item 2 of Proposition 4.19. Therefore, the statement is reduced to establish that the images of $\sigma_{\leq n}$ and $\sigma_{\geq n}$ are in $\mathscr{G}(\leq n)$ and $\mathscr{G}(\geq n)$ respectively, so that these functors are adjoints to the inclusions as in $\mathscr{F}$. Let $X \in \mathscr{G}$ and consider $\sigma_{\leq m}$. By item 3 of Proposition 4.19 and Remark 4.20 the following isomorphisms hold:

$$\begin{aligned} s^{-n}\operatorname{gr}^n(\sigma_{\leq m}X) &= s^{-n}\sigma_{\leq n}\sigma_{\geq n}\sigma_{\leq m}(X) \\ &\cong s^{-n}\sigma_{\leq n}\sigma_{\leq m}\sigma_{\geq n}(X) \\ &\cong s^{-n}\sigma_{\leq m}\sigma_{\leq n}\sigma_{\geq n}(X) \\ &\cong \sigma_{\leq m-n}s^{-n}\sigma_{\leq n}\sigma_{\geq n}(X). \end{aligned}$$

In particular, $s^{-n}\operatorname{gr}^n(\sigma_{\leq m}X) \cong \sigma_{\leq m-n}(A)$, where $A \in \mathscr{S}$. If $m-n \geq 0$, we have the following inclusions:

$$A \in \mathscr{S} \subset \mathscr{T} \subset \mathscr{F}(\leq 0) \subset \mathscr{F}(\leq m-n),$$

so $\sigma_{\leq m-n}(A) = A$. If $m-n<0$, being $A \in \mathscr{F}(\geq 0)$ it holds that $\operatorname{Hom}(A, \sigma_{\leq m-n}(A)) = 0$ by fcat3. In particular, item 2 of Proposition 4.19 entails that $\sigma_{\leq m-n}(A) = 0$. As wanted, $s^{-n}\operatorname{gr}^n(\sigma_{\leq m}X) \in \mathscr{S}$, so that $\sigma_{\leq m}X \in \mathscr{G}$. With a similar reasoning, one can prove that $\sigma_{\geq m}X \in \mathscr{G}$. □

The reason why filtered enhancements become of great interest is their relation with realization functors (see Definition 2.43).

4.22. Remark. In [3, Appendix], it is proven that every triangulated category with a filtered enhancement admits a realization functor for any heart. However, some authors point out that an additional requirement, called fcat7, may be necessary to provide the result (see [66, Appendix A] for further details). For the sake of completeness, let us state this new axiom using the same notation of Definition 4.18.

fcat7 Given any morphism $f : X \to Y$ in $\mathscr{F}$, the diagram

$$\begin{tikzcd} \sigma_{\geq 1}(X) \arrow[r] \arrow[d, "\alpha_{\sigma_{\geq 1}(Y)}\sigma_{\geq 1}(f)"] & X \arrow[r] \arrow[d, "\alpha_Y f"] & \sigma_{\leq 0}(X) \arrow[r, "\delta(X)"] \arrow[d, "\alpha_{\sigma_{\leq 0}(Y)}\sigma_{\leq 0}f"] & \sigma_{\geq 1}(X)[1] \arrow[d, "\alpha_{\sigma_{\geq 1}(Y)}\sigma_{\geq 1}(f)[1]"] \\ s(\sigma_{\geq 1}Y) \arrow[r] & s(Y) \arrow[r] & s(\sigma_{\leq 0}Y) \arrow[r] & s(\sigma_{\geq 1}Y)[1] \end{tikzcd}$$

can be extended to a 3×3-diagram whose rows and columns are distinguished triangles.

Once ensured that $\mathscr{F}$ satisfies fcat7, it is easy to prove that also $\mathscr{G}$ as defined in Proposition 4.21 fulfills fcat7. This will be key in what follows.

§4.4. Realized triangulated categories

This section revolves around the unconventional notion of realized triangulated categories. After the definition, we will give some large classes of examples studied in the literature and show that

Bondal's theorem [6, Theorem 6.2] can be generalized for this type of triangulated categories.

4.23. Definition. A triangulated category $\mathscr{T}$ is called *realized* if for every heart $\mathscr{A}$ of every triangulated subcategory $\mathscr{S} \subset \mathscr{T}$ there exists a realization functor real : $\mathcal{D}^{\mathrm{b}}(\mathscr{A}) \to \mathscr{T}$. Alternatively, $\mathscr{T}$ is realized if for every admissible abelian subcategory there exists a realization functor in $\mathscr{T}$ (cf. Lemma 2.49).

4.24. Example.

1. Triangulated categories with a filtered enhancement are realized, as discussed in Proposition 4.21 and Remark 4.22.
2. Algebraic triangulated categories are realized by [43, Theorem 3.2], where the first item is proved in detail in [37, Section 4]. In fact, every algebraic triangulated category has a filtered enhancement (see [18, Proposition 3.8]), but fcat7 has not been investigated.
3. Every triangulated category which is the underlying category of a stable derivator admits a filtered enhancement; this is the content of [54]. This applies, for instance, to topological triangulated categories obtained by stable combinatorial model categories by [26, Example 4.2].
4. Triangulated categories satisfying the axioms explained in [58] are realized by [58, Theorem 5.1]. An interesting example is the stable homotopy category (see [58, p. 249]), which is not algebraic, as proved in [44, §7.6].

4.25. Remark. A derived category does not have all the Ext groups of its hearts (recall Definition 2.40). Let us consider $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^1)$ with the notation introduced in Example 4.14. One can show that $\mathscr{A} = \{\mathscr{O}_{\mathbb{P}^1}^{\oplus a_0}[2] \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus a_1} \mid a_0, a_1 \geq 0\}$ gives a heart by applying Theorem 4.7. As highlighted in [53, Exercise 5.3], $\mathcal{D}^{\mathrm{b}}(\mathscr{A}) \cong \mathcal{D}^{\mathrm{b}}(\mathrm{pt})^{\oplus 2}$ is not equivalent to $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^1)$, so $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^1)$ cannot have all the Ext groups of $\mathscr{A}$ by Proposition 2.45.

With a different approach, notice that $\mathscr{A} \ni \mathscr{O}_{\mathbb{P}^1}[2] \to \mathscr{O}_{\mathbb{P}^1}(1)[2] \in \mathscr{A}[2]$ does not factor through an object in $\mathscr{A}[1]$, and therefore Corollary 2.41 proves that $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^1)$ does not have all the Ext groups of $\mathscr{A}$.

4.26. Remark. Let $\mathbb{K}$ be a field and consider a realized $\mathbb{K}$-linear triangulated category $\mathscr{T}$ with a full strong exceptional sequence $\langle E_1, \dots, E_n\rangle$. Then we can consider the heart $\mathscr{A}$ on $\mathscr{T}$ obtained according to Theorem 4.7 and Example 4.12, giving rise to a realization functor $\mathcal{D}^{\mathrm{b}}(\mathscr{A}) \to \mathscr{T}$. One would like to prove that such functor is in fact an equivalence using Proposition 2.45, so that [63, Corollary 1.9] can be applied to ensure the generalization of Bondal's result [6, Theorem 6.2]. However, when $n > 2$, it is not always true that $\mathscr{T}$ has all the Ext groups of $\mathscr{A}$: for instance, if $n = 3$,

$$\mathscr{A} \ni E_1[2] \xrightarrow{f} E_3[2] \in \mathscr{A}[2]$$

does not necessarily factor through $\mathscr{A}[1]$. In general, we would have $f \notin \operatorname{Ext}^2_{\mathscr{A}}(E_1[2], E_3)$ by item 1 of Proposition 2.35. For example, consider the quiver obtained by the following vertices and arrows:

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r] \arrow[rr, bend left, "f"] & \underset{2}{\bullet} & \underset{3}{\bullet} \end{tikzcd}$$

In order to resolve this issue, we recall what was already discussed in Remark 4.9. If the length of the exceptional sequence is 2, the heart obtained by Theorem 4.7 is a tilt of $\mathrm{mod}(A)$, where $A = \mathrm{End}(\bigoplus_{i=1}^{2} E_i)$. As we will see, the same idea can be used to prove the general case.

Let us prove a technical lemma concerning triangulated categories having all the Ext-groups.

4.27. Lemma. *Let $\mathscr{T}$ be a triangulated category with a semiorthogonal decomposition $\mathscr{T} = \langle \mathscr{T}_1, \mathscr{T}_2 \rangle$ and two compatible t-structures $\mathscr{T}_1^{\leq 0}$ and $\mathscr{T}_2^{\leq 0}$ on $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively. We denote with $\mathscr{A}_i$ the heart of $\mathscr{T}_i^{\leq 0}$. By Theorem 4.7, we obtain the heart of t-structure*

$$\mathscr{A} = \mathscr{A}_2 * \mathscr{A}_1[1].$$

We consider the following hypotheses:

1. *$\mathscr{T}_i$ has all the Ext groups of $\mathscr{A}_i$;*
2. *The map $f_{m,A,B} : \mathrm{Ext}^m_{\mathscr{A}}(A,B) \to \mathrm{Hom}_{\mathscr{T}}(A,B[m])$ defined in the statement of Proposition 2.35 is an isomorphism for every $A \in \mathscr{A}_1[1]$ and $B \in \mathscr{A}_2$.*

Then $\mathscr{T}$ has all the Ext groups of $\mathscr{A}$.

PROOF. Before starting the actual proof, let us remark that $\mathrm{Ext}^m_{\mathscr{A}}(A,B) = \mathrm{Ext}^m_{\mathscr{A}_2}(A,B)$ whenever $A,B \in \mathscr{A}_2$. Indeed, let

$$\mathbf{X} : 0 \to B \to X_1 \to X_2 \to \cdots \to X_n \to A \to 0$$

be an extension in $\mathscr{A}$ with $A,B \in \mathscr{A}_2$ and let $\sigma_2 : \mathscr{T} \to \mathscr{T}_2$ be the right adjoint of the inclusion functor $\iota : \mathscr{T}_2 \to \mathscr{T}$ (see Remark 4.2). Then we get

$$\begin{tikzcd} \iota\sigma_2\mathbf{X}: & 0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{id}"] & \iota\sigma_2 X_1 \arrow[r] \arrow[d] & \dots \arrow[r] & \iota\sigma_2 X_n \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ \mathbf{X}: & 0 \arrow[r] & B \arrow[r] & X_1 \arrow[r] & \dots \arrow[r] & X_n \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

which shows that $\iota\sigma_2\mathbf{X} \cong \mathbf{X}$ in $\mathrm{Ext}^m_{\mathscr{A}}(A,B)$ (recall the equivalence relation used to describe the Yoneda extensions in Definition 2.29). Since $\sigma_2\mathbf{X} \in \mathrm{Ext}^m_{\mathscr{A}_2}(A,B)$, we conclude that ι gives an isomorphism between $\mathrm{Ext}^m_{\mathscr{A}_2}(A,B)$ and $\mathrm{Ext}^m_{\mathscr{A}}(A,B)$ whenever $A,B \in \mathscr{A}_2$. In a similar way, considering the left adjoint of the inclusion $\mathscr{T}_1 \to \mathscr{T}$, one can prove that $\mathrm{Ext}^m_{\mathscr{A}}(A,B) = \mathrm{Ext}^m_{\mathscr{A}_1[1]}(A,B)$ if $A,B \in \mathscr{A}_1[1]$.

Given $A,B \in \mathscr{A}$, we consider two distinguished triangle $A_2 \to A \to A_1 \to A_2[1]$ and $B_2 \to B \to B_1 \to B_2[1]$ with $A_2,B_2 \in \mathscr{A}_2$ and $A_1,B_1 \in \mathscr{A}_1[1]$. We obtain the following hom-exact sequences

$$\ldots \longrightarrow \mathrm{Hom}(A_1,B[m]) \longrightarrow \mathrm{Hom}(A,B[m]) \longrightarrow \mathrm{Hom}(A_2,B[m]) \longrightarrow \ldots$$

$$\ldots \longrightarrow \mathrm{Hom}(A_1,B_2[m]) \longrightarrow \mathrm{Hom}(A_1,B[m]) \longrightarrow \mathrm{Hom}(A_1,B_1[m]) \longrightarrow \ldots$$

$$\ldots \longrightarrow \mathrm{Hom}(A_2,B_2[m]) \longrightarrow \mathrm{Hom}(A_2,B[m]) \longrightarrow \mathrm{Hom}(A_2,B_1[m]) = 0 \longrightarrow \ldots$$

By Proposition 2.35, these exact sequences have maps from the Ext groups. We proceed by induction on m. From the induction hypothesis and item 3 of Proposition 2.35 we deduce that

$$\mathrm{Ext}^m_{\mathscr{A}}(A_2,B_1) \subseteq \mathrm{Hom}(A_2,B_1[m]) = 0.$$

Therefore, using the third row, hypothesis 1 and the five lemma entail that $\mathrm{Hom}(A_2,B[m]) \cong \mathrm{Ext}^m_{\mathscr{A}}(A_2,B)$. The second row proves that $\mathrm{Hom}(A_1,B[m]) \cong \mathrm{Ext}^m_{\mathscr{A}}(A_1,B)$ by both hypotheses and the five lemma. From the first row, we conclude that $\mathrm{Hom}(A,B[m]) \cong \mathrm{Ext}^m_{\mathscr{A}}(A,B)$. □

4.28. Theorem. *Let $\mathbb{K}$ be a field and let $\mathscr{T}$ be a realized $\mathbb{K}$-linear triangulated category with a full strong exceptional sequence $\langle E_1,\ldots,E_n\rangle$ such that $\bigoplus_i \mathrm{Hom}(X,Y[i])$ is a finite-dimensional vector space for any $X,Y \in \mathscr{T}$. Then $\mathscr{T} \cong \mathcal{D}^{\mathrm{b}}(\mathrm{mod}(A))$, where $A = \mathrm{End}(\bigoplus_{i=1}^n E_i)$ and $\mathrm{mod}(A)$ is the category of finitely generated (right) A-modules.*

PROOF. We will prove the statement by induction on n, the length of the exceptional sequence. The base case $n = 2$ is already taken care of by Corollary 4.13.

If $n > 2$, we write $\mathscr{T} = \langle \tilde{\mathscr{T}}, E_n\rangle$ with $\tilde{\mathscr{T}} := \langle E_1,\ldots,E_{n-1}\rangle$. By induction hypothesis, there exists a triangulated equivalence $\varphi : \mathcal{D}^{\mathrm{b}}(\mathrm{mod}(\tilde{A})) \to \tilde{\mathscr{T}}$ with $\tilde{A} = \mathrm{End}(\bigoplus_{i=1}^{n-1} E_i)$. We divide the proof in two parts:

1. The t-structures associated to $\varphi(\mathrm{mod}(\tilde{A}))$ and E_n are compatible. By Theorem 4.7, we obtain a heart $\mathscr{A}$ on $\mathscr{T}$.
2. $\mathscr{T}$ has all the Ext groups of $\mathscr{A}$.

Once both items are ensured, Proposition 2.45 can be applied, proving that $\mathscr{T} \cong \mathcal{D}^{\mathrm{b}}(\mathscr{A})$, and an application of [63, Corollary 1.9] will complete the proof.

From (4.16), every object $X \in \mathrm{mod}(\tilde{A})$ has an associated filtration

$$0 = F^0X \hookrightarrow F^1X \hookrightarrow \cdots \hookrightarrow F^{n-2}X \hookrightarrow F^{n-1}X = X$$

where $F^kX/F^{k-1}X$ is a direct sum of copies of S_k. Moreover, for each P_k there is a short exact sequence $0 \to F^{k-1}P_k \to P_k \to S_k \to 0$ by (4.17). In particular, $S_1 = P_1$.

Let us deal with 1. It suffices to show that $\mathrm{Hom}(\varphi(X),E_n[m]) = 0$ for every $m \leq -1$ and $X \in \mathrm{mod}(\tilde{A})$. We proceed by induction on k, requiring $F^kX = X$. If $k = 1$, F^1X is in fact a direct sum of copies of $P_1 = \varphi^{-1}(E_1)$, so the claim holds because the sequence is strong.

If $k > 1$, notice that the short exact sequence $0 \to F^{k-1}P_k \to P_k \to S_k \to 0$ is associated to a distinguished triangle in $\mathscr{T}$, so it gives rise to the hom-sequence

$$\mathrm{Hom}(\varphi(F^{k-1}P_k)[1],E_n[m]) \to \mathrm{Hom}(\varphi(S_k),E_n[m]) \to \mathrm{Hom}(E_k,E_n[m]).$$

By induction, $\mathrm{Hom}(\varphi(F^{k-1}P_k)[1], E_n[m]) = 0$, while $\mathrm{Hom}(E_k, E_n[m]) = 0$ by hypothesis. Therefore, $\mathrm{Hom}(\varphi(S_k), E_n[m]) = 0$. We now consider $X = F^kX$ and the distinguished triangle

$$F^{k-1}X \to X \to X/F^{k-1}X \to F^{k-1}X[1]$$

obtained by the filtration. From the associated hom-sequence, $\mathrm{Hom}(\varphi(X), E_n[m]) = 0$ since the same holds for $F^{k-1}X$ and $X/F^{k-1}X$, the last one being a direct sum of copies of S_k.

It remains to prove item 2. According to Lemma 4.27, we will prove by induction on m that $\mathrm{Hom}(\varphi(X), E_n[m]) \cong \mathrm{Ext}^m_{\mathscr{A}}(\varphi(X), E_n)$ with $\varphi(X) \in \varphi(\mathrm{mod}(\tilde{A}))[1] \subset \mathscr{A}$. The cases $m = 0, 1$ are true since $\mathscr{A}$ is a heart. Let $m > 1$. By Proposition 2.35, it holds that $\mathrm{Ext}^m(E_k[1], E_n) \subset \mathrm{Hom}(E_k[1], E_n[m]) = 0$, and therefore $\mathrm{Ext}^m(E_k[1], E_n) = 0$. Let us consider the distinguished triangle $F^{k-1}P_k \to P_k \to S_k \to F^{k-1}P_k[1]$. Applying $\mathrm{Hom}(\varphi(-), E_n[m])$, we get the following commutative diagram with exact rows

(4.29)
$$\begin{array}{ccccccc}
\mathrm{Ext}^{m-1}(E_k[1], E_n) & \longrightarrow & \mathrm{Ext}^{m-1}(\varphi(F^{k-1}P_k)[1], E_n) & \longrightarrow & \mathrm{Ext}^m(\varphi(S_k)[1], E_n) & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \\
\mathrm{Hom}(E_k[2], E_n[m]) & \longrightarrow & \mathrm{Hom}(\varphi(F^{k-1}P_k)[2], E_n[m]) & \longrightarrow & \mathrm{Hom}(\varphi(S_k)[1], E_n[m]) & \longrightarrow & 0
\end{array}$$

proving that $\mathrm{Ext}^m(\varphi(S_k)[1], E_n) \cong \mathrm{Hom}(\varphi(S_k)[1], E_n[m])$ for every k by the five lemma.

Now, we proceed by induction on the length of the filtration. If $X = F^1X$, there is nothing to prove since F^1X is a sum of copies of $S_1 = E_1$, and therefore $\mathrm{Hom}(\varphi(F^1X)[1], E_n[m]) = 0$ since $m > 1$. If $X = F^kX$, we consider the short exact sequence $0 \to F^{k-1}X \to X \to X/F^{k-1}X \to 0$. Then we get the following commutative diagram with exact columns:

$$\begin{array}{ccc}
\mathrm{Ext}^{m-1}(\varphi(F^{k-1}X)[1], E_n) & \xrightarrow{\cong} & \mathrm{Hom}(\varphi(F^{k-1}X)[2], E_n[m]) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^{m}(\varphi(X/F^{k-1}X)[1], E_n) & \xrightarrow{\cong} & \mathrm{Hom}(\varphi(X/F^{k-1}X)[1], E_n[m]) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^{m}(\varphi(X)[1], E_n) & \xrightarrow{f_k} & \mathrm{Hom}(\varphi(X)[1], E_n[m]) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^{m}(\varphi(F^{k-1}X)[1], E_n) & \xrightarrow{\cong} & \mathrm{Hom}(\varphi(F^{k-1}X)[1], E_n[m]) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^{m+1}(\varphi(X/F^{k-1}X)[1], E_n) & \xrightarrow{g_k} & \mathrm{Hom}(\varphi(X/F^{k-1}X), E_n[m]).
\end{array}$$

To show that f_k is an isomorphism, it suffices to apply the five lemma whenever g_k is a monomorphism. More strongly, we claim that g_k is an isomorphism. The idea is exactly the one seen above with the diagram (4.29). In order to prove that

$$\mathrm{Ext}^{m+1}(E_k[1], E_n) \subset \mathrm{Hom}(E_k[1], E_n[m+1]) = 0,$$

we will check that $\mathrm{Ext}^m(E_k[1],Y) \cong \mathrm{Hom}(E_k[1],Y[m])$ for any $Y \in \mathscr{A}$, and conclude by item 3 of Proposition 2.35. This is in fact true. Indeed, notice that

$$\mathrm{Ext}^m(E_k[1],\varphi(X)[1]) = \mathrm{Hom}(E_k[1],\varphi(X)[m+1]) = 0$$

for any $X \in \mathrm{mod}(\tilde{A})$ because E_k is projective in $\varphi(\mathrm{mod}(\tilde{A}))$. Furthermore, as remarked before (4.29), $\mathrm{Ext}^m(E_k[1],E_n) = \mathrm{Hom}(E_k[1],E_n[m]) = 0$. We conclude that

$$\mathrm{Ext}^m(E_k[1],Y) = \mathrm{Hom}(E_k[1],Y[m]) = 0$$

since any $Y \in \mathscr{A}$ is the extension of a direct sum of copies of E_n and an object $\varphi(X)[1] \in \varphi(\mathrm{mod}(\tilde{A}))[1]$. $\square$

CHAPTER 5.

On strongly unique enhancements

In this chapter, we discuss the content of [50]. This research was firstly motivated by the study of $\mathrm{mod}(\mathbb{K})$, the category of finite-dimensional vector spaces over the field $\mathbb{K}$. This category is triangulated in a natural way, with shift the identity and distinguished triangles generated by short exact sequences. In [71], it is proved that $\mathrm{mod}(\mathbb{K})$ does not have a unique $\mathbb{Z}$-linear enhancement for $\mathbb{K} = \mathbb{F}_p$ with p prime (see [14, Corollary 3.10] for the DG-version of the result). However, when $\mathbb{K}$-linearity is assumed, the uniqueness of enhancements is ensured, and it follows from the fact that for any intrinsically formal graded ring B, $\mathrm{tr}(B)$ has a unique enhancement (this is the content of Proposition 5.1). For more details, we refer to Corollary 5.39 and Corollary 5.41.

Next, we wondered how intrinsic formality can be used to study the strong uniqueness of the enhancements. To this end, we relaxed the intrinsic formality, defining triangulated formal DG-categories. This new concept lies between the uniqueness of enhancements and the semi-strong uniqueness of enhancements. More precisely, if A is a triangulated formal DG-category, then $\mathrm{tr}(\mathsf{A})$ has a unique enhancement, and if A is pretriangulated, then $\mathrm{tr}(\mathsf{A})$ has a semi-strongly unique enhancement. Inspired by the notions of D-standard and K-standard categories introduced by Chen and Ye in [19], we also define formally standard DG-categories. Considering graded categories, we have the following.

5.33. Theorem. *Let* B *be a graded category. The following are equivalent:*

1. B *is triangulated formal and formally standard;*
2. $\mathrm{tr}(\mathsf{B})$ *has a strongly unique enhancement;*
3. $\mathsf{D}(\mathsf{B})^{\mathrm{c}}$ *has a strongly unique enhancement.*

As a matter of fact, the implication $1 \Rightarrow 2,3$ holds for more general DG-categories (see Proposition 5.28 and Remark 5.29). First simple applications of the result are discussed in §5.4. In Corollary 5.39, Example 5.40 and Corollary 5.41, we deal with particular cases of periodic triangulated categories, i.e. triangulated categories such that $[n] \cong \mathrm{id}$ for some integer n (see [70]).

In §5.5, we show that K-standardness and D-standardness are instances of formal standardness. Briefly, we recall that an exact category $\mathscr{E}$ is D-standard if any triangulated autofunctor F on $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ satisfying $F(\mathscr{E}) \subset \mathscr{E}$ and $F_{|\mathscr{E}} \cong \mathrm{id}_{\mathscr{E}}$ is naturally isomorphic to the identity (see Definition 5.44 and Lemma 5.46). For additive categories, one can analogously define K-standardness considering $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$.

5.47. Proposition. *An additive category $\mathscr{A}$ is K-standard if and only if $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$ has a strongly unique enhancement.*

This follows from the fact that $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$ has a semi-strongly unique enhancement for every choice of $\mathscr{A}$ (see Proposition 5.7). Interestingly, this result appeared in [51] in a slightly different fashion, but the authors did not show that $\mathrm{tr}(\mathscr{A}) \cong \mathcal{K}^{\mathrm{b}}(\mathscr{A})$ (recall Example 3.59).

Concerning bounded derived categories, an analogous result holds.

5.51. Theorem. *An exact category $\mathscr{E}$ is D-standard if and only if $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ has a strongly unique enhancement.*

As above, this holds true because $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ has a semi-strongly unique enhancement for any exact category $\mathscr{E}$ (see Proposition 5.10). In particular, all bounded derived categories of hereditary abelian categories have a strongly unique enhancement (Corollary 5.53). For instance, $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}(\mathbb{Z}))$, the bounded derived category of abelian groups, has a strongly unique enhancement. Moreover, all known geometric examples show D-standardness to conclude that the bounded derived category has a strongly unique enhancement (see [61, 62, 14, 60, 48]). This is done by exhibiting an almost ample set, which generalizes the ample sequences introduced by Orlov in [61]. In §5.A, we give a self-contained proof of the following.

5.58. Theorem. *Let $\mathscr{A}$ be an abelian category with an almost ample set. Then $\mathcal{D}^{\mathrm{b}}(\mathscr{A})$ has a strongly unique enhancement (cf. [13, Proposition 3.7]).*

We also prove the strong uniqueness of enhancements for algebraic triangulated categories with a full strong exceptional sequence (see Theorem 5.61). This result is already known for a broader class of examples, and the reader may refer to [17, Theorem 1.1], but it has never been stated in the context of exceptional sequences.

§5.1. Triangulated formality

Comparing Definition 1.19 and Definition 3.61, one may expect to find a connection between intrinsic formality and uniqueness of enhancements. This idea inspires the original notion of triangulated formal DG-categories, which mimics the behaviour of intrinsically formal graded rings in the sense of Proposition 5.1 below. In this section, we will discuss how this concept relates to the study of enhancements (see Figure 5.18 for an overview). We work under Convention 3.1. Recall Notation 3.10.

5.1. Proposition. *Let B be an intrinsically formal graded ring. For any enhancement (C, E) of* $\mathrm{tr}(B)$*, there exists a quasi-equivalence quasi-functor* $\mathsf{f} : B^{\mathrm{pretr}} \to \mathsf{C}$ *such that* $EH^0(\mathsf{f})(O_B) \cong O_B$.

PROOF. Without loss of generality, we assume that C is strongly pretriangulated. Consider $C \in \mathsf{C}$ such that $E(C) \cong O_B$. Since

$$\begin{aligned} H^n(\mathrm{Hom}_{\mathsf{C}}(C,C)) &\cong H^0(\mathrm{Hom}_{\mathsf{C}}(C,C)[n]) \cong H^0(\mathrm{Hom}_{\mathsf{C}}(C,C[n])) \\ &\cong \mathrm{Hom}_{H^0(\mathsf{C})}(C,C[n]) \cong \mathrm{Hom}_{\mathrm{tr}(B)}(O_B,O_B[n]) \cong H^n(B) = B^n, \end{aligned}$$

$\mathrm{Hom}_{\mathsf{C}}(C,C)$ has cohomology B (the product of B is recovered from the composition on $\mathrm{tr}(B)$). By hypothesis, we obtain a zig-zag of quasi-isomorphisms from B to $\mathrm{Hom}_{\mathsf{C}}(C,C)$ extending to a quasi-fully faithful quasi-functor $\mathsf{f} : B^{\mathrm{pretr}} \to \mathsf{C}$ by Remark 3.41 and Proposition 3.39. Since $EH^0(\mathsf{f})(O_B) \cong O_B$, Remark 3.69 shows that $H^0(\mathsf{f})$ is an equivalence, so f is a quasi-equivalence. □

The previous proposition motivates the following.

5.2. Definition. A DG-category A is *triangulated formal* if

TF For any enhancement (C, E) of $\mathrm{tr}(\mathsf{A})$, we have a quasi-equivalence quasi-functor $\mathsf{f} : \mathsf{A}^{\mathrm{pretr}} \to \mathsf{C}$ such that

$$EH^0(\mathsf{f})(X) \cong X \qquad \text{for all } X \in H^0(\mathsf{A}). \tag{5.3}$$

A DG-category A is *unbounded triangulated formal* if

uTF Given any enhancement (C, E) of $\mathrm{D}(\mathsf{A})$, there exists a quasi-equivalence quasi-functor $\mathsf{f} : \mathrm{SF}(\mathsf{A}) \to \mathsf{C}$ such that (5.3) holds.

5.4. Remark. Let A be a DG-category.

1. Triangulated formality is stable under quasi-equivalence. Indeed, if $\mathsf{g} : \mathsf{A} \to \mathsf{B}$ is a quasi-equivalence quasi-functor and B is triangulated formal, for any enhancement (C, E) of $\mathrm{tr}(\mathsf{A})$ we obtain an enhancement $(\mathsf{C}, H^0(\mathsf{g}^{\mathrm{pretr}})E)$ of $\mathrm{tr}(\mathsf{B})$. By assumption, we obtain $\mathsf{f} : \mathsf{B}^{\mathrm{pretr}} \to \mathsf{C}$ satisfying (5.3). Then $\mathsf{f}\mathsf{g}^{\mathrm{pretr}} : \mathsf{A}^{\mathrm{pretr}} \to \mathsf{C}$ satisfies (5.3) for A. The same is true also in the unbounded case by picking $\mathrm{SF}(\mathsf{g})$ instead of $\mathsf{g}^{\mathrm{pretr}}$.
2. If $\mathrm{tr}(\mathsf{A}) \cong \mathrm{tr}(\mathsf{A}')$ via the inclusion $\mathsf{A} \subset \mathsf{A}'$, then A is (unbounded) triangulated formal whenever A' is (unbounded) triangulated formal.
3. If A is triangulated formal, then $\mathrm{tr}(\mathsf{A})$ has a unique enhancement. Analogously, if A is unbounded triangulated formal, then $\mathrm{D}(\mathsf{A})$ has a unique enhancement.
4. If $\mathrm{tr}(\mathsf{A})$ has a semi-strongly unique enhancement, then A is triangulated formal by picking a $(\mathsf{A}^{\mathrm{pretr}}, \mathrm{id}) - (\mathsf{C}, E)$-semilift of the identity $\mathrm{id} : \mathrm{tr}(\mathsf{A}) \to \mathrm{tr}(\mathsf{A})$ for any enhancement (C, E) (we use Proposition 3.65). Analogously, if $\mathrm{D}(\mathsf{A})$ has a semi-strongly unique enhancement, then A is unbounded triangulated formal.
5. As a matter of fact, $\mathrm{tr}(\mathsf{A})$ has a semi-strongly unique enhancement if and only if $\mathsf{A}^{\mathrm{pretr}}$ is triangulated formal.

5.5. Example. Intrinsically formal graded rings are examples of triangulated formal graded categories with one object by Proposition 5.1. In particular, rings are triangulated formal by Proposition 1.20.

We now provide a wide range of meaningful examples of triangulated formal DG-categories.

5.6. Proposition. *Let $\mathscr{A}$ be a ($\Bbbk$-linear) category. Then it is triangulated formal (cf. [51, Proposition 2.6]).*

PROOF. The idea is to proceed analogously to the proof of a ring being intrinsically formal (cf. Proposition 1.20). Let (C,E) be an enhancement of $\mathrm{tr}(\mathscr{A})$. For the sake of simplicity, assume C to be strongly pretriangulated and consider $\mathsf{C}_{|\mathscr{A}}$ (recall Notation 3.66). Take $\tau_{\leq 0}(\mathsf{C}_{|\mathscr{A}})$ as described in Definition 3.18. We have two natural DG-functors given by truncation: $\tau_{\leq 0}(\mathsf{C}_{|\mathscr{A}}) \to \mathsf{C}_{|\mathscr{A}}$ and, by Remark 3.19, $\tau_{\leq 0}(\mathsf{C}_{|\mathscr{A}}) \to H^0(\mathsf{C}_{|\mathscr{A}}) \cong \mathscr{A}$ (the equivalence $H^0(\mathsf{C}_{|\mathscr{A}}) \cong \mathscr{A}$ is given by the restriction of E). It is easy to prove that these DG-functors are in fact quasi-equivalences, because $H^i(\mathsf{C}_{|\mathscr{A}}) = 0$ for $i \neq 0$. By Remark 3.41 and Proposition 3.39, we can extend the zig-zag $\mathscr{A} \leftarrow \tau_{\leq 0}(\mathsf{C}_{|\mathscr{A}}) \to \mathsf{C}_{|\mathscr{A}}$ to obtain a quasi-equivalence quasi-functor $\mathsf{f} : \mathscr{A}^{\mathrm{pretr}} \to \mathsf{C}$. The fact that $EH^0(\mathsf{f})(X) \cong X$ for all $X \in \mathscr{A}$ follows from the definition of f; indeed, $H^0(\mathsf{f})$ restricted to $\mathscr{A}$ is the inverse of E on objects. □

5.7. Proposition. *Let $\mathscr{A}$ be an additive category. Then $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$ has a semi-strongly unique enhancement.*

PROOF. We recall that $\mathcal{K}^{\mathrm{b}}(\mathscr{A}) \cong \mathrm{tr}(\mathscr{A})$ by Example 3.59, so $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$ has a unique enhancement from Proposition 5.6 and Remark 5.4. To simplify the notation, assume $\mathcal{K}^{\mathrm{b}}(\mathscr{A}) = \mathrm{tr}(\mathscr{A})$. We want to show item 3 of Proposition 3.65 for the enhancement $(\mathscr{A}^{\mathrm{pretr}}, \mathrm{id})$. Let F be an autoequivalence of $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$. Since $\mathscr{A}$ is triangulated formal by Proposition 5.6, considering the enhancement $(\mathscr{A}^{\mathrm{pretr}}, F)$ of $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$, we get a quasi-equivalence quasi-functor $\mathsf{f} : \mathscr{A}^{\mathrm{pretr}} \to \mathscr{A}^{\mathrm{pretr}}$ such that $FH^0(\mathsf{f})(X) \cong X$ for all $X \in \mathscr{A}$. Let $G := FH^0(\mathsf{f})$. Then $G_{|\mathscr{A}}$ gives an equivalence $\mathscr{A} \to \mathscr{A}$, so we can consider the DG-functor $\mathsf{g} := (G_{|\mathscr{A}})^{\mathrm{pretr}} : \mathscr{A}^{\mathrm{pretr}} \to \mathscr{A}^{\mathrm{pretr}}$. Of course, $GH^0(\mathsf{g}^{-1})$ is the identity when restricted to $\mathscr{A}$. By [19, Proposition 3.2], $GH^0(\mathsf{g}^{-1})(X) \cong X$ for all $X \in \mathcal{K}^{\mathrm{b}}(\mathscr{A})$. By recalling the definition of G and moving the equivalences around, we get $F(X) \cong H^0(\mathsf{g}\mathsf{f}^{-1})(X)$ for all $X \in \mathcal{K}^{\mathrm{b}}(\mathscr{A})$. This is exactly item 3 of Proposition 3.65, as wanted. □

Proposition 5.8 below is inspired by [43, Theorem 3.2]. In that article, the authors used the definition of algebraic triangulated categories via Frobenius categories (see [14, Proposition 3.1]). Using DG-categories, we are able to say something more, and provide a proof of uniqueness of enhancements for bounded derived categories of exact categories. Furthermore, the DG-category $\mathscr{E}_{\mathrm{DG}}$ associated is immediately triangulated formal.

5.8. Proposition. *Let $\mathscr{T}$ be an algebraic triangulated category and let $\mathscr{E}$ be an admissible exact subcategory. Then for any enhancement (C,E) of $\mathscr{T}$, there exists a realization functor* real :

$\mathcal{D}^{\mathrm{b}}(\mathscr{E}) \to \mathscr{T}$ admitting a $(\mathcal{D}^{\mathrm{b}}_{\mathrm{DG}}(\mathscr{E}), \mathrm{id}) - (\mathsf{C}, E)$-lift, where $\mathcal{D}^{\mathrm{b}}_{\mathrm{DG}}(\mathscr{E})$ was defined in Example 3.59.*

PROOF. As our reasoning will not be affected by the quasi-equivalence inclusion $\mathsf{y} : \mathsf{C} \to \mathsf{C}^{\mathrm{pretr}}$, for the sake of simplicity we assume C to be strongly pretriangulated, and consider the DG-functor $(\mathsf{C}_{|\mathscr{E}})^{\mathrm{pretr}} \to \mathsf{C}$ obtained by Proposition 3.39.

From the natural truncation $\tau_{\leq 0}\mathsf{C}_{|\mathscr{E}} \to \mathsf{C}_{|\mathscr{E}}$, and the quasi-equivalence $\tau_{\leq 0}\mathsf{C}_{|\mathscr{E}} \to H^0(\mathsf{C}_{|\mathscr{E}}) \cong \mathscr{E}$ (this is a quasi-equivalence because $\mathscr{E} \subset \mathscr{T}$ is admissible by assumption), Proposition 3.39 gives rise to a quasi-functor $\mathsf{f} : \mathscr{E}^{\mathrm{pretr}} \to (\mathsf{C}_{|\mathscr{E}})^{\mathrm{pretr}} \to \mathsf{C}$. At the homotopy level, this defines a triangulated functor $\mathcal{K}^{\mathrm{b}}(\mathscr{E}) \to \mathscr{T}$ (recall Example 3.59).

We now want to prove that $\mathrm{Ac}^{\mathrm{b}}(\mathscr{E}) \to \mathcal{K}^{\mathrm{b}}(\mathscr{E}) \to \mathscr{T}$ is the zero functor. Indeed, given any conflation $0 \to A \to B \to C \to 0$ in $\mathscr{E}$, we obtain a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[r] \arrow[d, "\cong"] & A[1] \arrow[d, equal] \\ A \arrow[r, "f"] & B \arrow[r] & C \arrow[r] & A[1] \end{tikzcd}$$

of distinguished triangles in $\mathscr{T}$. Since $\mathrm{Hom}(A[1], C) = \mathrm{Hom}(A, C[-1]) = 0$, the morphism $\mathrm{Cone}(f) \to C$ is determined by $B \to \mathrm{Cone}(f) \to C$, which is exactly the map appearing in the conflation. Looking at $\mathrm{Cone}(f) \to C$ in $\mathcal{K}^{\mathrm{b}}(\mathscr{E})$, its cone is the conflation $0 \to A \to B \to C \to 0$, and its image is zero since $\mathrm{Cone}(f) \to C$ is an isomorphism in $\mathscr{T}$. This shows that conflations are sent to zero via $\mathcal{K}^{\mathrm{b}}(\mathscr{E}) \to \mathscr{T}$. By Lemma 2.20, we conclude that $\mathrm{Ac}^{\mathrm{b}}(\mathscr{E}) \to \mathcal{K}^{\mathrm{b}}(\mathscr{E}) \to \mathscr{T}$ is the zero functor.

By Remark 3.44, at the DG-level we have that $\mathrm{Ac}^{\mathrm{b}}_{\mathrm{DG}}(\mathscr{E}) \to \mathsf{C}^{\mathrm{b}}_{\mathrm{DG}}(\mathscr{E}) \cong \mathscr{E}^{\mathrm{pretr}} \to \mathsf{C}$ is the trivial quasi-functor. Therefore, we obtain an induced quasi-functor $\mathsf{r} : \mathcal{D}^{\mathrm{b}}_{\mathrm{DG}}(\mathscr{E}) \to \mathsf{C}$ satisfying the statement. □

5.9. Corollary. *For any exact category $\mathscr{E}$, $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ has a unique enhancement. More precisely, the full DG-subcategory $\mathscr{E}_{\mathrm{DG}} := \mathcal{D}^{\mathrm{b}}_{\mathrm{DG}}(\mathscr{E})_{|\mathscr{E}}$ is triangulated formal.*

In addition, given any enhancement (C, E) of $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$, the pretriangulated closure of the truncation $\mathsf{p}_{\leq 0} : \tau_{\leq 0}\mathsf{C}_{|\mathscr{E}} \to \mathsf{C}_{|\mathscr{E}}$ gives rise to a DG-quotient.

PROOF. Notice that $\mathrm{tr}(\mathscr{E}_{\mathrm{DG}}) \cong \mathcal{D}^{\mathrm{b}}(\mathscr{E})$ since $\mathrm{tr}(\mathscr{E}_{\mathrm{DG}}) \subset H^0(\mathcal{D}^{\mathrm{b}}_{\mathrm{DG}}(\mathscr{E})) \cong \mathcal{D}^{\mathrm{b}}(\mathscr{E})$ and $\mathrm{tr}(\mathscr{E}_{\mathrm{DG}})$ is the triangulated envelope of $\mathscr{E}$. By applying Proposition 5.8 to $\mathscr{T} = \mathcal{D}^{\mathrm{b}}(\mathscr{E})$, we can construct a quasi-equivalence quasi-functor between any enhancement of $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ and $\mathcal{D}^{\mathrm{b}}_{\mathrm{DG}}(\mathscr{E})$ (the quasi-functor is a quasi-equivalence by Corollary 2.42 and Proposition 2.45). Moreover, this quasi-equivalence fixes $\mathscr{E}$, so (5.3) is satisfied. The last part of the statement follows from the construction of the realization functor in the proof of Proposition 5.8. □

5.10. Proposition. *Let $\mathscr{E}$ be an exact category. Then $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ has a semi-strongly unique enhancement.*

*For the sake of simplicity, we assume $\mathcal{D}^{\mathrm{b}}(\mathscr{E}) = H^0(\mathcal{D}^{\mathrm{b}}_{\mathrm{DG}}(\mathscr{E}))$.

PROOF. Mimicking the proof of Proposition 5.7 with the natural enhancement obtained by $\mathcal{D}^{\mathsf{b}}_{\mathrm{DG}}(\mathscr{E})$, the statement follows since the reasoning of [19, Proposition 3.2 and Proposition 3.7] can be adapted to this setting. The only detail to be precise about is the fact that $G_{|\mathscr{E}}$ gives an exact equivalence $\mathscr{E} \to \mathscr{E}$, which gives a DG-functor $\mathsf{C}^{\mathsf{b}}_{\mathrm{DG}}(\mathscr{E}) \to \mathsf{C}^{\mathsf{b}}_{\mathrm{DG}}(\mathscr{E})$ inducing a quasi-functor $\mathsf{g} : \mathcal{D}^{\mathsf{b}}_{\mathrm{DG}}(\mathscr{E}) \to \mathcal{D}^{\mathsf{b}}_{\mathrm{DG}}(\mathscr{E})$ via the property of DG-quotients. □

5.11. Remark. Recalling Remark 2.18, Proposition 5.10 in fact generalizes both Proposition 5.7 and [12, Remark 5.4], which shows that $\mathcal{D}^{\mathsf{b}}(\mathscr{A})$ has a semi-strongly unique enhancement for any abelian category $\mathscr{A}$.

Let us state some results relating triangulated formality with the uniqueness of enhancements. First of all, we motivate why we avoided the notion of triangulated formality for perfect complexes.

5.12. Proposition. *A DG-category* A *is triangulated formal if and only if the following holds*

cTF *For any enhancement* (C, E) *of* $\mathsf{D}(\mathsf{A})^{\mathsf{c}}$*, we can choose a quasi-equivalence quasi-functor* $\mathsf{f} : \mathrm{Perf}(\mathsf{A}) \to \mathsf{C}$ *satisfying* (5.3).

PROOF. TF $\Rightarrow$ cTF. Let (C, E) be an enhancement of $\mathsf{D}(\mathsf{A})^{\mathsf{c}}$ and consider the restriction $\mathsf{C}_{|\mathrm{tr}(\mathsf{A})}$ (see Notation 3.66). By triangulated formality, we get a quasi-equivalence quasi-functor $\mathsf{f}' : \mathsf{A}^{\mathrm{pretr}} \to \mathsf{C}_{|\mathrm{tr}(\mathsf{A})}$ satisfying (5.3). Denote by f the following composition:

$$\mathrm{Perf}(\mathsf{A}) \xrightarrow{\mathrm{Perf}(\mathsf{y})} \mathrm{Perf}(\mathsf{A}^{\mathrm{pretr}}) \xrightarrow{\mathrm{Perf}(\mathsf{f}')} \mathrm{Perf}(\mathsf{C}_{|\mathrm{tr}(A)}) \xrightarrow{\mathrm{Perf}(\mathrm{incl})} \mathrm{Perf}(\mathsf{C}) \xleftarrow{\ \mathsf{y}\ } \mathsf{C}$$

Notice that $EH^0(\mathsf{f})(X) \cong X$ for all $X \in H^0(\mathsf{A})$, and all DG-functors considered in the composition are quasi-fully faithful. Lemma 3.68 shows that f is a quasi-equivalence.

cTF $\Rightarrow$ TF. Given any enhancement (D, F) of $\mathrm{tr}(\mathsf{A})$, then $\mathrm{Perf}(\mathsf{D})$ is an enhancement of $\mathsf{D}(\mathsf{A})^{\mathsf{c}}$ with the unique extension of F (we recall Remark 3.52). Moreover, $\mathsf{D} \cong \mathrm{Perf}(\mathsf{D})_{|\mathrm{tr}(\mathsf{A})}$ via inclusion. By cTF, we have a quasi-equivalence quasi-functor $\mathsf{g} : \mathrm{Perf}(\mathsf{A}) \to \mathrm{Perf}(\mathsf{D})$ satisfying (5.3). Restricting g to $\mathsf{A}^{\mathrm{pretr}}$, by Remark 3.69 we get a quasi-equivalence $\mathsf{A}^{\mathrm{pretr}} \to \mathrm{Perf}(\mathsf{D})_{|\mathrm{tr}(\mathsf{A})}$ satisfying (5.3). □

5.13. Remark. From Proposition 5.12, $\mathsf{D}(\mathsf{A})^{\mathsf{c}}$ has a unique enhancement for any triangulated formal DG-category A.

5.14. Proposition. *A triangulated formal DG-category* A *is also unbounded triangulated formal.*

PROOF. Let (C, E) be an enhancement of $\mathsf{D}(\mathsf{A})$ and consider $\mathsf{C}' := \mathsf{C}_{|\mathsf{D}(\mathsf{A})^{\mathsf{c}}}$. By Proposition 5.12, we obtain a quasi-equivalence quasi-functor $\mathsf{f} : \mathrm{Perf}(\mathsf{A}) \to \mathsf{C}'$ satisfying (5.3). Let us define h as the composition

$$\mathrm{SF}(\mathsf{A}) \xrightarrow{\phi_{\mathrm{Perf}(\mathsf{A})}} \mathrm{SF}(\mathrm{Perf}(\mathsf{A})) \xrightarrow{\mathrm{SF}(\mathsf{f})} \mathrm{SF}(\mathsf{C}') \xleftarrow{\ \phi_{\mathsf{C}'}\ } \mathsf{C},$$

where $\phi_{\mathrm{Perf(A)}}$ and $\phi_{\mathsf{C}'}$ are quasi-functors described in [51, §1]. Then h is a quasi-equivalence as explained in the proof of item 2 of Proposition 3.72.

We are reduced to check that (5.3) is satisfied. In [51], $\phi_{\mathrm{Perf(A)}}$ and $\phi_{\mathsf{C}'}$ are obtained by a Yoneda embedding and a restriction functor, both of which do not affect the subcategory associated (Perf(A) and C' respectively). Therefore, since SF(f) is an extension of f, h fulfils (5.3). ☐

5.15. Remark. Notice that $\mathsf{D}(\mathsf{A})$ has a unique enhancement for any triangulated formal DG-category A by Proposition 5.14.

We can prove the converse of Proposition 5.14 in a special case.

5.16. Proposition. *An unbounded triangulated formal DG-category* A *is also triangulated formal if the following holds:*

EE *For any enhancement* (C,E) *of* $\mathrm{tr}(\mathsf{A})$*, E extends to a triangulated equivalence* $E' : \mathsf{D}(\mathsf{C}) \to \mathsf{D}(\mathsf{A})$ *up to natural isomorphism, i.e.* $E'_{|H^0(\mathsf{C})} \cong E$.

(EE stands for Extending Enhancements).

PROOF. Let (C,E) be an enhancement of $\mathrm{tr}(\mathsf{A})$. By assumption, we can consider the enhancement $(\mathrm{SF}(\mathsf{C}),E')$ of $\mathsf{D}(\mathsf{A})$ such that $E'_{|H^0(\mathsf{C})} \cong E$. Since A is unbounded triangulated formal, there exists $\mathsf{f} : \mathrm{SF}(\mathsf{A}) \to \mathrm{SF}(\mathsf{C})$ such that $E'H^0(\mathsf{f})(X) \cong X$ for all $X \in H^0(\mathsf{A})$.

We now want to show that $\mathrm{EssIm}(\mathsf{f}_{|\mathsf{A}^{\mathrm{pretr}}})$ is contained in $\bar{\mathsf{C}}$, the homotopy closure of C in $\mathrm{SF}(\mathsf{C})$. Let $Y \in \mathrm{EssIm}(\mathsf{f}_{|\mathsf{A}})$. Then there exists $X \in H^0(\mathsf{A})$ such that $Y \cong H^0(\mathsf{f})(X)$. By applying E', we have $E'Y \cong E'H^0(\mathsf{f})(X) \cong X$. Since $E'_{|H^0(\mathsf{C})} \cong E$, we can choose $Y' \in H^0(\mathsf{C})$ such that $E'Y' \cong X \cong E'Y$. In particular, Y is homotopy equivalent to Y'. From this, we have $\mathsf{f}_{|\mathsf{A}} : \mathsf{A} \to \bar{\mathsf{C}}$. Being $H^0(\bar{\mathsf{C}})$ triangulated, $\mathrm{EssIm}(\mathsf{f}_{|\mathsf{A}^{\mathrm{pretr}}}) \subseteq \bar{\mathsf{C}}$, as wanted.

Consider the quasi-functor $\mathsf{g} : \mathsf{A}^{\mathrm{pretr}} \to \bar{\mathsf{C}} \hookleftarrow \mathsf{C}$, where the first map is the quasi-functor $\mathsf{f}_{|\mathsf{A}^{\mathrm{pretr}}}$ and the second is a Yoneda embedding. Since $EH^0(\mathsf{g})(X) \cong EH^0(\mathsf{f}_{|\mathsf{A}^{\mathrm{pretr}}})(X) \cong X$ for any $X \in H^0(\mathsf{A})$, by Remark 3.69 we conclude that g is a quasi-equivalence, so A is triangulated formal. ☐

5.17. Remark. Assume A is a DG-category for which EE holds and $\mathsf{D}(\mathsf{A})$ has a semi-strongly unique enhancement. Then Perf(A) is unbounded triangulated formal because $\mathrm{SF}(\mathrm{Perf}(\mathsf{A})) \cong \mathrm{SF}(\mathsf{A})$ by [51, Proposition 1.17]. Since A satisfies EE, from [2, Theorem 1.5] one can prove that Perf(A) also satisfies EE. By Proposition 5.16, Perf(A) is triangulated formal, meaning that $\mathsf{D}(\mathsf{A})^{\mathrm{c}}$ has a semi-strongly unique enhancement by Remark 5.4.

5.18. Figure. Relation between triangulated formality and uniqueness of enhancements for a DG-category A.

$$
\begin{tikzcd}
\text{$\mathsf{D}(\mathsf{A})^c$ has a semi-strongly unique enhancement} \arrow[d, Rightarrow, "\text{Remark 3.71}"] & \\
\text{$\mathrm{tr}(\mathsf{A})$ has a semi-strongly unique enhancement} \arrow[d, Rightarrow, "\text{Remark 5.4}"] & \text{$\mathsf{D}(\mathsf{A})$ has a semi-strongly unique enhancement} \arrow[d, Rightarrow, "\text{Remark 5.4}"] \arrow[ul, dashed, bend right, "\text{Remark 5.17 +EE}"'] \\
\text{A is triangulated formal} \arrow[r, Rightarrow, "\text{Proposition 5.14}"] \arrow[d, Rightarrow, "\text{Remark 5.4}"] & \text{A is unbounded triangulated formal} \arrow[d, Rightarrow, "\text{Remark 5.4}"] \arrow[l, dashed, bend left, "\text{Proposition 5.16 +EE}"] \\
\text{$\mathrm{tr}(\mathsf{A})$ has a unique enhancement} \arrow[d, Rightarrow, "\text{Proposition 3.72}"'] & \text{$\mathsf{D}(\mathsf{A})$ has a unique enhancement} \\
\text{$\mathsf{D}(\mathsf{A})^c$ has a unique enhancement} \arrow[ur, Rightarrow, "\text{Proposition 3.72}"'] &
\end{tikzcd}
$$

§5.2. Formal standardness

We recall that strong uniqueness of enhancements can be rephrased by saying that all triangulated autoequivalences admit a lift once uniqueness is ensured (see Proposition 3.64). Roughly speaking, in order to obtain such property in the case of a triangulated formal DG-category A (in vision of Proposition 5.28), we need to force all equivalences fixing the objects of A to have a lift. This idea motivates the original definition of formal standardness. The name gets inspiration from the concepts of D-standardness and K-standardness, studied in §5.5, and from its relation to triangulated formal DG-categories.

5.19. Lemma. *Let $\mathscr{T}$ be a triangulated category and consider $\mathscr{S} \subset \mathscr{T}$ a full subcategory. Then any triangulated equivalence $F : \mathscr{T} \to \mathscr{T}$ such that $FX \cong X$ for $X \in \mathscr{S}$ is naturally isomorphic to a triangulated equivalence G such that $GX = X$ for $X \in \mathscr{S}$.*

PROOF. Let us consider the family of isomorphisms $\eta := (\eta_X)$, where $\eta_X : X \to FX$ is an isomorphism if $X \in \mathscr{S}$, while $\eta_X = \mathrm{id}_{FX}$ if $X \notin \mathscr{S}$.[†] We then define G as follows:

$$G(X) := \begin{cases} X & \text{if } X \in \mathscr{S} \\ F(X) & \text{otherwise} \end{cases} \quad \text{and} \quad G_{X,Y} := \eta_Y^{-1} F_{X,Y} \eta_X.$$

Notice that η becomes a natural isomorphism $G \to F$. □

5.20. Definition. Let A be a DG-category and consider a triangulated autoequivalence (F, η) of $\mathrm{tr}(\mathsf{A})$ such that

$$FX = X \quad \text{for any } X \in H^0(\mathsf{A}). \tag{5.21}$$

[†] This definition explicitly requires the axiom of choice, since we choose an isomorphism for every object in $\mathscr{S}$.

Its *graded restriction* is a graded functor $F^{gr}_{|H^*(\mathsf{A})} : H^*(\mathsf{A}) \to H^*(\mathsf{A})$ defined by $F^{gr}_{|H^*(\mathsf{A})}(X) = X$ and

$$\begin{tikzcd}
\operatorname{Hom}_{H^*(\mathsf{A})}(X,Y) \arrow[rr, "(F^{gr}_{|H^*(\mathsf{A})})_{X,Y}"] \arrow[d, "\cong"] & & \operatorname{Hom}_{H^*(\mathsf{A})}(X,Y) \arrow[d, "\cong"] \\
\bigoplus_i \operatorname{Hom}_{\mathrm{tr}(\mathsf{A})}(X,Y[i]) \arrow[r, "\oplus_i F_{X,Y[i]}"] & \bigoplus_i \operatorname{Hom}_{\mathrm{tr}(\mathsf{A})}(FX,F(Y[i])) \arrow[r, "\oplus_i \eta^i_Y"] & \bigoplus_i \operatorname{Hom}_{\mathrm{tr}(\mathsf{A})}(X,Y[i])
\end{tikzcd}$$

for any $X,Y \in \mathsf{A}$, where the vertical arrows are obtained via the following isomorphisms[‡]

$$\operatorname{Hom}_{H^*(\mathsf{A})}(X,Y) \cong \bigoplus_i H^i(\operatorname{Hom}_{\mathsf{A}}(X,Y)) \cong \bigoplus_i H^0(\operatorname{Hom}_{\mathsf{A}}(X,Y[i])) \cong \bigoplus_i \operatorname{Hom}_{H^0(\mathsf{A})}(X,Y[i]).$$

5.22. Definition. A DG-category A is

- *formally standard* if, given two triangulated equivalences $F,G : \mathrm{tr}(\mathsf{A}) \to \mathrm{tr}(\mathsf{A})$ satisfying (5.21) and $F^{gr}_{|H^*(\mathsf{A})} \cong G^{gr}_{|H^*(\mathsf{A})}$, there is a natural isomorphism $F \cong G$.
- *lifted* if for every triangulated equivalence $F : \mathrm{tr}(\mathsf{A}) \to \mathrm{tr}(\mathsf{A})$ for which (5.21) holds, we have a quasi-functor $\mathsf{f} : \mathsf{A} \to \mathsf{A}$ such that $H^*(\mathsf{f}) \cong F^{gr}_{|H^*(\mathsf{A})}$.

The notion of liftedness is introduced to treat at the same time the two following examples, which are crucial for applications.

5.23. Example.

- Every graded category B is lifted: indeed, given a triangulated equivalence $F : \mathrm{tr}(\mathsf{B}) \to \mathrm{tr}(\mathsf{B})$, the DG-functor f required is simply given by the graded restriction $F^{gr}_{|\mathsf{B}}$.
- Let $\mathscr{E}$ be an exact category and consider $\mathscr{E}_{\mathrm{DG}} := \mathcal{D}^{\mathsf{b}}_{\mathrm{DG}}(\mathscr{E})_{|\mathscr{E}}$ as in Corollary 5.9. We claim that $\mathscr{E}_{\mathrm{DG}}$ is lifted: indeed, for any triangulated equivalence $F : \mathrm{tr}(\mathscr{E}_{\mathrm{DG}}) \to \mathrm{tr}(\mathscr{E}_{\mathrm{DG}})$ satisfying (5.21), we obtain an exact equivalence $\mathscr{E} \to \mathscr{E}$. This induces an equivalence $\mathsf{C}^{\mathsf{b}}_{\mathrm{DG}}(\mathscr{E}) \to \mathsf{C}^{\mathsf{b}}_{\mathrm{DG}}(\mathscr{E})$; via quotient we get a quasi-functor $\mathcal{D}^{\mathsf{b}}_{\mathrm{DG}}(\mathscr{E}) \to \mathcal{D}^{\mathsf{b}}_{\mathrm{DG}}(\mathscr{E})$, and finally a quasi-functor $\mathsf{f} : \mathscr{E}_{\mathrm{DG}} \to \mathscr{E}_{\mathrm{DG}}$. Moreover, the exact equivalence $\mathscr{E} \to \mathscr{E}$ uniquely determines what happens on $H^*(\mathscr{E}_{\mathrm{DG}})$; this is Proposition 5.24 below. We conclude that $H^*(\mathsf{f}) \cong F^{gr}_{|H^*(\mathscr{E}_{\mathrm{DG}})}$.

5.24. Proposition. *Let $\mathscr{E}$ be an exact category, and consider $(F,\eta) : \mathcal{D}^{\mathsf{b}}(\mathscr{E}) \to \mathcal{D}^{\mathsf{b}}(\mathscr{E})$ a triangulated equivalence such that $F(X) = X$ for all $X \in \mathscr{E}$. Then $F_{|\mathscr{E}}$ determines uniquely $F^{gr}_{|H^*(\mathscr{E}_{\mathrm{DG}})}$, meaning that any other triangulated equivalence (G,μ) satisfying $G_{|\mathscr{E}} = F_{|\mathscr{E}}$ is such that $G^{gr}_{|H^*(\mathscr{E}_{\mathrm{DG}})} = F^{gr}_{|H^*(\mathscr{E}_{\mathrm{DG}})}$.*

PROOF. By Corollary 2.42, $H^*(\mathscr{E}_{\mathrm{DG}})$ is simply the category of the Ext-groups, i.e.

$$\operatorname{Hom}_{H^*(\mathscr{E}_{\mathrm{DG}})}(X,Y) = \bigoplus_i \operatorname{Ext}^i(X,Y)[-i]$$

[‡] The first and the last isomorphisms are given by definition, while the second one is obtained by iterated composition of the closed morphisms associated to the suspension functor.

for every $X,Y \in \mathscr{E}$. Since every morphism in $\mathrm{Hom}(X,Y[1])$ is associated to a conflation as explained in Definition/Proposition 2.32, given another triangulated autoequivalence (G,μ) such that $G_{|\mathscr{E}} = F_{|\mathscr{E}}$ (so $G(X) = X$ for all $X \in \mathscr{E}$ as well), we have the following isomorphism of distinguished triangles

$$\begin{tikzcd} Y \arrow[r, "Ff"] \arrow[d, "\mathrm{id}"] & Z \arrow[r, "Fg"] \arrow[d, "\mathrm{id}"] & X \arrow[r, "\eta_Y Fh"] \arrow[d, dashed] & Y[1] \arrow[d, "\mathrm{id}"] \\ Y \arrow[r, "Gf"] & Z \arrow[r, "Gg"] & X \arrow[r, "\mu_Y Gh"] & Y[1], \end{tikzcd}$$

where the first two vertical arrows are the identity because $G_{|\mathscr{E}} = F_{|\mathscr{E}}$. From the universal property of the cokernel X in $\mathscr{E}$, the dashed morphism is also the identity, so that $\eta_Y Fh = \mu_Y Gh$. As every extension is obtained by Yoneda products of elements in the first Ext-group (cf. the beginning of the proof of Proposition 2.35), a simple induction concludes the proof. □

5.25. Lemma. *A DG-category* A *is formally standard if and only if any triangulated equivalence* $F : \mathrm{tr}(\mathsf{A}) \to \mathrm{tr}(\mathsf{A})$*, such that* (5.21) *holds and* $F^{gr}_{|H^*(\mathsf{A})} \cong \mathrm{id}$*, is naturally isomorphic to the identity.*

PROOF. Of course, if A is formally standard, then any such F is naturally isomorphic to the identity. Conversely, let $F,G : \mathrm{tr}(\mathsf{A}) \to \mathrm{tr}(\mathsf{A})$ be two triangulated equivalences satisfying (5.21) such that $F^{gr}_{|H^*(\mathsf{A})} \cong G^{gr}_{|H^*(\mathsf{A})}$. Then

$$(G^{-1}F)^{gr}_{|H^*(\mathsf{A})} \cong (G^{gr}_{|H^*(\mathsf{A})})^{-1} F^{gr}_{|H^*(\mathsf{A})} \cong \mathrm{id}^{gr}_{|H^*(\mathsf{A})}.$$

We conclude that $G^{-1}F \cong \mathrm{id}$, so $F \cong G$, as wanted. □

5.26. Remark. Replacing $\mathrm{tr}(\mathsf{A})$ with $\mathrm{D}(\mathsf{A})^{\mathrm{c}}$ in Definition 5.22, one could be tempted to define *perfect formally standard* and *perfect lifted* DG-categories. However, this is not useful, since the equivalences $F : \mathrm{D}(\mathsf{A})^{\mathrm{c}} \to \mathrm{D}(\mathsf{A})^{\mathrm{c}}$ satisfying (5.21) are determined by $F_{|\mathrm{tr}(\mathsf{A})}$ up to natural isomorphism by [2, Theorem 1.5], and Remark 3.69 shows that the inclusion $\mathrm{tr}(\mathsf{A}) \subset \mathrm{EssIm}(F_{|\mathrm{tr}(\mathsf{A})})$ is an equivalence, so that $F_{|\mathrm{tr}(\mathsf{A})}$ can be thought of as an equivalence $\mathrm{tr}(\mathsf{A}) \to \mathrm{tr}(\mathsf{A})$. In other words, the only equivalences admitting a graded restriction on $\mathrm{D}(\mathsf{A})^{\mathrm{c}}$ are equivalences restricting to $\mathrm{tr}(\mathsf{A})$.

5.27. Remark. Given a DG-category A and a full DG-subcategory $\mathsf{A}' \subset \mathsf{A}$ such that the inclusion induces a DG-equivalence $\mathsf{A}'^{\mathrm{pretr}} \cong \mathsf{A}^{\mathrm{pretr}}$, the following implications hold.

1. If A is lifted, then A' is lifted.
2. If A' is formally standard, then A is formally standard.

Roughly speaking, these properties should suggest that a DG-category is lifted and formally standard when a good balance is achieved.

§5.3. Main result

In this section, we show how the new notions introduced so far (triangulated formality, formal standardness and liftedness) are connected to strong uniqueness of enhancements. In the case of graded categories B, we are able to provide a characterization of strong uniqueness of enhancements for $\mathrm{tr}(\mathsf{B})$ and $\mathrm{D}(\mathsf{B})^{\mathrm{c}}$ (see Theorem 5.33).

Let us start with a sufficient condition for strong uniqueness of enhancements.

5.28. Proposition. *Let* A *be a lifted, triangulated formal and formally standard DG-category. Then* $\mathrm{tr}(\mathsf{A})$ *has a strongly unique enhancement.*

PROOF. Let $F : \mathrm{tr}(\mathsf{A}) \to \mathrm{tr}(\mathsf{A})$ be any triangulated equivalence. Let us consider the enhancement $(\mathsf{A}^{\mathrm{pretr}}, F)$ of $\mathrm{tr}(\mathsf{A})$. By triangulated formality, there exists $\mathsf{f} : \mathsf{A}^{\mathrm{pretr}} \to \mathsf{A}^{\mathrm{pretr}}$ such that $FH^0(\mathsf{f})(X) \cong X$ for all $X \in H^0(\mathsf{A})$. Up to natural isomorphism, we can assume that $G := FH^0(\mathsf{f})$ satisfies (5.21) by Lemma 5.19. We aim to prove that G has an $(\mathsf{A}^{\mathrm{pretr}}, \mathrm{id})$-lift. This suffices to conclude that F has an $(\mathsf{A}^{\mathrm{pretr}}, \mathrm{id})$-lift as well. Since A is lifted, we have a quasi-functor $\mathsf{g} : \mathsf{A} \to \mathsf{A}$ such that $H^*(\mathsf{g}) \cong G^{gr}_{|H^*(\mathsf{A})}$. Up to natural isomorphism, notice that $H^0(\mathsf{g}^{\mathrm{pretr}})$ satisfies (5.21). It remains to prove that $G^{gr}_{|H^*(\mathsf{A})} \cong H^0(\mathsf{g}^{\mathrm{pretr}})^{gr}_{|H^*(\mathsf{A})}$, which follows from direct computations. Formal standardness implies that $G \cong H^0(\mathsf{g}^{\mathrm{pretr}})$.

Since any triangulated equivalence $F : \mathrm{tr}(\mathsf{A}) \to \mathrm{tr}(\mathsf{A})$ has an $(\mathsf{A}^{\mathrm{pretr}}, \mathrm{id})$-lift, item 3 of Remark 5.4 and Proposition 3.64 show that $\mathrm{tr}(\mathsf{A})$ has a strongly unique enhancement. □

5.29. Remark. By Proposition 5.12 and Remark 5.26, we can modify the proof of Proposition 5.28 to prove that $\mathrm{D}(\mathsf{A})^{\mathrm{c}}$ has a strongly unique enhancement under the same requirements.

We now aim to prove the converse implication of Proposition 5.28. In order to do so, we need to restrict to graded categories and state a technical result (Lemma 5.31). First, we recall the following.

5.30. Fact. [29, Theorem 1.2.10] and [74, Théorème 2.1, Remarque 1]. *Every DG-category is quasi-equivalent to a cofibrant*[§] *DG-category. Moreover, let* C *be a cofibrant DG-category. Every quasi-functor* $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ *can be represented by a DG-functor* $\mathsf{f}' : \mathsf{C} \to \mathsf{D}$*. In particular, this means that* $H^0(\mathsf{f}) \cong H^0(\mathsf{f}')$.

5.31. Lemma – Presentation via a cofibrant DG-category. *Let* A *be a DG-category and let* C *be a cofibrant DG-category with a quasi-equivalence quasi-functor* $\mathsf{f} : \mathsf{C} \to \mathsf{A}^{\mathrm{pretr}}$*. Then we can construct a quasi-equivalence* $\mathsf{h} : \mathsf{D} \to \mathsf{A}$*, where* $\mathsf{D} := \mathsf{D}^{H^0(\mathsf{f})}_{|H^0(\mathsf{A})}$.

PROOF. We use the notation of the statement. Let A' be a full pretriangulated DG-subcategory of $\mathsf{A}^{\mathrm{pretr}}$ such that $\mathsf{A} \subset \mathsf{A}'$ and all the homotopy equivalence classes of A in A' are represented

[§]One may refer to [75, §2] for the definition of a cofibrant DG-category. However, the lazy reader may rest assured that we are interested only in the property highlighted by this statement.

only by objects of A. In other words, if $Y \in \mathsf{A}'$ is homotopy equivalent to an object of A, then $Y \in \mathsf{A}$. Notice the functor $i : H^0(\mathsf{A}') \to H^0(\mathsf{A}^{\mathrm{pretr}}) = \mathrm{tr}(\mathsf{A})$, obtained from the inclusion $\mathsf{A}' \subset \mathsf{A}^{\mathrm{pretr}}$, is an equivalence by Lemma 3.68, so (A', i) is an enhancement of $\mathrm{tr}(\mathsf{A})$.

Let us now consider the quasi-equivalence quasi-functor $\mathsf{f}' : \mathsf{C} \to \mathsf{A}^{\mathrm{pretr}} \leftarrow \mathsf{A}'$. By Fact 5.30, we can assume f' to be a DG-functor. In $\mathsf{A}^{\mathrm{pretr}}$, the image of D via the quasi-functor f lies in the homotopy closure of A, so in general we may find an object of $\mathsf{f}(\mathsf{D})$ not belonging to A. The definition of A' ensures that this is not the case for f'; we conclude that $\mathsf{f}'(\mathsf{D}) \subset \mathsf{A}$. Finally, the restriction $\mathsf{f}'_{|\mathsf{D}} : \mathsf{D} \to \mathsf{A}$ is the wanted quasi-equivalence h. □

5.32. Proposition. *Let B be a graded category. If $\mathrm{tr}(\mathsf{B})$ has a strongly unique enhancement, then B is triangulated formal and formally standard.*

PROOF. By item 4 of Remark 5.4, we are reduced to check that B is formally standard. For this purpose, we will show that any autoequivalence F on $\mathrm{tr}(\mathsf{B})$ satisfying (5.21) is naturally isomorphic to $H^0((F^{gr}_{|\mathsf{B}})^{\mathrm{pretr}})$. Indeed, if this is true, given any autoequivalence G such that (5.21) holds, $F^{gr}_{|\mathsf{B}} \cong G^{gr}_{|\mathsf{B}}$ implies that $(F^{gr}_{|\mathsf{B}})^{\mathrm{pretr}} \cong (G^{gr}_{|\mathsf{B}})^{\mathrm{pretr}}$ by Proposition 3.39, from which $F \cong H^0((F^{gr}_{|\mathsf{B}})^{\mathrm{pretr}}) \cong H^0((G^{gr}_{|\mathsf{B}})^{\mathrm{pretr}}) \cong G$, as wanted. We divide our reasoning in two steps:

1. We choose a presentation A according Lemma 5.31 and a "well-behaved" associated enhancement $(\mathsf{C}, H^0(\mathsf{e}))$. The meaning of its well-behaviour will become clear in the second part of the proof.
2. We describe a DG-functor f', which is a $(\mathsf{B}^{\mathrm{pretr}}, \mathrm{id})$-lift of F. We conclude that $F \cong H^0(\mathsf{f}') \cong H^0((F^{gr}_{|\mathsf{B}})^{\mathrm{pretr}})$.

We prove item 1. Let (C, E) be a cofibrant enhancement of $\mathrm{tr}(\mathsf{B})$ and define $\mathsf{A} := \mathsf{C}^{E}_{|H^0(\mathsf{B})}$. We consider the DG-functors $\mathsf{j} : \mathsf{A}^{\mathrm{pretr}} \to \mathsf{C}^{\mathrm{pretr}}$, induced by the inclusion, and $\mathsf{h} : \mathsf{A} \to \mathsf{B}$, associated to a $(\mathsf{C}, E) - (\mathsf{B}^{\mathrm{pretr}}, \mathrm{id})$-lift of the identity (which exists by Proposition 3.64) as expressed in Lemma 5.31. We define e to be the (quasi-equivalence) quasi-functor given by the following composition

$$\mathsf{C} \xhookrightarrow{\mathsf{y}} \mathsf{C}^{\mathrm{pretr}} \xleftarrow{\mathsf{j}} \mathsf{A}^{\mathrm{pretr}} \xrightarrow{\mathsf{h}^{\mathrm{pretr}}} \mathsf{B}^{\mathrm{pretr}},$$

where y is the Yoneda embedding. We want to consider the enhancement $(\mathsf{C}, H^0(\mathsf{e}))$. The "well-behaviour" discussed above is motivated by the fact that $\mathsf{A} = \mathsf{C}^{H^0(\mathsf{e})}_{|H^0(\mathsf{B})}$. Let us prove it. By the definition of the functors, for every $X \in H^0(\mathsf{A})$ we have

$$\begin{aligned} H^0(\mathsf{e})(X) &\cong H^0(\mathsf{h}^{\mathrm{pretr}}) H^0(\mathsf{j})^{-1} H^0(\mathsf{y})(X) \cong H^0(\mathsf{h}^{\mathrm{pretr}}) H^0(\mathsf{j})^{-1}(X) \\ &\cong H^0(\mathsf{h}^{\mathrm{pretr}})(X) \cong Y \in H^0(\mathsf{B}). \end{aligned}$$

This implies that $\mathsf{A} \subset \mathsf{C}^{H^0(\mathsf{e})}_{|H^0(\mathsf{B})}$. Conversely, let $X \in \mathsf{C}^{H^0(\mathsf{e})}_{|H^0(\mathsf{B})}$. Then we have $H^0(\mathsf{e})(X) \cong Y \in H^0(\mathsf{B})$. Since $\mathsf{h} : \mathsf{A} \to \mathsf{B}$ is a quasi-equivalence, there exists $Z \in \mathsf{A}$ such that $H^0(\mathsf{h})(Z) \cong Y$. From the definitions of j and y, we have $H^0(\mathsf{e})(Z) \cong Y$. In particular, X and Z are homotopy equivalent. Since A is closed under homotopy equivalence by Lemma 3.67, $X \in \mathsf{A}$ as wanted.

We prove item 2. By Proposition 3.64, F has a $(\mathsf{C}, H^0(\mathsf{e}))$-lift, and by Fact 5.30, this lift can be chosen to be a DG-functor f. Since $\mathsf{A} = \mathsf{C}^{H^0(\mathsf{e})}_{|H^0(\mathsf{B})}$, we have that $\mathsf{f}(\mathsf{A}) \subseteq \mathsf{A}$ because $H^0(\mathsf{e})H^0(\mathsf{f})(X) \cong FH^0(\mathsf{e})(X) \cong H^0(\mathsf{e})(X)$ for every $X \in H^0(\mathsf{A})$. We define $\mathsf{f}_\mathsf{A} := \mathsf{f}_{|\mathsf{A}} : \mathsf{A} \to \mathsf{A}$. Notice that $\mathsf{f}^{\mathrm{pretr}}\mathsf{j} \cong \mathsf{j}\mathsf{f}_\mathsf{A}^{\mathrm{pretr}}$ by Proposition 3.39.

Moreover, since B is graded and $\mathsf{h} : \mathsf{A} \to \mathsf{B}$ is a quasi-equivalence, we have that $H^*(\mathsf{h}) : H^*(\mathsf{A}) \to H^*(\mathsf{B}) = \mathsf{B}$ is a graded equivalence (i.e. a DG-equivalence between graded categories). Therefore, for the sake of simplicity, we can replace B with $H^*(\mathsf{A})$ and h with $H^*(\mathsf{h})^{-1}\mathsf{h}$, so that $H^*(\mathsf{h}) = \mathrm{id}$. From the definition of H^*, we obtain the commutative diagram of DG-categories

$$
\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathsf{f}_\mathsf{A}"] \arrow[d, "\mathsf{h}"] & \mathsf{A} \arrow[d, "\mathsf{h}"] \\
\mathsf{B} \arrow[r, "H^*(\mathsf{f}_\mathsf{A})"] & \mathsf{B}
\end{tikzcd}
$$

which can be extended to the pretriangulated closures by Proposition 3.39. Let $\mathsf{f}' = (H^*(\mathsf{f}_\mathsf{A}))^{\mathrm{pretr}}$. We have the following situation

$$
\begin{tikzcd}
\mathsf{B}^{\mathrm{pretr}} \arrow[d, "\mathsf{f}'"] & \mathsf{A}^{\mathrm{pretr}} \arrow[l, "\mathsf{h}^{\mathrm{pretr}}"'] \arrow[r, "\mathsf{j}"] \arrow[d, "\mathsf{f}_\mathsf{A}^{\mathrm{pretr}}"] & \mathsf{C}^{\mathrm{pretr}} \arrow[d, "\mathsf{f}^{\mathrm{pretr}}"] & \mathsf{C} \arrow[l, hook', "\mathsf{y}"'] \arrow[r, squiggly, "H^0(\mathsf{e})"] \arrow[d, "\mathsf{f}"] \arrow[lll, bend right, "\mathsf{e}"'] & \mathrm{tr}(\mathsf{B}) \arrow[d, "F"] \\
\mathsf{B}^{\mathrm{pretr}} & \mathsf{A}^{\mathrm{pretr}} \arrow[l, "\mathsf{h}^{\mathrm{pretr}}"'] \arrow[r, "\mathsf{j}"] & \mathsf{C}^{\mathrm{pretr}} & \mathsf{C} \arrow[l, hook', "\mathsf{y}"'] \arrow[r, squiggly, "H^0(\mathsf{e})"] \arrow[lll, bend left, "\mathsf{e}"] & \mathrm{tr}(\mathsf{B}).
\end{tikzcd}
$$

In particular, this diagram shows that f' is a $(\mathsf{B}^{\mathrm{pretr}}, \mathrm{id})$-lift of F. We have a natural isomorphism $\eta : F \to H^0(\mathsf{f}')$, which gives a graded isomorphism $\mu : F^{gr}_{|\mathsf{B}} \to H^*(\mathsf{f}_\mathsf{A})$. Being B a DG-category with trivial differential, μ is a DG-natural isomorphism, so we can extend it to a unique DG-natural isomorphism on $\mathsf{B}^{\mathrm{pretr}}$ by Proposition 3.39. In particular, $(F^{gr}_{|\mathsf{B}})^{\mathrm{pretr}} \cong \mathsf{f}'$ up to DG-isomorphism, so $F \cong H^0((F^{gr}_{|\mathsf{B}})^{\mathrm{pretr}})$. □

5.33. Theorem. *Let* B *be a graded category. The following are equivalent:*

1. B *is triangulated formal and formally standard;*
2. $\mathrm{tr}(\mathsf{B})$ *has a strongly unique enhancement;*
3. $\mathsf{D}(\mathsf{B})^{\mathrm{c}}$ *has a strongly unique enhancement.*

PROOF. Proposition 5.28 (remembering Example 5.23) and Proposition 5.32 prove 1 ⇔ 2. Remark 5.29 deals with 1 ⇒ 3, while Proposition 3.70 shows 3 ⇒ 2. □

§5.4. Free generators

Our aim is to apply Theorem 5.33. We start with some simple examples, described by the following non-canonical definition.

5.34. Definition. A DG-category A is a *free generator* if every object $X \in \mathrm{D}(\mathsf{A})^{\mathrm{c}}$ is isomorphic to a direct summand of

$$\bigoplus_{i\in\mathbb{Z}}\left(\bigoplus_{j=1}^{n_i} C_{i,j}[i]\right) \qquad \text{for some } C_{i,j} \in H^0(\mathsf{A}) \text{ and } n_i \neq 0 \text{ for finitely many } i\text{'s}. \tag{5.35}$$

5.36. Example. In a trivial way, any pretriangulated DG-category is a free generator, but this example is far from our idea of application, since we want to study it for graded categories (i.e. DG-categories with trivial differential). Let us give some meaningful examples.

- The DG-category R given by a semisimple ring is a free generator. Indeed, a finitely generated R-module is a direct summand of a free R-module of finite rank. As a consequence, $\mathrm{D}(R)^{\mathrm{c}}$ is obtained by cones of closed morphisms
$$\bigoplus_{i\in\mathbb{Z}} R^{n_i}[i] \to \bigoplus_{j\in\mathbb{Z}} R^{m_j}[j],$$
which are simply given by kernels and cokernels of maps $R^{n_i} \to R^{m_i}$, again expressed via finitely generated R-modules.
- Consider an algebraic finite triangulated category $\mathscr{T}$ as defined in [55] and let Λ be the algebra of endomorphisms associated to a basic additive generator X. Given an enhancement (C, E) of $\mathscr{T}$, any object $Y \in E^{-1}(X)$, together with its endomorphisms, defines a free generator DG-category with one object (this follows from the fact that $\mathscr{T}$ is equivalent to the category of finitely generated projective (right) Λ-modules).

Roughly speaking, the following lemma tells us that whenever a ring is a free generator, then it is a free generator also for its associated periodic triangulated categories (a triangulated category is called *periodic* if $[n] \cong \mathrm{id}$ for some n; see [70] for a more thorough introduction).

5.37. Lemma. *Let R be a ring and consider $R[t,t^{-1}]$ with t homogeneous of positive degree. If R is a free generator, then so is $R[t,t^{-1}]$.*

PROOF. Let us consider the inclusion $R \to R[t,t^{-1}]$, which is a (differential) graded morphism. Then we can extend it to a DG-functor $\pi : \mathrm{Perf}(R) \to \mathrm{Perf}(R[t,t^{-1}])$. Set $n := \deg(t)$, and for any $k \in \mathbb{Z}$ use $\bar{k} \in \{0,\dots,n-1\}$ for the representative of k in $\mathbb{Z}/n\mathbb{Z}$.

The objects in $\mathrm{DGMod}(R)$ are functors $M : R^o \to \mathsf{C}_{\mathrm{DG}}(\mathrm{Mod}(\Bbbk))$, so they are DG-modules $M(R^o) := (M^k, d^k)_{k\in\mathbb{Z}}$. Similarly, an object $M_{/n} \in \mathrm{DGMod}(R[t,t^{-1}])$ can be thought of as the DG-module $M_{/n}(R[t,t^{-1}]^o) := (M^k_{/n}, d^k_{/n})_{k\in\mathbb{Z}}$. The morphism $M_{/n}(t) : M_{/n} \to M_{/n}$ has degree n

and an inverse, which shows that $M^k_{/n} \cong M^{k+jn}_{/n}$ for every $j \in \mathbb{Z}$. In particular, $M_{/n}$ is identified with $(M^{\bar{k}}_{/n}, d^{\bar{k}}_{/n})^{n-1}_{\bar{k}=0}$, with $d^{n-1}_{/n} : M^{n-1}_{/n} \to M^0_{/n}$.

Then π can be explicitly expressed by (cf. [70, Definition 3.7])

$$\pi(M)^k := \bigoplus_{\bar{\ell}=k} M^\ell, \qquad d^k_{\pi(M)} := \bigoplus_{\bar{\ell}=k} d^\ell_M$$

(the behaviour of π on morphisms is expressed accordingly). From this description, it is easy to notice that π is essentially surjective. Since π is also additive and preserves suspensions, the statement follows. $\square$

5.38. Proposition. *Let* A *be a free generator DG-category. Then* A *is formally standard.*

PROOF. Define $\mathsf{A}_{\mathrm{add}}$ the full DG-subcategory of $\mathsf{A}^{\mathrm{pretr}}$ whose objects are of the form (5.35). Notice that $H^0(\mathsf{A}_{\mathrm{add}}) \subset \mathrm{tr}(\mathsf{A})$. Let $F : \mathrm{tr}(\mathsf{A}) \to \mathrm{tr}(\mathsf{A})$ be a triangulated equivalence such that its graded restriction is naturally isomorphic to the identity. Therefore, $F_{|H^0(\mathsf{A}_{\mathrm{add}})}$ is naturally isomorphic to the identity. Let G be the composition

$$H^0(\mathsf{A}_{\mathrm{add}}) \xrightarrow{F_{|H^0(\mathsf{A}_{\mathrm{add}})}} \mathrm{tr}(\mathsf{A}) \hookrightarrow \mathrm{D}(\mathsf{A})^{\mathrm{c}}.$$

Since A is a free generator, $\mathrm{D}(\mathsf{A})^{\mathrm{c}}$ is the idempotent completion of $H^0(\mathsf{A}_{\mathrm{add}})$; [2, Proposition 1.3] gives a unique extension $H : \mathrm{D}(\mathsf{A})^{\mathrm{c}} \to \mathrm{D}(\mathsf{A})^{\mathrm{c}}$ of G, which is therefore an extension of F. Further, from the same proposition the natural isomorphism $F_{|H^0(\mathsf{A}_{\mathrm{add}})} \to \mathrm{id}$ extends to a natural isomorphism $H \to \mathrm{id}$. By restricting this natural isomorphism, F is naturally isomorphic to the identity. Lemma 5.25 concludes the proof. $\square$

5.39. Corollary. *Let* $\mathbb{K}$ *be a field. Given a free generator* $\mathbb{K}$*-algebra* Λ *with finite projective dimension* d *as a* Λ*-bimodule, then* $\mathrm{D}(\Lambda[t,t^{-1}])^{\mathrm{c}}$ *has a strongly unique enhancement for any* t *homogeneous of degree greater or equal than* d.

PROOF. Under these assumptions, Proposition 1.47 shows that $\Lambda[t,t^{-1}]$ is intrinsically formal. By Example 5.5, Lemma 5.37, Proposition 5.38 and Theorem 5.33, we conclude. $\square$

5.40. Example. Let $\mathbb{K}$ be a perfect field and let Λ be a semisimple finite-dimensional $\mathbb{K}$-algebra. In this case, Λ is a projective Λ-bimodule by [64, Corollary b, p. 192]. Then, by Example 5.36 and Corollary 5.39, $\mathrm{D}(\Lambda[t,t^{-1}])^{\mathrm{c}}$ has a strongly unique enhancement for any homogeneous t of positive degree.

5.41. Corollary. *Let* $\mathbb{K}$ *be a field. The triangulated category* $\mathrm{mod}(\mathbb{K})$ *with suspension the identity has a strongly unique* $\mathbb{K}$*-linear enhancement.*

PROOF. We claim that $\mathrm{mod}(\mathbb{K})$ is triangulated equivalent to $\mathrm{D}(\mathbb{K}[t,t^{-1}])^{\mathrm{c}}$ with t homogeneous of degree 1. First, let us consider the description of the DG-functor $\pi : \mathrm{Perf}(\mathbb{K}) \to \mathrm{Perf}(\mathbb{K}[t,t^{-1}])$ given in the proof of Lemma 5.37. At the homotopy level, we have an induced triangulated functor $F : \mathcal{K}^{\mathrm{b}}(\mathrm{mod}(\mathbb{K})) \to \mathrm{D}(\mathbb{K}[t,t^{-1}])^{\mathrm{c}}$ (recall Example 3.59). Since $\deg(t) = 1$, $F_{|\mathrm{mod}(\mathbb{K})}$ is essentially surjective and fully faithful. Therefore, $\mathrm{D}(\mathbb{K}[t,t^{-1}])^{\mathrm{c}} \cong \mathrm{mod}(\mathbb{K})$ as ($\mathbb{K}$-linear) additive categories. We now observe that the suspension functor in $\mathrm{D}(\mathbb{K}[t,t^{-1}])^{\mathrm{c}}$ is isomorphic to the identity because $\deg(t) = 1$ (cf. the proof of Lemma 5.37). This suffices to conclude that $\mathrm{mod}(\mathbb{K})$ is triangulated equivalent to $\mathrm{D}(\mathbb{K}[t,t^{-1}])^{\mathrm{c}}$, as noted in [71, §1.1]. The statement now follows from Corollary 5.39. □

5.42. Remark. Notice that Corollary 5.41 does not hold when $\mathbb{K} = \mathbb{F}_p$ (with p a prime) and we consider linearity over $\Bbbk = \mathbb{Z}$ (see [71] and [14, Corollary 3.10]).

Furthermore, since $\mathbb{F}_p[t,t^{-1}]$ is formally standard by Lemma 5.37 and Proposition 5.38, Theorem 5.33 shows that $\mathrm{D}(\mathbb{F}_p[t,t^{-1}])^{\mathrm{c}}$ has a strongly unique enhancement if and only if $\mathrm{mod}(\mathbb{F}_p) \cong \mathbb{F}_p[t,t^{-1}]$ is triangulated formal. Example 5.5 then ensures that $\mathbb{F}_p[t,t^{-1}]$ is not intrinsically formal as a $\mathbb{Z}$-linear graded ring.

5.43. Remark. Let us briefly discuss the example of a non-unique $\mathbb{K}$-linear enhancement provided by Rizzardo and Van den Bergh in [67].

Let $\mathbb{K}$ be a field and $\mathbb{F} := \mathbb{K}(x_1,\dots,x_n)$ with $n > 0$ even. Then $\mathrm{D}(\mathbb{F}[t,t^{-1}])^{\mathrm{c}}$ with $\deg(t) = n$ has a non-unique enhancement, as shown in [67]. We notice that this example carefully avoids any situation described above. As discussed in Example 5.36, $\mathbb{F}[t,t^{-1}]$ is a free generator because $\mathbb{F}$ is semisimple. However, it is not finite-dimensional, because, as seen in Example 5.40, this will not work for a perfect field.

As one may expect from the viewpoint depicted in this article, the proof in [67] shows explicitly that $\mathbb{F}[t,t^{-1}]$ is not intrinsically formal by deforming the graded algebra into a different minimal A_∞-algebra.

§5.5. D-standardness and K-standardness

In this section we consider the notions of D-standard and K-standard categories introduced in [19], and show that they are, in fact, equivalent to strong uniqueness of enhancements in a proper way. We emphasize that these results hold for $\Bbbk$-linearity, where $\Bbbk$ is any commutative ring.

5.44. Definition. Let $\mathscr{A}$ be an additive category. Then $\mathscr{A}$ is *K-standard* if the following implication holds:

(♠) Whenever $F : \mathcal{K}^{\mathrm{b}}(\mathscr{A}) \to \mathcal{K}^{\mathrm{b}}(\mathscr{A})$ is a triangulated equivalence such that $F(\mathscr{A}) \subseteq \mathscr{A}$ and $\eta_0 : F_{|\mathscr{A}} \to \mathrm{id}_{\mathscr{A}}$ is a natural isomorphism, there exists a natural isomorphism $\eta : F \to \mathrm{id}$ extending η_0.

Accordingly, an exact category $\mathscr{E}$ is *D-standard* if (♠) holds for $F : \mathcal{D}^{\mathrm{b}}(\mathscr{E}) \to \mathcal{D}^{\mathrm{b}}(\mathscr{E})$.

5.45. Remark. Because of Remark 2.18, K-standardness is just a specialized version of D-standardness. Nonetheless, we will treat them separately as the results for additive categories are easier to prove.

5.46. Lemma. *Let $\mathscr{A}$ be an additive category (resp. let $\mathscr{E}$ be an exact category). Then (♠) holds if and only if the following is satisfied.*

(♣) *Whenever $F : \mathcal{K}^{\mathrm{b}}(\mathscr{A}) \to \mathcal{K}^{\mathrm{b}}(\mathscr{A})$ is a triangulated equivalence such that $F(\mathscr{A}) \subseteq \mathscr{A}$ and $F_{|\mathscr{A}} \cong \mathrm{id}_{\mathscr{A}}$, then $F \cong \mathrm{id}$ (replace $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$ and $\mathscr{A}$ with $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ and $\mathscr{E}$ respectively in the case of exact categories).*

PROOF. This result is analogous to [19, Lemma 3.5]. The fact that (♠) implies (♣) is obvious. Conversely, let $\eta_0 : F_{|\mathscr{A}} \to \mathrm{id}_{\mathscr{A}}$. By (♣), there exists $\mu : F \to \mathrm{id}$. In particular, $\mu_{|\mathscr{A}} : F_{|\mathscr{A}} \to \mathrm{id}_{\mathscr{A}}$, so we can consider $\mathcal{K}^{\mathrm{b}}(\eta_0 \mu^{-1}_{|\mathscr{A}})\mu : F \to \mathrm{id}$: by definition, such natural isomorphism restricted to $\mathscr{A}$ is η_0. This concludes the proof. Analogously, we can show the result for $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$. □

5.47. Proposition. *An additive category $\mathscr{A}$ is K-standard if and only if $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$ has a strongly unique enhancement.*

PROOF. Remembering Example 3.59, Lemma 5.25 and Lemma 5.46 show that $\mathscr{A}$ is K-standard if and only if it is formally standard. Since $\mathscr{A}$ is triangulated formal by Proposition 5.6, Theorem 5.33 concludes the proof. □

5.48. Example. We recall that a Krull-Schmidt category $\mathscr{A}$ is an additive category in which every object decomposes into a finite direct sum of objects having local endomorphism rings.

A Krull-Schmidt category $\mathscr{A}$ is an *Orlov category* if

1. The endomorphism ring of each indecomposable is a division ring;
2. There is a degree function $\deg : \mathrm{ind}(\mathscr{A}) \to \mathbb{Z}$, where $\mathrm{ind}(\mathscr{A})$ is the set of all indecomposables, such that $\deg X \geq \deg Y$ implies $\mathrm{Hom}(X,Y) = 0$ whenever $X \ncong Y$.[¶]

As proved in [19, Proposition 4.6], an Orlov category $\mathscr{A}$ is K-standard, so $\mathcal{K}^{\mathrm{b}}(\mathscr{A})$ has a strongly unique enhancement. An example is given by $\mathrm{proj}(A)$ for A a triangular algebra (i.e. an algebra whose associated Ext-quiver does not admit oriented cycles), see [19, Example 4.7] and [17, Lemma 2.1].

5.49. Lemma. *Let $\mathscr{E}$ be an exact category, and let $\mathscr{E}_{\mathrm{DG}} := \mathcal{D}^{\mathrm{b}}_{\mathrm{DG}}(\mathscr{E})$ as in Example 5.23. Then $\mathscr{E}$ is D-standard if and only if $\mathscr{E}_{\mathrm{DG}}$ is formally standard.*

PROOF. By Lemma 5.46 and Lemma 5.19, $\mathscr{E}$ is D-standard if and only if any triangulated equivalence $F : \mathcal{D}^{\mathrm{b}}(\mathscr{E}) \to \mathcal{D}^{\mathrm{b}}(\mathscr{E})$ such that $F(X) = X$ for $X \in \mathscr{E}$ and $F_{|\mathscr{E}} \cong \mathrm{id}_{\mathscr{E}}$ is naturally isomorphic to the identity. By Lemma 5.25, we are reduced to prove that $F_{|\mathscr{E}} \cong \mathrm{id}_{\mathscr{E}}$ if and only if $F^{gr}_{|H^*(\mathscr{E}_{\mathrm{DG}})} \cong \mathrm{id}_{H^*(\mathscr{E}_{\mathrm{DG}})}$. This follows from Proposition 5.24. □

We now aim to show the analogous of Proposition 5.47 for derived categories.

5.50. Proposition. *Let $\mathscr{E}$ be an exact category, and consider F a triangulated autoequivalence of $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ such that $F(\mathscr{E}) \subset \mathscr{E}$ and $F_{|\mathscr{E}} \cong \mathrm{id}_{\mathscr{E}}$. Let (C, E) be any enhancement of $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$. If F has a (C, E)-lift, then F is naturally isomorphic to the identity.*

[¶] Here the degree function differs slightly from the definition in [19] (we used $\geq$ instead of $\leq$); the proof of Theorem 5.61 will motivate our choice.

Moreover, the identity of C *is the only quasi-functor lifting the identity of* $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$*. Consequently, any autoequivalence of* $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ *has at most one* (C, E)*-lift.*

PROOF. Without loss of generality, assume C is a cofibrant DG-category. By Fact 5.30, there exists a DG-functor $\mathsf{f} : \mathsf{C} \to \mathsf{C}$ which is a (C, E)-lift of F. We now consider $\mathsf{f}_{|\mathscr{E}} : \mathsf{C}_{|\mathscr{E}} \to \mathsf{C}_{|\mathscr{E}}$. Defined the quasi-equivalence quasi-functor

$$(\mathsf{C}_{|\mathscr{E}})^{\mathrm{pretr}} \xrightarrow{\mathsf{j}} \mathsf{C}^{\mathrm{pretr}} \xhookleftarrow{\mathsf{y}} \mathsf{C},$$

where j is induced by inclusion, notice that $\mathsf{y}\mathsf{f} \cong \mathsf{f}^{\mathrm{pretr}}\mathsf{y}$ and $\mathsf{j}(\mathsf{f}_{|\mathscr{E}})^{\mathrm{pretr}} \cong \mathsf{f}^{\mathrm{pretr}}\mathsf{j}$ by Proposition 3.39.

Moreover, we are able to construct the following commutative diagram

$$\begin{tikzcd} H^0(\mathsf{C}_{|\mathscr{E}}) \arrow[d, "\mathrm{id}"] & \tau_{\leq 0}\mathsf{C}_{|\mathscr{E}} \arrow[l, "\simeq"'] \arrow[r, "\mathsf{p}_{\leq 0}"] \arrow[d, "\tau_{\leq 0}\mathsf{f}_{|\mathscr{E}}"] & \mathsf{C}_{|\mathscr{E}} \arrow[d, "\mathsf{f}_{|\mathscr{E}}"] \\ H^0(\mathsf{C}_{|\mathscr{E}}) & \tau_{\leq 0}\mathsf{C}_{|\mathscr{E}} \arrow[l, "\simeq"'] \arrow[r, "\mathsf{p}_{\leq 0}"] & \mathsf{C}_{|\mathscr{E}} \end{tikzcd}$$

by assumption (indeed $H^0(\mathsf{f}_{|\mathscr{E}}) = F_{|\mathscr{E}} \cong \mathrm{id}$). The commutative diagram obtained by taking the pretriangulated closures shows that $(\mathsf{f}_{|\mathscr{E}})^{\mathrm{pretr}}$ is the identity quasi-functor by the universal property of the DG-quotient (see item 1 of Definition/Proposition 3.57 and Corollary 5.9). This suffices to show that f is the identity as well because $\mathsf{f}\mathsf{y}^{-1}\mathsf{j} \cong \mathsf{y}^{-1}\mathsf{j}(\mathsf{f}_{|\mathscr{E}})^{\mathrm{pretr}} \cong \mathsf{y}^{-1}\mathsf{j}$, where j and y are quasi-equivalences. In particular, $F \cong EH^0(\mathsf{f})E^{-1} \cong EE^{-1} \cong \mathrm{id}$, as wanted. □

5.51. Theorem. *Let* $\mathscr{E}$ *be an exact category. Then* $\mathscr{E}$ *is D-standard if and only if* $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ *has a strongly unique enhancement.*

PROOF. Recall that $\mathscr{E}_{\mathrm{DG}}$ is triangulated formal by Corollary 5.9. If $\mathscr{E}$ is D-standard, by Lemma 5.49 $\mathscr{E}_{\mathrm{DG}}$ is also formally standard. Proposition 5.28 shows that $\mathcal{D}^{\mathrm{b}}(\mathscr{E})$ has a strongly unique enhancement, since $\mathscr{E}_{\mathrm{DG}}$ is always lifted (see Example 5.23). The converse implication follows from Proposition 5.50 and Proposition 3.64. □

5.52. Proposition. *Let* $\mathscr{A}$ *be an abelian category with enough projective objects. We denote with* $\mathrm{Proj}(\mathscr{A})$ *its subcategory of projective objects. If* $\mathcal{K}^{\mathrm{b}}(\mathrm{Proj}(\mathscr{A}))$ *has a strongly unique enhancement, then* $\mathcal{D}^{\mathrm{b}}(\mathscr{A})$ *has a strongly unique enhancement.*

PROOF. It immediately follows from Proposition 5.47, [19, Theorem 6.1] and Theorem 5.51. □

5.53. Corollary. *Let* $\mathscr{A}$ *be a hereditary category. Then its bounded derived category* $\mathcal{D}^{\mathrm{b}}(\mathscr{A})$ *has a strongly unique enhancement (cf. [19, Corollary 5.6]).*

5.54. Example. Let R be any ring (recall Convention 1.3). If R is (right) hereditary, the bounded derived category of (all right) R-modules has a strongly unique enhancement. If R is (right) semihereditary and Noetherian, the bounded derived category of finitely generated R-modules has a strongly unique enhancement. Dually, the result holds for left modules.

By combining Corollary 5.53 with Hubery's Theorem 2.56, we have the following.

5.55. Corollary. *Any triangulated category with a hereditary heart is algebraic and has a strongly unique enhancement.*

§5.A. Appendix. Almost ample sets and exceptional sequences

In the context of bounded derived categories in algebraic geometry, it is common to consider Fourier-Mukai transforms. Roughly speaking, these are triangulated functors coming from a geometric perspective. When dealing with smooth projective varieties, it is possible to show that Fourier-Mukai transforms are precisely the triangulated functors admitting a lift. The interested reader may refer to [14, §6.3] for a brief overview of this connection; further suggested readings on the topic are [75], [52] and [72].

A well-known theorem by Lunts and Orlov states that the bounded derived category of coherent sheaves on a projective variety (with a technical assumption) has a strongly unique enhancement (see [51, Theorem 9.9], but also [61, Theorem 2.2] in the context of Fourier-Mukai transforms). The proof strategy is to show that D-standardness holds using the notion of ample sequence [61, Definition 2.12]. Here we consider a generalization due to Canonaco and Stellari [13, Definition 2.9], and prove explicitly the theorem for the sake of completeness.

We conclude the section by showing that an algebraic triangulated category with a full strong exceptional sequence has a strongly unique enhancement as well. This result is already known in a wider generality (see Example 5.48).

5.56. Definition. Given an abelian category $\mathscr{A}$ and a set I, we say that $\{P_i\}_{i\in I} \subset \mathscr{A}$ is an *almost ample set* if, for any $A \in \mathscr{A}$, there exists $i \in I$ such that

1. There is a natural number k and an epimorphism $P_i^{\oplus k} \to A$;
2. $\mathrm{Hom}(A, P_i) = 0$.

5.57. Example. Given an algebraic space X proper over an Artinian ring with depth ≥ 1 at every closed point, the category of coherent sheaves $\mathrm{Coh}(X)$ has an almost ample set (see [60, Lemma 3.3.2]). Another class of examples is given by [13, Proposition 2.12].

5.58. Theorem. *Let $\mathscr{A}$ be an abelian category with an almost ample set. Then $\mathcal{D}^{\mathrm{b}}(\mathscr{A})$ has a strongly unique enhancement (cf. [13, Proposition 3.7]).*

PROOF. We will show that $\mathscr{A}$ is D-standard, and conclude by Theorem 5.51. Let $(F, \eta) : \mathcal{D}^{\mathrm{b}}(\mathscr{A}) \to \mathcal{D}^{\mathrm{b}}(\mathscr{A})$ be an autoequivalence such that $F(\mathscr{A}) \subset \mathscr{A}$ and $F_{|\mathscr{A}} \cong \mathrm{id}_{\mathscr{A}}$. The idea is to follow the reasoning of [62, Proposition 3.4.6]. Notice that we are reduced to show the original proof from Step 4 forward, as the previous steps follows directly from assuming that F fixes $\mathscr{A}$. For clarity, we will use Σ instead of $[1]$ to indicate the shift functor.

Step 1. We extend the natural isomorphism $f : \mathrm{id}_{\mathscr{A}} \to F_{|\mathscr{A}}$ to the shifts of $\mathscr{A}$.

In order to extend f to a natural transformation between triangulated functors, we need to require that the diagram

$$\begin{tikzcd} \Sigma \arrow[r, "\mathrm{id}"] \arrow[d, "f\Sigma"] & \Sigma \arrow[d, "\Sigma f"] \\ F\Sigma \arrow[r, "\eta"] & \Sigma F \end{tikzcd}$$

is commutative. This forces us to define, for $X \in \mathscr{A}$, $f_{\Sigma X} := \eta_X^{-1}\Sigma(f_X)$ and, inductively, $f_{\Sigma^i X} := \eta^{-1}\Sigma(f_{\Sigma^{i-1}X})$ for any $i \geq 1$. In a similar fashion, $f_{\Sigma^{-i}X} = \Sigma^{-1}(\eta_{\Sigma^{-i}X} f_{\Sigma^{1-i}X})$.

Step 2. By induction on n, we extend f to the whole $\mathcal{D}^{\mathrm{b}}(\mathscr{A})$.

Let $\mathscr{D}_n \subset \mathcal{D}^{\mathrm{b}}(\mathscr{A})$ be the full subcategory of objects X for which there exists $a \in \mathbb{Z}$ such that $H^p(X) = 0$ for $p < a$ or $p > a+n$. For the sake of simplicity, assume $X = (X^i, d^i)$ with $H^p(X) = 0$ for $p < -n$ or $p > 0$. By assumption, there exist $i \in I$ and $k \in \mathbb{N}$ such that $s : P_i^{\oplus k} \to \ker d^0$ is an epimorphism and $\mathrm{Hom}(\ker d^0, P_i^{\oplus k}) = 0$. First of all, s induces a morphism $P_i^{\oplus k} \to X$ by composition. We obtain a distinguished triangle

$$\Sigma^{-1}(X_{n-1}) \longrightarrow P_i^{\oplus k} \longrightarrow X \longrightarrow X_{n-1}$$

where $H^p(X_{n-1}) = 0$ for $p < -n$ or $p > -1$, so $X_{n-1} \in \mathscr{D}_{n-1}$. Since $\mathrm{Hom}(\ker d^0, P_i^{\oplus k}) = 0$, the epimorphism $\ker d^0 \to H^0(X)$ implies that $\mathrm{Hom}(H^0(X), P_i^{\oplus k}) = 0$. Using the filtration of X, we conclude that

$$\mathrm{Hom}(X, F(P_i^{\oplus k})) \cong \mathrm{Hom}(X, P_i^{\oplus k}) \cong \mathrm{Hom}(H^0(X), P_i^{\oplus k}) = 0.$$

By [62, Lemma 3.1.1], we can choose a unique morphism $f_X : X \to F(X)$ completing the diagram

$$\begin{tikzcd} \Sigma^{-1}(X_{n-1}) \arrow[r] \arrow[d, "f_{\Sigma^{-1}(X_{n-1})}"] & P_i^{\oplus k} \arrow[r] \arrow[d, "f_{P_i^{\oplus k}}"] & X \arrow[r, "\pi"] \arrow[d, dashed, "f_X"] & X_{n-1} \arrow[d, "f_{X_{n-1}}"] \\ F(\Sigma^{-1}(X_{n-1})) \arrow[r] & F(P_i^{\oplus k}) \arrow[r] & F(X) \arrow[r, "F(\pi)"] & F(X_{n-1}). \end{tikzcd} \tag{5.59}$$

More strongly, f_X is the only morphism satisfying $f_{X_{n-1}}\pi = F(\pi) f_X$.

Step 3. The morphism f_X does not depend on the choice of s.

Assume there exists another $j \in I$ such that $\mathrm{Hom}(\ker d^0, P_j^{\oplus m}) = 0$ and $t : P_j^{\oplus m} \to \ker d^0$ is an epimorphism. We claim that we can find ℓ such that

$$\begin{tikzcd} P_\ell^{\oplus n} \arrow[r] \arrow[d] \arrow[dr, two heads, "u"] & P_i^{\oplus k} \arrow[d, two heads, "s"] \\ P_j^{\oplus m} \arrow[r, two heads, "t"] & \ker d^0 \end{tikzcd}$$

and $\mathrm{Hom}(\ker d^0, P_\ell^{\oplus n}) = 0$. Indeed, we just consider $P_i^{\oplus k} \times_{\ker d^0} P_j^{\oplus m}$: since $\{P_i\}_{i \in I}$ is an almost ample set, there exists an epimorphism $P_\ell^{\oplus n} \to P_i^{\oplus k} \times_{\ker d^0} P_j^{\oplus m}$ and $\mathrm{Hom}(P_i^{\oplus k} \times_{\ker d^0}$

$P_j^{\oplus m}, P_\ell^{\oplus n}) = 0$. Since s,t are epimorphisms, so is $P_i^{\oplus k} \times_{\ker d^0} P_j^{\oplus m} \to \ker d^0$: we conclude that $\mathrm{Hom}(\ker d^0, P_\ell^{\oplus n}) = 0$. From the previous step, we can define f_X^ℓ associated to $P_\ell^{\oplus n}$. In particular, the following diagram

$$\begin{tikzcd}
P_\ell^{\oplus n} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\mathrm{id}"] & X_{n-1}^\ell \arrow[r] \arrow[d, dashed, "v"] & \Sigma(P_\ell^{\oplus n}) \arrow[d] \\
P_i^{\oplus k} \arrow[r] & X \arrow[r, "\pi"] & X_{n-1} \arrow[r] & \Sigma(P_i^{\oplus k})
\end{tikzcd}$$

is commutative for some $v : X_{n-1}^\ell \to X_{n-1}$. By induction, we have that $f_{X_{n-1}}$ and $f_{X_{n-1}^\ell}$ are natural isomorphisms in $\mathscr{D}_{n-1}$, so the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f_X^\ell"] \arrow[rr, bend left, "\pi"] & X_{n-1}^\ell \arrow[r, "v"] \arrow[d, "f_{X_{n-1}^\ell}"] & X_{n-1} \arrow[d, "f_{X_{n-1}}"] \\
F(X) \arrow[r] \arrow[rr, bend right, "F(\pi)"'] & F(X_{n-1}^\ell) \arrow[r, "F(v)"] & FX_{n-1}
\end{tikzcd}$$

commutes (the composition of the horizontal morphisms are π and $F(\pi)$ from the choice of v). In particular, $f_{X_{n-1}}\pi = F(\pi) f_X^\ell$, but this property determines uniquely f_X, so $f_X = f_X^\ell$. Considered f_X^j associated to $P_j^{\oplus m}$, we can show that $f_X^j = f_X^\ell$ by the same reasoning. Finally, f_X does not depend on the choice of s.

Step 4. The morphism f_X extends the natural isomorphism in a natural way over $\mathscr{D}_n$.

Let $g : X \to Y$ be a morphism of $\mathscr{D}_n$. We want to prove that $f_Y g = F(g) f_X$. Since any morphism in $\mathcal{D}^{\mathrm{b}}(\mathscr{A})$ is a fraction of a quasi-isomorphism and a morphism of complexes, we can assume for the sake of simplicity that g is a morphism of complexes. Also, we can assume as before that $H^p(X) = 0$ for $p < -n$ and $p > 0$. Set c as the greatest integer such that $H^c(Y) \neq 0$ (if $Y \cong 0$, there is nothing to prove, so we can consider Y nonzero), and set the notation $Y = (Y^i, e^i)$. We distinguish the proof in two cases.

Case 1. $c < 0$. Consider the pullback $K := \ker d^0 \times_{\ker e^0} Y^{-1}$ given by the diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \ker d^0 \arrow[d] \\
Y^{-1} \arrow[r] & \ker e^0
\end{tikzcd}$$

and let $i \in I$ such that $P_i^{\oplus k} \to K$ is an epimorphism and $\mathrm{Hom}(K, P_i) = 0$. Since $Y^{-1} \to \ker e^0 = \operatorname{im} e^{-1}$ is an epimorphism (because $c < 0$), then $s : P_i^{\oplus k} \to K \to \ker d^0$ is an epimorphism and $\mathrm{Hom}(\ker d^0, P_i) = 0$. By Step 3, we can choose f_X induced by the morphism s just defined. We notice that $P_i^{\oplus k} \to \ker d^0 \to X \to Y$ factors through $\ker e^0$. Let us call $w : P_i^{\oplus k} \to \ker e^0 \subset Y^0$.

By definition of the epimorphism $P_i^{\oplus k} \to \ker d^0$, w factors through $w' : P_i^{\oplus k} \to Y^{-1}$, a morphism such that $w = e^{-1}w'$. This suffices to show that $P_i^{\oplus k} \to X \to Y$ is homotopy equivalent to 0, so it is 0 in $\mathcal{D}^{\mathrm{b}}(\mathscr{A})$. Finally, g factors through $g_{n-1} : X_{n-1} \to Y$, where $X_{n-1} \in \mathscr{D}_{n-1}$ is defined as in Step 2. The commutativity $f_Y g = F(g) f_X$ can now be checked by substituting X with X_{n-1}. Notice that if $X \in \mathscr{D}_k$, then $X_{n-1} \in \mathscr{D}_{k-1}$ for $k > 0$, and whenever $X \in \mathscr{D}_0$, then $X_{n-1} \in \mathscr{D}_0$ with $\Sigma^{-1}(X_{n-1})$ isomorphic to an object of $\mathscr{A}$. In this last case, passing from g to $\Sigma(g_{n-1})$, c increases by 1.

Case 2. $c \geq 0$. Choose an epimorphism $P_i^{\oplus k} \to \ker e^c \oplus H^c(X)$ and $\mathrm{Hom}(\ker e^c \oplus H^c(X), P_i^{\oplus k}) = 0$ (notice $H^c(X)$ is nonzero only if $c = 0$). Let $s : \Sigma^{-c}(P_i^{\oplus k}) \to Y$ the morphism obtained from the epimorphism above. As in Step 2, this gives a distinguished triangle with cone $Y_{n-1} \in \mathscr{D}_{n-1}$. Notice that $f_Y g = F(g) f_X$ follows once we prove that f_X is compatible with $g_{n-1} : X \to Y \to Y_{n-1}$. Indeed, in the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f_X"] & Y \arrow[r, "t"] \arrow[d, "f_Y"] & Y_{n-1} \arrow[d, "f_{Y_{n-1}}"] \\ FX \arrow[r, "F(g)"] & FY \arrow[r, "F(t)"] & FY_{n-1} \end{tikzcd}$$

the square on the right commutes by definition, so whenever the rectangle commutes, also the square on the left does, because

$$F(t)(f_Y g - F(g) f_X) = f_{Y_{n-1}} g_{n-1} - F(g_{n-1}) f_X = 0$$

implies that $f_Y g - F(g) f_X$ factors through $F(s)$, so it is 0 from

$$\mathrm{Hom}(X, F(\Sigma^{-c}(P_i^{\oplus k}))) \cong \mathrm{Hom}(X, \Sigma^{-c}(P_i^{\oplus k})) = 0.$$

(indeed, $\mathrm{Hom}(\Sigma^{-p}(H^p(X)), \Sigma^{-c}(P_i^{\oplus k})) = 0$ for all p). As in Case 1, if $Y \in \mathscr{D}_k$, then $Y_{n-1} \in \mathscr{D}_{k-1}$ for $k > 0$, and whenever $Y \in \mathscr{D}_0$, then $Y_{n-1} \in \mathscr{D}_0$ and, passing from g to g_{n-1}, c decreases by 1.

Finally, to check commutativity of f_X, let $X, Y \in \mathscr{D}_n$. We use either Case 1 or Case 2 to replace one of the two with an object in $\mathscr{D}_{n-1}$. By applying the same case a finite number of times, we can apply the other case. In particular, we end up with two objects in $\mathscr{D}_{n-1}$, for which the property has already been verified by induction. □

5.60. Remark. We shall notice that the natural isomorphism constructed is unique because it is the only one making (5.59) commutative. An abelian category with such a property is called *strongly D-standard*, see [19, Definition 5.1]. An example of a D-standard abelian category which is not strongly D-standard is $\mathrm{mod}(\mathbb{K}[\varepsilon])$ with $\varepsilon^2 = 0$ (see [19, Theorem 7.1]).

We now investigate the case of a triangulated category with a full strong exceptional sequence, defined in Example 4.12. The following result holds in fact for all derived categories of finitely generated modules over a triangular algebra, as discussed in Example 5.48.

Moreover, by recalling Theorem 4.28, we conclude that having a full strong exceptional sequence (with finite-dimensional hom-sets) forces any realized triangulated category to be algebraic with a strongly unique enhancement.

5.61. Theorem. *Let $\mathbb{K}$ be a field and consider an algebraic $\mathbb{K}$-linear triangulated category $\mathscr{T}$ with a full strong exceptional sequence $\langle E_1, \dots, E_n \rangle$ such that $\bigoplus_i \operatorname{Hom}(X, Y[i])$ is a finite-dimensional vector space for any $X, Y \in \mathscr{T}$. Then $\mathscr{T}$ has a strongly unique enhancement.*

PROOF. By [63, Corollary 1.9], we have $\mathscr{T} \cong \mathcal{D}^{\mathrm{b}}(\operatorname{mod}(A))$, where $A := \operatorname{End}(\bigoplus_{i=1}^n E_i)$. Furthermore, such A is a finite ordered quiver with relations, i.e. there exists a finite ordered quiver Q and an ideal $I \subset \mathbb{K}Q$ for which $A \cong \mathbb{K}Q/I$. We now consider the projective submodules P_i, for $i = 1, \dots, n$, as in §4.2. Notice $A = \bigoplus_i P_i$. By [9, Proposition 1.3.6], any indecomposable projective module is isomorphic to some P_i.

We claim that $\operatorname{proj}(A)$, the category of finite-dimensional projective modules of A, is an Orlov category because $\operatorname{Hom}(P_i, P_i) = \mathbb{K}$ and we can define a degree function

$$\deg : \operatorname{ind}(\operatorname{proj}(A)) \to \mathbb{Z} : P_i \mapsto i.$$

Indeed, if $j = \deg(X) \geq \deg(Y) = i$, then $X \cong P_j$, $Y \cong P_i$ and $\operatorname{Hom}(P_j, P_i) = 0$ for $j > i$. By [19, Proposition 4.6] (or [17, Proposition 2.2]), $\operatorname{proj}(A)$ is K-standard. Proposition 5.52 concludes the proof. $\square$

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